Problem. (Proposed by SABIR ILYASS, SAFI, MOROCCO).

Let $\alpha, a, x_1, x_2, \ldots, x_n$ be non-negative real numbers such that $x_1 \times x_2 \times \ldots \times x_n = \alpha^n$

Prove that :

$$\sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (a+x_j)} \geqslant \frac{n\alpha^n}{(a+\alpha)^{n-1}}$$

Solution.

Let $t>0$, according to **AM-GM** inequality we have for all $i \in [\![1, n]\!]$

$$\frac{x_i^n}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (a+x_j)} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a+x_j}{t^n} \geqslant n \frac{x_i}{t^{n-1}}$$

So,

$$\sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (a+x_j)} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a+x_j}{t^n} \geqslant n \sum_{i=1}^{n} \frac{x_i}{t^{n-1}}$$

Therefore

$$\begin{aligned} \sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (a+x_j)} &\geqslant \frac{1}{t^{n-1}} \left(n - \frac{n-1}{t} \right) \sum_{i=1}^{n} x_i - \frac{an(n-1)}{t^n} \\ &\geqslant \frac{n\alpha}{t^{n-1}} \left(n - \frac{n-1}{t} \right) - \frac{an(n-1)}{t^n} \end{aligned}$$

This inequality hold for all $t>0$, we conclude that

$$\sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (a+x_j)} \geqslant \max_{t>0} \varphi_{n,\alpha}(t)$$

With

$$\forall t>0,\, \varphi_{n,\alpha}(t) = \frac{n\alpha}{t^{n-1}} \left(n - \frac{n-1}{t} \right) - \frac{an(n-1)}{t^n}$$

We can easly prove that

$$\max_{t>0} \varphi_{n,\alpha}(t) = \frac{n\alpha^n}{(a+\alpha)^{n-1}}$$

The problem is completely solved, and he equality hold for $x_1 = x_2 = \cdots = x_n = \alpha$