

# Essential Maths for DTC DPhil Students

Michaelmas Term 2020

## Problem Sheet 12: linear algebra 1

### Introductory problems

1. Write the following system of equations:

$$x + y + z = 7$$

$$2x - y + z = 2$$

$$x - 2y + 2z = 5$$

in the form

a)  $A\mathbf{x} = \mathbf{b}$

b)  $\mathbf{y}B = \mathbf{c}$

Where  $A$  and  $B$  are  $3 \times 3$  matrices, and  $\mathbf{x}, \mathbf{y}, \mathbf{b}$  and  $\mathbf{c}$  are vectors whose size and shape you should carefully indicate.

Check that your answers make sense by expanding your expressions to ensure you get back to the original equations.

2. Given

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 4 \\ 7 & 2 \end{pmatrix}; \quad C = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}; \quad D = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \quad E = \begin{pmatrix} 2 & -1 \end{pmatrix};$$

- a) write down  $a_{21}$ ,  $b_{12}$  and  $c_{22}$

- b) calculate:

i.  $A + A$     ii.  $A - B$     iii.  $4C$

- c) calculate, where possible, or explain why the product is not defined:

i.  $AB$     ii.  $CD$     iii.  $DC$     iv.  $EE$

- d) Do  $A$  and  $B$  commute? Do  $A$  and  $C$  commute?

- e) Does  $(AB)C = A(BC)$ ? Does this either prove or disprove that matrix multiplication is associative?

- f) Does  $AC + BC = (A + B)C$ ? Does this either prove or disprove the distributive property of matrices?

### Main problems

1. If  $A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , find  $A^2$  and  $A^3$  and comment on your results.

2. Given  $A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & -1 & 1 \\ -5 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix},$

Find  $AB$ ,  $BA$ ,  $A^T B^T$ ,  $B^T A^T$ ,  $(AB)^T$ ,  $(BA)^T$ . Comment on your results.

3. Find the determinant,  $|A|$ , of the following matrices:

a)  $A = \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix}$

b)  $A = \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}$

4. Find the inverse,  $A^{-1}$ , of the following matrices:

a)  $A = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$

b)  $A = \begin{pmatrix} -3 & 2 \\ -1 & 7 \end{pmatrix}$

5. Use Python's `numpy.linalg.solve` to solve the following systems of equations:

	$x + 2y - 3z = 9,$	$x + 5y + 3z = 17,$	$2y + z = -8,$
a)	$2x - y + z = 0,$	b) $5x + y - 2z = 4,$	c) $x - 2y - 3z = 0,$
	$4x - y + z = 4.$	$x + 2y + z = 7.$	$-x + y + 2z = 3.$

```
# hint
import numpy as np

A = np.array([[1,2,-3], [2,-1,1], [4,-1,1]])
b = np.array([9,0,4])

x = np.linalg.solve(A, b)

print(x)
```

6. Given  $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $Y = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ ,

calculate  $XY$ ,  $YX$ ,  $X^{-1}$ ,  $Y^{-1}$ ,  $X^{-1}Y^{-1}$ ,  $Y^{-1}X^{-1}$ ,  $(XY)^{-1}$ ,  $(YX)^{-1}$ . Comment on your results.

### Extension problems

1. Show that

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \lambda_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \lambda_2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

where  $\lambda_1$  and  $\lambda_2$  are constants to be determined.