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### Problem: 1

1.a The Spin Singlet State for two e- be given by:  
 $S=0; m=0$  i.e  $\{|\frac{1}{2} \frac{1}{2}\rangle - |\frac{1}{2} -\frac{1}{2}\rangle\}/\sqrt{2}$

$$\text{or; } |0 0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

Now for  $S=0$ ; the Spin ~~matrices~~ matrix are  $(1 \times 1)$  & hence they are just numbers. So they will commute.

On other way we can say that for  $S=0$ :

$$S^2 |x\rangle = 0|x\rangle; S_z |x\rangle = 0|x\rangle$$

~~( $|x\rangle$ )~~ ( $|x\rangle$ ) = Spinor in 1 dimension

And using that  $S^2$  &  $S_z$  are both 1 dim Square matrix; we get  $S^2 = 0$ ;  $S_z = 0$

Similarly  $S_+ = 0$ ;  $S_- = 0$ ;  $S_m = 0$ ;  $S_j = 0$   
 $\& [S_+, S_-] = 2i S_z = 0$

Now for the given problem; the State  $|0 0\rangle$  can't be raised or lowered with  $S_{\pm}$ ; ~~so it makes~~ can't be part of the complete set in  $S=0$  space.

So;  $|0 0\rangle$  makes the complete set in  $S=0$  space.

So; for rotation of 'J' value is conserved & and ~~so does~~ the coefficient for different 'm'

~~(of the ket)  $|S m\rangle$ ) changes according to Wigner D matrix; here the state will be kept unchanged as there is only one possible m value.~~

$$\text{i.e } \mathcal{D}(R) |0 0\rangle = |0 0\rangle$$

We can also proof this mathematically. (2)

Here the rotation of axis is  $(\theta, \phi = 0)$

$$\therefore \hat{m} = \cos \theta \cdot \hat{z} + \sin \theta \cdot \hat{m}$$

$$\therefore \hat{s} \cdot \hat{m} = \sin \theta \hat{s}_x + \cos \theta \hat{s}_z$$

→ If the rotation angle be  $\beta$ ; then

$$D_{\beta}^{(0)}(R) = \exp \left\{ -\frac{i\beta}{\hbar} (\sin \theta \hat{s}_x + \cos \theta \hat{s}_z) \right\} = e^{-\frac{i\beta \hat{s} \cdot \hat{m}}{\hbar}}$$

$$= \exp \left\{ -\frac{i\beta}{\hbar} \sin \theta \hat{s}_x \right\} \cdot \exp \left\{ -\frac{i\beta}{\hbar} \cos \theta \hat{s}_z \right\}$$

(Only for  $S=0$  we can do this as  $[S_z, S_x] = 0$ )

$$S_0; D_{\beta}^{(0)}(R) |0 0\rangle$$

$$= \exp \left\{ -\frac{i\beta}{\hbar} \sin \theta \hat{s}_x \right\} \cdot \exp \left\{ -\frac{i\beta}{\hbar} \cos \theta \hat{s}_z \right\} |0 0\rangle$$

but  $|0 0\rangle$  being eigenstate of  $\hat{s}_z$  with eigenval 0:

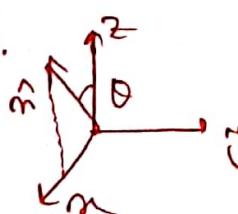
$$D_{\beta}^{(0)}(R) |0 0\rangle = \exp \left\{ -\frac{i\beta}{\hbar} (\sin \theta) \cdot \hat{s}_x \right\} \cdot \exp \left\{ -\frac{i\beta \cos \theta \cdot 0}{\hbar} \right\} |0 0\rangle$$

$$= \exp \left\{ -\frac{i\beta}{\hbar} \sin \theta \hat{s}_x \right\} |0 0\rangle$$

$$= \exp \left( -\frac{i\beta \sin \theta}{2\hbar} (S_+ + S_-) \right) |0 0\rangle$$

$$= \exp \left( -\frac{i\beta \sin \theta}{2\hbar} S_+ \right) \cdot \exp \left( -\frac{i\beta \sin \theta}{2\hbar} S_- \right) |0 0\rangle$$

( $\because [S_+, S_-] = 2\hbar S_z = 0$  for  $S=0$ )



③

$$= \exp\left(-\frac{i\beta \sin\theta S_+}{2\hbar}\right) \cdot \left(1 + \left(\frac{-i\beta \sin\theta S_+}{2\hbar}\right) + \frac{1}{2!} \left(\frac{-i\beta \sin\theta S_+}{2\hbar}\right)^2 + \dots\right) |00\rangle$$

$$= \exp\left(-\frac{i\beta \sin\theta S_+}{2\hbar}\right) |00\rangle \quad [\because S_- |00\rangle = 0]$$

$$= \left[1 + \left(\frac{-i\beta \sin\theta S_+}{2\hbar}\right) + \frac{1}{2!} \left(\frac{-i\beta \sin\theta S_+}{2\hbar}\right)^2 + \dots\right] |00\rangle$$

$$= |00\rangle \quad [\because S_+ |00\rangle = 0]$$

$$\text{So: } \mathcal{D}_\beta^{(0)}(R) |00\rangle = |00\rangle$$

i.e. the state is invariant under rotation.

1.b.i) For the state  $2p^3\}$ :

$$S_1 = S_2 = \frac{1}{2}; L_1 = L_2 = 1$$

$$\therefore S = 0, 1; L = 0, 1, 2.$$

$$\text{For } S = 0: L = 0, 1, 2 \Rightarrow j = 0, 1, 2$$

The states are  ${}^1S_0, {}^1P_1, {}^1D_2$

$$S = 1: L = 0 \Rightarrow j = 1: \text{State: } {}^3S_1$$

$$S = 1: L = 1 \Rightarrow j = 0, 1, 2: \text{States: } {}^3P_0, {}^3P_1, {}^3P_2$$

$$S = 1: L = 2 \Rightarrow j = 1, 2, 3: \text{States: } {}^3D_1, {}^3D_2, {}^3D_3$$

So there are 10 states:

$${}^1S_0, {}^1P_1, {}^1D_2, {}^3S_1, {}^3P_0, {}^3P_1, {}^3P_2, {}^3D_1, {}^3D_2, {}^3D_3$$

### Splitting:-

i)  $s-s$  splitting:  $s = 0, 1$ .

The  $\overline{\Delta V}$  for  $s-s$  splitting be given by:

$$\overline{\Delta V}_s = -\alpha [s(s+1) - s_1(s_1+1) - s_2(s_2+1)]$$

$$\text{for } s=0 : \overline{\Delta V}_s = -\alpha [0 - \frac{3}{4} - \frac{3}{4}] = \frac{3\alpha}{2}$$

$$s=1 : \overline{\Delta V}_s = -\alpha [2 - \frac{3}{4} \times 2] = \frac{\alpha}{2}$$

ii) for  $l-l$  splitting:  $\overline{\Delta V}_l = -m [l(l+1) - l_1(l_1+1) - l_2(l_2+1)]$

here  $l_1 = l_2 = l$ ;  $l = 0, 1, 2$ .

$$\text{for } l=0 : \overline{\Delta V} = -m [0 - 2 - 2] = 4m$$

$$l=1 : \overline{\Delta V} = -m [2 - 2 - 2] = 2m$$

$$l=2 : \overline{\Delta V} = -m [(-2) - 2 - 2] = -2m$$

iii) Splitting of  $sp$  orbital:  $\overline{\Delta V} = \chi [j(j+1) - l(l+1) - s(s+1)]$

So splitting of  $3p$  levels:  $s=1; l=1; j=0, 1, 2$

$$\therefore \overline{\Delta V}_{|j=0} = \chi [0 - 2 - 2] = -4\chi$$

$$\text{for } j=1 : \overline{\Delta V} = \chi [2 - 2 - 2] = -2\chi$$

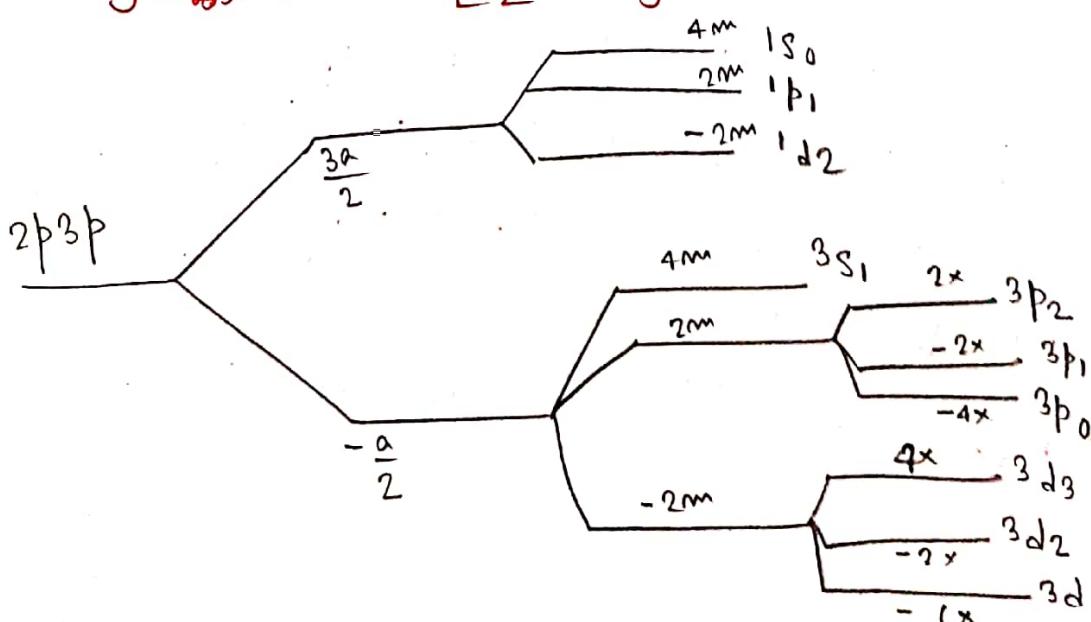
$$j=2 : \overline{\Delta V} = \chi [(-2) - 2 - 2] = +2\chi$$

for  $3d$  levels:  $s=1; l=2; j=1, 2, 3$ .

$$j=1 : \overline{\Delta V} = \chi [2 - 6 - 2] = -6\chi$$

$$j=2 : \overline{\Delta V} = \chi [(-6) - 6 - 2] = -2\chi$$

$$j=3 : \overline{\Delta V} = \chi [12 - 6 - 2] = +4\chi$$



$$\text{I.C. } \hat{H} = -\frac{eB}{m} \vec{S} \cdot \vec{B} = -\frac{eB\alpha}{m} S_\alpha$$

Clearly  $[\hat{H}, S_\alpha] = 0$  and hence the eigenstates

of  $\hat{H}$  be:  $|1_\alpha\rangle$ : with energy  $-\frac{e\gamma B}{2m}$

$|+\alpha\rangle$ : " "  $+\frac{e\gamma B}{2m}$

Given that  $|\alpha(0)\rangle = |\uparrow\rangle = \frac{|+\alpha\rangle + |-\alpha\rangle}{\sqrt{2}}$

$$\text{So } |\alpha(+)\rangle = |\alpha(0)\rangle \cdot e^{-\frac{iE_t}{\hbar}}$$

$$= \frac{1}{\sqrt{2}} \exp\left(-\frac{i\gamma B t}{2}\right) |+\alpha\rangle_m + \frac{1}{\sqrt{2}} \exp\left(\frac{i\gamma B t}{2}\right) |-\alpha\rangle_m$$

So the probability to find  $|\downarrow\rangle$  at a later time

is given by:

$$P = |\langle \downarrow | \alpha(+)\rangle|^2 = \left| \frac{1}{\sqrt{2}} (\langle +\alpha | - \langle -\alpha |) \right|^2$$

$$= \left| \left\{ \frac{\langle +\alpha | - \langle -\alpha |}{\sqrt{2}} \right\} \cdot \left\{ \frac{1}{\sqrt{2}} \exp\left(-\frac{i\gamma B t}{2}\right) |+\alpha\rangle_m + \frac{1}{\sqrt{2}} \exp\left(\frac{i\gamma B t}{2}\right) |-\alpha\rangle_m \right\} \right|^2.$$

$$= \left| \frac{1}{2} \exp\left(-\frac{i\gamma B t}{2}\right) - \frac{1}{2} \exp\left(\frac{i\gamma B t}{2}\right) \right|^2$$

$$= \left| \frac{1}{2} \exp\left(-\frac{i\gamma B t}{2}\right) - \frac{1}{2} \exp\left(\frac{i\gamma B t}{2}\right) \right|^2$$

$$= \sin^2\left(\frac{\gamma B t}{2}\right) = \sin^2\left(\frac{e B t}{2m}\right)$$

The spin will flip from  $|\uparrow\rangle$  to  $|\downarrow\rangle$  when the probability to find  $|\downarrow\rangle$  will become 1.

i.e.  $\sin^2\left(\frac{eBt}{2me}\right) = 1 = \sin^2\left(\frac{\pi}{2}\right)$ .

i.e.  $\frac{eBt}{2m} = \frac{\pi}{2}$  or  $t = \frac{\pi m}{eB}$ .

putting values:

$$t = \frac{\pi \times 9.11 \times 10^{-31}}{1.6 \times 10^{-19} \times 10^{-2}} \quad \left( \because 100 \text{ gauss} = 10^{-2} \text{ T} \right)$$

$$\approx 17.89 \times 10^{-10} \Delta$$

$$\approx 1.789 \times 10^{-9} \Delta.$$

I. d.  $J_1 = 1$  ;  $J_2 = \frac{1}{2}$

$$\text{So, } J = \frac{3}{2}, \frac{1}{2}.$$

VII Now for  $j = \frac{3}{2}$  :  $m = \pm \frac{3}{2}, \pm \frac{1}{2}$ .

$$|\frac{3}{2} \frac{3}{2}\rangle = |1 1\rangle |\frac{1}{2} \frac{1}{2}\rangle \quad (\text{obviously}) \checkmark$$

$$\Rightarrow J^- |\frac{3}{2} \frac{3}{2}\rangle = J_1^- |1 1\rangle |\frac{1}{2} \frac{1}{2}\rangle + J_2^- |1 1\rangle |\frac{1}{2} \frac{1}{2}\rangle$$

$$\Rightarrow \sqrt{(\frac{3}{2} + \frac{3}{2})(\frac{3}{2} - \frac{3}{2} + 1)} |\frac{3}{2} \frac{1}{2}\rangle = \sqrt{(1+1)(1-1+1)} |1 0\rangle |\frac{1}{2} \frac{1}{2}\rangle \\ + \sqrt{(\frac{1}{2} + \frac{1}{2})(\frac{1}{2} - \frac{1}{2} + 1)} |1 1\rangle |\frac{1}{2} - \frac{1}{2}\rangle$$

$$\Rightarrow \sqrt{3} |\frac{3}{2} \frac{1}{2}\rangle = \sqrt{2} |1 0\rangle |\frac{1}{2} \frac{1}{2}\rangle + |1 1\rangle |\frac{1}{2} - \frac{1}{2}\rangle$$

$$\Rightarrow |\frac{3}{2} \frac{1}{2}\rangle = \sqrt{\frac{2}{3}} |1 0\rangle |\frac{1}{2} \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |1 1\rangle |\frac{1}{2} - \frac{1}{2}\rangle \checkmark$$

and obviously:

$$|\frac{3}{2} - \frac{3}{2}\rangle = |1 -1\rangle |\frac{1}{2} - \frac{1}{2}\rangle \checkmark$$

$$\Rightarrow J^+ |\frac{3}{2} - \frac{3}{2}\rangle = J_1^+ |1 -1\rangle |\frac{1}{2} - \frac{1}{2}\rangle + J_2^+ |1 -1\rangle |\frac{1}{2} - \frac{1}{2}\rangle$$

$$\Rightarrow \sqrt{(\frac{3}{2} + \frac{3}{2})(\frac{3}{2} - \frac{3}{2} + 1)} |\frac{3}{2} - \frac{3}{2}\rangle = \sqrt{(1+1)(1+1-1)} |1 0\rangle |\frac{1}{2} - \frac{1}{2}\rangle \\ + \sqrt{(\frac{1}{2} + \frac{1}{2})(\frac{1}{2} - \frac{1}{2} + 1)} |1 -1\rangle |\frac{1}{2} - \frac{1}{2}\rangle$$

$$\Rightarrow \sqrt{3} |\frac{3}{2} - \frac{3}{2}\rangle = \sqrt{2} |1 0\rangle |\frac{1}{2} - \frac{1}{2}\rangle + |1 -1\rangle |\frac{1}{2} + \frac{1}{2}\rangle$$

$$\Rightarrow |\frac{3}{2} - \frac{3}{2}\rangle = \sqrt{\frac{2}{3}} |1 0\rangle |\frac{1}{2} - \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |1 -1\rangle |\frac{1}{2} + \frac{1}{2}\rangle. \checkmark$$

VIII for  $j = \frac{1}{2}$  :  $m = \pm \frac{1}{2}$ .

