Essential Maths for DTC DPhil Students

Michaelmas Term 2022

Problem Sheet 3: differentiation 1

Introductory problems

1. Use the formula $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)$ to calculate the derivatives of the functions below. Check your answers by using the standard rules for differentiation:
   a) $y = 3x + 3$
   b) $y = 4x^2 - 3x + 2$
   c) $y = 2x^3 - 5$
   d) $y = \dfrac{1}{x^2}$ (harder)

2. Find the gradient at the given points of the following curves:
   a) $y = x^3 - 4$ where $x = 1$
   b) $y = 3x^3 + 4x - 3$ where $x = -2$

3. Find the x and y coordinates of the points on the given curves at which the gradient is zero and find out whether they are maxima, minima or points of inflexion:
   a) $y = (x-3)(x-4)$
   b) $y = x^3 - 4x^2 + 2x - 2$
   c) $y = \dfrac{4x+1}{x}$
   d) $y = 16 - 2x^3$

Main problems

1. One hour after taking x mg of a drug, the body temperature, T, in °C of a patient is given by:

$$T = T_0 - 0.00625\, x^2(18 - x),$$

   where T_0 is the initial body temperature.
   a) Determine the value of x that produces the greatest drop in body temperature, and the magnitude of that temperature change.
   b) Sketch T as a function of the concentration.

2. The formula for the Lennard Jones potential between two non polar atoms is given in terms of the positive constants A and B and the internuclear distance, R, as:

$$V(R) = \frac{A}{R^{12}} - \frac{B}{R^6}$$

   a) Use this formula to calculate $\dfrac{\mathrm{d}V}{\mathrm{d}R}$ as a function of R.

b) Show that the potential where the gradient is zero is $V(R) = \dfrac{-B^2}{4A}$.
c) Find mathematically whether this point is a maximum, minimum or point of inflexion.

3. The number n (in thousands) of bacteria on an agar plate at time t days is given by the expression:

$$n = 21.35 + 1.34t - t^2$$

a) Draw a graph of the function $n(t)$ between $t = 0$ and $t = 7$ days. Give one reason why this function might be a reasonable model for the number of bacteria on the plate at time t. Are there any values of t for which this is probably *not* a good model?
b) Calculate the time at which the greatest number of bacteria are present on the plate and show that this must be a maximum number.
c) By finding the roots of the equation for n, find the two times at which the value of n is zero and say why only one of these times is physically reasonable. Mark and label the maximum point on your graph together with the point at which the number of bacteria is zero.
d) Find the rates at which the bacteria are growing when $t = 0.8$ and $t = 3.5$ days.

4. Being able to change direction rapidly helps fish to avoid predators. We can describe the distance x (in cm) travelled by time t seconds after a stimulus by

$$x = kt^b,$$

where for rainbow trout $k = 300\,\mathrm{cm\,s^{-1.6}}$, $b = 1.60$, and for green sunfish $k = 210\,\mathrm{cm\,s^{-1.71}}$, $b = 1.71$.

a) Compare the distance travelled, and the instantaneous velocity and acceleration for these species at $t = 0.1\,\mathrm{s}$.
b) Compare the velocities in fish lengths per second, given that the lengths of trout and sunfish are 14.4 cm and 8.0 cm respectively, commenting on your answer.

5. A researcher measured the concentration c of a protein *in vitro* and obtained the readings below:

time (min)	0	1	2	3	4	5	6
c (mM)	11.91	7.06	4.40	2.57	1.81	1.03	0.72

She surmised that the protein was being degraded according to the reaction scheme

$$A \xrightarrow{k}$$

under mass action kinetics.

a) Use a suitable transformation to draw a straight-line graph of the data, in order to test her hypothesis.
b) Find the maximum rate of decay and the time at which this occurs.
c) Find the concentration of protein remaining after 10 minutes.

