ED-2 ASSIGNMENT: 1; PART: 1.

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Problem: 1.2

For localized charge distⁿ we get:

$$\vec{\nabla} \times \vec{E} = 0 \quad ; \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho(\bar{r})$$

from the first eq; we can write $\vec{E}$ as a gradient of a scalar quantity or scalar field (for the constrain of a localized static charge distⁿ) called $\phi(\bar{r})$

i.e $\quad \vec{E} = -\vec{\nabla}\phi \quad \cdots (1)$

$$\therefore \quad \vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla}\phi) = -\nabla^2\phi = 4\pi\rho(r)$$

$$\therefore \quad \nabla^2\phi = -4\pi\rho(r) \qquad \text{(in field point)}$$

Clearly as the solution of the given eq is finite in any point in space (as $|E(r)| \to 0$ as $\hat{r} \to \infty$);

A unique Boundary condition in E will give a unique B.C in ϕ (except an additional constant) to satisfy eq (1).

Now let's take two different solutions of ϕ which satisfies the same B.C

i.e ϕ_1 & ϕ_2 so that:

$$\nabla^2\phi_1 = -4\pi\rho \quad ; \quad \nabla^2\phi_2 = -4\pi\rho$$

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As on the Boundary the soln is unique to satisfy the given B.C so; at boundary

$$\tilde{\phi} = \phi_1 - \phi_2 = 0 \quad \cdots (2)$$

Ans for all points in space

$$\nabla^2 \tilde{\phi} = \nabla^2 \phi_1 - \nabla^2 \phi^2 = -4\pi\rho + 4\pi\rho = 0. \quad (\because \rho \text{ is finite})$$

~~but~~ i.e $\tilde{\phi}$ satisfies Laplace's eq with it's value 0 on boundary.

As $\nabla^2 \tilde{\phi} = 0 \Rightarrow \sum_i \partial_i^2 \tilde{\phi} = 0$

So if at any point of $\tilde{\phi}$ has maxima or minima then at that point $\partial_i^2 \phi \lesseqgtr 0 \; ; \forall i$

i.e $\sum \partial_i^2 \tilde{\phi} = \nabla^2 \tilde{\phi} \lessgtr 0$

but since $\tilde{\phi}$ has no point with $\nabla^2 \tilde{\phi} \lessgtr 0$ and always $\nabla^2 \tilde{\phi} = 0$ so $\tilde{\phi}$ has no local maxima or minima and hence all maxima/minima of $\tilde{\phi}$ occured on boundary.

But on boundary: $\tilde{\phi} = 0$; i.e $(\tilde{\phi}_{inside})_{max/min} = 0$

$\therefore$ in all point enclosed by given boundary we must have $\tilde{\phi} = 0$

i.e ~~$\tilde{\phi}$ has uni~~ $\phi_1 - \phi_2 = 0$ i.e $\phi_1 = \phi_2$ (everywhere inside) i.e $-\vec{\nabla}\phi_1 = -\vec{\nabla}\phi_2$ or $\vec{E_1} = \vec{E_2}$

i.e we get a unique solⁿ for the given set of eqⁿs in the region.

proved

Problem: 1.3

Let's take a piece of linear conductor in which arbitrary / random flow of charge is going on & the motion of charge at any point be given by $v(\vec{r}, t)$

Now the maxwell eq gives:

$$\vec{\nabla} \times \vec{B} = \mu \vec{j} + \frac{1}{(\mu\epsilon)^{-1}} \left(\frac{\partial \vec{E}}{\partial t}\right)$$

(μ, ϵ = permeability & permittivity of medium)

$$\therefore \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 \text{ (obvious)} = \vec{\nabla} \cdot \left\{ \mu \vec{j} + \frac{1}{(\mu\epsilon)^{-1}} \frac{\partial \vec{E}}{\partial t} \right\}$$

$$\text{i.e } \mu \nabla \cdot \vec{j} = -\left(\frac{1}{\mu\epsilon}\right)^{-1} \frac{\partial}{\partial t} (\nabla \cdot \vec{E})$$

$$\text{i.e } \mu \nabla \cdot \vec{j} = -\mu\epsilon \times \frac{1}{\epsilon} \frac{\partial \rho}{\partial t} \quad \left(\because \nabla \cdot \vec{E} = \frac{\rho}{\epsilon}\right)$$

$$\therefore \boxed{\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}} \longrightarrow \text{local charge conservation eq / continuity eq.}$$

i.e for a finite space if we integrate:

$$\int (\nabla \cdot \vec{j})\, d\tau = -\int \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

i.e $\oint \vec{j} \cdot d\vec{A} = -\frac{\partial}{\partial t}\left(\int \rho \cdot d\tau\right)$ (Stoke's theorem)

i.e $-\frac{\partial Q_{int}}{\partial t} = \oint \vec{j} \cdot d\vec{A}$

i.e $\frac{\partial Q_{int}}{\partial t} = -\oint \vec{j} \cdot d\vec{A}$

i.e the amount of charge vanishing from a given finite volume element (τ) is the same amount of charge going out through the surface of that element. Which is the statement of charge conservation in global form. ⟶ <u>proved</u>

