AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

The Finite Difference Method

These slides are based on the recommended textbook: Culbert B. Laney. "Computational Gas Dynamics," CAMBRIDGE UNIVERSITY PRESS, ISBN 0-521-62558-0

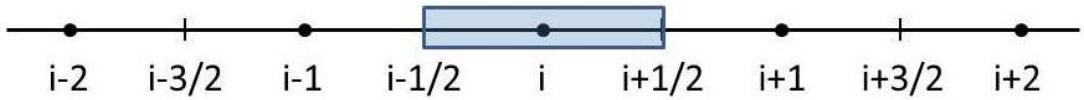
Outline

Note: The material covered in this chapter equally applies to scalar conservation laws and to the Euler equations, in one and multiple dimensions. In order to keep things as simple as possible, it is presented in most cases for scalar conservation laws: first in one dimension, then in multiple dimensions. It is also presented in two dimensions for the Euler equations.

- Recall that scalar conservation laws are simple scalar models of the Euler equations that can be written in strong conservation form as

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{1}$$

- Suppose that a 1D space is divided into grid points x_i and "cells" $[x_{i-1/2}, x_{i+1/2}]$, where $x_{i+1/2}$ is called a "cell edge"

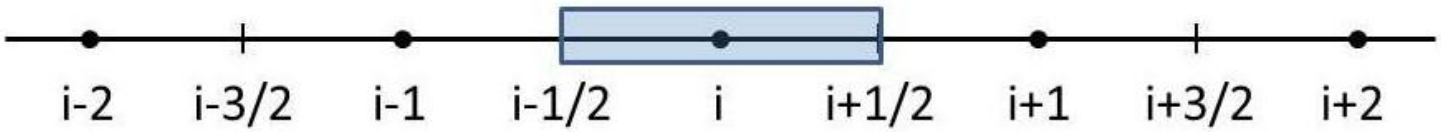


- Also suppose that time is divided into time-intervals $[t^n, t^{n+1}]$
- The *conservation form of a finite difference method* applied to the numerical solution of equation (1) is defined as follows

$$\boxed{\Delta t \widehat{\left(\frac{\partial u}{\partial t}\right)}_i^n = -\lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n)} \tag{2}$$

where the subscript i designates the point x_i, the superscript n designates the time t^n, a "hat" designates a time-approximation, and

$$\lambda = \frac{\Delta t}{\Delta x}, \quad \Delta t = t^{n+1} - t^n, \quad \Delta x = x_{i+1/2} - x_{i-1/2}$$

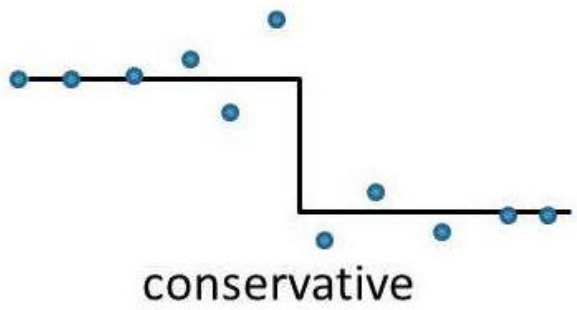


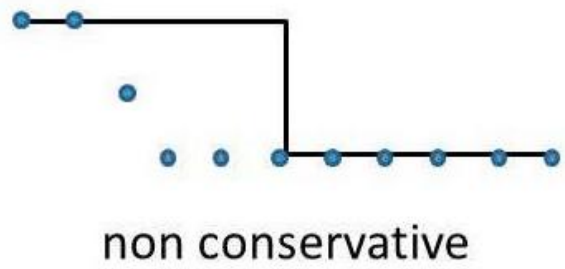
- One interpretation of the finite difference approach (2) and the conservation form label is the approximation of the following integral form of equation (1)

$$\frac{1}{\Delta x}\left(\int_{x_{i-1/2}}^{x_{i+1/2}}\left[u(x,t^{n+1})\,dx - u(x,t^{n})\right]dx\right)$$
$$= \quad -\left(\frac{\Delta t}{\Delta x}\right)\frac{1}{\Delta t}\left(\int_{t^n}^{t^{n+1}}\left[f\left(u(x_{i+1/2},t)\right) - f\left(u(x_{i-1/2},t)\right)\right]dt\right)$$

  which clearly describes a conservation law (hint: recall the mean value theorem and apply a first-order Taylor expansion to the integrand of the left term)

- Not every finite difference method can be written in conservation form: Those which can are called conservative and their associated quantities $\hat{f}^n_{i+1/2}$ are called *conservative numerical fluxes*
  - finite difference methods derived from the conservation form of the Euler equations or scalar conservation laws tend to be conservative
  - finite difference methods derived from other differential forms (for example, primitive or characteristic forms) of the aforementioned equations tend not to be conservative
  - conservative finite differencing implies correct shock and contact locations





- Like many approximation methods, conservative finite difference methods can be divided into *implicit* and *explicit* methods
  - in a typical implicit method

$$\begin{aligned} \widehat{\left(\frac{\partial u}{\partial t}\right)}_i^n &= \widehat{\left(\frac{\partial u}{\partial t}\right)}(u_{i-K_1}^n, ..., u_{i+K_2}^n; u_{i-L_1}^{n+1}, ..., u_{i+L_2}^{n+1}) \\ \hat{f}_{i+1/2}^n &= \hat{f}(u_{i-K_1+1}^n, ..., u_{i+K_2}^n; u_{i-L_1+1}^{n+1}, ..., u_{i+L_2}^{n+1}) \end{aligned} \tag{3}$$

so that from (2) one has

$$\boxed{u_i^{n+1} = u(u_{i-K_1}^n, ..., u_{i+K_2}^n; u_{i-L_1}^{n+1}, ..., u_i^{n+1}, ..., u_{i+L_2}^{n+1})} \tag{4}$$

$\Longrightarrow$ the solution of a system of equations is required at each time-step

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$$\widehat{\left(\frac{\partial u}{\partial t}\right)}_i^n = \widehat{\left(\frac{\partial u}{\partial t}\right)}(u^n_{i-K_1}, ..., u^n_{i+K_2}; u^{n+1}_{i-L_1}, ..., u^{n+1}_{i+L_2})$$
$$\hat{f}^n_{i+1/2} = \hat{f}(u^n_{i-K_1+1}, ..., u^n_{i+K_2}; u^{n+1}_{i-L_1+1}, ..., u^{n+1}_{i+L_2}) \quad (3)$$

so that from (2) one has

$$\boxed{u_i^{n+1} = u(u^n_{i-K_1}, ..., u^n_{i+K_2}; u^{n+1}_{i-L_1}, ..., u_i^{n+1}, ..., u^{n+1}_{i+L_2})} \quad (4)$$

$\Longrightarrow$ the solution of a system of equations is required at each time-step

Note: if $u^n_{i-K_1+1}$ $(u^n_{i-L_1+1})$ in (3) were written as $u^n_{i-K_1}$ $(u^n_{i-L_1})$, one would get the less convenient notation
$u_i^{n+1} = u(u^n_{i-K_1-1}, ..., u^n_{i+K_2}; u^{n+1}_{i-L_1-1}, ..., u^{n+1}_{i+L_2})$ instead of (4)

- Like many approximation methods, conservative finite difference methods can be divided into *implicit* and *explicit* methods (continue)
  - in a typical explicit method

$$\widehat{\left(\frac{\partial u}{\partial t}\right)}_i^n = \widehat{\left(\frac{\partial u}{\partial t}\right)}(u_{i-K_1}^n, ..., u_{i+K_2}^n; u_i^{n+1})$$
$$\hat{f}_{i+1/2}^n = \hat{f}(u_{i-K_1+1}^n, ..., u_{i+K_2}^n)$$

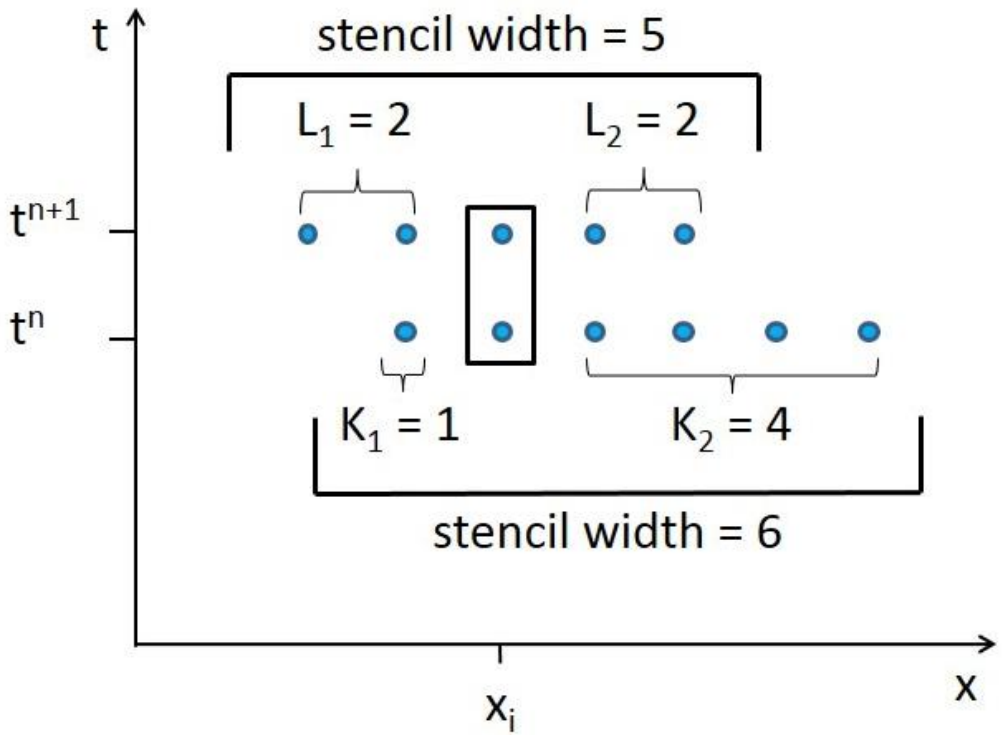
so that from (2) one has

$$\boxed{u_i^{n+1} = u(u_{i-K_1}^n, ..., u_{i+K_2}^n)}$$

$\Longrightarrow$ only function evaluations are incurred at each time-step

- $(u_{i-K_1}^n, ..., u_{i+K_2}^n)$ and $(u_{i-L_1}^{n+1}, ..., u_{i+L_2}^{n+1})$ are called the *stencil* or *direct numerical domain of dependence* of u_i^{n+1}
- $K_1 + K_2 + 1$ and $L_1 + L_2 + 1$ are called the *stencil widths*

- Summary: typical stencil diagram



- Like any proper numerical approximation, proper finite difference approximation becomes perfect in the limit $\Delta x \to 0$ and $\Delta t \to 0$
  - an approximate equation is *consistent* if it equals the true equation in the limit $\Delta x \to 0$ and $\Delta t \to 0$
  - a solution to an approximate equation is *convergent* if it equals the true solution of the true equation in the limit $\Delta x \to 0$ and $\Delta t \to 0$
- Hence, a conservative (finite difference) approximation is consistent when

$$\boxed{\hat{f}(u, ..., u) = f(u)}$$

  $\Longrightarrow$ in this case, the conservative numerical flux $\hat{f}$ is said to be *consistent* with the physical flux f
- A conservative numerical method — and therefore a conservative finite difference method — automatically *locates* shocks correctly (however, it does not necessarily reproduce their shapes correctly)
- Methods that explicitly enforce the Rankine-Hugoniot relations are called *shock-tracking* or *shock-fitting* methods
- Methods that do not explicitly enforce the Rankine-Hugoniot relations are called *shock-capturing* methods: They must be conservative and are the subject of this course

