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Prob - 9

Particle in a ring:-

a. The given Hamiltonian: $\hat{H} = -\frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial\phi^2}$.

\therefore T.I.S.E gives:

$$-\frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial\phi^2} \psi(\phi) = E \cdot \psi(\phi).$$

(The particle is effectively in 1-d with only generalized coordinate ϕ .)

$$\Rightarrow \frac{\partial^2 \psi}{\partial\phi^2} = -\frac{2mR^2 E}{\hbar^2} \psi = -K^2 \psi. \quad (\because E \geq 0)$$

$$\therefore \psi = A e^{\pm iK\phi}$$

$$\text{clearly B.C gives: } \psi(\phi) = \psi(\phi + 2\pi)$$

$$\Rightarrow e^{\pm iK \cdot 2\pi} = 1 \quad \Rightarrow K = 0, \pm 1, \pm 2, \dots$$

$$\text{but } K = \sqrt{\frac{2mR^2 E}{\hbar^2}}$$

$$\Rightarrow \frac{2mR^2 E}{\hbar^2} = m^2 \quad (m \text{ is an integer})$$

$$\Rightarrow E = \frac{m^2 \hbar^2}{2mR^2}$$

$$\text{Ans} \int_0^{2\pi} |\psi|^2 d\phi = 1 \quad \Rightarrow A^2 \times 2\pi = 1 \quad \Rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$\therefore \psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(i m \phi) \quad (m = 0, 1, 2, \dots)$$

with

$$\therefore \Psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad (m = 0, \pm 1, \pm 2, \dots)$$

with $E_m = \frac{m^2 \hbar^2}{2mr^2}$.

clearly each energy state is doubly degenerate.

b. In presence of Magnetic field; the Hamiltonian

goes as: $\hat{H} = \frac{1}{2m} (-i\hbar \vec{\nabla} - e\vec{A})^2$

Here $\vec{A} = \frac{BR}{2} \hat{\phi}$ (given)

& as the particle is constrained to move only on the ring; so replacing

$$-i\hbar \vec{\nabla} \rightarrow -\frac{i\hbar \hat{\phi}}{BR} \frac{\partial}{\partial \phi}$$

$$\therefore \hat{H} = \frac{1}{2m} \left[-\frac{i\hbar}{R} \frac{1}{\partial \phi} - \frac{eBR}{2} \right]^2$$

\therefore T.I.S.E gives:

$$\frac{1}{2m} \left[-\frac{i\hbar}{R} \frac{1}{\partial \phi} - \frac{eBR}{2} \right]^2 \Psi(\phi) = E \cdot \Psi(\phi)$$

$$\Rightarrow -\frac{\hbar^2}{R^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{e^2 B^2 R^2}{4} \Psi + \frac{i\hbar}{R} \cdot \frac{eBR}{2} \frac{\partial \Psi}{\partial \phi} = 2mE \Psi$$

$$\Rightarrow \frac{\partial^2 \Psi}{\partial \phi^2} - \frac{e^2 B^2 R^4}{4\hbar^2} \Psi - \frac{i\hbar eB \cdot R^2}{\hbar^2} \frac{\partial \Psi}{\partial \phi}$$

$$= -\frac{2mE R^2}{\hbar^2} \Psi$$

$$\Rightarrow \frac{d^2\psi}{d\phi^2} - \underbrace{\frac{i\epsilon eBR^2}{\hbar^2}}_{\beta} \frac{d\psi}{d\phi} + \left(\underbrace{\frac{2mER^2}{\hbar^2} - \frac{e^2B^2R^4}{4\hbar^2}}_{\gamma} \right) \psi = 0$$

$$\Rightarrow \psi'' + \beta\psi' + \gamma\psi = 0.$$

The trial soln is of the form $\psi = e^{im\phi}$.

$$\therefore m^2 + \beta m + \gamma = 0.$$

$$\therefore m = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

Using values of β & γ we get:

$$m = \frac{1}{2} \left[\frac{ieBR^2}{\hbar} \pm \sqrt{-\frac{e^2B^2R^4}{\hbar^2} - \frac{8mER^2}{\hbar^2} + \frac{e^2B^2R^4}{\hbar^2}} \right]$$

$$= \frac{i}{2} \left[\frac{eBR^2}{\hbar} \pm \frac{2R}{\hbar} \sqrt{2mE} \right]$$

$$\therefore \psi(\phi) = A \cdot \exp \left[\frac{i}{2} \left[\frac{eBR^2}{\hbar} \pm \frac{2R}{\hbar} \sqrt{2mE} \right] \right].$$

Using $\psi(\phi + 2\pi) = \psi(\phi)$ gives:

$$e^{2\pi m i} = 1 \quad \Rightarrow \quad m = 0, \pm i, \pm 2i, \dots$$

$$\text{i.e., } \frac{eBR^2}{2\hbar} \pm \frac{R}{\hbar} \sqrt{2mE} = n$$

$$m = 0, \pm 1, \pm 2, \dots$$

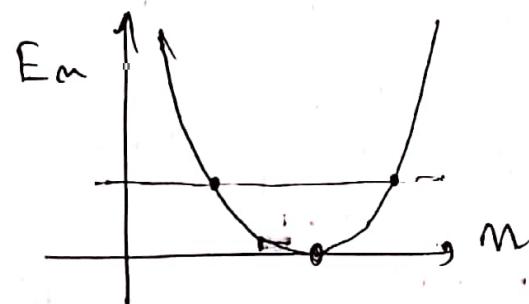
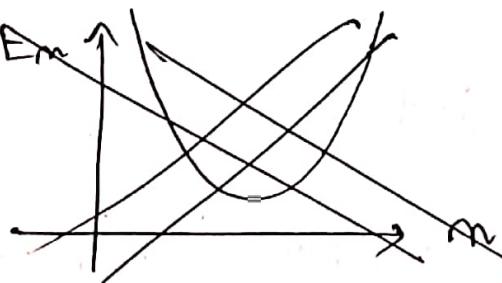
$$\text{i.e. } \frac{R}{\hbar} \sqrt{2mE} = \left(n - \frac{eBR^2}{2\hbar} \right)$$

$$\text{i.e } E = \frac{\hbar^2}{2mR^2} \left(m - \frac{eBR^2}{2\hbar} \right)^2 = E_m$$

Clearly the magnetic field breaks the degeneracy & splits the m^{th} level into different energy states.

At $B \rightarrow 0$: $E_m = \frac{m^2 \hbar^2}{2mR^2}$; which way problem.
the eigenstate of original Ans

c. The plot of E_m vs m be:



Clearly at a particular value of E_m there will be 2, m values iff $m \neq 0$ the ground state

where in the expression of

$$E_m = \frac{\hbar^2}{2mR^2} \left(m - \frac{eBR^2}{2\hbar} \right)^2$$

we get $\frac{eBR^2}{2\hbar}$ = an integer. = m
correspond to $m=1$.

The smallest value of B corresponds to $m=1$.

$$\therefore B_{\min} = \frac{1 \times 2\hbar}{eR^2} = \frac{e\hbar}{2R^2}$$

Ans

Prob-9

Interacting Spin System:-

Instead of $| \uparrow\downarrow \rangle$; I use $| + \rangle$ for notation.

Now here; in absence of external magnetic field; the conserves ~~quantities~~ quantity is neither S_1 nor S_2 but; $S = (S_1 + S_2)$

$$S = S_1 + S_2 \Rightarrow S^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2$$
$$\Rightarrow S_1 \cdot S_2 = \frac{S^2 - S_1^2 - S_2^2}{2}$$

$$\text{AD } S = \frac{\hbar}{2} \sigma \text{ Do; } \sigma_1 \cdot \sigma_2 = \frac{\sigma_1^2 - \sigma_2^2 - \sigma_1^2}{2}$$

$$\therefore \hat{H} = E_0 + \frac{A}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2$$
$$= E_0 + \frac{A}{8} (\sigma^2 - \sigma_1^2 - \sigma_2^2)$$

a. we have to express \hat{H} in the basis of $| + + \rangle$

$| + - \rangle, | - + \rangle, | - - \rangle$.

i.e we're to evaluate $\langle + - | H | - + \rangle, \dots$ etc quantities.

expressing $| + - \rangle$'s in the coupled basis we get; the coupled basis has + vectors

$$| 11 \rangle = | + + \rangle; | 10 \rangle = \frac{| + - \rangle + | - + \rangle}{\sqrt{2}}; | 00 \rangle = \frac{| + - \rangle - | - + \rangle}{\sqrt{2}}$$

$$| 1- \rangle = | - - \rangle.$$

$$\therefore |+\rangle = \frac{|10\rangle + |00\rangle}{\sqrt{2}} ; |-\rangle = \frac{|10\rangle - |00\rangle}{\sqrt{2}}$$

Here we get:

$$\langle 11111 \rangle_s = \langle 10110 \rangle = \langle 1-111 \rangle = \underset{=1}{\cancel{\langle 00000 \rangle}}$$

$$\text{any. } \langle 10111 \rangle = \langle 1-100 \rangle = \dots = 0.$$

$$\begin{aligned} & \cancel{\langle ++|\hat{P}|++\rangle} \quad (\hat{P} \equiv \sigma^2 - \sigma_1^2 - \sigma_2^2) \\ & = \cancel{\frac{1}{2} \langle ++|\sigma^2 - \sigma_1^2 - \sigma_2^2|++\rangle} \\ & = \frac{1}{2} (\langle 10 \rangle + \langle 00 \rangle) (\sigma^2 - \sigma_1^2 - \sigma_2^2) (|10\rangle + |00\rangle) \\ & \cancel{\text{but } \sigma^2 |10\rangle = 1(|11\rangle |10\rangle =} \end{aligned}$$

$$\begin{aligned} \text{So: } & \langle ++|\hat{P}|++\rangle \quad (\hat{P} \equiv \sigma^2 - \sigma_1^2 - \sigma_2^2) \\ & = \langle 11|\sigma^2 - \sigma_1^2 - \sigma_2^2|11\rangle \\ & = \langle 11|\left\{2 - 2 \times \frac{3}{4}\right\}|11\rangle \end{aligned}$$

$$\begin{aligned} \sigma^2 |11\rangle &= 1(|11\rangle |11\rangle = 2|11\rangle \} \text{ using tree.} \\ \sigma_1^2 |11\rangle &= \frac{1}{2}(\frac{1}{2}+1)|11\rangle = \frac{3}{4}|11\rangle \\ \sigma_2^2 |11\rangle &= \frac{1}{2}(\frac{1}{2}+1)|11\rangle = \frac{3}{4}|11\rangle \end{aligned}$$

$$= \langle 11|\frac{1}{2}|11\rangle = \frac{1}{2}.$$

$$\begin{aligned}
 & \langle ++ | \sigma^2 - \sigma_1^2 - \sigma_2^2 | + - \rangle = \\
 & = - \langle ++ | \sigma^2 - \sigma_1^2 - \sigma_2^2 + \cancel{\sigma_1^2} \rangle^* \\
 & = \langle +- | \sigma^2 - \sigma_1^2 - \sigma_2^2 | + + \rangle^* \\
 & = \left[\left\{ \frac{\langle 101 \rangle + \langle 001 \rangle}{\sqrt{2}} \right\} \frac{1}{2} \cdot | + + \rangle \right]^* \\
 & = 0
 \end{aligned}$$

Similarly $\langle ++ | \hat{p} | - + \rangle = 0 = \langle ++ | \hat{p} | - - \rangle$

And in the same way:

$$\begin{aligned}
 \langle -- | \hat{p} | - - \rangle &= \frac{1}{2} (\langle 100 \rangle - \langle 000 \rangle) \\
 \langle -- | \hat{p} | + - \rangle &= 0 = \langle -- | \hat{p} | - + \rangle \\
 \langle -- | \hat{p} | ++ \rangle &= 0
 \end{aligned}$$

Now, $\langle +- | \hat{p} | + - \rangle$

$$\begin{aligned}
 &= \left(\frac{\langle 101 \rangle + \langle 001 \rangle}{\sqrt{2}} \right) \cdot (\sigma^2 - \sigma_1^2 - \sigma_2^2) \left(\frac{|110\rangle + |100\rangle}{\sqrt{2}} \right) \\
 &= \frac{1}{2} (\langle 101 \rangle + \langle 001 \rangle) \left\{ 1(1+1) |110\rangle - \frac{1}{2} (\frac{1}{2}+1) |110\rangle \right. \\
 &\quad - \frac{1}{2} (\frac{1}{2}+1) |110\rangle + 0(0+1) |100\rangle - \\
 &\quad \left. \frac{1}{2} (\frac{1}{2}+1) |100\rangle - \frac{1}{2} (\frac{1}{2}+1) |000\rangle \right\}
 \end{aligned}$$

$$= \frac{1}{2} (\langle 10| + \langle 00|) \left\{ \left(2 - \frac{3}{4} - \frac{3}{4} \right) |110\rangle - \left(\frac{3}{4} + \frac{3}{4} \right) |100\rangle \right\}$$

$$= \frac{1}{2} (\langle 10| + \langle 00|) \left\{ \frac{1}{2} |110\rangle - \frac{3}{2} |100\rangle \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{2} - 0 + 0 - \frac{3}{2} \right\}$$

$$= -\frac{1}{2}$$

$$\langle + - | \hat{P} | - + \rangle = \langle - + | \hat{P} | + 0 \rangle^*$$

$$= \left\{ \frac{1}{2} (\langle 10| - \langle 00|) \cdot \left\{ \frac{1}{2} |110\rangle - \frac{3}{2} |100\rangle \right\} \right\}^*$$

↓
from previous result.