* Here the fact that all charges are moving with arbitrary ~~finit~~ velocity $\vec{v}(\vec{r},t)$ is ~~or~~ already imposed in the used Maxwell's eq $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. So we don't have to impose that anymore in the proof & use $\frac{d}{dt} \to \frac{\partial}{\partial t} + v \cdot \nabla$ as done in class starting from the electrostatics law & not from general electrodynamics law.

Problem: 1.8

a) The general Lagrangian of the EM field + charge system has two parts:

i) describing the energy stored in field + K.E of charge

ii) The interaction between field & charge.

The Lagrangian is given by: (treating for charges: $v \ll c$)

$$L = \underbrace{\sum_i \frac{m v_i^2}{2}}_{\text{K.E of charges}} - \underbrace{\frac{1}{4\mu_0}\int_V F_{\mu\nu} F^{\mu\nu} d^3x}_{\text{Lagrangian purely due to field}} \underbrace{- \sum q_i \phi + \sum q_i (\vec{A}\cdot\vec{v}_i)}_{\text{Interaction between field \& charge}}$$

(i = index for charge)

Answer

b) Derivation of the eq of motion of charge in the field:

Here we have to (sufficiently) do the variation of action on the Lagrangian of a single ~~part~~ charged particle.

For a single particle in the field; the Lagrangian be:

$$L = \frac{mv^2}{2} + q(\vec{A}\cdot\vec{v}) - q\phi$$

$$= \sum_{i=1}^{3} \frac{m\dot{x}_i^2}{2} + q\sum_{i=1}^{3}(A_i v_i) - q\phi \quad \left(\begin{array}{c} i = i\text{th} \\ \text{dimension} \end{array}\right)$$

for $i=1$; i.e x component; the eq of motion from variational principle gives:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

but $$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_1(x,y,z)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + q\left(\dot{x}\frac{\partial A_1}{\partial x} + \dot{y}\frac{\partial A_1}{\partial y} + \dot{z}\frac{\partial A_1}{\partial z}\right) + q\frac{\partial A_1}{\partial t}.$$

And; $$\frac{\partial L}{\partial x} = q\left(\dot{x}\frac{\partial A_1}{\partial x} + \dot{y}\frac{\partial A_2}{\partial x} + \dot{z}\frac{\partial A_3}{\partial x}\right) - q\left(\frac{\partial \phi}{\partial x}\right)$$

So:

$$m\ddot{x} + q\left(\vec{v}\cdot\vec{\nabla}A_1\right) + q\left(\frac{\partial A_1}{\partial t}\right) = q\left(\dot{x}\frac{\partial A_1}{\partial x} + \dot{y}\frac{\partial A_2}{\partial x} + \dot{z}\frac{\partial A_3}{\partial x}\right) - q\frac{\partial \phi}{\partial x}$$

$$\Rightarrow m\ddot{x} + q\frac{\partial A_1}{\partial t} = q\left[\dot{y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \dot{z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right] - q\left(\frac{\partial \phi}{\partial x}\right)$$

$$= q\left[\dot{y}\cdot\left(\vec{\nabla}\times\vec{A}\right)_z - \dot{z}\cdot\left(\vec{\nabla}\times\vec{A}\right)_y\right] - q\left(\frac{\partial \phi}{\partial x}\right)$$

i.e $$m\ddot{x} = +q\left\{-\frac{\partial \phi}{\partial x} - \frac{\partial A_1}{\partial t}\right\} + q\left\{v_y B_z - v_z B_y\right\}$$

i.e $$m\ddot{x} = q\left[\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right)_x + q\left(\vec{v}\times\vec{B}\right)_x\right]$$

i.e for 3d vector form:

$$m\ddot{\vec{r}} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \longrightarrow \underline{\underline{\text{Answer}}}$$

$$\left(\text{using } \vec{B} = \vec{\nabla}\times\vec{A} \; ; \; \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right)$$

<u>c</u> The field Lagrangian density is given by:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$$

Now $$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix} \text{Antisymmetric}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix} \text{Antisymmetric}$$

$$\therefore F_{\mu\nu}F^{\mu\nu} = \left(-\frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} - \frac{E_z^2}{c^2} + B_z^2 + B_y^2 + B_x^2 - \frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} - \frac{E_z^2}{c^2} + B_x^2 + B_y^2 + B_z^2\right)$$

$$= -\frac{2E^2}{c^2} + 2B^2.$$

$$\text{i.e } \mathcal{L} = \frac{\frac{1}{c^2}E^2 - B^2}{2\mu_0} - \rho\phi + \vec{j}\cdot\vec{A}$$