- Forward Time (FT) conservative finite difference methods correspond to the choices

$$\boxed{\Delta t \widehat{\left(\frac{\partial u}{\partial t}\right)}_i^n = u_i^{n+1} - u_i^n \quad \underline{\textbf{and}} \quad \hat{f}_{i+1/2}^n = \hat{f}(u_{i-K_1+1}^n, \ldots, u_{i+K_2}^n)}$$

  - with Forward Space (FS) approximation of the term $\frac{\partial u}{\partial x}(x_i, t^n)$, this leads to the FTFS scheme

$$u_i^{n+1} = u_i^n - \lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n), \quad \text{with} \quad \hat{f}_{i+1/2}^n = f(u_{i+1}^n)$$

  - with Backward Space (BS) approximation of the term $\frac{\partial u}{\partial x}(x_i, t^n)$, this leads to the FTBS scheme

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  - with Central Space (CS) approximation of the term $\frac{\partial u}{\partial x}(x_i, t^n)$, this leads to the FTCS scheme

$$u_i^{n+1} = u_i^n - \lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n), \text{ with } \hat{f}_{i+1/2}^n = \frac{1}{2}\left(f(u_{i+1}^n) + f(u_i^n)\right)$$

- Backward Time (BT) conservative finite difference methods correspond to the choices

$$\boxed{\Delta t \widehat{\left(\frac{\partial u}{\partial t}\right)}_i^n = u_i^{n+1} - u_i^n \quad \underline{\textbf{and}} \quad \hat{f}_{i+1/2}^n = \hat{f}(u_{i-L_1+1}^{n+1}, ..., u_{i+L_2}^{n+1})}$$

  - with Forward Space (FS) approximation of the term $\frac{\partial u}{\partial x}(x_i, t^n)$, this leads to the BTFS scheme

$$u_i^{n+1} = u_i^n - \lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n), \quad \text{with} \quad \hat{f}_{i+1/2}^n = f(u_{i+1}^{n+1})$$

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- Central Time (CT) conservative finite difference methods correspond to the choices

$$\Delta t \widehat{\left(\frac{\partial u}{\partial t}\right)}_i^n = \frac{1}{2}(u_i^{n+1} - u_i^{n-1}) \quad \underline{\textbf{and}} \quad \hat{f}_{i+1/2}^n = \hat{f}(u_{i-K_1+1}^n, ..., u_{i+K_2}^n)$$

  - with Forward Space (FS) approximation of the term $\frac{\partial u}{\partial x}(x_i, t^n)$, this leads to the CTFS scheme

$$u_i^{n+1} = u_i^{n-1} - 2\lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n), \quad \text{with} \quad \hat{f}_{i+1/2}^n = f(u_{i+1}^n)$$

  - with Backward Space (BS) approximation of the term $\frac{\partial u}{\partial x}(x_i, t^n)$, this leads to the CTBS scheme

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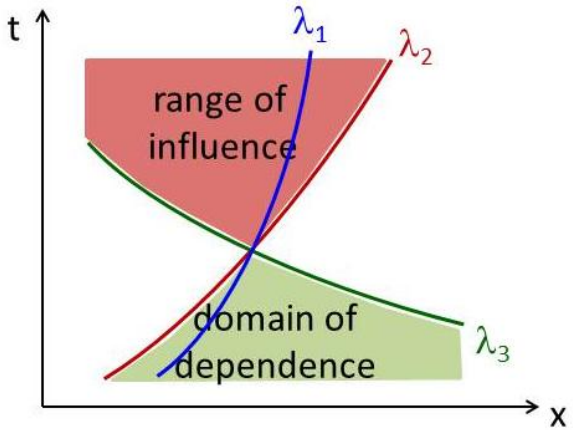
$$u_i^{n+1} = u_i^{n-1} - 2\lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n), \text{ with } \hat{f}_{i+1/2}^n = \frac{1}{2}(f(u_{i+1}^n) + f(u_i^n))$$

- Recall the theory of characteristics: A point in the $x - t$ plane is influenced only by points in a finite *domain of dependence* and influences only points in a finite *range of influence*



- Hence, the *physical domain of dependence* and *physical range of influence* are bounded on the right and left by the waves with the highest and lowest speeds
- In a well-posed problem, the range of influence of the initial and boundary conditions should exactly encompass the entire flow in the $x - t$ plane

- The *direct numerical domain of dependence* of a finite difference method is its stencil: For example, if the solution approximated by an implicit finite difference method can be written as

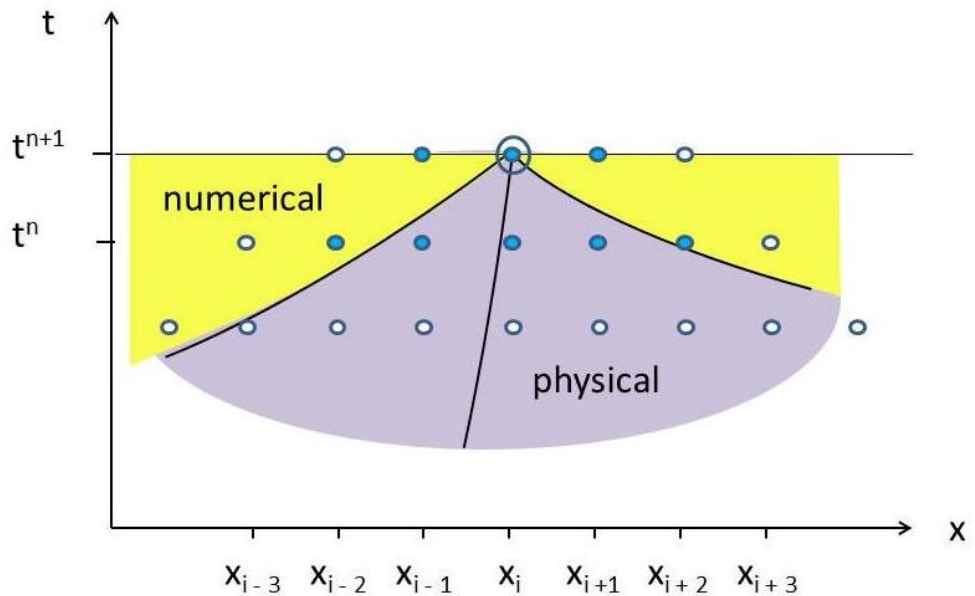
$$u_i^{n+1} = u(u_{i-K_1}^n, ..., u_{i+K_2}^n; u_{i-L_1}^{n+1}, ..., u_{i+L_2}^{n+1})$$

  its direct numerical domain of dependence is the region of the $x-t$ plane covered by the points $(u_{i-K_1}^n, ..., u_{i+K_2}^n; u_{i-L_1}^{n+1}, ..., u_{i+L_2}^{n+1})$
- Similarly, if the solution approximated by an explicit finite difference method can be written as

$$u_i^{n+1} = u(u_{i-K_1}^n, ..., u_{i+K_2}^n)$$

  its direct numerical domain of dependence is the region of the $x-t$ plane covered by the points $(u_{i-K_1}^n, ..., u_{i+K_2}^n)$
- The *full* (or *complete*) numerical domain of dependence of a finite difference method consists of the union of its direct numerical domain of dependence and the domain covered by the points of the $x-t$ plane upon which the numerical values in the direct numerical domain of dependence depend upon

- The Courant-Friedrichs-Lewy or (CFL) condition
  The full numerical domain of dependence must contain the physical domain of dependence



- Any numerical method that violates the CFL condition misses information affecting the exact solution and may blow up to infinity: For this reason, the CFL condition is *necessary* condition for numerical stability – however, it is *not* a *sufficient* condition for numerical stability

- Consider first the linear advection problem

$$\begin{aligned} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 0 \\ u(x,0) &= \begin{cases} 1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geq 0 \end{cases} \end{aligned}$$

- **Assume that** $a > 0$: The exact solution is

$$u(x,t) = u(x-at,0) = \begin{cases} 1 & \text{if} \quad x - at < 0 \\ 0 & \text{if} \quad x - at \geq 0 \end{cases}$$

- The FTFS approximation with $\Delta x = cst$ is

$$\begin{aligned} u_i^{n+1} &= (1+\lambda a)u_i^n - \lambda a u_{i+1}^n \\ u_i^0 &= u(i\Delta x, 0) = \begin{cases} 1 & \text{if} \quad i \leq -1 \\ 0 & \text{if} \quad i \geq 0 \end{cases} \end{aligned}$$

where as before, $\lambda = \dfrac{\Delta t}{\Delta x}$

- Then

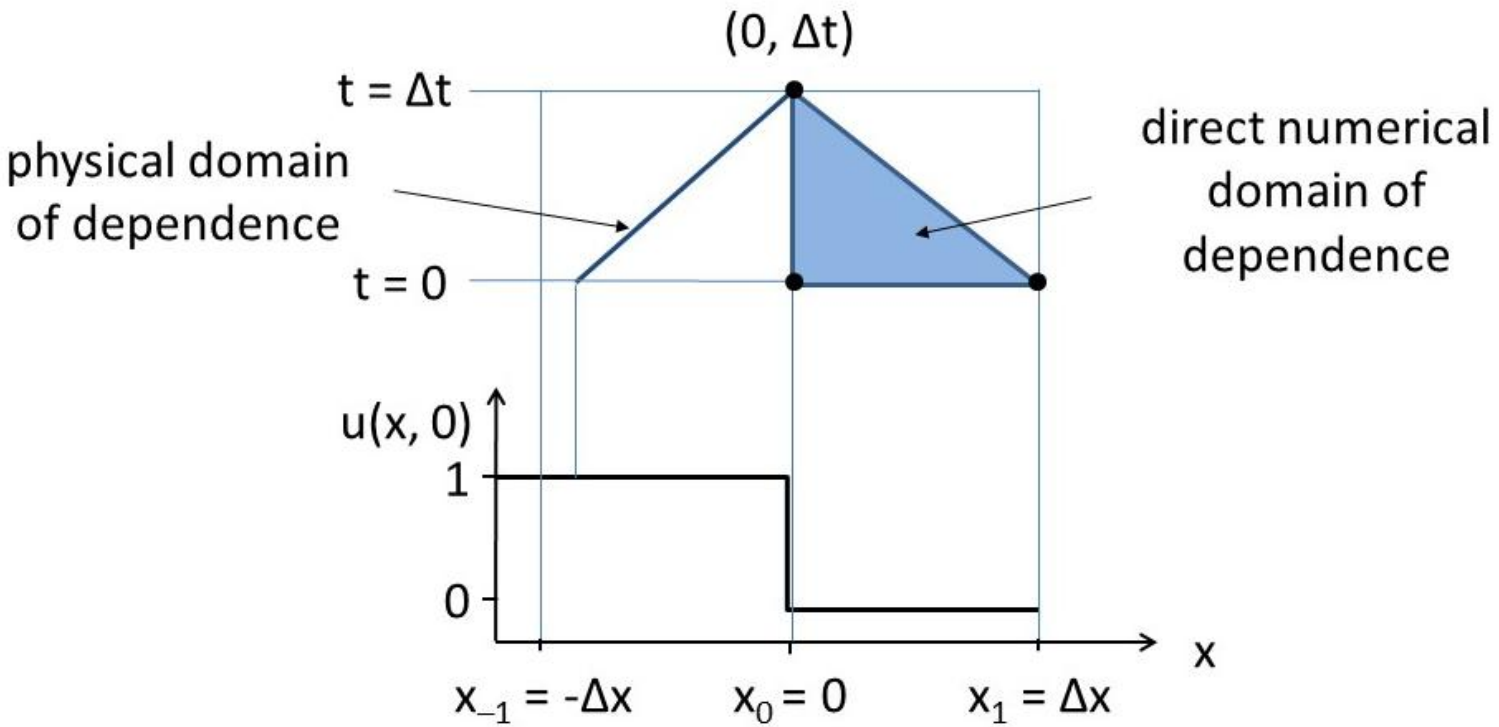
$$u_i^1 = \begin{cases} 1 & \text{if} \quad i \leq -2 \\ 1+\lambda a & \text{if} \quad i = -1 \\ 0 & \text{if} \quad i \geq 0 \end{cases}$$

$$u_i^2 = \begin{cases} 1 & \text{if} \quad i \leq -3 \\ (1+\lambda a)(1-\lambda a) & \text{if} \quad i = -2 \\ (1+\lambda a)(1+\lambda a) & \text{if} \quad i = -1 \\ 0 & \text{if} \quad i \geq 0 \end{cases}$$

