Quantum Assignment (Scattering)

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Problem 1 :

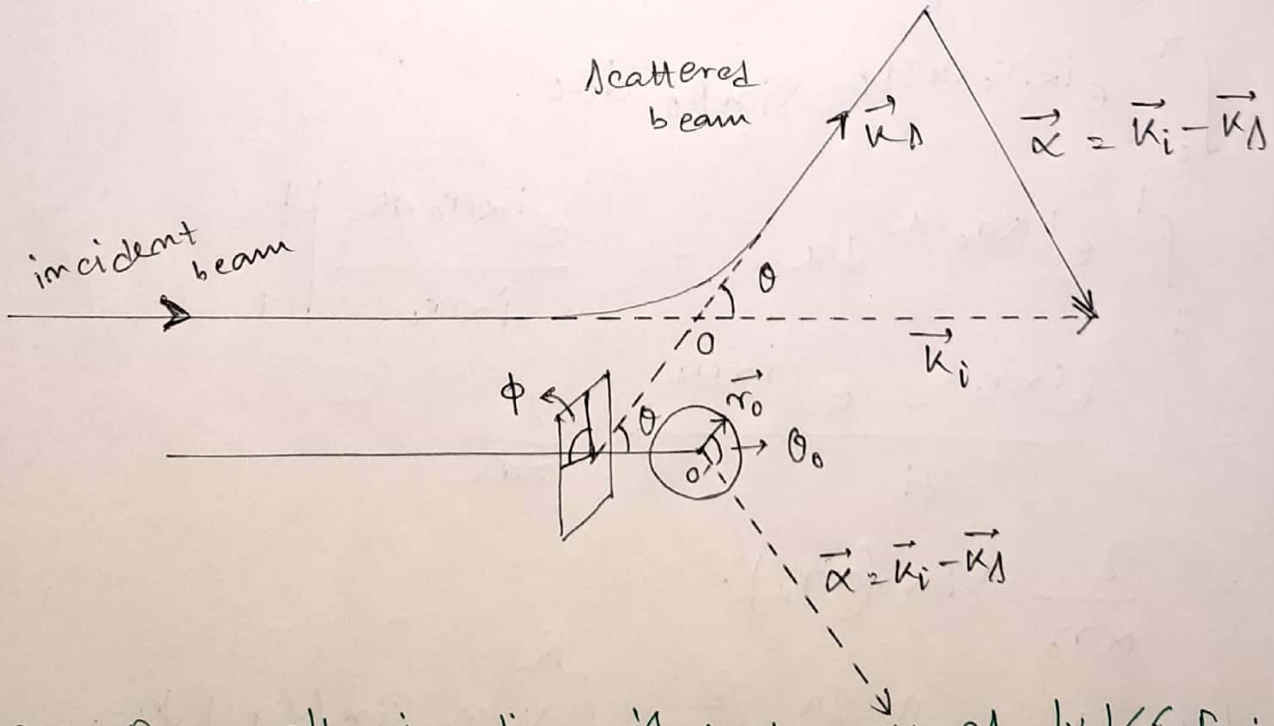
Soft sphere potential →

The given potential is : $V(r) = 0 \quad r > R$
$\qquad\qquad\qquad\qquad\quad = V_0 \quad r \leq R$

Clearly the potential is spherically symmetric.
Now from Born approximation we get for an incident wave of wavevector $\vec{k}_i = k\hat{z}$ & scattered wave $\vec{k}_s = k\hat{r}$; the scattering amplitude be given by:

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i(\vec{k}_i - \vec{k}_s)\cdot\vec{r}_0}\, V(\vec{r}_0)\, d^3\vec{r}_0$$

when $\vec{r}_0$ is measured from scatterer centre.



(Here Born approximation is ~~val~~ valid as $|V_0| << E$; i.e the incident wave dose'nt change much in scattering region.)

(and we can take $\psi(\vec{r_0}) = \psi_0(\vec{r_0})$ in Born appx.)

Letting $\vec{\alpha}(\theta,\phi) = \vec{k_i} - \vec{k_f}$; from fig we get

$|\vec{\alpha}| = 2k \sin(\theta/2)$

And if the polar angle for $\vec{r_0}$ from the origin be measured as the angle (θ_0) with $\vec{\alpha}$ then from fig:

$(\vec{k_i} - \vec{k_f}) \cdot \vec{r_0} = \alpha r_0 \cos\theta_0$.

So the integral gives:

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\alpha r_0 \cos\theta_0} \cdot V(r_0) \cdot r_0^2 \sin\theta_0 \, dr_0 \, d\theta_0 \, d\phi_0.$$

as $V(r_0) = 0$ for $r_0 > R$ so:

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int_0^R V_0(r_0) r_0^2 dr_0 \cdot \int_0^\pi e^{i\alpha r_0 \cos\theta_0} \sin\theta_0 d\theta_0 \int_0^{2\pi} d\phi_0.$$

The ϕ_0 integral gives 2π.

$$\int_0^\pi e^{i\alpha r_0 \cos\theta_0} \sin\theta_0 \, d\theta_0.$$

$$= \int_{-1}^{1} e^{i\alpha r_0 x} dx = \left.\frac{e^{i\alpha r_0 x}}{i\alpha r_0}\right|_{-1}^{1}$$

$$= \frac{e^{i\alpha r_0} - e^{-i\alpha r_0}}{2i} \times \frac{2}{\alpha r_0}$$

$$= \frac{2}{\alpha r_0} \sin(\alpha r_0).$$

$$\therefore f(\theta,\phi) = -\frac{m\, 2\pi}{2\pi\hbar^2} \cdot \frac{2}{\alpha} \int_0^R V_0 r_0 \sin(\alpha r_0) \, dr_0.$$

i.e $f(\theta,\phi) = -\dfrac{2mV_0}{\hbar^2\alpha}\displaystyle\int_0^R x \sin(\alpha x)\,dx$

$$= -\frac{2mV_0}{\hbar^2\alpha}\left[-\frac{x\cos(\alpha x)}{\alpha}\Big|_0^R + \int_0^R \frac{\cos(\alpha x)}{\alpha}\,dx\right]$$

$$= -\frac{2mV_0}{\hbar^2\alpha}\left[-\frac{R\cos(\alpha R)}{\alpha} + \frac{\sin(\alpha R)}{\alpha^2}\right]$$

Now $\alpha R = 2KR\sin(\theta/2)$ & from given info,
$KR \ll 1$ $\therefore$ $\alpha R \ll 1$. So by Taylor expansion
we get:

$$f(\theta,\phi) \cong -\frac{2mV_0}{\hbar^2\alpha}\left[\frac{1}{\alpha^2}\left(\alpha R - \frac{\alpha^3R^3}{3!}\right) - \frac{R}{\alpha}\left(1-\frac{\alpha^2R^2}{2}\right)\right]$$

$$\simeq -\frac{2mV_0}{\hbar^2\alpha}\left[\cancel{\frac{R}{\alpha}} - \frac{\alpha R^3}{6} - \cancel{\frac{R}{\alpha}} + \frac{\alpha R^3}{2}\right]$$

$$= -\frac{2mV_0}{\hbar^2\cancel{\alpha}}\cdot\frac{\cancel{\alpha}R^3}{2}\cdot\left(1-\frac{1}{3}\right)$$

$$= -\frac{2mR^3V_0}{3\hbar^2}.$$

$\therefore$ The differential cross section is given by:

$$\sigma(\theta,\phi) = \frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{4}{9}\frac{m^2V_0^2R^6}{\hbar^4}.$$

Clearly the differential cross section is isotropic. i.e not dependent on polar and azimuthal directions θ & ϕ. Proved

