ED - 2 MIDTERM EXAM

SAGAR DAM

DNAP.

Problem: 7

a. The Lagrangian density of the field is given by:

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\alpha A^\alpha$$

$$= -\frac{1}{16\pi} g_{\lambda\mu} g_{\nu\sigma} \left(\partial^\mu A^\sigma - \partial^\sigma A^\mu\right)\left(\partial^\lambda A^\nu - \partial^\nu A^\lambda\right) - \frac{1}{c} J_\alpha A^\alpha$$

$$\left(\because F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

Now varying the $\mathcal{L}$ w.r.t A^ν i.e the field quantity gives:

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{16\pi} g_{\lambda\mu} g_{\nu\sigma} \left[\delta^\mu_\beta \delta^\sigma_\alpha F^{\lambda\nu} - \delta^\sigma_\beta \delta^\mu_\alpha F^{\lambda\nu} + \delta^\lambda_\beta \delta^\nu_\alpha F^{\mu\sigma} - \delta^\nu_\beta \delta^\lambda_\alpha F^{\mu\sigma} \right]$$

As $g_{\mu\nu} = g_{\nu\mu}$; and $F^{\alpha\beta} = -F^{\beta\alpha}$ we get:

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} \cancel{= -\frac{1}{16\pi} g_{\lambda\alpha} g_{\nu\sigma}} \Big\{ F$$

$$= -\frac{1}{16\pi}\Big\{ g_{\lambda\beta}\, g_{\nu\alpha} F^{\lambda\nu} - g_{\lambda\alpha}\, g_{\nu\beta} F^{\lambda\nu} + g_{\beta\mu}\, g_{\alpha\sigma} F^{\mu\sigma} - g_{\alpha\mu}\, g_{\beta\sigma} F^{\mu\sigma} \Big\}$$

$$= -\frac{1}{16\pi}\Big\{ F_{\beta\alpha} - F_{\alpha\beta} + F_{\beta\alpha} - F_{\alpha\beta} \Big\}$$

$$\left(\because\ F_{\alpha\beta} = g_{\lambda\alpha}\, g_{\nu\beta} F^{\lambda\nu} \right)$$

$$= \frac{1}{4\pi} F_{\alpha\beta}.$$

i.e the field eq be: $\partial^\beta \left\{ \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} \right\} = \frac{\partial \mathcal{L}}{\partial A^\alpha}$

i.e $\boxed{\frac{1}{4\pi} \partial^\beta (F_{\alpha\beta}) = -\frac{1}{c} J_\alpha}$

i.e $\boxed{\frac{1}{4\pi} \partial^\beta (F_{\beta\alpha}) = \frac{J_\alpha}{c}}$ $\longrightarrow$ Ans.

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For $\alpha = 0$ this gives:

$$\frac{1}{4\pi} \partial^\beta \left(F_{0\beta} \right) = -\frac{1}{c} J_0$$

i.e $$\frac{1}{4\pi} \left(\partial^0 \cancel{F_{00}}^{\,0} + \partial^i F_{0i} \right) = -\frac{1}{c} \cdot \rho c$$

i.e $$\partial^1 \left\{ (\partial_0 A_1 - \partial_1 A_0) \right\} + \cancel{(\partial_0 A_2} + \partial^2 (\partial_0 A_2 - \partial_2 A_0)$$

$$+ \partial^3 (\partial_0 A_3 - \partial_3 A_0) = -4\pi\rho$$

i.e $$\partial_0 \left\{ \partial^i A_i \right\} - \partial^i \partial_i A_0 = -4\pi\rho$$

i.e $$+ \vec{\nabla} \cdot \left(\frac{\partial \vec{A}}{\partial t} \right) + \nabla^2 \phi = -4\pi\rho$$

i.e $$\boxed{\vec{\nabla} \cdot \left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} \right) = 4\pi\rho} \quad \cdots (1)$$

for $\alpha = 1$:

$$\frac{1}{4\pi} \left(\partial^0 F_{10} + \partial^1 \cancel{F_{11}}^{\,0} + \partial^2 F_{12} + \partial^3 F_{13} \right) = -\frac{J_1}{c}.$$

i.e $$\frac{1}{4\pi} \left\{ \partial^0 \left(\partial_1 A_0 - \partial_0 A_1 \right) + \partial^2 (\partial_1 A_2 - \partial_2 A_1) + \partial^3 (\partial_1 A_3 - \partial_3 A_1) \right\} = -\frac{J_1}{c}$$