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Here the Lagrangian is written in terms of both the real fields ($\vec{E}, \vec{B}$) And the Auxiliary fields ($\phi, \vec{A}$). So we need to rewrite $\mathcal{L}$ in terms of one single field quantity

We use

$$\left.\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}\right\} \quad \cdots (1)$$

to write $\mathcal{L}$ in terms of $(\vec{A}, \phi)$

i.e;

$$\mathcal{L} = \frac{1}{2\mu_0}\left\{\frac{1}{c^2}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)^2 - \left(\vec{\nabla}\times\vec{A}\right)^2\right\} - \rho\phi + \vec{J}\cdot\vec{A} \quad \cdots (2).$$

Now the Homogeneous Maxwell's eq is not derivable from the Lagrangian; as it is itself used (eq (1)) to write the Lagrangian from 4 field quantity ($\vec{E}, \vec{B}, \phi, \vec{A}$) to Auxiliary field ($\phi, \vec{A}$) quantity.

i.e from (1) we get:

$$\boxed{\begin{aligned} \vec{\nabla}\times\vec{E} &= \vec{\nabla}\times\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right) \\ \text{i.e } \vec{\nabla}\times\vec{E} &= 0 - \frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right) = -\frac{\partial \vec{B}}{\partial t} \end{aligned}}$$

And $\cancel{\vec{\nabla}\times\vec{B}}$ $\boxed{\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0}$

The other two eqn's have to be derived from the field Lagrangian from variation:

The E-L eq for field is given by:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x_\mu}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right\}$$

but $\frac{\partial \mathcal{L}}{\partial \phi} = -\rho$ (from (2))

And $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$ for $\mu = 0$

(as $\mathcal{L}$ is indp of $\dot{\phi}(t)$; i.e $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$)

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)} = \frac{\partial \mathcal{L}}{\partial \phi'(x)} \qquad \left(\phi'(x_i) = \frac{\partial \mathcal{L}}{\partial x_i} \cdots \text{etc}\right)$$

$$= \frac{1}{c^2}\frac{\partial}{\partial \phi'(x)}\left\{\frac{1}{2\mu_0}\left(-\hat{i}\phi'(x) - \hat{j}\phi'(y) - \hat{k}\phi'(z) - \frac{\partial \vec{A}}{\partial t}\right)^2\right\} + \underset{\text{from other terms}}{0}$$

$$= \frac{1}{2\mu_0 c^2}\left\{\left(-\hat{i}\phi'(x) - \hat{j}\phi'(y) - \hat{k}\phi'(z) - \frac{\partial \vec{A}}{\partial t}\right)\cdot(-\hat{i})\right\}\times 2$$

$$= \frac{1}{\cancel{2}\epsilon_0}\left(+\phi'(x) + \frac{\partial A_x}{\partial t}\right) \qquad \left(\because c^2 = \frac{1}{\mu_0\epsilon_0}\right)$$

$$\therefore \quad \frac{\partial}{\partial x_1}\left(\frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)}\right) = -\frac{\partial}{\partial x}\left\{\frac{1}{\epsilon_0^{-1}}\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\right)\right\}$$

$$= -\left(\frac{1}{\epsilon_0}\right)^{-1} \frac{\partial E_x}{\partial x}$$

$$\left(\because E_x = \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\right)$$

Similarly $\frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial(\partial_j \phi)}\right) = -\left(\frac{1}{\epsilon_0}\right)^{-1} \frac{\partial E_j}{\partial x_j}$

i.e E.L eq gives!

$$-\rho = 0 - \left(\frac{1}{\epsilon_0}\right)^{-1}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

i.e $\rho = \epsilon_0 \, \vec{\nabla} \cdot \vec{E}$

i.e $\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$ $\longrightarrow$ 1st inhomogeneous Maxwell's eq'n.