The total cross section is:

$$\sigma_{total} = \int d\sigma = \int_{\Omega} |f(\theta,\phi)|^2 \, d\Omega$$

$$= \frac{4}{9} \frac{m^2 v_0^2 R^6}{\hbar^4} \underbrace{\int d\Omega}_{\to 4\pi}.$$

$$\text{i.e } \sigma_{tot} = \frac{16\pi}{9} \frac{m^2 v_0^2 R^6}{\hbar^4}$$ proved

(iii) part 2 $\longrightarrow$ As $\alpha = 2k \sin(\theta/2)$ and $E = \frac{\hbar^2 k^2}{2m}$

i.e $k = \frac{\sqrt{2mE}}{\hbar}$ So for a slight increase in energy; we should take at least next higher order approximation in the expansion of $f(\theta,\phi)$ to get the dominant θ dependance of scattering amplitude.

$$\text{Now, } f(\theta,\phi) = - \frac{2mV_0}{\hbar^2 \alpha}\left[-\frac{R}{\alpha}\cos(\alpha R) + \frac{\sin(\alpha R)}{\alpha^2}\right]$$

$$= -\frac{2mV_0}{\hbar^2\alpha}\left[\frac{1}{\alpha^2}\left(\alpha R - \frac{\alpha^3 R^3}{6} + \frac{\alpha^5 R^5}{120}\right) - \frac{R}{\alpha}\left(1 - \frac{\alpha^2 R^2}{2} + \frac{\alpha^4 R^4}{24}\right)\right]$$

(Taking the expansion term upto $O(\alpha^5)$)
so that the max power term in $f(\theta,\phi)$ be $O(\alpha^2)$

$$\therefore f(\theta,\phi) = -\frac{2mV_0}{\hbar^2\alpha}\left[\cancel{\frac{R}{\alpha}} - \frac{R^3\alpha}{6} + \frac{R^5\alpha^3}{120} - \cancel{\frac{R}{\alpha}} + \frac{\alpha R^3}{2} - \frac{\alpha^3 R^5}{24}\right]$$

$$= -\frac{2mV_0}{\hbar^2}\alpha\left[\left(\frac{R^3}{2} - \frac{R^3}{6}\right) + \frac{R^5\alpha^2}{24}\left(\frac{1}{5} - 1\right)\right]$$

$$= -\frac{2mV_0}{\hbar^2}\left[\frac{2R^3}{3} - \frac{4}{5}\times\frac{\alpha^2 R^5}{24}\right]$$

$$= -\frac{2mV_0}{\hbar^2}\left(\frac{2}{3}R^3 - \frac{\alpha^2 R^5}{30}\right)$$

$$\therefore \frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

$$\simeq \frac{4m^2V_0^2}{\hbar^4}\left(\frac{4}{9}R^6 \cancel{+\alpha^4} - 2\frac{\alpha^2 R^5}{30}\cdot\frac{2}{3}R^3\right)$$

(keeping terms upto $O(\alpha^2)$ and neglecting $O(\alpha^4)$; which was also taken in the true expansion of $f(\theta,\phi)$)

$$\therefore \frac{d\sigma}{d\Omega} \simeq \frac{16m^2V_0^2R^6}{9\hbar^4} - \frac{16}{90}\frac{m^2V_0^2R^8}{\hbar^4}\cdot\alpha^2$$

~~but we got $\alpha = 2k\cos$~~

but we got $\alpha = 2k\sin(\theta/2)$.

$$\therefore \alpha^2 = 4k^2\sin^2(\theta/2) = 2k^2(1-\cos\theta).$$

$$\text{So; } \frac{d\sigma}{d\Omega} \simeq \frac{16m^2V_0^2R^6}{9\hbar^4} - \frac{8}{45}\frac{m^2V_0^2R^8}{\hbar^4}\cdot 2k^2(1-\cos\theta)$$

So: $\dfrac{d\sigma}{d\Omega} \simeq$

$$\underbrace{\frac{16\, m^2 v_0^2 R^6}{9\hbar^4}\left(1 - \frac{R^2k^2}{5}\right)}_{A} + \underbrace{\frac{16}{45}\frac{m^2 v_0^2 R^8 k^2}{\hbar^4}}_{B} \cos\theta.$$

i.e $\boxed{\dfrac{d\sigma}{d\Omega} \simeq A + B\cos\theta} \longrightarrow$ proved

We get $\dfrac{B}{A} = \dfrac{R^2k^2}{\cancel{8}\left(5 - R^2k^2\right)}$ (from expressions)

But $E = \dfrac{\hbar^2 k^2}{2m}$; i.e $k^2 = \dfrac{2mE}{\hbar^2}$.

So $\dfrac{B}{A} = \dfrac{2mE}{\hbar^2} \cdot \dfrac{R^2}{\left(5 - \dfrac{2mER^2}{\hbar^2}\right)}$

$$= \frac{R^2}{\left(\dfrac{5\hbar^2}{2mE} - R^2\right)}$$ Answer

Remark

Clearly for $E \to 0$ i.e long wave scattering; $\dfrac{B}{A} \to 0$ i.e $B \ll A$; yeilding the previous result.

problem: 4

From the integral representation of Schrodinger eq we get:

$$\psi(r) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik.|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} V(\vec{r}_1)\psi(\vec{r}_1)\, d^3\vec{r}_1$$

for large $\vec{r}$ comparing the 2nd term we get; the scattering amplitude be given by,

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{k}\cdot\vec{r}'}. V(\vec{r}')\psi(\vec{r}')\, d^3\vec{r}'$$

Now as $e^{i\vec{K}.\vec{r}'}$ is the scattered wave

so we get:

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2}\int \phi(\vec{r}')\, V(\vec{r}').\psi(\vec{r}')\, d^3r'$$

$$\left(\phi(\vec{r}') = e^{i\vec{k}.\vec{r}'}\right)$$
$$\vec{k} = k\hat{r}$$

$$\therefore f(\theta, \phi) = -\frac{m}{2\pi\hbar^2}\langle \phi|V|\psi\rangle$$

for the zeroth order Born approximation we take $|\psi\rangle = |\psi_0\rangle$ = incident wave (which is indeed true for large $\vec{r}$ limit)

So; $$f(\theta, \phi) \cong -\frac{m}{2\pi\hbar^2}\langle \psi_{scattered}|V|\psi_{incident}\rangle$$

for the given situation

$$\psi_{scattered}(\vec{r}, \vec{k}_0, \vec{r}') = e^{i\vec{k}_0\cdot\vec{r}'}\cdot \psi_0(\vec{r}')$$

$\psi_{incident} = e^{i\vec{k}\cdot\vec{r}} \cdot \psi_0(\vec{r}')$

when $\vec{k}_0 = k\hat{r}$; $\vec{k} = k\hat{z}$ (assuming the direction of incidence be $\hat{z}$) and as told; the atom remains in it's ground state before and after scattering process. The ground state be labeled by $\psi_0(\vec{r}')$

$\left(\begin{array}{l}\vec{r} = \text{coordinate of } e^- \text{ ~~frame~~ from lab} \\ \vec{r}' = \quad '' \quad '' \quad \text{atom} \quad '' \quad ''\end{array}\right)$

$$\therefore f(\theta,\phi) = -\frac{m}{2\pi\hbar^2} \int d^3r \, e^{i\vec{q}\cdot\vec{r}} \int d^3r' \, \psi_0^*(\vec{r}') \cdot \left[-\frac{e^2}{r} + \frac{e^2}{|\vec{r}-\vec{r}'|}\right] \psi_0(\vec{r}')$$

