AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Treatment of Boundary Conditions

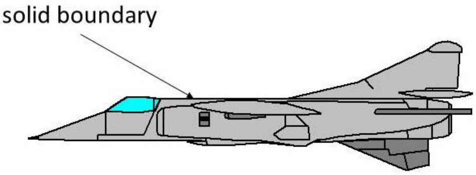
These slides are partially based on the recommended textbook: Culbert B. Laney. "Computational Gas Dynamics," CAMBRIDGE UNIVERSITY PRESS, ISBN 0-521-62558-0

Outline

1. Two Types of Boundaries
2. Two Types of Grids
3. General Results
4. Solid Boundaries
5. Far-Field Boundaries
   - How Many Flow Variables to Specify
   - Which Flow Variables to Specify
   - Free-Stream Conditions
   - Nonreflecting Boundary Conditions
   - Steger-Warming Flux Vector Splitting Method for the Euler Equations

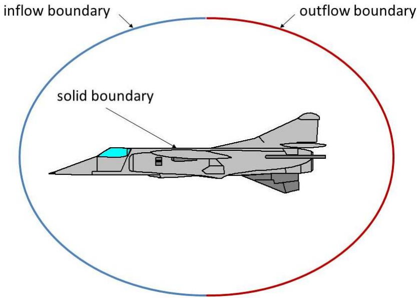
Note: This chapter considers only formally zero-, first-, and second-order accurate boundary treatments so that there is no need to distinguish between finite difference and finite volume schemes. For the sake of simplicity, it also focuses on one-dimensional flows and explicit time-integration methods. The generalizations of the presented material to multidimensional flows and implicit time-integration methods are straightforward.

- Solid (*surface*, *wall*, or *impermeable*) boundaries



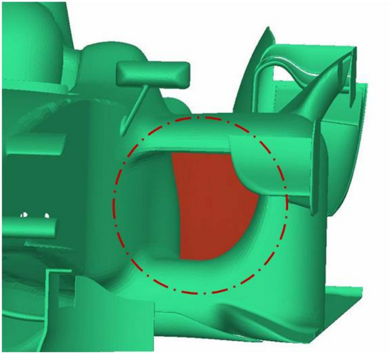
  - rigid stationary
  - rigid moving (not covered in this chapter)
  - flexible (or deformable) stationary (not covered in this chapter)
  - flexible (or deformable) dynamic (not covered in this chapter)
- Solid boundaries reflect existing waves

- Far-field (*open*, *artificial*, *absorbing*, *permeable*, or *remote*) boundaries
  - *inflow* boundaries
  - *outflow* boundaries

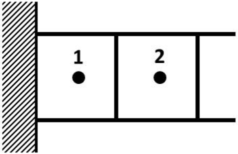


- Far-field boundaries absorb existing waves and emit new ones

- Other types of boundaries not covered in this chapter
  - *permeable* or *porous* boundaries
    - Brinkman-Forchheimer-extended Darcy model for radiator flows

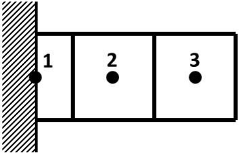


- Cell-centered type of grid (primal cells are shown)



  - most common type of grid for the finite volume method
  - the numerical flux is the focus entity in this case

- Vertex-based type of grid (dual cells are shown)

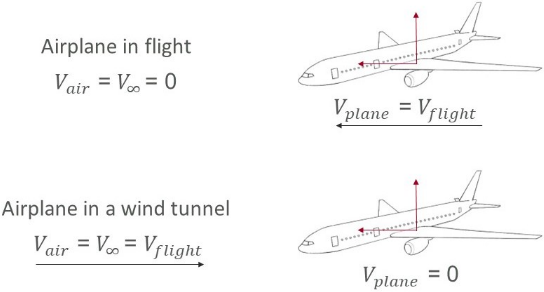


- most common type of grid for the finite difference method
- the flow variable is the focus entity in this case