Next start with $\vec{A}$ field; particularly A_x component.

$$\frac{\partial \mathcal{L}}{\partial A_x} = \frac{\partial}{\partial x_\mu}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_x)}\right\}$$

Now $\frac{\partial \mathcal{L}}{\partial A_x} = +j_x$

And $\frac{\partial \mathcal{L}}{\partial(\partial_0 A_x)} = \frac{\partial \mathcal{L}}{\partial \dot{A}_x} \times c$

$$= \frac{1}{2\mu_0 c^2} \times c \, \frac{\partial}{\partial \dot{A}_x} \left\{ -\vec{\nabla}\phi - \hat{i}\frac{\partial A_x}{\partial t} - \hat{j}\frac{\partial A_y}{\partial t} - \hat{k}\frac{\partial A_z}{\partial t} \right\}^2$$

$$= \frac{1}{2\mu_0 c} \left\{ \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \right) \cdot \left(-\hat{i} \right) \right\} \times 2$$

$$= \frac{1}{2\mu_0 c} \left(\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t} \right)_x \times 2 = -\frac{2E_x}{2\mu_0 c}$$

$$\therefore \frac{\partial}{\partial x_0} \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 A_x)} \right] = -\frac{2}{2\mu_0 c} \frac{\partial E_x}{\partial t} \times \frac{1}{c}$$

$$= -\frac{1}{\cancel{2}\mu_0 c^2} \frac{\partial E_x}{\partial t} = -\left(\frac{1}{\epsilon_0}\right)^{-1} \frac{\partial E_x}{\partial t}.$$

And, $$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x A_x)} \right) = \frac{\partial}{\partial x} \left\{ \frac{\partial \mathcal{L}}{\partial A'_x(x)} \right\} \qquad \begin{pmatrix} A'_x(x) = \partial_x A_x \\ A'_y(x) = \partial_x A_y \\ \cdots \text{etc} \end{pmatrix}$$

Now, $$\frac{\partial \mathcal{L}}{\partial(\partial_x A_x)} = -\frac{1}{2\mu_0} \frac{\partial}{\partial(\partial_x A_x)} \left(\vec{\nabla} \times \vec{A} \right)^2.$$

$$= -\frac{1}{2\mu_0} \times 2 \left(\vec{\nabla} \times \vec{A} \right) \cdot \frac{\partial}{\partial A'_x(x)} \left(\vec{\nabla} \times \vec{A} \right)$$

$$= -\frac{\vec{B}}{\mu_0} \cdot \frac{\partial}{\partial(\partial_x A_x)} \underbrace{\left\{ \hat{i}\left(\partial_y A_z - \partial_z A_y \right) - \hat{j}\left(\partial_x A_z - \partial_z A_x \right) + \hat{k}\left(\partial_x A_y - \partial_y A_x \right) \right\}}_{0}$$

$= 0.$

$$\frac{\partial}{\partial y}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_y A_x)}\right\} = \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial A'_{x}(y)}\right)$$

Now, $$\frac{\partial \mathcal{L}}{\partial(\partial_y A_x)} = -\frac{\vec{B}}{\mu_0}\cdot\frac{\partial}{\partial(\partial_y A_x)}(\vec{\nabla}\times\vec{A})$$

like previous way

$$= -\frac{\vec{B}}{\mu_0}\cdot\frac{\partial}{\partial(\partial_y A_x)}\left[\hat{i}(\partial_y A_z - \partial_z A_y) - \hat{j}(\partial_x A_z - \partial_z A_x) + \hat{k}(\partial_x A_y - \partial_y A_x)\right]$$

$$= -\frac{\vec{B}}{\mu_0}\cdot(-\hat{k}) = \frac{B_z}{\mu_0}.$$

i.e $$\frac{\partial}{\partial y}\left\{\left(\frac{\partial \mathcal{L}}{\partial(\partial_y A_x)}\right)\right\} = \frac{1}{\mu_0}\partial_y B_z.$$

And $$\frac{\partial}{\partial z}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_z A_x)}\right\} = \frac{\partial}{\partial z}\left(-\frac{\vec{B}}{\mu_0}\cdot(+\hat{j})\right) = \frac{\partial}{\partial z}\left(-\frac{B_y}{\mu_0}\right)$$

$$= -\frac{1}{\mu_0}\partial_z B_y$$

$$\therefore \frac{\partial \mathcal{L}}{\partial A_x} = \frac{\partial}{\partial x_\mu}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_x)}\right\} \text{ gives,}$$

$$j_x = +\frac{1}{\mu_0}(\partial_y B_z - \partial_z B_y) - \epsilon_0\frac{\partial E_x}{\partial t}.$$

i.e $$j_x = -\epsilon_0\left(\frac{\partial \vec{E}}{\partial t}\right)_x + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})_x$$

So in vector form (when we vary $\mathcal{L}$ for all A_j) we get:

$$\vec{j} = -\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\vec{\nabla} \times \vec{B}}{\mu_0}.$$

i.e $$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}} \longrightarrow \underline{\underline{\text{Answer}}}$$

(Which is 2nd inhomogeneous Maxwell's eq)

Thus varying the ~~field~~ Auxiliary field A^μ we get 4 inhomogeneous (1 in scalar form; Another 3 combines to the vector eq) Maxwell's eq'ns.