$\left(\begin{array}{l}\vec{q} = (\vec{k}-\vec{k}_0) \\ |q| = 2k\sin(\theta/2)\end{array}\right)$

$$\text{Now; } \int d^3\vec{r} \, \frac{e^{i\vec{q}\cdot\vec{r}}}{r} = \int_0^r r\,dr \int_0^\pi e^{iqr\cos\theta} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi \int_0^r 2r\,dr \cdot \left(\frac{\sin qr}{qr}\right) = \frac{4\pi}{q} \int_0^\infty \sin(qr)\,dr$$

$$\text{but } \int_0^\infty \sin(qr)\,dr = \lim_{\lambda\to 0} \int_0^\infty e^{-\lambda r} \sin(qr)\,dr$$

$$= \lim_{\lambda\to 0} \left(\frac{q}{\lambda^2+q^2}\right) = \frac{1}{q} \quad (q \neq 0)$$

$$\therefore \int d^3\vec{r}\, \frac{e^{i\vec{q}\cdot\vec{r}}}{r} = \frac{4\pi}{q^2}.$$

Using this result & using $\int d^3r'\, \psi_0^*(\vec{r}')\, \psi_0(\vec{r}') = 1$ we get finally:

$$f(\theta,\phi) = \frac{m}{2\pi\hbar^2}\left[\frac{4\pi e^2}{q^2} - \int d^3r\, e^{i\vec{q}\cdot\vec{r}} \int d^3\vec{r}'\,.\,\psi_0^*(\vec{r}') \left(\frac{e^2}{|\vec{r}-\vec{r}'|}\right)\cdot \psi_0(\vec{r}')\right]$$

The 2nd integral gives:

$$I = \int d^3r\, e^{i\vec{q}\cdot\vec{r}} \int d^3\vec{r}'\, \psi_0(\vec{r}')\, \frac{e^2}{|\vec{r}-\vec{r}'|}\, \psi_0(\vec{r}')$$

$$= e^2 \int d^3\vec{r}'\, \psi_0^*(\vec{r}')\, \psi_0(\vec{r}')\, e^{i\vec{q}\cdot\vec{r}'} \underbrace{\int d^3\vec{r}\,.\, \frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}}{|\vec{r}-\vec{r}'|}}$$

$$= \int d^3\vec{r}_0\, \frac{e^{i\vec{q}\cdot\vec{r}_0}}{r_0} \qquad (\vec{r}_0 = \vec{r}-\vec{r}')$$

$$= \frac{4\pi}{q^2} \quad \text{(previously evaluated)}$$

$$\therefore I = \frac{4\pi e^2}{q^2} \int d^3r'\, \psi_0(\vec{r}')\, e^{i\vec{q}\cdot\vec{r}'}\, \psi_0(\vec{r}')$$

but $\psi_0(\vec{r}') = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r'/a}$

So, $$I = \frac{1}{\pi a_0^3}\cdot\frac{4\pi e^2}{q^2} \int_0^\infty r^2 e^{-\frac{2r}{a_0}}\, dr \int_0^\pi e^{iqr\cos\theta}\,.\,\sin\theta\, d\theta \int_0^{2\pi} d\phi$$

(dropping the prime sign without loss)

$\therefore$ Using Mathematica the integral gives:

$$I = \frac{4e^2}{a_0^3 q^2} \times 2\pi \times \int_0^\infty 2 \frac{\sin(qr)}{qr} \cdot r^2 e^{-\frac{2r}{a_0}} \, dr.$$

$$= \frac{64 e^2 \pi}{q^2 (4 + q^2 a_0^2)^2}$$

So ultimately:

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2}\left[\frac{4\pi e^2}{q^2} + \frac{64 e^2 \pi}{q^2(4+q^2a_0^2)^2}\right]$$

$$= \frac{2me^2}{\hbar^2 q^2}\left[1 - \frac{16}{(4+q^2a_0^2)^2}\right]$$

$\therefore$ The scattering cross section:

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{4m^2e^4}{\hbar^4 q^4}\left[1 - \frac{16}{(4+q^2a_0^2)^2}\right]^2$$

proved

Problem: 5

Given $V(\vec{r}) = g e^{-\frac{r^2}{a^2}}$

Due to the spherically symmetric nature of V; like problem (1) we using Born approximation we get the scattering amplitude is given by:

$$f(\theta,\phi) = -\frac{2m}{\hbar^2 \alpha}\int_0^\infty r V(r) \sin(\alpha r)\, dr$$

$$\left(\vec{\alpha} = \vec{k}' - \vec{k} \; ; \; |\vec{\alpha}| = \alpha = 2k\sin(\theta/2) \; ; \; E = \frac{\hbar^2 k^2}{2m}\right)$$

$$\therefore f(\theta,\phi) = -\frac{2mg}{\hbar^2\alpha}\underbrace{\int_0^\infty r \sin(\alpha r)\, e^{-\frac{r^2}{a^2}}\, dr}_{I}$$

Using mathematica we get the integral to be:

$$I = \frac{1}{4\cdot\left(\frac{1}{a^2}\right)^{3/2}}\cdot \alpha\sqrt{\pi}\, e^{-\frac{a^2\alpha^2}{4}}$$

So $f(\theta,\phi) = -\frac{2mg}{\hbar^2\alpha} I = -\frac{e^{-\frac{a^2\alpha^2}{4}}\cdot gm\sqrt{\pi}\, a^3}{2\hbar^2}$

Scattering amplitude. $\left(\alpha = 2k\sin\left(\frac{\theta}{2}\right)\right)$

Ans Scattering cross section:

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{a^6 e^{-\frac{a^2\alpha^2}{2}} g^2 m^2 \pi}{4\hbar^4}$$

Ans

Total cross section is given by:

$$\sigma_{tot} = \int |f|^2 d\Omega = \int |f|^2 \sin\theta\, d\theta\, d\phi = 2\pi\int_0^\pi |f|^2 \sin\theta\, d\theta$$

$$= 2\pi\int_0^\pi \frac{a^6 \exp\left(-2a^2k^2\sin^2\left(\frac{\theta}{2}\right)\right) g^2 m^2 \pi}{4\hbar^4}\sin\theta\, d\theta$$

(using $q = 2k\sin(\theta/2)$)

Using mathematica the total cross section becomes

$$\sigma_{total} = \frac{a^4\left(1 - e^{-2a^2k^2}\right)\cdot g^2 m^2 \pi^2}{2k^2\hbar^4}$$

$$= \frac{a^4\left(1 - e^{-\frac{4mEa^2}{\hbar^2}}\right)\cdot g^2 m^2 \pi^2}{4mE\hbar^2}$$

Answer

(using $E = \frac{\hbar^2 k^2}{2m}$)

Problem: 8

For $l=0$ (s wave); the partial wave analysis tells; the wavefunction at the exterior region is given by,

$$\psi_{ext}(\vec{r}) = A\left[j_0(kr) + a_0 \frac{e^{ikr}}{r}\right] \quad (r > a)$$

(from $\psi(r,\theta) = A\sum_{l=0}^{\infty} i^l(2l+1)\left(j_l(kr) + ika_l h_l^{(1)}(kr)\right)P_l(\cos\theta)$
and putting $l=0$.)

i.e $\psi_{ext}(\vec{r}) = A\left[\frac{\sin(kr)}{r} + a_0 \frac{e^{ikr}}{r}\right]$

for internal region $(r<a)$ solving T.I.S.E for $l=0$ gives;

$$\frac{d^2u}{dr^2} = -\frac{2m(E+V_0)}{\hbar^2}u = -\eta^2 u$$

i.e $u = B\sin(\eta r) + C\cos(\eta r)$.

$$\therefore \psi^{l=0}_{int}(\vec{r}) = B\frac{\sin(\eta r)}{r} + C\frac{\cos(\eta r)}{r}.$$

To keep $\psi(0)$ = finite we must have $C=0$.

$$\therefore \psi^{l=0}_{int}(\vec{r}) = B\frac{\sin(\eta r)}{r}.$$

Continuity of ψ at $r=a$ gives:

$$A\left(\frac{\sin(ka)}{ka} + a_0\frac{e^{ika}}{a}\right) = B\frac{\sin(\eta a)}{a} \quad \cdots (1).$$

Continuity of $\psi'(r)$ (or $u'(r)$ as $R = ru$) at $r=a$

gives; $u'_{int}(a) = u'_{ext}(a)$.