$$\text{Let } |\frac{1}{2} \frac{1}{2}\rangle = C_1 |1 0\rangle |\frac{1}{2} \frac{1}{2}\rangle + C_2 |1 1\rangle |\frac{1}{2} - \frac{1}{2}\rangle$$

$$\Rightarrow J^+ |\frac{1}{2} \frac{1}{2}\rangle = 0 = C_1 J_1^+ |1 0\rangle |\frac{1}{2} \frac{1}{2}\rangle + C_2 J_2^+ |1 0\rangle |\frac{1}{2} \frac{1}{2}\rangle \\ + C_2 J_1^+ |1 1\rangle |\frac{1}{2} - \frac{1}{2}\rangle + C_1 J_2^+ |1 1\rangle |\frac{1}{2} - \frac{1}{2}\rangle$$

$$\Rightarrow \Psi = C_1 \sqrt{(1+0)(1+1-0)} |11\rangle |\frac{1}{2}\frac{1}{2}\rangle + 0 + 0 + \\ C_2 \sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}+1-\frac{1}{2})} |11\rangle |\frac{1}{2}\frac{1}{2}\rangle$$

$$\Rightarrow \Psi = C_1 \sqrt{2} + C_2 \Rightarrow C_2 = -C_1 \sqrt{2}$$

but  $C_1^2 + C_2^2 = 1 \Rightarrow C_1^2 + 2C_1^2 = 1 \Rightarrow C_1 = \frac{1}{\sqrt{3}}$  i.e.  $C_2 = -\sqrt{\frac{2}{3}}$ .

$$\therefore |\frac{1}{2}\frac{1}{2}\rangle = \frac{1}{\sqrt{3}} |10\rangle |\frac{1}{2}\frac{1}{2}\rangle + -\sqrt{\frac{2}{3}} |11\rangle |\frac{1}{2}-\frac{1}{2}\rangle \quad \checkmark$$

Ans Let:

$$|\frac{1}{2}-\frac{1}{2}\rangle = C_1 |10\rangle |\frac{1}{2}-\frac{1}{2}\rangle + C_2 |1-1\rangle |\frac{1}{2}\frac{1}{2}\rangle$$

$$\Rightarrow J^- |\frac{1}{2}-\frac{1}{2}\rangle = 0 = C_1 (J_1^- |10\rangle |\frac{1}{2}-\frac{1}{2}\rangle + J_2^- |10\rangle |\frac{1}{2}-\frac{1}{2}\rangle) \\ + C_2 (J_1^- |1-1\rangle |\frac{1}{2}\frac{1}{2}\rangle + J_2^- |1-1\rangle |\frac{1}{2}\frac{1}{2}\rangle)$$

$$= C_1 \sqrt{(1+0)(1+1-0)} |1-\rangle |\frac{1}{2}-\frac{1}{2}\rangle + C_2 \sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}+1-\frac{1}{2})} |1-\rangle |\frac{1}{2}-\frac{1}{2}\rangle$$

$$\Rightarrow \sqrt{2} C_1 + C_2 = 0 \Rightarrow C_2 = -C_1 \sqrt{2}$$

but  $C_1^2 + C_2^2 = 1 \Rightarrow C_1^2 + 2C_1^2 = 1 \Rightarrow C_1 = \frac{1}{\sqrt{3}}$  i.e.  $C_2 = -\sqrt{\frac{2}{3}}$

So;

$$|\frac{1}{2}-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}} |10\rangle |\frac{1}{2}-\frac{1}{2}\rangle + -\sqrt{\frac{2}{3}} |1-1\rangle |\frac{1}{2}\frac{1}{2}\rangle \quad \checkmark$$

~~Ques.~~ Finding the CG coefficients for  $J_1 = 1; J_2 = \frac{3}{2}$   
 Use  $|1\rangle$  for uncoupled state &  $|1\rangle|1\rangle$  for coupled state  
 $J_1 = 1; J_2 = \frac{3}{2}$ .

$$J = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}.$$

For,  $J = \frac{5}{2}; m = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}$ . (Obviously)

$$\Psi\left(\frac{5}{2}, \frac{5}{2}\right) = |11\rangle, |1\frac{3}{2}\frac{3}{2}\rangle$$

$$J^- |1\frac{5}{2}\frac{5}{2}\rangle = (J_1^- + J_2^-)(|11\rangle |1\frac{3}{2}\frac{3}{2}\rangle)$$

$$\Rightarrow \sqrt{\left(\frac{5}{2} + \frac{5}{2}\right) \cdot \left(\frac{5}{2} - \frac{5}{2} + 1\right)} |1\frac{5}{2}\frac{5}{2}\rangle = \sqrt{(1+1)(1-1+1)} |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle + \sqrt{2\left(\frac{3}{2} + \frac{3}{2}\right) \left(\frac{3}{2} - \frac{3}{2} + 1\right)} |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle$$

$$\Rightarrow \sqrt{5} |1\frac{5}{2}\frac{5}{2}\rangle = \sqrt{2} |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle + \sqrt{3} |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle$$

$$\Rightarrow |1\frac{5}{2}\frac{5}{2}\rangle = \frac{\sqrt{2}}{\sqrt{5}} |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle + \frac{\sqrt{3}}{\sqrt{5}} |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle$$

$$J^- |1\frac{5}{2}\frac{5}{2}\rangle = \frac{\sqrt{2}}{\sqrt{5}} \left( J_1^- |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle + J_2^- |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle \right) + \frac{\sqrt{3}}{\sqrt{5}} \left( J_1^- |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle + J_2^- |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle \right)$$

$$\sqrt{\left(\frac{5}{2} + \frac{3}{2}\right) \left(\frac{5}{2} - \frac{3}{2} + 1\right)} |1\frac{5}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{5}} \left( \sqrt{(1+0)(1-0+1)} |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle + \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right) \left(\frac{3}{2} - \frac{1}{2} + 1\right)} |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle \right)$$

$$\sqrt{\frac{3}{5}} \left( \sqrt{(1+1)(1-1+1)} |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle + \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right) \left(\frac{3}{2} - \frac{1}{2} + 1\right)} |11\rangle |1\frac{3}{2}\frac{-1}{2}\rangle \right)$$

$$\sqrt{24} \cdot \frac{4}{\chi} |1\frac{5}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{5}} \cdot \sqrt{2} |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle + \frac{\sqrt{2} \cdot \sqrt{3}}{\sqrt{5}} |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle$$

$$+ \frac{\sqrt{3} \cdot \sqrt{2}}{\sqrt{5}} |11\rangle |1\frac{3}{2}\frac{-1}{2}\rangle + \sqrt{\frac{3}{5}} \cdot 2 |11\rangle |1\frac{3}{2}\frac{-1}{2}\rangle$$

$$\Rightarrow \chi \sqrt{2} |1\frac{5}{2}\frac{1}{2}\rangle = \frac{2}{\sqrt{5}} |11\rangle |1\frac{3}{2}\frac{3}{2}\rangle + \frac{2\sqrt{6}}{\sqrt{5}} |11\rangle |1\frac{3}{2}\frac{1}{2}\rangle + 2 \cdot \sqrt{\frac{3}{5}} |11\rangle |1\frac{3}{2}\frac{-1}{2}\rangle$$

$$\left| \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \frac{1}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + \frac{\sqrt{6}}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{\sqrt{\frac{3}{10}}}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle$$

$$J_+ \left| \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \frac{1}{\sqrt{10}} \left[ J_+ \left| 1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + J_+ \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right]$$

$$+ \sqrt{\frac{6}{10}} \left[ J_+ \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + J_+ \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \right]$$

$$+ \sqrt{\frac{3}{10}} \left[ J_+ \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + J_+ \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle \right]$$

$$\Rightarrow \sqrt{\left( \frac{5}{2} + \frac{1}{2} \right) \left( \frac{5}{2} - \frac{1}{2} + 1 \right)} \left| \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \frac{1}{\sqrt{10}} \left[ 0 + \sqrt{\left( \frac{3}{2} + \frac{3}{2} \right) \left( \frac{3}{2} - \frac{3}{2} + 1 \right)} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right]$$

$$+ \sqrt{\frac{6}{10}} \left[ \sqrt{(1+0)(1-0+1)} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \sqrt{\left( \frac{3}{2} + \frac{1}{2} \right) \left( \frac{3}{2} - \frac{1}{2} + 1 \right)} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \right]$$

$$+ \sqrt{\frac{3}{10}} \left[ \sqrt{(1+1)(1-1+1)} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + \sqrt{\left( \frac{3}{2} - \frac{1}{2} \right) \left( \frac{3}{2} + \frac{1}{2} + 1 \right)} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle \right]$$

$$\cdot \sqrt{3 \cdot 3} \cdot \left| \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \frac{1}{\sqrt{10}} \cdot \sqrt{3} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{\sqrt{6}}{\sqrt{10}} \cdot \sqrt{2} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle$$

$$+ \sqrt{2 \cdot 2} \cdot \frac{\sqrt{6}}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + \sqrt{\frac{3}{10}} \cdot \sqrt{2} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

$$+ 0 \cdot \sqrt{\frac{3}{10}} \cdot \sqrt{3} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

$$\Rightarrow \left| \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle \cdot \sqrt{3} = \frac{1}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{2}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle$$

$$+ \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \left\{ \frac{2\sqrt{2}}{10} + \frac{\sqrt{2}}{10} \right\} + \sqrt{\frac{3}{10}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

$$\Rightarrow \left| \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \sqrt{\frac{3}{10}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{\sqrt{1}}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle$$

$$+ \frac{1}{\sqrt{10}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

$$J_- \left| \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \sqrt{\frac{3}{10}} \left[ J_- \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + J_- \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right]$$

$$+ \sqrt{\frac{6}{10}} \left[ J_- \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + J_- \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \right]$$

$$+ \sqrt{\frac{3}{10}} \left[ J_- \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle + J_- \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle \right]$$

$$\rightarrow \sqrt{\left(\frac{5}{2} - \frac{1}{2}\right)\left(\frac{5}{2} + \frac{1}{2} + 1\right)} \left| \frac{5}{2} - \frac{3}{2} \right\rangle = \sqrt{\frac{3}{10}} \left\{ 0 + \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right)\left(\frac{3}{2} + \frac{1}{2} + 1\right)} \left| 1,1 \right\rangle \right.$$

$$+ \sqrt{\frac{6}{10}} \left\{ \sqrt{(1+0)(1-0+1)} \left| 1,-1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)\left(\frac{3}{2} + \frac{1}{2} + 1\right)} \left| 1,0 \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle \right\}$$

$$+ \sqrt{\frac{1}{10}} \left\{ \sqrt{(1+1)(1-1+1)} \left| 1,0 \right\rangle \left| \frac{3}{2} + \frac{3}{2} \right\rangle + \cancel{\sqrt{6}}^0 \right\}$$

$$\rightarrow \sqrt{2 \cdot 4} \left| \frac{5}{2} - \frac{3}{2} \right\rangle = \sqrt{\frac{3}{10}} \cdot 2 \left| 1,-1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + \sqrt{\frac{6}{10}} \cdot \sqrt{2} \left| 1,-1 \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle$$

$$+ \sqrt{\frac{6}{10}} \cdot \sqrt{3} \left| 1,0 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle + \frac{1}{\sqrt{10}} \cdot \sqrt{2} \left| 1,0 \right\rangle \left| \frac{3}{2} + \frac{3}{2} \right\rangle$$

$$\Rightarrow \dots \cdot 2 \left| \frac{5}{2} - \frac{3}{2} \right\rangle = \left( \sqrt{\frac{6}{10}} + \sqrt{\frac{6}{10}} \right) \left| 1,-1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle$$

$$+ \left( \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} \cdot \left| 1,0 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle \right)$$

$$\rightarrow \left| \frac{5}{2} - \frac{3}{2} \right\rangle = \sqrt{\frac{6}{10}} \left| 1,-1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + \frac{2}{\sqrt{10}} \left| 1,0 \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle$$

And obviously.

$$\left| \frac{5}{2} - \frac{5}{2} \right\rangle = \left| 1,-1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

$J = \frac{3}{2}$ ;  ~~$m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$~~ ; Now let.

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle = c_1 |11\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + c_2 |10\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle$$

~~$\rightarrow \langle 11 | \left( \begin{array}{cc} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{array} \right) | 11 \rangle$~~

$$J_z | \frac{3}{2} \frac{3}{2} \rangle = 0 = c_1 \left[ J_1^+ |11\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right. \\ \left. + J_2^+ |11\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right] +$$

$$+ c_2 \left[ J_1^+ |10\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + J_2^+ |10\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle \right]$$

$$\Rightarrow 0 = c_1 \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)\left(\frac{3}{2} + \frac{1}{2} + 1\right)} |11\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + \\ c_2 \sqrt{(1-0)(1+0+1)} |10\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle$$

$$\Rightarrow c_1 \sqrt{1 \cdot 3} + c_2 \sqrt{2} = 0$$

$$\Rightarrow c_1 = -c_2 \cdot \sqrt{\frac{2}{3}} ; \quad c_1^2 + c_2^2 = 1$$

$$\Rightarrow c_2^2 + \frac{2}{3} c_2^2 = 1$$

$$\Rightarrow \frac{5}{3} c_2^2 = 1 \Rightarrow c_2 = -\sqrt{\frac{3}{5}}$$

$$\Rightarrow c_1 = \sqrt{\frac{2}{5}}$$

$$\therefore \left| \frac{3}{2} \frac{3}{2} \right\rangle = \sqrt{\frac{2}{5}} |11\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle - \sqrt{\frac{3}{5}} |10\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle$$

$$J^- | \frac{3}{2} \frac{3}{2} \rangle = \sqrt{\frac{2}{5}} \left[ J_1^- |11\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + J_2^- |11\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right] \\ - \sqrt{\frac{3}{5}} \left[ J_1^- |10\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + J_2^- |10\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle \right]$$

$$\sqrt{\left(\frac{3}{2} + \frac{3}{2}\right)\left(\frac{3}{2} - \frac{3}{2} + 1\right)} | \frac{3}{2} \frac{1}{2} \rangle = \sqrt{\frac{2}{5}} \left[ \sqrt{(1+1)(1-1+1)} |10\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right. \\ \left. + \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right)\left(\frac{3}{2} - \frac{1}{2} + 1\right)} |11\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \right]$$

$$- \sqrt{\left(\frac{3}{2} + \frac{3}{2}\right)\left(\frac{3}{2} - \frac{3}{2} + 1\right)} |10\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \sqrt{\left(\frac{3}{2} + \frac{3}{2}\right)\left(\frac{3}{2} - \frac{3}{2} + 1\right)} |10\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle$$

$$\Rightarrow \sqrt{3} \left| \begin{smallmatrix} 3 & 1 \\ 2 & 2 \end{smallmatrix} \right\rangle = \sqrt{\frac{2}{5}} \left[ \sqrt{2} \left| \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 3 & 1 \\ 2 & 2 \end{smallmatrix} \right\rangle + 2 \left| \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 3 & -1 \\ 2 & -2 \end{smallmatrix} \right\rangle \right]$$

$$= \sqrt{\frac{3}{5}} \left[ \sqrt{2} |1-1\rangle | \frac{3}{2} \frac{3}{2} \rangle + \sqrt{3} |10\rangle | \frac{3}{2} \frac{1}{2} \rangle \right]$$

$$\rightarrow \left| \frac{3}{2} \frac{1}{2} \right\rangle = \frac{\sqrt{3}}{\sqrt{15}} \left| 10 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{2\sqrt{2}}{\sqrt{15}} \left| 11 \right\rangle \left| \frac{3}{2} -\frac{1}{2} \right\rangle$$

$$-\frac{\sqrt{6}}{\sqrt{15}} |1-1\rangle \left| \frac{3}{2}, \frac{3}{2} \right\rangle$$

$$= -\frac{1}{\sqrt{15}} |10\rangle |1\rangle \left| \begin{smallmatrix} \frac{3}{2} & \frac{1}{2} \end{smallmatrix} \right\rangle + \frac{2\sqrt{2}}{\sqrt{15}} |11\rangle |1\rangle \left| \begin{smallmatrix} \frac{3}{2} & -\frac{1}{2} \end{smallmatrix} \right\rangle$$

$$- \sqrt{\frac{6}{15}} |11\rangle |1\rangle \left| \begin{smallmatrix} \frac{3}{2} & \frac{3}{2} \end{smallmatrix} \right\rangle$$

$$\begin{aligned} J_+ \left| \frac{3}{2}, \frac{1}{2} \right\rangle &= -\frac{1}{\sqrt{15}} \left[ J_1^+ \left| 1, 0 \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle + J_2^+ \left| 1, 0 \right\rangle \left| \frac{3}{2}, \frac{1}{2} \right\rangle \right] \\ &\quad + \frac{2\sqrt{2}}{\sqrt{15}} \left[ J_1^+ \left| 1, 1 \right\rangle \left| \frac{3}{2}, -\frac{1}{2} \right\rangle + J_2^+ \left| 1, 1 \right\rangle \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \right] \\ \Phi &= \sqrt{\frac{6}{15}} \left[ J_1^+ \cancel{\left| 1, -1 \right\rangle} \left| \frac{3}{2}, \frac{3}{2} \right\rangle + J_2^+ \left| 1, -1 \right\rangle \left| \frac{3}{2}, \frac{3}{2} \right\rangle \right] \end{aligned}$$

$$\Rightarrow \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right)\left(\frac{3}{2} - \frac{1}{2} + 1\right)} \left| \begin{matrix} \frac{3}{2} \\ \frac{1}{2} \end{matrix} \right\rangle = \frac{1}{\sqrt{15}} \left[ \sqrt{(1+0)(1-0+1)} \left| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle \left| \begin{matrix} \frac{3}{2} \\ \frac{1}{2} \end{matrix} \right\rangle \right.$$

$$+ \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right)\left(\frac{3}{2} - \frac{1}{2} + 1\right)} \left| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle \left| \begin{matrix} \frac{3}{2} \\ -\frac{1}{2} \end{matrix} \right\rangle \left. \right]$$

$$+ \frac{2\sqrt{2}}{\sqrt{15}} \left[ \sqrt{(1+1)(1-1+1)} \left| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle \left| \begin{matrix} \frac{3}{2} \\ -\frac{1}{2} \end{matrix} \right\rangle + \sqrt{\left(\frac{3}{2} + \frac{3}{2}\right)\left(\frac{3}{2} - \frac{3}{2} + 1\right)} \left| \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle \left| \begin{matrix} \frac{3}{2} \\ \frac{1}{2} \end{matrix} \right\rangle \right]$$

