SAGAR DAM

DNAP

Q: 1

a In this case $\rho(r) = \text{Const} = C$ (assume) $0<r<R$
$= 0$ o/w

i.e $\langle r^2 \rangle = \frac{\int r^2 \rho(r)\, d\tau}{\int \rho(r)\, d\tau}$

$$= \frac{c\int r^2 \times r^2 dr\, d\theta\, d\phi}{c\int r^2\, dr\, d\theta\, d\phi} = \frac{\int_0^R r^4 dr}{\int_0^R r^2 dr} = \frac{\left(\frac{R^5}{5}\right)}{\left(\frac{R^3}{3}\right)} = \frac{3R^2}{5}.$$

b Here $\rho(r) = \frac{\rho_0}{1+e^{\left(\frac{r-R}{a}\right)}}$

$$\therefore I_1 = \int_0^\infty r^2 \rho(r)\, dr = \int_0^\infty \frac{\rho_0 r^2}{1+e^{\left(\frac{r-R}{a}\right)}}\, dr \cancel{d\theta\, d\phi}$$

* The function in the integrand is not an elementary integral. We need to integrate it in expansion form.

i.e $\int_0^\infty r^2 \rho(r)\, dr = \int_0^R \frac{\rho_0 r^2\, dr}{1+e^{\frac{r-R}{a}}} + \int_R^\infty \frac{\rho_0 r^2\, dr}{1+e^{\frac{r-R}{a}}}$

$$= \int_0^R r^2 \rho_0 \left\{ 1 - e^{\frac{r-R}{a}} + e^{2\left(\frac{r-R}{a}\right)} - \cdots \right\} dr$$

$$+ \int_R^\infty r^2 e^{-\frac{r-R}{a}} \left\{ 1 - e^{-\frac{r-R}{a}} + e^{-2\left(\frac{r-R}{a}\right)} - \cdots \right\} dr$$

$$\cancel{= \int_0^R \rho_0 r^2 dr + \int_0^\infty \sum_{m=1}^{\infty} (-1)^m \cdot r^2 \left\{ e^{m\left(\frac{r-R}{a}\right)} - e^{-m\left(\frac{r-R}{a}\right)} \right\}}$$

$$= \int_0^R \rho_0 r^2 dr + \rho_0 \sum_{m=1}^{\infty} (-1)^m \left[\int_0^R r^2 e^{m\left(\frac{r-R}{a}\right)} dr - \int_R^\infty r^2 e^{m\left(\frac{R-r}{a}\right)} dr \right]$$

Using Mathematica to evaluate the 2nd integral; the result turns out to be:

$$= \rho_0 \left(\frac{R^3}{3}\right) + \rho_0 \sum_{m=1}^{\infty} (-1)^{m+1} \cdot \frac{2a^2\left(a e^{-\frac{mR}{a}} + 2mR\right)}{m^3}$$

And the other required integral is:

$$I_2 = \int_0^\infty r^4 \rho(r)\, dr = \rho_0 \int_0^R \frac{r^4\, dr}{1 + e^{\frac{r-R}{a}}} + \rho_0 \int_R^\infty \frac{r^4\, dr}{1 + e^{\frac{r-R}{a}}}$$

Following the same way like the previous integral

$$I_2 = \int_0^R \rho_0 r^4 dr + \rho_0 \sum_{m=1}^{\infty} (-1)^m \left[\int_0^R r^4 e^{m\left(\frac{r-R}{a}\right)} - \int_R^{\infty} r^4 e^{-m\left(\frac{r-R}{a}\right)} dr \right]$$

$$= \rho_0 \left(\frac{R^5}{5}\right) + \rho_0 \sum_{m=1}^{\infty} (-1)^{m+1} \cdot \frac{8a^2\left(3a^3 e^{-\frac{mR}{a}} + 6a^2 mR + m^3R^3\right)}{m^5}.$$

So finally the required quantity:

$$\langle r^2 \rangle = \frac{\int r^2 \rho(r)\, d\tau}{\int \rho(r)\, d\tau} = \frac{\int r^2 \rho(r) \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi}{\int \rho(r) \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

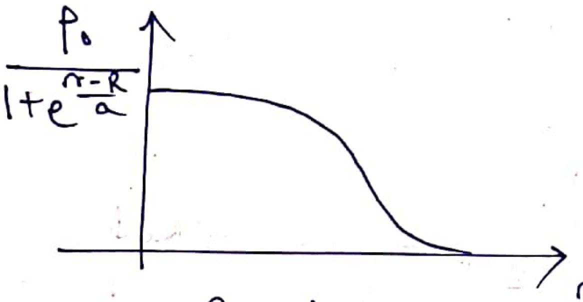
$$= \frac{\int r^4 \rho(r)\, dr}{\int r^2 \rho(r)\, dr} \quad (\because \rho(r) \text{ is indep of } \theta, \phi)$$

$$= \frac{I_2}{I_1}$$

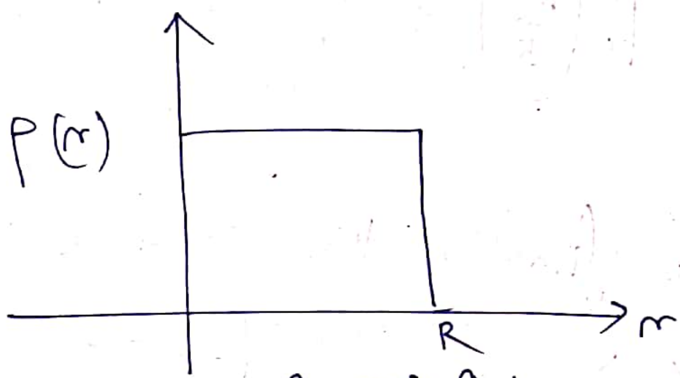
$$\boxed{= \frac{\frac{R^5}{5} + \sum_{m=1}^{\infty} (-1)^{m+1} \cdot \frac{8a^2\left(3a^3 e^{-\frac{mR}{a}} + 6a^2 mR + m^3R^3\right)}{m^5}}{\frac{R^3}{3} + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{2a^2\left(a e^{-\frac{mR}{a}} + 2mR\right)}{m^3}}}$$

Ans

Clearly when $a \to 0$; i.e the density is uniform upto $r = R$ and then goes to zero ;



$a \neq 0$



$a \to 0+$

the problem reduces to the case (a) in the question and we retrive the result as!

$$\langle r^2 \rangle = \lim_{a \to 0} \left(\frac{I_2}{I_1} \right) = \lim_{a \to 0} \frac{\frac{R^5}{5} + \sum_{m=1}^{\infty} (-1)^{m+1} \cdot 8a^2 \times (\#)}{\frac{R^3}{3} + \sum_{m=1}^{\infty} (-1)^{m+1} \cdot 2a^2 \cdot \times (\#)}$$

$$= \frac{3R^5}{5}$$

Which is same as before.