i.e $\eta B\cos(\eta a) = A\left(\cos(ka) + i a_0 k e^{ika}\right)$

Using (1) to eliminate A gives:

$$\cos(ka) + i a_0 k e^{ika} = \frac{\eta \alpha}{\sin(\eta a)}\left(\frac{\sin(ka)}{k\alpha} + a_0\frac{e^{ika}}{\alpha}\right).\cos\eta a$$

$$\Rightarrow \frac{\cos(ka)}{\cos(\eta a)} + \frac{i a_0 k e^{ika}}{\cos(\eta a)} = \frac{\eta \sin(ka)}{\sin(\eta a)} + \frac{\eta a_0 e^{ika}}{\sin(\eta a)}.$$

$$\Rightarrow a_0 e^{ika}\left(\frac{\eta}{\sin(\eta a)} - \frac{ik}{\cos(\eta a)}\right) = \frac{\cos(ka)}{\cos(\eta a)} - \frac{\eta \sin(ka)}{\sin(\eta a)}$$

$$\Rightarrow a_0 e^{ika}\left(\eta \cot(\eta a) - ik\right) = \cos(ka) - \frac{\eta}{k}\sin(ka)\cot(\eta a)$$

$$= \cos(ka) - i\sin(ka) - \frac{\eta}{k}\sin(ka)\cot(\eta a) + i\sin(ka).$$

$$= e^{-ika} - \frac{\sin(ka)}{k}\left(\eta \cot \eta a - ik\right)$$

So finally:

$$a_0 = \frac{e^{-2ika}}{\eta \cot(\eta a) - ik} - \frac{\sin(ka).\ e^{-ika}}{k}.$$

but for $l=0$; we get from $f(\theta) = \sum_l (2l+1) \cdot a_l \cdot P_l(\cos\theta)$

that $f_0(\theta) = a_0 =$ Scattering amplitude of S wave.

So;

$$f_0(\theta) = \frac{e^{-2ika}}{\eta \cot(\eta a) - ik} - \frac{e^{-ika}\sin(ka)}{k}$$

$$\left(\eta = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}\right)$$

Proved

Problem: 2

phase shift in spherical delta potential →

The given potential $V(\vec{r}) = \frac{\gamma\hbar^2}{2m}\,\delta(r-R)$

$$= \eta \cdot \delta(r-R) \quad \left(\eta = \frac{\gamma\hbar^2}{2m}\right)$$

For the interior region $(r<R)$ with s wave $(l=0)$ Solving T.I.S.E we get:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu$$

$$\therefore\ u = B\sin(kr) + C\cos(kr).$$

for $l=0$: $Y_l^m(\theta,\phi) = 1$ So $\psi_{int}^{l=0}(\vec{r}) = \frac{u}{r} = \frac{B\sin kr + C\cos kr}{r}.$

Now, $\lim_{r\to 0}\psi(\vec{r}) = \text{finite} \Rightarrow C=0.$

$$\therefore\ \psi_{int}^{l=0}(\vec{r}) = B\frac{\sin(kr)}{r}.$$

Now the required problem can be done with two equivalent methods.

Method - 1 → w.r.t transmitted part

If the phase shift be δ_0, then the exterior wave can be written as:

$$u_{ext}^{l=0}(r) = A\sin(kr+\delta_0)$$

Continuity of ψ at $r=R$ gives:

$$u_{int}(R) = u_{ext}(R)$$

$$\Rightarrow B\sin(kR) = A\sin(kR+\delta_0) \quad \cdots (1)$$

discontinuity of φ' (or u') at $r = R$ gives:

$$-\frac{\hbar^2}{2m}\int_{R-\epsilon}^{R+\epsilon}\frac{d^2u}{dr^2}\,dr + \int_{R-\epsilon}^{R+\epsilon}\eta\cdot\delta(r-R)\cdot u(r)\,dr = \int_{R-\epsilon}^{R+\epsilon}E\cdot u(r)\,dr.$$

at $\epsilon \to 0$ limit this gives:

$$-\frac{\hbar^2}{2m}\Delta u'(R) + \eta\, u(R) = 0.$$

$$\therefore \Delta u'(R) = \frac{2m\eta}{\hbar^2}u(R) \quad \cdots\cdots (2).$$

Using values of u at $r \gtrless R$ we get:

$$AK\cos(KR+\delta_0) - BK\cos(KR) = \frac{2m\eta}{\hbar^2}\times B\sin(KR).$$

$$\Rightarrow A\cos(KR+\delta_0) = B\left(\cos(KR) + \frac{2m\eta}{\hbar^2 K}\sin(KR)\right)$$

Using (1) to eliminate A gives:

$$\cos(KR) + \frac{2m\eta}{\hbar^2 K}\sin(KR) = \frac{\cos(KR+\delta_0)\sin(KR)}{\sin(KR+\delta_0)}$$

$$\Rightarrow \cot(KR+\delta_0) = \cot(KR) + \frac{2m\eta}{\hbar^2 K}.$$

$$\Rightarrow (KR+\delta_0) = \cot^{-1}\left(\cot(KR) + \frac{2m\eta}{\hbar^2 K}\right)$$

i.e. $$\underbrace{\delta_0 = -KR + \cot^{-1}\left(\cot(KR) + \frac{2m\eta}{\hbar^2 K}\right)}_{\text{Answer}}$$

$$\left(\eta = \frac{\gamma\hbar^2}{2m}\right)$$

i.e. $$\delta_0 = -KR + \cot^{-1}\left[\cot(KR) + \frac{\gamma}{K}\right]$$

Method 2 →

Here we first find the partial wave amplitude for $l=0$ (s wave).

~~Let~~ for $R<r$ part the partial wave function for $l=0$ gives the wave fn to be:

$$\psi_{ext}^{l=0}(\vec{r}) = A\left[j_0(kr) + ik a_0 h_0'(kr)\right] P_0(\cos\theta)$$

$$= A\left[\frac{\sin(kr)}{kr} + a_0 \frac{e^{ikr}}{r}\right]$$

Likewise the previous way continuity of ψ at $r=R$ gives: $\psi_{ext}^{l=0}(R) = \psi_{int}^{l=0}(R)$

$\left(\psi_{int}^{l=0}(R) = \frac{B\sin(kR)}{R}\right.$; previously evaluated$\left.\right)$

So; $$A\left[\frac{\sin(kR)}{kR} + a_0\frac{e^{ikR}}{R}\right] = B\frac{\sin(kR)}{R} \quad \cdots (3).$$

And previously we got $\Delta u' = \frac{2m\eta}{\hbar^2} u(R)$

So;

$$A\left[\cos(kR) + ia_0 k e^{ikR}\right] - Bk\cos(kR) = B\sin(kR)\times\frac{2m\eta}{\hbar^2}$$

$$\Rightarrow A\left[\cos(kR) + ia_0 k e^{ikR}\right] = \gamma B\left[\sin(kR) + \frac{k}{\gamma}\cos(kR)\right]$$

Using (3) we get: $\left(\because \eta = \frac{\gamma\hbar^2}{2m},\ \frac{2m\eta}{\hbar^2} = \gamma\right)$

$$\cos(kR) + i a_0 k e^{ikR} = \frac{\gamma}{\sin(kR)} \left(\frac{\sin(kR)}{k} + a_0 e^{ikR} \right) \times \left(\frac{k}{\gamma} \cos(kR) + \sin(kR) \right)$$

$$= \gamma \left(1 + \frac{k}{\gamma} \cot(kR) \right) \left(\frac{\sin(kR)}{k} + a_0 e^{ikR} \right)$$

i.e $$\cancel{\cos(kR)} + i a_0 k e^{ikR} = \frac{\gamma}{k} \sin(kR) + \cancel{\cos(kR)} + a_0 \gamma e^{ikR} + a_0 k e^{ikR} \cot(kR)$$

i.e $$a_0 e^{ikR} \left(\gamma + k\cot(kR) - ik \right) = -\frac{\gamma}{k} \sin(kR).$$

