Essential Maths for DTC DPhil Students

Michaelmas Term 2020

Problem Sheet 10: differential equations 2

Introductory problems

1. Find the general solutions of the following differential equations:

   a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 + e^x$

   b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2}{x}$

   c) $\dfrac{1}{x}\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{4x}$

   d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2xy$

   Check your answers by differentiating them.

Main problems

1. A circular patch of oil on the surface of some water has a radius r metres at time t minutes. When $t = 0$ minutes, $r = 1\,\mathrm{m}$ and when $t = 10$ minutes, $r = 2\,\mathrm{m}$.

   a) redict the value T of t when $r = 4\,\mathrm{m}$, using a simple model in which the rate of increase of r is taken to be constant. Find T for this model.
   b) In a more refined model, the rate of increase of r is taken to be proportional to $1/r$. Express this statement as a differential equation, and find the general solution. Find T for this model.
   c) Compare the two models used in a) and b), by sketching $r(t)$ on the same figure or plotting using Python. Comment on the differences seen during different time intervals.

2. A nuclear installation local to Oxford 'lost' 17g of Cobalt-60 between two inspections 6 months apart. A spokesperson told a newspaper reporter that the loss was due to 'natural radioactive decay' during that period.

   If this explanation was correct, what mass of Cobalt was stored on the site? (The half life of $^{60}\mathrm{Co} =$ 5.26 years.)

3. By making a substitution $z = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, solve the equation

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (1-3x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \quad \text{with} \quad y(2) = 0 \quad \text{and} \quad y'(2) = 1/6.$$

   Hint: you will need to use partial fractions for part of this question, to break up

   $$\frac{1}{x^3 - x} = \frac{-1}{x} + \frac{1/2}{x+1} + \frac{1/2}{x-1}.$$

4. Suppose a cell contains a chemical (the solute) dissolved in it at a concentration of $c(t)$, and the concentration of the same substance outside the cell is a constant k. By Fick's law, if $c(t)$ and k are unequal, solute moves across the cell wall at a rate proportional to the difference between $c(t)$ and k, towards the region of lower concentration.

   a) Write down a differential equation which is satisfied by $c(t)$.

b) Solve this differential equation with the initial condition $c(0) = c_0$.

c) Sketch the solutions for $c_0 > k$ and $k > c_0$.

d) Blood glucose concentration is 5.1 mM and the concentration inside the cell is at 0.1 mM. If glucose utilisation within the cell is totally inhibited, it takes 1 min for the intracellular concentration to reach 2.6 mM. How long would it take for the concentration to reach 5.0 mM?

e) Calculate the amount of glucose (moles) entering the cell to bring the concentration from 0.1 mM to 5 mM assuming the cell is spherical with a diameter of $10\,\mu$m.

Extension problems

1. Consider a simple model of the production and degradation of a protein, shown by the reaction chain

$$\xrightarrow{k_1} S \xrightarrow{k_2}$$

where k_1 and k_2 are mass action coefficients.

a) Form and solve a differential equation describing the change in protein concentration.

b) What concentration is reached after 'sufficient' time has elapsed?