- *Boundary treatments* are usually an issue when/if the chosen spatial approximation requires the value of the solution at non-existing points
- Typical boundary treatments
  - change the computational domain $\rightarrow$ *ghost cells* $\rightarrow$ ghost flow and reflection method
  - change the approximation method $\rightarrow$ *one-sided* approximations
  - enforce explicitly the appropriate boundary condition $\Rightarrow$ weak and strong enforcements
  - combine some of the above approaches
- Boundary treatments affect the accuracy, order of accuracy, and stability of the chosen numerical method on the boundaries and in the interior of the computational domain
- Gustafsson (1975): The formal order of accuracy on the boundaries may be lower than that in the interior (this is not surprising, considering that the order of accuracy can drop near shocks)

- An $(m-1)$-th order scheme on the boundaries is compatible with an m-th order scheme in the interior $\big($justification: The information passing through the mesh along the characteristic path is updated or solved for N times at interior points, with $N \approx L/\left(\Delta t \,(|v|+c)\right) \approx L/(CFL\ \Delta x)$ but only once at a "thin set" of (boundary, shock, etc.) points$\big)$
- However, accuracy on some boundaries can be crucial (for example, wall boundaries where lift and drag are computed)

- Wind tunnel and wind tunnel analogy
  - in the wind tunnel, the test model and supporting instruments remain sationary while air moves relative to them
  - wind tunnel analogy: formulating the CFD problem in an inertial frame of reference attached to a body cruising the constant speed V_∞ is equivalent to solving the corresponding wind tunnel problem, as illustrated below



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- Recall that the linear matrix stability analysis accounts for boundary treatments and therefore can determine their effects on numerical stability
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- At solid boundaries, the numerical domain of dependence must be
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- Hence, the CFL condition is automatically satisfied at solid boundaries

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General Results

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- By definition, the numerical domain of dependence is always completely contained within the computational domain
- Hence, if any waves enter the far-field boundaries, the CFL condition is automatically violated at far-field boundaries, unless the numerical method is equipped to know something about any waves entering through the far-field boundaries

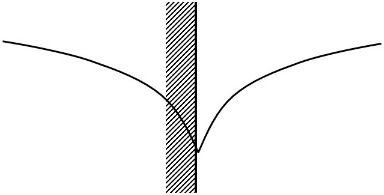
- At solid boundaries, inviscid flows are characterized by the *no-penetration* boundary condition: $v_x = 0$ (in mutliple dimensions, $\vec{v} \cdot \vec{\nu} = 0$, where $\vec{\nu}$ is the normal to the wall)
- In a numerical method, a solid boundary treatment should enforce the no-penetration condition without otherwise restricting the flow
- For simplicity, only left-hand solid surfaces positioned at $x = x_L$ are treated next (the case of right-hand solid surfaces is treated in a similar manner)

- Ghost flow and reflection method
  - treat the solid surface as a mirror

$$
\begin{aligned}
\rho(x_L - x, t) &= \rho(x_L + x, t) \\
v_x(x_L - x, t) &= -v_x(x_L + x, t) \\
p(x_L - x, t) &= p(x_L + x, t)
\end{aligned}
$$

  - $v_x(x_L - x, t) = -v_x(x_L + x, t) \Rightarrow v_x(x_L, t) = 0$ (assuming only that the velocity is continuous)
  - same reasoning can be performed using the conservative variables to obtain the same conclusion

- The flow properties are continuous across the real/ghost boundaries, but their first derivatives are not ⇒ cusped flows at solid boundaries, in theory



- An explicit numerical method with K_1 points to the left requires at least K_1 ghost cells
- For example, for $K_1 = 2$, the numerical approximation in the required ghost cells goes as follows
  - case of a cell-centered grid

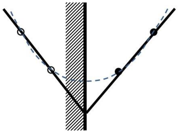
$$\begin{array}{rcl} \rho_0^n &=& \rho_1^n, \qquad \rho_{-1}^n = \rho_2^n \\ v_{x_0}^n &=& -v_{x_1}^n, \qquad v_{x_{-1}}^n = -v_{x_2}^n \\ p_0^n &=& p_1^n, \qquad p_{-1}^n = p_2^n \end{array}$$

  - case of a vertex-based grid

$$\begin{array}{rcl} \rho_0^n &=& \rho_2^n, \qquad \rho_{-1}^n = \rho_3^n \\ v_{x_0}^n &=& -v_{x_2}^n, \qquad v_{x_{-1}}^n = -v_{x_3}^n \\ p_0^n &=& p_2^n, \qquad p_{-1}^n = p_3^n \end{array}$$