Now from the partial wave - phase shift relationship

$$a_l = \frac{e^{i\delta_l} \sin(\delta_l)}{k} \xrightarrow{l=0} a_0 = \frac{e^{i\delta_0} \sin(\delta_0)}{k}$$

we get:

$$\frac{e^{ikR} e^{i\delta_0}}{k} \sin\delta_0 \left(\gamma + k\cot(kR) - ik \right) = -\frac{\gamma}{k} \sin(kR).$$

Taking modulus from both sides: (assuming $\sin(kR)$ is $-$ve; o/w a minus sign will be extra)

$$\frac{\sin\delta_0}{\cancel{k}} \sqrt{(\gamma + k\cot(kR))^2 + k^2} = -\frac{\gamma}{\cancel{k}} \sin(kR)$$

i.e $$\sin(\delta_0) \left(\gamma^2 + k^2 \operatorname{cosec}^2(kR) + 2\gamma k \cot(kR) \right)^{1/2} = -\gamma \sin kR$$

i.e $\delta_0 = -\sin^{-1}\left[\dfrac{\gamma \sin(KR)}{\sqrt{K^2 + (\gamma + K\cot(KR))^2}}\right]$ ... (4) Ans

This is the same result as previously calculated by method (1). That equivalence can be checked via the inverse fn relations. (except by some additional factor of 2π, ... etc due to the complex phase factor to real modulus conversion.)

b. Hard Sphere :→

For large γ value (i.e when no wave can pass through) we get from eq (4)

$$\delta_0 = -\lim_{\gamma\to\infty} \sin^{-1}\left[\frac{\gamma \sin(KR)}{\gamma\sqrt{\left(\frac{K}{\gamma}\right)^2 + \left(1 + \frac{K}{\gamma}\cot(KR)\right)^2}}\right]$$

$$= -\lim_{\gamma\to\infty} \sin^{-1}\left[\frac{\sin(KR)}{\sqrt{\left(\frac{K}{\gamma}\right)^2 + \left(1 + \frac{K}{\gamma}\cot KR\right)^2}}\right]$$

$$= -\sin^{-1}\sin(KR)$$

i.e $\delta_0 = -KR$

(ii) Which also comes from result of method (1)

$$\delta_0\Big|_{\gamma\to\infty} = \lim_{\gamma\to\infty}\left[-KR + \cot^{-1}\left\{\cot(KR) + \frac{\gamma}{K}\right\}\right]$$

$$= -kR + \lim_{\gamma \to \infty} \cot^{-1}\left(\frac{\gamma}{k} + \cot(kR)\right)$$

$$= -kR + 0 \qquad \left(\because \cot x|_{x \to 0+} = \infty\right)$$

i.e $\delta_0 = -kR$.

(ii) Now for the true hard sphere case the wave fn vanishes at $r = R$.

$\therefore \psi^{l=0}_{ext}(R) = 0$.

$$\therefore \lim_{r \to R} A\left[\frac{\sin(kr)}{kr} + a_0 \frac{e^{ikr}}{r}\right] = 0$$

(previously calculated)

i.e $\dfrac{\sin(kR)}{kR} + a_0 \dfrac{e^{ikR}}{R} = 0$.

i.e $a_0 = -\dfrac{\sin(kR)}{k\, e^{ikR}}$

but $a_0 = \dfrac{e^{i\delta_0} \cdot \sin \delta_0}{k}$ and hence

$$\frac{e^{i\delta_0} \sin \delta_0}{k} = -\frac{\sin(kR)}{k\, e^{ikR}}$$

taking modulus (assuming previous convention that $\sin(kR)$ is $-ve$)

$$\left|e^{i\delta_0} \sin \delta_0\right| = \left|-\frac{\sin(kR)}{e^{ikR}}\right| \Rightarrow \sin \delta_0 = -\sin(kR)$$

Proved

$\Rightarrow \delta_0 = -kR$

Yeilding same result as before for $\gamma \to \infty$

Resonance : From the exact expression of δ_0 we get for the resonating condition $\delta_0 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ etc gives.

$$\cot (kR + \delta_0) = \cot kR + \frac{\gamma}{k}$$

$$\Rightarrow \cot kR + \frac{\gamma}{k} = \frac{\cot (kR) \cot (\delta_0) - 1}{\cot (kR) + \cot (\delta_0)}$$

$$\cot (\delta_0) = 0 \text{ for } \delta_0 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\Rightarrow \cot (kR) + \frac{\gamma}{k} = -\frac{1}{\cot (kR)}$$

$$\Rightarrow \cot^2 (kR) + 1 + \frac{\gamma \cot(kR)}{k} = 0 \quad \ldots \text{(A)}$$

$$\Rightarrow \cancel{\operatorname{cosec}^2 (kR)} \operatorname{cosec}^2 kR + \frac{\gamma \cot (kR)}{k} = 0$$

$$\cancel{\cot (kR) =} \Rightarrow \frac{1}{\sin kR} + \frac{\gamma}{k} \cos kR = 0$$

$$\Rightarrow \frac{\gamma}{k} \sin kR \cos kR = -1. \quad \left(\because \frac{1}{\sin kR} \neq 0\right)$$

$$\Rightarrow \frac{\gamma}{2k} \sin (2kR) = -1.$$

$$\Rightarrow 2kR = -\sin^{-1} \left(\frac{2k}{\gamma}\right)$$

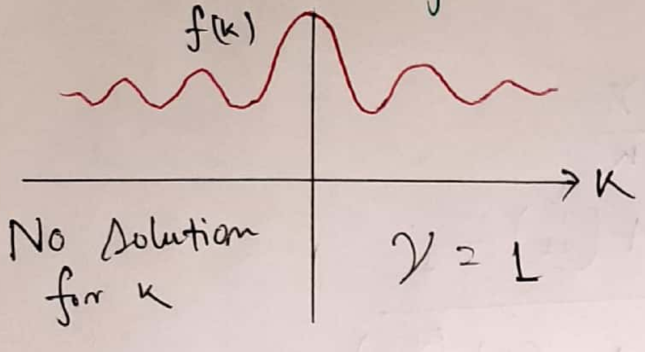
$$\text{or} \quad \frac{\gamma \sin (2kR)}{2k} + 1 = 0$$

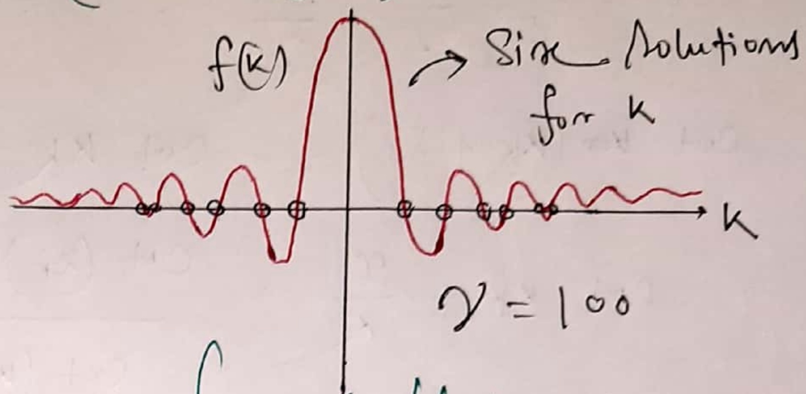
Answer

Any one of those be a condition for resonance.

If $f(k) = \frac{\gamma \sin(2kR)}{2k} + 1$ then the solution of $f(k) = 0$ are $k_{resonance}$ points that we need.