$\longrightarrow$ <u>proved</u>

<u>d.</u> For any classical system if H is the Hamiltonian & u be a fn of canonical variables & time i.e $u = u(q, p, t)$ then:

$$\frac{du}{dt} = \frac{\partial u}{\partial q} \dot{q} + \frac{\partial u}{\partial p} \dot{p} + \frac{\partial u}{\partial t}.$$

but the E.O.M from Hamilton's ~~to~~ least action principle gives:

$$\dot{q} = \frac{\partial H}{\partial p} \; ; \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

So; $\frac{du}{dt} = \frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial u}{\partial p} + \frac{\partial u}{\partial t}$

$$= [u, H]_{p.b} + \frac{\partial u}{\partial t}.$$

for $u = H$ we get:

$$\frac{dH}{dt} = [H, H]_{p.b} + \frac{\partial H}{\partial t}$$

but $[f(H), H] = 0$; so, $[H, H] = 0$

i-e $\frac{dH}{dt} = \frac{\partial H}{\partial t}$

Here

$$H = \sum_i \frac{1}{2m_i}\left(\vec{p_i} - \frac{q_i \vec{A}(\vec{r_i})}{c}\right)^2 + \int \frac{E^2 + B^2}{8\pi} d^3\vec{r}$$

As $\vec{A} = \vec{A}(\vec{r_i})$ so, $\partial_t \vec{A} = 0$

And as $\vec{B} = \vec{\nabla} \times \vec{A}$; $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ (~~for this~~)

So; $\partial_t \vec{B} = 0$; $\partial_t \vec{E} = 0$.

i-e $H = H(\vec{p_i}, \vec{r_i})$ only & $\partial_t H = 0$.

$$\therefore \boxed{\frac{dH}{dt} = \partial_t H(\vec{r_i}, \vec{p_i}) = 0} \longrightarrow \text{proved}$$

i-e total energy is conserved.

Problem: 1.4

Without magnetic charge: Maxwell's eq (in space) is:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_E}{\epsilon_0} \; ; \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 J_E + \mu_0\epsilon_0 \frac{\partial E}{\partial t} \; ; \quad \vec{\nabla}\cdot\vec{B} = 0.$$

Now if there were magnetic charge g & the magnetic charge density be ρ_g then the force law between two magnetic ~~char~~ charges will be

$$|F_g| = \alpha \frac{g_1 g_2}{r^2}$$ (coloumb's law for magnetic charge)

α is some constant like G in gravity or $\frac{1}{4\pi\epsilon_0}$ in electrostatics.

But corresponding to the force law

$\cancel{|F_E| = qE}$ $\quad \vec{F}_E = q_e\vec{E}$

we should also had

$$|\vec{F}| = g|\vec{B}|$$

$\therefore$ from dimensional analogy we get:

$$[\alpha]\frac{[g]^2}{[r^2]} = [g]\cdot[B]$$

i.e $$[\alpha]\frac{[g]}{L^2} = [B]$$

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We can (without loss of generality) make g with some dimension so that α is dimensionless. (it's one choice; not the only way to make the theory.)

In that case we get: $[g] = L^2[B]$

Now corresponding to maxwell's 1st law (which comes from $\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_E\, d\tau}{\lambda}\hat{\lambda}$) we should similarly get the same eq (Gauss law of magnetic charge) in same way. Just the factors will be different.

Now, $$\vec{E}_E = \frac{1}{4\pi}\int \frac{(\rho_E/\epsilon_0)\,d\tau}{\lambda^2}\hat{\lambda} \Rightarrow \nabla\cdot\vec{E} = \frac{\rho_E}{\epsilon_0}.$$

$$\vec{B} = \vec{E_B} = \alpha \int \frac{\rho_m\, d\tau}{\lambda^2}\hat{\lambda} \Rightarrow \nabla\cdot\vec{B} = 4\pi\alpha\rho_m$$

(coming from coloumb's law in magnetic charge)

Now we choose ρ_m's unit so that $\alpha = \frac{1}{4\pi}$

i.e $\vec{\nabla}\cdot\vec{B} = \rho_m$ or ρ_g (doesn't matter m/g is just label for magnetic charge)

Now electric current density has dimension

$$[j_E] = [A\cdot m^{-2}] = [A\,s\,m^{-2}\,s^{-1}]$$
$$= [q_e]\,L^{-2}T^{-1}$$

Similarly there will be some quantity magnetic current density: j_g/j_m so that $[j_m] = [g]\,L^{-2}T^{-1}$

Like wise

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_E + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

we ~~woud~~ would expect

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} - c \vec{J}_m$$

(c is some constant)

Now, $$[c] = \frac{[B] T^{-1}}{[J_m]} = \frac{[B] T^{-1}}{[g] L^{-2} T^{-1}}$$

$$= \frac{[B]}{[g] L^{-2}}$$

but $[g] = BL^2$ i.e $[c] = 1$

So c is dimensionless constant.

we can choose $c = 1$ and give unit of J_m accordingly.