Ans

Q.4

a) I'm deriving the result from the basic of quantum scattering theory:

Like the classical scattering; let a particle is moving towards the scattering centre (the direction is $\hat{z}$) and hence the incoming wave is given by:

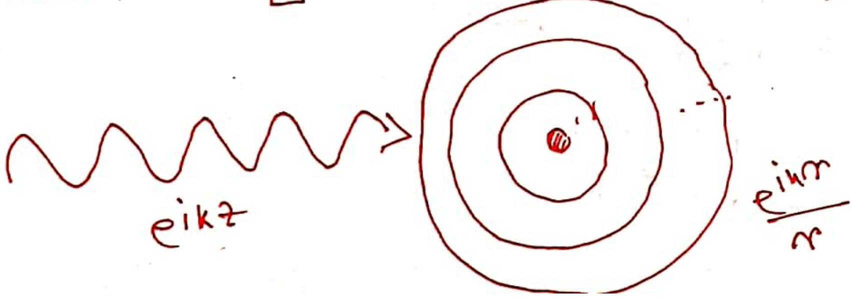
$$\psi_i = A e^{ikz} \quad \text{(no scattering)}$$

Now after scattering we assume the scattered wave will be some outgoing spherical wave with the amplitude a θ dependent form. But as the potential is azimuthally symmetric, there will be no ϕ term.

i.e $$\psi_{scatt} \sim \frac{f(\theta)\, e^{ikr}}{r} \quad (r \gg 0)$$

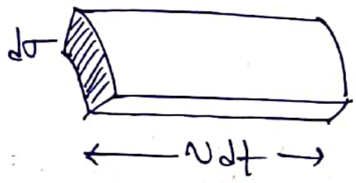
So as a whole the observer will see the ~~wa~~ particle as a superposition of ψ_i & ψ_{scatt} & we get

$$\psi(r,\theta) = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right] \quad (r \gg 0)$$



Now if we find the particle in a volume dV with ~~impact~~ cross section $d\sigma$ along the incident direction and the particle velocity is v then for a time instant dt; we get; the probability

$$dp = |\psi_i|^2 dV = |A|^2 v\,dt\, d\sigma$$



And if the wave at this region scatters at a solid angle $d\Omega$; then the probability to find it after scattering in that region is:

$$dp' = |\psi_{scatt}|^2 dV = \frac{|A|^2 |f(\theta)|^2 v' dt}{r^2} \times r^2 d\Omega$$

as we assume $r >> 0$ so the velocity will be equal i.e $v = v'$ as there will be little impact of the potential on the particle. i.e. $v = v'$

And by conservation of probability $dp = dp'$

i.e. $$d\sigma = |f(\theta)|^2 d\Omega$$

i.e $$\frac{d\sigma}{d\Omega} = |f(\theta)|^2.$$

Now; as the wave fn is separable for a radial potential we assume

$$\psi_l^m(r,\theta,\phi) = R(r) \cdot Y_l^m(\theta,\phi)$$

i.e T.I.S.E gives: (with $u = r \cdot R(r)$)

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left(V(r) + \frac{\hbar^2}{2m} \cdot \frac{l(l+1)}{r^2}\right)u = Eu$$

Assuming the potential falling off very fast (faster than $\frac{1}{r^2}$); for $r \gg 0$; it gives:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}u \approx Eu$$

i.e $$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u \approx -k^2u \quad \left(k^2 = \frac{2mE}{\hbar^2}\right)$$

The solutions are spherical Bessel functions

$$u \sim A j_l(kr) + B n_l(kr).$$

Now as at large r; $\psi \sim \frac{e^{ikr}}{r}$; we combine j_l & n_l to construct Hankle function; which gives the same nature.

i.e, $h_l^{(\pm)}(kr) = j_l(kr) \pm i n_l(kr)$

Among them $h_l^{(+)}$ is Outgoing wave at $r \gg 0$ and $h_l^{(-)}$ is ~~into~~ incoming. So we only take the (+) sign.

i.e $\psi_{scatt} \sim \sum_{l,m} A_{l,m} h_l^{(+)}(kr) Y_l^m(\theta, \phi)$

But for only r dependent potential, the scattering is azimuthally symmetric & we get only

$$\psi_{scatt}(r, \theta, \phi) = \psi_{scatt}(r, \theta)$$

by taking $m = 0$.

$\left(\text{as } Y_l^m(\theta, \phi) \approx P_l^m(\cos\theta) e^{im\phi} \right)$

i.e $\psi_{scatt} = \sum_l A_l h_l^{(+)}(kr) P_l(\cos\theta)$

taking $A_l = i^{(l+1)} k \sqrt{4\pi(2l+1)}\, a_l$ we get the whole wave f^n^ as:

$$\psi(r, \theta) = A\left[e^{ikz} + k \sum_{l=0}^{\infty} i^{l+1} (2l+1) a_l h_l^{(+)}(kr) P_l(\cos\theta) \right]$$

~~i.e comparing; we get:~~

$\cancel{f(\theta) =}$

As for large r we get $h_l^{(+)} \sim \frac{(-i)^{l+1} e^{ikr}}{kr}$

we get:

$$\psi = A\left[e^{ikz} + \sum_{l=0}^{\infty} \frac{(2l+1)\, a_l P_l(\cos\theta)\, e^{ikr}}{r}\right]$$

And hence by comparism; the differential ~~cross section~~ amplitude is:

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1)\, a_l P_l(\cos\theta)$$

i.e differential cross section:

$$D(\theta) = |f(\theta)|^2$$

$$= \sum_{l,l'} (2l+1)(2l'+1)\, a_l^* a_{l'} P_l(\cos\theta) P_{l'}(\cos\theta)$$

And total cross section

$$\sigma = \int D(\theta)\, d\Omega = \int \sum_{l,l'} (2l+1)(2l'+1)\, a_l^* a_{l'} P_l P_{l'} \sin\theta\, d\theta\, d\phi$$

$$= \sum_{l,l'} (2l+1)(2l'+1) \times \frac{a_l a_{l'} \times 2}{(2l+1)} \times 2\pi \times \delta_{ll'}$$

Scanned with CamScanner

i.e. $\sigma = 4\pi \sum_{l=0}^{\infty} (2l+1) |a_l|^2$.