i.e. $\partial^2 (\nabla\times A)_3 - \partial^3 (\nabla\times A)_2 - \frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)_1$

$$= -\frac{4\pi J_1}{c}$$

i.e $-\left\{\nabla\times(\nabla\times\vec{A})\right\}_1 + \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right)_1 = -\frac{4\pi J_1}{c}$

i.e $\left(\nabla\times(\nabla\times A)\right)_1 = \frac{4\pi J_1}{c} + \frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right)_1$

i.e for $\alpha = 2, 3$ we get combining in vector form

$$\boxed{\nabla\times(\nabla\times\vec{A}) = \frac{4\pi\vec{J}}{c} + \frac{\partial}{\partial t}\left\{-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right\}} \rightarrow (2)$$

Thus we get the source dependent Maxwell's eq from the Lagrangian.

Here if we do two substitutions

i.e $\nabla\times\vec{A} = \vec{B}$

And $-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} = \vec{E}$

i.e Assume these $\vec{E}, \vec{B}$ to be two new fields,

then we get,

$$\boxed{\begin{aligned} \nabla\cdot\vec{E} &= 4\pi\rho \\ \nabla\times\vec{B} &= \frac{4\pi\vec{J}}{c} + \frac{\partial\vec{E}}{\partial t} \end{aligned}}$$

i.e our familiar form in terms of $\vec{E}, \vec{B}$

if instead of this I've written $F^{\alpha\beta}$ in terms of $\vec{E}, \vec{B}$ totally i.e written ~~they~~ $\mathcal{L}(E,B)$ then we got these eqn's exactly. (I've done that ~~..~~ in Assignment)

Now the Homogeneous Maxwell's eqn's are not any new eq'n and they are assumed to be true here. when we consider

$$\left.\begin{aligned} &\vec{B} = \nabla\times\vec{A} \Rightarrow \nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0 \\ &\text{And } \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \Rightarrow \nabla\times\vec{E} = -\frac{\partial(\nabla\times\vec{A})}{\partial t} = -\frac{\partial\vec{B}}{\partial t} \end{aligned}\right\} \underline{\text{Ans}}$$

<u>b</u>: $\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c} A_\alpha J^\alpha.$

stays EOM invariant under Gauge transform

$$A^\mu \longrightarrow A^\mu + \partial^\mu \lambda \quad (\lambda = \text{Gauge parameter})$$

Now, $\mathcal{L} - \mathcal{L}'$

$$= -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \cancel{\frac{1}{c} A_\alpha J^\alpha} + \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta.$$
$$+ \cancel{\frac{1}{c} A_\beta J^\beta}$$

$$= -\frac{1}{8\pi}\left(\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} - \partial_\alpha A_\beta \partial^\alpha A^\beta\right)$$

$$= -\frac{1}{8\pi}\left(\frac{1}{2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \partial_\alpha A_\beta \partial^\alpha A^\beta\right)$$

$$= -\frac{1}{8\pi}\left(\frac{1}{2}\left(\cancel{\partial_\alpha A_\beta \partial^\alpha A^\beta} - \partial_\beta A_\alpha \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha + \cancel{\partial_\beta A_\alpha \partial^\beta A^\alpha}\right) - \cancel{\partial_\alpha A_\beta \partial^\alpha A^\beta}\right)$$

$$= -\frac{1}{8\pi} \times \left(-\partial_\alpha A_\beta \partial^\beta A^\alpha\right)$$

$$= \frac{\partial_\alpha A_\beta \partial^\beta A^\alpha}{8\pi} = \frac{1}{8\pi} \partial_\alpha \left(A_\beta \partial^\beta A^\alpha\right)$$

$$= \frac{1}{8\pi} \partial_\alpha \left\{(A \cdot \partial) A^\alpha\right\}$$

($A \cdot \partial$ = 4 dim inner product) i.e $\Delta \mathcal{L} = \frac{1}{8\pi} \partial_\alpha J^\alpha$

