Essential Maths for DTC DPhil Students

Michaelmas Term 2021

Problem Sheet 9: differential equations 1

Introductory problems

1. Find the general solutions of the following differential equations:

   a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x$

   b) $\dfrac{\mathrm{d}r}{\mathrm{d}t} = -\sin(\pi t)$

   c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = bx^2$

   d) $(x-4)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3y$

   e) $u\,\dfrac{\mathrm{d}u}{\mathrm{d}v} = v+6$

   f) $7e^x\,\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{y}$

   Check your answers by differentiating them.

2. Find the solution to the following differential equations subject to the specified boundary conditions:

   a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x} \quad \text{with} \quad y(2) = 0$

   b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y \quad \text{with} \quad y(0) = 1$

   Use Python's `scipy.integrate.odeint` to verify your solutions

```python
# hint
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt

# you need a function that calculates dy/dt
def dydx(y,x):
    return 1 / x

y0 = 0  # <-- the y-value of the initial condition
x0 = 2  # <-- the x-value of the initial condition

# the x-values at which to calculate the solution
x = np.linspace(x0, x0 + 10, 1000)

# solve ODE numerically
y = odeint(dydx, y0, x)

# plot the numerical solution and your hand-calculated
# solution, and check that they agree
```

Main problems

1. The number of bacteria present in a given culture increases at a rate proportional to the number present. When first observed, the culture contained n_0 bacteria, and two hours later it contained n_1.

   a) Find the number present t hours after observations began.

   b) How long did it take for the number of bacteria to triple?

   c) Sketch a curve of the solution to the equation that you derive.

   d) What assumptions are implicit in this model of bacterial growth?

2. Solve:

   a) $y^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2}{3}x \quad \text{with} \quad y(\sqrt{2}) = 1$

   b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\beta}{x} \quad \text{with} \quad y(1) = 0$. Find β such that $y(e^3) = 1$.

   c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = a + bx + cx^2 + dx^3 + ex^4 \quad \text{with} \quad y(0) = \pi$

3. In a certain chemical reaction, substance A is transformed into product P. The mass of A at any given time, t, is m_t, and the rate of transformation of A at time t is proportional to m_t. Given that the original mass of A is 130g, and that 50g has been transformed after 150 seconds:

   a) Form and solve the differential equation relating m_t to t.

   b) Find the mass of A transformed over a 300s period.

   c) Sketch a graph of m_t versus t.

4. Newton's law of cooling states the the rate of decrease of the temperature of a body is proportional to the amount by which its temperature exceeds the temperature of its surroundings. If T_0 is the initial temperature of a body, T_s is the temperature of its surroundings, and T is the temperature of the body at time t:

   a) Form a differential equation for Newton's Law of cooling.

   b) Show that $T - T_s = (T_0 - T_s)\, e^{-kt}$, where k is a constant, and state the units of the constant k.

   c) Glycerol is to be added to a protein sample prior to storage.
   The glycerol is heated to 65 °C to aid accurate pipetting. To avoid denaturation of the sample, the glycerol must then be allowed to cool to below 29 °C before being added to the protein. If the ambient temperature is 22 °C, the glycerol cools to $T = 59$ °C at time $t = 2$ minutes. At what time can the glycerol be added to the protein?

   d) Using a choice of axes that will allow you easily to predict the temperature of the glycerol, sketch a graph of the anticipated variation of the glycerol temperature with time.

   e) Once the glycerol has been added to the protein, will the rate of cooling be described by the same constant k? Give reasons for your answer.

5. The amount of ^{14}C (radioactive carbon-14) in a sample is measured using a Geiger counter, which records each disintegration of an atom. The rate at which ^{14}C decays is proportional to the amount present.

The half-life of ^{14}C is about 5730 years. This means that half of the sample will have disintegrated after 5730 years.

In living tissue, ^{14}C disintegrates at a rate of about 13.5 atoms per minute per gram of carbon. Because living tissue is constantly exchanging carbon with its environment, the proportion of ^{14}C among its carbon atoms remains constant over time. Once the tissue is no longer living, this constant exchange of carbon ceases and the fraction of ^{14}C among its carbon atoms begins to get smaller. Consequently, the disintegration rate drops.

In 1977 a charcoal fragment found at Stonehenge on the Salisbury Plain recorded 8.2 disintegrations per minute per gram of carbon: about 60% of that for living tissue. Assuming that the charcoal was formed during the building of the site, use this information to estimate the date at which Stonehenge was built.

Extension problems

1. The *absorbance* A of a solution is given by the equation:

$$A = \log_{10}\left(\frac{I_o}{I}\right)$$

where I_o is the intensity of the light impinging on the solution (incident light) and I is the intensity of the light emerging from it (transmitted light). The Beer-Lambert law states that

$$A = \epsilon \cdot c \cdot l$$

where ϵ is the absorbance of the solute, c is the concentration of the solute and l is the distance that the light has travelled through the solution.

a) The *transmittance* T is defined as the fraction of incident light transmitted through the solution ($T = \dfrac{I}{I_o}$). Derive an expression relating the transmittance, T, of the solution to ϵ, c and l.

b) The *attenuation* Q of the light beam is defined as the difference between the intensities of the incident and the transmitted light ($Q = I_o - I$). Derive an expression for the attenuation of the light beam when a beam of light intensity I_o traverses a distance l through a solution of fixed concentration c. Sketch a graph showing the dependence of Q on l in a solution of fixed concentration.

c) ATP has a molar absorbtion of $15.7 \times 10^3\,\mathrm{M}^{-1}\mathrm{cm}^{-1}$. Calculate the initial rate (in watts/cm) at which light intensity is attenuated when a light beam of intensity 200 watts enters a $10\mu\,\mathrm{M}$ solution of ATP. What would happen to this rate if

i. the concentration of ATP is doubled;

ii. the intensity of the incident light is doubled;

iii. the length of the cell holding the solution is doubled?