Clearly Now we only need a_l to find out the required expression in the question.

Now; we see the problem from different point of view. Let there is some incoming & scattered wave.

If we write the incoming wave in terms of different angular momentum components which is decomposable in terms of ~~radially~~ radial waves; then we assume that each angular momentum component is also ~~reflected~~ scattered back with probability conservation (as the angular momentum is conserved for central potential) as some outgoing wave with some phase shift.

~~Mathemat~~ Mathematically:

$$\left|\psi_l^{incoming}\right|^2 = \left|\psi_l^{scattering}\right|^2$$

i.e. $\psi_l^{scattering} = e^{i\delta_l} \psi_l^{incoming}$

$(\psi_l = \psi_l(r, \theta, \phi))$

So. we must to decompose ψ_l in radially ~~into~~ incoming or outgoing waves.

clearly $\psi_{incoming} = A e^{ikz}$

$$= \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

(expansion form taken from textbook)

$$= \sum_{l=0}^{\infty} i^l (2l+1) \frac{h^{(+)}(kr) + h^{(-)}(kr)}{2} P_l(\cos\theta)$$

$$\approx \sum_{l=0}^{\infty} \frac{i^l (2l+1)}{2} \times \frac{1}{kr}\left\{(-i)^{l+1} e^{ikr} + (i)^{l+1} e^{-ikr}\right\} P_l(\cos\theta)$$

(at $r \gg 0$)

(using asymptotic form of Hankle function)

i.e if $V(r) = 0$ then

$$\psi = \psi_i = \sum_{l=0}^{\infty} \frac{A.(2l+1)}{2ikr}\left(e^{ikr} - (-1)^l e^{-ikr}\right) P_l(\cos\theta)$$

Scanned with CamScanner

In presence of the scattering potential the incoming wave remains same but the outgoing wave takes a phase factor of $2\delta_l$

i.e $\psi \approx \sum \psi_l$

when $\psi_l = \frac{A(2l+1)}{2ikr}\left[e^{i(kr+2\delta_l)} - (-1)^l e^{-ikr}\right] P_l(\cos\theta)$

coming from scattering event; $\delta_l = 0$ if $V=0$

But in terms of a_l the wave fn is written as,

$$\psi_l = A\left[e^{ikz} + \frac{(2l+1)}{r} a_l e^{ikr} P_l(\cos\theta)\right]$$

$$= A\left[\underbrace{\frac{(2l+1)}{2ikr}\left(e^{ikr} - (-1)^l e^{-ikr}\right)} + \underbrace{\frac{(2l+1) a_l e^{ikr}}{r}}\right] P_l(\cos\theta)$$

coming from $V=0$ i.e no scattering

coming from scattering effect.

So we compare the both expressions of ψ_l & get:

$$\frac{2l+1}{2ikr} + \frac{(2l+1)a_l}{r} = \frac{(2l+1)e^{2i\delta_l}}{2ikr}$$

i.e $a_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{e^{i\delta_l}}{k} \sin(\delta_l)$

i.e the cross section (derived earlier)

$$\sigma = 4\pi \sum_{l=0}^{\infty} (2l+1) |a_l|^2$$

$$= \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

for only s wave scattering i.e $l=0$ we get the cross section to be:

$$\sigma_{l=0} = \frac{4\pi}{k} \times (2 \times 0 + 1) \sin^2 \delta_0$$

i.e

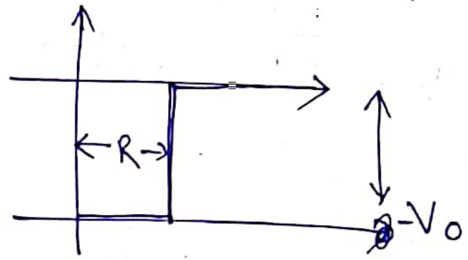
$$\boxed{\sigma = \frac{4\pi \sin^2(\delta_0)}{k}}$$

proved

b) Spin dependence of nuclear potential: →

As we know that nuclear force is short ~~range~~ ranged, when we calculate for Deuteron; we assume that the two body system is in a short central potential & reduce the problem in effective mass problem with the form of the potential as a finite spherical well. i.e.

$$V(r) = -V_0 \quad \text{if } r<R;$$
$$= 0 \quad \text{if } r>R$$



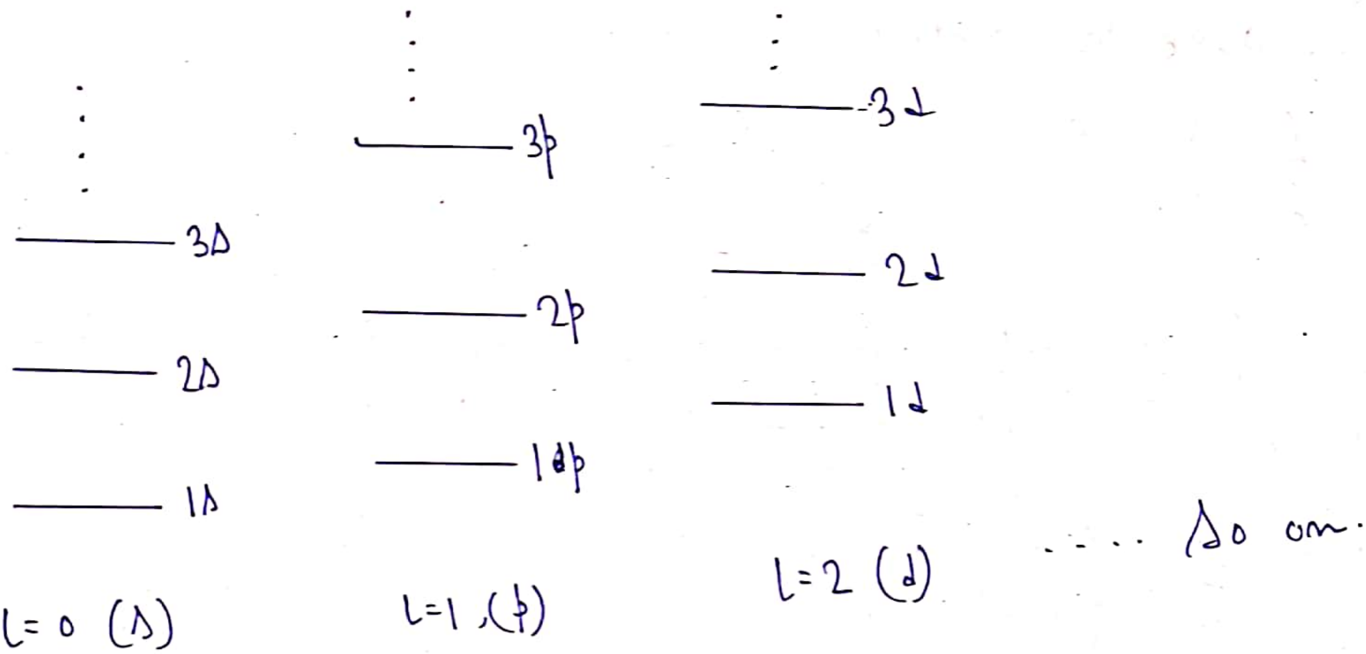
With this; the Radial eq. becomes:

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + \left\{-V_0 + \frac{l(l+1)\hbar^2}{2\mu r^2}\right\}u(r) = Eu; \quad r<R$$

And, $$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + \frac{l(l+1)\hbar^2 u}{2\mu r^2} = Eu \quad ; r>R.$$

The solutions are however takes some complicated form of spherical Bessel functions $(j_L(r))$; we get (for each value of $l = 0, 1, 2, \cdots$) some or no possible bound state for the system.

For one possible value of l; these states are like different principle quantum number (n) states. However the states look like:



However the number of allowed states are dependent on R and V_0; we see that the ground state among all is 1s (i.e $n=1$, $L=0$)

Now in case of Deutron we find that there is weakly only one bound state and it must be the ground state. Deuteron doesn't have any excited state.

Here the ~~Main~~ story comes:

The ~~bound~~ ground state of the potential well is 1s.

i.e Deuteron should have $L = 0$

Now the total spin qm number of neutron and proton ~~is~~ if combined is:

$$s = (s_1 + s_2) \text{ to } |s_1 - s_2| \quad \text{i.e } s = 0, 1$$

So the total nuclear angular ~~m~~ number I can take the values

if $s = 0$; $I = l = 0$

if $s = 1$; $I = 1$

But from experimental data we have found that all bound deuterons are of $I = 1$.

Reversely when $I = 1$ we get the possible l values are (according to addition algebra) $l = 0, 1, 2$.

Among them $l = 0$ is the only bound (ground) state. for $l = 0$ we must $s = 1$ (to ~~mat~~ make $I = 1$) & this would give us 3 different spin states

$$l = 0 \; ; \; s = 1 \; ; \; m_s = \pm 1, 0$$

But as we have assumed the central potential we should get one more $l = 0$ state available which is $l = 0, s = 0, m_s = 0$ & give $I = 0$.

i.e There must be 4 types of ^2H with probabilities

$l = 0$; $s = 1$; $m_s = 1$; $I = 1 \longrightarrow 25\%$

$l = 0$; $s = 1$; $m_s = -1$; $I = 1 \longrightarrow 25\%$

$l = 0$; $s = 1$; $m_s = 0$; $I = 1 \longrightarrow 25\%$

$l = 0$; $s = 0$; $m_s = 0$; $I = 0 \longrightarrow 25\%$

i.e $I = 1 \rightarrow 75\%$ & $I = 0 \rightarrow 25\%$

As this contradicts the experiment ($I = 1 \rightarrow 100\%$)

we conclude that $S = 0$ is not possible and only ~~t=1~~ $S = 1$ contributes to the true bound state of deuteron..

i.e $S = 0$ & $S = 1$ are two different energy states,

So

$\boxed{\text{Nuclear force is non central \& spin dependent}}$

proved

Scanned with CamScanner

Prob: 5

a) $$\vec{\mu} = \vec{\mu_L} + \vec{\mu_s}$$

i.e $$\mu_z = \mu_{Lz} + \mu_{sz}$$

$$= g_l L_z + g_s S_z.$$

i.e $$\langle \mu_z \rangle = g_l \langle L_z \rangle + g_s \langle S_z \rangle$$
$$= g_l m_l + g_s m_s. \qquad (\hbar = 1)$$

So after the angular momentum addition we go to $|l, s, m_l, m_s\rangle \longrightarrow |j, l, s, m_j\rangle$ basis & we get similarly

$$\langle \mu_z \rangle = g_j m_j$$ (not $g_j j$ that given in question)

Now we need to derive the expression of g_j as given.

We start with:

$$g_l \vec{L} + g_s \vec{S} = g_j \vec{J}$$

i.e $$g_l (\vec{J} - \vec{S}) + g_s \vec{S} = g_j \vec{J}$$

i.e. $$g_l \vec{J} + (g_s - g_l) \vec{S} = g_j \vec{J}$$

i.e. $$g_l \vec{J}\cdot\vec{J} + (g_s - g_l) \vec{S}\cdot\vec{J} = g_j \vec{J}\cdot\vec{J}$$

Now, $\vec{L} + \vec{S} = \vec{J}$ i.e $\vec{J} - \vec{S} = \vec{L}$

i.e $(\vec{J} - \vec{S})^2 = L^2 = J^2 + S^2 - 2\vec{S}\cdot\vec{J}$

i.e. $\langle(\vec{S}\cdot\vec{J})\rangle = \dfrac{\langle J^2 + S^2 - L^2\rangle}{2} = \dfrac{j(j+1) + s(s+1) - l(l+1)}{2}$

So $g_l \langle J^2\rangle + (g_s - g_l)\langle \vec{S}\cdot\vec{J}\rangle = g_j \langle J^2\rangle$

i.e. $g_l\, j(j+1) + (g_s - g_l)\dfrac{j(j+1) + s(s+1) - l(l+1)}{2} = g_j\, j(j+1)$

i.e. $g_j = g_l + (g_s - g_l)\dfrac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}$

