AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Pseudo-Time Integration

Outline

- From Chapter 7, it follows that the semi-discretization on unstructured (or for that matter structured) grids of the unsteady Euler equations in multiple dimensions can be written as

$$\frac{d\overline{W}_i}{dt} = -\sum_{\star} \frac{\widehat{\mathcal{F}}_{i\star}}{\|C_i\|}, \qquad i = 1,\ 2,\ \cdots,\ N$$

  where N denotes the total number of finite volume cells

- An alternative expression of the above equations is

$$\|C_i\| \frac{d\overline{W}_i}{dt} + \sum_{\star} \widehat{\mathcal{F}}_{i\star} = 0, \qquad i = 1,\ 2,\ \cdots,\ N$$

- Let

$$\mathbf{W} = [\overline{W}_1^T \ \overline{W}_2^T \ \cdots \ \overline{W}_N^T]^T$$

and

$$\mathbf{F}(\mathbf{W}) = [\sum_{\star} \widehat{\mathcal{F}}_{1\star}^T \ \sum_{\star} \widehat{\mathcal{F}}_{2\star}^T \ \cdots \ \sum_{\star} \widehat{\mathcal{F}}_{N\star}^T]^T$$

- And let $\boldsymbol{\Omega} = \text{diag}(\Omega_1, \ \Omega_2, \ \cdots, \ \Omega_N)$ denote the diagonal matrix of cell volumes, where $\Omega_i = \|C_i\|\mathbb{I}_m$ ($m = 3, 4, 5$ for one-, two-, and three-dimensional problems, respectively)
- Then, the semi-discretization in multiple dimensions of the unsteady Euler equations can be written in vector form as

$$\boxed{\boldsymbol{\Omega}\frac{d\mathbf{W}}{dt} + \mathbf{F}(\mathbf{W}) = 0}$$

- At the steady-state, the above semi-discrete equation simplifies to

$$\mathbf{F}(\mathbf{W}) = 0$$

- However, because $\mathbf{F}(\mathbf{W})$ is in general a highly nonlinear function of $\mathbf{W}$ whose solution by Newton's method is *difficult to initialize*, the above nonlinear algebraic equation is transformed into the following nonlinear ordinary differential equation

$$\boxed{\mathbf{U}\frac{d\mathbf{W}}{d\tau} + \mathbf{F}(\mathbf{W}) = 0}$$

  where $\mathbf{U}$ is a *positive definite* matrix, and τ is a *fictitious*, *global* time
- If the above problem is hyperbolic everywhere, then, regardless of the initial condition, as long as $\mathbf{U}$ is positive definite, the above system converges to a steady-state solution as $\tau \to \infty$
- In particular, the specific choice of a positive definite matrix $\mathbf{U}$ is unimportant, except that it may change the convergence point if there are multiple steady-state solutions for $\mathbf{W}$, which is assumed here not to be the case

$$\mathbf{U}\frac{d\mathbf{W}}{d\tau} + \mathbf{F}(\mathbf{W}) = 0$$

- Assume that a different *local* time τ_i is chosen for each cell C_i
- Then, there exists a Jacobian matrix J_i for the transformation from τ in every cell C_i to τ_i: $J_i = \dfrac{d\tau}{d\tau_i}\mathbb{I}_m$
- Let $\mathbf{J} = \text{diag}(J_1,\ J_2,\ \cdots,\ J_N)$
- Note that both $\mathbf{\Omega}$ and $\mathbf{J}$ are positive definite matrices
- Assume next that $\mathbf{U}$ is chosen as

$$\begin{aligned}\mathbf{U} = \mathbf{\Omega}\mathbf{J} &= \text{diag}(\Omega_1 J_1,\ \Omega_2 J_2,\ \cdots,\ \Omega_N J_N) \\ &= \text{diag}(\|C_1\|\frac{d\tau}{d\tau_1}\mathbb{I}_m,\ \|C_2\|\frac{d\tau}{d\tau_2}\mathbb{I}_m,\ \cdots,\ \|C_N\|\frac{d\tau}{d\tau_N}\mathbb{I}_m)\end{aligned}$$