$\left(J^\alpha = (A \cdot \partial) A^\alpha \right)$

i.e $\mathcal{L}$ differs by a 4 divergence

proves

c: ~~The added 4 divergence do not change the EOM.~~

$$\Delta S = \int d^4x \, (\Delta \mathcal{L})$$

$$= \int \cancel{8} \left(\partial_\alpha J^\alpha \right) d^4x$$

$$= \underset{4\text{ surface}}{\oiint} \vec{J} \cdot \vec{dS}$$

provided that $\vec{J}$ vanishes on boundary

$\Delta S = 0$

i.e The variation of S' is same and gives same E.O.M. $\longrightarrow$ proves

Problem: 8

Ans: $M \rightarrow m_1 + m_2$

In the rest frame of decaying particle the 4 momentum be:

$$P = M_0 \gamma (c, \vec{v})$$

but $\gamma = 1$; $\vec{v} = 0$ & let $c = 1$ i.e

$$P = M(1, \vec{0})$$

If (E_1, E_2); $(\vec{P}_1, \vec{P}_2)$ be the energies & momentum of produced particle then the 4-momentum be:

$$p_1^\mu = (E_1, \vec{P}_1) \quad ; \quad P_2^\mu = (E_2, \vec{P}_2)$$

Now; $P = p_1 + p_2$.

i.e $p_2^2 = (P - p_1)^2 = P^2 + p_1^2 - 2P \cdot p_1$

but $P^2 = M^2$; $p_1^2 = m_1^2$; $p_2^2 = m_2^2$; $P \cdot p_1 = ME_1$

i.e $m_2^2 = M^2 + m_1^2 - 2E_1 M$

i.e $\boxed{E_1 = \dfrac{(M^2 + m_1^2 - m_2^2)}{2M}} \longrightarrow$ proved

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b. When the ~~decy~~ decay is continuous i.e M goes to m ($\to \infty$) particles:

$$p = p_1 + p_2 + \cdots$$

And from momentum conservation $\sum \vec{p}_i = 0$.

Now, $T_1 = E_1 - m_1$

$$= \frac{M^2 + m_1^2 - m_2^2}{2M} - m_1 \quad \text{(for 2 particle)}$$

$$= \frac{M^2 - 2m_1 M + m_1^2 - m_2^2}{2M}$$

$$= \frac{(M - m_1)^2 - m_2^2}{2M}$$

$$= \frac{(M - m_1 + m_2)(M - m_1 - m_2)}{2M}$$

$$= \frac{\Delta M}{2M}(2M - 2m_1 - M + m_1 + m_2)$$

($\Delta M = M - m_1 - m_2$ = change of total mass to energy)

$$= \Delta M \left(1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\right)$$

Here I've assumed m_2 to be sum of all other masses. So for ith particle in general:

$$T_i = \Delta M \left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right) \rightarrow \underline{\text{proved}}$$

Problem: 9

~~Let~~ without loss of generality; let the Lorentz Boost is given along x direction with velocities v_1 & v_2. Consider the L.T matrix (only x-t part is important here.

$$M_1 = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 \\ -\gamma_1\beta_1 & \gamma_1 \end{pmatrix}$$

(I've not written other terms as those don't affect here)

$$M_2 = \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 \\ -\gamma_2\beta_2 & \gamma_2 \end{pmatrix}$$

Now if the rapidity is ϕ then $e^{\phi} = \gamma(1+\beta)$

i.e $e^{-\phi} = \frac{1}{\gamma(1+\beta)} = \frac{\gamma(1-\beta)}{\gamma^2(1-\beta^2)} = \gamma(1-\beta)$

i.e $\gamma = \dfrac{e^{\phi} + e^{-\phi}}{2} = \cosh\phi$

$\gamma\beta = \dfrac{e^{\phi} - e^{-\phi}}{2} = \sinh\phi$

$\beta = \tanh\phi$

$= \tanh(\ln \gamma(1+\beta))$

$\therefore$ The LT matrix be written as,

$$M_{\phi} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

i.e here the whole transformation is:

$$M = M_2 M_1 = \begin{pmatrix} \cosh\phi_2 & -\sinh\phi_2 \\ -\sinh\phi_2 & \cosh\phi_2 \end{pmatrix} \begin{pmatrix} \cosh\phi_1 & -\sinh\phi_1 \\ -\sinh\phi_1 & \cosh\phi_1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\phi_2\cosh\phi_1 + \sinh\phi_2\sinh\phi_1 & -(\cosh\phi_2\sinh\phi_1 + \sinh\phi_2\cosh\phi_1) \\ (\ldots) & \cosh\phi_2\cosh\phi_2 + \sinh\phi_2\sinh\phi_1 \end{pmatrix}$$