(choosing $c = -1$ would give a '+' sign in the formula. But doesn't matter; it ~~woud~~ would be just a transformation $\vec{J}_m \to -\vec{J}_m$)

So

$$\vec{\nabla} \times \vec{E} = - \vec{J}_m - \frac{\partial \vec{B}}{\partial t}.$$

i.e in presence of magnetic charge g (magnetic charge density ρ_m / ρ_g; current density $\vec{J}_m / \vec{J}_g$;) the Maxwell's eqn's be like: (in space)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_E}{\epsilon_0} \;;\; \vec{\nabla}\times\vec{E} = -\vec{J}_m - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = \rho_m \;;\; \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_E + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

(According to the definition of g & $\vec{J}_m$ as given previously.)

Ans

Problem: 1.9

The field part of E.M Lagrangian density is given by:

$$\mathcal{L}_{field} = \mathcal{L}_f = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu$$

purely field term — interaction term of field & Source

The given Gauge transformation

$$\vec{A} \longrightarrow \vec{A} + \nabla\lambda$$
$$\phi \longrightarrow \phi - \frac{1}{c}\frac{\partial\lambda}{\partial t}$$

is equivalent to $A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu\lambda$

Now the field tensor $F^{\mu\nu}$ is a rank 2 antisymmetric tensor & is made up totally by gauge invariant quantities. E_i & B_i i.e gauge invariant quantities.

$$\left(\because \vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \underbrace{\vec{\nabla}\times(\nabla\lambda)}_{0} = \vec{\nabla}\times\vec{A} = \vec{B}\right.$$

$$\left.\vec{E}' = -\nabla\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t} = -\nabla\phi + \cancel{\frac{1}{c}\frac{\partial}{\partial t}(\nabla\lambda)} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}(\nabla\lambda)}\right)$$

$$= \vec{E}$$

$\therefore F'_{\mu\nu} = F_{\mu\nu}$ ~~under~~ (under Gauge transformation)

now; $\mathcal{L}' = -\frac{1}{4\mu_0} F'^{\mu\nu} F'_{\mu\nu} - j^\mu A'_\mu$

$$= -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu - j^\mu (\partial_\mu \lambda)$$

$\left(\because j^\mu = (\rho c, \vec{j}) = j'^\mu\right)$

$$\therefore \Delta \mathcal{L} = \mathcal{L}' - \mathcal{L} = -j^\mu (\partial_\mu \lambda)$$

So the Lagrangian for a finite spacetime region the total change is:

$$\Delta L = \int_V d^3x \int_{t_1}^{t_2} dt \cdot (\Delta \mathcal{L})$$

Now we make two assumptions:

1) $j, \nabla\lambda$ vanishes on the boundary of V

ii) $\rho, \partial_t \lambda$ " at t_1, t_2.

$$\therefore \Delta L = -\int_V d^3x \int_{t_1}^{t_2} dt \cdot j^\mu (\partial_\mu \lambda)$$

$$= -\int_V d^3x \int_{t_1}^{t_2} dt \; \partial_\mu (j^\mu \lambda) + \cancel{\int_V d^3x \int_{t_1}^{t_2} dt \, (\partial_\mu j^\mu) \lambda}^{\,0}.$$

The 2nd term is zero as from continuity eq we get: $\partial_\mu j^\mu = 0$. i.e $\nabla \cdot j = -\frac{\partial \rho}{\partial t}$.

$$\text{i.e } \Delta L = -\int d^3x \int_{t_1}^{t_2} dt \; \partial_\mu (j^\mu \lambda)$$

$$= -\int d^3x \int_{t_1}^{t_2} dt \left\{ \frac{\partial}{\partial t}(j^0 \lambda) + \vec{\nabla}\cdot(f\vec{j}) \right\}$$

$$= -\cancel{\int d^3x \left[\lambda \rho\right]_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2} dt \int_V d^3x\, \vec{\nabla}\cdot(f\vec{j})$$

(but $\rho\big|_{t_1, t_2} = 0$ i.e the 1st term is zero)

$$= -\int_{t_1}^{t_2} dt \oint_S \lambda \vec{j}\cdot d\vec{S}$$

$$= 0$$

($\because$ $\vec{j}$ vanishes on the boundary surface
i.e $\oint_S \lambda \vec{j}\cdot d\vec{S} = 0$; $\forall t_1, t_2$.)

i.e The Lagrangian is invariant under Gauge transformation

i.e if $A'^\mu = A^\mu + \partial^\mu \lambda$

i.e $A^\mu \to A'^\mu$

then

$\tilde{\mathcal{L}} =$ Lagrangian density $\longrightarrow \tilde{\mathcal{L}}' = \tilde{\mathcal{L}} + j^\mu(\partial_\mu \lambda)$

