

# Essential Maths for DTC DPhil Students

Michaelmas Term 2022

## Problem Sheet 14: systems of differential equations 1

### Introductory problems

1. Find the general solution to the following system of ODEs:

$$\frac{dx}{dt} = x, \quad \text{and} \quad \frac{dy}{dt} = y.$$

Sketch the form of the solution in the  $x, y$  plane, using arrows to indicate where the solution moves over time.

2. Take the general decoupled linear system

$$\frac{dx}{dt} = ax, \quad \text{and} \quad \frac{dy}{dt} = by.$$

- a) Integrate the two equations separately to solve for  $x$  and  $y$  in terms of  $t$ .
- b) If you start at  $t = 0$ ,  $x(0) = 0$ ,  $y(0) = 0$  what happens to the solution over time?
- c) If you start at a general position  $x(0) = x_0$ ,  $y(0) = y_0$  what happens to the solution as  $t \rightarrow \infty$ ? What if  $a$  and  $b$  are both negative? What if only one of  $a$  or  $b$  is negative? What if either  $x_0$  or  $y_0$  is negative?
- d) Either by eliminating  $t$  from the original equations or by eliminating  $t$  from your solutions to part a) find a general solution of the system. (Why not try both methods?) Sketch this solution on the phase plane for
  - i.  $a > 0$ ,  $b > 0$ ,  $a = b$
  - ii.  $a > 0$ ,  $b < 0$ ,  $a = -b$ .

### Main problems

1. By reformulating the following system as one first order equation (i.e eliminating  $t$ ), find the general solution to:

$$\frac{dx}{dt} = -y, \quad \text{and} \quad \frac{dy}{dt} = x.$$

Sketch the form of the solutions in the  $x, y$  plane.

2. Again by eliminating  $t$  and reformulating the system as one first order equation, find the general solution to the following system of ODEs:

$$\frac{dx}{dt} = y, \quad \text{and} \quad \frac{dy}{dt} = x.$$

Sketch the form of the solutions in the  $x, y$  plane.

3. Find the eigenvalues and two independent eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  of the matrix  $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ .
  - a) Put the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  as the columns of a  $2 \times 2$  matrix  $P$ . Find  $P^{-1}$  and show (by calculation) that  $P^{-1}AP$  is diagonal. What are the entries of this matrix? What do they correspond to?
  - b) Find the general solution of the system  $\frac{dx}{dt} = x + 4y$ , and  $\frac{dy}{dt} = x + y$ .
  - c) Find the particular solution subject to  $x(0) = 0$  and  $y(0) = 2$ .

- d) Sketch the trajectory (the  $x(t)$ ,  $y(t)$  coordinates over time) in the  $x, y$  plane. Draw the eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  on the same figure. What happens as  $t \rightarrow \infty$ ? What about  $t \rightarrow -\infty$ ? What is  $\frac{dy}{dx}$  at a general point on the  $y$ -axis?

### Extension problems

1. The force on a damped harmonic oscillator is  $f = -kx - m\nu \frac{dx}{dt}$ , where  $x$  is a displacement,  $k > 0$  is a spring force constant,  $m > 0$  is the mass and  $\nu > 0$  is the strength of the damping.
  - a) Use Newton's 2nd law of motion to write down an equation for the acceleration  $\frac{d^2x}{dt^2}$ .
  - b) Make the substitution  $y = \frac{dx}{dt}$  (and hence  $\frac{dy}{dt} = \frac{d^2x}{dt^2}$ ) to obtain a system of two first-order linear ODEs.