$$\left. \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)\left(\frac{3}{2} + \frac{1}{2} + 1\right)} \left| \begin{matrix} 0 \\ 1 \end{matrix} \right\rangle \left| \begin{matrix} \frac{3}{2} \\ -\frac{3}{2} \end{matrix} \right\rangle \right]$$

$$- \frac{4}{\sqrt{15}} \left[ 0 + \sqrt{\left(\frac{3}{2} + \frac{3}{2}\right)\left(\frac{3}{2} - \frac{3}{2} + 1\right)} \left| \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle \left| \begin{matrix} \frac{3}{2} \\ \frac{1}{2} \end{matrix} \right\rangle \right]$$

$$|\psi\rangle = -\frac{1}{\sqrt{15}} \left[ \sqrt{2} |1-\rangle |\frac{3}{2}\frac{1}{2}\rangle + 2 |10\rangle |\frac{3}{2}-\frac{1}{2}\rangle \right]$$

$$+ \frac{2\sqrt{2}}{\sqrt{15}} \left[ \sqrt{2} |1,0\rangle |\frac{3}{2}, -\frac{1}{2}\rangle + \sqrt{3} |1,1\rangle |\frac{3}{2}, -\frac{3}{2}\rangle \right] \oplus$$

$$= \sqrt{\frac{6}{15}} \left( \frac{1}{\sqrt{3}} |1\rangle + |1\rangle \right) \left| \frac{3}{2} \right\rangle \frac{1}{2} \rangle$$

$$|0\rangle = |1\downarrow\rangle |1\frac{3}{2}\frac{1}{2}\rangle \left\{ -\frac{\sqrt{2}}{\sqrt{15}} - \frac{\sqrt{19}}{\sqrt{15}} \right\}$$

$$+ |110\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \left\{ \frac{4}{\sqrt{15}} \right\} - \frac{2}{\sqrt{15}} \}$$

$$+ \frac{2\sqrt{6}}{\sqrt{15}} \left(1, 1\right) \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$$

$$, 8 \left| \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right. - \left| \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right. \right) = \left( 1 \rightarrow \left| \begin{smallmatrix} 3 & 1 \\ 2 & 2 \end{smallmatrix} \right. \right) \left\{ - \frac{1}{\sqrt{30}} - \frac{3}{\sqrt{30}} \right\}$$

$$+ |10\rangle | \frac{3}{2} - \frac{1}{2} \rangle \left\{ - \frac{\sqrt{2}}{\sqrt{30}} \right\} + \frac{2\sqrt{13}}{\sqrt{30}} |11\rangle | \frac{3}{2} - \frac{3}{2} \rangle$$

$$|2\rangle = \frac{1}{\sqrt{30}} |1-\rangle |\frac{3}{2}, \frac{1}{2}\rangle + \frac{\sqrt{2}}{\sqrt{30}} |1+\rangle |\frac{3}{2}, -\frac{1}{2}\rangle$$

$$+ \frac{2\sqrt{3}}{\sqrt{30}} |1+> | \frac{1}{2} - \frac{3}{2} >$$

$$J = \left( \begin{matrix} 3 & 2 \\ 2 & 2 \end{matrix} \right) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Am. Vet:

$$\text{Ansatz: } c_1 |1\downarrow\rangle \left| \begin{smallmatrix} \frac{3}{2} & -\frac{1}{2} \end{smallmatrix} \right\rangle + c_2 |1\cdot 0\rangle \left| \begin{smallmatrix} \frac{3}{2} & -\frac{3}{2} \end{smallmatrix} \right\rangle$$

$$\left| \frac{3}{2} - \frac{3}{2} \right\rangle = G_1 \left( \left| 1+1 \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \left| 1-1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \right) + G_2 \left( \left| 1-1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + \left| 1+1 \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle \right)$$

$$J = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = G_1 \left\{ \begin{array}{l} J_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \\ J_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right.$$

$$+ C_2 \left[ J_1 |10\rangle \left| \begin{smallmatrix} 3 & -3 \\ 2 & 2 \end{smallmatrix} \right\rangle + J_2 |10\rangle \left| \begin{smallmatrix} 2 & -2 \\ 2 & 2 \end{smallmatrix} \right\rangle \right]$$

$$= \boxed{(3-1) \left( \frac{3}{2} + \frac{1}{2} + 1 \right)} |1-1\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

$$0 = g \cdot \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)\left(\frac{3}{2} + \frac{1}{2} + 1\right)} |1\rangle \langle 1| - g \cdot \sqrt{(1+0)(1-0+1)} |1\rangle \langle \frac{3}{2} - \frac{3}{2}|$$

$$2 C_1 \sqrt{\frac{3}{2}} = -C_2 \sqrt{2} \Rightarrow C_2 = \frac{\sqrt{3} C_1}{\sqrt{2}}$$

$$3. \quad C_1 \sqrt{3} = -C_2 \sqrt{2} \Rightarrow C_1 = -C_2 \sqrt{\frac{2}{3}}$$

$$C_1^2 + C_2^2 = 1 \Rightarrow \frac{2}{3} C_2^2 + C_2^2 = 1 \Rightarrow \frac{5}{3} C_2^2 = 1$$

$$\Rightarrow C_2 = \sqrt{\frac{3}{5}} ; \quad C_1 = -\sqrt{\frac{2}{5}}$$

$$\therefore \left| \frac{3}{2} - \frac{3}{2} \right\rangle = -\sqrt{\frac{2}{5}} \left| 1+1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle + \sqrt{\frac{3}{5}} \left| 10 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

Now for  $J = \frac{1}{2}$ ;  $m = \pm \frac{1}{2}$ . Let.

$$\left| \frac{1}{2} \frac{1}{2} \right\rangle = C_1 \left| 10 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + C_2 \left| 1-1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle \\ + C_3 \left| 101 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle$$

$$J_z \left| \frac{1}{2} \frac{1}{2} \right\rangle = 0 = g \left[ J_1^+ \left| 10 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + J_2^+ \left| 10 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right]$$

$$+ C_2 \left[ J_1^+ \left| 1-1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + J_2^+ \left| 1-1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle \right]$$

$$+ C_3 \left[ J_1^+ \left| 10 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + J_2^+ \left| 10 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \right]$$

$$= g \left[ \sqrt{(1-0)(1+0+1)} \left| 11 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \sqrt{(\frac{3}{2} - \frac{1}{2})(\frac{3}{2} + \frac{1}{2} + 1)} \left| 10 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \right]$$

~~$$C_2 \left[ \sqrt{(1+1)(1+1-1)} \left| 11 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle \right]$$~~

$$+ C_3 \left[ \sqrt{(\frac{3}{2} + \frac{1}{2})(\frac{3}{2} - \frac{1}{2} + 1)} \left| 11 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle \right]$$

$$= (\sqrt{2} C_1 + 2 C_3) \left| 11 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle +$$

$$(\sqrt{2} C_2 + \sqrt{3} C_1) \left| 10 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle$$

$$S_0: \sqrt{2}G_1 + 2G_3 = 0 \Rightarrow G_3 = -\frac{G_1}{\sqrt{2}}$$

$$\sqrt{2}G_2 + \sqrt{3}G_1 = 0 \Rightarrow G_2 = -\frac{\sqrt{3}G_1}{\sqrt{2}}$$

$$G_1^2 + G_2^2 + G_3^2 = 1$$

$$\Rightarrow G_1^2 + \frac{G_1^2}{2} + \frac{3G_1^2}{2} = 1 \Rightarrow 3G_1^2 = 1 \Rightarrow G_1 = \frac{1}{\sqrt{3}}$$

$$\therefore G_2 = -\frac{1}{\sqrt{2}}; G_3 = -\frac{1}{\sqrt{6}}$$

$$\therefore \left| \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right\rangle = \frac{1}{\sqrt{3}} |10\rangle \left| \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \end{array} \right\rangle + -\frac{1}{\sqrt{2}} |11\rangle \left| \begin{array}{c} \frac{3}{2} \\ \frac{3}{2} \end{array} \right\rangle$$

$$- \frac{1}{\sqrt{6}} |111\rangle \left| \begin{array}{c} \frac{3}{2} \\ -\frac{1}{2} \end{array} \right\rangle \quad \checkmark$$

And finally let:

$$\left| \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \end{array} \right\rangle = G_1 |11\rangle \left| \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \end{array} \right\rangle + G_2 |10\rangle \left| \begin{array}{c} \frac{3}{2} \\ -\frac{1}{2} \end{array} \right\rangle + G_3 |101\rangle \left| \begin{array}{c} \frac{3}{2} \\ -\frac{3}{2} \end{array} \right\rangle$$

$$\begin{aligned} J^- \left| \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \end{array} \right\rangle &= 0 = G_1 [J_1^- |11\rangle \left| \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \end{array} \right\rangle + J_2^- |11\rangle \left| \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \end{array} \right\rangle] \\ &\quad + G_2 [J_1^- |10\rangle \left| \begin{array}{c} \frac{3}{2} \\ -\frac{1}{2} \end{array} \right\rangle + J_2^- |10\rangle \left| \begin{array}{c} \frac{3}{2} \\ -\frac{1}{2} \end{array} \right\rangle] \\ &\quad + G_3 [J_1^- |101\rangle \left| \begin{array}{c} \frac{3}{2} \\ -\frac{3}{2} \end{array} \right\rangle + J_2^- |101\rangle \left| \begin{array}{c} \frac{3}{2} \\ -\frac{3}{2} \end{array} \right\rangle] \end{aligned}$$

~~$$= G_1 [0 + \sqrt{(1-1)(1+1-1)} |11\rangle \left| \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \end{array} \right\rangle +$$~~

~~$$+ G_2 [\sqrt{(1-0)(1+0+1)} \left| \begin{array}{c} \frac{3}{2} \\ -\frac{1}{2} \end{array} \right\rangle \left( \frac{3}{2} + \frac{1}{2} + 1 \right) +$$~~

$$\begin{aligned}
 &= C_1 \left[ 0 + \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right) \left(\frac{3}{2} + 1 - \frac{1}{2}\right)} |1\rightarrow\rangle |\frac{3}{2} - \frac{1}{2}\rangle \right] + \\
 &C_2 \left[ \sqrt{(1-0)(1+1+0)} |1\rightarrow\rangle |\frac{3}{2} - \frac{1}{2}\rangle + \right. \\
 &\quad \left. \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right) \left(\frac{3}{2} + 1 + \frac{1}{2}\right)} |1\circlearrowleft\rangle |\frac{3}{2} - \frac{3}{2}\rangle \right] \\
 &+ C_3 \left[ \sqrt{(1+1)(1+1-1)} |1\circlearrowleft\rangle |\frac{3}{2} - \frac{3}{2}\rangle + 0 \right]
 \end{aligned}$$

$$\Rightarrow 0 = |1\rightarrow\rangle |\frac{3}{2} - \frac{1}{2}\rangle \left[ C_1 \cdot 2 + \sqrt{2} C_2 \right] \\
 \quad + |1\circlearrowleft\rangle |\frac{3}{2} - \frac{3}{2}\rangle \left[ \sqrt{2} C_3 + \sqrt{3} C_2 \right]$$

$$\left. \begin{array}{l} \text{So;} \quad 2C_1 + \sqrt{2} C_2 = 0 \\ \quad \sqrt{2} C_3 + \sqrt{3} C_2 = 0 \end{array} \right\} \text{ie } \begin{array}{l} C_1 = -\frac{C_2}{\sqrt{2}} \\ C_3 = -\frac{C_2 \sqrt{3}}{\sqrt{2}} \end{array} \right\}$$

$$\begin{aligned}
 \text{But } &|C_1|^2 + |C_2|^2 + |C_3|^2 = 1 \\
 \Rightarrow &\frac{C_2^2}{2} + \frac{3C_2^2}{2} + C_2^2 = 1 \Rightarrow 3C_2^2 = 1 \\
 \Rightarrow &C_2 = \frac{1}{\sqrt{3}} ; \text{ ie } C_1 = -\frac{1}{\sqrt{6}} ; C_3 = \frac{1}{\sqrt{2}}.
 \end{aligned}$$

$$\begin{aligned}
 \text{So;} \quad &|\frac{1}{2} - \frac{1}{2}\rangle = -\frac{1}{\sqrt{6}} |1\rightarrow\rangle |\frac{3}{2} - \frac{1}{2}\rangle + \frac{1}{\sqrt{3}} |1\circlearrowleft\rangle |\frac{3}{2} - \frac{1}{2}\rangle \\
 &\quad - \frac{1}{\sqrt{2}} |1\circlearrowleft\rangle |\frac{3}{2} - \frac{3}{2}\rangle. \quad \checkmark
 \end{aligned}$$

I was doing the problem informally in a rough work in my notebook. However ~~as~~ all the calculations are ~~so~~ large & messy. I'm not going to copy them again on any paper. This is the rough I've done. However I'm finally rewriting all the C.G.C.D in a fresh way.

i.e.) Stern Gerlach coefficients (finally) for  $J_1 = 1; J_2 = \frac{3}{2}$

IV For  $J = \frac{5}{2}$  : ( $m = \pm \frac{5}{2}, \pm \frac{3}{2}, \pm \frac{1}{2}$ )

$$i) \left| \frac{5}{2} \frac{5}{2} \right\rangle = \left| 1 1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle$$

$$ii) \left| \frac{5}{2} \frac{3}{2} \right\rangle = \sqrt{\frac{2}{5}} \left| 1 0 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + \sqrt{\frac{3}{5}} \left| 1 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle$$

$$iii) \left| \frac{5}{2} \frac{1}{2} \right\rangle = \sqrt{\frac{1}{10}} \left| 1 -1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle + \sqrt{\frac{6}{10}} \left| 1 0 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \\ + \sqrt{\frac{3}{10}} \left| 1 1 \right\rangle \left| \frac{3}{2} -\frac{3}{2} \right\rangle$$

$$iv) \left| \frac{5}{2} -\frac{1}{2} \right\rangle = \sqrt{\frac{3}{10}} \left| 1 -1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \sqrt{\frac{6}{10}} \left| 1 0 \right\rangle \left| \frac{3}{2} -\frac{1}{2} \right\rangle \\ + \frac{1}{\sqrt{10}} \left| 1 1 \right\rangle \left| \frac{3}{2} -\frac{3}{2} \right\rangle$$

$$v) \left| \frac{5}{2} -\frac{3}{2} \right\rangle = \sqrt{\frac{6}{10}} \left| 1 -1 \right\rangle \left| \frac{3}{2} -\frac{1}{2} \right\rangle + \frac{2}{\sqrt{10}} \left| 1 0 \right\rangle \left| \frac{3}{2} -\frac{3}{2} \right\rangle$$

$$vi) \left| \frac{5}{2} -\frac{5}{2} \right\rangle = \left| 1 -1 \right\rangle \left| \frac{3}{2} -\frac{3}{2} \right\rangle$$

VII For  $J = \frac{3}{2}$  : ( $m = \pm \frac{3}{2}, \pm \frac{1}{2}$ ) :

$$i) \left| \frac{3}{2} \frac{3}{2} \right\rangle = \sqrt{\frac{2}{5}} \left| 1 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle \mp \sqrt{\frac{3}{5}} \left| 1 0 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle$$

$$ii) \left| \frac{3}{2} \frac{1}{2} \right\rangle = -\frac{1}{\sqrt{15}} \left| 1 0 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{2\sqrt{2}}{\sqrt{15}} \left| 1 1 \right\rangle \left| \frac{3}{2} -\frac{1}{2} \right\rangle \\ - \sqrt{\frac{6}{15}} \left| 1 -1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle$$

$$iii) \left| \frac{3}{2} -\frac{1}{2} \right\rangle = \cancel{\sqrt{\frac{2}{30}} \left| 1 0 \right\rangle \left| \frac{3}{2} -\frac{1}{2} \right\rangle} \quad \sqrt{\frac{2}{30}} \left| 1 0 \right\rangle \left| \frac{3}{2} -\frac{1}{2} \right\rangle$$

$$\bullet -\frac{4}{\sqrt{30}} \left| 1 -1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{2\sqrt{3}}{\sqrt{30}} \left| 1 1 \right\rangle \left| \frac{3}{2} -\frac{3}{2} \right\rangle$$

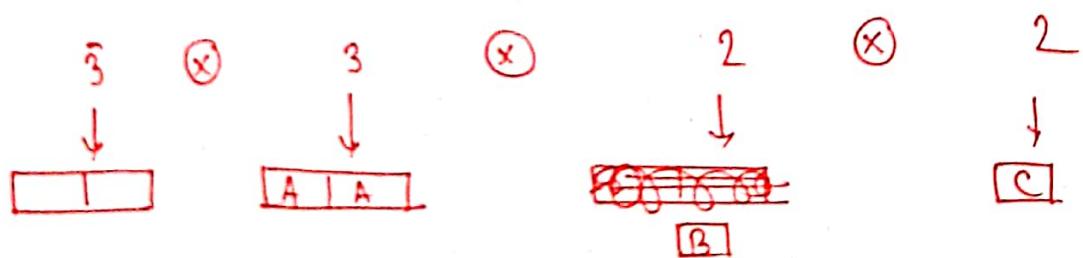
$$1) \left| \frac{3}{2} - \frac{3}{2} \right\rangle = -\sqrt{\frac{2}{5}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle + \sqrt{\frac{3}{5}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

2)  $J = \frac{1}{2}; m = \pm \frac{1}{2}$

$$3) \left| \frac{1}{2} \frac{1}{2} \right\rangle = \frac{1}{\sqrt{3}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle - \frac{1}{\sqrt{2}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{3}{2} \right\rangle \\ - \frac{1}{\sqrt{6}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle$$

$$4) \left| \frac{1}{2} - \frac{1}{2} \right\rangle = -\frac{1}{\sqrt{6}} \left| 1 \right\rangle \left| \frac{3}{2} \frac{1}{2} \right\rangle + \frac{1}{\sqrt{3}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{1}{2} \right\rangle \\ - \frac{1}{\sqrt{2}} \left| 1 \right\rangle \left| \frac{3}{2} - \frac{3}{2} \right\rangle$$