Extension problems

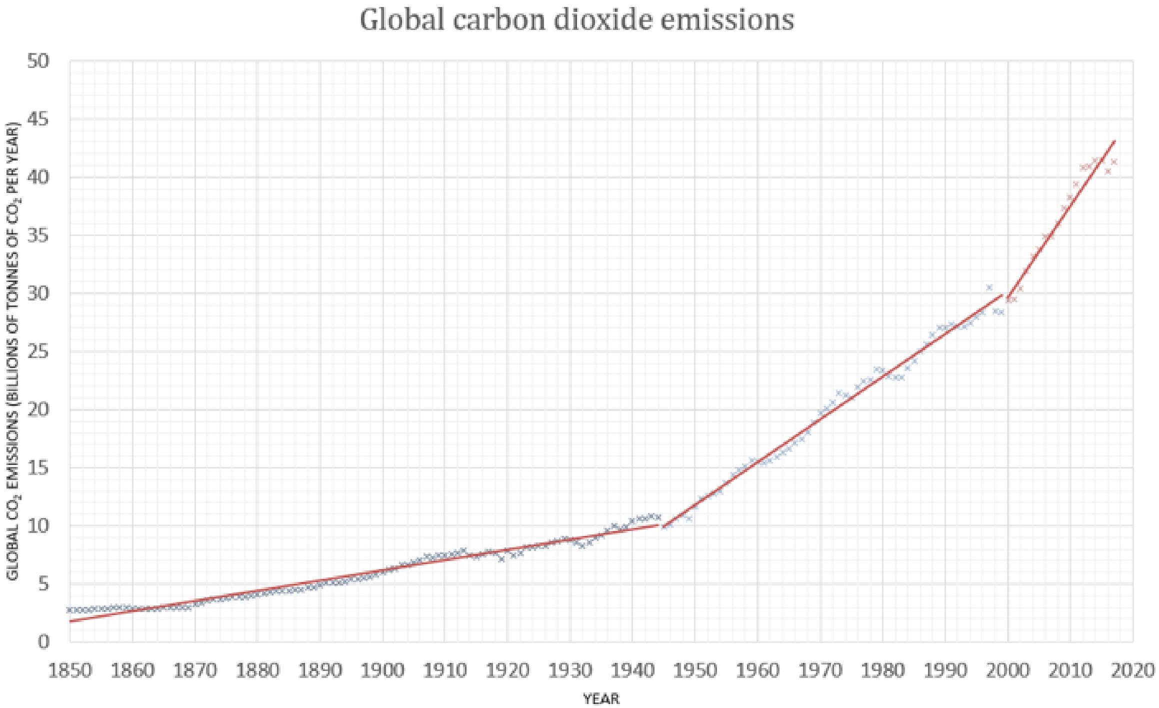
1. Find the values of x for which the following functions have stationary values and using your results, sketch a graph of each function:

a) $y = 2e^x - x$

b) $y = e^x - 2x - 1$

Use Python to check your results.

2. The graph shows the rate of CO_2 emissions per year since 1800, with three fitted lines in red.



A climate scientist thinks that a quadratic curve could be a better fit to the data, with the x-axis as years since 1850, and the y-axis as the rate of CO_2 emissions per year. The curve would have the equation $y = Ax^2 + Bx + C$

a) By using the points $(10, 3), (106, 14), (166, 42)$ taken from the best fit lines, evaulate the coefficients A, B, and C in this model.

b) Find the minima of this quadratic curve. Use this to assess the suitability of the quadratic model as a fit for the data.

3. A protein degrades according to the formula

$$p(t) = \frac{1}{2kt + \frac{1}{P_0}}$$

where p is the protein concentration, P_0 is the initial concentration, k is a constant, and t is time.

Find the rate for this reaction and deduce a plausible reaction schema. Hint: you may find it useful to express the derivative in terms of $p(t)$.

4. From first principles (as in the first question on this sheet) prove the formulas for differentiation of sums, differences and scalar multiples.