Symm

~~i.e the effective γ be~~ :

$$= \begin{pmatrix} \cosh(\phi_1+\phi_2) & -\sinh(\phi_1+\phi_2) \\ & \cosh(\phi_1+\phi_2) \end{pmatrix} \text{ symm.}$$

but $(\phi_1+\phi_2)$ is the whole rapidity.

i.e
$$\left.\begin{aligned} \gamma &= \cosh(\phi_1+\phi_2) \\ \gamma\beta &= \sinh(\phi_1+\phi_2) \end{aligned}\right\}$$

i.e $\beta = \tanh(\phi_1+\phi_2) = \tanh\left\{\ln \gamma_1(1+\beta_1) + \ln \gamma_2(1+\beta_2)\right\}$

$\cancel{= \tanh \ln \left\{\gamma_1\gamma_2(1+\beta_1)(1+\beta_2)\right\}} = \tanh\left(\ln \gamma_1\gamma_2(1+\beta_1)(1+\beta_2)\right) \quad \ldots (1)$

$\cancel{\text{i.e } \gamma\beta = \gamma_1\gamma_2(1+\beta_1)(1+\beta_2)}$

~~i.e~~ $\cancel{\beta = 1 + \frac{v_1}{c} + \frac{v_2}{c} + \frac{v_1 v_2}{c^2}}$

i.e
$$\left.\begin{aligned} \gamma &= \gamma_1\gamma_2 + \gamma_1\gamma_2\beta_1\beta_2 \\ \gamma\beta &= \gamma_1\gamma_2\beta_2 + \gamma_1\gamma_2\beta_1 \end{aligned}\right\}$$

i.e
$$\beta = \frac{\gamma_1\gamma_2(\beta_1+\beta_2)}{(1+\beta_1\beta_2)\gamma_1\gamma_2}$$

i.e
$$\frac{v_{eff}}{c} = \frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1 v_2}{c^2}}$$

i.e
$$\boxed{v_{eff} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}}$$

i.e the addition is a new L.T with velocity: $\boxed{v_{eff} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}} \longrightarrow$ Ans

Or from the rapidity: (or from eq (1)):

as $e^{\phi_1} \cdot e^{\phi_2} = e^{\phi_1 + \phi_2} = e^{\phi}$

i.e $\gamma_1 (1+\beta_1) \gamma_2 (1+\beta_2) = \gamma (1+\beta)$

$\gamma_1^2 (1+\beta_1)^2 \gamma_2^2 (1+\beta_2)^2 = \gamma^2 (1+\beta)^2$.

i.e $$\frac{1}{\left(1-\frac{v_1^2}{c^2}\right)} \cdot \frac{1}{\left(1-\frac{v_2^2}{c^2}\right)} \left(1+\frac{v_1}{c}\right)^2 \left(1+\frac{v_2}{c}\right)^2 = \frac{1}{\left(1-\frac{v^2}{c^2}\right)} \left(1+\frac{v}{c}\right)^2.$$

i.e $$\frac{1+\frac{v_1}{c}}{1-\frac{v_1}{c}} \cdot \frac{1+\frac{v_2}{c}}{1-\frac{v_2}{c}} = \frac{1+\frac{v}{c}}{1-\frac{v}{c}}$$

i.e $$\frac{(c+v_1)(c+v_2)}{(c-v_1)(c-v_2)} = \frac{c+v}{c-v}$$

i.e $$\frac{(c+v)-(c-v)}{(c+v)+(c-v)} = \frac{(c+v_1)(c+v_2)-(c-v_1)(c-v_2)}{(c+v_1)(c+v_2)+(c-v_1)(c-v_2)}$$

i.e $$\frac{v}{c} = \frac{2c(v_1+v_2)}{2(c^2+v_1v_2)}$$

i.e $$\boxed{V = V_{eff} = \frac{V_1 + V_2}{1 + \frac{V_1 V_2}{C^2}}} \longrightarrow \underline{\underline{\text{proved}}}$$

∴ This is effective velocity of the two successive L.T in same direction

2. $\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c}(\vec{E}\cdot\vec{J}) + \frac{1}{c}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$.