Q 1.f Let the representations of the  $3 \otimes 3 \otimes 2 \otimes 2$  blocks be given by:



$$i) 3 \otimes 3$$

$$= \boxed{\square \square} \otimes \boxed{A \mid A} = \boxed{\square \quad \boxed{AA}} \oplus \boxed{\begin{array}{c} A \\ A \end{array} \mid A} \oplus \boxed{\begin{array}{c} A \\ A \end{array} \mid \begin{array}{c} A \\ A \end{array}}$$

$D=5 \qquad D=3 \qquad D=1$

$$ii) 3 \otimes 3 \otimes 2$$

$$= \left[ \boxed{\square \quad \boxed{AA}} \oplus \boxed{\begin{array}{c} A \\ A \end{array} \mid A} \oplus \boxed{\begin{array}{c} A \\ A \end{array} \mid \begin{array}{c} A \\ A \end{array}} \right] \otimes \boxed{B}$$

$$= \left\{ \begin{array}{|c|c|c|c|} \hline & A & A & B \\ \hline D = 6 & \end{array} \right\} \oplus \left\{ \begin{array}{|c|c|c|} \hline & A & A \\ \hline B & \end{array} \right\} \oplus \left\{ \begin{array}{|c|c|c|} \hline A & B \\ \hline A & \end{array} \right\} \oplus \left\{ \begin{array}{|c|c|c|} \hline A & B \\ \hline A & B \\ \hline \end{array} \right\} \quad D = 2$$

$$\oplus \begin{array}{|c|c|c|} \hline & & B \\ \hline A & A \\ \hline \end{array}$$

$$D = 2$$

$$(iii) 3 \otimes 3 \otimes 3 \otimes 2$$

$$= \left[ \begin{array}{|c|c|c|c|} \hline & A & A & B \\ \hline & & B \\ \hline \end{array} \right] \oplus \begin{array}{|c|c|c|} \hline & A & A \\ \hline B & \end{array} \oplus \begin{array}{|c|c|c|} \hline & A & B \\ \hline A & \end{array} \oplus \begin{array}{|c|c|c|} \hline A & B \\ \hline A & B \\ \hline \end{array}$$

$\oplus \begin{array}{|c|c|c|} \hline & & B \\ \hline A & A \\ \hline \end{array} \right] \otimes C$

$$= \left[ \begin{array}{|c|c|c|c|} \hline & A & A & B \\ \hline & & B \\ \hline & & C \\ \hline \end{array} \right] \oplus \cancel{\begin{array}{|c|c|c|c|} \hline & A & A & C \\ \hline & & B \\ \hline & & C \\ \hline \end{array}} \quad D = 5$$

$$\oplus \begin{array}{|c|c|c|c|} \hline & A & A & C \\ \hline & & B \\ \hline & & C \\ \hline \end{array} \quad D = 5$$

$$\oplus \begin{array}{|c|c|c|} \hline & A & A \\ \hline B & C \\ \hline \end{array} \quad D = 3$$

$$\oplus \begin{array}{|c|c|c|} \hline & A & B \\ \hline A & \end{array} \quad D = 5$$

$$\oplus \begin{array}{|c|c|c|} \hline & A & B \\ \hline A & C \\ \hline \end{array} \quad D = 3$$

$$\begin{array}{|c|c|c|} \hline & A & C \\ \hline A & B \\ \hline \end{array} \quad D = 3$$

$$\oplus$$

$$\begin{array}{|c|c|c|} \hline & & A \\ \hline A & B & C \\ \hline \end{array} \quad D = 1$$

$$\oplus$$

$$\begin{array}{|c|c|c|} \hline & & C \\ \hline A & A \\ \hline \end{array} \quad D = 3$$

$$\oplus \begin{array}{|c|c|c|} \hline & & B \\ \hline A & A & C \\ \hline \end{array} \quad D = 1$$

Ans

~~This~~ These are the irreducible combinations of  $3 \otimes 3 \otimes 3 \otimes 2$ . Using Young tableau.

1.9. Two identical  $\frac{5}{2}$  Spins are combining.

$$\therefore \frac{5}{2} \text{ has } 6 \text{ m values: } \begin{aligned} \pm \frac{5}{2} &\rightarrow [a], [d] \\ \pm \frac{3}{2} &\rightarrow [b], [e] \\ \pm \frac{1}{2} &\rightarrow [c], [f] \end{aligned} \quad \left. \right\}$$

The Symmetrical States are given by:

[aa]	[bb]	[cc]	[dd]	[ee]	[ff]
[ab]	[bc]	[cd]	[de]	[ef]	
[ac]	[bd]	[ce]	[df]		
[ad]	[be]	[cf]			
[ae]	[bf]				
[af]					

Total no of States  
 $1+2+\dots+6 = 21.$

Antisymmetric States:

[a]	[b]	[c]	[d]	[e]
[b]	[c]	[d]	[e]	
[a]	[d]	[e]		
[a]	[c]	[b]		
[a]				

Total States  
 $1+2+\dots+5 = 15.$

for Spin  $j$ : total possible <sup>no of</sup> values of  $m = (2j+1)$

All Symmetric States are given ~~comin~~ combinations

from ~~[+][+]~~ ... ~~[-][+]~~ ... ~~[-][+]~~ So no of  
 Symmetric States:  $1+2+\dots+(2j+1) = \frac{(2j+1)(2j+2)}{2} = (j+1)(2j+1)$

For Antisymmetric States the sum goes from

$$1+2+3+\dots+2j = \frac{2j(2j+1)}{2} = j(2j+1).$$

Ans

## Question: 2

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$$\underline{2.a} \quad \boxed{\hat{H}|m, l, j, m\rangle = E_m |m, l, j, m\rangle}$$

with  $E_m = -\frac{m^2 e^4}{2\hbar^2 \times 16\pi^2 \epsilon_0^2 m^2} = -\frac{m^2 e^4}{32\hbar^2 \pi^2 \epsilon_0^2 m^2}$

but fine structure const  $\alpha = \frac{e^2}{4\pi \epsilon_0 \hbar c}$

$$\text{So, } \alpha^2 c^2 = \frac{e^4}{16\pi^2 \epsilon_0^2 \hbar^2}$$

$$\therefore E_m = -\frac{m^2}{2m^2} \times \frac{e^4}{16\pi^2 \hbar^2 \epsilon_0^2} = -\frac{m^2 \alpha^2 c^2}{2m^2}$$

Ans

$$\boxed{\hat{L}^2|m, l, j, m\rangle = l(l+1)\hbar^2 |m, l, j, m\rangle}$$

Here the eigenvalue of  $\hat{L}^2$  is already expressed in terms of those quantities. i.e  $\boxed{l(l+1)\hbar^2}$

$$\boxed{\hat{j}^2|m, l, j, m\rangle = j(j+1)\hbar^2 |m, l, j, m\rangle}$$

$\therefore$  eigenvalue of  $\hat{j}^2$  is:  $\boxed{j(j+1)\hbar^2}$

$$\boxed{\hat{J}_z|m, l, j, m\rangle = mh |m, l, j, m\rangle}$$

$\therefore$  eigenvalue of  $\hat{J}_z$ :  $\boxed{mh}$ .

2.b. Here  $m$  can take any value from 1 to  $\infty$

for  $m=1$ ;  $E_m = -13.6 \text{ ev}$ ; for  $m \rightarrow \infty$ ;  $E_m \rightarrow 0$

i.e the electron becomes free at  $m=\infty$ . It makes the scattering continuum

For a particular value of  $m$ ; the value of  $l$  can change from 0 to  $m-1$ .  
i.e  $l = 0, 1, 2, \dots, m-1$ . (total  $m$  values)

Here ~~J~~  $J = L + S$ ; So according to addition of

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Angular momentum;  $J$  can take values of

$$J = L - \frac{1}{2} \neq L + \frac{1}{2} \quad \begin{cases} \text{if } L \neq 0 \\ \text{if } L = 0 \end{cases} \quad \left. \begin{array}{l} \text{(AS } S = \frac{1}{2} \text{)} \\ \text{for } e^- \end{array} \right\}$$
$$= \frac{1}{2}$$

For a particular value of  $J$ ;  $m$  can take  $(2j+1)$  values of:

$$m = -j, -j+1, \dots, j-1, j$$

These are the bound~~s~~ on  $(n, l, j, m)$

If an  $e^-$  stay at  $2p_1^3$  state

$$n=2; l=1; j = \frac{3}{2}, \frac{1}{2} \quad \left. \begin{array}{l} \text{for } j = \frac{3}{2}: m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \\ \text{for } j = \frac{1}{2}: m = -\frac{1}{2}, \frac{1}{2} \end{array} \right\}$$

or lower

2.c. We know  $J_{\pm}$  lift~~s~~, the  $m_j$  value by 1.

$$\text{so: i) } \langle 3, 1, \frac{3}{2}, \frac{3}{2} | J_+ | 3, 1, \frac{3}{2}, -\frac{1}{2} \rangle$$

$$= \hbar \sqrt{\left(\frac{3}{2} + \frac{1}{2}\right)\left(\frac{3}{2} - \frac{1}{2} + 1\right)} \underbrace{\langle 3, 1, \frac{3}{2}, \frac{3}{2} |}_{0} \langle 3, 1, \frac{3}{2}, \frac{1}{2} \rangle$$

$$= 0$$

$$(\because \langle m, l, j, m' | m' \cdot l' | j' m' \rangle = \delta_{mm'} \delta_{ll'} \delta_{jj'} \delta_{mm'})$$

$$\text{ii) } \langle 3, 1, \frac{3}{2}, \frac{3}{2} | J_+ | 3, 1, \frac{3}{2}, \frac{1}{2} \rangle$$

$$= \hbar \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)\left(\frac{3}{2} + \frac{1}{2} + 1\right)} \underbrace{\langle 3, 1, \frac{3}{2}, \frac{3}{2} |}_{1} \langle 3, 1, \frac{3}{2}, \frac{3}{2} \rangle$$

$$= \hbar \sqrt{1 \times 3}$$

$$= \hbar \sqrt{3}.$$

$$^{(3)}) \quad \left\langle 2, 1, \frac{3}{2}, \frac{3}{2} \mid p_z \mid 2, 1, \frac{3}{2}, \frac{1}{2} \right\rangle =$$

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$$\text{iv) } \langle 2, 1, \frac{1}{2}, -\frac{1}{2} | L^2 | 2, 1, \frac{1}{2}, -\frac{1}{2} \rangle$$

$$= 1(1+1) \hbar^2 \langle 2, 1, \frac{1}{2}, -\frac{1}{2} | 2, 1, \frac{1}{2}, -\frac{1}{2} \rangle$$

$$= 2\pi^2$$

$$v) \left\langle 3, 2, \frac{3}{2}, -\frac{1}{2} \right| J^2 \left| 3, 2, \frac{3}{2}, -\frac{1}{2} \right\rangle$$

$$= \frac{3}{2} \left( \frac{3}{2} + 1 \right) \hbar^2 \left\langle 3, 2, \frac{3}{2}, -\frac{1}{2} \mid 3, 2, \frac{3}{2}, -\frac{1}{2} \right\rangle$$

$$= \frac{15}{4} h^2$$

$$vi) \langle 3, 1, \frac{3}{2}, \frac{3}{2} | J_2 | 3, 1, \frac{3}{2}, \frac{1}{2} \rangle$$

$$= \frac{1}{2} \hbar \left\langle 3, 1, \frac{3}{2}, \frac{3}{2} \mid 3, 1, \frac{3}{2}, \frac{3}{2} \right\rangle = \frac{\hbar}{2}$$

$$2-2. \quad \langle 10\frac{1}{2}\frac{1}{2} | \hat{p}_i \hat{p}_j | 10\frac{1}{2}\frac{1}{2} \rangle$$

$\psi_{ij}$  can be written as sum of tensor operators of rank

$$q = 0, 1, 2$$

but for  $r=2; j_1 = \frac{1}{2} \Rightarrow C.G$  coefficient will be 0 since

$2 \neq \frac{1}{2}$  can give  $\vec{v} = \frac{3}{2}, \frac{5}{2}$

$$\therefore q = 0,1$$

$$\text{for } q=1: \langle 10\frac{1}{2}\frac{1}{2} | T_0 | 10\frac{1}{2}\frac{1}{2} \rangle = \langle 10\frac{1}{2}\frac{1}{2} | 1\frac{1}{2} 0\frac{1}{2} \rangle_{\text{cusp}}^*$$

$$= 0$$

$$\stackrel{?}{=} e \langle m' l' j' m' | \vec{\sigma} \cdot \vec{r} | m l j m \rangle$$

$\vec{\sigma} \cdot \vec{r}$  is a rank 0 tensor. (Scalar)

∴ By Wigner & Eckart theorem:

$$\langle m' l' j' m' | \vec{\sigma} \cdot \vec{r} | m l j m \rangle \propto \langle j_1^0 j_2^0 j_3^0 m' | j_1^0 j_2^0 m_2^0 m_3^0 \rangle$$

$$\neq 0 \text{ iff } j' = j; m' = m + 0 = m$$

Also  $\vec{\sigma} \cdot \vec{r}$  under parity goes to  $-\vec{\sigma} \cdot \vec{r}$ .

¶ under parity  $|l\rangle \rightarrow (-1)^L |l\rangle$

∴ The overall parity must be (+) for the matrix element to be non zero.

$$\text{i.e. } (-1)^{L+L'+l} = 1 \Rightarrow L+L'+l = \text{even.}$$

$$\therefore L+L' = \text{odd.}$$

$$\text{but } \vec{\sigma} = \vec{\sigma}_L + \vec{\sigma}_S \quad \& \quad j' = j \Rightarrow \begin{cases} L' = L \\ S' = S = \frac{1}{2} \end{cases} \quad \begin{cases} \text{either } L' = L \\ \text{or } L - L' = \pm 1. \end{cases}$$

But  $L = L'$  is not possible as  $L+L' = \text{odd.}$

So the condition is:

$$j' = j, m' = m \quad \& \quad L' - L = \pm 1. \quad \underline{\text{Ans}}$$

Question : 3

$$3.a \quad \langle \psi | \psi \rangle = 1 = \cancel{N} \cancel{\int R_2(r) d\tau}$$

$$\text{but } \psi(r, \theta, \phi) = N \left\{ R_{21}(r) \left( 2i|21\rangle + (2+i)|210\rangle + 3i|211\rangle \right) \right\}$$

$$\therefore \psi | \psi \rangle = N \left\{ 2i|21\rangle + (2+i)|210\rangle + 3i|211\rangle \right\}$$

$$\text{so, } \langle \psi | \psi \rangle = N^2 \left\{ |2i|^2 + |2+i|^2 + |3i|^2 \right\}$$

$$= N^2 \left( 4 + (2^2 + i^2) + 9 \right) = N^2 (4 + 9 + 5) = 18 N^2$$

$$\text{so; } 18 N^2 = 1 \quad \therefore N = \frac{1}{\sqrt{18}} \quad (\text{N being real + ve})$$

$$3.b \quad \langle L_z \rangle = \langle \psi | L_z | \psi \rangle$$

$$= \frac{1}{18} \left[ (-2i)\langle 21\rangle + (2-i)\langle 210\rangle - 3i\langle 211\rangle \right] \cdot L_z$$

$$(2i|21\rangle + (2+i)|210\rangle + 3i|211\rangle)$$

$$= \frac{1}{18} \left[ (-2i)\langle 21\rangle + (2-i)\langle 210\rangle - 3i\langle 211\rangle \right] \cdot$$

$$(-2i|21\rangle + 0 + 3i|211\rangle)$$

$$= \frac{\hbar}{18} [(-2i)(-2i) + (-3i)(3i)]$$

$$= \frac{\hbar}{18} [9 - 4] = \frac{5\hbar}{18}$$

$$3.c \quad \langle L^2 \rangle = \frac{1}{18} \left[ (-2i)\langle 21\rangle + (2-i)\langle 210\rangle - 3i\langle 211\rangle \right] L^2$$

$$(2i|21\rangle + (2+i)|210\rangle + 3i|211\rangle)$$

$$= \frac{\hbar^2}{18} \left[ (-2i)\langle 21\rangle + (2-i)\langle 210\rangle - 3i\langle 211\rangle \right] \times$$

$$[(1+i)(2i|21\rangle + (2+i)|210\rangle + 3i|211\rangle)]$$

[ $\because$  for each state  $l=1$  so give taken  $1(l+1) = 1(1+1) = 2$   
as common factor.]

$$= \frac{\hbar^2}{9} [ |z_1|^2 + |z_2|^2 + |z_3|^2 ] = \frac{18\hbar^2}{9} = 2\hbar^2.$$

which is obvious as for each  $m$ ;  $|z_i| = 1$ . So we get  
 $|z|^2 = 1(1+1)\hbar^2 = 2\hbar^2$ .