~~$$= \frac{1}{2} (\langle 01| - \langle 00|)$$~~

$$= \left[\frac{1}{2} \left\{ \frac{1}{2} + 0 + 0 + \frac{3}{2} \right\} \right]^* = 1.$$

Ans $\langle - + | \hat{P} | - + \rangle$

$$= \frac{1}{2} (\langle 10| - \langle 00|) (\sigma^2 - \sigma_1^2 - \sigma_2^2) (|110\rangle - |100\rangle)$$

$$= \frac{1}{2} (\langle 01| - \langle 00|) \left\{ 2 |110\rangle - \frac{3}{4} |110\rangle - \frac{3}{4} |110\rangle \right. \\ \left. - |100\rangle + \frac{3}{4} |100\rangle + \frac{3}{4} |100\rangle \right\}$$

$$= \frac{1}{2} (\langle 101 - \langle 001 \rangle) \left(\frac{1}{2} |110\rangle + \frac{3}{2} |000\rangle \right)$$

$$= \frac{1}{2} \left(\frac{1}{2} + 0 + 0 - \frac{3}{2} \right) = -\frac{1}{2}.$$

As \hat{P} is Hermitian; So these are sufficient to construct \hat{P} ; Clearly:

$$\hat{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{array}{c} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{array}$$

$$\langle ++| \quad \langle +-| \quad \langle -+| \quad \langle --|$$

$$\text{So: } \hat{H} = E_0 + \frac{A}{4} \sigma_1 \cdot \sigma_2$$

$$= E_0 + \frac{A}{4} \cdot \hat{P} = E_0 \cdot I_4 + \frac{A}{4} \hat{P}$$

(As E_0 is constant; So that is some unit matrix)

$$\Rightarrow \hat{H} = \begin{pmatrix} E_0 + \frac{A}{16} & 0 & 0 & 0 \\ 0 & E_0 - \frac{A}{16} & \frac{A}{4} & 0 \\ 0 & \frac{A}{4} & E_0 - \frac{A}{16} & 0 \\ 0 & 0 & 0 & E_0 + \frac{A}{16} \end{pmatrix}$$

Ans

in the given basis -

9.6 $\text{AD } \hat{E}_0 = E_0 F_0 = \text{constant matrix}$
 any ~~or~~ in the form of unit matrix; so
 all vectors are eigen vectors to it.

$\sigma_1, \sigma_2 \sim \frac{\sigma^2 - \sigma_1^2 - \sigma_2^2}{2}$ So the eigen-
 basis of \hat{H} will be those ~~basis~~ vectors
 which are eigenvectors of $\sigma^2, \sigma_1^2, \sigma_2^2$
 simultaneously.

Clearly those common eigenbasis are the
 coupled basis ket vectors of σ_1 & σ_2 .
 (which I've already used in part a.)

The basis set is: $((j, m_j) \text{ form})$

$$\begin{aligned} |1+1\rangle &= |++\rangle \\ |1-0\rangle &= \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} \\ |1-1\rangle &= |-\rangle \\ |0+0\rangle &= \frac{|+-\rangle - |-\rangle}{\sqrt{2}} \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Any}$$

$|j, m_j\rangle$

j is the added up angular momenta
 m_j is the Z component of total angular momentum

Q.C Given $|\psi(0)\rangle = \alpha|++\rangle + \beta|+-\rangle$

Now, $|++\rangle = |11\rangle$

$$|+-\rangle = \frac{|10\rangle + |00\rangle}{\sqrt{2}}$$

$$\Rightarrow |\psi(0)\rangle = \alpha|11\rangle + \frac{\beta}{\sqrt{2}}|10\rangle + \frac{\beta}{\sqrt{2}}|00\rangle$$

Expressed in terms of eigenbasis of H.

Q.2 Let at time t: $|\psi(t)\rangle$ is given by:

$$|\psi(t)\rangle = \alpha_1|11\rangle + \alpha_2|10\rangle + \alpha_3|1-\rangle + \alpha_4|00\rangle$$

$$\hat{H}|\psi(t)\rangle = E_0|\psi(t)\rangle + \frac{A}{8}(s^2 - \sigma_1^2 - \sigma_2^2)|\psi(t)\rangle$$

$$= E_0|\psi(t)\rangle + \frac{A}{8} \left\{ \alpha_1 \left(2 - \frac{3}{2} \right) |11\rangle + \alpha_2 \left(2 - \frac{3}{2} \right) |10\rangle \right.$$

$$\left. + \alpha_3 \left(2 - \frac{3}{2} \right) |1-\rangle + \alpha_4 \left(0 - \frac{3}{2} \right) |00\rangle \right\}$$

$$= E_0|\psi(t)\rangle + \frac{A}{8} \left\{ \frac{1}{2} \alpha_1 |11\rangle + \frac{\alpha_2}{2} |10\rangle + \frac{\alpha_3}{2} |1-\rangle - \frac{3\alpha_4}{2} |00\rangle \right\}$$

$$= \alpha_1 \left(E_0 + \frac{A}{16} \right) |11\rangle + \alpha_2 \left(E_0 + \frac{A}{16} \right) |10\rangle$$

$$+ \alpha_3 \left(E_0 + \frac{A}{16} \right) |1-\rangle + \alpha_4 \left(E_0 - \frac{3A}{16} \right) |00\rangle$$

$$\text{but } \hat{H}|\psi(t)\rangle = i\hbar \frac{\partial |\psi\rangle}{\partial t} = i\hbar \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \\ \dot{a}_4 \end{pmatrix}$$

So by comparing we get: $a_1(t)$

$$i\hbar \dot{a}_1 = a_1 \left(E_0 + \frac{A}{16} \right) \Rightarrow \int \frac{da_1}{a_1} = -i \frac{\left(E_0 + \frac{A}{16} \right)}{\hbar} \int dt$$

$$\therefore a_1(t) = \alpha \exp \left[-i \frac{\left(E_0 + \frac{A}{16} \right)}{\hbar} t \right]$$

$$\text{Similarly } a_2(t) = \frac{\beta}{\sqrt{2}} \cdot \exp \left[-i \frac{\left(E_0 + \frac{3A}{16} \right)}{\hbar} t \right]$$

$$a_3(t) = 0 \quad \left[\because a_3(t=0) = 0 \right]$$

$$a_4(t) = \frac{\beta}{\sqrt{2}} \exp \left[-i \frac{\left(E_0 - \frac{3A}{16} \right)}{\hbar} t \right]$$

So in the eigenbasis of \hat{H} ; $|\psi(t)\rangle$ is given by:

$$|\psi(t)\rangle = \begin{pmatrix} \alpha \exp \left[-i \frac{\left(E_0 + \frac{A}{16} \right)}{\hbar} t \right] \\ \frac{\beta}{\sqrt{2}} \exp \left[-i \frac{\left(E_0 + \frac{3A}{16} \right)}{\hbar} t \right] \\ 0 \\ \frac{\beta}{\sqrt{2}} \exp \left[-i \frac{\left(E_0 - \frac{3A}{16} \right)}{\hbar} t \right] \end{pmatrix}$$

Ans

7. $l=1$ Hydrogen State with variational method.

In the true wave fm of H atom; the radial part varies as:

$$R_{nl}(r) \sim e^{-\frac{r}{na}} \left(\frac{2r}{na}\right)^l L_{n-l}^l \left(\frac{2r}{na}\right)$$

Now the associated Laguerre poly be given by:

$$L_{q-p}^p(x) \sim (-1)^p \frac{d^p}{dx^p} L_q(x)$$

$$\text{where; } L_q(x) = \sum_{j=0}^q c_j x^j \quad (\text{for some non-zero } c_j)$$

as in $L_q(x)$ all powers of x upto $j=q$ if present; while in its derivative; there will be at least one term which is constant zero as a whole.

$$\text{i.e. } L_q(x) = \sum_{j=0}^q c_j x^j$$

$$\Rightarrow \frac{d^m}{dx^m} L_q(x) = 0 \quad \text{if } m > q \quad \text{nonzero}$$

& $\frac{d^m}{dx^m} L_q(x)$ has one constant term in expansion if $m \leq q$ i.e. it's not zero as whole.

So for any physical normalizable wave fm in Hydrogen for an l state we get

$$\text{Let } R_{nl}(r) \sim \lim_{r \rightarrow 0} e^{-\frac{r}{na}} \left(\frac{2r}{na}\right)^l \left(c_0 + c_1 r + \dots + c_n r^n\right)$$

for some $\kappa > 0$ & we get

$$c_0, c_1, \dots, c_k \neq 0$$

$$\text{So; } \lim_{r \rightarrow 0} R_{nl}(r) \sim \cancel{A} e^0 \cdot r^l \cdot (c_0 + 0 + \dots + 0) \\ \sim r^l.$$

Here for the ~~given~~ trial state we
use that information.
more over at $r \rightarrow \infty$; $e^{-\kappa r} \cdot r^l \rightarrow 0$
for all κ . So the good trial fm can be
of the form:

$$R_{nl} = A r^l e^{-\alpha r} = A r e^{-\alpha r} = \psi_\alpha (\text{say})$$

As the Spherical Harmonics are also normalized
so we get:

$$I = \int_0^\infty r^2 R^2 dr = \int_0^\infty A^2 r^2 \cdot r^2 \cdot e^{-2\alpha r} dr \\ = A^2 \cdot \int_0^\infty \frac{z^4}{2^4 \alpha^4} \cdot e^{-2z} \frac{dz}{2\alpha} \quad (2\alpha r = z) \\ = \frac{A^2}{32 \alpha^5} \Gamma(5) = \frac{A^2 \times 4 \times 3 \times 2}{32 \alpha^5} = \frac{3A^2}{4 \alpha^5}.$$

$$\therefore A = \sqrt{\frac{8\alpha^5}{3 \times 2}}.$$

Now R_{nl} satisfies the eq!

$$-\frac{\hbar^2}{2m} \cdot \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \left[\frac{\hbar^2}{2m} \cdot \frac{l(l+1)}{r^2} + V(r) \right] R = ER$$

$$\text{Here } V(r) = -\frac{e^2}{r}; \quad l=1 \text{ gives:}$$

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \left(\frac{\hbar^2}{m} \frac{1}{r^2} - \frac{e^2}{r} \right) R = ER.$$

i.e. $\hat{H}_R = -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \left(\frac{\hbar^2}{m} \frac{1}{r^2} - \frac{e^2}{r} \right)$

\hat{T}_R \hat{V}_R

At the ~~at~~-trial state ψ_α taken;

$$\langle H_R \rangle = \langle \hat{T}_R \rangle + \langle \hat{V}_R \rangle$$

where $\langle \hat{A}_R \rangle = \int_0^\infty (\psi_\alpha \hat{A}_R \psi_\alpha) \cdot r^2 dr$.