Using product rule:

$$\vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B}) = \frac{4\pi}{c}(\vec{E}\cdot\vec{J}) + \frac{1}{c}\cdot\frac{1}{2}\frac{\partial(E^2)}{\partial t}$$

i.e $-\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \nabla\cdot(\vec{E}\times\vec{B}) = 4 \quad 4$

$$\left(\because \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right)$$

i.e $\nabla\cdot(\vec{E}\times\vec{B}) = -\frac{4\pi}{c}(\vec{E}\cdot\vec{J}) - \frac{1}{2c}\frac{\partial}{\partial t}(E^2+B^2)$

Now, $\vec{E}\cdot\vec{J} = \rho\,\vec{E}\cdot\vec{v} = \rho(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v}$

$= \rho\cdot\vec{F}_I\cdot\vec{v} = \vec{F}_\rho\cdot\vec{v}$

$= \frac{dw}{dt}$ (w is the work done on charges in unit vol)

$\frac{1}{4\pi}\frac{E^2+B^2}{2}$ is the energy density of the E.M field.

i.e

$$\frac{4\pi}{c}\frac{dW}{dt} = -\frac{d}{dt}\left(\frac{E^2+B^2}{2c}\right) - \nabla\cdot\left(\vec{E}\times\vec{B}\right)$$

i.e $$\int \frac{dW}{dt}\, d\tau = -\frac{d}{dt}\int \frac{1}{4\pi}\left(\frac{E^2+B^2}{2}\right) d\tau - \frac{c}{4\pi}\int \nabla\cdot\left(\vec{E}\times\vec{B}\right) d\tau$$

i.e $$\frac{dW}{dt} = -\frac{d}{dt}\int \frac{1}{4\pi}\left(\frac{E^2+B^2}{2}\right) d\tau - \frac{c}{4\pi}\oint_S \left(\vec{E}\times\vec{B}\right)\cdot d\vec{S}$$

i.e the ~~too~~ loss of energy by the charge (work done by charge) is transmitted either to the field (E^2+B^2 term) or carried out by the field by $\vec{E}$ & $\vec{B}$ along the direction of $(\vec{E}\times\vec{B})$

i.e
work done rate on charge = decrease of energy stored in $\vec{E},\vec{B}$ fields

— energy going out through surface by $\vec{E},\vec{B}$ in direction $(\vec{E}\times\vec{B})$

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4.a from thermol energy distribution law we get:

$$U(\nu) = \frac{2h\nu^3}{c^3\left(e^{\frac{h\nu}{kT}} - 1\right)}$$

Now for a moving frame λ (i-e ν) changes due to two reasons.

1) Spacetime transformation ($\alpha \cdot T$)
11) Doppler shift.

1) by spacetime contraction (i.e $\alpha \cdot T$) the contraction is:

$$\lambda'_f \longrightarrow \frac{\lambda_i}{\gamma}$$

11) by doppler shift λ farther reduces to

$$\lambda_f \longrightarrow \frac{\lambda'_f}{1 + \vec{v}/c \cdot \hat{n}}$$

i.e $$\lambda_f \longrightarrow \frac{\lambda'_f}{1 + \frac{v}{c}\cos\theta}$$

i.e combining both gives: $$\lambda_f = \frac{\lambda_i}{\gamma\left(1 + \frac{v\cos\theta}{c}\right)}$$

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i.e $$\nu_{observed} = \frac{\sqrt{(1 - v^2/c^2)}^{-1}\, \nu_{true}}{\left(1 + \frac{v\cos\theta}{c}\right)^{-1}}$$

($\because \nu \sim \lambda^{-1}$ I've first written wrong. hence the inverse is added later)

$$\therefore U(\nu_{observed}) = \frac{2}{c^3} \cdot \frac{h\nu_{obs}^3}{e^{\frac{h\nu_{obs}}{kT'}} - 1}$$

$$= \frac{2}{c^3} \cdot \frac{h\left(\frac{\sqrt{1 - v^2/c^2}}{1 + \frac{v}{c}\cos\theta}\right)^{-3} \cdot \nu_{true}^3}{e^{\frac{h}{kT'} \cdot \left(\frac{\sqrt{1 - v^2/c^2}}{1 + \frac{v\cos\theta}{c}}\right)^{-1} \nu_{true}} - 1}$$

$$= \frac{2}{c^3} \frac{h\left(\frac{1 + \frac{v}{c}\cos\theta}{\sqrt{1 - v^2/c^2}}\right)^3 \nu_{true}^3}{e^{\frac{h\nu_{true}}{kT'} \cdot \left(\frac{1 + \frac{v\cos\theta}{c}}{\sqrt{1 - v^2/c^2}}\right)} - 1}$$