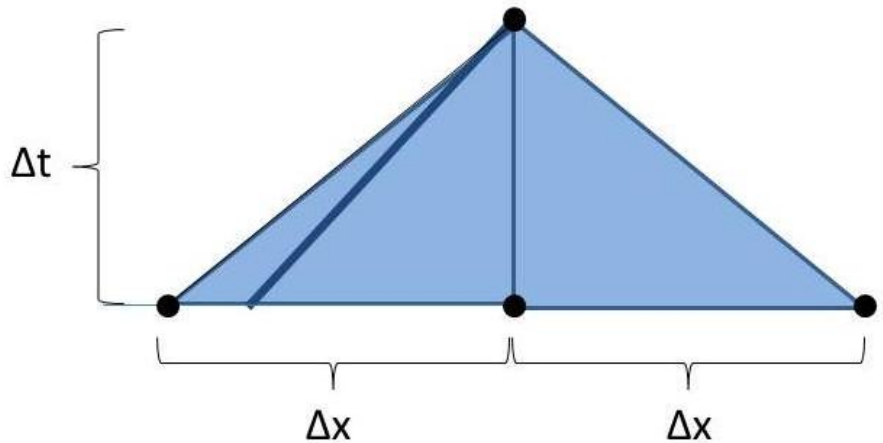
  and so forth
- The first two time-steps reveal that FTFS moves the jump in the wrong direction (left rather than right!) and produces spurious oscillations and overshoots
- Furthermore, the exact solution yields $u(0, \Delta t) = 1$, but FTFS yields $u_0^1 = 0$!

- This is because for $a > 0$, FTFS violates the CFL condition

$$u_i^{n+1} = (1 + \lambda a)u_i^n - \lambda a u_{i+1}^n$$



- FTCS satisfies the CFL condition if $\lambda \leq \dfrac{1}{|a|} \Rightarrow \boxed{-1 \leq \lambda a \leq 1}$



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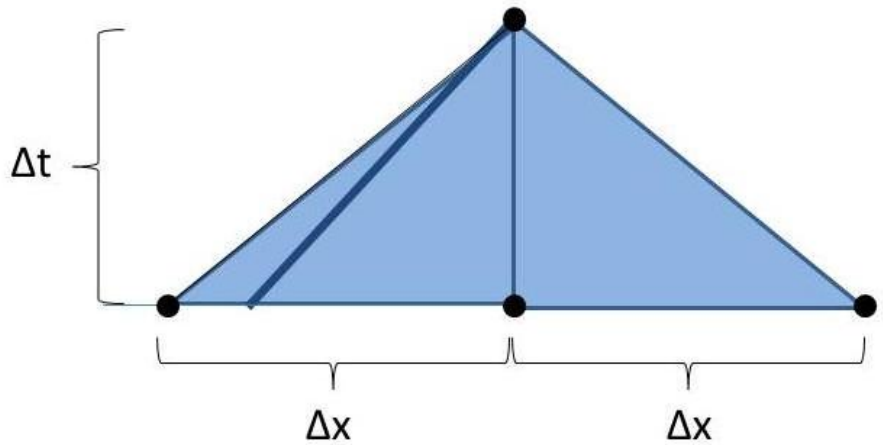

- however, it almost always blow up (as will be seen in a homework): This illustrates the fact that the CFL condition is a necessary but not sufficient condition for numerical stability

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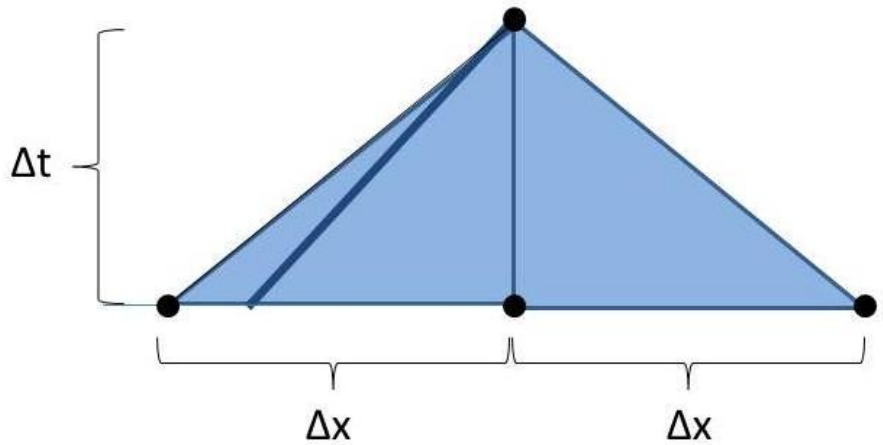
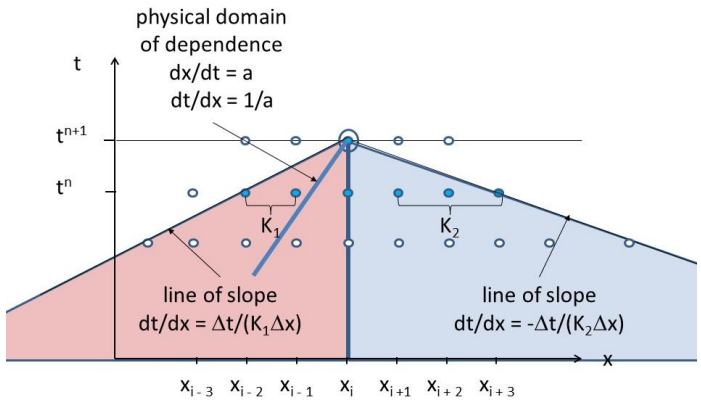


  - however, it almost always blow up (as will be seen in a homework): This illustrates the fact that the CFL condition is a necessary but not sufficient condition for numerical stability

- You can also check that when applied to the solution of any scalar conservation law, the BTCS method always satifies the CFL condition

- For scalar conservation laws, the CFL condition translates into a simple inequality restricting the time-step size
  - linear advection equation and explicit forward-time method with $u_i^{n+1} = u(u_{i-K_1}^n, \ldots, u_{i+K_2}^n)$
    - in the $x-t$ plane, the physical domain of dependence is the line of slope $1/a$
    - in the $x-t$ plane, the full numerical domain of dependence of u_i^{n+1} is bounded on the left by a line of slope $\dfrac{\Delta t}{K_1 \Delta x} = \dfrac{\lambda}{K_1}$ and on the right by a line of slope $-\dfrac{\Delta t}{K_2 \Delta x} = -\dfrac{\lambda}{K_2}$



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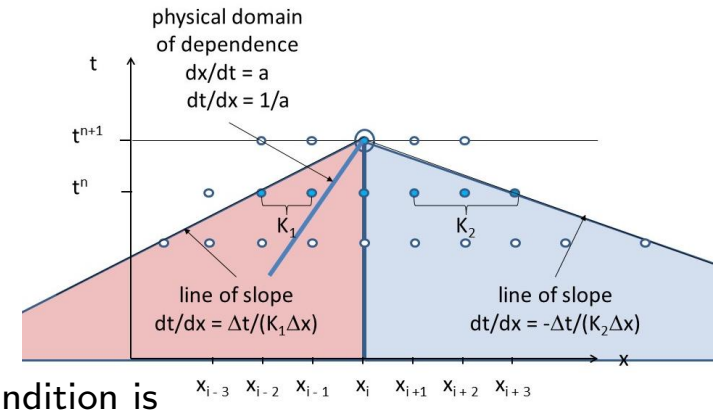
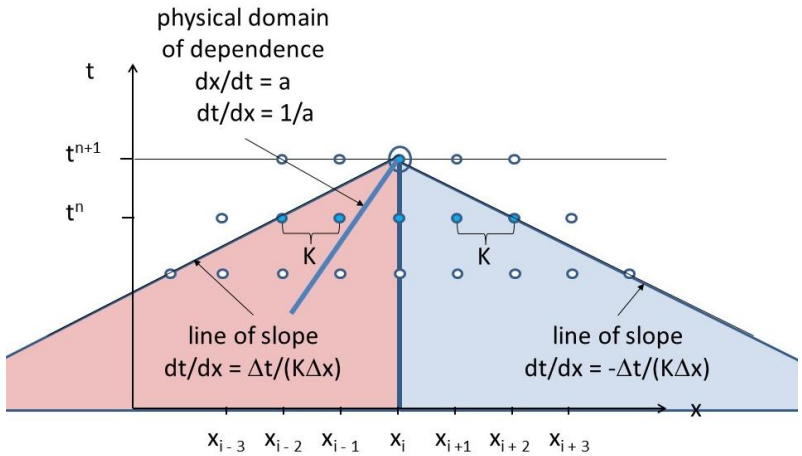


    - hence, the CFL condition is

$$\boxed{-\frac{K_2}{\lambda} \leq a \leq \frac{K_1}{\lambda} \Leftrightarrow -K_2 \leq \lambda a \leq K_1 \Leftrightarrow -K_2 \Delta x \leq a \Delta t \leq K_1 \Delta x}$$

    which requires that waves travel no more than K_1 points to the right or K_2 points to the left during a single time-step

- For scalar conservation laws, the CFL condition translates into a simple inequality restricting the time-step size (continue)
  - linear advection equation and explicit forward-time method with $u_i^{n+1} = u(u_{i-K_1}^n, \ldots, u_{i+K_2}^n)$ (continue)



  - if $K_1 = K_2 = K$, the previous CFL condition becomes

$$\boxed{\lambda|a| \leq K \Leftrightarrow \Delta t \leq K\frac{\Delta x}{|a|}} \tag{5}$$

  - for this reason, λa is called the *CFL number* or the *Courant number*
  - keep in mind however that in general, $a = a(u)$ and therefore the CFL number depends in general on the solution's range

- For scalar conservation laws, the CFL condition translates into a simple inequality restricting the time-step size (continue)
  - linear advection equation and implicit backward-time method with $u_i^{n+1} = u(u_{i-K_1}^n, ..., u_{i+K_2}^n; u_{i-L_1}^{n+1}, ..., u_{i+L_2}^{n+1})$
    - if $L_1 > 0$ and $L_2 = 0$, the full numerical domain of dependence of u_i^{n+1} includes everything to the left of $x = x_i$ and beneath $t = t^{n+1}$ in the $x - t$ plane
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    - if $L_1 = 0$ and $L_2 > 0$, the full numerical domain of dependence of u_i^{n+1} includes everything to the right of $x = x_i$ and beneath $t = t^{n+1}$ in the $x - t$ plane
    - if $L_1 > 0$ and $L_2 > 0$, the full numerical domain of dependence of u_i^{n+1} includes everything in the entire $x - t$ plane beneath $t = t^{n+1}$

conclusion: as long as their stencil includes one point to the left and one to the right, implicit methods avoid CFL restrictions by using the entire computational domain (hence, this includes BTCS but not BTFS or BTBS)

- In 1D, the Euler equations have three families of waves that define the physical domain of dependence
- For each family of waves, a CFL condition can be established for a given numerical method as in the case of a scalar conservation law: Then, the overall CFL condition is the most restrictive of all established CFL conditions
- For example, if $K_1 = K_2 = K$, A is the Jacobian matrix of the conservative flux vector, and $\rho(A)$ denotes its spectral radius $\left(\rho(A) = \max\left(|v_x - a|, |v_x|, |v_x + a|\right)\right.$, the CFL condition of an *explicit forward-time method* becomes $\left(\text{recall } (5)\right)$

$$\boxed{\lambda\rho(A) \leq K \Leftrightarrow \Delta t \leq K\frac{\Delta x}{\rho(A)}}$$

- $\lambda\rho(A)$ is called the CFL number or the *Courant number*

$$\lambda\rho(A) \leq K$$

- For supersonic flows, all waves travel in the same direction, either left or right $\Rightarrow$ the minimum stencil allowed by the CFL condition contains either W_{i-1}^n and W_i^n for right-running supersonic flow, or W_i^n and W_{i+1}^n for left-running supersonic flow
- For subsonic flows, waves travel in both directions, and the minimum stencil should always contain W_{i-1}^n, W_i^n, and W_{i+1}^n

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Hence, a *smart* or *adaptive* stencil can be useful for the case of the Euler equations!