∴ Here $\langle \hat{T}_R \rangle$

$$= - \int_0^\infty A^2 r e^{-\alpha r^2} \left(\frac{\hbar^2}{2m} \right) \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} (r e^{-\alpha r}) \right) \cdot r^2 dr.$$

I've evaluated the integrals in mathematica
which gives:

$$\langle \hat{T}_R \rangle_{\psi_\alpha} = \frac{\gamma}{\beta} \cdot \frac{\alpha^2 \hbar^2}{m \alpha^4} \cdot \frac{\gamma \alpha^8}{\beta \times 2} \quad \begin{matrix} \text{(Using value)} \\ \text{of } A \end{matrix}$$

~~(Crossed out)~~

$$= \frac{\alpha^2 \hbar^2}{2m}$$

$$\text{And } \langle \hat{V}_R \rangle_{\psi_\alpha} = \int_0^\infty \left(\frac{\hbar^2}{2mr^2} - \frac{e^2}{r} \right) \cdot m^2 A^2 r^2 e^{-2\alpha r^2} dr$$

$$= - \frac{\gamma}{\beta} \cdot \frac{m \hbar^2}{m \alpha^4} \cdot \frac{\gamma \alpha^8}{\beta \times 2} = - \frac{\alpha e^2}{2}$$

$$\text{So; } \langle H \rangle_{\psi_\alpha} = \frac{\alpha^2 \hbar^2}{2m} - \frac{\alpha e^2}{2}$$

Varying over α gives:

$$\frac{d\langle H \rangle_{\alpha}}{d\alpha} = 0 = \frac{2\alpha\hbar^2}{2m} - \frac{e^2}{2}$$

$$\Rightarrow \alpha = \frac{me^2}{2\hbar^2} = \alpha_0$$

$$\begin{aligned}\therefore \langle H \rangle_{\alpha} \Big|_{\alpha=\alpha_0} &= \left(\frac{\hbar^2 \alpha^2}{2m} - \frac{me^2}{2} \right) \Big|_{\alpha=\frac{me^2}{2\hbar^2}} \\ &= \frac{\hbar^2}{2m} \cdot \frac{m^2 e^4}{4\hbar^4} - \frac{e^2}{2} \cdot \frac{me^2}{2\hbar^2} \\ &= -\frac{me^4}{8\hbar^2}.\end{aligned}$$

$$\text{So } E_{l=1}^{\text{SD}} \leq -\frac{me^4}{8\hbar^2}. \quad \underline{\text{Ans}}$$

III The actual energy is given by:

$$E_m = -\frac{me^4}{2m^2\hbar^2} \quad (\text{taking } \frac{1}{4\pi\epsilon_0} = 1)$$

for $l=1$ we have $m \geq 2$. (obvious for H)

$$\therefore E_m^{\text{true}} = -\frac{me^2}{2m^2\hbar^2} = -\frac{me^2}{8\hbar^2}.$$

The result is true for $m=2$.

i.e. for all m values ($m=2, 3, 4, \dots$) ~~for~~

possible for $l=1$; the ground state ($m=2$)

exactly matches the energy. This is because
of the form of α_α . Which is equal
to the form of $R_{m=2}^{(n)}$ state.

Clearly the ~~value~~ value of m only takes part for determining 'Angular' part. ($y_i^m(\theta\phi)$) Ans

Problem:- II

Given: $H_0 = \frac{1}{2m_0} (\vec{p} - q\vec{A})^2 I$

Now using the traditional shorthand for

$$\vec{p} - q\vec{A} = \vec{\pi}$$

we get $[\pi_i, \pi_j] = [p_i - qA_i, p_j - qA_j]$
 $= [p_i, p_j] + q^2 [A_i, A_j] - q[A_i, p_j] - q[p_i, A_j]$
 $(\because A_i = A_i(x, y, z) \quad [x_i, x_j] = 0)$

$$= -i\hbar q \frac{\partial A_i}{\partial x_j} + i\hbar q \frac{\partial A_j}{\partial x_i}$$

$$= -i\hbar q \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) = -i\hbar q \epsilon_{ijk} B_k.$$

Here $q = -e$ so:

$$[\pi_i, \pi_j] = i\hbar e \epsilon_{ijk} B_k.$$

Now using $(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B)$

we get:

~~$\therefore (\vec{p} + e\vec{A}) [\sigma \cdot (\vec{p} + e\vec{A})]^2 = (\sigma \cdot \vec{\pi})^2 = (\vec{\sigma} \cdot \vec{\pi})(\vec{\sigma} \cdot \vec{\pi})$~~

$$= \vec{\pi} \cdot \vec{\pi} + i\sigma \cdot (\vec{\pi} \times \vec{\pi})$$

$$= I(\vec{p} + e\vec{A})^2 + i\sigma \cdot -i\hbar e \vec{B}$$

$$\left(\because \vec{\pi} \times \vec{\pi} \Big|_{\vec{\pi}_i} = \epsilon_{ijk} \pi_j \pi_k = [\pi_j, \pi_k] \times (\pm 1) \right)$$

according to circular order of j or k

$$= (\vec{p} + e\vec{A})^2 \cdot \vec{I} + 2m_e \cdot \mu_B \cdot \vec{B} \cdot \vec{\sigma}$$

(where $\mu_B = \frac{e\hbar}{2m_0}$) proved

Problem :- 6

a. For Spin 1 particle the states are $|l, m\rangle$ form
be denoted by:

$$|1+1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |10\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad |1-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{when } \hat{S}_z |l, m\rangle = m\hbar |l, m\rangle$$

$$\therefore (\hat{S}_z)_{ij} = \langle l, m_i | S_z | l, m_j \rangle = m_i \hbar \delta_{ij}$$

$$\text{So; } [\hat{S}_z] = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (l=1)$$

$$\text{Now; } S_+ |l, m\rangle = \hbar \sqrt{(l-m)(l+m+1)} |l, m+1\rangle \quad (l=1)$$

$$\therefore (S_+)_{ij} = \hbar \sqrt{(l-m_i)(l+m_j)} \quad \cancel{\delta_{ij}} \quad \delta_{i,j+1}$$

$$\text{So; } [\hat{S}_+] = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_- |l, m\rangle = \hbar \sqrt{(l+m)(l-m)} \quad \cancel{\delta_{ij}} \quad \delta_{i,j-1}$$

$$\therefore (S_-)_{ij} = \hbar \sqrt{(l+m_j)(l-m_j)} \quad \delta_{i,j-1}$$

$$\therefore [\hat{S}_z] = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\therefore \hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\hat{S}_- = \frac{\hat{S}_+ - \hat{S}_-}{2i} = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$\therefore S_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$\hat{S}_y^2 = -\frac{\hbar^2}{4} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$= -\frac{\hbar^2}{4} \begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$\therefore \hat{H} = \hbar^2 \left[A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + B \left\{ \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \right\} \right]$$

$$= \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

Now we find eigenvalue of \hat{H} (or $\frac{\hat{H}}{\hbar^2}$):

$$\det \left(\frac{H}{\hbar^2} - \lambda I \right) = 0 \quad \text{gives}$$

$$\det \begin{pmatrix} A - \lambda & 0 & B \\ 0 & -\lambda & 0 \\ B & 0 & A - \lambda \end{pmatrix} = 0$$

$$\Rightarrow (A - \lambda)^2 \cdot (-\lambda) + B^2 \lambda = 0$$

$$\text{i.e. } \lambda = 0 = \lambda_0$$

$$\text{or; } (A - \lambda)^2 = B^2 \quad \text{i.e. } A - \lambda = \pm B$$

$$\text{i.e. } \lambda = (A \pm B) = \lambda_{\pm}$$

Let the eig state be $x = (x \ y \ z)^T$.

for $\lambda = \lambda_0$

$$\begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} Ax + Bz = 0 \\ By + Az = 0 \end{cases} \quad \begin{array}{l} x = 0 = z \\ (A \neq B) \end{array}$$

$$\therefore x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

for $\lambda = A + B$

$$\begin{pmatrix} -B & 0 & B \\ 0 & -A-B & 0 \\ B & 0 & -B \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} Bz - Bx = 0 \\ (A+B)y = 0 \\ Bx - Bz = 0 \end{cases} \quad \begin{array}{l} x = y \neq z \\ y = 0 \end{array} \quad \begin{array}{l} (\text{assuming}) \\ B \neq 0 \\ A \neq -B \end{array}$$

$$\therefore \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

for $\lambda = A - B$:

$$\begin{pmatrix} +B & 0 & B \\ 0 & -A+B & 0 \\ B & 0 & B \end{pmatrix} \begin{pmatrix} \chi \\ \gamma \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left. \begin{array}{l} B(\lambda + z) = 0 \\ (B-A)\gamma = 0 \\ B(\lambda + z) = 0 \end{array} \right\} \begin{array}{l} \lambda = -2 \\ \gamma = 0 \end{array}$$

$$\therefore \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

\therefore The normalized eig-state of \hat{H} with eigenvalues are given by:

$$|\chi_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \lambda_1 = 0$$

$$|\chi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \lambda_2 = +\hbar^2(A+B)$$

$$|\chi_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rightarrow \lambda_3 = +\hbar^2(A-B).$$

Ans

$$\text{b. } \Theta S_i^2 \Theta^{-1} = \Theta S_i \Theta^{-1} \Theta S_i \Theta \quad (i = x, y, z)$$

(Θ = time reversal operator.)

$$= (-S_i) \cdot (-S_i) = S_i^2.$$

$$\text{Using } \Theta S_i \Theta = -S_i \text{ or } S_i \rightarrow -S_i \text{ (under } \Theta)$$

$$\begin{aligned}\hat{\theta}^\dagger \hat{\theta} &= [A(\theta s_z^2) + B(\theta s_n^2 - \theta s_g^2)] \theta^\dagger \\ &= \underbrace{A \theta s_z^2 \theta^\dagger}_{s_z^2} + B \left(\underbrace{\theta s_n^2 \theta^\dagger}_{s_n^2} - \underbrace{\theta s_g^2 \theta^\dagger}_{s_g^2} \right) \\ &= A s_z^2 + B(s_n^2 - s_g^2)\end{aligned}$$

Ams

$\Rightarrow H$

$\therefore \hat{H}$ is invariant under time reversal.

$$\underline{c.} \quad \theta |x_1\rangle = \theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \theta |1 0\rangle = (-1)^0 |1 0\rangle$$

(using $\theta |ijm\rangle = (-1)^m |j -m\rangle$)

$$\therefore \theta |x_1\rangle = |1 0\rangle = |x_1\rangle \quad \underline{\text{Ans}}$$

$$\begin{aligned}\theta |x_2\rangle &= \theta \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \theta \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \\ &= \frac{1}{\sqrt{2}} \theta \{ |1 1\rangle + |1 -1\rangle \} = \frac{-1}{\sqrt{2}} \underbrace{\{|1 -1\rangle + |1 1\rangle\}}_{-|x_2\rangle}\end{aligned}$$

$$\Rightarrow \theta |x_2\rangle = -|x_2\rangle \quad \underline{\text{Ans}}$$

$$\begin{aligned}\theta |x_3\rangle &= \frac{1}{\sqrt{2}} \theta \{ |1 1\rangle - |1 -1\rangle \} = \frac{1}{\sqrt{2}} \{ |1 -1\rangle - |1 1\rangle \} \\ &= \cancel{\theta} |x_3\rangle\end{aligned}$$

$$\therefore \theta |x_3\rangle = |x_3\rangle \quad \underline{\text{Ans}}$$

Problem:- 10

Here for $m=2$ State there are (ignoring Spin) m^2 degenerate States.

$$\text{Now: } E(t) = \hat{R} E_0 e^{-t^2/\alpha^2}$$

$$\therefore V(t) = \int E(t) dz = -\hat{R} Z E_0 e^{-t^2/\alpha^2}$$

$$(\text{that satisfies } E = -\frac{\partial V}{\partial z} - \nabla V)$$

$$\text{Now } \langle m_l m_z | m' l' m' \rangle$$

$$= \int r^3 R_{ml}(r) R_{m'l'}(r) dr \int Y_l^m(\theta, \phi) \cdot Y_l^{m'}(\theta, \phi) \cos \theta_r d\theta d\phi$$

$$(\because z = r \cos \theta)$$

The ϕ integral gives:

$$\langle m_l m_z | m' l' m' \rangle = (\text{r integral}) \times (\theta \text{ integral})$$

$$\times \int_0^{2\pi} e^{i(m_m - m'_m)} \phi d\phi$$

$$\neq 0 \text{ iff } m = m'$$

for $m=2$; the l values are $0 \neq 1$.

