# Probability that $l$ integers are relatively prime.

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Let $l \in \mathbb{N}^{\star}$, denote $\Omega = (\mathbb{N}^*)^l$ , we consider the probability space $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ , where $\mathbb{P}$ is the probability mass function defined by:

$\mathbb{P} : I \in \mathcal{P}(\Omega) \longmapsto \lim\limits_{n \to +\infty} \frac{\text{card}(I \bigcap \Omega_n)}{n^l} \in [0, 1] \quad \text{où } \Omega_n = [\![1, n]\!]^l$

denote

$$A_l = \left\{ (a_1, a_2, \ldots, a_l) \in \mathbb{N}_{\star}^l, \bigwedge_{k=1}^{l} a_k = 1 \right\}$$

and for all $n \geqslant 1$,

$$A_{l,n} = \left\{ (a_1, a_2, \ldots, a_l) \in [\![1, n]\!]^l, \bigwedge_{k=1}^{l} a_k = 1 \right\}$$

Let $p_1, \ldots, p_k$ be prime numbers less than $n$, and $(U_i)_{i \in [\![1,k]\!]}$ a set family defined by:

$$\forall i \in [\![1, k]\!], U_i = \{(a_1, a_2, \ldots, a_l) \in [\![1, n]\!]^l, \forall j \in [\![1, l]\!], p_i | a_j\}$$

We can easily notice that:

$$A_{l,n} = \overline{\bigcup_{i=1}^{k} U_i}$$

To calculate the cardinality, using this formula, we will use the inclusion-exclusion principale.

### Lemma 1. (Poincare's formula)

Let $E_1, \ldots, E_n$ be $n$ finite sets, we have the following equality:

$$\#\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (-1)^{k-1} \#\left(\bigcap_{j=1}^{k} E_{i_j}\right)$$

**Proof.** $\square$

Let $n \geqslant 2$, the proof for $n=2$ is seen above
Suppose that the formula is true for $n$, we show it for $n+1$,
First apply the $n=2$ case,then the distributivity of instersections:

$$\#\left(\bigcup_{i=1}^{n+1} E_i\right) = \#\left(\left(\bigcup_{i=1}^{n} E_i\right) \bigcup E_{n+1}\right)$$

Therefore

$$\#\left(\bigcup_{i=1}^{n+1} E_i\right) = \#\left(\bigcup_{i=1}^{n} E_i\right) + \#(E_{n+1}) - \#\left(\left(\bigcup_{i=1}^{n} E_i\right) \bigcap E_{n+1}\right)$$

This can be compactly written as

$$\#\left(\bigcup_{i=1}^{n+1} E_i\right) = \#\left(\bigcup_{i=1}^{n} E_i\right) + \#(E_{n+1}) - \#\left(\left(\bigcup_{i=1}^{n} (E_i \bigcap E_{n+1}\right)\right)$$

The first and the last terms are n-unions, for which we assumed the formula to hold. Therefore

$$\#\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (-1)^{k-1} \#\left(\bigcap_{j=1}^{k} E_{i_j}\right)$$

And

$$\#\left(\left(\bigcup_{i=1}^{n} (E_i \bigcap E_{n+1}\right)\right) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (-1)^{k-1} \#\left(\bigcap_{j=1}^{k} (E_{i_j} \bigcap E_{n+1})\right)$$

So

$$\#\left(\bigcup_{i=1}^{n+1} E_i\right) \quad = \quad \sum_{k=1}^{n} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} (-1)^{k-1}\#\left(\bigcap_{j=1}^{k} E_{i_j}\right) \quad + \quad \#(E_{n+1}) \quad +$$

$$\sum_{k=1}^{n} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} (-1)^{k}\#\left(\bigcap_{j=1}^{k} (E_{i_j} \bigcap E_{n+1})\right)$$

the right hand side can be rewritten to

$\sum\limits_{k=1}^{n} \sum\limits_{1\leqslant i_1<\cdots<i_k\leqslant n} (-1)^{k-1}\#\left(\bigcap\limits_{j=1}^{k} E_{i_j}\right) + \#(E_{n+1})$ is equal to:

$$\sum_{k=1}^{n+1} \sum_{\substack{1\leqslant i_1<\cdots<i_k\leqslant n+1 \\ i_k\neq n+1}} (-1)^{k-1}\#\left(\bigcap_{j=1}^{k} E_{i_j}\right) + \sum_{k=1}^{n+1} \#(E_k)$$

And

$\sum\limits_{k=1}^{n} \sum\limits_{1\leqslant i_1<\cdots<i_k\leqslant n} (-1)^{k}\#\left(\bigcap\limits_{j=1}^{k} (E_{i_j} \bigcap E_{n+1})\right)$ is equal to:

$$\sum_{k=2}^{n+1} \sum_{\substack{1\leqslant i_1<\cdots<i_k<i_k\leqslant n+1 \\ i_k=n+1}} (-1)^{k-1}\#\left(\bigcap_{j=1}^{k} E_{i_j}\right)$$