$$\text{Q.E.D. } \langle \hat{p}_{2m}^2 \rangle = \langle E - v \rangle \quad \left[ \because \frac{\hat{p}^2}{2m} = E - v \right]$$

$$= \langle K \cdot E \rangle = \langle E \rangle - \langle v \rangle$$

Now all the states (superpositions) being  $m=2$ ;  
we get:

$$\langle E \rangle = \langle E \rangle \Big|_{m=2} = E_2 = -\frac{13.6}{2^2} \text{ ev}$$

$$= -\frac{13.6}{4} \text{ ev}$$

$$\text{or in terms of } \alpha; \hbar; c; e: E_2 = -\frac{m\alpha^2 c^2}{2m^2} \Big|_{m=2}$$

$$= -\frac{m\alpha^2 c^2}{4}$$

$$\text{Now; } \langle v \rangle = \left\langle \frac{-1}{4\pi\epsilon_0} \frac{e^2}{r} \right\rangle$$

but for state with principal quantum number  $m$ ; we get

$$\langle v \rangle_m = -\frac{e^2}{4\pi\epsilon_0} \langle \frac{1}{r} \rangle_m = 2E_m$$

$$\therefore \text{for } m=2: \langle v \rangle_2 = \frac{2E_2}{2} = -\frac{4m\alpha^2 c^2}{2 \times 2 \times 2} = -\frac{m\alpha^2 c^2}{4}$$

$$\therefore \langle T \rangle = \frac{m\alpha^2 c^2}{4} - \frac{m\alpha^2 c^2}{4} = \frac{m\alpha^2 c^2}{4}$$

Question:- 4.

$$\text{a) } \hat{P} = \frac{1}{2} (A\mathbb{I} + \sigma \cdot B) = \frac{1}{2} \left[ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \underbrace{\begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix}}_{\sigma \cdot B} \right]$$

$$= \begin{pmatrix} \frac{A+B_3}{2} & \frac{B_1-iB_2}{2} \\ \frac{B_1+iB_2}{2} & \frac{A-B_3}{2} \end{pmatrix}$$

Now  $\hat{P}$  being a density matrix we get:

$$\text{Tr}(\hat{P}) = 1$$

$$\text{i.e. } \frac{A+B_3}{2} + \frac{A-B_3}{2} = 1 \quad \text{i.e. } A = 1.$$

$$\text{So; } \hat{P} = \begin{pmatrix} \frac{1+B_3}{2} & \frac{B_1-iB_2}{2} \\ \frac{B_1+iB_2}{2} & \frac{1-B_3}{2} \end{pmatrix}$$

$$\therefore \hat{P}^2 = \begin{pmatrix} \frac{1+B_3}{2} & \frac{B_1-iB_2}{2} \\ \frac{B_1+iB_2}{2} & \frac{1-B_3}{2} \end{pmatrix}^2 = \begin{pmatrix} \left(\frac{1+B_3}{2}\right)^2 + \frac{B_1^2+B_2^2}{4} & \frac{B_1-iB_2}{2} \\ \frac{B_1+iB_2}{2} & \left(\frac{1-B_3}{2}\right)^2 + \frac{B_1^2+B_2^2}{4} \end{pmatrix}$$

But for a pure ensemble we get

$$\hat{P}^2 = \hat{P} \quad \text{i.e. } \text{Tr}(\hat{P}^2) = \text{Tr}(\hat{P}) = 1$$

$$\text{i.e. } \frac{(1+B_3)^2}{4} + \frac{(1-B_3)^2}{4} + \frac{B_1^2+B_2^2}{2} = 1$$

$$\text{So; } \frac{2(1+B_3^2)}{4} + \frac{B_1^2+B_2^2}{2} = 1$$

$$\text{i.e. } 1 + B_1^2 + B_2^2 + B_3^2 = 2$$

$$\Rightarrow B_1^2 + B_2^2 + B_3^2 = 1. \quad \dots (1)$$

$$\text{So; } -1 \leq B_1, B_2, B_3 \leq 1 \quad \dots (2) \quad (\text{obviously})$$

$\hookrightarrow$  as  $B_i \in \mathbb{R}$ .

So the vector  $\vec{B}$  is a real unit vector in 3D space (along any direction  $\hat{m}$ ) 20

And the conditions on  $A \neq \vec{B}$  be given by }.

$$A = 1 ; \vec{B} = \hat{m} \text{ (in 3D space)}$$

to make  $\hat{\rho}$  a pure state density matrix. } Any

4.b.  $\langle \hat{S}_y \rangle = \text{Tr}(\hat{\rho} \cdot \hat{S}_y)$  (when both are given by same representations)

Here  $\hat{\rho}$  is given in  $|1\rangle_2, |1\rangle_2$  representation.

In the same basis;  $\hat{S}_y$  be given by:

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\begin{aligned} \therefore \hat{\rho} \cdot \hat{S}_y &= \frac{\hbar}{2 \times 2} \begin{pmatrix} 1+B_3 & B_1-iB_2 \\ B_1+iB_2 & 1-B_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= \frac{\hbar}{4} \begin{pmatrix} B_2+iB_1 & -i(1+B_3) \\ i(1-B_2) & B_2-iB_1 \end{pmatrix} \end{aligned}$$

~~so;  $\langle \hat{S}_y \rangle = \text{Tr}(\hat{\rho} \cdot \hat{S}_y) = \frac{\hbar}{4} \times 2B_2 = \frac{B_2 \hbar}{2}$ .~~

~~Clearly  $\langle S_y \rangle_{\max} = \frac{\hbar}{2}$  (as  $-1 \leq B_2 \leq 1$ )~~

4.c AS for all (pure/mixed) density matrices the essential condition be  ~~$\text{Tr}(\hat{\rho}) = 1$~~  So we get

~~$A = 1$~~

~~But for mixed state we don't have to get~~

$$\text{so } \langle \hat{S}_y \rangle = \text{Tr}(\hat{\rho} \hat{S}_y) = \frac{\hbar}{4} \times 2B_2 = \frac{B_2 \hbar}{2}.$$

Now  $-1 \leq B_2 \leq 1$ ; So the max possible value

~~of  $\langle S_y \rangle$  be  $\langle S_y \rangle_{\max} = \frac{\hbar}{2}$  with  $B_2 = 1$~~

But as  $B_1^2 + B_2^2 + B_3^2 = 1$  so for  $B_2 = 1$  we get  
 $B_1 = 0 = B_3$ .

Ans hence the vector  $\vec{B}$  be given as:

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$$\vec{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{\text{Ans}}$$

4.c. For all (pure/mixed) state we must have

$\text{Tr}(\hat{P}) = 1$  which gives the condition  $A=1$ .

But for mixed state we don't necessarily

need  $P^2 = P$  and hence the 2nd

condition is been violated.

$$\text{i.e } B_1^2 + B_2^2 + B_3^2 \neq 1$$

i.e  $\vec{B}$  is not an unit vector in 3D

i.e  $\vec{B}$  is not an unit vector in 3D

real space. Ans

### Question no 5:-

5.a (AD 3 marks) is given; I'm doing this explicitly)  
 in a long way.

Let the given state (i.e at  $t=0$ ) be given by

$$|\alpha(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \text{ (represented in basis of } |\uparrow_z\rangle \text{ & } |\downarrow_z\rangle)$$

Now this is an eigenstate of  $S_m$  with eig-val  $\frac{\hbar}{2}$ .

The representation of  $S_m$  in  $|\uparrow_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

be given by:

$$S_m = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So from given information:  $S_m |\alpha(0)\rangle = \frac{\hbar}{2} |\alpha(0)\rangle$

$$\text{i.e. } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\therefore$  by normalization  $\langle \alpha(0) | \alpha(0) \rangle = 1$  we get

$$a = b = \frac{1}{\sqrt{2}}. \text{ i.e. } \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{So } |\alpha(0)\rangle = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

but  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  being  $|\uparrow_z\rangle \neq |\downarrow_z\rangle$ :

$$|\alpha(0)\rangle = |\uparrow_m\rangle = \frac{|\uparrow_z\rangle + |\downarrow_z\rangle}{\sqrt{2}}$$

$$\underline{5.b} \text{ Here } \hat{H} = -\vec{\mu} \cdot \vec{B} = -\gamma B \hat{S}_z$$

clearly eigen state of  $\hat{H}$  = eigen state of  $\hat{S}_z$   
 i.e  $[\hat{H}, \hat{S}_z] = 0$ .

$$④ \hat{H} |\uparrow\rangle = -\gamma B S_z |\uparrow\rangle = -\frac{\gamma B \hbar}{2} |\uparrow\rangle$$

$$\hat{H} |\downarrow\rangle = -\gamma B S_z |\downarrow\rangle = \frac{\gamma B \hbar}{2} |\downarrow\rangle$$

So; if  $|\alpha(0)\rangle = c_+ |\uparrow\rangle + c_- |\downarrow\rangle$

then  $|\alpha(t)\rangle = c_+ e^{\frac{i\gamma B \hbar t}{2\hbar}} |\uparrow\rangle + c_- e^{-\frac{i\gamma B \hbar t}{2\hbar}} |\downarrow\rangle$

here  $c_{\pm} = \frac{1}{\sqrt{2}}$ . So;

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}} \left\{ e^{\frac{i\gamma B t}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-\frac{i\gamma B t}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Ans

$$\begin{aligned} 5.c. \quad & \langle S_y(t) \rangle = \langle \alpha(t) | S_y | \alpha(t) \rangle \\ &= \frac{1}{\sqrt{2}} \left( e^{-\frac{i\gamma B t}{2}} | e^{+\frac{i\gamma B t}{2}} \rangle \cdot \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i\gamma B t}{2}} \\ e^{-\frac{i\gamma B t}{2}} \end{pmatrix} \right. \\ &= \frac{\hbar}{4} \left( e^{i\gamma B t} e^{+\frac{i\gamma B t}{2}} \right) \begin{pmatrix} -i e^{-\frac{i\gamma B t}{2}} \\ i e^{\frac{i\gamma B t}{2}} \end{pmatrix} \\ &= \cancel{\frac{\hbar}{4} \left( i e^{\frac{i\gamma B t}{2}} + i e^{-\frac{i\gamma B t}{2}} \right)} = \frac{\hbar}{4} \left( i e^{\frac{i\gamma B t}{2}} - i e^{-\frac{i\gamma B t}{2}} \right) \\ &= -\frac{\hbar}{2} \left( \frac{e^{i\gamma B t} - e^{-i\gamma B t}}{2i} \right) = -\frac{\hbar}{2} \sin(\gamma B t) \end{aligned}$$

So;  $\langle S_y(t) \rangle = -\frac{\hbar}{2} \sin(\gamma B t)$  Ans

5. J. The Spin operator in the given direction is:

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$$\begin{aligned} S_m &= \vec{S} \cdot \hat{m} = \sin\theta \cdot S_x + \cos\theta \cdot S_z \\ &= \frac{\hbar}{2} \left[ \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix} \right] \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \end{aligned}$$

The given state is a eigenstate of  $S_m$  with eig-val  $-\frac{\hbar}{2}$ .

$$S_0 \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\text{i.e. } \cancel{\alpha \cos\theta} \begin{pmatrix} \alpha \cos\theta + \beta \sin\theta \\ \alpha \sin\theta - \beta \cos\theta \end{pmatrix} = - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\text{which gives: } \left. \begin{array}{l} \alpha(1+\cos\theta) + \beta \sin\theta = 0 \\ \alpha \sin\theta + \beta(1-\cos\theta) = 0 \end{array} \right\}$$

$$\alpha \sin\theta + \beta(1-\cos\theta) = 0$$

$$\Rightarrow \alpha = -\frac{\beta \sin\theta}{1+\cos\theta} = \frac{-2\beta \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\beta \tan\frac{\theta}{2}$$

$$\therefore |\uparrow\rangle = \alpha \begin{pmatrix} 1 \\ -\cot\frac{\theta}{2} \end{pmatrix}$$

Normalizing, we get:  $\alpha^2 \cos^2\frac{\theta}{2} \approx 1 \text{ i.e. } \alpha = \sin\frac{\theta}{2}$

$$\therefore |\uparrow\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = \sin\left(\frac{\theta}{2}\right) |\uparrow\rangle + \cos\left(\frac{\theta}{2}\right) |\downarrow\rangle$$

IN

Here from the given expression of  $|\psi(t)\rangle$  calculated in previous part we get

$$\langle S_z \rangle = \langle \psi(t) | S_z | \psi(t) \rangle = \frac{\hbar}{4} \left( e^{-\frac{i\gamma B t}{2}} e^{\frac{i\gamma B t}{2}} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{i\gamma B t}{2}} \\ e^{-\frac{i\gamma B t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2} \times \frac{1}{2} \left( e^{-\frac{i\gamma B t}{2}} e^{\frac{i\gamma B t}{2}} \right) \begin{pmatrix} e^{\frac{i\gamma B t}{2}} \\ -e^{-\frac{i\gamma B t}{2}} \end{pmatrix} = \frac{\hbar}{4} (1 - 1) = 0$$

And which is also clear from the physical view of the

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problem. The initial Spin is in  $\hat{x}$  direction (classical view) due to the  $\hat{z}$  magnetic field; it precesses around  $\hat{z}$  on my plane. So the given direction will only give the Spin-angular momentum value as  $(\pm \frac{\hbar}{2})$  if only possible iff  $\theta = \frac{\pi}{2}$  (for all value of  $\phi$ ). However, if the eigenvalue of ~~the~~ Spin angular momentum along given direction be  $-\frac{\hbar}{2}$ , then the state will be the calculated one.

5. e. The required probability:

$$P = |\langle \psi(t) | \chi_m^- \rangle|^2 \quad \text{when } |\chi_m^- \rangle = \begin{pmatrix} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \begin{pmatrix} e^{-i\gamma B t} & e^{i\gamma B t} \\ e^{\frac{i\gamma B t}{2}} & e^{\frac{i\gamma B t}{2}} \end{pmatrix} \begin{pmatrix} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix} \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \sin \frac{\theta}{2} \cdot e^{-\frac{i\gamma B t}{2}} + \cos \frac{\theta}{2} e^{\frac{i\gamma B t}{2}} \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \sin \frac{\theta}{2} \left( \cos \frac{\gamma B t}{2} - i \sin \frac{\gamma B t}{2} \right) + \cos \frac{\theta}{2} \left( \cos \frac{\gamma B t}{2} + i \sin \frac{\gamma B t}{2} \right) \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \cos \left( \frac{\gamma B t}{2} \right) \cdot \left( \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right) + i \sin \left( \frac{\gamma B t}{2} \right) \left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left\{ \cos^2 \left( \frac{\gamma B t}{2} \right) \cdot \left( \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)^2 + \sin^2 \left( \frac{\gamma B t}{2} \right) \cdot \left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right)^2 \right\}$$

if  $\theta = 0$  then  $P = \frac{1}{2} \left\{ \cos^2 \frac{\gamma B t}{2} + \sin^2 \frac{\gamma B t}{2} \right\} = 1/2$   
 i.e. the prob to find the Spin value as  $-\frac{\hbar}{2}$ ; when the Spin was oriented along my plane is  $\frac{1}{2}$ ; as expected from  $|\downarrow\rangle_2 = (|\uparrow_m\rangle - |\downarrow_m\rangle)/\sqrt{2}$ .

### Problem: 6

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$$\underline{\text{G.a.}} \quad \vec{B} = B_z \hat{z}; \quad \text{so; } H = -\frac{g\mu_B}{\hbar} \vec{S} \cdot \vec{B} = -\frac{g\mu_B}{\hbar} B S_z$$

$i.e. [H, S_z] = 0$ ; So the eigenstates of  $H$  are eigenstates of  $S_z$  also, i.e.  $| \pm \rangle$  with eigenvalue  $\mp \frac{g\mu_B B}{\hbar} \frac{\hbar}{2}$  i.e.  $\mp \frac{g\mu_B B}{2}$

The energy difference between states:  $\Delta E = g\mu_B B$ .

$$\underline{\text{G.b.}} \quad \text{given } |\psi(0)\rangle = |+_m\rangle$$

$$\text{In terms of } |\pm\rangle: |\psi(0)\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$\text{clearly } |\psi(t)\rangle = |\psi(0)\rangle \cdot e^{-\frac{iEt}{\hbar}}$$

$$= \frac{1}{\sqrt{2}} |+\rangle \cdot e^{\frac{i g\mu_B B t}{2\hbar}} + \frac{1}{\sqrt{2}} |-\rangle e^{-\frac{i g\mu_B B t}{2\hbar}}$$

~~Q2~~ If  $|\pm\rangle$  be given by  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  then:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi t} \\ e^{-i\phi t} \end{pmatrix} \quad (\phi = \frac{g\mu_B B t}{2\hbar})$$

$$\underline{\text{G.c.}} \quad \langle S_m \rangle_t = \langle \psi(t) | S_m | \psi(t) \rangle = \frac{\hbar}{2 \times 2} (e^{-i\phi t} e^{+i\phi t}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi t} \\ e^{-i\phi t} \end{pmatrix}$$

$$= \frac{\hbar}{4} (e^{-i\phi t} e^{i\phi t}) \begin{pmatrix} e^{-i\phi t} \\ e^{i\phi t} \end{pmatrix} = \frac{\hbar}{4} (e^{-2i\phi t} + e^{2i\phi t})$$

$$= \frac{\hbar}{2} \cos(2\phi t) = \frac{\hbar}{2} \cdot \cos\left(\frac{g\mu_B B t}{\hbar}\right)$$

$$\boxed{\text{Q2}} \quad \langle S_{\frac{1}{2}(+)} \rangle = \langle \psi(t) | S_{\frac{1}{2}(+)} | \psi(t) \rangle = \frac{\hbar}{4} (e^{-i\phi t} e^{i\phi t}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi t} \\ e^{-i\phi t} \end{pmatrix}$$

$$= \frac{\hbar}{4} (e^{-i\phi t} e^{i\phi t}) \begin{pmatrix} e^{i\phi t} \\ -e^{-i\phi t} \end{pmatrix} = \frac{\hbar}{4} (1 - 1) = 0$$

$$\boxed{\text{Q3}} \quad \langle S_{\frac{1}{2}(-)} \rangle = \langle \psi(t) | S_{\frac{1}{2}(-)} | \psi(t) \rangle = \frac{\hbar}{4} (e^{-i\phi t} e^{i\phi t}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\phi t} \\ e^{-i\phi t} \end{pmatrix}$$

$$= \frac{\hbar}{4} (e^{-i\phi t} e^{i\phi t}) \begin{pmatrix} -i e^{-i\phi t} \\ i e^{i\phi t} \end{pmatrix} = \frac{\hbar}{4} (i e^{2i\phi t} - i e^{-2i\phi t})$$

$$= -\frac{\hbar}{2} \frac{(e^{2i\phi t} - e^{-2i\phi t})}{2i} = -\frac{\hbar}{2} \sin(2\phi t) = -\frac{\hbar}{2} \sin\left(\frac{g\mu_B B t}{\hbar}\right)$$