$$\langle 100 | z | 200 \rangle$$

$$\sim \int r^3 R_{10}(r) R_{20}(r) dr \int Y_0^0(\theta, \phi) \cdot Y_0^0(\theta, \phi) \sin \theta \cos \theta d\theta d\phi$$

$$\sim I(R) \cdot \int_0^\pi \sin \theta \cos \theta d\theta \cdot \int_0^{2\pi} d\phi$$

(excluding normalization const)

$$\text{but } \int_0^\pi \sin \theta \cos \theta d\theta = \frac{1}{2} \int_0^{2\pi} \sin \theta d\theta = 0$$

$$\text{So: } \langle 100 | z | 200 \rangle = 0$$

$$\text{by } m \neq m' \text{ reason: } \langle 100 | \underset{\sim}{z} | 21\pm 1 \rangle = 0$$

$$\begin{aligned} \text{And } \langle 100 | z | 210 \rangle &= \int r^3 R_{10}(r) R_{21}(r) dr \int Y_0^0 Y_1^0 \sin \theta \cos \theta d\phi \\ &= \int_0^\infty 2a^{-3/2} e^{-r/a} r^3 \cdot \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a} dr \\ &\quad \times \int_0^\pi \frac{1}{\sqrt{4\pi}} \cdot \sqrt{\frac{3}{4\pi}} \sin \theta \cos^2 \theta d\theta \cdot \int_0^{2\pi} d\phi. \\ &= \frac{2a^3}{\sqrt{24} a} \cdot \frac{256 a^8}{81} \times \cancel{\frac{1}{4\pi}} \times \frac{1}{2\sqrt{3}\pi}. \end{aligned}$$

(evaluated)

$$= \frac{2^7 \sqrt{2}}{3^5} a \quad \left(\begin{array}{l} a = \text{Bohr radius} \\ \text{radius} \end{array} \right)$$

$$H'(t) = -qV(t) = -qE_0 e^{-t^2/\tau^2}$$

By previous calculation; we get: the charge can only go to $|210\rangle$ state.

The transition amplitude be (using formula for non degenerate time dependent perturbation theory which we can use as only one state with that particular energy with $m=2$ state is available here)

$$\begin{aligned} C_b &= -\frac{i}{\hbar} \int \langle 100 | z | 210 \rangle \cdot (-qE_0 e^{-t^2/\tau^2}) \cdot e^{i\omega t} dt \\ &= \frac{i}{\hbar} qE_0 \frac{2^7 \sqrt{2} a}{3^5} \int_0^\infty e^{i\omega t} \cdot e^{-t^2/\tau^2} dt \end{aligned}$$

$$\left(\omega_0 = \frac{E_{m=2} - E_{m=1}}{\hbar} = \frac{E_{2lm} - E_{100}}{\hbar} \right)$$

$$\Rightarrow G_b = \frac{i}{\hbar} \frac{2^7 \sqrt{2}}{3^5} q E_0 a \cdot \pi \sqrt{\pi} e^{-\frac{\omega_0^2 r^2}{4}}$$

So transition probability (required quantity)

$$|G_b|^2 = P = \left(\frac{e E_0}{\hbar}\right)^2 \cdot \left(\frac{2^{15} a_0^2}{3^{10}}\right) \pi r^2 e^{-\frac{\omega_0^2 r^2}{2}}.$$

Proved

④ The timely varying \vec{E} field induces a \vec{B} field given by Maxwell's eq. The spin of the e^- will interact with that \vec{B} field and add another perturbation

$$H'' = -\frac{eB(t)}{m} \cdot \hat{S}_z$$

\therefore The answer will change in presence of Spin.

Problem: 45

Linear Stark effect

a: This problem is quite similar (by calculation) with problem 10 (which I've done earlier.)

The perturbing potential be given by,

$$H' = V = -e z \cdot \vec{E} \quad (\text{so that } e\vec{E} = -\nabla V)$$

Now for $m=2$ states we get $\begin{cases} l=0; m=0 \\ l=1; m=0, \pm 1 \end{cases}$

i.e there are 4th order degeneracy.

\therefore The matrix elements of V (or H') be given

by: $(H'_{4 \times 4})_{ij} = \langle i | H' | j \rangle$

where $|i\rangle, |j\rangle = |200\rangle, |211\rangle, |210\rangle, |21-\rangle$

As: $H \approx z$ so $\langle 2l^m | z | 2l'^{m'} \rangle$

$$\sim \int_0^\infty R_{2l} R_{2l'} r^2 r dr \cdot \int_0^\pi Y_l^m(\theta \text{ part}) \cdot Y_{l'}^{m'}(\theta \text{ part}) \cdot \cos \theta \sin \theta d\theta$$
$$\times \int_0^{2\pi} e^{im(\phi - \phi')} d\phi.$$

$\neq 0$ iff $m = m'$

So in H' ; there are only two non zero elements $\langle 200 | z | 210 \rangle$ & h.c.

The diagonal terms are zero by the selection

rule on ℓ ; that tells $\langle m_1 m_1' | z | m_1' m_1' \rangle \neq 0$
iff $\ell = \ell' \pm 1$.

As here we need only $\langle H' \rangle$ to evaluate 1st order perturbation; so by the previous reasoning the only two states connected via the given field are $|200\rangle$ & $|210\rangle$.

b. $\langle 200 | z | 210 \rangle$ is given by:

$$H'_{12} \approx \int_0^{\infty} R_{20}^{(r)} R_{21}^{(r)} \cdot r^3 dr \int_0^{\pi} Y_0^* Y_1 \cdot \cos \theta \sin \theta d\theta$$

$$\approx \int_0^{\infty} r^3 \cdot \frac{1}{\sqrt{2}} \alpha^{3/2} \left(1 - \frac{r}{2\alpha} \right) e^{-r/2\alpha} \cdot \frac{1}{\sqrt{24}} \alpha^{3/2} \left(\frac{r}{\alpha} \right) e^{-r/2\alpha} dr.$$

$$\times \int_0^{\pi} \frac{1}{\sqrt{4\pi}} \cdot \sqrt{\frac{3}{4\pi}} \cos^2 \theta \sin \theta d\theta \cdot \int_0^{\pi} d\phi.$$

$$= \frac{2\pi}{\sqrt{3}\pi} \cdot \frac{1}{\sqrt{24}} \cdot (-3\sqrt{3}\alpha) = -3\alpha. = \langle 210 | z | 200 \rangle$$

$$\therefore H'_{12} = 3\alpha |E| e$$

$$\therefore H = \begin{pmatrix} 0 & 3\alpha |E| e & 0 & 0 \\ 3\alpha |E| e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} |200\rangle \\ |210\rangle \\ |211\rangle \\ |211'\rangle \end{array}$$

Here we only need to diagonalize the upper left block of \hat{H} , which contains the interaction term between $|200\rangle$ & $|210\rangle$.

$$H_{\text{upper left}} \approx \sim A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |200\rangle \langle 210|$$

As it is corresponding to Γ_0 ; So the eigenvectors will be $\frac{1}{\sqrt{2}}(|200\rangle \pm |210\rangle)$ {like for $\sigma_m: \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$ }

So the good linear combinations are

$$\alpha_1 = \frac{1}{\sqrt{2}} (|200\rangle + |210\rangle) \quad \left. \right\}$$

$$\alpha_2 = \frac{1}{\sqrt{2}} (|200\rangle - |210\rangle) \quad \left. \right\}$$

$$2D, \frac{1}{\sqrt{2}} (|200\rangle - |210\rangle) \quad \left. \right\}$$

$$|21\pm\rangle$$

$$\frac{1}{\sqrt{2}} (|200\rangle + |210\rangle) \quad \left. \right\}$$

E. Let $|\psi(t)\rangle = \underline{c_a(t)} c_a(t) |200\rangle + c_b(t) |210\rangle$ for $t < \frac{\hbar}{V}$ the ~~perturbation~~ given the H' matrix.

$$c_a(0) = 1; c_b(0) = 0 \quad (\text{given}).$$

first order perturbation fixed.

$$\dot{C}_a(t) = -\frac{i}{\hbar} H'_{12} e^{-i\omega_0 t} C_b^0(t) = 0$$

$$\Rightarrow C_a^1(t) - C_a(0) = 0 \Rightarrow C_a^1(t) = 1 = C_a(0).$$

$$\dot{C}_b^1(t) = -\frac{i}{\hbar} \cdot 3ae|E| e^0 \cdot 1 \quad (\because \omega_0 = \frac{E_b - E_a}{\hbar} = 0)$$

$$\Rightarrow C_b^1(t) - C_b^1(0) = -\frac{3iae|E|t}{\hbar}$$

$$\Rightarrow C_b^1(t) = -\frac{3iae|E|t}{\hbar}$$

2nd order for $C_a(t)$:

$$\begin{aligned}\dot{C}_a^2(t) &= -\frac{i}{\hbar} \cdot 3ae|E| \cdot e^0 \cdot -\frac{3iae|E|}{\hbar} + \\ &= -\left(\frac{3ae|E|}{\hbar}\right)^2 + \\ \therefore C_a^2(t) &= 1 - \left(\frac{3ae|E|}{\hbar}\right)^2 \cdot \frac{t^2}{2}.\end{aligned}$$

$$\therefore |\psi(t)\rangle \approx \left(1 - \left(\frac{3ae|E|}{\hbar}\right)^2 \cdot \frac{t^2}{2}\right) |200\rangle - \frac{3iae|E|t}{\hbar} |210\rangle$$

Answer

5. d.) prob to find the e- in different $m=2$ states,

$$\begin{aligned}P_{|200\rangle} &= |C_a|^2 = \left\{1 - \left(\frac{3ae|E|}{\hbar}\right)^2 \cdot \frac{t^2}{2}\right\}^2 \\ P_{|210\rangle} &= |C_b|^2 = \left(\frac{3iae|E|t}{\hbar}\right)^2.\end{aligned}$$

$$P_{|21\pm 1\rangle} = 0$$

Clearly $\sum P_i^2 = 1 + \underbrace{O(t^4)}_{\text{of error up to 2nd order theory}}$