- Unstable solutions exhibit *significant* spurious oscillations and/or overshoots
- Unstable solutions of linear problems exhibit *unbounded* spurious oscillations: Their errors grow to infinity as $t \to \infty$
- Hence the concept of instability discussed here for the solution of linear problems is that of *ubounded growth*

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- Hence the concept of instability discussed here for the solution of linear problems is that of *ubounded growth*

> Since unstable solutions typically oscillate, it makes sense to describe the solution of a linear problem such as a linear advection problem as a Fourier series (sum of oscillatory trigonometric functions)

- The Fourier series for the *continuous* (in space) solution $u(x, t^n)$ on any spatial domain $[a, b]$ is

$$u(x, t^n) = a_0^n + \sum_{m=1}^{\infty} a_m^n \cos\left(2\pi m \frac{x-a}{b-a}\right) + \sum_{m=1}^{\infty} b_m^n \sin\left(2\pi m \frac{x-a}{b-a}\right) \tag{6}$$

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- For a discrete solution $u(x_i, t^n)$, the Fourier series is obtained by sampling (6) as follows

$$u(x_i, t^n) = a_0^n + \sum_{m=1}^{\infty} a_m^n \cos\left(2\pi m \frac{x_i-a}{b-a}\right) + \sum_{m=1}^{\infty} b_m^n \sin\left(2\pi m \frac{x_i-a}{b-a}\right) \tag{7}$$

- Assume $x_{i+1} - x_i = \Delta x = cst$, $x_0 = a$, and $x_N = b \Rightarrow x_i - a = i\Delta x$ and $b - a = N\Delta x$: This transforms (7) into

$$u(x_i, t^n) = a_0^n + \sum_{m=1}^{\infty} \left(a_m^n \cos\left(2\pi m \frac{i}{N}\right) + b_m^n \sin\left(2\pi m \frac{i}{N}\right)\right) \tag{8}$$

- Recall that samples can only support wavelengths of $2\Delta x$ or longer – the Nyquist sampling theorem states that samples spaced apart by Δx perfectly represent functions whose shortest wavelengths are $4\Delta x$ ($N/m \geq 2$): Hence (8) is truncated as follows

$$u(x_i, t^n) \approx a_0^n + \sum_{m=1}^{N/2} \left(a_m^n \cos\left(\frac{2\pi mi}{N}\right) + b_m^n \sin\left(\frac{2\pi mi}{N}\right)\right) \qquad (9)$$

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  which is called a *discrete* Fourier series
- An equivalent expression in the complex plane using I as the notation for the pure imaginary number ($I^2 = -1$) is

$$u(x_i, t^n) \approx \sum_{m=-N/2}^{N/2} C_m^n e^{I\frac{2\pi mi}{N}} = \sum_{m=-N/2}^{N/2} u_{i_m}^n \qquad (10)$$

- From (9), (10), and Euler's formula $e^{I\theta} = \cos\theta + I\sin\theta$ it follows that

$$C_0^n = a_0^n, \quad C_m^n = \frac{a_m^n - Ib_m^n}{2}, \quad C_{-m}^n = \frac{a_m^n + Ib_m^n}{2}$$

- Hence, each term of the Fourier series can be written as

$$u_{i_m}^n = C_m^n e^{\left(I\frac{2\pi mi}{N}\right)} = C_m^n e^{I\phi_m i}$$

where $\phi_m = \dfrac{2\pi m}{N}$ and $m = -N/2, \cdots, N/2$

- Because of linearity, the amplification factor

$$\boxed{G_m = \frac{C_m^{n+1}}{C_m^n} = G_m(\lambda)}$$

does not depend on n: However, it depends on λ (since u_i^{n+1} and u_i^n are produced by the numerical scheme being analyzed), which itself depends on Δt

- Hence, each term of the Fourier series can be expressed as

$$u_{i_m}^n = \frac{C_m^n}{C_m^{n-1}} \cdots \frac{C_m^2}{C_m^1}\frac{C_m^1}{C_m^0} C_m^0 e^{I\phi_m i} = G_m \cdots G_m C_m^0 e^{I\phi_m i} = G_m^n C_m^0 e^{I\phi_m i}$$

where $G_m^n = G_m^n(\lambda) = (G_m(\lambda))^n$

- Finally, assume that $C_m^0 = 1$ (for example): This leads to

$$\boxed{u_{i_m}^n = G_m^n(\lambda) e^{I\phi_m i}}$$

- Conclusions
  - the linear approximation is linearly stable if $|G_m(\lambda)| < 1$ for all m
  - it is neutrally linearly stable if $|G_m(\lambda)| \leq 1$ for all m and $|G_m(\lambda)| = 1$ for some m
  - it is linearly unstable if $|G_m(\lambda)| > 1$ for some m
- Each of the above conclusion can be re-written in terms of $\lambda = \Delta t/\Delta x$
- Application (in class): apply the von Neumann analysis to determine the stability of the FTFS scheme for the linear advection equation

- Shortcomings of the von Neumann stability analysis method
  - requires the solution to be periodic ($u_0^n = u_N^n$)
  - requires constant spacing Δx
  - does not account for the boundary conditions

- Shortcomings of the von Neumann stability analysis method
  - requires the solution to be periodic ($u_0^n = u_N^n$)
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  - does not account for the boundary conditions
- Alternative method: so-called Matrix (eigenvalue analysis) Method based on the fact that for a linear problem and a linear approximation method, one has

$$u^{n+1} = M(\lambda)u^n, \qquad \text{where} \qquad u^n = (u_o^n \;\; u_1^n \;\; \cdots \;\; u_N^n)^T$$

  and M is an amplification matrix which depends on the stencil of the approximation scheme and on λ
- This implies

$$\boxed{u^n = M^n(\lambda)u^0}$$

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- Suppose that M is diagonalizable

$$M(\lambda) = Q^{-1}\Lambda(\lambda)Q, \qquad \Lambda = \text{diag}\left(\lambda_1(\lambda), \cdots, \lambda_N(\lambda)\right)$$

- Let $v = Qu$
- Then

$$M^n(\lambda) = Q^{-1}\Lambda^n Q \Rightarrow u^n = Q^{-1}\Lambda^n(\lambda)Qu^0 \Rightarrow Qu^n = \Lambda^n(\lambda)Qu^0$$

$$\Rightarrow \boxed{v^n = \Lambda^n(\lambda)v^0}$$

- Conclusions
  - the linear approximation is linearly stable if $\rho\left(M(\lambda)\right) < 1$
  - it is neutrally linearly stable if $\rho\left(M(\lambda)\right) = 1$
  - it is linearly unstable if $\rho\left(M(\lambda)\right) > 1$

Linear Stability Analysis

Matrix Method

- Advantages of the Matrix Method for (linear) stability analysis
  - does not require the solution to be periodic
  - does not require constant grid spacing
  - incorporates the effects of the boundary conditions
- Shortcoming: in general, the analytical computation of $\rho\left(M(\lambda)\right)$ is not trivial

- Advantages of the Matrix Method for (linear) stability analysis
  - does not require the solution to be periodic
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  - incorporates the effects of the boundary conditions
- Shortcoming: in general, the analytical computation of $\rho\left(M(\lambda)\right)$ is not trivial
- However, the above shortcoming is a non issue when the objective is to prove the unconditional stability of an (implicit) scheme
  - re-write the linear version of equation (2) before time-discretization in matrix form as

$$\frac{du}{dt} + B(\Delta x)u = 0 \tag{11}$$

  - suppose that B is diagonalizable and transform equation (11) into the set of independent *scalar* equations

$$\frac{dv_m}{dt} + \mu_m(\Delta x)v_m = 0, \qquad \mu_m > 0, \qquad m = 1, \cdots, N$$

  - focus on one of the above equations and discretize it in time
  - apply the scalar form of the Matrix Method for stability analysis: if the conclusion turns out to be independent of $\mu_m \Delta t$, the aforementioned shortcoming is a non issue
  - example (in class): apply the Matrix Method to determine the stability of a BT scheme for the linear advection equation
- Similarly, the above shortcoming is a non issue when only the maximum eigenvalue μ_m^{max} is needed to conclude the stability analysis – example (in class): apply the Matrix Method to determine the stability of an FT scheme for the same equation

- Formal order of accuracy measures the orders of accuracy of the individual space and time approximations separately
  - Taylor series expansions
  - modified linear equations
- However due to instability, formal order of accuracy may not be indicative of the actual performance of a method: For example, recall that a necessary stability condition is

$$\lambda\rho(A) \leq K \Rightarrow \Delta t\rho(A) \leq K\Delta x$$

and observe that such a stability condition prevents, for example, fixing Δt and studying the order of accuracy of the individual space approximation when $\Delta x \to 0$

- Besides formal order of accuracy, one way to measure the order of accuracy is to reduce Δx and Δt simultaneously while maintaining $\lambda = \dfrac{\Delta t}{\Delta x}$ constant and fixing the initial and boundary conditions
- In this case, a method is said to have *global* p-th order of accuracy (in space and time) if

$$\boxed{\|e\|_\infty \leq C_x \Delta x^p = C_t \Delta t^p, \qquad e_i = u(x_i, t^n) - u_i^n}$$

  for some constant C_x (independent of Δx) and the related constant $C_t = \dfrac{C_x}{\lambda^p}$ (independent of Δt)
- Other error measures can be obtained by using the 1-norm, 2-norm, or any vector norm, or if the error is measured pointwise

- Determining analytically the global order of accuracy defined above can be challenging: For this reason, it is usually predicted by comparing two different numerical solutions obtained using the same numerical method but two different values of Δx

$$\boxed{p = \frac{\log(\|e_2\|_\infty / \|e_1\|_\infty)}{\log(\Delta x_2 / \Delta x_1)} = \frac{\log(\|e_2\|_\infty / \|e_1\|_\infty)}{\log(\Delta t_2 / \Delta t_1)}}$$

where $e_{\ell_i} = u(x_i, t^n) - u_i^n$ is the error for Δx_ℓ and $\Delta t_\ell = \lambda \Delta x_\ell$, $\ell = 1,\ 2$

- Another way to measure the order of accuracy is to assume that the solution is perfect at time t^n — that is, $u_i^n = u(x_i, t^n) \, \forall i$, which is usually true for $n = 0$ — and measure the *local (in time) truncation error induced by a single time-step*