7. We know  $S_1^z + S_2^z = S_z$

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$$\therefore S_1 \cdot S_2 = S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z$$

$$\text{Now; } H = J(S_1^x S_2^x + S_1^y S_2^y + K S_1^z S_2^z) + \mu(S_1^z + S_2^z)B.$$

$$= JS_1 \cdot S_2 + (K\gamma) JS_1^z S_2^z + \mu BS_z.$$

$$\text{But } S = S_1 + S_2 \Rightarrow S^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2$$

$$\Rightarrow S_1 \cdot S_2 = \frac{(S^2 - S_1^2 - S_2^2)}{2}$$

$$\therefore H = \frac{J}{2}(S^2 - S_1^2 - S_2^2) + (K\gamma) JS_1^z S_2^z + \mu BS_z.$$

Now for two spin  $\frac{1}{2}$  particles: (in  $|mm\rangle$  form)

The Symmetric States are:  $|110\rangle, |111\rangle, |11\rangle$

and the antiSymmetric State:  $|00\rangle$

$$\text{with } |110\rangle = \frac{|+-\rangle + |-+\rangle}{2}; \quad |00\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

So the energy measurement on different states give as following:

$$\textcircled{1} \hat{H}|11\rangle = \frac{J}{2}(S^2 - S_1^2 - S_2^2)|11\rangle + (K\gamma) JS_1^z S_2^z |++\rangle + \mu BS_z |11\rangle$$

(using  $|11\rangle \equiv |++\rangle$  in  $|mm\rangle$  or  $|m_1 m_2\rangle$  form)

$$= \frac{J\hbar^2}{2} \left(2 - \frac{3}{4} - \frac{3}{4}\right)|11\rangle + (K\gamma) J \cdot \frac{\hbar}{2} \cdot \frac{\hbar}{2} |++\rangle + \mu B \hbar |11\rangle$$

$$= \left(\frac{J\hbar^2}{4} + (K\gamma) \frac{J\hbar^2}{4}\right)|11\rangle + \mu B \hbar |11\rangle$$

$$= \left(\frac{J\hbar^2}{4} + \mu B \hbar\right)|11\rangle$$

$$\textcircled{2} \hat{H}|10\rangle = \frac{J}{2}(S^2 - S_1^2 - S_2^2)|10\rangle + (K\gamma) JS_1^z S_2^z \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} + \mu BS_z |10\rangle$$

$$= \frac{J\hbar^2}{2} \left(2 - \frac{3}{4} - \frac{3}{4}\right) |10\rangle + \frac{(K-1)J}{\sqrt{2}} \left(\frac{\hbar}{2} \cdot \left(\frac{\hbar}{2}\right) |+-\rangle + \frac{\hbar}{2} \cdot \left(-\frac{\hbar}{2}\right) |-+\rangle\right) + 0$$

$$= \frac{J\hbar^2}{2} \times \frac{1}{2} |10\rangle - \frac{(K-1)J\hbar^2}{4} \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}$$

$$= \left(\frac{J\hbar^2}{4} - \frac{(K-1)J\hbar^2}{4}\right) |10\rangle = \frac{J\hbar^2}{4} (2-K) |10\rangle$$

⑥  $\hat{H}|11\rangle = \cancel{\frac{J}{2}} (\cancel{S^2} - S_1^2 - S_2^2) |11\rangle + (K-1)JS_1^2S_2^2 |-\rangle + \mu_B S^2 |11\rangle$

$$= \frac{J\hbar^2}{2} \left(2 - \frac{3}{4} - \frac{3}{4}\right) + (K-1)J \cdot \left(-\frac{\hbar}{2}\right)^2 |-\rangle - \mu_B \hbar |11\rangle$$

$$= \left(\frac{KJ\hbar^2}{4} - \mu_B \hbar\right) |11\rangle$$

⑦  $\hat{H}|00\rangle = \cancel{\frac{J}{2}} (\cancel{S^2} - \cancel{S_1^2} - \cancel{S_2^2}) \cancel{+^2}$

$$= \frac{J}{2} (S^2 - S_1^2 - S_2^2) |00\rangle + (K-1)JS_1^2S_2^2 \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} + \mu_B S^2 |00\rangle$$

$$= \frac{J\hbar^2}{2} \left(0 - \frac{3}{4} - \frac{3}{4}\right) + \frac{(K-1)J}{\sqrt{2}} \left\{ -\frac{\hbar^2}{4} \cdot \frac{|+\rangle}{\sqrt{2}} - \left(-\frac{\hbar^2}{4}\right) \cdot \frac{|-\rangle}{\sqrt{2}} \right\} + 0$$

↳ (like  $|10\rangle$ )

$$= \frac{J\hbar^2}{2} \times \left(-\frac{3}{2}\right) + (K-1)J \cdot \left(-\frac{\hbar^2}{4}\right) \cdot \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

$$= \left(-\frac{3J\hbar^2}{4} - (K-1)\frac{J\hbar^2}{4}\right) |00\rangle$$

$$= -\left(\frac{J\hbar^2}{2} + \frac{KJ\hbar^2}{4}\right) |00\rangle = -\frac{J\hbar^2}{4} (2+K) |00\rangle$$

So the energy measurement on different states give,

$$\begin{aligned} |11\rangle &\rightarrow \frac{KJ\hbar^2}{4} \pm \mu_B \hbar \} \text{symmetric state} \\ |10\rangle &\rightarrow \frac{J\hbar^2}{4} (2-K) \end{aligned}$$

$$|00\rangle \rightarrow -\frac{J\hbar^2}{4} (2+K) \rightarrow \text{anti-symmetric state.}$$

Q. The given Hamiltonian:

$$\hat{H} = -t \left\{ (|1\uparrow, 1\downarrow\rangle\langle 1\uparrow, 2\downarrow| + |1\uparrow, 2\downarrow\rangle\langle 1\uparrow, 1\downarrow|) + \right. \\ (|2\uparrow, 2\downarrow\rangle\langle 1\uparrow, 2\downarrow| + |1\uparrow, 2\downarrow\rangle\langle 2\uparrow, 2\downarrow|) + \\ (|1\uparrow, 1\downarrow\rangle\langle 1\uparrow, 2\uparrow| + |1\downarrow, 2\uparrow\rangle\langle 1\uparrow, 1\downarrow|) + \\ \left. (|2\uparrow, 2\downarrow\rangle\langle 1\uparrow, 2\uparrow| + |1\downarrow, 2\uparrow\rangle\langle 2\uparrow, 2\downarrow|) \right\} \\ + V \left\{ |1\uparrow, 1\downarrow\rangle\langle 1\uparrow, 1\downarrow| + |2\uparrow, 2\downarrow\rangle\langle 2\uparrow, 2\downarrow| \right\}$$

The only states present in the  $\hat{H}$  are:

$|1\uparrow, 1\downarrow\rangle, |1\uparrow, 2\downarrow\rangle, |2\uparrow, 2\downarrow\rangle, |1\downarrow, 2\uparrow\rangle$   
 (while  $|1\uparrow 2\uparrow\rangle, |2\uparrow 2\uparrow\rangle, \dots$  etc are not)  
 So those 4 vectors forming the eigenbasis of the  $\hat{H}$ .

So the Hamiltonian be given by:

$$\hat{H} = \begin{pmatrix} V & 0 & -t & -t \\ 0 & V & -t & -t \\ -t & -t & 0 & 0 \\ -t & -t & 0 & 0 \end{pmatrix} \begin{array}{c} |1\uparrow 1\downarrow\rangle \\ |2\uparrow 2\downarrow\rangle \\ |1\uparrow 2\downarrow\rangle \\ |1\downarrow 2\uparrow\rangle \end{array}$$

$$\begin{array}{c} \langle 1\uparrow 1\downarrow | \\ \langle 2\uparrow 2\downarrow | \\ \langle 1\uparrow 2\downarrow | \\ \langle 1\downarrow 2\uparrow | \end{array}$$

$$= V \begin{pmatrix} 1 & 0 & -\gamma_u & -\gamma_u \\ 0 & 1 & -\gamma_u & -\gamma_u \\ -\gamma_u & -\gamma_u & 0 & 0 \\ -\gamma_u & -\gamma_u & 0 & 0 \end{pmatrix} \begin{array}{l} \text{(forgive for the} \\ \text{small u in ( )} \\ \text{in place of V} \\ \text{by mistake.)} \end{array}$$

but  $V \gg t$  so  $\gamma_u \approx 0$ .

$$\text{so; } \hat{H} \approx V \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Being in the diagonal form; we get  
 $\hat{H}$  have two eigenvalues  $\{U, 0\}$  with two fold degeneracy.  
 with eigenvectors  $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow U$ ;  $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow U$ ;  $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \nmid \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow 0$ .

So the ground states are

$$\left. \begin{array}{l} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \equiv |1\uparrow, 2\downarrow\rangle \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv |1\downarrow 2\uparrow\rangle \end{array} \right\} \text{with energy } 0.$$

And the first excited states:

$$\left. \begin{array}{l} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv |1\uparrow 1\uparrow\rangle \\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \equiv |2\uparrow 2\uparrow\rangle \end{array} \right\} \text{with energy } U.$$

Now instead of doing; try if I'd take  $t/U \neq 0$  & then find out the eigenvalues & eigenvectors of  $\hat{H}$  then ~~I'll~~ ~~want~~ I'll get some different result. Obviously  $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \dots$  are not eigenvectors of  $\hat{H}$ . But due to the  $4 \times 4$  matrix it is very hard to do by hand.

Using python I've found out the eigenvalues for  $t/U = 0.01$ . The eig vals are

$$\lambda = U(1.0004, 1.0, -3.999 \times 10^{-4}, -2.040 \times 10^{-33})$$

The coupling term splits the degeneracy & we get 4 different energy levels.

Q. Two spin  $\frac{1}{2}$  particle will stay in spin singlet state iff 31  
the composite state be given by:

$$|+\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Now for calculation; I assume that  $\hat{m}_a$  &  $\hat{m}_b$  are  
directions along  $(\theta_1, \phi_1)$  &  $(\theta_2, \phi_2)$ .

The spin operator along  $\hat{m}$  be given by:

$$\hat{S}_{\hat{m}} = \sin\theta \cos\phi \cdot \hat{S}_x + \sin\theta \sin\phi \hat{S}_y + \cos\theta \hat{S}_z$$

$$= \frac{\sin\theta \cos\phi}{2} (S_+ + S_-) + \frac{\sin\theta \sin\phi}{2i} (S_+ - S_-)$$

$$+ \cos\theta \cdot S_z$$

And magnetic moment factors

$$= \frac{S_+}{2} \sin\theta e^{-i\phi} + \frac{S_-}{2} \sin\theta e^{i\phi} + \cos\theta \cdot S_z.$$

Now we get:

$$\hat{S}_{\hat{m}} |+\rangle = S_+ \frac{\sin\theta e^{-i\phi}}{2} |+\rangle + \frac{\sin\theta e^{i\phi}}{2} S_- |+\rangle$$

$$+ \cos\theta \cdot S_z |+\rangle$$

$$= 0 + \frac{\sin\theta}{2} e^{i\phi} |-\rangle + \frac{\hbar}{2} \cos\theta |+\rangle$$

Similarly:

$$\hat{S}_{\hat{m}} |-\rangle = \frac{\hbar}{2} \sin\theta \cdot e^{-i\phi} |+\rangle - \frac{\hbar}{2} \cos\theta |-\rangle$$

Now for the given problem:

$$(S^a \cdot \hat{m}_a) (S^b \cdot \hat{m}_b) |+\rangle = (S_{\hat{m}_a}^a S_{\hat{m}_b}^b) \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$= \frac{S_{\hat{m}_a}^a |+\rangle_a S_{\hat{m}_b}^b |-\rangle - S_{\hat{m}_a}^a |-\rangle_a S_{\hat{m}_b}^b |+\rangle}{\sqrt{2}}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2}} \left[ \left\{ \frac{\hbar \sin \theta_1 e^{i\phi_1}}{2} |-\rangle_a + \frac{\hbar \cos \theta_1}{2} |+\rangle_a \right\} - \left\{ \frac{\hbar \cos \theta_2}{2} |-\rangle_b - \frac{\hbar \sin \theta_2 e^{-i\phi_2}}{2} |+\rangle_b \right\} \right] \\
 &\quad \left( \frac{\hbar \sin \theta_2 e^{+i\phi_2}}{2} |-\rangle_b + \frac{\hbar \cos \theta_2}{2} |+\rangle_b \right) \\
 &= \frac{\hbar^2}{4\sqrt{2}} \left[ \begin{array}{ll} \sin \theta_1 \sin \theta_2 e^{i(\phi_1 - \phi_2)} |--\rangle & - \\ \sin \theta_1 \cos \theta_2 e^{i\phi_1} & |--\rangle + \\ \cos \theta_1 \sin \theta_2 e^{-i\phi_2} & |++\rangle - \\ \cos \theta_1 \cos \theta_2 & |+-\rangle - \\ \sin \theta_1 \sin \theta_2 e^{i(\phi_2 - \phi_1)} |+-\rangle & - \\ \sin \theta_1 \cos \theta_2 e^{-i\phi_1} & |++\rangle + \\ \cos \theta_1 \sin \theta_2 e^{i\phi_2} & |--\rangle + \\ \cos \theta_1 \cos \theta_2 & |-+\rangle \end{array} \right]
 \end{aligned}$$

So the given ~~is~~ immer product be:

$$\langle \hat{S}_a^\alpha \hat{S}_b^\beta \rangle = \langle \psi | \hat{S}_a^\alpha \hat{S}_b^\beta | \psi \rangle$$

~~(+ -)~~ ~~(- +)~~ [Third term]

$$= \frac{\hbar^2}{4 \times 2} (\langle + - | - + \rangle)$$

(using  $\langle ++|++\rangle = 1$ ;  $\langle +-|+-\rangle = 1$ )

$$= \frac{\hbar^2}{8} \left[ -\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 e^{i(\phi_2 - \phi_1)} \right. \\
 \left. - \sin \theta_1 \sin \theta_2 e^{i(\phi_1 - \phi_2)} - \cos \theta_1 \cos \theta_2 \right]$$

$$\left( \text{using } \langle ++|++\rangle = 1; \langle +-|+-\rangle = 1 \right. \\
 \left. \langle + - | - + \rangle = 0; \langle + - | - - \rangle = 0 \dots \text{etc} \right)$$

$$= -\frac{\hbar^2}{8} \left[ 2 \cos \theta_1 \cos \theta_2 + 2 \sin \theta_1 \sin \theta_2 e^{\frac{i(\phi_1 - \phi_2)}{2}} + e^{-\frac{i(\phi_1 - \phi_2)}{2}} \right]$$

$$= -\frac{\hbar^2}{8} \left[ 2 \cos \theta_1 \cos \theta_2 + 2 \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2) \right]$$

So; in terms of  ~~$(\theta_1, \phi_1)$~~   $(\theta_1, \phi_1) \equiv \hat{m}_a$  and  
 $(\theta_2, \phi_2) \equiv \hat{m}_b$ ; the value of the given  
expectation be:

$$\langle \hat{m}_a \hat{m}_b \rangle = -\frac{\hbar^2}{4} \left( \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2) \right)$$

Ans

10. The spherical tensor of rank  $K$  be defined as:

$$u^+(R) T_q^K u(R) = \sum_{q'=-K}^K D_{qq'}^{KK} T_{q'}^K$$

Here the components of the Tensor + Change according to this definition.