We conclude that

$$\#\left(\bigcup_{i=1}^{n+1} E_i\right) \quad = \quad \sum_{k=1}^{n+1} \#(E_k) \quad + \quad \sum_{k=1}^{n+1} \left[ \sum_{\substack{1\leqslant i_1<\cdots<i_k\leqslant n+1 \\ i_k\neq n+1}} (-1)^{k-1}\#\left(\bigcap_{j=1}^{k} E_{i_j}\right) \quad + \right.$$

$$\left. \sum_{\substack{1\leqslant i_1<\cdots<i_k<i_k\leqslant n+1 \\ i_k=n+1}} (-1)^{k-1}\#\left(\bigcap_{j=1}^{k} E_{i_j}\right) \right]$$

So

$$\#\left(\bigcup_{i=1}^{n+1} E_i\right) = \sum_{k=1}^{n+1} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n+1} (-1)^{k-1}\#\left(\bigcap_{j=1}^{k} E_{i_j}\right)$$

which justifies the formula for $n+1$.

**Definition 1.(The Möbius function)**
Let $n \in \mathbb{N}^{\star}$, denote $\mu(n)$ the integer defined by :

$$\mu(n) = \begin{cases} 0 \quad \text{if } n \text{ has } a \text{ squared prime factor.} \\ 1 \text{ if } n \text{ is } a \text{ square} - \text{free positive integer with an even number of prime} \\ \text{factor}s. \\ -1 \text{ if } n \text{ is } a \text{ square} - \text{free positive integer with an odd number of prime} \\ \text{factor}s. \end{cases}$$

According to this lemma, we have

$$\#\left(\bigcup_{i=1}^{k} U_i\right) = \sum_{j=1}^{k} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant k} (-1)^{j-1} \#\left(\bigcap_{m=1}^{j} U_{i_m}\right)$$

To conclude, it suffices to calculate $\bigcap_{m=1}^{j} U_{i_m}$, for all $\leqslant i_1 < \cdots < i_j \leqslant k$

Let $I \subset [\![1,k]\!]$ not empty, the cardinal of the intersection $\bigcap_{i \in I} U_i$ is equal to the number of the $l$-tuples of strictly positive multiples of $\prod_{i \in I} p_i$ less than or equal to $n$, this cardinal is equal to: $\left\lfloor \frac{n}{\prod_{i \in I} p_i} \right\rfloor^l$

The Poincare's formula gives :

$$\#\left(\bigcup_{i=1}^{k} U_i\right) = \sum_{j=1}^{k} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant k} (-1)^{j-1} \left\lfloor \frac{n}{\prod_{m=1}^{j} p_{i_m}} \right\rfloor^l$$

Therefore

$$\# A_{l,n} = n^l - \#\left(\bigcup_{i=1}^{k} U_i\right) = \sum_{d=1}^{n} \mu(d) \left\lfloor \frac{n}{d} \right\rfloor^l$$

So

$$\frac{\#(A_{l,n})}{n^l} = \frac{1}{n^l} \sum_{d=1}^{n} \mu(d) \left\lfloor \frac{n}{d} \right\rfloor^l$$

To continue the proof, we need a fundamental property of Möbius function.

**Proposition 1.**

For all integer $n \neq 1$, we have

$$\sum_{d|n} \mu(d) = 0$$

**Proof.** $\square$

Method 1 : Let $n = \prod_{i=1}^{m} p_i^{a_i}$ lthe prime factorization of $n$,

and if $d \in \mathbb{N}$, we have :

$d|n$ and $\mu(d) \neq 0$ if and only if $d = \prod_{i \in J} p_i^{a_i}$ with $J \subset [\![1, m]\!]$ so $\mu(d) = (-1)^{\#J}$, we conclude that

$$\sum_{d|n} \mu(d) = \sum_{J \subset [\![1,m]\!]} (-1)^{\#J} = (1-1)^m = 0 \,(\text{because } m > 0)$$

Method 2 :Let $n \geqslant 2$, According to the fundamental theorem of arithmetic we have the existence of $(p_1, \ldots, p_r) \in \mathcal{P}^r$ and $\alpha_1, \ldots, \alpha_r \geqslant 1$ such that $n = \prod_{i=1}^{r} p_i^{\alpha_i}$

We have

$$\sum_{d|n} \mu(d) = \sum_{k_1=0}^{\alpha_1} \sum_{k_2=0}^{\alpha_2} \cdots \sum_{k_r=0}^{\alpha_r} \mu\left(\prod_{i=1}^{r} p_i^{k_i}\right)$$

Therefore:

$$\sum_{d|n} \mu(d) = \sum_{\substack{(k_1, \ldots, k_r) \in \prod_{i=1}^{r} [\![0, \alpha_i]\!] \\ \exists i_0 \in [\![1,r]\!]\, k_{i_0} \geqslant 2}} \mu\left(\prod_{i=1}^{r} p_i^{k_i}\right) + \sum_{(k_1, \ldots, k_r) \in [\![0,1]\!]^r} \mu\left(\prod_{i=1}^{r} p_i^{k_i}\right)$$

since for all $(k_1, \ldots, k_r) \in \prod_{i=1}^{r} [\![0, \alpha_i]\!]$ such as $\exists i_0 \in [\![1, r]\!]\, k_{i_0} \geqslant 2$