Now the plot of $f(k)$ be: (For $R = 1$)





Clearly we get $K_{resonance}$ for sufficiently large value of γ (for particular R)

The exact condition can be treated out from eq (A) in previous page.

$$\cot(kR) = \left(-\frac{\gamma}{k} \pm \sqrt{\frac{\gamma^2}{k^2} - 4}\right)/2$$

So for ~~rect~~ real solution we must have

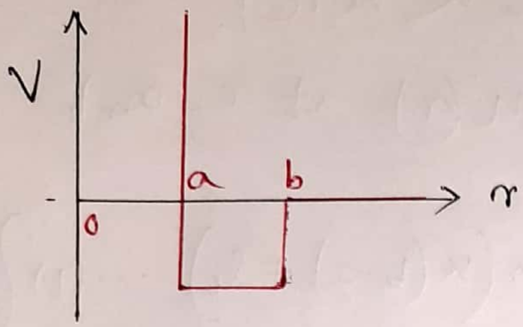
$$\frac{\gamma^2}{k^2} - 4 > 0 \quad \text{i.e} \quad \boxed{\gamma^2 > 4k^2}$$

Answer

Problem: 9

Infinite barrier with finite well :-

The given potential:
$$V(r) = \begin{cases} \infty & r < a \\ -V_0 & a < r < b \\ 0 & r > b \end{cases}$$



Clearly we can write $\psi(\vec{r}) = 0$ for $r < a$.

Now for the intermediate region $r \in (a, b)$

Solving radial eq for S wave $(l=0)$ we get:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_0 u = Eu$$

$$\Rightarrow \frac{d^2u}{dr^2} = -\frac{2m(V_0+E)}{\hbar^2}u = -\eta^2 u$$

$$\therefore u = u_{int}(r) = A\sin(\eta r) + B\cos(\eta r).$$

but $u_{int}(a) = 0$ from Boundary condition

$$\therefore A\sin\eta a + B\cos\eta a = 0.$$

$$\therefore u_{int}(r) = A\left(\sin\eta r - \frac{\sin\eta a}{\cos\eta a}\cos\eta r\right)$$

$$= A'\sin(\eta(r-a)) \qquad \left(A' = \frac{A}{\cos\eta a}\right)$$

For the exterior region $(r>b)$ ~~the~~ let the solution be ahead of a phase factor δ_0.

$\therefore$ ~~ext~~ $u_{ext}(r) = C \sin(k(r-a)+\delta_0)$.

($\because$ The wave number will be k here).

Continuity of ~~u_{int}~~ $u(r)$ & $u'(r)$ at $r = b$ gives:

$$\left.\begin{aligned} A' \sin(\eta(b-a)) &= C \sin(k(b-a)+\delta_0) \quad \cdots (1) \\ \eta A' \cos(\eta(b-a)) &= kC \cos(k(b-a)+\delta_0) \quad \cdots (2) \end{aligned}\right\}$$

dividing (1) by (2) gives,

$$\frac{\tan(\eta(b-a))}{\eta} = \frac{\tan(k(b-a)+\delta_0)}{k}$$

$$\Rightarrow \tan(k(b-a)+\delta_0) = \frac{k}{\eta}\tan(\eta(b-a))$$

$$\therefore \delta_0 = k(a-b) + \tan^{-1}\left[\frac{k}{\eta}\cdot\tan(\eta(b-a))\right]$$

$$\left(\text{with } \eta = \frac{\sqrt{2m(E+V_0)}}{\hbar}\right)$$

Answer

Problem 6 :→ (Scattering of identical particles)

In the CM frame we get the scattering cross section for spacial symmetric & anti-symmetric states of identical particles be given by:

$$\left.\begin{aligned}\frac{d\sigma_S}{d\Omega} &= |f(\theta) + f(\pi-\theta)|^2 \\ \frac{d\sigma_A}{d\Omega} &= |f(\theta) - f(\pi-\theta)|^2\end{aligned}\right\} \quad f = \text{scattering amplitude}$$

(S or A for spacially symm or AntiSymm state)

Now here $v(\vec{r}) = g\,\dfrac{e^{-\mu r}}{r}$ so by Born appr for spherically symmetrical potential we get:

$$f(\theta,\phi) \simeq -\frac{2m}{\hbar^2\alpha}\int_0^\infty r\,V(r)\,\sin(\alpha r)\,dr$$

$$= -\frac{2mg}{\hbar^2\alpha}\int_0^\infty e^{-\mu r}\sin(\alpha r)\,dr \qquad \left(\begin{aligned}\alpha &= 2k\sin(\theta/2) \\ \vec{\alpha} &= \vec{k}'-\vec{k}\end{aligned}\right)$$

$$= -\frac{2mg}{\hbar^2(\mu^2+\alpha^2)} \qquad \text{(using mathematica)}$$

$$\Rightarrow f(\theta) = -\frac{2mg}{\hbar^2(\mu^2 + 2^2k^2\sin^2(\theta/2))}$$

$$\therefore f(\pi-\theta) = -\frac{2mg}{\hbar^2(\mu^2 + 2^2k^2\cos^2(\theta/2))}$$

Now for e^- as the over all wave fn is antisymmetric; hence Spacial Symmetry / Anti-Symmetry chooses for Spin Antisymmetry / Symmetry.

i.e $\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{Spin}\\ \text{Sym}}} = \frac{d\sigma_A}{d\Omega}$; $\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{Spin}\\ \text{Antisym}}} = \frac{d\sigma_S}{d\Omega}$.

So the required quantities:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{Spin}\\ \text{Sym}}} = \frac{4m^2g^2}{\hbar^4}\left[\frac{1}{\mu^2+4k^2\sin^2\frac{\theta}{2}} - \frac{1}{\mu^2+4k^2\cos^2\frac{\theta}{2}}\right]^2$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{Spin}\\ \text{Antisym}}} = \frac{4m^2g^2}{\hbar^4}\left[\frac{1}{\mu^2+4k^2\sin^2\frac{\theta}{2}} + \frac{1}{\mu^2+4k^2\cos^2\frac{\theta}{2}}\right]^2$$

Ans

Ans as there are 3 symmetric spin configurations ($|11\rangle, |10\rangle, |1-1\rangle$) & one antisymm spin configuration ($|00\rangle$); So the differential cross section for unpolarized beam be given by:

$$\frac{d\sigma}{d\Omega} = \frac{3}{4}\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{Spin}\\ \text{Symm}}} + \frac{1}{4}\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{Spin}\\ \text{antisym}}}$$

$$= \frac{m^2g^2}{\hbar^4}\left[3\left\{\frac{1}{\mu^2+4k^2\sin^2\frac{\theta}{2}} - \frac{1}{\mu^2+4k^2\cos^2\theta/2}\right\}^2 + \left\{\frac{1}{\mu^2+4k^2\sin^2\theta/2} + \frac{1}{\mu^2+4k^2\cos^2\theta/2}\right\}^2\right]$$

Ans

Problem : 7

The problem is similar to the previous one; except the potential contains $(\vec{\sigma}_1 \cdot \vec{\sigma}_2)$ term.
Now this time as the total spin be conserved
So the good basis is the eig-basis of σ^2.

Now, $\vec{\sigma} = \vec{\sigma}_1 + \vec{\sigma}_2$

$$\Rightarrow \sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2$$

$$\Rightarrow \vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{\sigma^2 - \sigma_1^2 - \sigma_2^2}{2}$$

$$\therefore V(\vec{r}, \vec{\sigma}_i) = \frac{g}{2}\left(\sigma^2 - \sigma_1^2 - \sigma_2^2\right) \frac{e^{-\mu r}}{r}.$$

Now for the spin-symmetric state the scattering amplitude be $(S=1)$:

$$f(\theta) - f(\pi - \theta) = f_{S=1}(\theta)$$

but $f(\theta) = -\frac{2mg}{\hbar^2 \alpha} \int_0^\infty r \frac{(\sigma^2 - \sigma_1^2 - \sigma_2^2)}{2} \frac{e^{-\mu r}}{r} \sin(\alpha r)\, dr$

but $\sigma = 1$; $\sigma_1 = \sigma_2 = 1/2$. for symmetric spin state of 2 spin $\frac{1}{2}$ particles.