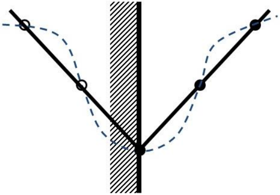
- The true derivatives of the flow properties may be non-zero near solid walls, but certain numerical approximations might predict them to be vanishing at solid walls
- For example, on cell-centered grids, an FS approximation says

$$\frac{\partial \rho}{\partial x} \approx \frac{\rho_1^n - \rho_0^n}{\Delta x} = 0$$



- For the Euler equations, from the conservation of momentum in primitive variable form $\dfrac{\partial v_x}{\partial t} + v_x \dfrac{\partial v_x}{\partial x} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = 0$ and the non-penetration condition at solid walls, it follows that $\partial p/\partial x(x_L, t) = 0$

- If the numerical method senses the cusp at the boundary, it may produce spurious oscillations



- Such oscillations can be combatted in exactly the same way as oscillations and instabilities caused by shocks are combatted

- Note: If the numerical method accurately enforces the no-penetration condition, it should ensure at solid boundaries
  - case of a cell-centered grid

$$\widehat{\mathcal{F}}^n_{x_{1/2}} = [0,\ p^n_{1/2},\ 0]^T$$

  - case of a vertex-based grid

$$W^n_1 = [\rho^n_1,\ 0,\ E^n_1]^T$$

- Alternatives to the method of ghost flow and reflection
  - change the approximation method $\rightarrow$ *one-sided* approximations $\Rightarrow$ may not effectively enforce the no-penetration boundary condition
  - explicit enforcement of the no-penetration boundary condition $\Rightarrow$ comes in two flavors: weak and strong enforcements

- Weak enforcement of the no-penetration boundary condition
  - example: finite volume method and cell-centered grid

$$\widehat{\mathcal{F}}^n_{x_{i+1/2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathcal{F}_x\left(W(x_{i+1/2}, t)\right) dt$$

    - $v_x(x_{1/2}, t) = 0 \Rightarrow$ set $\mathcal{F}_x\left(W(x_{1/2}, t)\right) = [0,\ p(x_{1/2}, t)\ ,0]^T$

$$\Longrightarrow \widehat{\mathcal{F}}^n_{x_{1/2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} [0,\ p(x_{1/2}, t)\ ,0]^T dt$$

    - extrapolate $p(x_{1/2}, t)$, for example, as

$$p(x_{1/2}, t) = p_1^n + O\left(\mathbf{\Delta x^2}\right) + O\left(t - t^n\right) \qquad \text{(constant extrapolation)}$$

or as

$$p(x_{1/2}, t) = \frac{3}{2}p_1^n - \frac{1}{2}p_2^n + O\left(\Delta x^2\right) + O\left(t - t^n\right) \qquad \text{(linear extrapolation)}$$

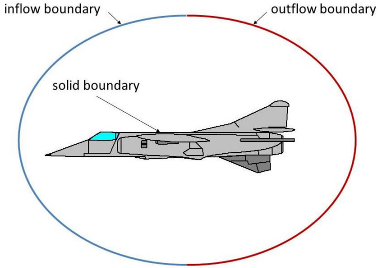
  - for the usual reason, the constant extrapolation may become superior to the linear one as a shock nears the solid boundary

- Strong enforcement of the no-penetration boundary condition
  - example: finite volume method and vertex-based grid
    - $v_x(x_1, t) = 0 \Rightarrow$ set $W_1^n = [\rho_1^n,\ 0, E_1^n]^T$
    - focus on the second component of the advanced fluid state vector

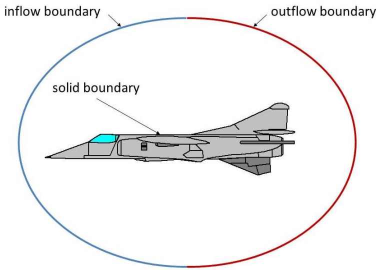
$$(W_1^{n+1})_2 = (W_1^n)_2 - \lambda[(\widehat{\mathcal{F}}_{x_{3/2}})_2 - (\widehat{\mathcal{F}}_{x_{1/2}})_2]$$