Ans

Problem:- 12

~~~~~

a)  $\vec{J} = \vec{L}_1 + \vec{L}_2$

so  $j$  varies from  $(l_1+l_2)=2$  to  $|l_1-l_2|=0$ .

clearly for  $j=2$ ;  $m_j = \pm 2, \pm 1, \mp 0$

$$|22\rangle = |11\rangle|11\rangle$$

$$J_z|22\rangle = J_1^-|11\rangle|11\rangle + J_2^-|11\rangle|11\rangle$$

$$\Rightarrow 2|2\rangle = \sqrt{2}|110\rangle|11\rangle + \sqrt{2}|111\rangle|10\rangle$$

$$\therefore |21\rangle = \frac{|110\rangle|11\rangle}{\sqrt{2}} + \frac{|111\rangle|10\rangle}{\sqrt{2}}$$

- so;  $J_z|21\rangle = \frac{1}{\sqrt{2}}(J_1^-|110\rangle|11\rangle + J_2^-|110\rangle|11\rangle)$   
 $+ \frac{1}{\sqrt{2}}(J_1^-|111\rangle|10\rangle + J_2^-|110\rangle|11\rangle)$

$$\therefore \sqrt{3 \times 2} |20\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{1 \times 2} |1-1\rangle|11\rangle + \sqrt{2}|110\rangle|10\rangle + \sqrt{1 \times 2} |111\rangle|10\rangle + \sqrt{2}|110\rangle|11\rangle \right)$$

$$\therefore |20\rangle = \frac{1}{\sqrt{6}} (|1-1\rangle|11\rangle + |11\rangle|1-1\rangle + 2|10\rangle|10\rangle)$$

Ans;  $|2-2\rangle = |11\rangle|11\rangle$

$$\Rightarrow J^+|2-2\rangle = J_1^+|11\rangle|11\rangle + J_2^+|11\rangle|11\rangle$$

$$\Rightarrow |2-1\rangle = \frac{1}{\sqrt{2}} (|1-1\rangle|10\rangle + |10\rangle|1-1\rangle)$$

Using property that for  $m \neq m_1 + m_2$  the C.G  
coefficients  $j_i$  we let: (for  $j=1$ )

$$|11\rangle = a|10\rangle|11\rangle + b|11\rangle|10\rangle$$

$$\Rightarrow J^+|11\rangle = a(J_1^+|10\rangle|11\rangle + J_2^+|10\rangle|11\rangle) \\ + b(J_1^+|11\rangle|10\rangle + J_2^+|11\rangle|10\rangle)$$

~~2 3 0 \* 3 1~~ Using  $J^+|jj\rangle = 0$ :

$$\Rightarrow 0 = a\sqrt{1 \cdot 2}|11\rangle|11\rangle + 0 + 0 \\ + b\sqrt{2}|11\rangle|11\rangle$$

$$\therefore a = -b; \text{ now } a^2 + b^2 = 1 \Rightarrow 2a^2 = 1$$

$$\text{So; } a = \frac{1}{\sqrt{2}}; b = -\frac{1}{\sqrt{2}}.$$

$$\therefore |11\rangle = \frac{1}{\sqrt{2}}(|10\rangle|11\rangle - |11\rangle|10\rangle)$$

Similarly for  $|11\rangle$  gives:

$$|11\rangle = \frac{1}{\sqrt{2}}(|10\rangle|11\rangle - |11\rangle|10\rangle)$$

Applying  $J^+$  on both sides:

$$J^+|11\rangle = \frac{1}{\sqrt{2}}(J_1^+ + J_2^+)(|10\rangle|11\rangle - |11\rangle|10\rangle)$$

$$\Rightarrow \sqrt{2}|10\rangle = \frac{1}{\sqrt{2}}(\sqrt{2}|11\rangle|1\rangle - \sqrt{2}|1\rangle|11\rangle)$$

$$\Rightarrow |10\rangle = \frac{1}{\sqrt{2}}|11\rangle|1\rangle - \frac{1}{\sqrt{2}}|1\rangle|11\rangle$$

Now let: (for  $j=0$ )

$$|10\rangle = a|11\rangle|11\rangle + b|11\rangle|10\rangle + c|10\rangle|10\rangle$$

applying  $J_+$  on ~~both~~ both sides:

$$0 = a J_+ |11\rangle|11\rangle + b J_+ |11\rangle|10\rangle + 0 + 0 \\ + c J_+ |10\rangle|10\rangle + c J_+ |10\rangle|10\rangle$$

$$= a\sqrt{2}|10\rangle|11\rangle + b\sqrt{2}|11\rangle|10\rangle \\ + c\sqrt{1.2}|11\rangle|10\rangle + c\sqrt{2}|10\rangle|11\rangle$$

$$\Rightarrow (a+c)|10\rangle|11\rangle + (b+c)|11\rangle|10\rangle = 0$$

$$\Rightarrow a = -c; \quad b = -c$$

$$\text{but } a^2 + b^2 + c^2 = 1 \Rightarrow c^2 = \frac{-1}{\sqrt{3}}.$$

$$\therefore a = b = \frac{1}{\sqrt{3}}.$$

$$\therefore |10\rangle = \frac{1}{\sqrt{3}}(|11\rangle|11\rangle + |11\rangle|10\rangle - |10\rangle|10\rangle)$$

These are the expansion of coupled state  
in the eigenstate of  ~~$\hat{J}_z$~~ ,  $J_1, J_2$

Now here the two particle state is given by

$$l_1 = l_2 = 1; m_1 = m_2 = 0$$

$$\text{i.e. } |10\rangle|10\rangle.$$

so from the previous eq we have to expand  $|10\rangle|10\rangle$  in eigenstate of  $J^2$ ; i.e in coupled state term.

$$\text{Let } |10\rangle|10\rangle = a|20\rangle + b|10\rangle + c|00\rangle$$

from the previous relations; multiplying

$|j0\rangle$  & the right side expansion by ~~left~~  $\langle j0|$  we get:

$$a = (\langle 10| \langle 10| 20 \rangle)^* = \langle 20| 10 \rangle|10\rangle$$

$$= \frac{2}{\sqrt{6}}.$$

$$b = (\langle 10| \langle 10| 10 \rangle)^* = 0$$

$$c = (\langle 10| \langle 10| 00 \rangle)^* = -\frac{1}{\sqrt{3}}.$$

$$\text{So; } \underbrace{|10\rangle|10\rangle}_{\text{given 2 particle state}} = \frac{2}{\sqrt{6}}|20\rangle - \frac{1}{\sqrt{3}}|00\rangle$$

$\underbrace{\quad\quad\quad}_{\text{expansion in basis of } J^2}$

$\therefore$  ~~0~~  $j$  values 0 & 2 have non zero amplitude in the expansion.

The probabilities of finding those  $j^2$  be:

$$\left. \begin{aligned} P(j=0) &= \frac{1}{3} \\ P(j=2) &= \frac{2}{3} \end{aligned} \right\} \text{on a measurement of } J^2. \quad \underline{\text{Ans}}$$

12.b Here there is no external field. So the conserved quantities are  $L_1^2, L_2^2 \neq J^2, J_z$  but not  $(m_1, m_{12})$ .

$$\# \quad \vec{J} = \vec{L}_1 + \vec{L}_2 \Rightarrow J^2 = L_1^2 + L_2^2 + 2\vec{L}_1 \cdot \vec{L}_2$$

$$\Rightarrow \vec{L}_1 \cdot \vec{L}_2 = \frac{1}{2}(J^2 - L_1^2 - L_2^2)$$

∴ Eigenstate of  $H$  are simultaneously eigenstates of  $J^2, L_1^2, L_2^2$ . But these are the coupled basis:  $|22\rangle, |21\rangle, \dots, |00\rangle$   
(evaluated in part a.)

Let in terms of the eigenbasis of  $\hat{H}$ ; the state vector at time  $t$  be given by:

$$|\psi(t)\rangle = \underbrace{a_1|20\rangle + a_2|00\rangle}_{\text{which were present in } |\psi\rangle \text{ at } t=0} + \underbrace{b_1|22\rangle + b_2|21\rangle + b_3|2-1\rangle + \dots \text{ etc}}_{\text{which were not present in } |\psi(t=0)\rangle}$$

$$\begin{aligned} \text{Now } \hat{H}|jm_j\rangle &= \frac{\hbar^2}{2}(J^2 - L_1^2 - L_2^2)|jm_j\rangle \\ &= \frac{r\hbar^2}{2}(j(j+1) - l(l+1) - 1(1+1))|jm_j\rangle \\ &= \frac{r\hbar^2}{2}(j(j+1) - 4)|jm_j\rangle \end{aligned}$$

So here we get:

$$\hat{H}|\psi(t)\rangle = \frac{m\hbar^2}{2} \left[ (2x_3 - 4)|20\rangle a_1 + (0-4)|00\rangle a_2 + b'_1|22\rangle + b'_2|21\rangle + \dots \text{etc} \right]$$

$$[b'_i = b_i (j(j+1) - 4)]$$

$$= \frac{m\hbar^2}{2} \left[ 2|20\rangle a_1 - 4|00\rangle a_2 + b'_1|22\rangle + b'_2|21\rangle + \dots \text{etc} \right]$$

but  $\hat{H}|\psi(t)\rangle = i\hbar \frac{\partial \psi}{\partial t}$

$$= i\hbar \left( \dot{a}_1|20\rangle + \dot{a}_2|00\rangle + b'_1|22\rangle + b'_2|21\rangle + \dots \text{etc} \right)$$

so comparing coefficients gives

$$i\hbar \dot{a}_1 = \frac{m\hbar^2}{2} \cdot 2a_1 = a_1 m\hbar^2.$$

$$\Rightarrow \int \frac{da_1}{a_1} = -i\hbar \int dt \quad \text{but } a_1(0) = \sqrt{\frac{2}{3}}.$$

$$\Rightarrow a_1(t) = \sqrt{\frac{2}{3}} \exp(-i\hbar t)$$

~~$$\text{Similarly } \dot{a}_2 = a_2(0) \exp$$~~

Similarly:  $i\hbar \dot{a}_2 = -2m\hbar^2 a_2$

$$\Rightarrow \int \frac{da_2}{a_2} = \int_0^t 2i\hbar m dt \quad a_2(0) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow a_2(t) = \frac{1}{\sqrt{3}} \exp(2i\hbar m t)$$

for  $b_1, b_2, \dots$  etc

we get:

$$b_i(t) = b_i(0) \exp(\text{some complex factor})$$

$$\text{but } b_i(0) = 0 \text{ for all } i \}$$

$$\Rightarrow b_i(t) = 0 \text{ for all } i \}$$

$$\text{i.e. } |\psi(t)\rangle = \frac{\sqrt{2}}{\sqrt{3}} \exp(-irt\hbar) |20\rangle$$

$$+ \frac{1}{\sqrt{3}} \exp(2irt\hbar) |00\rangle$$

$$\text{So: } |\langle \psi(t) | \psi(0) \rangle|^2$$

$$= \left| \left( \sqrt{\frac{2}{3}} \exp(+irt\hbar), \frac{1}{\sqrt{3}} \exp(2irt\hbar) \right) \cdot \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right|^2$$

$$= \left| \frac{2}{3} \exp(+irt\hbar) + \left( \frac{1}{\sqrt{3}} \right)^2 \exp(-2irt\hbar) \right|^2.$$

$$= \left| \left\{ \frac{2}{3} \cos(rt\hbar) + \frac{1}{3} \cos(2rt\hbar) \right\} + \right.$$

$$\left. \left\{ \frac{2i}{3} \sin(rt\hbar) - \frac{i}{3} \sin(2rt\hbar) \right\} \right|^2.$$