Now for $j = l + \frac{1}{2}$; (and putting $s = 1/2$):

$$g_j = g_l + \frac{(g_s - g_l)}{2}\left[1 + \frac{\frac{3}{4} - l^2 - l}{(l+\frac{1}{2})(l+\frac{3}{2})}\right]$$

$$= g_l + \frac{g_s - g_l}{2}\left[1 - \frac{(l-\frac{1}{2})\cancel{(l+3/2)}}{(l+\frac{1}{2})\cancel{(l+3/2)}}\right]$$

$$= g_l + \frac{g_s - g_l}{2} \times \frac{1}{(l+\frac{1}{2})} = g_l + (g_s - g_l)\cdot\frac{1}{2l+1}$$

Similarly for $j = l - 1/2$ we get: $g_j = g_l - (g_s - g_l)\cdot\dfrac{1}{2l+1}$

So we get the required result: $\boxed{g_j = g_l \pm \dfrac{g_s - g_l}{2l+1}}$ $(j = l \pm 1/2)$

proved

Scanned with CamScanner

b) $^{17}_{8}O_9$: $j = \frac{5}{2}$; $\pi = +1$ for last unpaired n^0

neutron config: $\frac{\uparrow\downarrow}{1s_{1/2}}$ $\frac{\uparrow\downarrow\ \uparrow\downarrow}{1p_{3/2}}$ $\frac{\uparrow\downarrow}{1p_{1/2}}$ $\frac{\uparrow}{1d_{5/2}}$

i.e. $l = 2$; And hence we take $j = \frac{5}{2} = 2 + \frac{1}{2} = l + 1/2$

All protons are paired.

i.e. $\mu_z = j\left(g_l + \frac{(g_s - g_l)}{2l+1}\right)\mu_N$

$g_s = -3.82$; $g_l = 0$ for n^0

i.e using values: $\mu_z = \frac{\cancel{5}}{2}\left(\frac{-3.82}{\cancel{5}}\right)\mu_N = -1.91\ \mu_N$

from NNDC : $\mu_N = -1.89\ \mu_N$.

$^{17}_{9}F_8$; neutrons are paired.

proton config: $\frac{\uparrow\downarrow}{1s_{1/2}}$ $\frac{\uparrow\downarrow\ \uparrow\downarrow}{1p_{3/2}}$ $\frac{\uparrow\downarrow}{1p_{1/2}}$ $\frac{\uparrow}{1d_{5/2}}$

Similarly $j = l + 1/2$; $g_l = 1$; $g_s = 5.58$ for proton

i.e. $\mu_z = \frac{5}{2}\left(1 + \frac{4.48}{5}\right)\mu_N = 4.79\ \mu_N$

from NNDC : $\mu_z = 4.7223\ \mu_N$.

Prob: 6

Let's take $\omega_1 = \omega_2 = \omega$; $\omega_3 = \Omega$

Now TISE gives:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{m(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)}{2} = E\psi$$

i.e $$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{m\omega^2}{2}(x^2+y^2)\psi + \frac{m\Omega^2 z^2}{2}\psi = E\psi$$

In cylindrical coordinate system; we get:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

$$+ \frac{m\omega^2\rho^2}{2}\psi + \frac{m\Omega^2 z^2\psi}{2} = E\psi.$$

Letting $\psi = R(\rho)\,T(\phi)\,Z(z)$; we get:

$$-\frac{\hbar^2}{2m}\left(TZ\frac{d^2R}{d\rho^2} + \frac{TZ}{\rho}\frac{dR}{d\rho} + \frac{RZ}{\rho^2}\frac{d^2T}{d\phi^2} + RT\frac{d^2Z}{dz^2}\right)$$

$$+ \frac{m\omega^2\rho^2}{2}RTZ + \frac{m\Omega^2 z^2 RTZ}{2} = ERTZ$$

i.e. $$-\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 T}\frac{d^2T}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right)$$

$$+ \frac{m\omega^2\rho^2}{2} + \frac{m\Omega^2 z^2}{2} = E$$

Taking the separation of variables, we get,

$$-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{m\Omega^2 z^2}{2}=E_z \quad \cdots\cdots (1)$$

And

$$-\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d^2R}{d\rho^2}+\frac{1}{\rho R}\frac{dR}{d\rho}+\frac{1}{\rho^2 T}\frac{dT}{d\phi^2}\right)+\frac{m\omega^2\rho^2}{2}=E_{\rho\phi}$$
$$\cdots\cdots (2)$$

Now eq (1) is 1-d Harmonic oscillator eq for z; with the solution known (not deriving the same 1-d L.H.O story here) as:

$$\boxed{Z(z)=\frac{1}{\pi^{1/4}\times\sqrt{2^{n_z}\cdot n_z!\, z_0}}H_{n_z}\left(\frac{z}{z_0}\right)\exp\left(-\frac{z^2}{2z_0^2}\right)}$$

(when $n_z = 0,1,2,\cdots$
And $z_0=\sqrt{\frac{\hbar}{m\Omega}}$; H_n = n'th order Hermite polynomial)

Eq (2) gives:

$$-\frac{\hbar^2}{2m}\left(\frac{\rho^2}{R}\frac{d^2R}{d\rho^2}+\frac{\rho}{R}\frac{dR}{d\rho}+\frac{1}{T}\frac{dT}{d\phi^2}\right)+\left(\frac{m\omega^2\rho^2}{2}-E_{\rho\phi}\right)\rho^2=0$$

We further take the separation to get:

$$-\frac{\hbar^2}{2m}\frac{1}{T}\frac{d^2T}{d\phi^2}=\alpha^2 \quad \cdots (3)$$

i.e. $$\frac{d^2T}{d\phi^2}=-\frac{2m\alpha^2}{\hbar^2}T=-K^2T$$

i.e. $T(\phi) = A_{\pm}^{(k)} e^{\pm ik\phi}$

clearly $\psi(\phi) = \psi(\phi + 2\pi)$

i.e. $2\pi k = 2\pi \times$ some integer.

i.e $k =$ some integer. $= 0, 1, 2, \ldots$

(don't need negative one as $-k$ will serve the purpose)

And we left with ρ eq:

$$-\frac{\hbar^2}{2m}\left(\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho}\right) + \left(\frac{m\omega^2\rho^2}{2} - E_{\rho\varphi}\right)\rho^2 = -\alpha^2$$

Scanned with CamScanner

i.e.