- Then, the above equation becomes

$$\boxed{\mathbf{\Omega}\mathbf{J}\frac{d\mathbf{W}}{d\tau} + \mathbf{F}(\mathbf{W}) = 0}$$

$$\mathbf{\Omega}\mathbf{J}\frac{d\mathbf{W}}{d\tau} + \mathbf{F}(\mathbf{W}) = 0$$

- The above equation can be re-written as

$$\begin{aligned}
\|C_i\| J_i \frac{dW_i}{d\tau} + \sum_{\star} \widehat{\mathcal{F}}_{i\star}(\mathbf{W}) &= \|C_i\| \frac{d\tau}{d\tau_i}\frac{dW_i}{d\tau} + \sum_{\star} \widehat{\mathcal{F}}_{i\star}(\mathbf{W}) \\
&= \|C_i\| \frac{dW_i}{d\tau_i} + \sum_{\star} \widehat{\mathcal{F}}_{i\star}(\mathbf{W}) \\
&= 0, \qquad i = 1,\ 2,\ \cdots,\ N
\end{aligned}$$

- It can also be expressed as

$$\frac{dW_i}{d\bar{\tau}_i} + \sum_{\star} \widehat{\mathcal{F}}_{i\star}(\mathbf{W}) = 0, \qquad i = 1,\ 2,\ \cdots,\ N$$

where

$$\bar{\tau}_i = \frac{\tau_i}{\|C_i\|}$$

$$\boxed{\frac{dW_i}{d\bar{\tau}_i} + \sum_{\star} \widehat{\mathcal{F}}_{i\star}(\mathbf{W}) = 0, \quad \bar{\tau}_i = \frac{\tau_i}{\|C_i\|}, \quad i = 1,\ 2,\ \cdots,\ N}$$

- The above equation suggests that the original semi-discrete equation

$$\mathbf{\Omega}\frac{d\mathbf{W}}{dt} + \mathbf{F}(\mathbf{W}) = 0$$

can be solved using any preferred ordinary differential equation solver and a *local* time-step

- To understand the benefit of a local time-step, consider the case of a 1D linear advection equation (speed $= a$) and a FT or BT approximation
- In this case, for a given CFL number CFL, the local and global time-steps are given by

$$\Delta t_i^{\ell} = \frac{CFL}{a} \Delta x_i \quad \text{and} \quad \Delta t_i^{g} = \frac{CFL}{a} \Delta x_{min}$$

  respectively
- It follows that

$$\Delta \overline{t}_i^{\ell} = \frac{\Delta t_i^{\ell}}{\Delta x_i} = \frac{CFL}{a} \quad \text{and} \quad \Delta \overline{t}_i^{g} = \frac{\Delta t_i^{g}}{\Delta x_i} = \frac{CFL}{a} \frac{\Delta x_{min}}{\Delta x_i}$$

Steady-State Solution

Local Time-Stepping

- Note that

$$\boxed{\begin{array}{c} \Delta \bar{t}_i^{\ell} = \dfrac{\Delta t_i^{\ell}}{\Delta x_i} = \dfrac{CFL}{a} \quad \text{and} \quad \Delta \bar{t}_i^{g} = \dfrac{\Delta t_i^{g}}{\Delta x_i} = \dfrac{CFL}{a}\dfrac{\Delta x_{min}}{\Delta x_i} \\ \Rightarrow \quad \Delta \bar{t}_i^{\ell} = \Delta \bar{t}_i^{g} \dfrac{\Delta x_i}{\Delta x_{min}} \end{array}}$$

- Hence, time-integrating the governing semi-discrete equations using a local time-step Δt_i^{ℓ} — a process also known as pseudo-time integration — advances the solution in each cell towards the steady-state at the same *scaled* pace
- Comparatively, time-integrating the governing semi-discrete equations using a global time-step Δt_i^{g} — as in the genuinely unsteady case — slows down convergence toward the steady-state solution
- What happens in the case of a 1D nonlinear advection equation?