But for small velocity $\left(\frac{1 + \frac{v}{c}\cos\theta}{\sqrt{1 - v^2/c^2}}\right)^3 \sim 1 + O\left\{\left(\frac{v}{c}\right)^3\right\}$

≈ 1

i.e $$U(\nu_{obs}) = \frac{2}{c^3} \frac{h\nu_{true}^3}{e^{\frac{h\nu}{kT'}\left(\frac{1 + v\cos\theta/c}{\sqrt{1 - v^2/c^2}}\right)} - 1}$$

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i.e equivalently we ~~say~~ can say that temperature transforms like:

$$T' = \frac{T\sqrt{1-v^2/c^2}}{\left(1+\frac{v\cos\theta}{c}\right)} \longrightarrow \underline{\underline{\text{Ans}}}$$

b. $I_\nu = C\,U_\nu$

i.e $I_{max} = C\,U_\nu(max)$; $I_{min} = C\,U_\nu(min)$.

from part (A) we get:

I will be maximum for $\theta = \pi$ & minimum for $\theta = 0$

i.e $$I_{max} = \frac{2}{c^2}\frac{h\nu_{cmb}^3}{e^{\frac{h\nu_{cmb}}{kT}\cdot\sqrt{\frac{1-v/c}{1+v/c}}}-1}$$

$$I_{min} = \frac{2}{c^2}\frac{h\nu_{cmb}^3}{e^{\frac{h\nu_{cmb}}{kT}\sqrt{\frac{1+v/c}{1-v/c}}}-1}$$

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(ν, T given here).

Now, $\quad$ Isotropy $= 10^{-3} = \dfrac{I_{max} - I_{min}}{I_{max} + I_{min}}$

i.e
$$\frac{\dfrac{1}{e^{\frac{h\nu}{kT}\sqrt{\frac{1-v/c}{1+v/c}}}-1} - \dfrac{1}{e^{\frac{h\nu}{kT}\sqrt{\frac{1+v/c}{1-v/c}}}-1}}{\dfrac{1}{e^{()\sqrt{\frac{1-v/c}{1+v/c}}}-1} + \dfrac{1}{e^{()\sqrt{\frac{1+v/c}{1-v/c}}}-1}} = 10^{-3}$$

i.e
$$\frac{e^{()\sqrt{\frac{1+v/c}{1-v/c}}} - e^{()\sqrt{\frac{1-v/c}{1+v/c}}}}{e^{()\sqrt{\frac{1+v/c}{1-v/c}}} + e^{()\sqrt{\frac{1-v/c}{1+v/c}}} - 2} = 10^{-3}$$

i.e ~~for~~ taking Taylor expansion upto $O\left(v/c\right)$:

$$\frac{\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}}}{\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}}} = 10^{-3}$$

i.e $\sqrt{\dfrac{1+\beta}{1-\beta}} \Big/ \sqrt{\dfrac{1-\beta}{1+\beta}} = \dfrac{1+10^3}{-1+10^3}$

i.e $\dfrac{1+\beta}{1-\beta} = \dfrac{1001}{999}$

i.e $\beta^{-1} \approx \dfrac{1001+999}{1001-999} \approx \dfrac{2\times 1000}{2}$

i.e $\beta \approx \dfrac{1}{1000}$

i.e $V \approx \dfrac{c}{1000} \approx 3\times 10^5$ m/s.

i.e The approximate value of earth's velocity w.r.t CMB is 3×10^5 m/s.

Problem: 3.

Let dN = no of points in phase space.

dv = volume element in phase space.

Now for L.T along x direction..

$$x' = \gamma(x - vt) \qquad p_x' = \gamma(p_x - \beta p^0)$$
$$y' = y \qquad p_y' = p_y$$
$$z' = z \qquad p_z' = p_z$$

~~Clearly $f = \frac{dN}{dV}$ will be conserved under L.T iff $dv = \prod dx_i\, dp_i$ = invariant. (as N is unaffected under L.T.)~~