    - $(W_1^{n+1})_2 = (W_1^n)_2 = 0 \Rightarrow (\widehat{\mathcal{F}}_{x_{1/2}})_2 = (\widehat{\mathcal{F}}_{x_{3/2}})_2$

$\Longrightarrow$ trivial task, at least in 1D



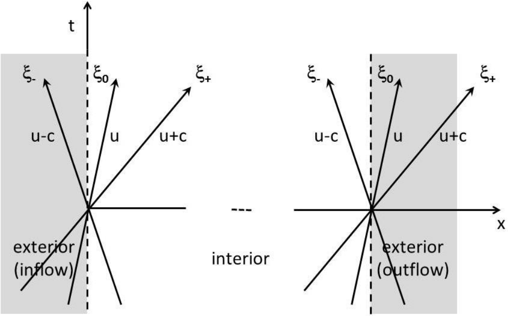
- Should allow waves to travel freely "in" and "out" of the computational domain $\Rightarrow$ corresponding boundary treatment should *specify incoming waves* and *prevent reflection of outgoing waves*



- Should allow waves to travel freely "in" and "out" of the computational domain $\Rightarrow$ corresponding boundary treatment should *specify incoming waves* and *prevent reflection of outgoing waves*
- Incoming waves carry information from the exterior $\Rightarrow$ boundary treatment should know "something" about the exterior (truly complete information about the exterior could require explicitly modeling the exterior, which is typically not desirable)

- Major steps in designing a far-field boundary treatment
  - determine how many flow variables must be specified (0, 1, 2, or 3)
  - choose which flow variables to specify
  - assign values to them (most difficult step), which completes the boundary treatment
- Design of boundary treatment is typically guided by characteristic theory

- Example: *Free-stream* boundaries (more details later)



- subsonic inflow: two quantities associated with ξ_0 and ξ_+ must be specified
- subsonic outflow: one quantity associated with ξ_- must be specified

- Distinguish between inflow and outflow boundaries
- Distinguish between subsonic and supersonic boundaries
- Inflow boundary
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- *Ignoring the above analysis leads to duplicative information at best, and contradictory information at worst*, and is bound in general to generate ill-conditioning and/or numerical instability

- Primitive, characteristic, or conservative variables can be specified, as well as combinations of such variables
  - characteristic variables are the most natural candidates, but involve the solution of differential equations for which analytic solutions may or may not be available
  - primitive variables are the most practical candidates because: (a) they are the easiest to measure when experimental data is involved, and (b) when they can be obtained, characteristic variables can be expressed in terms of primitive variables

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- Hence, primitive variables are usually specified with special care to avoid
  - overconstraining the problem at a given far-field boundary (specified variables determine outgoing characteristic variables)
  - underconstraining the problem at a given far-field boundary (specified variables do not specify the incoming characteristic variables)
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- Conflicts cannot arise when 0 or 3 quantities must be specified $\Rightarrow$ subsonic flow is the main case where special care is required

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  - recall that 2 quantities must be specified at a subsonic inflow far-field boundary (ξ_0 and ξ_+)
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$$d\xi_0 = d\rho - \frac{dp}{c^2}, \quad d\xi_+ = dv_x + \frac{dp}{\rho c}, \quad d\xi_- = dv_x - \frac{dp}{\rho c}$$

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    - $((d\rho, dp), d\xi_-)$: dp and $d\xi_- \Rightarrow dv_x \Rightarrow (d\rho, dp, dv_x) \Rightarrow$ admissible
    - $((d\rho, dv_x), d\xi_-)$: dv_x and $d\xi_- \Rightarrow dp \Rightarrow (d\rho, dp, dv_x) \Rightarrow$ admissible
  - result: at a subsonic inflow boundary, *one can specify* (ρ, p) *or* (ρ, v_x) *but not* (p, v_x) — for example, at a free-stream subsonic inflow boundary (more details later) one can specify $(\rho = \rho_\infty, p = p_\infty)$ or $(\rho = \rho_\infty, v_x = v_{x_\infty})$

