Bioen 7310 PS. 1 Samuel Colby

1. Write the derivation of the central slice theorem or 'projection?'

The theorum states that we can take a two-dimensional function, project it one-dimensionally, take the FT of it. This result would be equal to if we took the same two dimensional function, and take a 2-D FT of it, then slice thru the origin, parallel to the original projection. E.G.

$P(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$, where $P(x)$ is the projection of $f(x,y)$ onto x axis.

$$F(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i x k_x} e^{-2\pi i y k_y}\,dx\,dy.$$

The slice thru the origin is then given by:

$$F(k_x, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i x k_x}\,dx\,dy = \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)}_{P(x)} e^{-2\pi i x k_x}\,dx$$

$$F(k_x, 0) = \int_{-\infty}^{\infty} P(x) e^{-2\pi i x k_x}\,dx = FT\{P(x)\} \quad \checkmark$$

2. Find closed form of $S = \sum_{j=0}^{N-1} x^j = 1 + x + x^2 + x^3 + \dots + x^{N-1}$

$\Rightarrow xS = x + x^2 + x^3 + \dots + x^N$

$\Rightarrow S - xS = (1 + \cancel{x} + \cancel{x^2} + \dots + \cancel{x^{N-1}}) - (\cancel{x} + \cancel{x^2} + \cancel{x^3} + \dots + \cancel{x^{N-1}} + x^N)$

$S(1-x) = (1 - x^N)$

$$S \equiv \frac{(1-x^N)}{1-x}.$$

For case $|x|>1$, this series converges.

Samuel Colby

3. Taking the discrete IFT of a continuous FT.

CFT: $G(k) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i x k} dx$. dIFT: $g(n) = \frac{1}{N}\sum_{k=0}^{N-1} G(k) e^{\frac{i2\pi nk}{N}}$

$$\Rightarrow G(k) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i x k} dx \,, \quad \sum_{j'=0}^{(N-1)} G(j'\Delta k)$$

the inverse: $g(j'\Delta x) = \sum_{j'=0}^{N-1} G(j'\Delta k) e^{2\pi \frac{j'j}{N}}$

$$\Rightarrow g(j'\Delta x) = \sum_{j'=0}^{N-1} \left(\int_{-\infty}^{\infty} g(x) e^{-2\pi i x j'\Delta k} dx \right) e^{\frac{2\pi i j'j}{N}}$$

$$\Rightarrow \int_{-\infty}^{\infty} g(x) \sum_{j'=0}^{N-1} e^{-2\pi i x j'\Delta k} e^{2\pi i j \Delta x j'\Delta k} dx = \int_{-\infty}^{\infty} g(x) \sum_{j'=0}^{N-1} e^{-2\pi i j'\Delta k(x - j\Delta x)} dx$$

$$\Rightarrow \sum_{j'=0}^{N-1} e^{b(x-\Delta x)} \qquad b = i2\pi\Delta k$$

$$= \frac{e^{Nb(x-\Delta x)} - 1}{e^{b(x-\Delta x} - 1} \,, \quad b = i2\pi\Delta k \quad = \underline{q(x-\Delta x)}$$

$\Rightarrow$ upon solving, $g(j\Delta x) = \int g(x)\, q(x - j\Delta x) dx$.

The definition of convolution

L/L Problems Samuel Colby

$\boxed{2.18}$ let $S(K) = FT\{P(x)\}$

(a) Show hermitian symmetry: If is real, then $S(K) = S^*(-K)$.

if $S(K) = FT\{P(x)\} = \int_{-\infty}^{\infty} P(x) e^{-2\pi i Kx} dx$,

$$S^*(K) = \left(FT\{P(x)\}\right)^* = \left(\int_{-\infty}^{\infty} P(x) e^{-2\pi i kx} dx\right)^* = \underline{\int_{-\infty}^{\infty} P(x)^* e^{+2\pi i kx} dx}$$

by replacing K with $-K$, we get

$S^*(-K) = \int_{-\infty}^{\infty} P(x)^* e^{-2\pi i kx} dx$. When $P(x)$ is a real function, $\underline{P(x) = P^*(x)}$.

$$\therefore S^*(-k) = \int_{-\infty}^{\infty} P(x) e^{-2\pi i kx} dx = S(K) \quad ✓$$