$$= \left( \frac{2}{3} \cos(k) + \frac{1}{3} \cos(2k) \right)^2 + \left( \frac{2}{3} \sin k - \frac{1}{3} \sin 2k \right)^2$$
$$(k = rt\hbar)$$

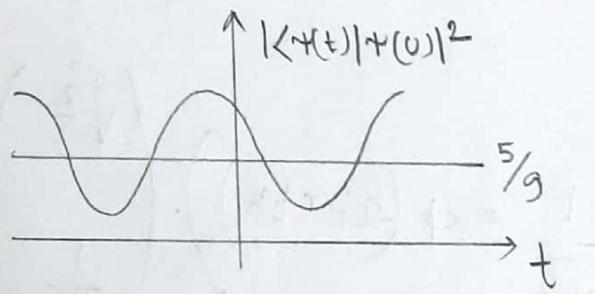
$$= \frac{4}{9} (\cos^2 k + \sin^2 k) + \frac{1}{9} (\cos^2 (2k) + \sin^2 (2k)) \\ + 2 \cdot \frac{2}{3} \cdot \frac{1}{3} (\cos 2k \cos k - \sin 2k \sin k)$$

$$= \frac{5}{9} + \frac{4}{9} \cos(3k).$$

~~$$\text{Clearly } = \frac{5}{9} + \frac{4}{9} \cos(3\pi t)$$~~

clearly  $\max(|\langle \psi(t) | \psi(0) \rangle|^2) = 1$ ; which is obvious.

At  $t = 0$ :  $|\langle \psi(t) | \psi(0) \rangle|^2 = 1$ ; again obvious.



The quantity is  
periodic with a period

$$T = \frac{2\pi}{3\pi}$$

Ans

At  $t = T/2$ :

$$|\langle \psi(t) | \psi(0) \rangle|^2 = \frac{5}{9} + \frac{4}{9} \cos \pi$$

$$= \frac{1}{9} = \text{minimum of the quantity.}$$

Ans

## 2. Quantum H.O. im E-M field:-

The original Hamiltonian:  $H_0 = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} m \omega_0^2 r^2$ .

In presence of magnetic field  $\vec{B}' = B_0 \hat{z} \sin(\omega t)$

I choose the gauge field:

$$\vec{A} = (g B_0 \sin(\omega t), 0, 0)$$

i. The new  $\hat{H}$  gives:

$$\hat{H} = \left( \frac{p_m - g \gamma B_0 \sin \omega t}{2m} \right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2} m \omega_0^2 r^2$$

$$= \underbrace{\frac{p_m^2 + p_y^2 + p_z^2}{2m}}_{H_0} + \frac{m \omega_0^2 r^2}{2} + \underbrace{\frac{g^2 B_0^2 \gamma^2 \sin^2 \omega t}{2m}}_{H'} - \frac{p_m g \gamma B_0 \sin \omega t}{2m}$$

$$(\because [g, p_m] = 0)$$

$$\therefore H' = \frac{g^2 B_0^2 \sin^2 \omega t}{2m} \cdot \gamma^2 - \frac{g B_0 \sin \omega t}{2m} \cdot (g p_m)$$

$$\text{Now, } \langle m_1, m_2, m_3 | H' | 000 \rangle$$

$(| m_1, m_2, m_3 \rangle)$  be quantum state of the actual oscillator with energy  $(m_1 + m_2 + m_3 + \frac{3}{2})\hbar\omega$

$$= \langle m_1, m_2, m_3 | \alpha \gamma^2 - \beta g p_m | 000 \rangle$$

( $\alpha, \beta$  are those coefficients)

$$\text{Now; } \hat{\gamma} = \sqrt{\frac{\hbar}{2m\omega_0}} (a_g^+ + a_g^-); \quad b_m = i\sqrt{\frac{\hbar m\omega}{2}} (a_m^+ - a_m^-)$$

$$\therefore \hat{\gamma}^2 = \frac{\hbar}{2m\omega} (a_g^{+2} + a_g^{-2} + \cancel{a_g^+ a_g^- + a_g^- a_g^+})$$

$$\text{So; } \langle m_1 m_2 m_3 | H' | 000 \rangle$$

$$= \langle m_1 m_2 m_3 | \alpha' (a_g^{+2} + a_g^{-2} + a_g^+ a_g^- + a_g^- a_g^+) | 000 \rangle$$

$$- \langle m_1 m_2 m_3 | \beta' (a_g^+ a_m^+ + a_g^- a_m^- - a_g^+ a_m^- - a_g^- a_m^+) | 000 \rangle$$

$$(\hat{\gamma} b_m = \frac{i\hbar}{2} \cancel{(a_g^+ a_m^+ - a_g^+ a_m^- - a_g^- a_m^+ + a_g^- a_m^-)})$$

$$= \alpha' \left\{ \langle m_1 m_2 m_3 | \sqrt{1 \cdot 2} | 020 \rangle + 0 + 0 \right. \\ \left. + \langle m_1 m_2 m_3 | 000 \rangle \right\}$$

$$- \beta' \left\{ \langle m_1 m_2 m_3 | 110 \rangle + 0 + 0 + 0 \right\}$$

Using  $a_i^+ |m_i m_j m_k\rangle = \sqrt{m_i+1} |m_i+1 m_j m_k\rangle$   
 $a_i^- |-\dots\rangle = \sqrt{m_i} |m_i-1 \dots\rangle$

So except  $\langle m_1 m_2 m_3 | = \langle 000 |$ ; the  
 only two non-zero terms present are.

given by  $\langle 110 | \neq \langle 020 |$

So; the allowed transitions from  $|000\rangle$  (ground state) be:

$$\left. \begin{array}{l} |000\rangle \rightarrow |020\rangle \\ |000\rangle \rightarrow |110\rangle \end{array} \right\} \text{due to perturbation of 1st order.}$$

ii) Transition amplitude

$$\begin{aligned} \langle_{|000\rangle}^{|020\rangle} &= -\frac{i}{\hbar} \int_0^t \langle_{|020\rangle}^{|000\rangle} H' e^{i\omega_c t} dt \\ &\quad \left( \omega_c = \frac{E_2 - E_1}{\hbar} = 2 \frac{\hbar \omega_0}{\hbar} = 2\omega_0 \right) \\ &= -\frac{i}{\hbar} \int_0^t \sqrt{2} \cdot \frac{\hbar}{2m\omega_0} \cdot \frac{q^2 B_0^2}{2m} \sin^2 \omega_0 t e^{2i\omega_0 t} dt \\ &= -\frac{\sqrt{2} q^2 B_0^2}{4m^2 \omega_0} \frac{i}{2} \int_0^t (1 - \cos 2\omega_0 t) e^{2i\omega_0 t} dt \\ &= -\frac{\sqrt{2} q^2 B_0^2 i}{8m^2 \omega_0} \left\{ \int_0^t e^{2i\omega_0 t} dt - \frac{1}{2} \int_0^t e^{2i(\omega + \omega_0)t} dt \right. \\ &\quad \left. - \frac{1}{2} \int_0^t e^{2i(\omega - \omega_0)t} dt \right\} \\ &= -\frac{\sqrt{2} q^2 B_0^2 i}{8m^2 \omega_0} \left\{ \frac{e^{2i\omega_0 t} - 1}{\omega_0} - \frac{1}{2} \frac{e^{2i(\omega + \omega_0)t} - 1}{\omega + \omega_0} \right. \\ &\quad \left. - \frac{1}{2} \frac{e^{2i(\omega - \omega_0)t} - 1}{\omega - \omega_0} \right\} \end{aligned}$$

So, the transition rate be given by:

$$\tilde{R}_{1020} = \frac{P_{1020}}{t} = \frac{|\psi_{1020}|^2}{t}$$

(This is very long algebra; I've found out the value using Mathematica)

$$P_{1020}(t)$$

evaluated avg  $\tilde{R}$  instead of

$$\tilde{R}_0 = \frac{dP(t)}{dt}$$

$$\begin{aligned} \tilde{R}_{1020} &= \frac{\beta^2}{16\sqrt{2}m^2\omega_0^2 t} \left[ \left( \frac{2}{\omega_0} + \frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0} \right. \right. \\ &\quad + \frac{\cos(2t(\omega - \omega_0))}{\omega_0 - \omega} + \frac{\cos(2t(\omega_0 + \omega))}{\omega_0 + \omega} \left. \left. \right)^2 \right. \\ &\quad \left. - \frac{2 \sin(2\omega_0 t)}{\omega_0} \right]^{1/2} \end{aligned}$$

The actual (instantaneous) transition rate at time  $t$  be

probability  
← Prob to find the system in  $|1020\rangle$  at time t  
be given by:

$$P_{|1020\rangle}(t) = |\psi_{1020}\rangle|^2$$

And the transition rate at time t:

$$\tilde{R}_{|1020\rangle} = \frac{\partial}{\partial t} P_{|1020\rangle}(t)$$

using Mathematica (as it is very complicated by hand.) the expression of  $\tilde{R}_{|1020\rangle}$  be given by:

$$\begin{aligned} \tilde{R}_{|1020\rangle} &= \frac{B_0^4 g^4}{512 m^4 \omega_0^2} \left[ 2 \left\{ \left( \frac{2}{\omega_0} + \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} - \frac{\cos(2\omega_0 t)}{\omega_0} \right. \right. \right. \\ &+ \frac{\cos(2t(\omega_0 - \omega))}{\omega_0 - \omega} + \frac{\cos(2t(\omega_0 + \omega))}{\omega_0 + \omega} \times \left( 4 \sin(2\omega_0 t) - \right. \\ &\left. \left. \left. 2 \sin(2t(\omega_0 - \omega)) - 2 \sin(2t(\omega_0 + \omega)) \right) \right\} + \\ &\left\{ 2 \left( 2 \cos(2t(\omega_0 - \omega)) + 2 \cos(2t(\omega_0 + \omega)) - 4 \cos(2\omega_0 t) \right) \times \right. \\ &\left. \left( \frac{\sin(2t(\omega_0 + \omega))}{\omega_0 + \omega} + \frac{\sin(2t(\omega_0 - \omega))}{\omega_0 - \omega} - \frac{2 \sin(2\omega_0 t)}{\omega_0} \right) \right] \end{aligned}$$

Ans

Similarly for  $|110\rangle$   
transition amplitude:

$$\psi_{|110\rangle} = -\frac{i}{\hbar} \int_0^t \langle 110 | H' | 000 \rangle e^{i\omega_c t} dt$$

$$\omega_c = \frac{E(|110\rangle) - E(|000\rangle)}{\hbar} = 2\omega_0 \text{ (again)}$$

$$= \frac{ik}{2} \cdot \frac{qB_0}{2m} \cdot i \int_0^t \sin(\omega t) e^{i\omega_0 t} dt.$$

Again using mathematics I found out:

$$P_{|1110\rangle} = |\psi_{1110}\rangle|^2 \text{ And } \tilde{R}_{|1110\rangle}^{(+)} = \frac{\partial \psi_{1110}}{\partial t}^{(+)}$$

$$\tilde{R}_{|1110\rangle}^{(+)} =$$

$$\frac{B^2 q^2}{4m^2 (\omega^2 - 4\omega_0^2)^2} \left[ \left( \omega^2 \cos(2\omega_0 t) \sin(\omega t) - 4\omega_0^2 \cos(2\omega_0 t) \sin(\omega t) \right) \times \left( \omega - \omega \cos(2\omega_0 t) \cos(\omega t) - 2\omega_0 \sin(2\omega_0 t) \sin(\omega t) \right) \right]$$

Ans

### Problem - 3

a) Due to L-S coupling; the coupling Hamiltonian gives:  $H \sim \vec{L} \cdot \vec{S}$

$$\text{but } \vec{J} = \vec{L} + \vec{S} \Rightarrow \vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L} \cdot \vec{S}$$