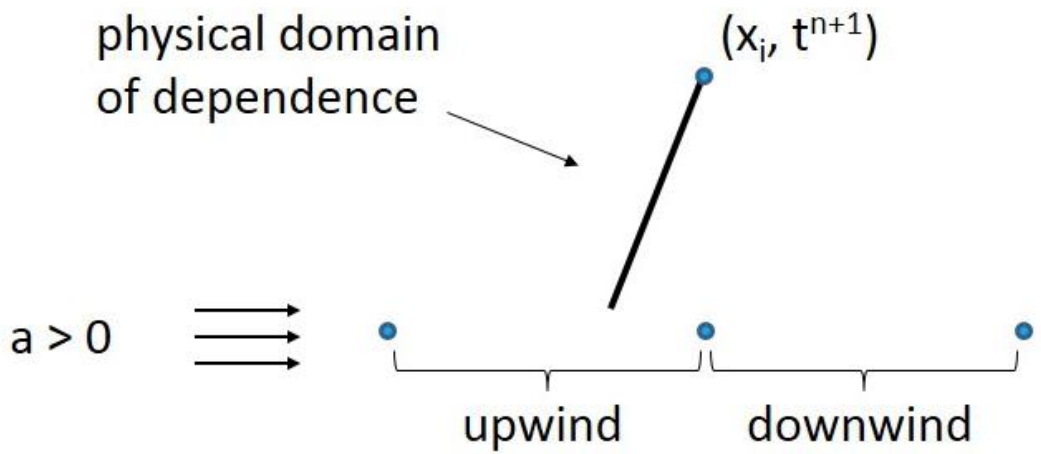
$$\boxed{\bar{e}_i = \frac{u(x_i, t^{n+1}) - u_i^{n+1}}{\Delta t}}$$

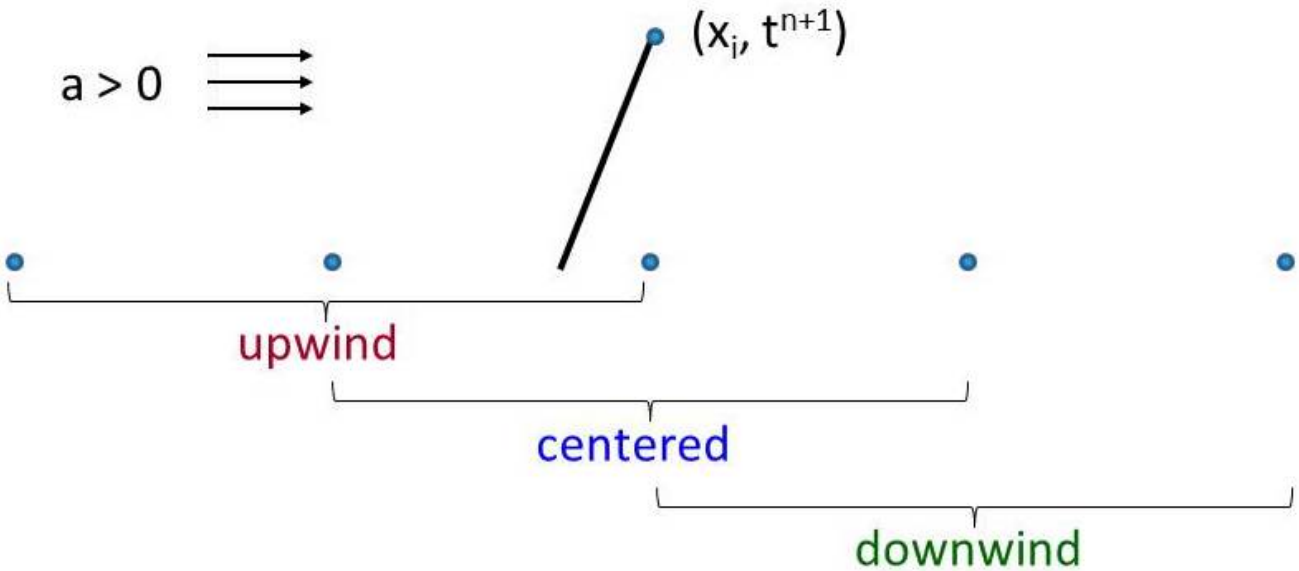
- Now, let $\Delta t \to 0$ and $\Delta x \to 0$ while maintaining $\lambda = \dfrac{\Delta t}{\Delta x}$ constant, and the initial and boundary conditions fixed: Then, a method is said to have *local* p-th order of accuracy (in space and time) if

$$\boxed{\|\bar{e}\|_\infty \leq C_x \Delta x^p = C_t \Delta t^p}$$

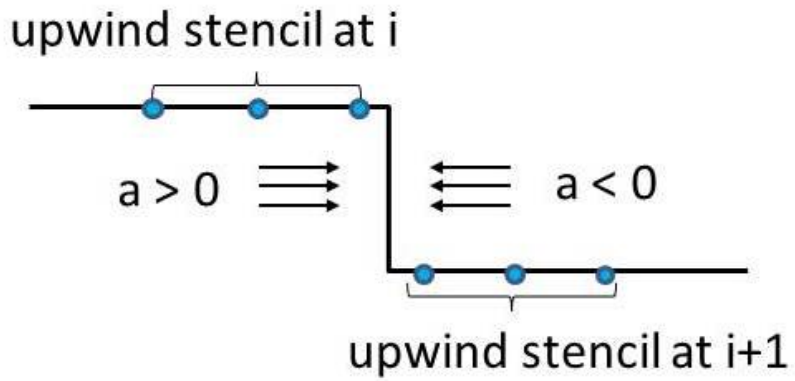
  for some constant C_x and the related constant $C_t = \dfrac{C_x}{\lambda^p}$
- The local order of accuracy is easy to determine analytically
- Example (in class): determine analytically the local order of accuracy of FTFS for linear advection

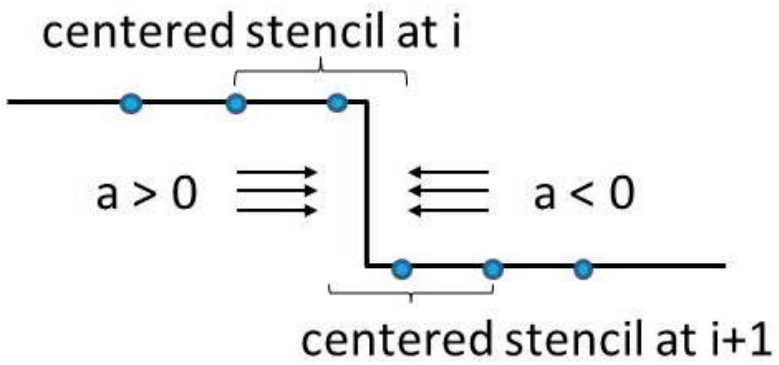
- Consider a nonlinear scalar conservation law
- In 1D, there are right-running waves and left-running waves: For right-running waves, right is the *downwind* direction and left is the *upwind* direction
- Similarly for left-running waves, left is the *downwind* direction and right is the *upwind* direction
- Then, every numerical approximation to a scalar conservation law can be described as
  - *Centered*: if its stencil contains equal numbers of points in both directions
  - *Upwind*: if its stencil contains more points in the upwind direction
  - *Downwind*: if its stencil contains more points in the downwind direction
- Upwind and downwind stencils are *adjustable* or *adaptive* stencils: Upwind and downwind methods test for wind direction and then, based on the outcome of the tests, select either a right- or a left-biased stencil


physical domain
of dependence
(x_i, t^{n+1})
a > 0
upwind
downwind

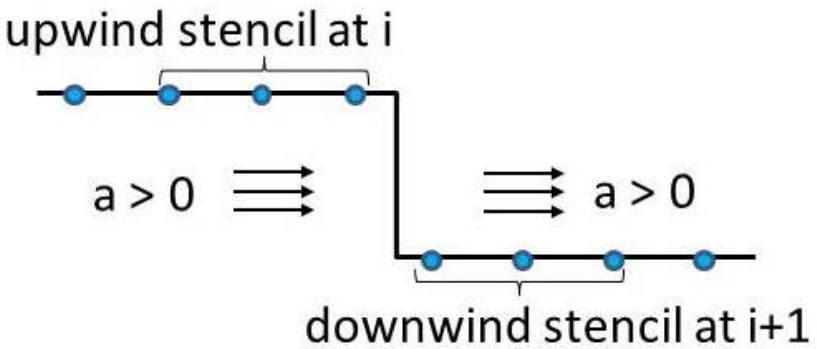

a > 0
(x_i, t^{n+1})
upwind
centered
downwind

- Upwinding ensures shock avoidance if the shock reverses the wind, whereas central differencing does not





- Upwinding does not ensure shock avoidance if the shock does not reverse the wind



  - downwinding on the right above avoids the shock but violates the CFL condition and thus would create larger errors than crossing the shock would

- General remarks
  - upwind methods are popular because of their excellent shock capturing ability
  - among simple FT or BT methods, upwind methods outdo centered methods: However, higher-order upwind methods often have no special advantages over higher-order centered methods
- Sample techniques for designing methods with upwind and adaptive stencils
  - flux averaging methods
  - flux splitting methods*
  - wave speed splitting methods*

- Flux splitting is defined as

$$\boxed{\begin{aligned} f(u) &= f^+(u) + f^-(u) \\ \frac{df^+}{du} &\geq 0, \qquad \frac{df^-}{du} \leq 0 \end{aligned}}$$

- Hence, $f^+(u)$ is *associated* with a right-running wave and $f^-(u)$ is associated with a left-running wave
- Using flux splitting, the governing conservation law becomes

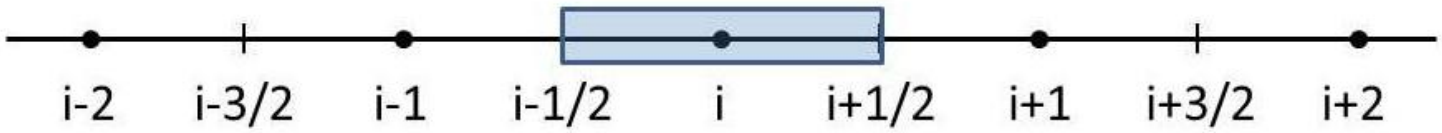
$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f^+}{\partial x} + \frac{\partial f^-}{\partial x} = 0}$$

  which is called the *flux split form*
- Then, $\dfrac{\partial f^+}{\partial x}$ can be discretized conservatively using at least one point to the left, and $\dfrac{\partial f^-}{\partial x}$ can be discretized conservatively using at least one point to the right, thus obtaining conservation and satisfaction of the CFL condition

- Unfortunately, because in general

$$\frac{\partial f^+}{\partial u} \neq \underbrace{\left(\frac{\partial f}{\partial u}\right)^+}_{a(u)^+} \quad \text{and} \quad \frac{\partial f^-}{\partial u} \neq \underbrace{\left(\frac{\partial f}{\partial u}\right)^-}_{a(u)^-}$$

  flux splitting cannot describe the true connection between fluxes and waves, unless all waves run in the same direction
  - if all waves are right-running, the unique physical flux splitting is $f^+ = f$ and $f^- = 0$
  - if all waves are left-running, the unique physical flux splitting is $f^- = f$ and $f^+ = 0$
- All waves run in the same direction only for (nonlinear) scalar conservation laws away from sonic points, and for the Euler equations in the supersonic regime



- Assume that $\dfrac{\partial f^+}{\partial x}$ is discretized with a leftward bias so that the approximation at $x = x_i$ is centered or biased towards $x = x_{i-1/2}$

$$\left(\frac{\partial f^+}{\partial x}\right)_i \approx \frac{\Delta \hat{f}^+_{i-1/2}}{\Delta x} \quad \text{for some } \Delta \hat{f}^+_{i-1/2}$$

- Assume that $\dfrac{\partial f^-}{\partial x}$ is discretized with a rightward bias so that the approximation at $x = x_i$ is centered or biased towards $x = x_{i+1/2}$

$$\left(\frac{\partial f^-}{\partial x}\right)_i \approx \frac{\Delta \hat{f}^-_{i+1/2}}{\Delta x} \quad \text{for some } \Delta \hat{f}^-_{i+1/2}$$

- Using forward Euler to perform the time-discretization leads to

$$\boxed{u_i^{n+1} = u_i^n - \lambda(\Delta \hat{f}^{+^n}_{i-1/2} + \Delta \hat{f}^{-^n}_{i+1/2})}$$

which is called the *flux split form* of the numerical approximation

- A method in flux split form is conservative if and only if

$$\boxed{\Delta \hat{f}^{+^n}_{i+1/2} + \Delta \hat{f}^{-^n}_{i+1/2} = g^n_{i+1} - g^n_i \text{ for some } g^n_i} \tag{12}$$