(b) Prove the modulation Property:

$$FT\{P(x)\cos(2\pi K_0 x)\} = \frac{1}{2}\left[S(K+K_0) + S(K-K_0)\right]$$

Through inverse euler's formula:

$$FT\{P(x)\cos(2\pi K_0 x)\} = FT\{P(x) \tfrac{1}{2}(e^{-2\pi K_0 x} + e^{2\pi K_0 x})\} = FT\{\tfrac{1}{2}(P(x)e^{-2\pi K_0 x} + P(x)e^{2\pi K_0 x})\}$$

Through linearity + shifting theorum:

$$FT\{\tfrac{1}{2}(P(x)e^{-2\pi K_0 x} + P(x)e^{2\pi K_0 x})\} = \tfrac{1}{2}\left(S(K-K_0) + S(K+K_0)\right) \quad ✓$$

$\boxed{2.19}$ Prove Convolution theorem: $P_1(x)P_2(x) \rightleftharpoons FT\{P_1\}(k) * FT\{P_2\}(k)$

Taking the FT: Say $FT(P_N(x)) = S_N(k)$

$$FT\{P_1(x)P_2(x)\} = \int_{-\infty}^{\infty} P_1(x)P_2(x)\, e^{-i2\pi kx}\, dx$$

w/ def of IFT:

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} S_1(k')\, e^{i2\pi k'x}\, dk'\right] P_2(x)\, e^{-i2\pi kx}\, dx$$

$$= \int_{-\infty}^{\infty} S_1(k')\left[\int_{-\infty}^{\infty} P_2(x)\, e^{(i2\pi k'x - i2\pi kx)}\, dx\right] dk'$$

$$= \int_{-\infty}^{\infty} S_1(k')\left[\int_{-\infty}^{\infty} P_2(x)\, e^{-i2\pi(k-k')x}\, dx\right] dk'$$

$$= \int_{-\infty}^{\infty} S_1(k')\, S_2(k-k')\, dk' = S_1(k) * S_2(k)$$

$$= FT\{P_1\}(k) * FT\{P_2\}(k) \checkmark$$

$\boxed{2.20}$ Calculate $FT\{\operatorname{sinc}^2(\pi a x)\}$ based on convolution theorem.

$$FT\{\operatorname{sinc}(\pi a x)\operatorname{sinc}(\pi a x)\} = \underbrace{FT\{\operatorname{sinc}(\pi a x)\} * FT\{\operatorname{sinc}(\pi a x)\}}_{\text{Convolution theorem}}$$

from Appendix A.6: $\Pi(t) \rightleftharpoons \operatorname{sinc}(\pi f)$.

$$\therefore \underline{\Pi(at) \rightleftharpoons \operatorname{sinc}(\pi a x)}$$

$$\Rightarrow \Pi(at) * \Pi(at) \doteq \frac{d}{dx}\left[\Pi(ax) * \Pi(ax)\right] = \left(\frac{d}{dx}\Pi(ax)\right) * \Pi(ax)$$

Derivative Property

$$= \left[\delta(x + 1/2a) - \delta(x - 1/2a)\right] * \Pi(ax) = \Pi(ax + 1/2a) - \Pi(ax - 1/2a)$$

$$\Rightarrow \Pi(ax) * \Pi(ax) = \int_{-\infty}^{ax}\left(\Pi(a(\tau + 1/2)) - \Pi(a(\tau - 1/2))\right) d\tau$$

$$\Rightarrow \begin{cases} 1 - |ax| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases} = \underline{\Lambda(ax)}$$

2.23 Show that a periodic function $f(t)$ w/ period T can be written as:

$$f(t) = f_T(t) * \frac{1}{T}\text{comb}\left(\frac{t}{T}\right)$$

where $f_T(t) = 1$ period of $f(t)$

The comb function can be defined as

$$\frac{1}{T}\text{comb}(t/T) = \sum_{n=-\infty}^{\infty} \delta(t-nT).$$

Taking the convolution...

$$\Rightarrow f_T(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT) = f_T(t) * (\delta(t+nT)+\dots+\delta(t)+\delta(t-T)+\dots+\delta(t-nT))$$

$$= f_T(t)*\delta(t+nT)+\dots+\delta(t)+\delta(t-T)+\dots+\delta(t-nT) = f_T(t+nT)+\dots+f_T(t)+f_T(t-T)+\dots+f_T(t-nT)$$

By definition of convolution:

$$g(x) * \delta(x-a) = g(a).$$

$$\Rightarrow \underline{\sum_{n=-\infty}^{\infty} f_T(t-nT)} = \underline{\underline{f(t)}}$$

The expression is the infinite summation of a period of $f(t)$ at every integer period of $f(t)$. $\therefore$ it is $f(t)$.