- Example: Subsonic far-field boundaries — Part II: outflow boundary
  - recall that at a subsonic inflow boundary, one can specify (ρ, p) or (ρ, v_x) but not (p, v_x)
  - recall that at a subsonic outflow boundary, only one quantity must be specified (ξ_-)

- Example: Subsonic far-field boundaries — Part II: outflow boundary
  - recall that at a subsonic inflow boundary, one can specify (ρ, p) or (ρ, v_x) but not (p, v_x)
  - recall that at a subsonic outflow boundary, only one quantity must be specified (ξ_-)
  - hence, the main concern is to avoid redundancy with what is chosen to be specified at the inflow boundary
  - since in this example both inflow and outflow boundaries are assumed to be subsonic and $d\xi_- = dv_x - dp/\rho c$:
    - if (ρ, p) is specified at the inflow boundary $\Rightarrow$ v_x must be specified at the outflow boundary
    - if (ρ, v_x) is specified at the inflow boundary $\Rightarrow$ p must be specified at the outflow boundary

- In aeronautics, the *physical* far-field is often a region of space known as the *free-stream*, where the flow is steady and uniform, and its properties are designated by the subscript ∞
- For this reason, after the flow variables to be specified at far-field inflow and outflow boundaries are chosen, one is tempted to set their values to the free-stream conditions

- In practice
  - if the computational domain is sufficiently (and often prohibitively) large — for example, when the *artificia*l far-field boundaries delimiting the computational domain associated with the flow past an airfoil are located away from the wall boundaries at distances of the order of 50 times the chord (in the case of a wing) — the above temptation is reasonable, not necessarily because at these distances the flow is at the free-stream conditions, but because the outgoing waves will be heavily dissipated by discretization errors (artificial viscosity) by the time they reach these artificial far-field boundaries
  - otherwise, the above temptation is unreasonable and will cause the outgoing waves (especially shock waves) to partially reflect from these artificial far-field boundaries and pollute the numerical solution in the interior of the computational domain

- Rather than relying on artificial viscosity, it is possible to construct special wave-absorbing buffer zones to damp outgoing waves before they can reach the far-field; or artificial far-field boundary conditions — also known as *nonreflecting*, *absorbing*, *silent*, *transparent*, or *one-way* boundary conditions — that absorb outgoing waves at the far-field and therefore do not allow them to reflect back into the interior of the computational domain
- There are two approaches for achieving the above objective
  - setting the characteristic variables or their derivatives equal to constants
  - modeling the outgoing waves to prevent their reflection using some sort of asymptotic analysis

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- There are two approaches for achieving the above objective
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  - modeling the outgoing waves to prevent their reflection using some sort of asymptotic analysis
- Both approaches are simpler to design and implement when the flow is linearized about the free-stream conditions in the neighborhood of the artificial far-field boundaries

Linearization About the Steady Uniform Free-Stream Conditions

- If the artificial far-field boundaries are placed sufficiently far away so that the far-field flow is nearly but not exactly equal to the steady, uniform, free-stream flow, the Euler (or for that matter the Navier-Stokes) equations can be linearized at the far-field boundaries
- Let

$$\rho = \rho_\infty + \rho', \quad v_x = v_{x_\infty} + v'_x, \quad p = p_\infty + p'$$

- Substituting the above equations into the Euler equations (primitive variables) and neglecting all second-order terms yields

$$\frac{\partial V}{\partial t} + A_\infty \frac{\partial V}{\partial x} = 0 \qquad (1)$$

where

$$V = \begin{pmatrix} \rho' \\ v'_x \\ p' \end{pmatrix} \quad \text{and} \quad A_\infty = \begin{pmatrix} v_{x_\infty} & \rho_\infty & 0 \\ 0 & v_{x_\infty} & \frac{1}{\rho_\infty} \\ 0 & \rho_\infty c_\infty^2 & v_{x_\infty} \end{pmatrix}$$

Setting the Characteristic Variables

- Consider now the characteristics of the linearized Euler equations (1)

$$d\xi_0 = d\rho - \frac{dp}{c_\infty^2}, \quad d\xi_+ = dv_x + \frac{dp}{\rho_\infty c_\infty}, \quad d\xi_- = dv_x - \frac{dp}{\rho_\infty c_\infty}$$