$$\Rightarrow \vec{L} \cdot \vec{S} \therefore \frac{\vec{J}^2 - \vec{L}^2 - \vec{S}^2}{2}$$

Here for  $m=3$ :  $L = 0, 1, 2$ ;  $S$  is always  $\frac{3}{2}$ .

$$\text{Now, } H' = \underbrace{\frac{e^2}{q\pi\epsilon_0}}_{\propto \chi} \frac{1}{m^2 c^2 r^3} \vec{S} \cdot \vec{L} = \frac{e^2 (\vec{J}^2 - \vec{L}^2 - \vec{S}^2)}{q\pi\epsilon_0 m^2 c^2 r^3 \times 2}.$$

$\hookrightarrow \chi = \text{const.} (kt)$ .

$$\begin{aligned} \Rightarrow \langle H' \rangle_{lmj} &= \chi \cdot \left\langle \frac{1}{r} \right\rangle \cdot \langle \vec{S} \cdot \vec{L} \rangle \\ &= \chi_0 \frac{j(j+1) - l(l+1) - s(s+1)}{l(l+1)(l+\frac{1}{2})} \quad (s = \frac{3}{2}) \\ &= \chi_0 \frac{j(j+1) - l(l+1) - \frac{15}{4}}{l(l+\frac{1}{2})(l+1)} = (E_1)_{s0}. \end{aligned}$$

$\therefore$  For  $l=0$ :

$$j = \frac{3}{2}; E'_1 = 0$$

for  $l=1$ :  $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

$$j = \frac{1}{2} \Rightarrow E'_1 = -\frac{10}{6} \chi_0$$

$$j = \frac{3}{2} \Rightarrow E'_1 = -\frac{2}{3} \chi_0.$$

$$j = \frac{5}{2} \Rightarrow E'_1 = \chi_0$$

for  $l = 2$ :  $j = 7/2, \dots, \cancel{10/2}, 1/2$ .

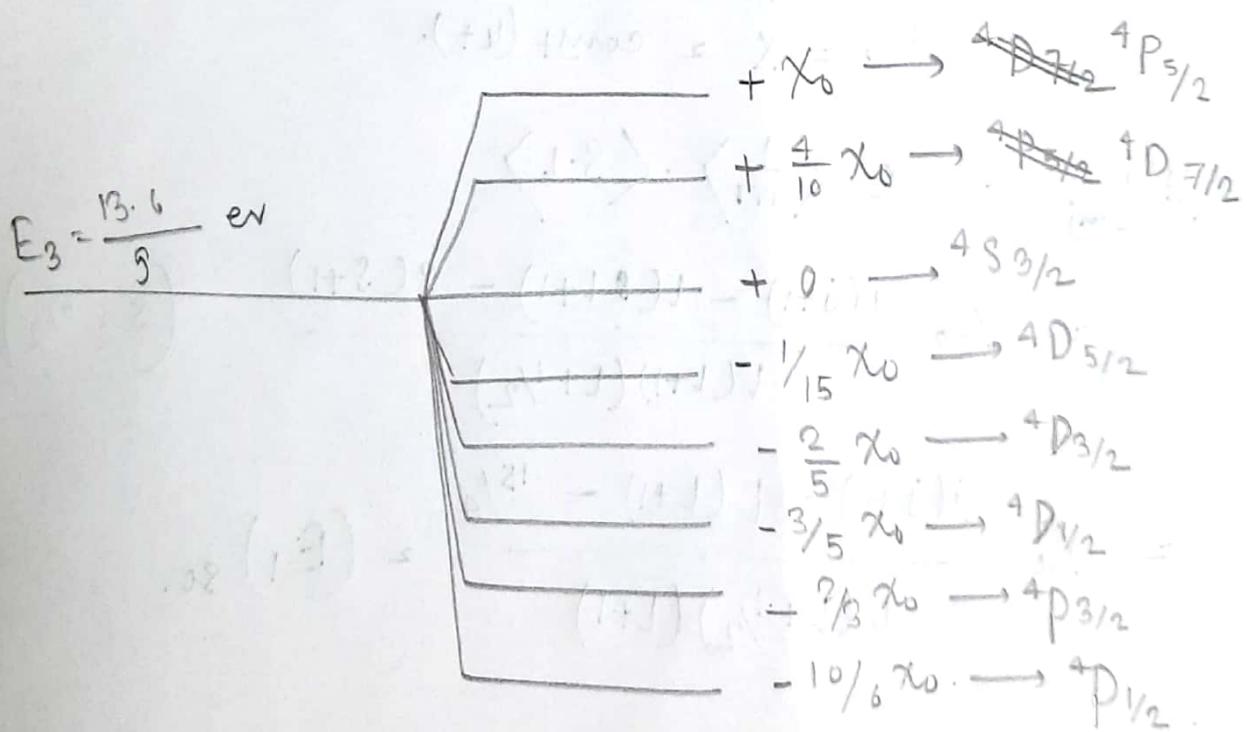
$$\text{for } j = 1/2 : E'_1 = -\frac{6}{10} \chi_0 \quad \left. \right\}$$

$$j = 3/2 : E'_1 = -\frac{4}{10} \chi_0 \quad \left. \right\}$$

$$j = 5/2 : E'_1 = -\frac{1}{15} \chi_0 \quad \left. \right\}$$

$$j = 7/2 : E'_1 = -\frac{4}{10} \chi_0 \quad \left. \right\}$$

So the level  $E_3 (m=3)$  splits into 8 lines.



3.b. For multi e- atom; there are L.S coupling between different e- present in the atom.

In our universe S is  $\frac{1}{2}$ .

$\therefore j$  changes from  $|l + \frac{1}{2}\rangle$  to  $|l - \frac{1}{2}\rangle$

In that universe S is  $\frac{3}{2}$ .

$\therefore j$  changes from  $|l + \frac{3}{2}\rangle$  to  $|l - \frac{3}{2}\rangle$

So depending on value of l; there will be some extra term.

Here total  $S = S_1 + S_2 = 0, 1, 0, 2, 3$ .  
 In fact universe  $S = 0, 1, 2, 3$ .

So for the extra 2  $S$  values (2, 3);  
 Some more states will appear in parallel  
 universe.

Ans

3.c  $S = 3/2 \Rightarrow (2S+1) = 4$ .

So for given  $m, l$ , there are 4 spin  
 state.

for given  $m$  there are  $4m^2$  electronic  
 configuration. As ~~fully~~ filled shell gives  
 inert gas; so the gases will have  $e^-$  no:  
 $\sum 4m^2$  (~~max~~  $m = 1, 2, \dots$ ) (All shells upto  $m_{\text{max}}$   
 are filled.)

1  $f+$  inert gas:  $4m^2 = 4 \times 1^2 = 4 \rightarrow \text{Be}$

$$2m_1 \quad : 4m^2 = 4 \times 2^2 = 16$$

2  $m_2$  inert gas:  $4m_1^2 + 4m_2^2 |_{(1,2)}$

$$= 4 \times 1^2 + 4 \times 2^2$$

$$= 4 + 16 = 20 e^- \rightarrow \text{Ca}$$

Ans

Problem: 4

A.i) For the state  $2p_3\}$ :

$$S_1 = S_2 = \frac{1}{2}; L_1 = L_2 = 1$$

$$\therefore S = 0, 1; L = 0, 1, 2.$$

For  $S = 0: L = 0, 1, 2 \Rightarrow j = 0, 1, 2$

The states are  $^1S_0, ^1P_1, ^1D_2$

$S = 1: L = 0 \Rightarrow j = 1: \text{State: } ^3S_1$

$S = 1: L = 1 \Rightarrow j = 0, 1, 2: \text{States: } ^3P_0, ^3P_1, ^3P_2$

$S = 1: L = 2 \Rightarrow j = 1, 2, 3: \text{States: } ^3D_1, ^3D_2, ^3D_3$ .

So there are 10 states:

$^1S_0, ^1P_1, ^1D_2, ^3S_1, ^3P_0, ^3P_1, ^3P_2, ^3D_1, ^3D_2, ^3D_3$ .

## Splitting:-

i)  $S-J$  Splitting:  $S = 0, 1.$

The  $\overline{\Delta V}$  for  $J-J$  splitting be given by:

$$\overline{\Delta V}_S = -a [J(J+1) - J_1(J_1+1) - J_2(J_2+1)]$$

$$\text{for } J=0 : \overline{\Delta V}_J = -a \left[ 0 - \frac{3}{4} - \frac{3}{4} \right] = \frac{3a}{2}$$

$$J=1 : \overline{\Delta V}_J = -a \left[ 2 - \frac{3}{4} \times 2 \right] = \frac{a}{2}$$

ii) for  $L-L$  splitting,  $\overline{\Delta V}_L = -m [L(L+1) - l_1(l_1+1) - l_2(l_2+1)]$

here  $l_1 = l_2 = l; L = 0, 1, 2.$

$$\text{for } l=0, \overline{\Delta V} = -m [0 - 2 - 2] = 4m$$

$$l=1 : \overline{\Delta V} = -m [2 - 2 - 2] = 2m$$

$$l=2 : \overline{\Delta V} = -m [(-2) - 2 - 2] = -2m$$

iii) Splitting of  $sp$  orbital:  $\overline{\Delta V} = x [J(J+1) - l(l+1) - s(s+1)]$

So splitting of  $3p$  levels:  $S=1; l=1; J=0, 1, 2$

$$\therefore \overline{\Delta V}|_{J=0} = x [0 - 2 - 2] = -4x$$

$$\text{for } J=1 : \overline{\Delta V} = x [2 - 2 - 2] = -2x$$

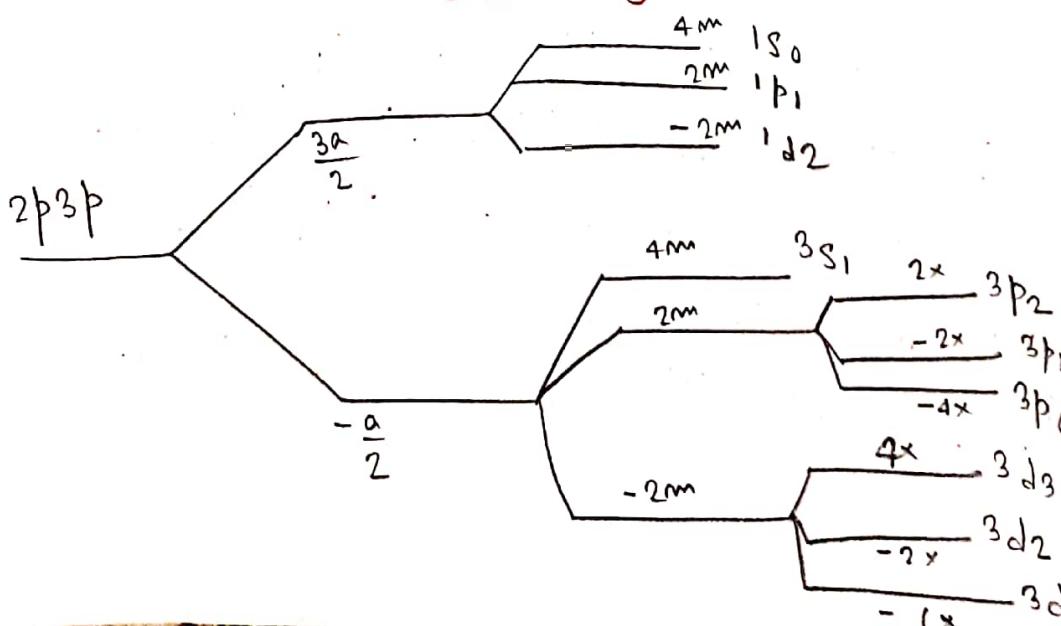
$$J=2 : \overline{\Delta V} = x [(-2) - 2 - 2] = +2x$$

for  $3d$  levels:  $S=1; l=2; j=1, 2, 3.$

$$J=1 : \overline{\Delta V} = x [2 - 6 - 2] = -6x$$

$$J=2 : \overline{\Delta V} = x [(-6) - 2] = -2x$$

$$J=3 : \overline{\Delta V} = x [12 - 6 - 2] = +4x$$



4.b.

Electric dipole Selection rule here given by:

$$\Delta l = \pm 1 ; \Delta m = 0, \pm 1, \Delta s = 0$$

for  $j=0$  no transition is possible here.

4.c. According to Selection rule the transition from  ${}^3p_1$  possible are:

$${}^3S_1 \text{ & } {}^3D_{1,2,3}$$

And the forbidden are:

$${}^1S_0, {}^1D_2, {}^3P_{0,1,2}, {}^1P_1 : \underline{\text{Any}}$$



## Problem:- 1

a. Here parity is being violated.  
When we apply  $\pi$  on  $\vec{S}$ ; it does not change sign. But  $\vec{p}$  changes sign.

$$\pi(\vec{S} \cdot \vec{p}) = \vec{S} \cdot \pi(\vec{p}) = -\vec{S} \cdot \vec{p}$$

So in the reverse world;  $\vec{p}$  is anti parallel to its momentum; i.e violates parity.

b. i)  $\langle 210 | \pi | 200 \rangle$

$$= \int_0^{\infty} r^3 R_{21}(r) \cdot R_{20}(r) dr \cdot \int_{0, \phi}^{Y_{10}^*} Y_{10} \cdot Y_{00} \cdot \sin \theta \cos^3 \phi d\phi.$$

The  $\phi$  integral gives:  $\int_0^{2\pi} \cos^3 \phi d\phi = 0$ .

So,  $\langle 210 | \pi | 200 \rangle = 0$ .

$|210\rangle$  doesn't have  $\phi$  term in it. So it gives  $Y_{10}(0, \phi)$ . But as  $\pi$  containing  $\cos^3 \phi$  so, this ~~can~~ can be written as in terms of  $Y_{1\pm 1}(0, \phi)$ .

$$+ Y_{1\pm 1}(0, \phi).$$

All Spunics Harmonics are orthogonal fm.

So, the whole  $\pi$  integral gives 0.

$$b. ii) \hat{p}_z = -i\hbar \frac{\partial}{\partial \phi}$$

in terms of  $(r, \theta, \phi)$  this can be written as:

$$\frac{\partial}{\partial \phi} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$$

$$\therefore \langle 210 | \hat{p}_z | 200 \rangle = -i\hbar \langle 210 | \cos \theta \frac{\partial}{\partial r} | 200 \rangle$$

$$+ i\hbar \langle 210 | \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} | 200 \rangle$$

$$\text{for the 1st } \langle \dots \rangle \text{ i.e. } \langle 210 | \cos \theta \frac{\partial}{\partial r} | 200 \rangle$$

the  $\theta$  integral gives:

$$\sim \int_0^\pi P_{10}(\cos \theta) \cdot P_{00}(\cos \theta) \cdot \cos \theta \sin \theta d\theta$$

the  $r$  integral gives:

$$\sim \int_0^\infty R_{21}(r) \left( \frac{\partial}{\partial r} R_{20}(r) \right) r^2 dr$$

$$= \int_0^\infty r^3 \exp\left(-\frac{r}{2a}\right) \cdot \frac{\partial}{\partial r} \left( \left(1 - \frac{r}{2a}\right) e^{-r^2/2a} \right) dr$$

Using mathematica, I find the integral to be

Zero.

$$\therefore \langle 210 | \cos \theta \frac{\partial}{\partial r} | 200 \rangle = 0.$$

the 2nd integral gives:

$$\langle 210 | \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} | 200 \rangle$$

$$\text{The } \theta \text{ integral gives } \sim \int_0^\pi P_{10}(\cos \theta) \cdot \sin^2 \theta \frac{\partial}{\partial \theta} P_{00}(\cos \theta) \cdot d\theta$$

But  $P_{00}(\cos\theta) = 1$

$$\therefore \frac{\partial}{\partial \theta}(P_{00}) = 0 \quad i.e. \langle 210 | \frac{\sin \theta}{\pi} \frac{\partial}{\partial \theta} | 200 \rangle = 0.$$

So;  $\langle 210 | p_z | 200 \rangle = 0.$  Ans

(iii)  $j = \frac{9}{2}; l = 4; s = \frac{1}{2}$  (obvious)