We use the spherical tensor  $T$  as general form of spherical harmonics  $y_i^m$  while replacing  $\hat{m}$  by arbitrary vector  $\vec{v}$  with components  $(v_x, v_y, v_z)$

$$\text{So: } T_q^K = y_i^m(\vec{v})$$

Now expression given in form of S.T!

$$\begin{aligned} \text{i) } \partial y_i^m &\rightarrow y_2^{\pm 2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta \cdot e^{\mp 2i\phi} \\ &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(m \pm i\phi)^2}{r^2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{m^2 - \phi^2 \pm 2im\phi}{r^2} \end{aligned}$$

$$\text{So: } T_{\pm 2}^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \left( v_x^2 - v_y^2 \pm 2iv_x v_y \right)$$

$$(v_i = \frac{ri}{r})$$

$$\text{i.e. } T_2^2 - T_{-2}^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} (4iv_x v_y)$$

$$\text{or, } v_x v_y = i \sqrt{\frac{2\pi}{15}} \cdot (T_{-2}^2 - T_2^2)$$

To get  $v_y$  we have to choose  $\vec{v} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{z}$

$$\text{So: } v_y = i \sqrt{\frac{2\pi}{15}} (T_{-2}^2 - T_2^2)$$

ii) From the previous one:

$$T_{-2}^2 + T_2^2 = \frac{1}{4} \cdot \sqrt{\frac{2\pi}{15}} 2(v_x^2 - v_y^2)$$

$$\Rightarrow v_x^2 - v_y^2 = \sqrt{\frac{8\pi}{15}} (T_2^2 + T_{-2}^2)$$

Choosing  $\vec{v} = \vec{r}$  we get:

$$v_x^2 - v_y^2 = \sqrt{\frac{8\pi}{15}} (T_2^2 + T_{-2}^2)$$

11)  $xz \rightarrow$

$$\begin{aligned}
 Y_2^{\pm 1} &= \mp \frac{1}{2} \sqrt{\frac{15}{2\pi}} e^{\pm i\phi} (\sin \theta \cos \phi) \\
 &= \mp \frac{1}{2} \sqrt{\frac{15}{2\pi}} \underbrace{(\alpha \pm i\gamma) z}_{r^2} \\
 &= \mp \frac{1}{2} \sqrt{\frac{15}{2\pi}} (V_2 V_2 \pm i V_3 V_2)
 \end{aligned}$$

$$S_0: T_1^2 - T_7^2 = -\frac{1}{2} \sqrt{\frac{15}{2n}} 2V_0 V_2$$

$$\Rightarrow V_2 V_2 = \sqrt{\frac{2\pi}{15}} (T_1^2 - T_2^2)$$

Taking  $\bar{V} = \bar{r}$  we get!

$$m_2 = \sqrt{\frac{2\pi}{15}} (T_1^2 - T_1^2)$$

■ We have to calculate ~~Design~~ ~~Design~~

$$\langle \alpha_{im} | x^2 - y^2 | \alpha_{ij} \rangle$$

Now from Wigner-Eckart theorem:

$$\langle \alpha' j' m' | T_q^k | \alpha j m \rangle = \langle j k m q | j k i m' \rangle \frac{\langle \alpha' j' | T^k | \alpha j \rangle}{\sqrt{2j+1}}$$

Now; the evaluating term.

$$Q = \langle \alpha_{jm'} | (m^2 - j^2) | \alpha_{jj} \rangle = \sqrt{\frac{8\pi}{15}} \langle \alpha_{jm'} | T_{-2}^2 + T_2^2 | \alpha_{jj} \rangle$$

$$= \sqrt{\frac{8\pi}{15}} \cdot \frac{\langle \alpha_j || T^2 || \alpha_j \rangle}{\sqrt{2j+1}} \left[ \langle j_2, j_{j-2} | j_2, j_{m'} \rangle + \langle j_2, j_2 | j_2, j_{m'} \rangle \right]$$

But  $\langle j_2, j_2 | j_2, j_m \rangle = 0$  since  $m' = j+2$  is not possible.

$$\therefore Q = \sqrt{\frac{8\pi}{15}} \cdot \left\langle \frac{d(j||T^2||^{\alpha_j})}{\sqrt{2j+1}} \right\rangle_{(j_2, j_3, \dots | j_2, j_m)}$$

$$\text{Now } \gamma_2^o = \sqrt{\frac{5}{16\pi}} \left( \frac{3z^2 - r^2}{r^2} \right) \rightarrow \gamma^2 = \sqrt{\frac{5}{16\pi}} (3z^2 - r^2)$$

$$\therefore (3z^2 - r^2) = \sqrt{\frac{16\pi}{5}} T_0^2$$

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$$\begin{aligned}
 & \text{QED} : \langle \alpha j m | \beta z^2 - r^2 | \alpha j m \rangle \\
 &= \sqrt{\frac{16\pi}{5}} \langle \alpha j m | T_0^2 | \alpha j m \rangle \\
 &= \sqrt{\frac{16\pi}{5}} \underbrace{\langle \alpha j || T^2 || \alpha j \rangle}_{\sqrt{2j+1}} \langle j_2, j_0 | j_2, j j \rangle
 \end{aligned}$$

$$\begin{aligned}
 & \text{So, } \langle \alpha j m' | \alpha^2 - y^2 | \alpha j j \rangle \\
 &= \frac{\alpha}{\sqrt{2}} \frac{\langle j_2, j-2 | j_2, j m' \rangle}{\langle j_2, j_0 | j_2, j j \rangle} \underline{\text{Ans}}
 \end{aligned}$$

ii) Instead of  $| \uparrow\downarrow \rangle$ , I prefer to write  $| +-\rangle$  for writing.

$$\text{Now, } S = S_1 + S_2 = \text{total Spin.} \Rightarrow S^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2$$

$$\Rightarrow S_1 \cdot S_2 = \frac{S^2 - S_1^2 - S_2^2}{2}$$

So for  $t > 0$ ; the Hamiltonian be given by:

$$H = \frac{2S}{\hbar^2} S_1 \cdot S_2 = \frac{2S}{\hbar^2} (S^2 - S_1^2 - S_2^2)$$

Clearly the eigenstates of  $H$  are simultaneous eigenstates of  $S^2, S_1^2, S_2^2$ . But for two Spin  $\frac{1}{2}$  particle; we get 4 two type of states. i.e

$$| 11 \rangle = | ++ \rangle; | 10 \rangle = \frac{| +\rangle + | -\rangle}{\sqrt{2}}; | 1- \rangle = | -\rangle; | 00 \rangle = \frac{| +\rangle - | -\rangle}{\sqrt{2}}$$

which form a complete orthonormal set.

Let at a later time  $t > 0$ ; the state be given by:

$$|\psi(t)\rangle = a_1 | 11 \rangle + a_2 | 10 \rangle + a_3 | 1- \rangle + a_4 | 00 \rangle \quad (a_i = a_i(t))$$

$$\begin{aligned} H|\psi(t)\rangle &= \frac{2S}{\hbar^2} (S^2 - S_1^2 - S_2^2) [a_1 | 11 \rangle + a_2 | 10 \rangle + a_3 | 1- \rangle + a_4 | 00 \rangle] \\ &= \frac{2S}{\hbar^2} \cdot \hbar^2 [a_1 (1(1+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)) | 11 \rangle + \end{aligned}$$

$$\begin{aligned} &a_2 (1(1+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)) | 10 \rangle + (1(1+1) - \frac{1}{2}(1+\frac{1}{2}) - \frac{1}{2}(\frac{1}{2}+1)) | 1- \rangle \cdot a_3 \\ &+ a_4 (0(0+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)) | 00 \rangle] \end{aligned}$$

$$= 2S \left[ \frac{1}{2} a_1 | 11 \rangle + \frac{a_2}{2} | 10 \rangle + \frac{a_3}{2} | 1- \rangle - \frac{3}{2} a_4 | 00 \rangle \right]$$

$$= S \left[ a_1 | 11 \rangle + a_2 | 10 \rangle + a_3 | 1- \rangle - 3a_4 | 00 \rangle \right] \dots (1)$$

$$\text{Now; } i\hbar \frac{d|\psi(t)\rangle}{dt} = i\hbar \left[ \dot{a}_1 | 11 \rangle + \dot{a}_2 | 10 \rangle + \dot{a}_3 | 1- \rangle + \dot{a}_4 | 00 \rangle \right] \dots (2)$$

As;  $H|\psi\rangle = i\hbar \frac{d|\psi\rangle}{dt}$ ; so comparing (1) & (2) we get

for coefficients of orthonormal states given:

$$i\hbar \dot{a}_1 = Sa_1 \Rightarrow \frac{\dot{a}_1}{a_1} = \frac{-iS}{\hbar}, \quad \int_{a_1^0}^{a_1} \frac{a_1 dt}{a_1} = -\frac{iS t}{\hbar}$$

$$\Rightarrow a_1 = a_1^0 e^{-\frac{iSt}{\hbar}}$$

Similarly  $a_2 = a_2^0 e^{-\frac{i\delta t}{\hbar}}$ ;  $a_3 = a_3^0 e^{-\frac{i\delta t}{\hbar}}$ .

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$$a_4 = a_4^0 e^{\frac{3i\delta t}{\hbar}}.$$

$$\text{So; } |\psi(t)\rangle = (a_1^0 |11\rangle + a_2^0 |10\rangle + a_3^0 |11\rangle + a_4^0 e^{\frac{3i\delta t}{\hbar}} |00\rangle)$$
$$e^{-\frac{i\delta t}{\hbar}}$$

The initial state ( $t=0$ ) be given by  $|\psi(0)\rangle = |+\rangle = \frac{|10\rangle + |00\rangle}{\sqrt{2}}$ .

$$\text{Plugging in: } a_1^0 = 0; a_2^0 = \frac{1}{\sqrt{2}}; a_3^0 = 0; a_4^0 = \frac{1}{\sqrt{2}}.$$

$$\text{So; } |\psi(t)\rangle = \frac{1}{\sqrt{2}} \left( e^{-\frac{i\delta t}{\hbar}} |10\rangle + e^{\frac{3i\delta t}{\hbar}} |00\rangle \right)$$

In terms of  $|++\rangle, |+-\rangle, |-\rangle, |--\rangle$  etc we get:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \left( e^{-\frac{i\delta t}{\hbar}} (|+\rangle + |-\rangle) + e^{\frac{3i\delta t}{\hbar}} (|+\rangle - |-\rangle) \right)$$
$$= \frac{1}{2} [|+\rangle \cdot \left\{ e^{\frac{3i\delta t}{\hbar}} + e^{-\frac{i\delta t}{\hbar}} \right\} + |-\rangle \left\{ e^{-\frac{i\delta t}{\hbar}} - e^{\frac{3i\delta t}{\hbar}} \right\}]$$
$$= \frac{1}{2} [2|+\rangle \cdot e^{\frac{i\delta t}{\hbar}} \underbrace{\left( e^{\frac{2i\delta t}{\hbar}} + e^{-\frac{2i\delta t}{\hbar}} \right)}_{2i} - 2i|-\rangle e^{\frac{i\delta t}{\hbar}} \underbrace{\left( e^{\frac{2i\delta t}{\hbar}} - e^{-\frac{2i\delta t}{\hbar}} \right)}_{2i}]$$
$$= e^{\frac{i\delta t}{\hbar}} \left[ \cos\left(\frac{2\delta t}{\hbar}\right) |+\rangle - i \sin\left(\frac{2\delta t}{\hbar}\right) |-\rangle \right]$$

So the probabilities to get  $|++\rangle, |+-\rangle, |-\rangle, |--\rangle$

as a function of time be given by respectively:

$$0, \cos^2\left(\frac{2\delta t}{\hbar}\right), \sin^2\left(\frac{2\delta t}{\hbar}\right), 0.$$

Ans

12.a. If  $A \in SV(N)$  then the conditions on  $A$  be:

$$\det(A) = 1 ; A^T A = I.$$

Now; if  $A \in SU(N)$  then obviously  $A^+ \in SU(N)$ .

$$I \in SU(N)$$

$$I^+ = I = e \text{ (in } SU(N))$$

$\therefore$  For any element  $x$  in  $SU(N)$   $\exists$  some  $y: y = x^{-1}$   
here  $y = A^+$  for  $x = A$ .

If  $A, B \in SU(N)$  then let  $x = A; y = B$

$$\therefore x \cdot y = A \cdot B = z \text{ (lt)}$$

$$\therefore z^{-1} = (AB)^+ = B^+ A^+$$

$$\text{Ans } z^{-1} z = (B^+ A^+) \cdot (AB) = B^+ I B = B^+ B = I.$$

$\therefore x \cdot y \in \text{the given set.}$

$$\text{Ans } \det(AB) = \det(A) \det(B) = 1 \times 1 = 1$$

so it follows the closure property.

$\therefore$  The matrices in  $SU(N)$  form a group.

b: i)  $SU(N)$ : The dimension of unitary matrix comes from exponentiating a Hermitian matrix  $H$  by the form  $\exp(iH)$

$$\text{AD if: } U = e^{iH}$$

$$\text{then } UTU = e^{-iH} \cdot e^{iH} = I.$$

Now for a general Hermitian matrix  $H$ :

$$H = \begin{pmatrix} & & & \\ & & & \\ & & n & \\ & & & \end{pmatrix} \xrightarrow{\frac{n^2-n}{2}}$$

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There are  $m$  real no's along the diagonal.  
 for the upper triangular part there are  $\frac{m^2 - m}{2}$   
 complex no's. i.e.  $(m^2 - m)$  indp parameters.

The lower ~~triangular~~ part is just ~~complex~~ conjugate of upper triangle as  $H^+ = H$ .

∴ Total no of ~~indp~~ parameters:

$$\frac{m^2 - m}{2} \times 2 + m = m^2.$$

but for  $SU(N)$ ; there is another constraint of  
~~det~~ determinant.  ~~$\text{Tr}(iH) = 1$~~

~~$\text{Tr}(e^{iH}) = \text{Tr}(SU) = e^{\text{Tr}(iH)} = 1$~~

~~$\det(e^{iH}) = (\det(SU)) = e^{\text{Tr}(iH)} = 1.$~~

$$\text{So; } \text{Tr}(iH) = 0$$

This gives another constraint.

So no of indp parameters of  $SU(N)$

= no of generators in  $SU(N)$

~~$= N^2 - 1 \rightarrow \text{from constraint } \text{Tr}(iH) = 0$~~

↳ from  $U(N)$

$$= (N^2 - 1) \quad \underline{\text{Ans}}$$

ii)  $O(N)$ : If  $R \in O(N)$ ; then  $R$  follows

the rule  $R^T R = R R^T = I$ . ... (ii)

So  $R R^T$  is a symmetric real matrix with  $N^2$  items.

$$R R^T = \begin{pmatrix} \nearrow & \downarrow \\ N & \end{pmatrix} \xrightarrow{\frac{N(N-1)}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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As the lower part is equal to the upper triangular part of a symmetric real matrix; so no of constraints from  $RR^T = I$  is

$$N + \frac{N(N-1)}{2} = \frac{N^2 + N}{2}$$

$\therefore$  No of independent parameters = No of generators

$$= N^2 - \frac{N^2 + N}{2} = \frac{N^2 - N}{2} = \frac{N(N-1)}{2}. \quad \underline{\text{Ans}}$$

iii)  $SO(N)$  is special type of orthogonal matrix with the constraint  $\det(R) = 1$ .

Now it was actually in the definition of  $O(N)$  too.

$$\text{As: } RR^T = I \Rightarrow \det(RR^T) = 1$$

$$\Rightarrow \det(R) \cdot \det(R^T) = \det(R) \cdot \det(R) = (\det(R))^2 = 1$$

$$\Rightarrow \det(R) = \pm 1.$$

This is a disjoint condition; so doesn't bring a constrain to  $O(N)$  [As from  $U(N)$  to  $SU(N)$  one comes from  $e^{i\pi} = 1$ ; here we can't get like that.]

So No of generators in  $SO(N)$

$$= \dots \cdot O(N)$$

$$= \frac{N(N-1)}{2}. \quad \underline{\text{Ans}}$$

iv) As  $SO(N)$  represents rotation in  $N$  dimension; and the rotation (in a plane) can be chosen in  $N$  ways in  $N$  dimension (as one plane need two axes) from  $N$  i.e. in  ${}^N C_2$  ways; so no of generators (geometrically) in  $SO(N) = {}^N C_2 = \frac{N(N-1)}{2} \quad \underline{\text{Ans}}$

12.C.  Previously I only know that the Gell-Mann matrices are the generators of  $SU(3)$ . Obviously there will be  $3^2 - 1 = 8$  matrices. But I don't know how to derive the generators of  $SU(N)$  generally for any  $N$ . (except that I have to use the basic rules like trace will be zero; they will be Hermitian etc....). My friend Soufarma knew this in a long ~~and~~ rigorous way. Generally other time we try to discuss among each other any them solve in own way. But now as we are at home so that type of discussion was not possible and I just have taken the problem from him. The complete credit goes ~~to~~ from Soufarma and the evaluation completely depends on the ~~grader~~ grader.

Using the Hermitian property, traceless property and the knowledge of  $SU(2)$ ; the 1st matrix be

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Another choice of the diagonal form  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$  will be possible iff  $a=2$ .

$$\therefore T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

After this I can't do from my own. I was going through his calculation. ~~But it is~~ However the final result (collected from wiki) are the

Sagar

~~Gell-Mann~~ Gell-Mann matrices are given by:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↓  
The assumed one

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Ans

$$\underline{13.a)} \quad \psi(\vec{r}) = (\alpha + \gamma + 3z) \cdot f(r)$$

$$= r \cdot f(r) \cdot [\sin\theta \cos\phi + \sin\theta \sin\phi + 3 \cos\theta].$$

Now from table

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \Rightarrow 3 \cos\theta = \sqrt{12\pi} Y_1^0(0, \phi)$$

$$\text{Ans. } Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta (\cos\phi + i \sin\phi)$$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta (\cos\phi - i \sin\phi).$$

$$\therefore Y_1^1 - Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \times 2 \cos\phi$$

$$\therefore \sin\theta \cos\phi = \sqrt{\frac{2\pi}{3}} (Y_1^1 - Y_1^{-1})$$

$$\text{Similarly } \sin\theta \sin\phi = \sqrt{\frac{2\pi}{3}} \frac{(Y_1^1 + Y_1^{-1})}{i}$$

$$\therefore \psi(\vec{r}) = r \cdot f(r) \cdot \left[ \sqrt{\frac{2\pi}{3}} (Y_1^1 - Y_1^{-1}) - i \sqrt{\frac{2\pi}{3}} (Y_1^1 + Y_1^{-1}) + \sqrt{12\pi} Y_1^0 \right]$$

$$= r \cdot f(r) \cdot \left[ Y_1^1 (1-i) \cdot \sqrt{\frac{2\pi}{3}} - Y_1^1 (1+i) \sqrt{\frac{2\pi}{3}} + \sqrt{12\pi} Y_1^0 \right]$$

So the  $l=1$  terms are present in the wavefn  
and hence the possible values of  $l$  only present

If  $l=1$ :

So The state is ~~an~~ an eigenstate of  $L^2$  with

$$L^2 |\psi(\vec{r})\rangle = 1(1+1)h^2 |\psi(\vec{r})\rangle = 2h^2 |\psi(\vec{r})\rangle.$$

13.b. From the ~~the~~ calculated form of the wavefn  
in terms of  $Y_l^m(\theta, \phi)$  we get ~~different~~  
probabilities of different ~~values~~ values:

$$\text{For } m=0 : P(m=0) = \frac{12\pi}{12\pi + \frac{2\pi}{3} \{ |1+i|^2 + |1-i|^2 \}}$$

$$= \frac{12}{12 + \frac{2}{3} \{2+2\}} = \frac{9}{11}$$

$$P(m=1) = \frac{|1+i|^2 \cdot \frac{2\pi}{3}}{12 + \frac{2\pi}{3} \{ |1+i|^2 + |1-i|^2 \}} = \frac{\frac{2}{3} \times 2}{12 + \frac{2}{3} \times 4} = \frac{1}{11}$$

$$P(m=-1) = \frac{\frac{2\pi}{3} |1-i|^2}{12 + \frac{2\pi}{3} \{ |1+i|^2 + |1-i|^2 \}} = \frac{\frac{2}{3} \times 2}{12 + \frac{8}{3}} = \frac{1}{11}$$

Clearly  $P(m=0) + P(m=1) + P(m=-1) = \frac{9+1+1}{11} = 1.$

13. iii) To find the energy eigenvalue we've to follow the steps:

i) For a given  $f(r)$  we have to normalize  $\psi(\bar{r}).$

ii) The radial part will be  $\Psi \sim \text{const} \times r f(r).$

(normalization is not essential)

iii)  $u = \frac{\text{A} \Psi f(r)}{r} = A \cdot f(r)$  is an eigenstate function

of the radial eqn.  