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- Proof

$$u^{n+1}_i = u^n_i - \lambda(\Delta \hat{f}^{+^n}_{i-1/2} + g^n_i - g^n_i + \Delta \hat{f}^{-^n}_{i+1/2})$$

  - compare with the conservation form $u^{n+1}_i = u^n_i - \lambda(\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2})$

$$\implies \hat{f}^n_{i+1/2} = \Delta \hat{f}^{-^n}_{i+1/2} + g^n_i, \qquad \hat{f}^n_{i-1/2} = -\Delta \hat{f}^{+^n}_{i-1/2} + g^n_i$$

$$\implies \underline{\Delta \hat{f}^{-^n}_{i+1/2} = \hat{f}^n_{i+1/2} - g^n_i}, \qquad \underline{\Delta \hat{f}^{+^n}_{i-1/2} = -\hat{f}^n_{i-1/2} + g^n_i}$$

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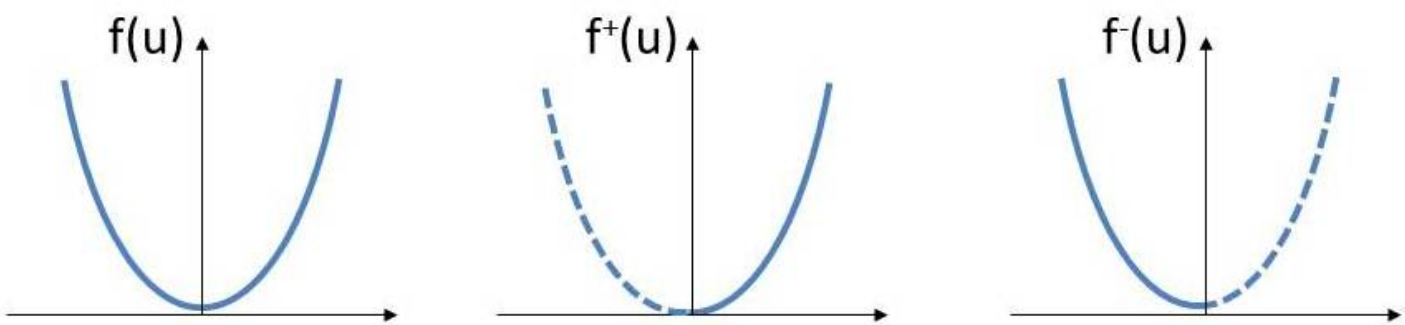
  - require now that
$\hat{f}^n_{i+1/2} = \hat{f}^n_{(i+1)-1/2} \Rightarrow \Delta \hat{f}^{-^n}_{i+1/2} + g^n_i = -\Delta \hat{f}^{+^n}_{i+1/2} + g^n_{i+1}$

$$\implies \Delta \hat{f}^{+^n}_{i+1/2} + \Delta \hat{f}^{-^n}_{i+1/2} = g^n_{i+1} - g^n_i$$

- Since there are no restrictions on g^n_i, every conservative method has infinitely many flux split forms useful for **nonlinear stability** analysis

- Example: Design a first-order upwind method for Burgers' equation using flux splitting then re-write it in conservation form
  - for Burgers' equation, the unique physical flux splitting is

$$f(u) = \frac{u^2}{2} = \underbrace{\max(0,u)\frac{u}{2}}_{f^+(u)} + \underbrace{\min(0,u)\frac{u}{2}}_{f^-(u)}$$



- Example: Design a first-order upwind method for Burgers' equation using flux splitting then re-write it in conservation form (continue)
  - a flux split form of Burgers' equation is

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(\max(0,u)u\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\min(0,u)u\right) = 0$$

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  - a backward-space approximation of $\dfrac{\partial f^+}{\partial x}$ gives

$$\left(\frac{1}{2}\frac{\partial}{\partial x}\left(\max(0,u)u\right)\right)_i^n \approx \frac{\Delta \hat{f}^{+^n}_{i-1/2}}{\Delta x} = \frac{\max(0,u_i^n)u_i^n - \max(0,u_{i-1}^n)u_{i-1}^n}{2\Delta x}$$

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  - a forward-space approximation of $\dfrac{\partial f^-}{\partial x}$ gives

$$\left(\frac{1}{2}\frac{\partial}{\partial x}\left(\min(0,u)u\right)\right)_i^n \approx \frac{\Delta \hat{f}^{-\,n}_{i+1/2}}{\Delta x} = \frac{\min(0,u_{i+1}^n)u_{i+1}^n - \min(0,u_i^n)u_i^n}{2\Delta x}$$

- Example: Design a first-order upwind method for Burgers' equation using flux splitting then re-write it in conservation form (continue)
  - a flux split form of Burgers' equation is

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(\max(0,u)u\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\min(0,u)u\right) = 0$$

  - a backward-space approximation of $\frac{\partial f^+}{\partial x}$ gives

$$\left(\frac{1}{2}\frac{\partial}{\partial x}\left(\max(0,u)u\right)\right)_i^n \approx \frac{\Delta \hat{f}^{+\,n}_{i-1/2}}{\Delta x} = \frac{\max(0,u_i^n)u_i^n - \max(0,u_{i-1}^n)u_{i-1}^n}{2\Delta x}$$

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  - combining these with an FT approximation yields

$$\begin{aligned} u_i^{n+1} &= u_i^n - \frac{\lambda}{2}\left(\max(0,u_i^n)u_i^n - \max(0,u_{i-1}^n)u_{i-1}^n\right) \\ &- \frac{\lambda}{2}\left(\min(0,u_{i+1}^n)u_{i+1}^n - \min(0,u_i^n)u_i^n\right) \end{aligned}$$

- Example: Design a first-order upwind method for Burgers' equation using flux splitting then re-write it in conservation form (continue)
  - the reader can check that the first-order upwind method described in the previous page can be re-written in conservation form using

$$g_i^n = \frac{1}{2}(u_i^n)^2$$

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which means that $f(u)$ is a homogeneous function of degree 1 (recall Euler's theorem which states that a differentiable function $f(u)$ is a homogeneous function of degree p if and only if $(df/du)\ u = pf(u)$): This property makes flux splitting and wave speed splitting closely related

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- Then, $a^+ \dfrac{\partial u}{\partial x}$ can be discretized conservatively using at least one point to the left, and $a^- \dfrac{\partial u}{\partial x}$ can be discretized conservatively using at least one point to the right, thus obtaining satisfaction of the CFL condition

- For vector conservation laws such as the Euler equations, one can split the Jacobian matrix as follows

$$A(W) = A^+(W) + A^-(W)$$

where the eigenvalues of A^+ are non negative and those of A^- are non positive

$$A^+ \geq 0, \qquad A^- \leq 0$$

(recall that A^+ and A^- are obtained by computing and splitting the eigenvalues of A)

- The wave speed split form of the Euler equations can then be written as

$$\frac{\partial W}{\partial t} + A^+ \frac{\partial W}{\partial x} + A^- \frac{\partial W}{\partial x} = 0$$

- Again, $A^+ \dfrac{\partial W}{\partial x}$ can then be discretized conservatively using at least one point to the left, and $A^- \dfrac{\partial W}{\partial x}$ using at least one point to the right, thus obtaining satisfaction of the CFL condition

- Back to scalar conservation laws
- If $f(u)$ is a homogeneous function of degree 1, then from Euler's theorem it follows that

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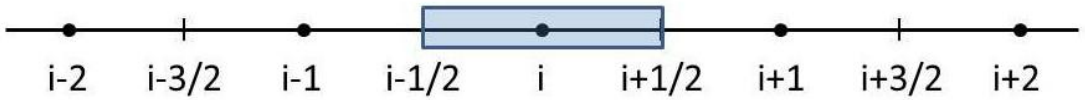
- Why?



- Assume that $a^+ \dfrac{\partial u}{\partial x}$ is discretized with a leftward bias so that the approximation at $x = x_i$ is centered or biased towards $x = x_{i-1/2}$

$$\left(a^+ \frac{\partial u}{\partial x} \right)_i^n \approx a^{+^n}_{i-1/2} \frac{u_i^n - u_{i-1}^n}{\Delta x}$$

- Assume that $a^- \dfrac{\partial u}{\partial x}$ is discretized with a rightward bias so that the approximation at $x = x_i$ is centered or biased towards $x = x_{i+1/2}$

$$\left(a^- \frac{\partial u}{\partial x} \right)_i^n \approx a^{-^n}_{i+1/2} \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

- Using forward Euler to perform the time-discretization leads to

$$\boxed{u_i^{n+1} = u_i^n - \lambda a^{-^n}_{i+1/2}(u_{i+1}^n - u_i^n) - \lambda a^{+^n}_{i-1/2}(u_i^n - u_{i-1}^n)}$$

which is called the *wave speed split form* of the numerical approximation

- The flux split form and wave speed form are connected via

$$\boxed{\Delta \hat{f}^{\pm^n}_{i+1/2} = a^{\pm^n}_{i+1/2}(u^n_{i+1} - u^n_i)}$$

- From the above relation and equation (12), it follows that a method in wave speed split form is conservative if and only if

$$\boxed{(a^{+^n}_{i+1/2} + a^{-^n}_{i+1/2})(u^n_{i+1} - u^n_i) = g^n_{i+1} - g^n_i \text{ for some flux function } g^n_i} \tag{13}$$

- Hence, the transformation from conservation form to wave speed form and vice versa is

$$\boxed{\hat{f}^n_{i+1/2} = a^{-^n}_{i+1/2}(u^n_{i+1} - u^n_i) + g^n_i, \quad \hat{f}^n_{i-1/2} = -a^{+^n}_{i-1/2}(u^n_i - u^n_{i-1}) + g^n_i} \tag{14}$$

$$u_i^{n+1} = u_i^n - \lambda \underbrace{a^{-n}_{i+1/2}(u_{i+1}^n - u_i^n)}_{\Delta \hat{f}^{-n}_{i+1/2}} - \lambda \underbrace{a^{+n}_{i-1/2}(u_i^n - u_{i-1}^n)}_{\Delta \hat{f}^{+n}_{i-1/2}}$$

- The above notation for the wave speed split form is the standard notation when wave speed splitting is used to derive new (forward Euler) approximation methods

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- The above notation for the wave speed split form is the standard notation when wave speed splitting is used to derive new (forward Euler) approximation methods
- Wave speed split form is also often used as a preliminary step in nonlinear stability analysis, in which case the standard notation is

$$\boxed{u_i^{n+1} = u_i^n + C_{i+1/2}^{+^n}(u_{i+1}^n - u_i^n) - C_{i-1/2}^{-^n}(u_i^n - u_{i-1}^n)}$$

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- Hence

$$\boxed{C_{i+1/2}^{+^n} = -\lambda a_{i+1/2}^{-^n}, \qquad C_{i-1/2}^{-^n} = \lambda a_{i-1/2}^{+^n} \Leftrightarrow C_{i+1/2}^{-^n} = \lambda a_{i+1/2}^{+^n}} \tag{15}$$

- From (15), it follows that if a method is derived using wave speed splitting and not just written in wave speed split form, the splitting underlying (13) can also be written as

$$\boxed{\lambda a(u) = \lambda a^+(u) + \lambda a^-(u) = C^-(u) - C^+(u), \quad C^+(u) \geq 0, C^-(u) \geq 0}$$

- Then, the conservation condition (13) becomes

$$\boxed{(C^{-^n}_{i+1/2} - C^{+^n}_{i+1/2})(u^n_{i+1} - u^n_i) = \lambda(g^n_{i+1} - g^n_i)}$$

- And equations (14) become

$$\lambda \hat{f}^n_{i+1/2} = -C^{+^n}_{i+1/2}(u^n_{i+1} - u^n_i) + \lambda g^n_i, \quad \lambda \hat{f}^n_{i-1/2} = -C^{-^n}_{i-1/2}(u^n_i - u^n_{i-1}) + \lambda g^n_i$$

