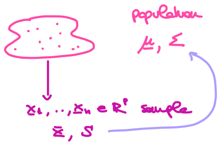


Gaussian Model

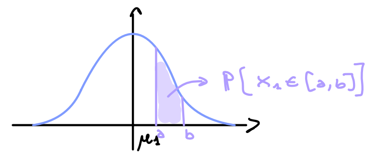
$\underline{\mu} \in \mathbb{R}^p$, Σ $p \times p$ positive def

$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$ $\quad \Sigma = [\sigma_{ij}]$ $\quad i,j = 1, \dots, p$

$\underline{p=1}$ $\quad \mu_1 \in \mathbb{R}$ $\quad \sigma_{11} > 0$

$\mathbb{R} \ni X_1 \sim N_1(\mu_1, \sigma_{11})$

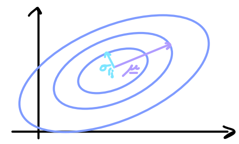
if density of X_1 is $\phi(t) = \frac{1}{\sqrt{2\pi\sigma_{11}}} e^{-\frac{(t-\mu_1)^2}{\sigma_{11}}}$ $\quad t \in \mathbb{R}$



$\underline{\text{General } p}$

$\mathbb{R}^p \ni \underline{X} \sim N_p(\underline{\mu}, \Sigma)$

if $\phi(t) = \frac{1}{\sqrt{(2\pi)^p \operatorname{Det}(\Sigma)}} \exp\left[-\frac{1}{2} \underbrace{(\underline{t} - \underline{\mu})\Sigma(\underline{t} - \underline{\mu})}_{d^2_\Sigma(\underline{t}, \underline{\mu})}\right]$



Proposition

$\underline{X} \sim N_p(\underline{\mu}, \Sigma) \iff \forall \underline{a} \in \mathbb{R}^p, \quad \underline{a}' \underline{X} \sim N_1(\underline{a}'\underline{\mu}, \underline{a}' \Sigma \underline{a})$

Corollary

$\underline{X} \sim N_p(\underline{\mu}, \Sigma) \implies X_i \sim N_1(\mu_i, \sigma_{ii}) \quad i = 1, \dots, p$

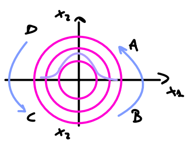
Proof: $\underline{u}_i = (0 \dots 0\, 1\, 0 \dots 0)'$

$\underline{u}_i' \underline{X} \underset{\hookrightarrow \text{prop}}{\sim} N_1(\underline{u}_i' \underline{\mu}_i, \underline{u}_i' \Sigma \underline{u}_i) = N_1(\mu_i, \sigma_i)$

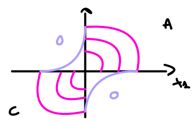
Explanation $\not\Leftarrow$

i.e. $X_i \sim N_1(\mu_i, \sigma_{ii}) \quad \forall i = 1, \dots, p \quad \not\Longrightarrow \quad \underline{X} \sim N_p(\underline{\mu}, ?)$

Take $\underline{X} \sim N_2(\underline{0}, I)$



Symetry w.r.t. x_1 & x_2



Same marginal, different mass

NOT GAUSSIAN!

(same volume as before)

Corollary

$\left.\begin{array}{l} \underline{X} \sim N_p(\underline{\mu}, \Sigma) \\ A \quad q \times p \text{ matrix} \end{array}\right\} \implies A\underline{X} \sim N_q(A\underline{\mu}, A\Sigma A')$

Proof: Need to prove: $\forall \underline{a} \in \mathbb{R}^q \quad \underline{a}'(A\underline{X}) \sim N_1(\underline{a}' A \underline{\mu}, \underline{a}' A \Sigma A' \underline{a})$

but, $\underline{a}'(A\underline{X}) = (\underline{a}'A)\underline{X} = \underbrace{(A'\underline{a})'}_{\in \mathbb{R}^p} \underline{X} \overset{\text{Prop}}{\sim} N_1((A'\underline{a})'\underline{\mu}_1, (A'\underline{a})\Sigma(A'\underline{a})) = N_1(\underline{a}'(A\underline{x})\underline{\mu}, a'A\Sigma A'\underline{a})$

$\overset{\text{Prop}}{\Longrightarrow} A\underline{X} \sim N_p(A\underline{\mu}, A\Sigma A')$

Corollary

$\underline{d} \in \mathbb{R}^p$, $\underline{X} \sim N_p(\underline{\mu}, \Sigma) \implies \underline{X} + \underline{d} \sim N_p(\underline{\mu} + \underline{d}, \Sigma)$

Obs

$z_1, \dots, z_p$ iid $\sim N_1(0,1)$ $\quad$ density of z_i: $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$

$\underline{z} = (z_1 \dots z_p)'$ $\quad$ density of $\underline{z}$ $\quad \phi_p(\underline{t}) = \phi_1(t_1)\phi_2(t_2) \dots \phi_p(t_p) = \frac{1}{\sqrt{(2\pi)^p}} \prod_{i=1}^{p} e^{-\frac{1}{2}t_i^2} = \frac{1}{\sqrt{(2\pi)^p}} e^{-\frac{1}{2}\sum_1^p t_i^2} = \frac{1}{\sqrt{(2\pi)^p}} e^{-\frac{1}{2}\underline{t}' I \underline{t}}$

$\implies \underline{z} \sim N(0, I)$

$\Sigma \quad p \times p$ $\qquad \Sigma^{1/2}\underline{z} \sim N_p(\underline{0}, \underbrace{\Sigma^{1/2} I \Sigma^{1/2}}_{\Sigma})$

$\Sigma^{1/2}\underline{z} + \underline{\mu} \sim N_p(\underline{\mu}, \Sigma)$

If $\underline{x} \sim N_p(\underline{x}, \Sigma) \implies \underline{z} = \Sigma^{-1/2}(\underline{x} - \underline{\mu}) \sim N_p(0, I)$

$\begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} \implies$ i.e. $z_1, \dots, z_p$ iid $\sim N_1(0,1)$

$\Sigma = \sum_{i=1}^{p} \lambda_i \underline{e}_i \underline{e}_i' \implies \Sigma^{1/2} = \sum_i^p \lambda_i^{r} \underline{e}_i \underline{e}_i'$

$\implies \Sigma^{-1/2} = \sum_i^p \frac{1}{\sqrt{\lambda_i}} \underline{e}_i \underline{e}_i'$

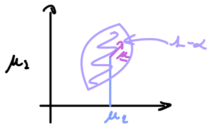
Obs

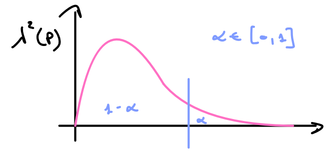
$\underline{x} \sim N_p(\underline{\mu}, \Sigma)$