- These differential equations have the following analytic solutions

$$\xi_0 = \rho - \frac{p}{c_\infty^2}, \quad \xi_+ = v_x + \frac{p}{\rho_\infty c_\infty}, \quad \xi_- = v_x - \frac{p}{\rho_\infty c_\infty}$$

- Then, at a subsonic inflow, one can set

$$\xi_0 = \rho - \frac{p}{c_\infty^2} = cst \left(= \rho_\infty - \frac{p_\infty}{c_\infty^2} \right) \quad \text{and} \quad \xi_+ = v_x + \frac{p}{\rho_\infty c_\infty} = cst' \left(= v_{x_\infty} + \frac{p_\infty}{\rho_\infty c_\infty} \right)$$

  and obtain the third information from?
- And at a subsonic outflow, one can set

$$\xi_- = v_x - \frac{p}{\rho_\infty c_\infty} = cst \left(= v_{x_\infty} - \frac{p_\infty}{\rho_\infty c_\infty} \right)$$

  and obtain the second and third information from?

Setting the Characteristic Variables

- This works well for 1D flows
- Multidimensional flows require multidimensional characteristics but otherwise a similar approach

Alternative Approach: Modeling the Outgoing Waves (Optional Material)

- Besides the (linearized) Euler equations, the outgoing waves typically satisfy a (linear) differential equation separate from that satisfied by the incoming waves or the total combination of outgoing and incoming waves
- In many cases, this other differential equation has an analytic solution that gives the outgoing waves a general functional form
- Either the aforementioned differential equation or the aforementioned general functional form can be used to *model* the outgoing waves
- A single wave equation or general functional form for pressure suffices for subsonic inlets as they allow only a single outgoing family of waves

Alternative Approach: Modeling the Outgoing Waves (Optional Material)

- In the far-field, to a first approximation, outgoing waves are planar, one-dimensional and satisfy the following linear advection equation

$$\frac{\partial p}{\partial t} - c\frac{\partial p}{\partial x} = 0$$

- At a left-hand far-field boundary, an equivalent model of the outgoing pressure wave is in this case the general functional form

$$p = p(x + ct)$$

Alternative Approach: Modeling the Outgoing Waves (Optional Material)

- Alternatively, the outgoing waves are *approximately* cylindrical in the far-field and satisfy the following 2D differential equation

$$\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial r} + \frac{c}{2r}p = 0$$

- In this case, an equivalent model of the outgoing pressure wave is the general functional form

$$p = \frac{1}{\sqrt{r}}p(r - ct)$$

Alternative Approach: Modeling the Outgoing Waves (Optional Material)

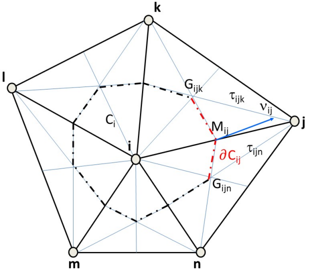
- A 3D model can be built similarly assuming that the outgoing waves are approximately spherical in the far-field
- Then, the chosen model — whether in differential equation or general functional form — can be discretized to solve for the outgoing pressure wave at the artificial far-field boundary
- Note: In general, the waves in the far-field are neither exactly planar, cylindrical, nor spherical, and therefore these characterizations are just the first terms in a longer series approximating the outgoing waves in the far-field

- Problems (equations of state) for which the physical flux $\mathcal{F}(\mathcal{W})$ is a homogeneous function of W of degree 1 — that is

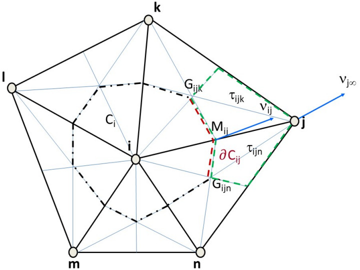
$$\forall i = 1, m \quad \sum_{j=1}^{m} \frac{\partial \mathcal{F}_i}{\partial W_j} W_j = \mathcal{F}_i(W_1, \ \cdots \ , \ W_m) \Rightarrow \mathcal{F} = A(W)W$$