- Focus is set here on explicit FT difference approximations
- Recall that unstable solutions exhibit *significant* spurious oscillations and/or overshoots
- Recall also that linear stability analysis focuses on these oscillations and relies on the Fourier series representation of the numerical solution: It requires only that this solution should not blow up, or more specifically, that each component in its Fourier series representation should not increase to infinity
  - because of linearity, this is equivalent to requiring that each component in the Fourier series should shrink by the same amount or stay constant at each time-step

- Similarly, nonlinear stability analysis focuses on the spurious oscillations of the numerical solution, but without representing it by a Fourier series

[1] The total variation (TV) of f defined on $[a, b] \subset \mathbb{R}$ is a measure of the 1D arclength of the curve with parametric equation $x \to f(x)$, for $x \in [a, b]$. In systems described by PDEs such as $\partial u/\partial t + a(u)\partial u/\partial x = 0$ however, the TV is given by $TV(u(\cdot, t)) = \int |\partial u/\partial x| dx$

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  - however, whereas not blowing up and shrinking are equivalent notions for linear equations, these are different notions for nonlinear equations: In particular, TVD implies TVB but TVB does not necessarily imply TVD

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- Monotonocity preservation was first suggested by the Russian scientist Godunov in 1959: It is a nonlinear stability condition, but not a great one for the following reasons:
  - it does not address the case of nonmonotone solutions
  - it is a too strong condition:
    - it does not allow even an *insignificant* spurious oscillation that does not threaten numerical stability
    - attempting to purge all oscillatory errors, even the small ones, may cause much larger nonoscillatory errors

- **Godunov's theorem:** *For linear (explicit) schemes*, that is, schemes that can be described by

$$u_j^{n+1} = \sum_{m=-K_1}^{K_2} \gamma_m u_{j+m}^n$$

(NOT to be confused with linear problems), *monotonicity preservation leads to first-order accuracy at best*

- TVD was first proposed by the american applied mathematician Amiram Harten in 1983 as a nonlinear stability condition
- The total variation of the exact solution may be defined as follows

$$TV\left(u(\cdot,t)\right) = \sup_{\text{all possible sets of samples } x_i} \sum_{i=-\infty}^{\infty} |u(x_{i+1},t) - u(x_i,t)|$$

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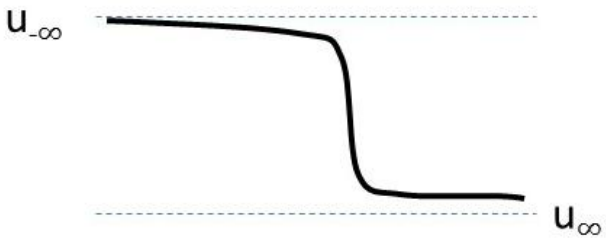
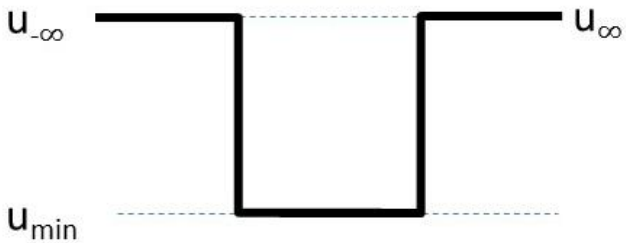
- Laney and Caughey (1991):

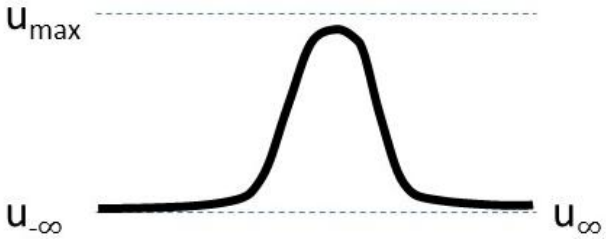
The total variation of a function on an infinite domain is a sum of extrema — maxima counted positively and minima counted negatively — with the two infinite boundaries always treated as extrema and counting each once, and every other extrema counting twice.



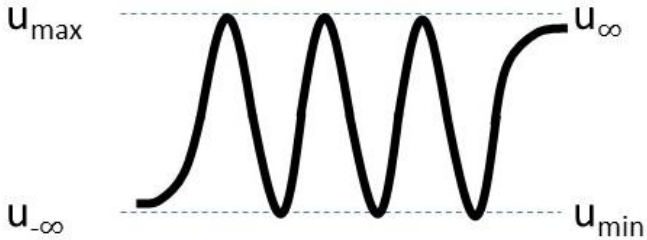
$$TV = u_{-\infty} - u_{\infty}$$



$$TV = -2\,u_{min} + u_{-\infty} + u_{\infty}$$

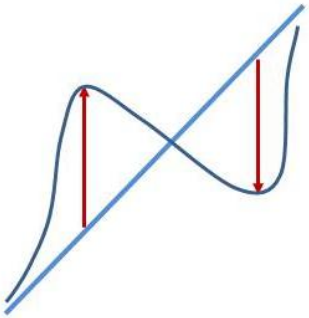


$$TV = 2\,u_{max} - u_{-\infty} - u_{\infty}$$

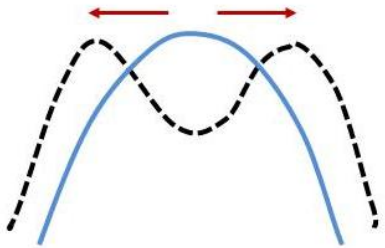


$$TV = 6\,u_{max} - 6\,u_{min} + u_{\infty} - u_{-\infty}$$
$$= 7\,u_{max} - 7\,u_{min}$$

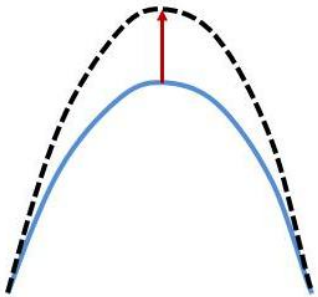
- Numerical effects that can cause the total variation to increase



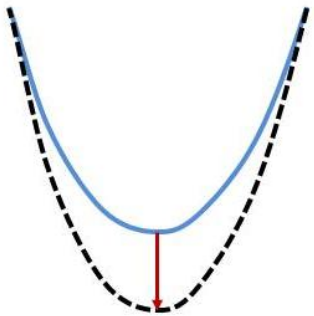
transforming a monotone region into one with a maximum and a minimum



splitting a maximum into two maxima and a minimum



amplifying a maximum



decreasing a minimum

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- The total variation of a numerical approximation at time t^n may be equally defined as

$$\boxed{TV(u^n) = \sum_{i=-\infty}^{\infty} |u_{i+1}^n - u_i^n|}$$

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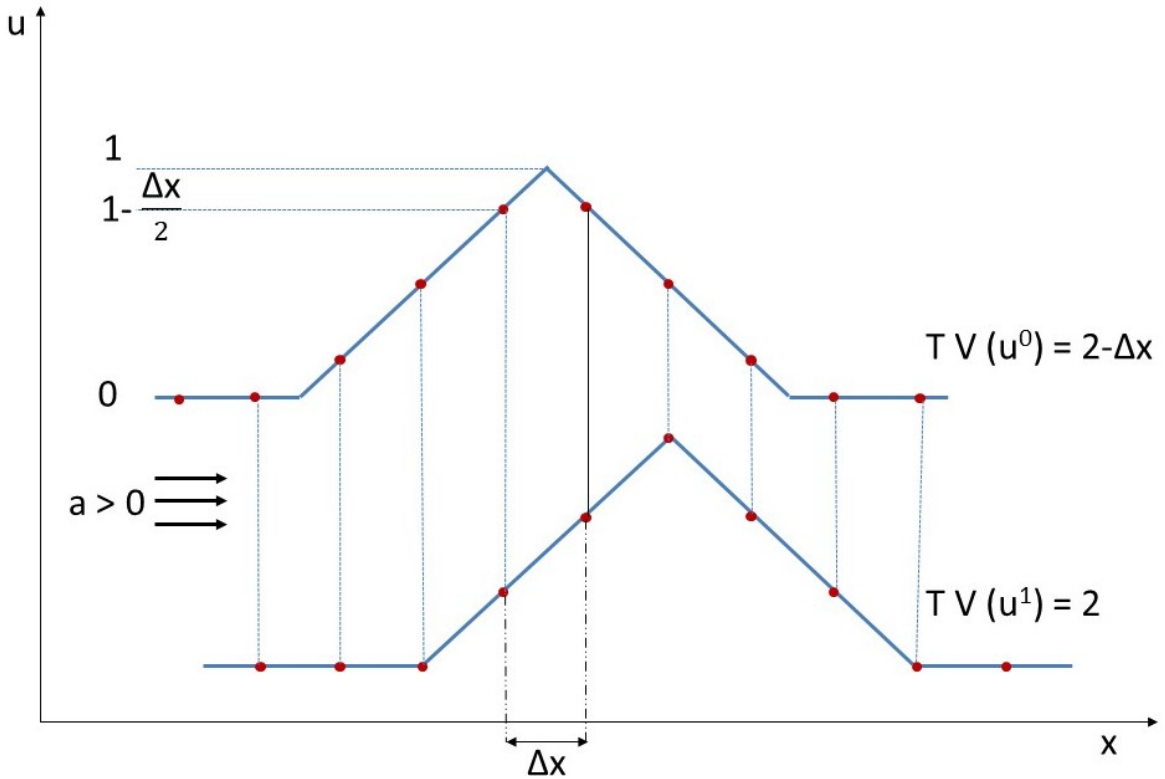
- Now, a numerical approximation inherits the TVD property if

$$\forall n,\ TV\left(u^{n+1}\right) \leq TV(u^n)$$

- Important result: *TVD implies monotonicity preservation* and therefore implies nonlinear stability
- Proof: Suppose that the initial condition is monotone
  - the TV of the initial condition is $u_\infty - u_{-\infty}$ if it is monotone increasing and $u_{-\infty} - u_\infty$ if it is monotone decreasing
  - if the numerical solution remains monotone, TV $= cst$; otherwise, it develops new maxima and minima causing the TV to increase
  - if the approximation method is TVD, this cannot happen and therefore the numerical solution remains monotone
- TVD can be a stronger nonlinear stability condition than the monotonicity preserving condition

- Drawback: Clipping phenomenon (illustrated with the linear advection of a triangle-shaped initial condition)