We have $\prod_{i=1}^{r} p_i^{k_i}$ is divisible by $p_{i_0}^2$ so $\mu\left(\prod_{i=1}^{r} p_i^{k_i}\right) = 0$

which implies that

$$\sum_{\substack{(k_1,\dots,k_r)\in\prod_{i=1}^{r}[\![0,\alpha_i]\!]\\ \exists i_0\in[\![1,r]\!]\,k_{i_0}\geqslant 2}} \mu\left(\prod_{i=1}^{r} p_i^{k_i}\right) = 0$$

Therefore

$$\sum_{d|n}\mu(d) = \sum_{(k_1,\dots,k_r)\in[\![0,1]\!]^r} \mu\left(\prod_{i=1}^{r} p_i^{k_i}\right)$$

For all $(k_1,\dots,k_r)\in[\![0,1]\!]^r$, we have $\sum_{i=1}^{r} k_i$ is the number of distinct prime factors of $\prod_{i=1}^{r} p_i^{k_i}$, and $\prod_{i=1}^{r} p_i^{k_i}$ is not divisible by the square of a prime number, then

$$\sum_{d|n}\mu(d) = \sum_{(k_1,\dots,k_r)\in[\![0,1]\!]^r} (-1)^{\sum_{i=1}^{r} k_i} = \prod_{i=1}^{r}\left(\sum_{k_1=0}^{1}(-1)^{k_i}\right) = (1-1)^r = 0$$

For the asymptotic study of $\frac{\#(A_{l,n})}{n^l}$, it seems natural to replace the term $\frac{1}{n^l}\left\lfloor\frac{n}{d}\right\rfloor^l$ by its equivalent $\frac{1}{d^l}$. The difference between the two sum is written

$$\left|\frac{\#(A_{l,n})}{n^l} - \sum_{d=1}^{n}\frac{\mu(d)}{d^l}\right| = \left|\sum_{d=1}^{n}\mu(d)\left(\frac{1}{n^l}\left\lfloor\frac{n}{d}\right\rfloor^l - \frac{1}{d^l}\right)\right|$$

As $\left\lfloor\frac{n}{d}\right\rfloor > \frac{n}{d} - 1$, We have $\sum_{k=1}^{l}\binom{l}{k}\frac{1}{d^k n^{l-k}} = \left(\frac{1}{d}-\frac{1}{n}\right)^l - \frac{1}{d^l} < \frac{1}{n^l}\left\lfloor\frac{n}{d}\right\rfloor^l - \frac{1}{d^l} \leqslant 0$
Which give

$$\left|\frac{\#(A_{l,n})}{n^l} - \sum_{d=1}^{n}\frac{\mu(d)}{d^l}\right| \leqslant \sum_{d=1}^{n}\sum_{k=1}^{l}\binom{l}{k}\frac{1}{d^k n^{l-k}} = \sum_{k=1}^{l}\binom{l}{k}\frac{1}{n^{l-k}}\left(\sum_{d=1}^{n}\frac{1}{d^k}\right)$$

$$\sum_{k=1}^{l}\binom{l}{k}\frac{1}{n^{l-k}}\left(\sum_{d=1}^{n}\frac{1}{d^k}\right) \underset{n\to+\infty}{\sim} \binom{l}{1}\frac{1}{n^{l-1}}\log(n) + \sum_{k=2}^{l}\binom{l}{k}\frac{1}{n^{l-k}}\zeta(k) = O\left(\frac{1}{n^{l-1}}\log(n)\right)$$

So,

$$\mathbb{P}(A_l) = \lim_{n\to+\infty}\frac{\#(A_{l,n})}{n^l} = \sum_{d=1}^{+\infty}\frac{\mu(d)}{d^l}$$

**Definition 2.**

We define the Riemann zeta function as

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}, \text{ when } \mathfrak{Re}(z) \geqslant 1$$

**Proposition 2.**

For all complex number $z$ such that $\mathfrak{Re}(z) \geqslant 1$, we have:

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^z}$$

**Proof.** $\square$

Let $z \in \mathbb{C}$, with $\mathfrak{Re}(z) \geqslant 1$,we have according to proposition 1. :

$$\zeta(z).\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^z} = \left(\sum_{n=1}^{+\infty} \frac{1}{n^z}\right)\left(\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^z}\right) = \sum_{n,d \geqslant 1} \frac{\mu(d)}{(n.d)^z} = \sum_{n \geqslant 1} \sum_{d|n} \frac{\mu(d)}{n^z} = 1$$

We conclude that

$$\boxed{\mathbb{P}(A_l) = \frac{1}{\zeta(l)}}$$