- Recall flux splitting

$$\begin{array}{rcl} \mathcal{F}(W) & = & \mathcal{F}^+(W) + \mathcal{F}^-(W) \\ \dfrac{\partial \mathcal{F}^+}{\partial W} & \geq & 0, \qquad \dfrac{\partial \mathcal{F}^-}{\partial W} \leq 0 \\ \Longrightarrow & & \mathcal{F}(W) = A^+W + A^-W \\ \Longrightarrow & & \text{upwinding for } A^+W \text{ and downwinding for } A^-W \\ \Longrightarrow & & \text{for example, BS for } A^+W \text{ and FS for } A^-W \\ \Longrightarrow & & \widehat{\mathcal{F}}^n_{x_{i+1/2}} = A^+(W_i)W_i + A^-(W_{i+1})W_{i+1} \\ \text{and} & & \widehat{\mathcal{F}}^n_{x_{i-1/2}} = A^+(W_{i-1})W_{i-1} + A^-(W_i)W_i \end{array} \qquad (2)$$



- Recall also that $\widehat{\mathcal{F}}^n_{i\star} \approx \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \left(\int_{\partial C_{i\star}} \overrightarrow{\mathcal{F}} \cdot \vec{\nu}_{i\star} d\Gamma_i \right) dt$ is constructed in 2D exactly like $\widehat{\mathcal{F}}^n_{x_{i+1/2}} \approx \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \mathcal{F}_x \left(W(x_{i+1/2}, t)\right) dt$ is constructed in 1D
- However, the upwind method to be described here will not be based on $\widehat{\mathcal{F}}^n_{ij} = \overrightarrow{\mathcal{F}} \left(W_{RIEMANN}(M_{ij}, t)\right) \cdot \vec{\nu}_{ij}$



- Steger and Warming, 1981 (see (2) for 1D problems)

$$\begin{aligned}
\widehat{\mathcal{F}}_{ij}^n &= \widehat{\mathcal{F}}_{ij}^n\left(W_i, W_j, \vec{\nu}_{ij}\right) = A^+\left(W_i, \vec{\nu}_{ij}\right) W_i + A^-\left(W_j, \vec{\nu}_{ij}\right) W_j \\
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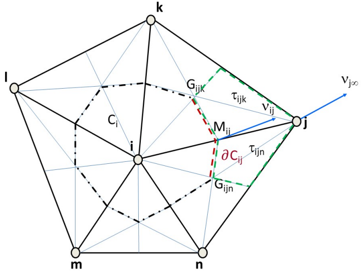

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- Steger-Warming flux at the far-field boundaries (**half cells and outward normals**)

$$\widehat{\mathcal{F}}_{j\infty}^n = \widehat{\mathcal{F}}_{j\infty}^n\left(W_j, W_\infty, \vec{\nu}_{j\infty}\right) = A^+\left(W_j, \vec{\nu}_{j\infty}\right) W_j + A^-\left(W_j, \vec{\nu}_{j\infty}\right) W_\infty$$

- Recall from Chapter 7 that the eigenvalues of $A(\vec{\nu}_{j\infty}) = A_{\vec{\nu}_{j\infty}}$ are: $v_{\vec{\nu}_{j\infty}} \pm c$, where $v_{\vec{\nu}_{j\infty}} = \vec{v} \cdot \vec{\nu}_{j\infty}$; and $v_{\vec{\nu}_{j\infty}}$ with multiplicity 3
- Hence, with the outward normal convention at a far-field boundary
  - at a subsonic inflow boundary, $v_{\vec{\nu}_{j\infty}} - c < 0$, $v_{\vec{\nu}_{j\infty}} < 0$, and $v_{\vec{\nu}_{j\infty}} + c > 0$
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- Analysis: if the far-field boundaries are located sufficiently far from the flow obstacle, at both subsonic inflow and outflow boundaries, the boundary treatment

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  - the term $A^-\left(W_j, \vec{\nu}_{j\infty}\right) W_\infty$ determines the correct number and chooses the correct combination of variables to specify, and assigns to them the correct values

$\Longrightarrow$ is a good example of a boundary treatment guided by the characteristic theory

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- note the limit "$j \to \infty$"