And the Lagrangian for a finite spacetime region

$$L' = \int \tilde{\mathcal{L}}' d^4x = \int \tilde{\mathcal{L}}\, d^4x + \cancel{\int j^\mu(\partial_\mu\lambda)\, d^4x}^{\,0}$$

$$= L$$

i.e Lagrangian (field part) is invariant under Gauge transformation $\longrightarrow$ proved

Problem: 1.7

The Hamiltonian is explicit fn of (p, q, t)
i.e $H = H(p, q, t)$ defined as $H = p\dot{q} - L$; i.e $L = p\dot{q} - H$

$\therefore$ The Action is given by: $S = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (p\dot{q} - H)\,dt$

According to Hamiltons Stationary action integral principles, if t_1, t_2 be fixed then:

$$\delta S = 0 \quad \text{i.e} \quad \delta\left(\int_{t_1}^{t_2} (p\dot{q} - H)\,dt\right) = 0$$

$$\text{i.e} \quad \int_{t_1}^{t_2}\left(\dot{q}\,\delta p + p\,\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial t}\delta t\right)dt = 0$$

And end points are fixed hence: $\delta t = 0$

$$\text{And} \quad \int_{t_1}^{t_2} p\,\delta\dot{q}\,dt = \int_{t_1}^{t_2} p\,\frac{d}{dt}(\delta q)\,dt \qquad \left(\because \frac{d}{dt}(\delta q) = \frac{\delta(dq)}{dt} = \delta\dot{q}\right)$$

$$= \cancel{\left(p\,\delta q\right)_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2} \dot{p}\,\delta q\,dt$$

[$\because$ At end point; the variations are zero]

$$\therefore \delta S = 0 \Rightarrow \int_{t_1}^{t_2}\left\{\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p + \left(-\dot{p} - \frac{\partial H}{\partial q}\right)\delta q\right\}dt = 0$$

δp, δq's coefficients are then separately zero for holonomic system; as q, p are indp variable in H. i.e $H(p, q, t)$.

$\therefore$ we must have (As Hamiltons eq'n of motion)

$$\left.\begin{aligned} \dot{q} - \frac{\partial H}{\partial p} &= 0 \quad \text{i.e} \quad \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} + \frac{\partial H}{\partial q} &= 0 \quad \text{i.e} \quad \dot{p} = -\frac{\partial H}{\partial q} \end{aligned}\right\} \text{Answer}$$

for a system of particles; the eq for jth particle be:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \; ; \; \dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad \left(\begin{aligned} \text{When } H &= H(q_j, p_j, t)\ \forall j \\ &= \sum p_j \dot{q}_j - L(q_j, \dot{q}_j, t) \end{aligned}\right)$$

Answer

Problem: 1.1

$\vec{F}_i$ = electrostatic force on ith charged particle

$= - \vec{\nabla}_i \{q_i \phi(\vec{r}_i)\}$

ϕ is the scalar potential due to all other charges.

Now; $\vec{F}_i = \sum_j \vec{F}_{ij}(\vec{r}_i - \vec{r}_j) = \vec{F}_{ii}(\vec{r}_i - \vec{r}_i) + \sum_{j \neq i} \vec{F}(\vec{r}_i - \vec{r}_j)$

Now we can give two reasons:

1) $\vec{F}_{ii} = \vec{F}_{ii}(\vec{r}_i - \vec{r}_i) = \vec{F}_{ii}(\vec{0})$

i.e the self interaction term does not depend on any specific direction. So from symmetry & homogenity of space it ~~....~~ must be a null vector.

i.e $\vec{F}_{ii} = \vec{0}$ i.e self interaction term does not contribute to the field. → proved

2). From Newton's 3rd law: $\vec{F}_{ij} = -\vec{F}_{ji}$

(when $\vec{F}_{ij}$ acts along joining line i.e $\vec{F}_{ij} = \vec{F}_{ij}(\vec{r}_i - \vec{r}_j)$)

for $j = i$ we get: $\vec{F}_{ii} = -\vec{F}_{ii}$

i.e $2\vec{F}_{ii} = \vec{0}$ or $\vec{F}_{ii} = \vec{0}$

$\therefore$ self interaction does not contribute to the field. → proved

Problem: 1.5

For notational confusion; I use $A(t) \to \tilde{A}(t)$.

