Problems 18: Probability Distributions

1 Probability

Exercise 1. Suppose a collector has 1000 postage stamps, 300 of which are pretty and 100 rare, with 30 being both pretty and rare. Calculate P(rare|pretty) and P(rare) within his collection. Now suppose he puts on a display of the stamps which are either rare or pretty (or both). What are the probabilities above within just this display? Comment on your results.

Exercise 2. (Assessed) A squirrel caches nuts for the winter in two locations. He has a 56% chance of remembering the location of the first cache, and an independent 25% chance of recalling the location of the second.

(a) What is the probability that he will recover both caches?

(b) What is the probability that he will recover at least one cache?

(c) If he recovers at least one cache, what is the probability that it was the first?

(d) (Extension) What is the expected value of the number of caches recovered?

Exercise 3. In a large population, 49% of the organisms express a phenotype known to be recessive, while the remainder display the phenotype of the dominant allele; there are only 2 alleles for the gene. A mother who expresses the recessive trait has mated with a male expressing the dominant trait. Assuming breeding is random,

(a) What proportion of the population is likely to be heterozygous?

(b) How likely is the male to be heterozygous?

(c) What is the probability that their first offspring will express the recessive trait?

(d) If the first offspring does not, what is the probability that the second offspring will?

Exercise 4. (Extension) A researcher is saving up to buy a sequencing machine costing N units of money. He has k units where $0 < k < N$, and tries to win the remainder by the following gamble with his bank manager. He tosses a fair coin repeatedly; if it comes up heads then the manager pays him one unit of money, but if it comes up tails then he pays the manager one unit. He plays the game repeatedly until either he runs out of money or is able to purchase the sequencer. What is the probability that he is eventually able to buy the sequencer?

2 Probability Distributions

Exercise 5. Consider a standard, fair, six-sided die

(a) Write the probability function of a random variable describing the possible 'number of spots' observed on a single roll of the die.

(b) What is the probability of obtaining a 6?

(c) What is the probability that the number of spots is even?

(d) What are the mean and variance of the possible rolls of the die?

(e) Calculate the probability of obtaining exactly 2 'sixes' in 4 throws of the die.

(f) Calculate the probability of obtaining at least 2 'sixes' in 4 throws of the die.

Exercise 6. Consider the continuous Uniform probability distribution for a random variable $X \sim U(a, b)$

$$f(x|a,b) = \begin{cases} \frac{1}{b-a} & \text{if } a < x \leq b \\ 0 & \text{otherwise} \end{cases},$$

Given two points x_1 and x_2 that lie between a and b, find the probability $P(x_1 \leq X \leq x_2)$, that a sample of X lies in the interval $x_1 \leq x \leq x_2$.

Exercise 7. The variable X can have any value in the continuous range $0 \leq x \leq 1$ with probability density function $\rho(x) = 6x(1-x)$.

(a) Derive an expression for the probability, $P(X \leq a)$, that the value of x is not greater than a.

(b) Confirm that $P(0 \leq X \leq 1) = 1$.

(c) Find the expectation $E(X)$ and the variance σ^2

(d) Find the probability that $E(X) - \sigma \leq x \leq E(X) + \sigma$

Exercise 8. The height of a species of plant grown from seed in the lab is normally distributed with mean 136 cm and standard deviation 12 cm.

(a) What proportion of these plants will be greater than 150 cm tall?

(b) What is the probability of a randomly sampled plant being more than 120 cm tall?

(c) After thinning, in which all plants shorter than 120 cm are removed, what is the probability that one of the remaining plants will be taller than 150 cm?

Exercise 9. Find the probability that none of the 3 bulbs in a traffic signal must be replaced during the first 1200 hours of operation, if the probability that a bulb must be replaced is a random variable X with density function $f(x) = 6(0.25 - (x-1.5)^2)$ when $1 \leq X \leq 2$ (and $f(x) = 0$ otherwise), where x is time measured in multiples of 1000 hours.

Exercise 10. In the radioactive decay of a particular element, the time until the next event is exponentially distributed, i.e. it has the probability density function $f(t) = Ae^{-3t}$, where $t \geq 0$ is measured in minutes.

(a) Determine the value of A, and hence the expected time until the next event.

(b) How does A relate to the half life $t_{1/2}$ of the material (Hint: $P(X > t_{1/2}) = \frac{1}{2}$)?

(c) What is the probability that no disintegrations will occur in the next minute?

(d) (Extension) If there are 1,000,000 atoms of the material present initially, how many do you expect to decay in the first 10 minutes?

Exercise 11. A fish searches a river bed for food, covering an area of 30 m^2 in 15 minutes. Suppose there are 2 pieces of food (A and B) randomly positioned within this area, and that consuming food takes negligible time.

(a) What is the cumulative probability function that the fish finds piece A after t seconds? What about for piece B?

(b) What is the expected time to find food A?

(c) What is the cumulative probability that the fish finds *both* pieces of food within t seconds? What is the expected time taken to find both pieces?

(d) Repeat part (c) for finding either piece of food.