$W = d^2_{\Sigma^{-1}}(\underline{x}, \underline{\mu}) = (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu}) = \underbrace{(\underline{x} - \underline{\mu}) \Sigma^{-1/2}}_{\underline{z}'} \underbrace{\Sigma^{-1/2} (\underline{x} - \underline{\mu})}_{\underline{z}} =$

$= \underline{z}'\underline{z} = \sum_i z_i^2 \sim \chi^2(p)$ $\qquad z_i \sim N_1(0,1)$

$z_i^2 \sim \chi^2(1)$





$\mathbb{P}[d^2_{\Sigma^{-1}}(\underline{x}, \underline{\mu}) \le \lambda^2_{1-\alpha}(p)] = 1-\alpha$

$\mathbb{P}[(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu}) \le \lambda^2(p)] = 1-\alpha$

$\mathbb{R}^p \ni \underline{x} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} \begin{matrix} \in \mathbb{R}^q \\ \in \mathbb{R}^{p-q} \end{matrix} \qquad q < p$

$\underline{\mu} = \begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix} \begin{matrix} \in \mathbb{R}^q \\ \in \mathbb{R}^{p-q} \end{matrix}$

$\Sigma = \begin{array}{cc|c} q & p-q & \\ \hline \Sigma_{11} & \Sigma_{12} & q \\ \Sigma_{21} & \Sigma_{22} & p-q \end{array}$ $\longrightarrow$ Cov between comp of $\underline{x}_1$ and comp of $\underline{x}_2$

$\Sigma_{12} = \Sigma_{21}'$

$q \times q$ $\text{Cov}(\underline{x}_1)$

$(p-q)\times(p-q)$ $\text{Cov}(\underline{x}_2)$

Corollary

$\underline{x} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} \sim N_p\left(\begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right) \implies \underline{x}_1 \sim N_q(\underline{\mu}_1, \Sigma_{11})$

Proof: $A = \overbrace{[I_q \quad \underline{0}]}^{p}\}q \qquad q > p$

$\implies A\underline{x} = \underline{x}_1 \sim N_q(\underline{\mu}_1, \underbrace{A\Sigma A'}_{=\Sigma_{11}})$

Proposition

$\underline{x} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} \sim N_p\left(\begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}, \begin{array}{c} \begin{array}{cc} q & p-q \end{array} \\ \begin{pmatrix} \Sigma_{11} & \underline{0} \\ \underline{0} & \Sigma_{22} \end{pmatrix} \end{array} \begin{matrix} q \\ p-q \end{matrix}\right)$ $\to \Sigma$

i.e. $\Sigma_{12} = \Sigma_{21} = \underline{0} \implies \underline{x}_1 \perp\!\!\!\perp \underline{x}_2$

$\phi_{\underline{x}}(\underline{t}) = \phi_{\underline{x}_1}(\underline{t}_1)\,\phi_{\underline{x}_2}(\underline{t}_2) \implies \underline{t} = \begin{pmatrix} \underline{t}_1 \\ \underline{t}_2 \end{pmatrix} \begin{matrix} \in \mathbb{R}^q \\ \in \mathbb{R}^{p-q} \end{matrix}$

Teorem

$\underline{x} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} \sim N_q\left(\begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$

$\implies \underline{x}_1 \mid \underline{x}_2 = \underline{x}_2 \sim N_q(\underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

Observation

$\underline{x}_1 \sim N_q(\underline{\mu}, \Sigma_{11})$

$\underline{x}_1 \mid \underline{x}_2 = \underline{x}_2 \sim N_p(\underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$ $\longrightarrow$ partial cov. known before knowing $\underline{x}_2$

Note: if $\Sigma_{12} = \underline{0} \implies \underline{x}_1 \mid \underline{x}_2 = \underline{x}_2 \sim N_q(\underline{\mu}_1, \Sigma_{11})$

$\mathbb{E}[\underline{x}_1 \mid \underline{x}_2 = \underline{x}_2] = \underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2)$ $\qquad$ regulation factor

Proof: Let $A = \begin{array}{cc|c} q & p-q & \\ \hline I & -\Sigma_{12}\Sigma_{22}^{-1} & q \\ \underline{0} & I & p-q \end{array} \qquad p \times p$

$A\begin{pmatrix} \underline{x}_1 - \underline{\mu}_1 \\ \underline{x}_2 - \underline{\mu}_2 \end{pmatrix} = \begin{pmatrix} \underline{x}_1 - \underline{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2) \\ \underline{x}_2 - \underline{\mu}_2 \end{pmatrix} \sim N_p\left(\begin{pmatrix} \underline{0} \\ \underline{0} \end{pmatrix}, \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \underline{0} \\ \underline{0} & \Sigma_{22} \end{pmatrix}\right)$

$\implies \underline{x}_1 - \underline{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2) \perp\!\!\!\perp \underline{x}_2 - \underline{\mu}_2 \implies \underline{x}_1 - \underline{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2) \mid \underline{x}_2 = \underline{x}_2 \sim N_q(\underline{0}, \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

$\implies \underline{x}_1 - \underline{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2) \mid \underline{x}_2 = \underline{x}_2 \sim N_p(\underline{0}, \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

$\implies \underline{x}_1 \mid \underline{x}_2 = \underline{x}_2 \sim N_q(\underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

Example

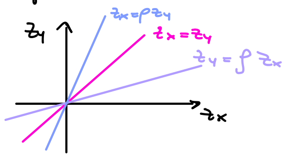
$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_Y \\ \mu_X \end{pmatrix}, \begin{bmatrix} \sigma_{YY} & \sigma_{YX} \\ \sigma_{XY} & \sigma_{XX} \end{bmatrix}\right)$

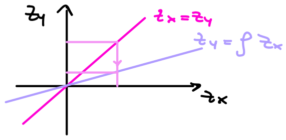
Before obs. x: $\quad Y \sim N_1(\mu_Y, \sigma_{YY})$

After this $X = x$: $\quad Y \mid X = x \sim N_1\left(\mu_Y + \frac{\sigma_{XY}}{\sigma_{XX}}(x - \mu_X), \sigma_{YY} - \frac{\sigma_{XY}^2}{\sigma_{XX}}\right) = N_1\left(\mu_Y + \rho\frac{\sqrt{\sigma_{YY}}}{\sqrt{\sigma_{XX}}}(x - \mu_X), \sigma_{YY}(1-\rho^2)\right)$

Reg fct: $\mathbb{E}[Y \mid X = x] = \mu_Y + \rho \frac{\sqrt{\sigma_{YY}}}{\sqrt{\sigma_{XX}}}(x - \mu_X) = \mu_Y - \rho \frac{\sqrt{\sigma_{YY}}}{\sqrt{\sigma_{XX}}}\mu_X + \rho \frac{\sqrt{\sigma_{YY}}}{\sqrt{\sigma_{XX}}} x = \beta_0 + \beta_1 x$ reg. line

