A. Finite Element Matrices in 1D Burgers' Equation

In the context of the 1D Burgers' equation, we need to compute several matrices using the Finite Element Method (FEM). Two key matrices are the mass matrix $\mathbf{M}$ and the convection matrix $\mathbf{C}(\mathbf{U})$.

A.1. Transformation to the Reference Element

To simplify integration, we map the physical element $[x_1, x_2]$ to a reference element $[-1, 1]$ using a linear transformation:

$$x(\xi) = \frac{(1-\xi)}{2}x_1 + \frac{(1+\xi)}{2}x_2 \tag{1}$$

where $\xi \in [-1, 1]$ is the coordinate in the reference element.
The integral over the physical element then transforms into an integral over the reference element:

$$\int_{x_1}^{x_2} f(x)\, dx = \int_{-1}^{1} f(\xi) \left| \frac{dx}{d\xi} \right| d\xi \tag{2}$$

where the Jacobian $\frac{dx}{d\xi}$ accounts for the transformation from ξ to x.

A.2. The Jacobian

In 1D, for a linear element, the Jacobian J is constant and relates the differential $d\xi$ in the reference element to the differential dx in the physical element:

$$J = \frac{dx}{d\xi} = \frac{x_2 - x_1}{2} \tag{3}$$

Here, x_1 and x_2 are the coordinates of the element's endpoints in the physical domain, and ξ is the coordinate in the reference element. The Jacobian J represents the stretching or compression of the reference element to fit the physical element.

A.3. Gaussian Quadrature

Gaussian quadrature is a numerical integration method used to approximate integrals over the interval $[-1, 1]$. The integral is approximated as a weighted sum of the function values at specific points (Gauss points):

$$\int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{k=1}^{n_{\text{gauss}}} w_k f(\xi_k) \tag{4}$$

where:

- ξ_k are the Gauss points, the locations in the reference element where the function is evaluated.
- w_k are the corresponding weights, which scale the contribution of each Gauss point to the integral.

In the implementation we have $n_{\text{gauss}} = 2$:

$$\text{Gauss points: } \xi_1 = -\frac{\sqrt{3}}{3}, \quad \xi_2 = \frac{\sqrt{3}}{3}$$

$$\text{Weights: } w_1 = w_2 = 1$$

These are represented in the code by the arrays:

$$\text{Gauss points: } = \left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right]$$

$$\text{Weights: } = [1, 1]$$

The shape functions $N_i(\xi)$ and their derivatives $\frac{dN_i(\xi)}{d\xi}$ are evaluated at these Gauss points for the integration. For instance, the shape functions in your code are defined as:

$$N(\xi) = \left[\frac{1-\xi}{2}, \frac{1+\xi}{2}\right]$$

and their derivatives are:

$$\frac{dN(\xi)}{d\xi} = \left[-\frac{1}{2}, \frac{1}{2}\right]$$

These arrays and the corresponding operations provide the necessary components to compute integrals in the FEM discretization process accurately.

A.4. Mass Matrix in FEM

The mass matrix $\mathbf{M}$ is defined as:

$$\mathbf{M}_{ij} = \int_\Omega N_i(x) N_j(x)\, dx \tag{5}$$

where $N_i(x)$ and $N_j(x)$ are the shape functions associated with the ith and jth nodes, respectively.
Using the transformations and Gaussian quadrature:

$$\mathbf{M}_{ij} \approx \sum_{k=1}^{n_{\text{gauss}}} w_k N_i(\xi_k) N_j(\xi_k) |J| \tag{6}$$

Here:

- $N_i(\xi_k)$ and $N_j(\xi_k)$ are the values of the shape functions evaluated at the Gauss points.
- $|J|$ is the absolute value of the Jacobian, accounting for the scaling between the reference and physical element.
- w_k is the weight associated with the Gauss point.