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \frac{2m}{\hbar^2}\left(E_\rho - \frac{m\omega^2\rho^2}{2}\right)R - \frac{K^2R}{\rho^2} = 0$$

or

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \frac{2m}{\hbar^2}\left(E_\rho - \frac{m\omega^2\rho^2}{2} - \frac{\hbar^2\alpha^2}{2m\rho^2}\right)R = 0$$

..... (3)

If $\rho \to \infty$; we get:

$$\frac{d^2R}{d\rho^2} \approx \frac{m^2\omega^2\rho^2}{\hbar^2}R$$

the approximate solution in this limit is:

$$R \sim e^{-\frac{m\omega^2\rho^2}{2\hbar^2}}$$

and at $\rho \to 0$: $\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{\alpha^2R}{\rho^2} = 0$

~~taking $R \sim \rho$; we get:~~

i.e. $\rho^2\frac{d^2R}{d\rho^2} + \frac{\rho^2}{\rho}\frac{dR}{d\rho} - \alpha^2R = 0.$

taking $R \sim \rho^\gamma$ we get:

$$\gamma(\gamma-1)\rho^\gamma + \gamma\rho^\gamma - \alpha^2\rho^\gamma = 0$$

i.e. $\gamma^2 - \gamma + \gamma - \alpha^2 = 0$

i.e. $\gamma = \pm\alpha.$

$\therefore$ the acceptable solution is $R \sim \rho^\alpha$ (i.e. $\gamma = +\alpha$)

So we take the form of the soln:

$$R \sim \rho^{\gamma} e^{-\frac{m\omega\rho^2}{2\hbar}} \cdot h(\rho)$$

let $h(\rho)$ takes the series form $h(\rho) = \sum_k a_k \rho^k$ and putting in (3)

Evaluating values of R'', R' (done in rough; I'm just putting final result) gives:

$$\sum_k a_k \rho^{(k+\alpha)} e^{-\frac{m\omega\rho^2}{2\hbar}} \left[\frac{m\omega^2\rho^2}{\hbar^2} - \frac{m\omega}{\hbar}(2k+2\alpha+1) + (k+\alpha)(k+m-1)\frac{1}{\rho^2} - \frac{m\omega}{\hbar} + (k+\alpha)\frac{1}{\rho^2} + \frac{2\alpha E\rho m}{\hbar^2} - \frac{\alpha^2}{\rho^2} - \frac{m\omega^2\rho^2}{\hbar^2} \right] = 0$$

i.e. $$\sum_k a_k \rho^{(k+\alpha)} e^{-\frac{m\omega\rho^2}{2\hbar}} \left[\frac{2\alpha E\rho m}{\hbar^2} - \frac{2m\omega}{\hbar}\left(2k+2\alpha+1\right) \right]$$

$$+ \sum_k a_{k+2} \rho^{k+\alpha} e^{-\frac{m\omega\rho^2}{2\hbar}} \left((k+\alpha+2)^2 - \alpha^2 \right) = 0$$

equating the coefficient of $\rho^{k+\alpha}$ gives:

$$a_{k+2} = \frac{\frac{2m\omega}{\hbar}(2k+2\alpha+1) - \frac{2\alpha E\rho m}{\hbar^2}}{(k+\alpha+2)^2 - \alpha^2} a_k .$$

for even & odd k we get two different recursion.
i.e even or odd series of $h(k)$.

Now at large k

$$a_{k+2} \sim \frac{2m\omega}{\hbar} \times \frac{2}{k} \quad \text{i.e.} \sim \left(\frac{2m\omega}{\hbar}\right)^{k/2} \times \frac{a_{initial}}{(k/2)!}$$

$$\text{i.e } h(\rho) \sim \sum \frac{1}{(k/2)!} \left(\frac{2m\omega}{\hbar}\right)^{k/2} \rho^k$$

$$\sim \sum \frac{1}{k!} \left(\frac{m\omega\rho^2}{\hbar}\right)^{k}$$

$$\sim e^{+\frac{m\omega\rho^2}{\hbar}}$$

$$\text{i.e } R(\rho) \sim e^{+\frac{m\omega\rho^2}{2\hbar}} \longrightarrow \infty$$

So the series must end at some maximum $K = \tilde{K}$

So that:

$$\frac{\cancel{2}m\omega}{\cancel{\hbar}} (2\tilde{k} + 2\alpha + 1) - \frac{\cancel{2}\alpha E_p \cancel{m}}{\hbar \cancel{2}} = 0$$

$$\text{i.e. } E_p = (2\tilde{K} + 2\alpha + 1)$$

The solution is not however easy.
After finding in some paper I came to know that the solution of the radial eq (eq (3)) is called ~~associated~~ Generalized Laguerre polynomial. which will take the same form I derives.

Scanned with CamScanner

With the proper normalization constant taken from the paper (given in reference); the solution is:

$$R_{nl}(\rho) = \sqrt{\frac{2p!}{(p+|l|)!} \times \frac{m\omega}{\hbar}} \times \left(\rho\sqrt{\frac{m\omega}{\hbar}}\right)^{|l|} \cdot e^{-\frac{m\omega\rho^2}{2\hbar}} \times L_p^l\left(\frac{m\omega\rho^2}{2\hbar}\right)$$

when $p = \frac{n-|l|}{2}$

L_p^l is the generalized Laguerre polynomial.

So collecting all; the solution looks like:

$$\boxed{\psi(\rho,\phi,z) = A H_{n_z}\left(\frac{z}{z_0}\right) e^{-\left(\frac{z^2}{2z_0^2}\right)} \times \rho^{|l|} e^{-\frac{m\omega\rho^2}{2\hbar}} L_p^l\left(\frac{m\omega\rho^2}{\hbar}\right) e^{\pm il\phi}}$$

(A = some normalization constant)

The lowest order state $\psi_{ground} \sim e^{-\rho^2/2} \cdot e^{-\frac{z^2}{2}}$

i.e a cylindrically symmetric Gaussian density falling off

Ans

Prob: 2

Given $S_{12} = \left(V_0(r) + V_1(r)\hat{\tau}\cdot\hat{\tau}'\right)\left[\frac{(\hat{r}\cdot\hat{\sigma})(\hat{r}\cdot\hat{\sigma}')}{r^2} - \frac{\hat{\sigma}\cdot\hat{\sigma}'}{3}\right]$

The first bracketed part is not dependent on θ, ϕ; i.e it's a radially symmetric fn & hence we don't even need to find the integral over whole space. ~~It is zero by symmetry.~~ The term itself will go out of the integral for angular coordinates.