Reg line: $y = \beta_0 + \beta_1 x \longrightarrow y = \mu_Y + \rho \frac{\sqrt{\sigma_{YY}}}{\sqrt{\sigma_{XX}}}(x - \mu_X) \longrightarrow \frac{y - \mu_Y}{\sqrt{\sigma_{YY}}} = \rho \frac{x - \mu_X}{\sqrt{\sigma_{XX}}} \longrightarrow z_Y = \rho z_X$





regression line

$-1 < \rho < 1$

$\rho = \frac{\sigma_{XY}}{\sqrt{\sigma_{XX}\sigma_{YY}}}$ Cov(X,Y)

Estimators for μ and Σ of a $N_p(\mu, \Sigma)$

Data

$$\mathbb{X} = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} \quad \begin{matrix} x_1 \in \mathbb{R}^p \rightsquigarrow \text{realiz. } X_1 \\ \vdots \\ x_n \in \mathbb{R}^p \rightsquigarrow \text{realiz. } X_n \end{matrix}$$

$X_1, \dots, X_n$ iid $\sim N_p(\mu, \Sigma)$

μ, Σ unknown

Obvious estimates: $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ sample mean for μ

$S = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})'$ for Σ

Short excursus on MLE

$X_1, X_2, \dots, X_n$ iid Bernoulli(p), $p \in [0,1]$

$P[X_i = 1] = p$ $\quad \mathbb{P}[X_i = 0] = 1 - p$

Data: $x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1,\ x_5 = 0$

$\hat{p} = 3/5$

$\mathbb{P}[X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 1, X_5 = 0] = p^3(1-p)^2$

$L(p) = p^3(1-p)^2$

$\ell(p) = 3 \log p + 2 \log(1-p)$ maximizing this

$\hat{p} = \arg\max L(p) = \arg\max \ell(p) = \frac{3}{5}$

Exercise

$X_1, \dots, X_n$ iid $\sim$ Bern(p)

$p \in [0,1]$ unknown

$x_1, \dots, x_n$ data

$L(p \mid x_1, \dots, x_n) = p^{\sum x_i}(1-p)^{n - \sum x_i}$

$\ell(p) = \sum x_i \log p + (n - \sum x_i)\log(1-p)$

$\arg\max_{p \in [0,1]} \ell(p) = \frac{1}{n}\sum x_i$

Invariance property of MLE

$\theta \in \mathbb{R}^k$

Data $x_1, \dots, x_n$ $\Longrightarrow$ $\hat{\theta} = \hat{\theta}(x_1, \dots x_n)$ MLE

Model $\quad h: \mathbb{R}^k \to \mathbb{R}^d$

$\eta = h(\theta)$ function of interest $\quad \hat{\eta}$ MLE $\Longrightarrow \hat{\eta} = \eta(\hat{\theta}) \Longrightarrow \boxed{\widehat{h(\theta)} = h(\hat{\theta})}$

EXAMPLE

Assume $\hat{\Sigma} = \frac{1}{n}\sum_{i}^{n}(x_i - \bar{x})(x_i - \bar{x})'$ MLE for Σ

$\Sigma = \sum \lambda_i e_i e_i'$ $\quad$ estimate λ_i, e_i ?

$\hat{\Sigma} = \sum \hat{\lambda}_i \hat{e}_i \hat{e}_i'$ $\quad$ estimate λ_i with $\hat{\lambda}_i$ and e_i with $\hat{e}_i$

Back to $X_1, \dots, X_n$ iid $\sim N_p(\mu, \Sigma)$

$L(\mu, \Sigma \mid x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{(2\pi)^p \mathrm{Det}(\Sigma)}} \exp\left(-\frac{1}{2}(x_i - \mu)'\Sigma^{-1}(x_i - \mu)\right)$

$\ell(\mu, \Sigma) = n \log \frac{1}{\sqrt{(2\pi)^p \mathrm{Det}(\Sigma)}} - \frac{1}{2}\sum_i (x_i - \mu)'\Sigma^{-1}(x_i - \mu)$

$\arg\max_{\substack{\mu \in \mathbb{R}^p \\ \Sigma\ p \times p \text{ posdef}}} \ell(\mu, \Sigma) = (\hat{\mu}, \hat{\Sigma})$

Theorem

$\underline{x}_1, \dots, \underline{x}_n$ iid $N_p(\mu, \Sigma)$

MLE for μ : $\hat{\mu} = \frac{1}{n}\sum_{i}^{n} x_i$ (unbiased)

MLE for Σ : $\hat{\Sigma} = \frac{1}{n}\sum_{i}^{n}(\underline{x}_i - \bar{\underline{x}})(\underline{x}_i - \bar{\underline{x}})'$ (unbiased)

What dist's for $\bar{\underline{x}}, \hat{\Sigma}, S'$ if $\underline{x}_1, \dots, \underline{x}_n$ iid $\sim N_p(\mu, \Sigma)$

Proposition

$\underline{x}_1, \dots, \underline{x}_n$ iid $N_p(\mu, \Sigma) \implies \bar{\underline{x}} \sim N_p(\mu, \frac{1}{n}\Sigma)$

Proof:

$$\tilde{\underline{x}} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{pmatrix} \in \mathbb{R}^{np} \qquad \tilde{\underline{x}} \sim N_{np}\left(\begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}, \begin{pmatrix} \Sigma & & 0 \\ & \ddots & \\ 0 & & \Sigma \end{pmatrix}\right)$$

$$A = \underbrace{\left[\begin{array}{ccc} \overbrace{1\,0\,\cdots}^{p} & \overbrace{1\,0\,\cdots}^{p} & \overbrace{1\,0\,\cdots}^{p} \\ 0\,1\,\cdots & 0\,1\,\cdots & 0\,1\,\cdots \\ \vdots & \vdots & \vdots \\ 0\,\cdots\,1 & 0\,\cdots\,1 & 0\,\cdots\,1 \end{array}\right]}_{np} \Big\} p \qquad A \;\; p \times np \qquad \frac{1}{n} A \tilde{\underline{x}} = \begin{bmatrix} \frac{1}{n}\sum_{i}^{n} x_{i1} \\ \frac{1}{n}\sum_{i}^{n} x_{i2} \\ \vdots \\ \frac{1}{n}\sum_{i}^{n} x_{ip} \end{bmatrix}$$

$$\tilde{\underline{x}} \sim N_p\left(\frac{1}{n} A \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix}, \frac{1}{n^2} A \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} A'\right) = N_p\left(\mu, \frac{1}{n}\Sigma\right)$$

Dist's of S or $\hat{\Sigma}$

Definition

Let $\underline{z}_1, \dots, \underline{z}_m$ iid $\sim N_p(\underline{0}, \Sigma)$. $A = \sum_{i}^{m} \underline{z}_i \underline{z}_i' \sim \text{Wishart}(\Sigma, m)$