$$\therefore j = \left(4 + \frac{1}{2}\right) \rightarrow m_j = \frac{9}{2}, \frac{7}{2}, \dots, -\frac{9}{2}.$$

clearly  $\left| \frac{9}{2} \frac{9}{2} \right\rangle = |4+4\rangle | \frac{1}{2} \frac{1}{2} \rangle$

$$\Rightarrow J^- \left| \frac{9}{2} \frac{9}{2} \right\rangle = L^- |4+4\rangle + S^- |4+4\rangle | \frac{1}{2} \frac{1}{2} \rangle$$

$$\Rightarrow \sqrt{9 \times \left( \frac{9}{2} + 1 - \frac{9}{2} \right)} \left| \frac{9}{2} \frac{7}{2} \right\rangle = \sqrt{(4+4)(4-4+1)} |43\rangle | \frac{1}{2} \frac{1}{2} \rangle \\ + \sqrt{1 \times \left( \frac{1}{2} + 1 - \frac{1}{2} \right)} |44\rangle | \frac{1}{2} - \frac{1}{2} \rangle$$

$$\Rightarrow 3 \left| \frac{9}{2} \frac{7}{2} \right\rangle = \sqrt{8} |43\rangle | \frac{1}{2} \frac{1}{2} \rangle + |44\rangle | \frac{1}{2} - \frac{1}{2} \rangle$$

$$\therefore \left| \frac{9}{2} \frac{7}{2} \right\rangle = \frac{2\sqrt{2}}{3} |43\rangle | \frac{1}{2} \frac{1}{2} \rangle + \frac{1}{3} |44\rangle | \frac{1}{2} - \frac{1}{2} \rangle$$

This state if  $j = \frac{9}{2}, m_j = \frac{7}{2}.$

$$\therefore L_z \left| \frac{9}{2} \frac{7}{2} \right\rangle = \frac{2\sqrt{2}}{3} \times 2\hbar |43\rangle | \frac{1}{2} \frac{1}{2} \rangle + \frac{4\hbar}{3} |44\rangle | \frac{1}{2} - \frac{1}{2} \rangle$$

$$\therefore \langle L_z \rangle = \langle \frac{9}{2} \frac{7}{2} | L_z | \frac{9}{2} \frac{7}{2} \rangle = \cancel{\left( \frac{2\sqrt{2}}{3} + \frac{4}{3} \right) \hbar} \\ = \frac{2\sqrt{2} \times 2}{3} \hbar + \frac{4\hbar}{3} \times \frac{1}{3} = \hbar \left( \frac{4\sqrt{2}}{3} + \frac{4}{3} \right) \text{ Ans}$$

$$\text{iv. Singlet} = \frac{|+\rightarrow -\downarrow +\downarrow\rangle}{\sqrt{2}}; \text{ triplet } (m=0) = \frac{|+\rightarrow +\downarrow -\downarrow\rangle}{\sqrt{2}}$$

Now, if Singlet & triplet be represented by:

$$|S_{\text{Singlet}}\rangle = |S_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \quad \left. \right\}$$

$$|S_{\text{triplet}}\rangle = |S_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \left. \right\}$$

$$(|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; |+\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; |-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; |-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$$

$$\therefore S_z^{e^-} = S_z^{e^-} \otimes \mathbb{I}^{e^+} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_z^{e^+} = \mathbb{I}^{e^+} \otimes S_z^{e^+} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \underbrace{S_z^{e^-} - S_z^{e^+}}_{\downarrow} = \frac{\hbar}{2} \begin{pmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle S_0 | \hat{A} | S_1 \rangle = \frac{\hbar}{4} (0 \ 1 \ -1) \begin{pmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{4} (0 \ 1 \ -1) \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = \frac{\hbar}{4} \times \cancel{(0+2)} \cancel{(0-2)} (2+2)$$

$$= \hbar \underline{\text{Ans}}$$

$$\underline{\underline{V}} \quad \vec{S}_1 \otimes \vec{S}_2 = S_{x_1} \otimes S_{x_2} + S_{y_1} \otimes S_{y_2} + S_{z_1} \otimes S_{z_2}$$

$$S_{y_1} \otimes S_{x_2} = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{4 \times 4}$$

$$S_{y_1} \otimes S_{y_2} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$S_{z_1} \otimes S_{z_2} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{So; } \vec{S}_1 \otimes \vec{S}_2 = \sum_i S_{xi_1} \otimes S_{xi_2}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In ground state (singlet configuration  $\frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$ )

$$\text{i.e } |X\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} : \langle S_1 \otimes S_2 \rangle$$

$$= \frac{\hbar^2}{4} \times \frac{1}{2} (0 \ 1 \ 1 \ 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{8} (0 \ 1 \ 1 \ 0) \begin{pmatrix} 0 \\ -3 \\ 3 \\ 0 \end{pmatrix} = \frac{\hbar^2}{8} (-1) = -\frac{3\hbar^2}{4} \underline{\text{Ans}}$$

$$\underline{\underline{c})} \quad (3z^2 - r^2) \sim T_0^2 \quad (\because z^2, r^2 \text{ are power 2})$$

$$\therefore \langle m'_l' m'_i' m'_s' | 3z^2 - r^2 | m_l m_i m_s \rangle$$

$$\sim \langle m'_l' m'_i' m'_s' | T_0^2 | m_l m_i m_s \rangle$$

$\neq 0$  for  $l' = l+2$ ;  $m'_i = m_i$ ;  $m'_s = m_s$ .

$\therefore$  Selection rule:  $\Delta l = 2$ ;  $\Delta m_i = \Delta m_s = 0$ .

$$ii) \quad \pi_T \sim T_2^2 - T_{-2}^2$$

$$= \langle m'_l' m'_i' m'_s' | \pi_T | m_l m_i m_s \rangle$$

$$\sim \langle m'_l' m'_i' m'_s' | T_2^2 - T_{-2}^2 | m_l m_i m_s \rangle$$

$\neq 0$  if  $l' = l+2$ ;  $m'_i = m_i \pm 2$ ;  $m'_s = m_s$ .

$\therefore$  Selection rule  $\Delta l = 2$ ;  $\Delta m_i = \pm 2$ ;  $\Delta m_s = 0$ .

$$\underline{d)} \quad \pi_T = \frac{1}{2i} (T_2^2 - T_{-2}^2)$$

$$\therefore \langle 322 | \pi_T | 300 \rangle = \frac{1}{2i} \langle 322 | T_2^2 - T_{-2}^2 | 300 \rangle$$

$$= \frac{1}{2i} \left\{ \langle 02; 02 | 02; 22 \rangle \cdot \frac{\langle 32 | T^2 | 30 \rangle}{\sqrt{1}} - \right.$$

$$\left. \langle 02; 02 | 02; 22 \rangle \cdot \frac{\langle 32 | T^2 | 30 \rangle}{\sqrt{1}} \right\}$$

$$\therefore \langle 32 | T^2 | 30 \rangle = \frac{2Ai}{\langle 02 02 | 02 22 \rangle - \langle 02 02 | 02 22 \rangle}$$

$$= \chi_A(\text{day})$$

$$\chi = \frac{2i}{\langle 02|02|0222 \rangle - \langle 02|0-2|0222 \rangle}$$

8v; i)  $\langle 32m|300 \rangle \langle 32m|\pi_4|300 \rangle$

$$= \frac{1}{2i} \langle 32m|T_2^2 - T_{-2}^2|300 \rangle$$

$$= \frac{1}{2i} \left[ \langle 02|02|022m \rangle \frac{\langle 32|T^2|30 \rangle}{1} - \langle 02|0-2|022m \rangle \langle 32|T^2|30 \rangle \right]$$

$$= (\langle 02|02|022m \rangle - \langle 02|0-2|022m \rangle) \cdot \chi_A$$

ii)  $\langle 32m|\pi_2|300 \rangle = \frac{1}{2} \langle 32m|T_1^2 - T_{-1}^2|300 \rangle$

$$= \frac{1}{2} \left[ \langle 02|0-1|022m \rangle \frac{\langle 32|T^2|30 \rangle}{1} - \langle 02|01|022m \rangle \langle 32|T^2|30 \rangle \right]$$

$$= (\langle 02|0-1|022m \rangle - \langle 02|01|022m \rangle) \chi_{Ai}$$

iii)  $\langle 32m|g_2|300 \rangle = -\frac{1}{2i} \langle 32m|T_1^2 + T_{-1}^2|300 \rangle$

$$= -\frac{1}{2i} \left[ \langle 02|01|022m \rangle \langle 32|T^2|30 \rangle + \langle 02|0-1|022m \rangle \langle 32|T^2|30 \rangle \right]$$

$$= -\chi_A [\langle 02|01|022m \rangle + \langle 02|0-1|022m \rangle]$$

$$\begin{aligned}
 \text{iv) } & \langle 32m | n^2 | 300 \rangle = \sqrt{\frac{2\pi}{15}} \langle 32m | T_2^2 + T_{-2}^2 | 300 \rangle \\
 &= \sqrt{\frac{2\pi}{15}} \cdot 2i \cdot \frac{1}{2i} \left[ \langle 02 02 | 02 2m \rangle + \langle 02 -2 | 02 2m \rangle \right] \langle 32 | T^2 | 30 \rangle \\
 &= 2i \sqrt{\frac{2\pi}{15}} \chi_A \left[ \langle 02 02 | 02 2m \rangle + \langle 02 -2 | 02 2m \rangle \right]
 \end{aligned}$$

~~④~~  $\langle 32m | 02 | 300 \rangle$

$$\text{v) } \langle 32m | z^2 | 300 \rangle \rightarrow \left( \frac{2\sqrt{\pi}}{3} T_0^0 + \frac{4\sqrt{\pi}}{3\sqrt{5}} T_{20}^2 \right)$$

$$\therefore \text{vi) } \langle 32m | z^2 | 300 \rangle$$

$$\begin{aligned}
 &= \frac{2\sqrt{\pi}}{3} \langle 32m | T_0^0 | 300 \rangle + \frac{4\sqrt{\pi}}{3\sqrt{5}} \langle 32m | T_{20}^2 | 300 \rangle \\
 &= \frac{2\sqrt{\pi}}{3} \cdot 2i \cdot \chi_A \cdot \left( \langle 00 00 | 00 0m \rangle + \langle 02 00 | 00 0m \rangle \right) \\
 &\quad + \frac{4\sqrt{8\pi}}{3\sqrt{5}} \cdot 2i \cdot \chi_A \cdot \left( \langle 02 20 | 02 2m \rangle + \langle 02 -20 | 02 2m \rangle \right) \\
 &= \frac{4\sqrt{\pi}i\chi_A}{3} \langle 00 00 | 00 0m \rangle \\
 &\quad + \frac{16\sqrt{\pi}i\chi_A}{3\sqrt{5}} \langle 02 20 | 02 2m \rangle
 \end{aligned}$$