So ~~we~~ we need the avg for the 2nd term.

Now, $\frac{(\hat{r}\cdot\hat{\sigma})(\hat{r}\cdot\hat{\sigma}')}{r^2} = \frac{(x\hat{\sigma}_x + y\hat{\sigma}_y + z\hat{\sigma}_z)(x\hat{\sigma}_x' + y\hat{\sigma}_y' + z\hat{\sigma}_z')}{r^2}.$

i.e. 2nd term becomes:

$$= \left(\frac{x^2}{r^2} - \frac{1}{3}\right)\hat{\sigma}_x\hat{\sigma}_x' + \left(\frac{y^2}{r^2} - \frac{1}{3}\right)\hat{\sigma}_y\hat{\sigma}_y' + \left(\frac{z^2}{r^2} - \frac{1}{3}\right)\sigma_z\sigma_z'$$

$$+ \text{ cross terms like } \left(\frac{xy}{r^2}, \frac{yz}{r^2}, \ldots\right) \times \sigma \text{ operators.}$$

Now $$\int_{\text{fixed } r} \frac{xy}{r^2}\, d\theta\, d\phi = \int_{\text{fixed } r} \frac{yz}{r^2}\, d\theta\, d\phi = \int_{\text{fixed } r} \frac{zx}{r^2}\, d\theta\, d\phi = 0$$

(as $x \to -x$ (or $y \to -y$); the integral changes sign) i.e the avg 2nd term reduces:

$$\langle S_{12}\rangle_{\theta,\phi} \propto \sum_{i=1}^{3}\left(\frac{\langle x_i^2\rangle}{r^2}-\frac{1}{3}\right)\sigma_{x_i}\sigma'_{x_i}\Big|_{\text{avg over all } \theta,\phi}$$

now due to symmetry $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle$

but $\langle x^2+y^2+z^2\rangle_{\text{angular}} = \sum\langle x_i^2\rangle_{\phi,\theta} = \langle r^2\rangle_{\theta,\phi} = r^2$

i.e $\langle x^2\rangle_{\theta,\phi} = \langle y^2\rangle_{\theta,\phi} = \langle z^2\rangle_{\theta,\phi} = \frac{r^2}{3}$

i.e $\frac{\langle x^2\rangle_{\theta,\phi}}{r^2} = \frac{\langle y^2\rangle_{\theta,\phi}}{r^2} = \frac{\langle z^2\rangle_{\theta,\phi}}{r^2} = \frac{1}{3}$

$$\text{i.e. } \langle S_{12}\rangle_{\theta,\phi} \propto \sum_{i=1}^{3}\underbrace{\left\{\frac{\langle x_i^2\rangle_{\theta,\phi}}{r^2}-\frac{1}{3}\right\}}_{=0}\sigma_{x_i}\sigma'_{x_i}$$

i.e $\boxed{\langle S_{12}\rangle_{\theta,\phi} = 0}$ proved.

Prob: 9

The β^- & β^+ decay modes are given by,

$$\left.\begin{array}{l} A(Z,N) \rightarrow A(Z+1, N-1) + \beta^- + \overline{\nu_e} \\ A(Z,N) \rightarrow A(Z-1, N+1) + \beta^+ + \nu_e \end{array}\right\}$$

Taking ν_e & $\overline{\nu_e}$ as massless: we find the mass difference or Q_β factor of the required reactions as:

$$Q_{\beta\pm} = M(Z,N) - M(Z\pm1, N\mp1) - m_e$$

Now, for ${}^{128}I$; $\beta\pm$ decay

$${}^{128}_{52}I \longrightarrow {}^{128}_{52}Te + \beta^+ + \nu_e; \quad {}^{128}_{53}I \rightarrow {}^{128}_{54}Xe + \beta^- + \overline{\nu_e}$$

$$Q_{\beta+} = M\left({}^{128}_{53}I\right) - M\left({}^{128}_{52}Te\right) - M_e$$

$$= \left(127.905 - 127.904 - 5.48\times10^{-4}\right)u$$

$$> 0$$

$$Q_{\beta-} = M\left({}^{128}_{53}I\right) - M\left({}^{128}_{54}Xe\right) - M_e$$

$$= \left(127.905 - 127.903 - 5.49\times10^{-4}\right)u$$

$$> 0$$

So both $\beta\pm$ are favourable for ${}^{128}_{53}I$ from energy perspective.

for $^{137}_{55}Cs$:

$^{137}_{55}Cs \longrightarrow {}^{137}_{54}Xe + \beta^{+} + \nu_e$ | $^{137}_{55}Cs \longrightarrow {}^{137}_{56}Ba + \beta^{-} + \overline{\nu_e}$

$$Q_{\beta^+} = M\left({}^{137}_{55}Cs\right) - M\left({}^{137}_{54}Xe\right) - M_e$$

$$= \left(136.907 - 136.911 - 5.48\times10^{-4}\right)u$$

$$< 0$$

And $$Q_{\beta^-} = M\left({}^{137}_{55}Cs\right) - M\left({}^{137}_{56}Ba\right) - M_e$$

$$= \left(136.907 - 136.905 - 5.48\times10^{-4}\right)u.$$

$$> 0$$

i.e. for $^{137}_{55}Cs$: β^+ is not possible

but β^- decay is possible.