Proposition 1

$$\begin{matrix} A_1 \sim \text{Wish}(\Sigma, m_1) \\ A_2 \sim \text{Wish}(\overset{\shortparallel}{\Sigma}, m_2) \end{matrix} \quad \text{if } A_1 \perp\!\!\!\perp A_2 \implies A_1 + A_2 \sim \text{Wish}(\Sigma, m_1 + m_2)$$

Proof:

$A_1 = \sum_{i}^{m_1} \underline{z}_i \underline{z}_i'$ $\qquad \underline{z}_1, \dots, \underline{z}_{m_1}$ iid $\sim N_p(\underline{0}, \Sigma)$

$A_2 = \sum_{i}^{m_2} \tilde{\underline{z}}_i \tilde{\underline{z}}_i'$ $\qquad \tilde{\underline{z}}_1, \dots, \tilde{\underline{z}}_{m_2}$ iid $\sim N_p(\underline{0}, \Sigma)$

$\underset{\underline{z}_1}{\underline{w}_1} \cdots \underset{\underline{z}_{m_1}}{\underline{w}_{m_1}}, \underset{\tilde{\underline{z}}_1}{\underline{w}_{m_1+1}}, \dots, \underset{\tilde{\underline{z}}_{m_2}}{\underline{w}_{m_1+m_2}}$ iid $\sim N_p(\underline{0}, \Sigma)$

$A_1 + A_2 = \sum_{i}^{m_1+m_2} \underline{w}_i \underline{w}_i' \sim \text{Wish}(\Sigma, m_1 + m_2)$

Proposition 2

$A \sim \text{Wish}(\Sigma, m)$ $\quad C$ $k \times p$ matrix $\implies CAC' \sim \text{Wish}(C\Sigma C', m)$

Proof: $A = \sum_{i}^{m} \underline{z}_i \underline{z}_i'$ $\qquad \underline{z}_1, \dots, \underline{z}_m$ iid $\sim N_p(\underline{0}, \Sigma)$

$CAC' = \sum_{i}^{m} \underbrace{C\underline{z}_i}_{\underline{w}_i} \underbrace{\underline{z}_i' C'}_{\underline{w}_i'}$ $\qquad \underline{w}_i \sim N_p(\underline{0}, C\Sigma C')$ indep. $\implies CAC' \sim \text{Wish}(C\Sigma C', m)$

Proposition 3

$A \sim \text{Wish}(\Sigma, m)$ $\quad \sigma^2 > 0 \implies \sigma^2 A \sim \text{Wish}(\sigma^2\Sigma, m)$

Proof: $\sigma^2 A = \sigma^2 \sum_{i=1}^{m} \underline{z}_i \underline{z}_i' = \sum_{i}^{m} \underbrace{\sigma \underline{z}_i}_{\underline{w}_i} \underbrace{\underline{z}_i' \sigma}_{\underline{w}_i'}$ $\qquad \underline{w}_1, \dots, \underline{w}_n$ iid $\sim N_p(\underline{0}, \sigma^2\Sigma)$

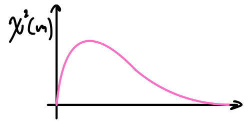
Proposition 4

Let $p = 1 \implies \text{Wishart}(\overbrace{\Sigma}^{= \Sigma = [\sigma^2]}, m) = \sigma^2 \chi^2_{(m)}$

Proof: $p = 1$, $\Sigma = [\sigma^2]$ $\qquad A \sim \text{Wish}(\Sigma, m)$ $\qquad A = \sum_{i}^{m} z_i z_i'$ $\qquad z_1, \dots, z_m$ iid $\sim N_1(0, \sigma^2)$

$\frac{1}{\sigma^2} A = \sum_{i}^{m} \frac{z_i}{\sigma} \frac{z_i'}{\sigma} = \sum_{i}^{m} \left(\frac{z_i}{\sigma}\right)^2$ $\qquad \frac{z_i}{\sigma} \sim N_1(0, 1)$ $\qquad \left(\frac{z}{\sigma}\right)^2 \sim \chi^2(1)$

$\frac{1}{\sigma^2} A \sim \chi^2(m)$
$\Updownarrow$
$A \sim \sigma^2 \chi^2(m) \quad = \quad W \sim \sigma \cdot N_1(0,1) \iff \frac{W}{\sigma} \sim N(0,1)$



Exercise

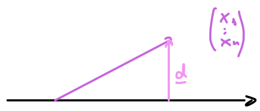
$A \sim Wish(\Sigma, m) \quad p \times p \qquad c \in \mathbb{R}^p \qquad ?\ c'Ac$

$c'Ac \sim Wis(c'\Sigma c, m) \qquad$ (prop. take $C = [c']$ $1 \times p$)

$c'\Sigma c > 0 \qquad c'Ac \sim (c'\Sigma c)\chi^2(m) \qquad$ i.e. $\frac{c'Ac}{c'\Sigma c} \sim \chi^2(m)$

Theoreme

$x_1, \dots, x_n$ iid $\sim N_p(\mu, \Sigma) \implies \sum_{i=1}^{m} (x_i - \bar{x})(x_i - \bar{x})' \sim Wish(\Sigma, n-1)$



Corollary

$x_1 \dots x_n$ iid $\sim N_p(\mu, \Sigma)$

$\hat{\Sigma} = \frac{1}{n} \sum_i^n (x_i - \bar{x})(x_i - \bar{x})' \sim Wish(\frac{1}{n}\Sigma, n-1)$

$S = \frac{1}{n-1} \sum_i^n (x_i - \bar{x})(x_i - \bar{x})' \sim Wish(\frac{1}{n-1}\Sigma, n-1)$

Theoreme

$x_1, \dots, x_n$ iid $\sim N_p(\mu, \Sigma)$

1. $\bar{x} \sim N_p(\mu, \frac{1}{n}\Sigma)$
2. $(n-1)S \sim Wish(\Sigma, n-1)$
3. $\bar{x} \perp\!\!\!\perp S$

Recall

CLT: $\mathbb{R}^p \ni x_1, \dots, x_n$ iid $\sim F$ (s.t. $\mathbb{E}[x] = \mu$, $\Sigma = Cov(x_i)$ exist)
then $\sqrt{n}(\bar{x} - \mu) \sim A\ N_p(0, \Sigma)$
In particular, for n large $\bar{x} \sim N_p(\mu, \frac{1}{n}\Sigma)$

LLN: $x_1, \dots, x_n$ iid $\sim F(\mu, \Sigma)$
$\bar{x} \xrightarrow{P} \mu$
$S \xrightarrow{P} \Sigma \qquad$ as $n \to +\infty$

