## 1. Weak Form of the 1D Burgers' Equation

To derive the weak form of the 1D Burgers' equation, we start with the strong form:

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \nu \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \tag{1}$$

*Multiplying by a Test Function and Integrating Over the Domain*

We multiply by a test function $v(x)$ and integrate over the domain $[0, L]$:

$$\int_0^L \left( \frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \nu \frac{\partial^2 u(x,t)}{\partial x^2} - f(x,t) \right) v(x)\, dx = 0 \tag{2}$$

*Applying Integration by Parts*

We handle the second-order derivative term $-\nu \frac{\partial^2 u}{\partial x^2}$ using integration by parts:

$$-\nu \int_0^L \frac{\partial^2 u}{\partial x^2} v\, dx = \left[ -\nu \frac{\partial u}{\partial x} v \right]_0^L + \nu \int_0^L \frac{\partial u}{\partial x}\frac{\partial v(x)}{\partial x}\, dx \tag{3}$$

Assuming homogeneous Dirichlet boundary conditions (i.e., $u = 0$ on $\partial\Omega$), the boundary term vanishes:

$$\nu \frac{\partial u}{\partial x} v \bigg|_0^L = 0 \tag{4}$$

Thus, we obtain:

$$\int_0^L \left( \frac{\partial u}{\partial t} v + u \frac{\partial u}{\partial x} v + \nu \frac{\partial u}{\partial x}\frac{\partial v(x)}{\partial x} - f(x,t) v \right) dx = 0 \tag{5}$$

The weak form of the 1D Burgers' equation with viscosity and a source term is:

$$\int_0^L \left( \frac{\partial u}{\partial t} v + u \frac{\partial u}{\partial x} v + \nu \frac{\partial u}{\partial x}\frac{\partial v(x)}{\partial x} \right) dx = \int_0^L f(x,t) v\, dx \tag{6}$$

This equation must be satisfied for all test functions $v(x)$ in the appropriate function space.