So; $f_{S=1}(\theta) = -\frac{2mg}{\hbar^2 \alpha} \times \frac{1 - \frac{1}{4} - \frac{1}{4}}{2} \int_0^\infty e^{-\mu r} \sin(\alpha r)\, dr$

i.e $f_{S=1}(\theta) = -\dfrac{mg}{\hbar^2(\mu^2+\alpha^2)}$ $\left(\alpha = 2k\sin\theta/2\right)$

$$= -\frac{mg}{\hbar^2\left(\mu^2 + 4k^2\sin^2\theta/2\right)}$$ Ans

for $S=0$; spin antisymmetric state

$$f_{S=0}(\theta) = -\frac{2mg}{\hbar^2\alpha}\int_0^\infty r.\frac{\left(0^2-\frac{1}{4}-\frac{1}{4}\right)}{2}\,\frac{e^{-\mu r}}{r}\cdot\sin(\alpha r)\,dr$$

$$= \frac{mg}{2\hbar^2(\mu^2+\alpha^2)}$$ Ans $= \dfrac{mg}{2\hbar^2\left(\mu^2+4k^2\sin^2\frac{\theta}{2}\right)}$

for an unpolarized beam all states among $|1\pm1\rangle, |1\,0\rangle, |0\,0\rangle$ are equally possible. Now the differential cross section for spin symmetric state/space antisymmetric state be:

$$\frac{d\sigma_A}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_{\substack{\text{Spin}\\ \text{Sym}}} = \left|f_{S=1}(\theta) - f_{S=1}(\pi-\theta)\right|^2 = \frac{d\sigma}{d\Omega}\bigg|_{\sigma=1}$$

$$= \frac{m^2g^2}{\hbar^4}\left[\frac{1}{\mu^2+4k^2\sin^2\frac{\theta}{2}} - \frac{1}{\mu^2+4k^2\cos^2\frac{\theta}{2}}\right]^2.$$ Ans

And the differential cross section for spin antisymmetric/space symmetric state be:

$$\frac{d\sigma_S}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_{\substack{\text{Spin}\\ \text{Antisym}}} = \left|f_{S=0}(\theta) + f_{S=0}(\pi-\theta)\right|^2 = \frac{d\sigma}{d\Omega}\bigg|_{\sigma=0}$$

$$= \frac{m^2g^2}{4\hbar^2}\left[\frac{1}{\mu^2+4k^2\sin^2\frac{\theta}{2}} + \frac{1}{\mu^2+4k^2\cos^2\theta/2}\right]^2.$$ Ans

Where as in the unpolarized scattered beam there are $\frac{3}{4}$ th amount of spin symmetric & $\frac{1}{4}$ amount of spin antisymmetric ~~state~~ state & hence the net differential cross section be,

$$\frac{d\sigma}{d\Omega} = \frac{3}{4}\frac{d\sigma_A}{d\Omega} + \frac{1}{4}\frac{d\sigma_S}{d\Omega}$$

$$= \frac{m^2 g^2}{\hbar^2}\left[\frac{3}{4}\left\{\frac{1}{\mu^2 + 4k^2 \sin^2(\theta/2)} - \frac{1}{\mu^2 + 4k^2\cos^2(\theta/2)}\right\}^2\right.$$

$$\left. + \frac{1}{16}\left\{\frac{1}{\mu^2 + 4k^2 \sin^2\theta/2} + \frac{1}{\mu^2 + 4k^2\cos^2\theta/2}\right\}^2\right]$$

Answer

Problem: 10

a) Given $\delta_l \cong 0$ for $l \geq 2$.

Clearly the total cross section is given by:

$$\sigma = \sum_{l=0}^{\infty} \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l = \frac{4\pi}{k^2}\sum_{l=0}^{1}(2l+1)\sin^2\delta_l$$

$$= \frac{4\pi}{k^2}\left(\sin^2\delta_0 + 3\sin^2\delta_1\right)$$

$$\simeq \frac{4\pi}{k^2}\left(\frac{1}{4} + 3\times 0.03\right)$$

Now, $k = \frac{\sqrt{2mE}}{\hbar}$; here $E = 5$ MeV.

$$\therefore k^2 = \frac{2\times M_{neutron}\times E}{\hbar^2} = \frac{2\times 939.565\times 5\times(1.6\times10^{-13})^2}{(1.054\times10^{-34}\times3\times10^{8})^2}$$

$$= 2.4057\times10^{29}\ m^{-2}.$$

$$\therefore \sigma = \cancel{5.22}\ \frac{4\pi}{2.4057\times10^{29}}\left(\frac{1}{4}+3\times0.03\right)$$

$$\simeq 5.223\times10^{-29}\ m^{-2}.$$ Ans

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{1}(2l+1)\,e^{i\delta_l}\sin\delta_l\cdot P_l(\cos\theta).$$

$$= \frac{1}{k}\left[e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cdot(\cos\theta)\right]$$

$$= \frac{1}{k}\Big[\left\{(\cos\delta_0\cdot\sin\delta_0) + 3\cos\delta_1\sin\delta_1\cos\theta\right\}$$
$$+ i\left\{\sin^2\delta_0 + 3\sin^2\delta_1\cos\theta\right\}\Big]$$

$$\therefore |f(\theta)|^2 = \frac{1}{k^2}\Big[\left(\cos\delta_0 \cdot \sin\delta_0 + 3\cos\delta_1 \sin\delta_1 \cos\theta\right)^2 + \left(\sin^2\delta_0 + 3\sin^2\delta_1 \cos\theta\right)^2 \Big]$$

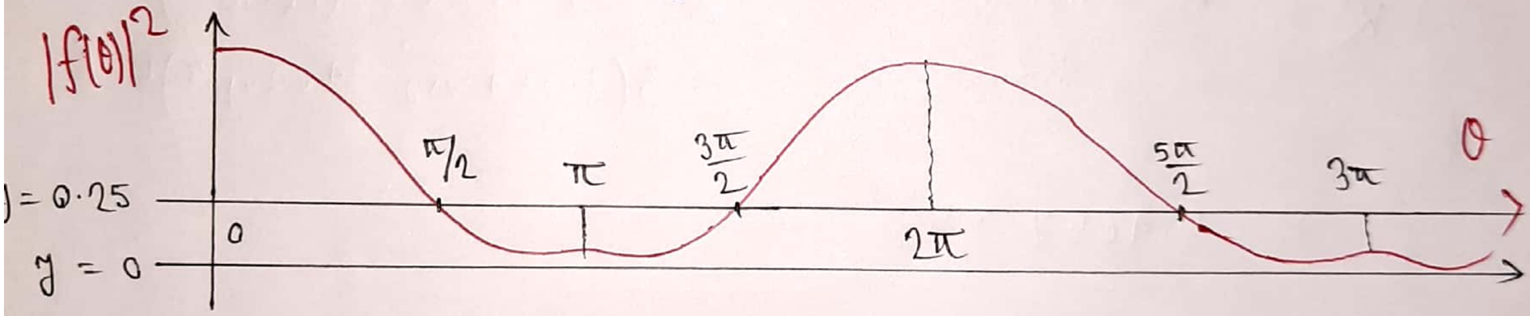
Using values we get:

$$|f(\theta)|^2 = \frac{1}{k^2}\Big[(0.433 + 0.513\cos\theta)^2 + (0.25 + 0.09\cos\theta)^2 \Big]$$

$$\simeq \frac{1}{k^2}\left[0.25 + 0.27\cos^2\theta + 0.49\cos\theta\right]$$

but $\dfrac{d\sigma}{d\Omega} = |f(\theta)|^2 = \dfrac{(0.27\cos^2\theta + 0.49\cos\theta + 0.25)}{k^2}$

$$= 4.15\times10^{-30} \cdot \left(0.27\cos^2\theta + 0.49\cos\theta + 0.25\right)$$



$$\therefore \left.\frac{d\sigma}{d\Omega}\right|_{\theta=30^\circ} = |f(30^\circ)|^2 = 3.64\times10^{-30}\ m^{-2}\cdot sr^{-1}$$