Inference for the mean $\mu \in \mathbb{R}^p$

$x_1, \dots, x_n$ iid $\sim F(\mu, \Sigma)$

$\bar{x} = \frac{1}{n} \sum_i^n x_i$

- n is very large ($n >> p$): $\mathbb{R}^p \ni \sqrt{n}(\bar{x} - \mu) \sim A\ N_p(0, \Sigma)$ ie $\bar{x} \sim N_p(\mu, \frac{1}{n}\Sigma)$ (approx)

$\implies (\bar{x} - \mu)'(\frac{1}{n}\Sigma)^{-1}(\bar{x} - \mu) \sim \chi^2(p)$

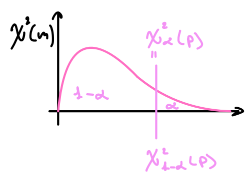
$(*)\ n(\bar{x} - \mu)'\Sigma^{-1}(\bar{x} - \mu) \sim \chi^2(p)$

Obs: if Σ is known $(*)$ is pivotal
if Σ is unknown $S \xrightarrow{P} \Sigma$ (LLN) $\implies n(\bar{x} - \mu)' S^{-1}(\bar{x} - \mu) \sim \chi^2(p)$

For $\alpha \in [0,1]$. $Pr[n(\bar{x} - \mu)' S^{-1}(\bar{x} - \mu) \le \chi^2_{1-\alpha}(p)] = 1 - \alpha$

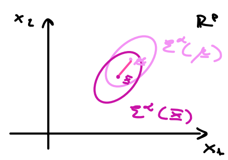
$\Updownarrow$

$Pr[d^2_{(\frac{1}{n}S)^{-1}}(\bar{x}, \mu) \le \chi^2_{1-\alpha}(p)] = 1 - \alpha$



Consider: $\mathcal{E}^\alpha_{\chi^2_{1-\alpha}(p)}(\mu) = \{x \in \mathbb{R}^p : d^2_{(\frac{1}{n}S)^{-1}}(x, \mu) \le \chi^2_{1-\alpha}(p)\}$

$\mathcal{E}^\alpha_{\chi^2_{1-\alpha}(p)}(\bar{x}) = \{\eta \in \mathbb{R}^p : d^2_{(\frac{1}{n}S)^{-1}}(\eta, \bar{x}) \le \chi^2_{1-\alpha}(p)\}$

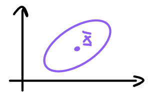


$\bar{x} \in \mathcal{E}^\alpha_{\chi^2_{1-\alpha}(p)}(\mu)$
$\Updownarrow$
$\mu \in \mathcal{E}^\alpha_{\chi^2_{1-\alpha}(p)}(\bar{x})$

Equivalent formulations for $(*)$: $P[\underset{\text{Random}}{\bar{x}} \in \mathcal{E}^\alpha_{\chi^2_{1-\alpha}(p)}(\mu)] = 1 - \alpha = P[\mu \in \underbrace{\mathcal{E}^\alpha_{\chi^2_{1-\alpha}(p)}(\bar{x})}_{\text{Random}}]$

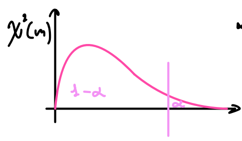
Definition (confidence region)

$CR_{1-\alpha}(\mu) = \{ \mu \in \mathbb{R}^p : n(\mu - \bar{x})' S^{-1} (\mu - \bar{x}) \leq \chi_{1-\alpha}(p) \}$



Test

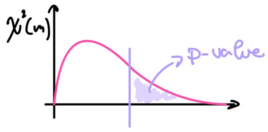
$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases}$ $\quad n(\bar{x} - \mu_0)' S^{-1} (\bar{x} - \mu_0) = T_0^2$



$\alpha \in [0,1]$
usually $\alpha = 0{,}05,\ 0{,}01$

Reject H_0 if T_0^2 is large. How large? $\quad T_0^2 \sim \chi^2(p)$
Reject if $\{ T_0^2 > \chi^2_{1-\alpha}(p) \}$

Let T_0^2 observed value



p-value $\leq \alpha \iff T_0^2 > \chi^2_{1-\alpha}(p)$
Reject H_0 if p-value is small (p-val $< 0{,}05$)

Observation

$H_0 : \mu = \mu_0$ is rejected if $T_0^2 > \chi^2_{1-\alpha}(p)$
$n(\bar{x} - \mu_0) S^{-1} (\bar{x} - \mu_0) > \chi^2_{1-\alpha}(p)$
$d^2_{(\frac{1}{n}S)^{-1}}(\bar{x}, \mu_0) > \chi^2_{1-\alpha}(p) \iff \mu_0 \notin CR_{1-\alpha}(\mu)$
i.e. $CR_{1-\alpha}(\mu)$ identifies all values of μ_0 for which you cannot reject $H_0 : \mu = \mu_0$

If n is not large?

$x_1, \dots, x_n$ iid $\sim N_p(\mu, \Sigma)$
? $n(\bar{x} - \mu) S^{-1} (\bar{x} - \mu) \sim$?

Definition

$Y \sim \chi^2(n)$, $W \sim \chi^2(m)$, $Y \perp\!\!\!\perp W$
Distribution of $\frac{Y/n}{W/m} \sim F(n,m)$

Observation

1. $t \sim t(n)$ $\quad t = \frac{Z}{\sqrt{\frac{W}{m}}}$ $\quad Z \sim N(0,1)$, $W \sim \chi^2(m)$ $\quad Z \perp\!\!\!\perp W$

$t^2 = \frac{\overset{\sim \chi^2(1)}{Z^2}}{\underset{\sim \chi^2(m)}{\frac{W}{m}}} \sim F(1, m)$

2. $F(n,m) \xrightarrow[m \to +\infty]{} \frac{1}{n} \chi^2(n)$

$\frac{Y/n}{W/m} \sim F(n,m)$ $\quad \frac{W}{m} \xrightarrow[m \to +\infty]{} ?$ $\quad W \sim \chi^2(m)$
$W = \sum_i^m Z_i^2$ $\quad Z_1, \dots, Z_m$ iid $\sim N(0,1)$

$\frac{1}{m} \sum_i^m Z_i^2 \xrightarrow[m \to +\infty]{P} \mathbb{E}[Z_i^2] = 1$

hence $\frac{Y/n}{W/m} \xrightarrow[m \to +\infty]{\text{in distr.}} Y/n \sim \frac{1}{n} \chi^2(n)$

Theorem (Hotelling)

Assume: - $x \sim N_p(\mu, \Sigma)$
- $mW \sim \text{Wishart}(\Sigma, m)$
- $x \perp\!\!\!\perp W$

$\implies \frac{m-p+1}{mp} (x - \mu)' W^{-1} (x - \mu) \sim F(p, m-p+1)$

Corollary

$X_1, \dots, X_n$ iid $\sim N_p(\mu, \Sigma) \implies \underbrace{n(\bar{X} - \mu)' S^{-1} (\bar{X} - \mu)}_{= d^2_{(\frac{1}{n}S)^{-1}}(\bar{X}, \mu)} \sim \frac{(n-1)p}{n-p} F(p, n-p)$

