Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 15: systems of differential equations 2

We will return to these questions on the next sheet.

Introductory problems

1. Find the fixed points of the following linear systems:

   a) $\dot{x} = x + 3y, \qquad \dot{y} = -6x + 5y;$

   b) $\dot{x} = x + 3y + 4, \qquad \dot{y} = -6x + 5y - 1;$

   c) $\dot{x} = x + 3y + 1, \qquad \dot{y} = -6x + 5y.$

2. Find the fixed points of the following nonlinear systems:

   a) $\dot{x} = -4y + 2xy - 8 \qquad \dot{y} = 4y^2 - x^2;$

   b) $\dot{x} = y - x^2 + 2, \qquad \dot{y} = 2(x^2 - y^2).$

Main problems

1. Consider the chemical reaction network

$$\xrightarrow{k_0} A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B \xrightarrow{k_2}$$

   a) Write down the system of two linear ODEs which describe the evolution of the concentrations of A and B in this system under the law of mass action.

   b) Find the ratio of concentrations of A and B for which this system is in steady state: that is the concentrations do not change over time.

2. Consider the reversible enzyme reaction

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} P + E$$

   Verify the Haldane relation, which states that when the reaction is in equilibrium,

$$\frac{p}{s} = \frac{k_1 k_2}{k_{-1} k_{-2}},$$

   where p and s are the concentrations of P and S, respectively.

3. The population of a host, $H(t)$, and a parasite, $P(t)$, are described approximately by the equations

$$\frac{\mathrm{d}H}{\mathrm{d}T} = (a - bP)H, \qquad \frac{\mathrm{d}P}{\mathrm{d}T} = (c - \frac{dP}{H})P, \qquad H > 0,$$

   where a, b, c, d are positive constants. By a suitable change of scales show that these equations may be put in the simpler form

$$\dot{y} = (1 - x)y, \qquad \dot{x} = \alpha x(1 - \frac{x}{y}),$$

where $\alpha = \dfrac{c}{a}$.

Sketch the phase flow across the following lines:

a) $y = x$;

b) $x = 0$;

c) $y = 0$;

d) $x = 1$;

e) $y = \beta x$, for β greater than and less than 1.

4. Consider a lake with some fish attractive to anglers. We wish to model the fish-angler interaction under the following assumptions:

   - the fish population grows logistically in the absence of fishing;
   - the presence of anglers depresses the fish growth rate at a rate jointly proportional to the size of the fish and angler populations;
   - anglers are attracted to the lake at a rate directly proportional to the number of fish in the lake;
   - anglers are discouraged from the lake at a rate directly proportional to the number of anglers already there.

   a) Write down a mathematical model for this situation, clearly defining your terms.

   b) Use a suitable scaling to show that a non-dimensionalised version of the model is

$$\dot{x} = rx(1 - x) - xy, \qquad \dot{y} = \beta x - y$$