i.e $\tilde{A}(t) = Re(A e^{i\omega t})$; $\tilde{B}(t) = Re(B e^{i\omega t})$

$$\therefore \langle \tilde{A}\tilde{B} \rangle = \frac{1}{T}\int_0^T \tilde{A}\tilde{B}\, dt \quad \left(T = \frac{2\pi}{\omega}\right)$$

$$= \frac{1}{2T}\int_0^T 2AB\cos^2(\omega t)\, dt = \frac{AB}{2T}\int_0^T \left(1 + \cos(2\omega t)\right) dt$$

$$= \frac{AB}{2T}\left(t + \frac{\sin(2\omega t)}{2\omega}\right)_0^T = \frac{AB}{2T} \times (T + 0 - 0 - 0)$$

$$= \frac{AB}{2}.$$

But $Re(\tilde{A}^*\tilde{B}) = Re(\tilde{A}\tilde{B}^*) = \frac{AB}{2}.$

So: $\langle \tilde{A}\tilde{B} \rangle = \frac{1}{2} Re(\tilde{A}^*\tilde{B}) = \frac{1}{2} Re(\tilde{A}\tilde{B}^*)$

$\vec{E} = \hat{x} E_0 \cos(kz - \omega t)$; $\vec{B} = \hat{y} B_0 \cos(kz - \omega t)$

(for some E.M wave of freq ω)

We know; $\frac{\omega}{k} = c = \frac{E_0}{B_0} = \frac{|\vec{E}|}{|\vec{B}|}$

i.e $U = U_E + U_M = \frac{1}{2}\int \left(\frac{B^2}{\mu_0} + \epsilon_0 E^2\right) dv$

$$= \frac{1}{2}\int dv \left(\frac{E^2}{\mu_0 c^2} + \epsilon_0 E^2\right) = \int \epsilon_0 E^2\, dv.$$

$\therefore$ Energy density $= u = \frac{\partial U}{\partial V} = \epsilon_0 E^2.$

Now the Poynting vector for monochromatic wave,

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = c\,\epsilon_0 E_0^2 \cos^2(kz - \omega t) \cdot \hat{z}$$

$\therefore |\vec{S}| = c\,\epsilon_0 E^2 = cu$ i.e $\frac{|\vec{S}|}{u} = c =$ Speed of light $\to$ proved

Prob: 1.6

$$L = \frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2 - V(r,t).$$

$$S = \int_{t_1}^{t_2} L\, dt$$

i.e $$\delta S = \int_{t_1+\delta t_1}^{t_2+\delta t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)\, dt - \int_{t_1}^{t_2} L(q,\dot{q},t)\, dt.$$

$$= \int_{t_1+\delta t_1}^{t_2+\delta t_2} \left(L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) dt - \int_{t_1}^{t_2} L(q,\dot{q},t)\, dt$$

$$= \int_{t_1+\delta t_1}^{t_2+\delta t_2} L\, dt - \int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) dt.$$

Now; $$\int_{t_1+\delta t_1}^{t_2+\delta t_2} = \int_{t_1+\delta t_1}^{t_2} + \int_{t_2}^{t_2+\delta t_2}$$

$$= -\int_{t_2}^{t_1} - \int_{t_1}^{t_1+\delta t_1} + \int_{t_2}^{\delta t_2+t_2}.$$

$$\therefore \delta S = \int_{t_2}^{t_2+\delta t_2} L\, dt - \int_{t_1}^{t_1+\delta t_1} L\, dt + \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\delta q\, dt$$

$$= L(\delta t_2 - \delta t_1) + \frac{\partial L}{\partial \dot{q}}(\delta q - \dot{q}\,\delta t)\Big|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right).\delta q.\, dt.$$

Since δq is arbitrary we get: for $\delta S = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \longrightarrow \text{Newton's EOM in E.L form}$$

i.e

$$\delta S = \underbrace{\left(L - \frac{\partial L}{\partial \dot{q}} \cdot \dot{q}\right)}_{-H} \delta t \Big|_{t_1}^{t_2} + \left(\underbrace{\frac{\partial L}{\partial \dot{q}}}_{p} \delta q\right)\Big|_{t_1}^{t_2}$$

$$= \left(p\,\delta q - H\,\delta t\right)\Big|_{t_1}^{t_2}$$

$$= m\dot{r}\,\delta r - \left(\frac{m\dot{r}^2}{2} + V(r,t)\right)\delta t.$$

$= G$ = generator of the transformation

$\longrightarrow$ proved.

If we take only time variation; i.e

$$\delta q\Big|_{t_1}^{t_2} = 0 \quad \& \quad \delta t(t_1) = \delta t_1 \ ; \ \delta t(t_2) = \delta t_2.$$

then:

$$\delta S = -\left(H\,\delta t\right)\Big|_{t_1}^{t_2} = \left(H(t_1) - H(t_2)\right)\delta t.$$

i.e $\delta S = 0$ iff $H(t_1) = H(t_2)$.

i.e total energy is conserved

$\longrightarrow$ proved.