Proof: $\sqrt{n}(\bar{X}-\mu) \sim N_p(0, \Sigma)$
$(n-1)\,S \sim \text{Wishart}(\Sigma, n-1)$
$\sqrt{n}(\bar{X}-\mu) \perp\!\!\!\perp S$

$\Longrightarrow$ with $m = n-1$
$n(\bar{X}-\mu)' S^{-1}(\bar{X}-\mu) \sim \frac{(n-1)p}{n-p} F(p, n-p)$
(pivotal)

$\alpha \in [0,1]$: $CR_{1-\alpha}(\mu) = \left\{ \eta \in \mathbb{R}^p : n(\bar{X}-\eta)' S^{-1}(\bar{X}-\eta) \le \overbrace{\frac{(n-1)p}{n-p} F_{1-\alpha}(p, n-p)}^{\text{radius}^2} \right\}$

n large: $\text{radius}^2 = \chi^2_{1-\alpha}(p)$
n small, Gaussian dist.: $\text{radius}^2 = \frac{(n-1)p}{n-p} F_{1-\alpha}(p, n-p)$

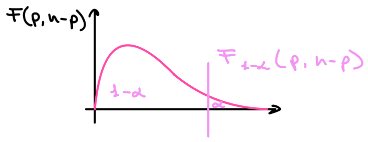
Obs: For $n \to +\infty$ $\quad \frac{(n-1)p}{n-p} F_{1-\alpha}(p, n-p) \longrightarrow \chi^2(p)$

$\frac{(n-1)p}{n-p} \to 1$, $\quad F_{1-\alpha}(p, n-p) \to \frac{1}{p}\chi^2(p)$

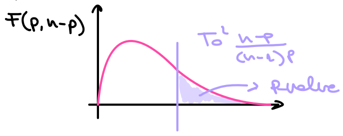
Test

$\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu \ne \mu_0 \end{cases}$ $\qquad$ test statistic $\quad T_0^2 = n(\bar{X}-\mu_0) S^{-1}(\bar{X}-\mu_0)$

$\alpha \in [0,1]$ $\quad$ Reject H_0 if:
If H_0 is true: $T_0^2 \sim \frac{(n-1)p}{n-p} F(p, n-p)$
hence reject if: $T_0^2 > \frac{(n-1)p}{n-p} F_{1-\alpha}(p, n-p)$



p-value: observed T_0^2



Obs: $CR_{1-\alpha}(\mu)$ is the set of μ_0 that you cannot reject at level α

Exercise

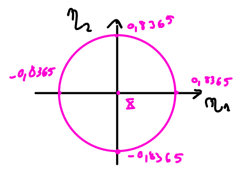
$n = 10 \quad x_1, \dots, x_n$ iid $\sim N_2(\mu, \Sigma)$

$\bar{X} = 0, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$CR_{1-\alpha}(\mu) = \left\{ \eta \in \mathbb{R}^2 : 10(\eta - 0)'(\eta - 0) \le \frac{(10-1)2}{10-2} F_{1-\alpha}(2, 8) \right\}$

$\alpha = 0,1 \qquad F_{0,9}(2,8) = 3,11$

$CR_{0,9}(\mu) = \left\{ \eta \in \mathbb{R}^2 : 10\,\eta'\eta \le \frac{18}{8}(3,11) \right\} = \left\{ \eta : \eta_1^2 + \eta_2^2 \le 0,6997 \right\}$



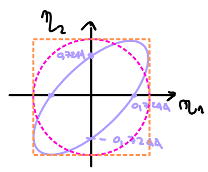
now suppose

$\bar{X} = 0, \quad S = \begin{bmatrix} 1 & 0,5 \\ 0,5 & 1 \end{bmatrix}$

$CR_{0,9}(\mu) = \left\{ \eta : \eta' S^{-1} \eta \le 0,6997 \right\}$

different geometry $\quad$ same numbers

$= \left\{ \eta : \frac{4}{3}\eta_1^2 - \frac{4}{3}\eta_1\eta_2 + \frac{4}{3}\eta_2^2 \le 0,6997 \right\}$



Inference for linear combination of μ

$X_1, \dots, X_n$ iid $\sim N_p(\mu, \Sigma)$, $\quad \mu$ estimator is $\bar{X}$
Which can be estimated for $a'\mu = a_1\mu_1 + a_2\mu_2 + \dots + a_p\mu_p$ $\quad$ for $a \in \mathbb{R}^p$?

It can be: $a'\bar{X}$ but I want to consider uncertainty, I need an interval.

- $a'\bar{X} \sim N_1(a'\mu, \frac{1}{n} a'\Sigma a)$
- $\frac{\sqrt{n}(a'\bar{X} - a'\mu)}{\sqrt{a'\Sigma a}} \sim N_1(0,1) \quad \perp\!\!\!\perp \quad \frac{(n-1)\,a' S a}{a'\Sigma a} \sim \chi^2(p)$
- $(n-1)\,S' \sim \text{Wish}(\Sigma, n-1)$
  $(n-1)\,a' S a \sim \text{Wish}(a'\Sigma a, n-1) \sim (a'\Sigma a)\chi^2(p)$
- $N(0,1) \sim \dfrac{\dfrac{\sqrt{n}(a'\bar{X} - a'\mu)}{\sqrt{a'\Sigma a}}}{\sqrt{\dfrac{(n-1)a'Sa}{a'\Sigma a}\dfrac{1}{n-1}}} \sim t(n-1)$

  $\perp\!\!\!\perp$

  $\sqrt{\dfrac{\chi^2(p)}{n}}$

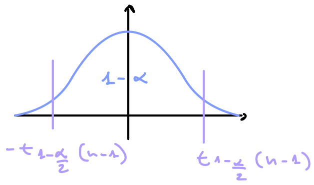
  we can rewrite $\quad \dfrac{\sqrt{n}(a'\bar{X} - a'\mu)}{\sqrt{a' S a}} \sim t(n-1)$
  (Pivotal)

$\alpha \in [0,1]$

$$\Pr\left[\frac{\sqrt{n}\,|\underline{a}'\bar{\underline{x}} - \underline{a}'\underline{\mu}|}{\sqrt{\underline{a}'S\underline{a}}} < t_{1-\frac{\alpha}{2}}(n-1)\right] = 1-\alpha$$

$$\Pr\left[|\underline{a}'\bar{\underline{x}} - \underline{a}'\underline{\mu}| < t_{1-\frac{\alpha}{2}}(n-1)\cdot\sqrt{\frac{\underline{a}'S\underline{a}}{n}}\right] = 1-\alpha$$

$$CI_{1-\alpha}(\underline{a}'\underline{\mu}) = \left[\underline{a}'\bar{\underline{x}} \pm t_{1-\frac{\alpha}{2}}(n-1)\sqrt{\frac{\underline{a}'S\underline{a}}{n}}\right]$$