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = E u.$$

with  $l=1$

iv) every term except  $V$  is known.

$$\text{So: } V u(r) = \left[ E - \frac{\hbar^2}{2m} \cdot \frac{1(1+1)}{r^2} \right] u + \frac{\hbar^2}{2m} u''(r)$$

$$\Rightarrow V = \underbrace{\left[ E - \frac{\hbar^2}{mr^2} \right] \cdot u(r)}_{u(r)} + \frac{\hbar^2}{2m} u''(r) = V(r).$$

And we get the form of  $V = V(r).$

Ans

74. In  $|j, m_j\rangle$  notation the given state is  $|2 0\rangle$  and the rotated state be  $\mathcal{D}(R)|2 0\rangle$ .  
 In the usual Euler angle angles;  $(\theta, \phi, \gamma)$  which are taken w.r.t  $z, y', z'$  axes; the values for this problem be given by  $\theta, \gamma = 0; \phi = \alpha$ .

$$\text{Now let: } \mathcal{D}(R)|2 0\rangle = \sum_m \mathcal{D}(R)|2 0\rangle \cdot |2^m\rangle \times |2^m\rangle \\ = \sum_m \underbrace{\langle 2^m | \mathcal{D}(R)|2 0\rangle}_{C_m} \cdot |2^m\rangle \dots (1)$$

However  $\langle 2^m | 2 0\rangle$  be given by  $y_2^{m*}(\theta=0, \phi)$   
 (I'm not going to derive the whole thing as the complete proof is not necessary. I'm just using the result from Sakurai.)

but  $y_2^{m*}(\theta=0, \phi)$  be given by

$$: \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \left|_{l=2}^0 \cdot S_m^0 \right. \quad (\because \text{it gives only non-zero value for } m=0) \\ = \sqrt{\frac{2l+1}{4\pi}} \left|_{l=2}^0 S_m^0 \right. \quad (\because P_1(1)=1)$$

So we get:

$$\mathcal{D}(R)|2 0\rangle = \sum_m \mathcal{D}_m$$

$$\langle 2^m' | \mathcal{D}(R)|2 0\rangle = \sum_m \cancel{\langle 2^m' | \mathcal{D}(R)|2^m\rangle} \quad \cancel{\langle 2^m | 2 0\rangle} \\ = \sum_m \mathcal{D}_{m'm}(R) \cdot \sqrt{\frac{2l+1}{4\pi}} S_m^0 \Big|_{l=2} \\ = \mathcal{D}_{m'0}(R) \cdot \sqrt{\frac{2l+1}{4\pi}} \Big|_{l=2}$$

So from (1) we get:

$$C_m = \langle 2m | \mathcal{D}(R) | 20 \rangle = D_{mo}(R) \sqrt{\frac{2l+1}{4\pi}} \Big|_{l=2} = \sqrt{\frac{5}{4\pi}} D_{mo}(R)$$

$$\text{Now } \langle 2m | \mathcal{D}(R) | 20 \rangle = Y_2^{m*}(\theta, \phi) = \sqrt{\frac{5}{4\pi}} D_{mo}(R)$$

$$\text{So, } D_{mo}(R) = \sqrt{\frac{4\pi}{5}} Y_2^{m*}(\theta, \phi)$$

$$\therefore \langle 2 | \mathcal{D}(0, \alpha, 0) | 20 \rangle = \sum_{m=-2}^2 D_{mo}(0, \alpha, 0) \cdot | 2m \rangle$$

∴ The probability to find new state in  $| 2m \rangle$

State be:  $| D_{mo}(0, \alpha, 0) \rangle^2$

$$= \frac{4\pi}{5} | Y_2^m(0, \alpha, \phi=0) |^2$$

$$\text{So; } \langle 20 | \mathcal{D}(R) | 20 \rangle = \frac{4\pi}{5} \cdot | Y_2^0(0=\alpha, \phi=0) |^2 \\ = \frac{4\pi}{5} \cdot \frac{5}{16\pi} \cdot (3 \cos^2 \alpha - 1)^2 = \frac{1}{4} (3 \cos^2 \alpha - 1)^2$$

$$\langle 2\pm 1 | \mathcal{D}(R) | 20 \rangle = \frac{4\pi}{5} | Y_2^{\pm 1}(\alpha, 0) |^2 \\ = \frac{4\pi}{5} \cdot \frac{15}{16\pi} \sin^2 \alpha \cos^2 \alpha = \frac{3}{2} \sin^2 \alpha \cos^2 \alpha.$$

$$\langle 2\pm 2 | \mathcal{D}(R) | 20 \rangle = \frac{4\pi}{5} | \langle Y_2^{\pm 2}(\alpha, 0) \rangle |^2$$

$$= \frac{4\pi}{5} \cdot \frac{15}{32\pi} \sin^4 \alpha = \frac{3}{8} \sin^4 \alpha$$

$$15. \text{ a.) } \vec{\mu} = \gamma_1 \vec{j}_1 + \gamma_2 \vec{j}_2 \quad (\vec{A} = \text{quantum vector operator})$$

$$\text{So: } \mu_m = \gamma_1 j_1^m + \gamma_2 j_2^m$$

$$= \gamma_1 \left( \frac{j_1^+ + j_1^-}{2} \right) + \gamma_2 \left( \frac{j_2^+ + j_2^-}{2} \right)$$

Now the given state  $|jmij_1j_2\rangle$  is a linear combination of the states  $|j, m_1 j_1 m_2\rangle$

$$\text{i.e. } |jmij_1j_2\rangle = \sum C_{m_1 m_2}^{jmij_1j_2} |j, m_1 j_1 m_2\rangle$$

$$\therefore \langle \mu_m \rangle = \langle jmij_1j_2 | \mu_m | jmij_1j_2 \rangle$$

$$= \left( \sum C_{m_1 m_2}^{jmij_1j_2} \langle j, m_1 j_1 m_2 | \right) \left( \gamma_1 \frac{j_1^+ + j_1^-}{2} + \gamma_2 \frac{j_2^+ + j_2^-}{2} \right)$$

$$\left( \sum C_{m_1 m_2}^{jmij_1j_2} |j, m_1 j_1 m_2\rangle \right)$$

(such that  $m_1 + m_2 = m = m'_1 + m'_2$ )

$$= \left( \sum C_{m_1 m_2}^{jmij_1j_2} \langle j, m_1 j_1 m_2 | \right) \cancel{\left( \begin{matrix} \cancel{j_1^+} \\ \cancel{j_2^+} \end{matrix} \right)} \times$$

$$\sum C_{m_1 m_2}^{jmij_1j_2} \left[ \frac{\gamma_1}{2} \alpha'_1 |j_1, m'_1 + 1, j_2 m'_2\rangle \right]$$

$$+ \frac{\gamma_1}{2} \alpha'_2 |j_1, m'_1 - 1, j_2 m'_2\rangle$$

$$+ \frac{\gamma_2}{2} \alpha'_3 |j_1, m'_1 j_2 m'_2 + 1\rangle$$

$$+ \frac{\gamma_2}{2} \alpha'_4 |j_1, m'_1 j_2 m'_2 - 1\rangle \Big]$$

$$\left( \alpha = \sqrt{(j+m)(j+m+1)} = \text{const} \right)$$

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Now for all states in the bra we get

$$m_1 + m_2 = m$$

And for the large ket part:  $m'_1 + m'_2 \pm 1 = m \pm 1$

As the bra is the clebsch gordan expansion of  
 $|jmij_1j_2\rangle$  and the ket composed of such states  
for which  $m'_1 + m'_2 \neq m$ ; so each state  
in the bra is orthogonal to each states in the

ket.

$$\text{e.g. } \langle i_1 m_1 i_2 m_2 | j_1^+ j_2^- | 10110 \rangle$$

$$= \langle 1011 | (j_1^+ + j_2^-) | 1710 \rangle$$

$$= \alpha_1 \underbrace{\langle 1011 | 1010 \rangle}_0 + \alpha_2 \underbrace{\langle 1011 | 1010 \rangle}_{m'_1 m'_2 + 1}$$

$$= 0$$

This situation happens for every term in the summation present in the expansion of  $\langle Mm \rangle$ .

So by orthogonality we get:

$$\langle Mm \rangle = 0.$$

VII Now for  $\langle Mg \rangle$  the operator be given by:

$$\langle Mg \rangle = \frac{\gamma_1}{2i} (J_1^+ + J_2^-) + \frac{\gamma_2}{2i} (J_1^+ - J_2^-)$$

The calculation is totally similar like  $\langle Mm \rangle$   
with just 2 in denominator replaced by  $2i$ .

So by similar calculation we get

$$\langle Mg \rangle = \langle jmij_1j_2 | f(j_1^\pm, j_2^\pm) | jmij_1j_2 \rangle$$

$$= 0$$

$$\text{III} \quad M_z = \gamma_1 J_1^z + \gamma_2 J_2^z = \frac{\gamma_1 + \gamma_2}{2} (J_1^z + J_2^z) + \frac{\gamma_1 - \gamma_2}{2} (J_1^z - J_2^z)$$

but  $J_1^z + J_2^z = J_z$ .

So;  $M_z = \frac{(\gamma_1 + \gamma_2)}{2} J_z + \frac{(\gamma_1 - \gamma_2)}{2} (J_1^z - J_2^z)$

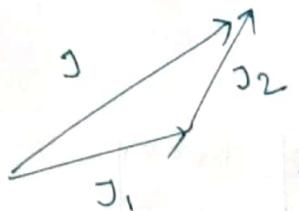
$|jm j_1 j_2\rangle$  are simultaneously eigenstates of  $J^2, J_z, J_1^2, J_2^2$ ; So.

$$\langle M_z \rangle = \frac{\gamma_1 + \gamma_2}{2} \langle J_z \rangle + \frac{\gamma_1 - \gamma_2}{2} (\langle J_1^z \rangle - \langle J_2^z \rangle)$$

but  $\langle J_z \rangle = m\hbar$ ; So:

$$\langle M_z \rangle = \frac{\gamma_1 + \gamma_2}{2} \cdot m\hbar + \frac{\gamma_1 - \gamma_2}{2} (\langle J_1^z \rangle - \langle J_2^z \rangle) \dots \text{(A)}$$

Here comes the crucial part.



$$\text{AD } \langle J_1^x \rangle, \langle J_1^y \rangle = 0; \text{ So we get}$$

$$\langle J_1 \rangle = \underbrace{\langle J_1^x \rangle}_0 + \underbrace{\langle J_1^y \rangle}_0 + \langle J_1^z \rangle = \langle J_1^z \rangle$$

But from fig.  ~~$\langle J_1 \rangle = \frac{(J_1 \cdot J)}{J^2} \cdot J = \langle J_1^z \rangle$~~

~~but by comparing coefficient  $\langle J^z \rangle = \frac{(J \cdot J)}{J^2} \langle J_z \rangle$~~

~~$$\langle J_1^z \rangle = \frac{(J_1 \cdot J)}{J^2} \cdot J^2$$~~

AD  $J^z$  is conserved or better to say  $|jm j_1 j_2\rangle$  is an eigenstate of  $J^z$ ; So; the arg of  $J_1$  is the arg of  $J$  along 'J'

$$\text{i.e. } \langle J_1 \rangle = \left\langle \frac{J_1 \cdot J}{J^2} \cdot J \right\rangle \dots (1)$$

but  $\langle J_1 \rangle = \langle J_1^z \rangle$  : So comparing the z components from (1) we get:

$$\langle J_1^z \rangle = \left\langle \frac{J_1 \cdot J}{J^2} \right\rangle \langle J_2 \rangle$$

$$\text{Now, } J_2 = J - J_1 \Rightarrow J_2^2 = J^2 + J_1^2 - 2 J_1 \cdot J$$

$$\Rightarrow J_1 \cdot J = \frac{J^2 + J_1^2 - J_2^2}{2} \Rightarrow \langle J_1 \cdot J \rangle = \left\langle \frac{J^2 + J_1^2 - J_2^2}{2} \right\rangle$$

$$\text{Similarly } \langle J_2 \cdot J \rangle = \left\langle \frac{J^2 - J_1^2 + J_2^2}{2} \right\rangle$$

$$\text{So, } \langle J_1^z \rangle = \left\langle \frac{(J^2 + J_1^2 - J_2^2)}{2J^2} \cdot J_2 \right\rangle$$

Now  $|jm_{j_1, j_2}\rangle$  is an eigenstate of every operator in the right parenthesis.

$$\begin{aligned} \therefore \langle J_1^z \rangle &= \left\langle \frac{(J^2 + J_1^2 - J_2^2)}{2J^2} J_2 \right\rangle |jm_{j_1, j_2}\rangle \\ &= \frac{\{j(j+1) + j_1(j_1+1) - j_2(j_2+1)\}}{2j(j+1)} \hbar^2 \cdot m \hbar \\ &= \frac{\{j(j+1) + j_1(j_1+1) - j_2(j_2+1)\}}{2j(j+1)} \cdot m \hbar \end{aligned}$$

$$\text{Similarly: } \langle J_2^z \rangle = \frac{\{j(j+1) - j_1(j_1+1) + j_2(j_2+1)\}}{2j(j+1)} m \hbar.$$

So from eq (A) we get:

$$\begin{aligned}\langle M_z \rangle &= \frac{\gamma_1 + \gamma_2}{2} \cdot m_h + \frac{\gamma_1 - \gamma_2}{2} \cdot \frac{m_h}{2j(j+1)} \left\{ j(j+1) + j_1(j_1+1) \right. \\ &\quad \left. - j_2(j_2+1) - j(j+1) + j_1(j_1+1) - j_2(j_2+1) \right\} \\ &= \frac{(\gamma_1 + \gamma_2)}{2} \cdot m_h + \frac{(\gamma_1 - \gamma_2)}{2} \cdot \frac{m_h \times \chi}{2j(j+1)} (j_1(j_1+1) - j_2(j_2+1)) \\ &= m_h \left[ \frac{\gamma_1 + \gamma_2}{2} + \frac{\gamma_1 - \gamma_2}{2} \cdot \frac{j_1(j_1+1) - j_2(j_2+1)}{j(j+1)} \right]\end{aligned}$$

Proved

15.b. excluding the factor of  $h$  in the previous expression, the term gives the factor of how many times the magnetic moment comes of the fundamental unit. (i.e Bohr magneton or nuclear magneton.)

i.e if  $\mu = \alpha M_{\text{fundamental}}$  then.

$$\alpha = m \left[ \frac{\gamma_1 + \gamma_2}{2} + \frac{\gamma_1 - \gamma_2}{2} \cdot \frac{j_1(j_1+1) - j_2(j_2+1)}{j(j+1)} \right]$$

The given state:  ${}^{2S+1}L_J = {}^2P_{1/2} \equiv |j \ m \ j_1 \ j_2\rangle$

So;  $S = \frac{1}{2} \Rightarrow j_1 = \frac{1}{2}$ ;  $L = 1 \Rightarrow j_2 = \frac{1}{2}$ ;  $J = \frac{1}{2} \Rightarrow m = \pm \frac{1}{2}$ .

Ans;  $\gamma_1 = \gamma_2 = 5.6$  (for proton);  $\gamma_2 = 1$ .

$$\text{So } \alpha_p = \pm \frac{1}{2} \left[ \frac{6.6}{2} + \frac{4.4}{2} \cdot \frac{\frac{1}{2}(\frac{1}{2}+1) - 1(1+1)}{\frac{1}{2}(\frac{1}{2}+1)} \right]$$

$$= \pm \frac{1}{2} \left[ 3.3 - 2.3 \times \frac{2 - \frac{3}{4}}{\frac{3}{4}} \right] = \pm \frac{1}{2} \left[ 3.3 - 2.3 \times \frac{5}{3} \right]$$

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$$= \pm 0.2666 \dots$$

$$\therefore \langle \mu_z \rangle_p = \pm 0.266 \mu_{\text{nuclear magneton}}$$

c) for e<sup>-</sup> the only difference is:  $\gamma_1 = 2$ .

$$S_0 \propto e = \pm \frac{1}{2} \left[ \frac{3}{2} - \frac{1}{2} \times \frac{5}{3} \right]$$

$$= \pm \frac{1}{2} \times 0.166 = \pm 0.33$$

$$\therefore \langle \mu_z \rangle_e = \pm 0.33 \mu_{\text{Bohr}}$$

$$\simeq \pm \frac{1}{3} \mu_{\text{Bohr.}} \quad \underline{\text{proved}}$$