A.5. Convection Matrix in FEM

The convection matrix $\mathbf{C}(\mathbf{U})$ arises from the advection term and is defined as:

$$\mathbf{C}_{ij}(\mathbf{U}) = \int_{\Omega} U(x,t) \frac{\partial N_j(x)}{\partial x} N_i(x)\, dx \tag{7}$$

Using the same transformations and Gaussian quadrature:

$$\mathbf{C}_{ij}(\mathbf{U}) \approx \sum_{k=1}^{n_{\text{gauss}}} w_k U(\xi_k) \frac{\partial N_j(\xi_k)}{\partial \xi} N_i(\xi_k) |J| \tag{8}$$

Here:

- $U(\xi_k)$ is the value of the solution field at the Gauss points.
- $N_i(\xi_k)$ and $N_j(\xi_k)$ are the values of the shape functions evaluated at the Gauss points.
- $\frac{\partial N_j(\xi_k)}{\partial \xi}$ is the derivative of the shape function $N_j(x)$ with respect to ξ.
- $|J|$ is the absolute value of the Jacobian.
- w_k is the weight associated with the Gauss point.

A.6. Diffusion Matrix in FEM

The diffusion matrix **K** arises from the diffusion term in the 1D Burgers' equation and is defined as:

$$\mathbf{K}_{ij} = \int_{\Omega} \frac{dN_i(x)}{dx} \frac{dN_j(x)}{dx} \, dx \tag{9}$$

where $\frac{dN_i(x)}{dx}$ and $\frac{dN_j(x)}{dx}$ are the derivatives of the shape functions associated with the ith and jth nodes, respectively. Using the transformations and Gaussian quadrature:

$$\mathbf{K}_{ij} \approx \sum_{k=1}^{n_{\text{gauss}}} w_k \frac{dN_i(\xi_k)}{d\xi} \frac{dN_j(\xi_k)}{d\xi} |J| \tag{10}$$

Here:

- $\frac{dN_i(\xi_k)}{d\xi}$ and $\frac{dN_j(\xi_k)}{d\xi}$ are the values of the derivatives of the shape functions evaluated at the Gauss points.
- $|J|$ is the absolute value of the Jacobian, which accounts for the scaling between the reference and physical element.
- w_k is the weight associated with the Gauss point.

A.7. Load Vector (Source Term) in FEM

The load vector $\mathbf{F}$, which arises from the source term $f(x,t)$ in the 1D Burgers' equation, is defined as:

$$\mathbf{F}_i = \int_{\Omega} f(x,t) N_i(x)\, dx \tag{11}$$

where $N_i(x)$ is the shape function associated with the ith node, and $f(x,t)$ is the source term.
Using the transformations and Gaussian quadrature:

$$\mathbf{F}_i \approx \sum_{k=1}^{n_{\text{gauss}}} w_k f(\xi_k) N_i(\xi_k) |J| \tag{12}$$

Here:

- $f(\xi_k)$ is the value of the source term evaluated at the Gauss points.
- $N_i(\xi_k)$ is the value of the shape function evaluated at the Gauss points.
- $|J|$ is the absolute value of the Jacobian, accounting for the scaling between the reference and physical element.
- w_k is the weight associated with the Gauss point.

A.8. Local to Global Matrix Assembly

Once the matrices for each element are computed, these local matrices are assembled into the global matrices by summing up the contributions from all elements.

A.9. Summary

- **Mass Matrix**: Represents the distribution of mass in the system.
- **Convection Matrix**: Represents the effect of advection in the system.
- **Diffusion Matrix**: Represents the effect of diffusion (viscosity) in the system.
- **Load Vector (Source Term)**: Accounts for external sources or forcing terms applied to the system.
- **Transformation to Reference Element**: Simplifies integration by mapping the physical element to a standard reference element.
- **Jacobian Determinant**: A scaling factor that accounts for the transformation from reference to physical coordinates.
- **Gaussian Quadrature**: A method to numerically approximate integrals using weighted sums at specific points.
- **Assembly**: The local matrices and vectors from individual elements are combined to form the global matrices and vectors for the entire domain.