Ex: $\underline{a} = [0 \dots 0\,1\,0 \dots 0]$ $\quad i = 1, \dots, p$

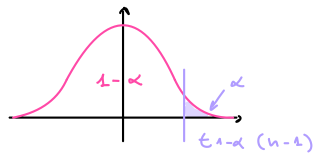
$$CI_{1-\alpha}(\mu_i) = \left[\bar{x}_i \pm t_{1-\frac{\alpha}{2}}(n-1)\sqrt{\frac{S_{ii}}{n}}\right]$$

$$CI_{1-\alpha}(\mu_i - \mu_j) = \left[\bar{x}_i - \bar{x}_j \pm t_{1-\frac{\alpha}{2}}(n-1)\sqrt{\frac{S_{ii} - 2S_{ij} + S_{jj}}{n}}\right]$$

Test

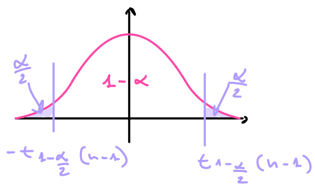
$\begin{cases} H_0: \underline{a}'\underline{\mu} \le \delta_0 \\ H_1: \underline{a}'\underline{\mu} > \delta_0 \end{cases}$ test stat: $\dfrac{\underline{a}'\bar{\underline{x}} - \delta_0}{\sqrt{\underline{a}'S\underline{a}}}\sqrt{n} < t_0$

Reject H_0 if: $t_0 > t_{1-\alpha}(n-1)$



$\begin{cases} H_0: \underline{a}'\underline{\mu} = \delta_0 \\ H_1: \underline{a}'\underline{\mu} \neq \delta_0 \end{cases}$ $\quad t_0 = \dfrac{\underline{a}'\bar{\underline{x}} - \delta_0}{\sqrt{\underline{a}'S\underline{a}}}\sqrt{n}$

Reject H_0 if: $t_0 > t_{1-\frac{\alpha}{2}}(n-1)$
or $t_0 < -t_{1-\frac{\alpha}{2}}(n-1)$



$\alpha \in [0,1]$ $\quad \forall \underline{a} \in \mathbb{R}^p: \quad CI_{1-\alpha}(\underline{a}'\underline{\mu}) = \left[\underline{a}'\bar{\underline{x}} \pm t_{1-\frac{\alpha}{2}}(n-1)\sqrt{\frac{\underline{a}'S\underline{a}}{n}}\right]$

$\forall \underline{a} \in \mathbb{R}^p: \quad \Pr\left[\underline{a}'\underline{\mu} \in \left[\underline{a}'\bar{\underline{x}} \pm t_{1-\frac{\alpha}{2}}(n-1)\sqrt{\frac{\underline{a}'S\underline{a}}{n}}\right]\right] = 1-\alpha$ $\quad$ "given an $\underline{a}$ interval, it's prob is"

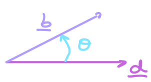
? $\Pr\left[\underline{a}'\underline{\mu} \in \left[\underline{a}'\bar{\underline{x}} \pm t_{1-\frac{\alpha}{2}}(n-1)\sqrt{\frac{\underline{a}'S\underline{a}}{n}}\right],\ \forall \underline{a} \in \mathbb{R}^p\right] = 1-\alpha$ $\quad$ "given all $\underline{a}$ interval, this prob is for all possible trial simultaneously"

→ it cant be this distribution

A bit of Algebra

$\underline{b}, \underline{d} \in \mathbb{R}^p$

$\cos\theta = \dfrac{\underline{b}'\underline{d}}{\|\underline{b}\|\|\underline{d}\|}$



$(\underline{b}'\underline{d})^2 = \|\underline{b}\|^2\|\underline{d}\|^2\cos^2\theta$

$(\underline{b}'\underline{d})^2 \le \|\underline{b}\|^2\|\underline{d}\|^2$ $\quad$ (equality if $\theta = 0$, $\theta = \pi$, meaning $\underline{b} = k\underline{d}$)

$\left(\sum_1^p b_i d_i\right)^2 \le \left(\sum_1^p b_i\right)\left(\sum_1^p d_i\right)$ $\quad$ CHAUCHY – SCHWARTZ

EXTENDED C.S. INEQUALITY

B: $p \times p$ pos. def., $\forall \underline{b}, \underline{d} \in \mathbb{R}^p \implies (\underline{b}'\underline{d})^2 \le (\underline{b}'B\underline{b})(\underline{d}'B^{-1}\underline{d})$ $\quad$ (with the equality if $\underline{b} \in \mathcal{L}(B^{-1}\underline{d})$)

$(\underline{b}'\underline{d})^2 = (\underline{b}'B^{1/2}B^{-1/2}\underline{d})^2 \le \|B^{1/2}\underline{b}\|^2\,\|B^{-1/2}\underline{d}\|^2 =$ $\quad$ with equality if $B^{1/2}\underline{b} \in \mathcal{L}(B^{-1/2}\underline{d}) \iff \underline{b} \in \mathcal{L}(B^{-1}\underline{d})$

$= (\underline{b}'B^{1/2}B^{1/2}\underline{b})(\underline{d}'B^{-1/2}B^{-1/2}\underline{d}) =$

$\overset{!}{=} (\underline{b}'B\underline{b})(\underline{d}'B^{-1}\underline{d})$

Lemma

B $p \times p$ pos. def $\quad \underline{d} \in \mathbb{R}^p \quad \max\limits_{\substack{\underline{x} \in \mathbb{R}^p \\ \underline{x} \neq \underline{0}}} \dfrac{(\underline{x}'\underline{d})^2}{\underline{x}'B\underline{x}} = \underline{d}'B\underline{d}$

Proof: $(\underline{x}'\underline{d}) \le (\underline{x}'B\underline{x})(\underline{d}'B\underline{d})$ $\quad$ C.S. eq holds if $\underline{x} \in \mathcal{L}(B^{-1}\underline{d})$

Since $\underline{x} \neq \underline{0}$ $\quad \underline{x}'B\underline{x} > 0 \quad \dfrac{(\underline{x}'\underline{d})^2}{\underline{x}'B\underline{x}} \le (\underline{d}'B^{-1}\underline{d}) \quad \forall \underline{x} \neq 0$

$\implies \max\limits_{\underline{x} \in \mathbb{R}^p} \dfrac{(\underline{x}'\underline{d})^2}{\underline{x}'B\underline{x}} = \underline{d}'B^{-1}\underline{d}$