(using calculator)

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=45^\circ} = |f(45^\circ)|^2 = 3.036\times10^{-30}\ m^{-2}\cdot sr^{-1}$$

Ans

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=90^\circ} = |f(90^\circ)|^2 = 1.037\times10^{-30}\ m^{-2}\cdot sr^{-1}$$

b. We know $a_l = e^{i\delta_l}\dfrac{\sin\delta_l}{k}$ i.e. $|a_l| \sim \sin\delta_l$

$\therefore$ for $\delta_l \simeq 0$ for $l \geq 2$ means ~~~~ mody of $l = 2, 3, \ldots$ are absent in the true

Scattered wave. So when solving TISE to write

$$\psi \sim e^{ikz} + k\sum_{l=0}^{\infty} i^{(l+1)}(2l+1)\,a_l h_l^{(1)}(kr)\,P_l(\cos\theta)$$

we mostly get $l=0,1$ waves as outgoing waves. It means the potential drops out sufficiently rapidly, keeping $l \geq 2$ waves undisturbed.

i.e the potential range is such that ~~the repulsive part~~ it can't act on high energy incoming particles (l = high) and only affects for low l terms.

Ans

(The calculation for effective range of potential (i.e after which $V(r)$ effectively dies out) needs some numerical technique & more regorous way to evaluate.)

Problem: 3

Let $|\phi\rangle$ & $|\psi\rangle$ are wave functions for incident and scattered particles respectively.

$\therefore$ Transition amplitude

$$A = -\frac{i}{\hbar}\int_0^t \langle\phi|V(\vec{r}, t')|\psi\rangle e^{i\omega_0 t'} dt'$$

$$\left(\omega_0 = \frac{E_f - E_i}{\hbar}\right)$$

$$\therefore A = -\frac{i}{\hbar}\langle\phi|V(\vec{r})|\psi\rangle \int_0^t e^{i\omega_0 t'}\cos(\omega t')dt'$$

$$= -\frac{i}{2\hbar}\langle\phi|V(\vec{r})|\psi\rangle \int_0^t \left\{e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}\right\}dt'$$

$$= -\frac{i}{2\hbar}\langle\phi|V(\vec{r})|\psi\rangle \left(\frac{e^{i(\omega_0+\omega)t'}}{i(\omega_0+\omega)} + \frac{e^{i(\omega_0-\omega)t'}}{i(\omega_0-\omega)}\right)_0^t$$

for large t we get:

$$A \sim -\frac{1}{2\hbar}\langle\phi|V(\vec{r})|\psi\rangle\left[\delta(\omega_0+\omega) + \delta(\omega_0-\omega)\right]$$

So the transition amplitude is zero except $\omega_0 = \pm\omega$.

$$\therefore E_f = E_i \pm \hbar\omega_0 = E_i \pm \hbar\omega.$$

$\therefore$ Energy of scattered particle increased / decreased by an amount of $\hbar\omega$.

proved

Problem : 11

Finite Spherical well :→

Given $V(r) = -V_0 \quad r \leq r_0$
$\qquad = 0 \quad r > r_0$

For the interior region $(r \leq r_0)$ for $l=0$ (S wave)
Solution of T.I.S.E gives :

$$u_{int}(r) = A \sin(k_1 r) \qquad \left(k_1 = \frac{\sqrt{2m(E+V_0)}}{\hbar}\right)$$

For the exterior region $(r > r_0)$; the solution is:

$$u_{ext}(r) = B' \sin(kr) + C' \cos(kr)$$
$$= B \sin(kr + \delta_0) \qquad \left(k = \frac{\sqrt{2mE}}{\hbar}\right)$$

Continuity of ψ & ψ' (better to say u & u')
at $r = r_0$ gives:

$$\left.\begin{aligned} A \sin(k_1 r_0) &= B \sin(kr_0 + \delta_0) \\ A k_1 \cos(k_1 r_0) &= Bk \cos(kr_0 + \delta_0) \end{aligned}\right\} \begin{aligned} &\cdots (1) \\ &\cdots (2) \end{aligned}$$

dividing (1) by (2) gives:

$$\frac{\tan(k_1 r_0)}{k_1} = \frac{\tan(kr_0 + \delta_0)}{k}$$

$$\Rightarrow \tan(kr_0 + \delta_0) = \frac{k}{k_1} \tan(k_1 r_0)$$

$$\Rightarrow \delta_0 = -kr_0 + \tan^{-1}\left(\frac{k}{k_1} \tan(k_1 r_0)\right)$$

Proved

VIII For small kr_0 we get

$$\delta_0 \simeq \tan^{-1}\left(\frac{k}{k_1}\tan k_1 r_0\right)$$

Now for $k_1 = \frac{(2n+1)\pi}{2r_0}$ we get:

$$\delta_0 \simeq \tan^{-1}\left(\frac{k}{k_1}\tan\frac{(2n+1)\pi}{2}\right) \sim \tan^{-1}(\pm x)$$

when $x \to \infty$

i.e $\delta_0 \simeq \frac{(2n+1)\pi}{2} = \delta_{resonance}$.

So, the resonance condition be given by

$$k_1 = \frac{(2n+1)\pi}{2r_0}.$$ proved

Now, for near resonance situation if

$k_1 = \frac{(2n+1)\pi}{2r_0} + \epsilon$ $\quad (\epsilon \simeq 0)$

$(k = (k+\epsilon) = k_0(\text{let}))$

then we get:

$$\delta_0 \simeq - k_0 r_0 + \tan^{-1}\left[\frac{k_0}{k_1}\tan\left\{\frac{(2n+1)\pi}{2} + \epsilon r_0\right\}\right]$$

$$= - k_0 r_0 + \tan^{-1}\left[\frac{k_0}{k_1}\cdot(-\cot \epsilon r_0)\right]$$

$$\simeq - k_0 r_0 + \tan^{-1}\left[-\frac{k_0}{k_1 \epsilon r_0}\right]$$

$\left(\because \cot(\epsilon r_0) \simeq \frac{1}{\epsilon r_0} \text{ for } \epsilon \simeq 0\right)$

~~where $k_1' = k_1 + \epsilon$~~

$$= -K_0 r_0 + \tan^{-1}\left[\frac{k_0}{-\epsilon k_1 r_0}\right]$$

Now, $E = \dfrac{\hbar^2 k_1^2}{2m} + V_0 \Rightarrow \Delta E = \dfrac{2\hbar^2 k_1 \Delta k_1}{2m}$

but $\Delta k_1 = \epsilon \Rightarrow \Delta E = \dfrac{\hbar^2 k_1 \epsilon}{m}$

$$\therefore \epsilon = \frac{m \Delta E}{\hbar^2 k_1} = \frac{m(E - E_0)}{\hbar^2 k_1}.$$

So, $\delta_0 \simeq -k_0 r_0 + \tan^{-1}\left[\dfrac{k_0}{r_0 \cdot \dfrac{m(E_0 - E)}{\hbar^2}}\right]$

$$\simeq -k_0 r_0 + \tan^{-1}\left(\frac{\hbar^2 k_0}{m r_0 (E_0 - E)}\right)$$

i.e $\underbrace{\delta_0 \simeq -k_0 r_0 + \tan^{-1}\left[\dfrac{\Gamma/2}{E_0 - E}\right]}_{\text{proved}}$ $\left(\dfrac{\Gamma}{2} = \dfrac{\hbar^2 k_0}{m r_0}\right)$