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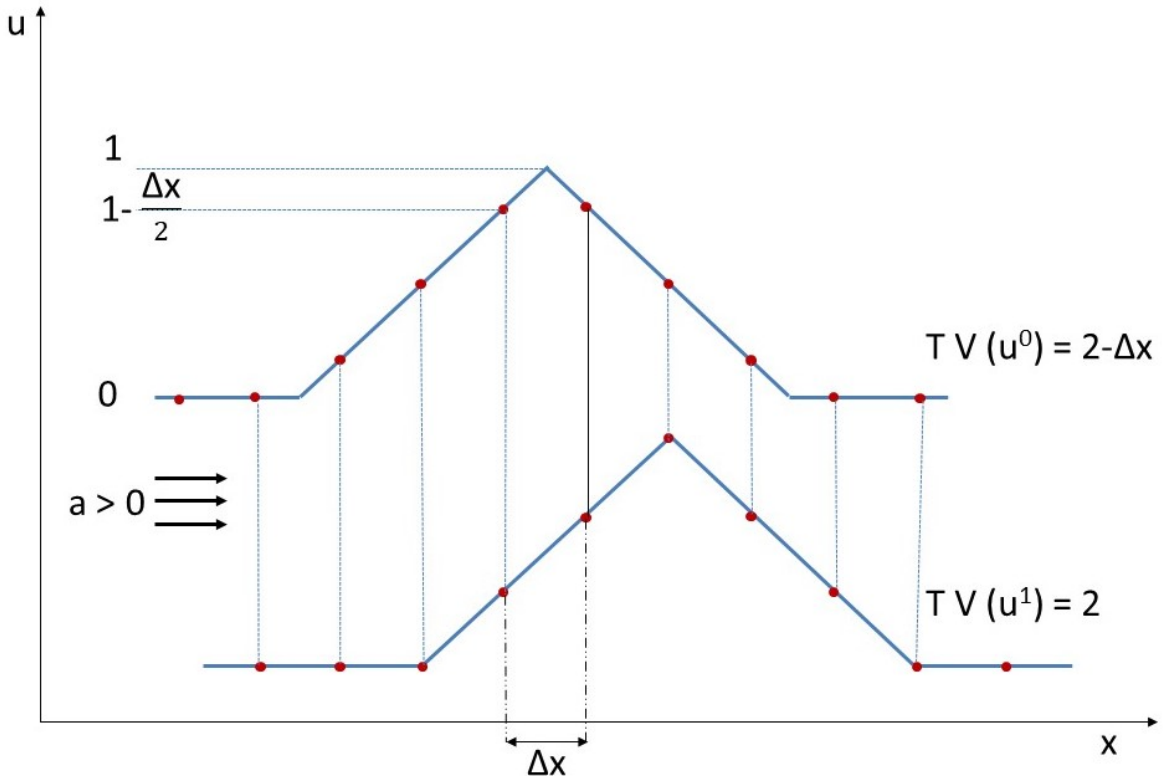


- The TV should increase by Δx between the considered time-steps but a TVD scheme will not allow this $\Rightarrow$ clipping error (here this error is $O(\Delta x)$ because it happens at a nonsmooth maximum, but for most smooth extrema it is $O(\Delta x^2)$)

- Summary of what should be known about TVD
  - TVD implies monotonicity preservation: This is desirable when monotonicity preservation is too weak but less desirable when monotonicity preservation is too strong, given that TVD can be stronger
  - TVD tends to cause clipping errors at extrema: In theory, clipping does not need to occur at every extrema — since, for example, the local maximum could increase provided that a local maximum decreased or a local minimum increased or a local maximum-minimum pair disappeared somewhere else — and may be only moderate when it occurs: However, in practice, most TVD schemes clip all extrema to between first- and second-order accuracy
  - in theory, TVD may allow large spurious oscillations but in practice it rarely does — in any case, it does not allow the unbounded growth type of instability
  - in practice, most attempts at constructing a TVD scheme end up enforcing stronger nonlinearity stability conditions such as the *positivity condition* discussed next

- Recall that the wave speed split form of a FT scheme is given by

$$\begin{aligned} u_i^{n+1} &= u_i^n + C_{i+1/2}^{+^n}(u_{i+1}^n - u_i^n) - C_{i-1/2}^{-^n}(u_i^n - u_{i-1}^n) \\ C_{i+1/2}^{+^n} &\geq 0 \quad \text{and} \quad C_{i+1/2}^{-^n} \geq 0 \end{aligned}$$

- Suppose that a given FT numerical scheme *can be written* in wave speed split form with

$$C_{i+1/2}^{+^n} \geq 0, \quad C_{i+1/2}^{-^n} \geq 0 \quad \text{and} \quad C_{i+1/2}^{+^n} + C_{i+1/2}^{-^n} \leq 1 \ \forall i \qquad (16)$$

- Condition (16) above is called the *positivity condition* (also proposed first by Harten in 1983)
- What is the connection between the positivity condition and the nonlinear stability of a scheme?

- The answer is: The positivity condition implies TVD

- The answer is: The positivity condition implies TVD
- Example: FTFS applied to the nonlinear advection equation $\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0$ is positive if $-1 \leq \lambda a^n_{i+1/2} \leq 0$
- Proof:
  - FTFS *can be written in wave speed split form for the purpose of nonlinear stability analysis* as follows

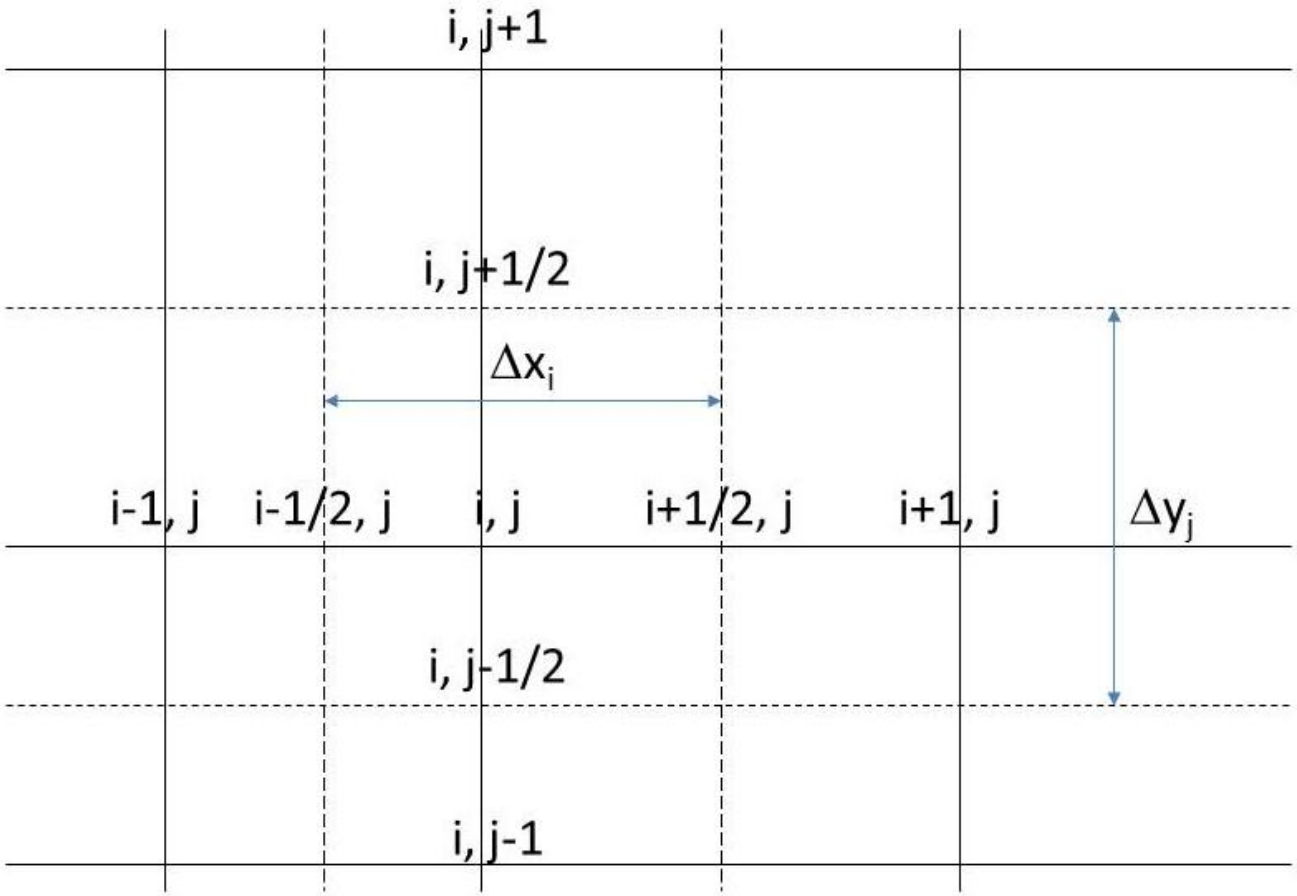
$$u_i^{n+1} = u_i^n + C^{+^n}_{i+1/2}(u^n_{i+1} - u^n_i) - C^{-^n}_{i-1/2}(u^n_i - u^n_{i-1})$$

    where $C^{+^n}_{i+1/2} = -\lambda a^n_{i+1/2}$ and $C^{-^n}_{i-1/2} = 0$
  - $\lambda a^n_{i+1/2} \leq 0 \Rightarrow C^{+^n}_{i+1/2} = -\lambda a^n_{i+1/2} \geq 0$
  - $C^{+^n}_{i+1/2} + C^{-^n}_{i+1/2} = -\lambda a^n_{i+1/2}$ and therefore the condition (16) becomes in this case $-1 \leq \lambda a^n_{i+1/2} \leq 0$
  - also, note that the positivity condition is in this case equivalent to the CFL condition

- The extension to multiple dimensions of the *computational* part of the material covered in this chapter may be tedious in some cases but is straightforward (except perhaps for the characteristic theory)
- The expressions of the Euler equations in 2D and 3D can be obtained from Chapter 2 (as particular cases of the expression of the Navier-Stokes equations in 3D)
- For simplicity, the focus is set here on the 2D Euler equations

$$\frac{\partial W}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x}(W) + \frac{\partial \mathcal{F}_y}{\partial y}(W) = 0$$

- 2D *structured grid*



$$\frac{\partial W}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x}(W) + \frac{\partial \mathcal{F}_y}{\partial y}(W) = 0$$

- For the above 2D Euler equations, the equivalent of equation (2) on a 2D structured grid is

$$\boxed{\Delta t \widehat{\left(\frac{\partial W}{\partial t}\right)}_{i,j}^{n} = -\lambda_x(\widehat{\mathcal{F}}^n_{x_{i+1/2,j}} - \widehat{\mathcal{F}}^n_{x_{i-1/2,j}}) - \lambda_y(\widehat{\mathcal{F}}^n_{y_{i,j+1/2}} - \widehat{\mathcal{F}}^n_{y_{i,j-1/2}})}$$

where

$$\begin{aligned} \lambda_x &= \frac{\Delta t}{\Delta x_i}, \quad \lambda_y = \frac{\Delta t}{\Delta y_j} \\ \Delta x_i &= x_{i+1/2,j} - x_{i-1/2,j} \ \forall j, \quad \Delta y_j = y_{i,j+1/2} - y_{i,j-1/2} \ \forall i \end{aligned}$$

and $\widehat{\mathcal{F}}_{x_{i+1/2,j}}$ and $\widehat{\mathcal{F}}_{y_{i,j+1/2}}$ are constructed exactly like $\hat{f}_{i+1/2}$ in 1D

- For example, a 2D version of FTCS has the following conservative numerical fluxes

$$
\begin{aligned}
\widehat{\mathcal{F}}^n_{x_{i+1/2,j}} &= \frac{1}{2}\left(\mathcal{F}_x(W^n_{i+1,j}) + \mathcal{F}_x(W^n_{i,j})\right) \\
&= \frac{1}{2}\begin{pmatrix} (\rho v_x)_{i+1,j} + (\rho v_x)_{i,j} \\ (\rho v_x^2)_{i+1,j} + (\rho v_x^2)_{i,j} + p_{i+1,j} + p_{i,j} \\ (\rho v_x v_y)_{i+1,j} + (\rho v_x v_y)_{i,j} \\ (E v_x)_{i+1,j} + (E v_x)_{i,j} + (p v_x)_{i+1,j} + (p v_x)_{i,j} \end{pmatrix} \\
\widehat{\mathcal{F}}^n_{y_{i,j+1/2}} &= \frac{1}{2}\left(\mathcal{F}_y(W^n_{i,j+1}) + \mathcal{F}_y(W^n_{i,j})\right) \\
&= \frac{1}{2}\begin{pmatrix} (\rho v_y)_{i,j+1} + (\rho v_y)_{i,j} \\ (\rho v_x v_y)_{i,j+1} + (\rho v_x v_y)_{i,j} \\ (\rho v_y^2)_{i,j+1} + (\rho v_y^2)_{i,j} + p_{i,j+1} + p_{i,j} \\ (E v_y)_{i,j+1} + (E v_y)_{i,j} + (p v_y)_{i,j+1} + (p v_y)_{i,j} \end{pmatrix}
\end{aligned}
$$