$\underline{a} \in \mathbb{R}^p$ $\quad \frac{\underline{a}'\overline{\underline{X}} - \underline{a}'\underline{\mu}}{\sqrt{\underline{a}' S \underline{a}}} \sqrt{n}$ $\quad$ Let's square it and maximize $\quad \max_{\underline{a} \in \mathbb{R}^p} \frac{(\underline{a}'\overline{\underline{X}} - \underline{a}'\underline{\mu})^2}{\underline{a}' S \underline{a}} n = n(\overline{\underline{X}} - \underline{\mu})' S^{-1} (\overline{\underline{X}} - \underline{\mu}) \sim \frac{(n-1)p}{(n-p)} F(p, n-p)$

$$Pr\left[\underline{a}'\underline{\mu} \in \left[\underline{a}'\overline{\underline{x}} \pm \overset{?}{c} \sqrt{\frac{\underline{a}' S \underline{a}}{n}}\right], \ \forall \underline{a} \in \mathbb{R}^p \right] = 1 - \alpha \qquad \alpha \in [0,1]$$

$$Pr\left[\frac{(\underline{a}'(\overline{\underline{X}} - \underline{\mu}))^2}{\underline{a}' S \underline{a}} n \le c^2, \ \forall \underline{a} \in \mathbb{R}^p \right] = Pr\left[\max_{\underline{a} \in \mathbb{R}^p} \frac{(\underline{a}'(\overline{\underline{X}} - \underline{\mu}))^2}{\underline{a}' S \underline{a}} n \le c^2 \right] = 1 - \alpha$$

$$\Longrightarrow \quad c^2 = \frac{(n-1)p}{n-p} F_{1-\alpha}(p, n-p)$$

Hence $\boxed{Pr\left[\underline{a}'\underline{\mu} \in \left[\underline{a}'\overline{\underline{x}} \pm \sqrt{\frac{(n-1)p}{n-p} F_{1-\alpha}(p,n-p)} \sqrt{\frac{\underline{a}' S \underline{a}}{n}} \right], \ \forall \underline{a} \in \mathbb{R}^p \right] = 1 - \alpha}$

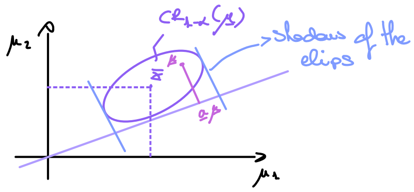
$\alpha \in [0,1]$

For $\underline{a} \in \mathbb{R}^p$ $\quad CI_{1-\alpha}(\underline{a}'\underline{\mu}) = \left[\underline{a}'\overline{\underline{X}} \pm \underbrace{t_{1-\frac{\alpha}{2}}(n-1)}_{\text{smaller}} \sqrt{\frac{\underline{a}' S \underline{a}}{n}} \right]$ $\quad$ one-at-the-time

simultaneously-interval-conf.: $\quad SimCI_{1-\alpha}(\underline{a}'\underline{\mu}) = \left[\underline{a}'\overline{\underline{X}} \pm \underbrace{\sqrt{\frac{(n-1)p}{n-p} F_{1-\alpha}(p,n-p)}}_{\text{bigger}} \sqrt{\frac{\underline{a}' S \underline{a}}{n}} \right]$

$\alpha \in [0,1]$

$$CR_{1-\alpha}(\underline{\mu}) = \left[\underline{q} \in \mathbb{R}^p : n(\underline{q} - \overline{\underline{X}})' S^{-1} (\underline{q} - \overline{\underline{X}}) \le \frac{(n-1)p}{n-p} F_{1-\alpha}(p, n-p) \right]$$



Let $\underline{a}_1, \dots, \underline{a}_k \in \mathbb{R}^p$

$\beta \in [0,1]$ $\quad CI_{1-\beta} \left[\underline{a}_i'\overline{\underline{X}} \pm t_{1-\frac{\beta}{2}}(n-1) \sqrt{\frac{\underline{a}_i' S \underline{a}_i}{n}} \right]$ for all $i = 1, \dots, k$ is an one-at-time CI for $\underline{a}_i'\underline{\mu}$ of level $1-\beta$

$$\alpha \in [0,1] \quad Pr\left[\bigcap_{i=1}^{k} \left[\underline{a}_i'\underline{\mu} \in CI_{1-\beta}(\underline{a}_i'\underline{\mu}) \right] \right] = 1 - \alpha \quad = 1 - Pr\left[\bigcup_{i=1}^{k} \left[\underline{a}_i'\underline{\mu} \notin CI_{1-\beta}(\underline{a}_i'\underline{\mu}) \right] \right] = 1 - \alpha$$
$$\ge 1 - \sum_{i=1}^{k} Pr\left[\underline{a}_i'\underline{\mu} \notin CI_{1-\beta}(\underline{a}_i'\underline{\mu}) \right] = 1 - k\beta$$

Hence take $\beta = \alpha / k$

Bonferroni sim conf int for $\underline{a}_i'\underline{\mu}$ $\quad i = 1, \dots, k$: $\quad BCI(\underline{a}_i'\underline{\mu}) = \left[\underline{a}_i'\overline{\underline{X}} \pm t_{1-\frac{\alpha}{2k}}(n-1) \sqrt{\frac{\underline{a}_i' S \underline{a}_i}{n}} \right]$

Testing $\quad \underline{a}_1, \dots, \underline{a}_k \in \mathbb{R}^p$

$$H_0: \begin{cases} \underline{a}_1'\underline{\mu} = \delta_1 \\ \underline{a}_2'\underline{\mu} = \delta_2 \\ \vdots \\ \underline{a}_k'\underline{\mu} = \delta_k \end{cases} \quad vs \quad H_1: \exists i \in \{1, \dots, k\} \text{ s.t. } \underline{a}_i'\underline{\mu} \ne \delta_i$$

$\alpha \in [0,1]$ $\quad$ Reject H_0 at level α if for at least one $i \in \{1, \dots, k\}$

$$\frac{|\underline{a}_i'\overline{\underline{X}} - \delta_i|}{\sqrt{\underline{a}_i' S \underline{a}_i}} \sqrt{n} > t_{1-\frac{\alpha}{2k}}(n-1) \qquad \text{(Bonferroni)}$$

$$P[\text{Reject } H_0 \mid H_0 \text{ is true}] = Pr\left[\bigcup_{i=1}^{k} \left\{ \frac{|\underline{a}_i'\overline{\underline{X}} - \delta_i|}{\sqrt{\underline{a}_i' S \underline{a}_i}} \sqrt{n} > t_{1-\frac{\alpha}{2k}}(n-1) \right\} \Big| H_0 \right]$$
$$\le \sum_{i=1}^{k} \underbrace{Pr\left[\frac{|\underline{a}_i'\overline{\underline{X}} - \delta_i|}{\sqrt{\underline{a}_i' S \underline{a}_i}} \sqrt{n} > t_{1-\frac{\alpha}{2k}}(n-1) \Big| H_0 \right]}_{\frac{\alpha}{k}} = \alpha$$