From NNDC data we also check this:

	Q_{β^+}	Q_{β^-}
^{128}I	2123.4 kev (possible) 93.1% decay mode.	234.4 kev (possible) 6.9% decay mode
^{137}Cs	− 5184.43 kev (not possible) 0% decay mode.	1175.63 kev (possible) 100% decay mode

Ans

Prob: 7

a) $^{3}_{1}H$:

$p: \underline{\uparrow}\ 1S_{1/2}$

$n: \underline{\uparrow\downarrow}\ 1S_{1/2}$

Spin = 1/2

parity = $(-1)^0 = +1$

b) $^{3}_{2}He$

$p: \underline{\uparrow\downarrow}\ 1S_{1/2}$

$n: \underline{\uparrow}$

Spin = 1/2

parity = $(-1)^0 = +1$

c) $^{9}_{4}Be \longrightarrow$ even even nucleus i.e $p\ \underline{\uparrow\downarrow\ \ \uparrow\downarrow}$

$n\ \underline{\uparrow\downarrow\ \ \uparrow\downarrow}$

$1S_{1/2}\ \ 1p_{3/2}$

i.e Spin = 0

parity = +1

d) $^{11}_{5}B$: $p: \underline{\uparrow\downarrow}\ \ \underline{\uparrow\downarrow\ \uparrow}$

$n: \underline{\uparrow\downarrow}\ \ \underline{\uparrow\downarrow\ \uparrow\downarrow}$

$1S_{1/2}\ \ 1p_{3/2}$

Spin = 3/2

parity = $(-1)^1 = -1$

e) $^{13}_{6}C$: $p: \underline{\uparrow\downarrow}\ \ \underline{\uparrow\downarrow\ \uparrow\downarrow}$

$n: \underline{\uparrow\downarrow}\ \ \underline{\uparrow\downarrow\ \uparrow\downarrow}\ \ \underline{\uparrow}$

$1S_{1/2}\ \ 1p_{3/2}\ \ 1p_{1/2}$

Spin = 1/2

parity = $(-1)^1 = -1$

f) $^{43}_{20}Ca$: proton number is even.

neutron is filled upto $1d_{3/2}$ i.e 20 n°

after that: $n\ \underline{\uparrow\downarrow\ \uparrow}$

$1f_{7/2}$

Spin: 7/2

parity: $(-1)^3 = -1$

g) $^{123}_{50}Sn_{73}$: proton number is even

neutron is filled upto $3S_{1/2}$ (70 n°)

Then: n° $\underline{\uparrow\downarrow\ \uparrow}$ $\longrightarrow$ Spin: 4/2

$1h_{11/2}$ parity: $(-1)^5 = -1$

h) ${}^{116}_{49}In_{67}$: n: $\underline{\uparrow\downarrow\ \uparrow}$ $2d_{3/2}$

p: $\underline{\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow}$ $1g_{9/2}$

Spin: $\left(\frac{9}{2}-\frac{3}{2}\right)$ to $\left(\frac{9}{2}+\frac{3}{2}\right)$

i.e 3, 4, 5, 6

parity: $(-1)^2 \times (-1)^4 = +1$

i) ${}^{152}_{63}Eu_{89}$: n: $\underline{\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow}$ $1h_{9/2}$

p: $\underline{\uparrow\downarrow\ \uparrow\downarrow\ \uparrow}$ $2d_{5/2}$

Spin: $\left(\frac{9}{2}-\frac{5}{2}\right)$ to $\left(\frac{9}{2}+\frac{5}{2}\right)$

i.e 2, 3, 4, 5, 6, 7

parity: $(-1)^2 \times (-1)^5 = -1$

j) ${}^{197}_{79}Au_{118}$ n: even number

p: $\underline{\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow}$ $1h_{11/2}$

Spin: $11/2$

π : $(-1)^5 = -1$

k) ${}^{207}_{82}Pb_{125}$ p: even number

n: $\underline{\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow\ \uparrow}$ $1i_{3/2}$

Spin: $3/2$

π : $(-1)^6 = +1$

l) ${}^{252}_{98}Cf_{154}$: even-even

Spin: 0

π : +1

(Shell structure taken from online.)

Q:8

The gs of $^{17}_{9}F_{8}$ has proton config: $1d_{5/2}$
The next shells are: $1d_{3/2}$, $2S_{1/2}$, $1f_{7/2}$.

i.e

State	My calculation	NNDC
1st	$s = \frac{1}{2}$, $\pi = +1$	$1/2$ +
2nd	$s = 3/2$, $\pi = -1$	$1/2$ −
3rd	$s = 7/2$; $\pi = -1$	$5/2$ −

for $^{41}_{20}Ca_{21}$: gs filled up proton with 20 p upto $1d_{3/2}$.
neutron: $1d^4_{3/2}\, 1f^1_{7/2}$

next excited states:

State	calc	NNDC
1st	$3/2$; -1	$3/2$, -1
2nd	$5/2$; -1	$5/2$, -1
3rd	$1/2$; -1	$3/2$, -1

for $^{209}_{83}Bi_{126}$:

State	calc	NNDC
1st	$7/2$, -1	$7/2$, -1
2nd	$13/2$, $+1$	$13/2$, $+1$
3rd	$3/2$, -1	$1/2$, $+1$

Q:10

$$^{238}U + n^0_1 \longrightarrow {}^{239}U$$

neutron energy $= M(^{239}U) - M(^{238}U) - M(n^0_1)$

Now from NNDC:

$\frac{BE}{A}$ for $^{238}U = 7.57$ Mev

for $^{239}U = 7.56$ Mev.

i.e

$$M(^{238}U) = -7.57 \times 238 + 92 M_p + 146 M_n$$

$$M(^{239}U) = 92 M_p + 147 M_n - 7.56 \times 239$$

i.e $E_{(n)} = (7.57 \times 238 - 7.56 \times 239)$ Mev

$= -5.18$ Mev.

~~Clearly this means the reaction is not possible~~
~~or if we give the neutron a huge K.E~~

This means we should not need any K.E for the neutron for the reaction. We will get some excess energy rather.

But the activation energy for the fission is 6.2 Mev (from online data)

i.e req energy $= (6.2 - 5.18)$ Mev

$= 1.02$ Mev.

Now ^{239}U is not stable & goes:

$$^{239}U \longrightarrow {}^{239}Pb^{*} + 2\beta^{-}$$

$\downarrow$ fission

$$X + Y + n^{\circ}\ (5.93\ \text{Mev})$$

i.e the reaction should sustain as a chain reaction.

$$^{208}_{82}Pb_8 + n \longrightarrow {}^{209}Pb$$

$$E_{(n)} = BE_{^{208}Pb} - BE_{^{209}Pb}$$

$$= (7.867 \times 208 - 7.848 \times 209)\ \text{Mev}$$

$$= -3.896\ \text{Mev}.$$

I haven't found out the activation energy of the ~~neutron~~ reaction.

If Act. En $>$ 3.896 Mev; the n° K.E > 0

if Act En $<$ 3.896 Mev; the n° K.E $= 0$

Ans

Scanned with CamScanner