

# Essential Maths for DTC DPhil Students

Michaelmas Term 2021

## Problem Sheet 7: integration 2

### Introductory problems

1. By using suitable substitutions, evaluate the following integrals:

a)  $\int x^2(x^3 + 4)^2 \, dx$

b)  $\int e^{-x}(5 - 4e^{-x}) \, dx$

c)  $\int (1 + x)\sqrt{(4x^2 + 8x + 3)} \, dx$

d)  $\int 3xe^{(x^2+1)} \, dx$

2. Find the indefinite integrals, with respect to  $x$ , of the following functions:

a)  $x e^{3bx}$

b)  $x^3 e^{-3x}$

c)  $x \cos(x)$

d)  $e^{bx} \sin(x)$

3. Sketch the curve  $y = (x - 2)(x - 5)$  and calculate by integration the area under the curve bounded by  $x = 2$  and  $x = 5$ .

### Main problems

1. Evaluate the following indefinite and definite integrals:

a)  $\int \frac{6}{(7 - x)^3} \, dx$

b)  $\int 13x^3(9 - x^4)^5 \, dx$

c)  $\int_2^5 5 \log(x) \, dx$

d)  $\int x^x (1 + \log(x)) \, dx$

2. Suppose the area  $A(t)$  (in  $\text{cm}^2$ ) of a healing wound changes at a rate

$$\frac{dA}{dt} = -4t^{-3},$$

where  $t$ , measured in days, lies between 1 and 10, and the area is  $2 \text{ cm}^2$  after 1 day. What will the area of the wound be after 10 days?

3. A rocket burns fuel, so its mass decreases over time. If it burns fuel at a constant rate  $\rho \text{ kg/s}$ , and if the exhaust velocity relative to the rocket is a constant  $v_e \text{ m/s}$ , then there will be a constant force of magnitude  $\rho v_e$  propelling it. The rocket starts burning fuel at  $t = 0 \text{ s}$  with total mass of  $m_0 \text{ kg}$ , and runs out of fuel at a later time  $t = t_f \text{ s}$ , with a final mass of  $m_f \text{ kg}$ .

a) Newton's second law tells us that the instantaneous acceleration  $a$  of the rocket at time  $t$  is equal to the force propelling it at that time, divided by its mass at that time. Write down an expression for  $a$  as a function of  $t$ .

b) By integrating this expression, show that the rocket's total change in velocity is given by  $v_e \ln \left( \frac{m_0}{m_f} \right)$ .

4. The flow of water pumped upwards through the xylem of a tree,  $F$ , is given by:

$$F = M_0(p + qt)^{3/4},$$

where  $t$  is the tree's age in days,  $p$  and  $q$  are positive constants, and  $M_0 p^{3/4}$  is the mass of the tree when planted (i.e. at  $t = 0$ ). Determine the total volume of water pumped up the tree in its tenth year (ignoring leap years) if  $p = 10$ ,  $q = 0.01 \text{ day}^{-1}$ , and  $M_0 = 0.921 \text{ day}^{-1}$ .

### Extension problems

- Express  $\frac{1}{x(x^2 - 16)}$  in the form  $\frac{A}{x} + \frac{B}{(x + 4)} + \frac{C}{(x - 4)}$ . Hence calculate  $\int \frac{1}{x(x^2 - 16)} dx$ .
- The probability that a molecule of mass  $m$  in a gas at temperature  $T$  has speed  $v$  is given by the Maxwell-Boltzmann distribution:

$$f(v) = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT}$$

where  $k$  is Boltzmann's constant. Find the average speed:

$$\bar{v} = \int_0^\infty v f(v) dv.$$

- Baranov developed expressions for commercial yields of fish in terms of lengths,  $L$ , of the fish. His formula gave the total number of fish of length  $L$  as  $k e^{-cL}$ , where  $c$  and  $k$  are constants ( $k$  is positive).
  - Give a sketch of the graph  $f(L) = k e^{-cL}$ . (Something decreasing, concave upward and asymptotic to horizontal axis will do.) On your sketch, introduce marks on the horizontal axis that represent lengths  $L = 1, L = 2, L = 3, L = 4$  and  $L = 5$ . Now draw a rectangle on your sketch that represents the number of fish whose lengths are between  $L = 3$  and  $L = 4$ .
  - Explain how we can represent the total number of fish  $N$  as an area. Show that this number equals  $k/c$ .
  - Only fish longer than  $L_0$  count as commercial. Hence, assuming that the fish are all similar in shape (i.e. their width and breadth scales with their length) and of equal density  $\rho$ , show that the weight,  $W$ , of the commercial fish population is

$$W = \int_{L_0}^{+\infty} a k \rho L^3 e^{-cL} dL,$$

and hence that

$$W = \frac{N a \rho e^{-cL_0}}{c^3} ((cL_0)^3 + 3(cL_0)^2 + 6cL_0 + 6),$$

where  $a$  is a constant.