Essential Maths for DTC DPhil Students

Michaelmas Term 2022

Problem Sheet 16: systems of differential equations 3

Main problems

These questions are extensions of questions on the previous sheet.

1. Classify the fixed points and discuss stability of the following linear systems:

   a) $\dot{x} = x + 3y, \qquad \dot{y} = -6x + 5y;$

   b) $\dot{x} = x + 3y + 4, \qquad \dot{y} = -6x + 5y - 1;$

   c) $\dot{x} = x + 3y + 1, \qquad \dot{y} = -6x + 5y.$

2. Classify the fixed points and discuss stability of the following nonlinear systems:

   a) $\dot{x} = -4y + 2xy - 8 \qquad \dot{y} = 4y^2 - x^2;$

   b) $\dot{x} = y - x^2 + 2, \qquad \dot{y} = 2(x^2 - y^2).$

3. The population of a host, $H(t)$, and a parasite, $P(t)$, are described approximately by the equations

$$\frac{\mathrm{d}H}{\mathrm{d}T} = (a - bP)H, \qquad \frac{\mathrm{d}P}{\mathrm{d}T} = (c - \frac{dP}{H})P, \qquad H > 0,$$

   where a, b, c, d are positive constants. Previously, by a change of scales, these equations were put in the simpler form

$$\dot{y} = (1 - x)y, \qquad \dot{x} = \alpha x(1 - \frac{x}{y}),$$

   where $\alpha = \dfrac{c}{a}$.

   Find and classify the fixed points of these simplified equations. Sketch the phase flow diagram including these fixed points and the information from the previous sketch. That is, include the flow across:

   a) $y = x;$

   b) $x = 0;$

   c) $y = 0;$

   d) $x = 1;$

   e) $y = \beta x$, for β greater than and less than 1.

4. In the previous parts of this question a model of fish and anglers was developed and simplified. This question analyses the simplified model.

   The simplified version of the model is

$$\dot{x} = rx(1 - x) - xy, \qquad \dot{y} = \beta x - y$$

   where x and y represent fish and angler populations, respectively. The $\dot{x}$ notation represents the derivative with respect to non-dimensional time.

a) Calculate the steady states of the system.

b) Determine the stability of the fixed points in the case $\beta = r = 4$.

c) Draw the phase plane, including the nullclines and phase trajectories.

Extension problems

1. A population F of foxes feeds on a population H of hares. A model for the changes in the population is given by

$$\frac{\mathrm{d}H}{\mathrm{d}T} = aH - bHF \qquad (H > 0),$$
$$\frac{\mathrm{d}F}{\mathrm{d}T} = cHF - dF \qquad (F > 0).$$

a) Define the four variables a, b, c, d.

b) Find the fixed point of the system and describe the motion in its neighbourhood.

c) What does this mean in terms of the population of hares and foxes?.