The phase space volume transforms like:

$$dv \rightarrow dv' = J\left(x_i p_i \rightarrow x_i' p_i'\right) \cdot dv.$$

but $J = \det \left| \frac{\partial x_i'}{\partial x_j} \right|_{\text{all } i,j}$ $\quad$ ($x_i \in$ space & momentum)

i.e the transformation matrix is:

$$= \begin{pmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \cdots & \frac{\partial x_1'}{\partial p_1} & \cdots \\ \frac{\partial x_2'}{\partial x_1} & & & & \\ \vdots & & & & \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & & & & 0 & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \gamma & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

i.e $J = \det \left| \frac{\partial x_i}{\partial x_j} \right|_{\forall i,j} = \gamma^2.$

i.e $dV \rightarrow \gamma^2 dV. = dV'$

Now, $dN = \prod dn_{x_i} \, dn_{p_i}$

dn_{x_i} = no of particle between x & $dx + x$

dn_{p_i} = " " " momentum in p_i & $p_i + dp_i$

i.e dn_i's also have the same transformation like x_i & p_i

i.e $\frac{\partial m x_i}{\partial m x_j'}$ is similar to $\frac{\partial x_i}{\partial x_j'}$

(This is because L.T does not affect dN by changing no of particles ; but affect by changing definition of / measurement of dx, dp etc)

i.e $dN' = \gamma^2 dN$ (similarly).

$\therefore \; f' = \frac{dN'}{dV'} = \frac{\gamma^2 (dN)}{\gamma^2 (dV)} = \frac{dN}{dV} = f$

i.e phase space density is conserves

$\longrightarrow$ proves

b) $U(\nu) = \frac{2h\nu^3}{c^3 \left(e^{\frac{h\nu}{KT}} - 1\right)}$

And, $I(\nu) = \frac{2h}{c^3} \frac{\nu^3}{e^{\frac{h\nu}{KT}} - 1} \cdot \frac{dE}{dt}$.

5.a The definition of $\delta(x-x')$ is given by integral eq:

$$\int_{\text{all space}} \delta(x-x')\, f(x')\, dx' = f(x)$$

i.e for $f(x')=1$: $\int \delta(x-x')\, dx' = 1$.

In some coordinate system:

$$dx\,dy\,dz \longrightarrow |J|\, du\, dv\, dw.$$

where $|J|$ is the Jacobian & related to the metric tensor g as: $|J| = \sqrt{\det(g)}$

i.e $dx\,dy\,dz \longrightarrow \sqrt{\det(g)}\; du\, dv\, dw.$

i.e $$\int_{\text{all space}} f(\vec{x}')\, \delta^3(\vec{x}'-\vec{x})\, d^3x = f(\vec{x})$$

$$\Rightarrow \int_{\text{all space}} \tilde{f}(\vec{u}')\, \delta^3(\vec{u}-\vec{u}')\, \sqrt{\det(g)}\, d^3\vec{u} = \tilde{f}(\vec{u})$$

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but for $f(\bar{x}) = 1$ for all space

$\tilde{f}(\bar{u}) = 1$ " " "

i.e $\int\limits_{\text{all space}} \delta(\vec{u} - u') \sqrt{\det(g)}\, d^3\vec{u} = 1$

$\longrightarrow$ Ans

b. for cylindrical system:

$x = \rho\cos\phi \;;\; y = \rho\sin\phi \;;\; z = z$

$$\therefore |J| = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

i.e $\det(g) = \rho$.

$$\therefore \delta(\bar{u} - \bar{u}') = \frac{1}{\rho}\, \delta(\rho - \rho')\, \delta(\phi - \phi')\, \delta(z - z')$$

$\longrightarrow$ Ans

Problem: 6

Let $\vec{E} = E\hat{x}$

& $\vec{v}(t=0) \parallel \hat{y}$.

Now, $\vec{F} = qE\hat{x}$

i.e $\frac{dp_x}{dt} = qE$; $\frac{dp_y}{dt} = 0$; $\frac{dp_z}{dt} = 0$

Clearly $p_z = p_z(t=0) = 0$ | $p_x = qEt$

$p_y = p_y(t=0) = p_0$

Now $E = \sqrt{p^2c^2 + m^2c^4}$

$= \sqrt{c^2 q^2 E^2 t^2 + m^2c^4 + p_0^2 c^2}$

but $E^2(0) = m^2c^4 + p_0^2c^2$

i.e $E = \sqrt{E^2(0) + q^2E^2c^2t^2}$.

$\therefore V_x = \frac{p_x c^2}{E} = \frac{c^2 qE t}{\sqrt{E^2(0) + q^2E^2c^2t^2}}$

$$v_y = p_y \frac{c^2}{E} = \frac{p_0 c^2}{\sqrt{E^2(0) + q^2 c^2 E^2 t^2}}$$

$$v_z = 0$$

$$\frac{dx}{dt} = \frac{c^2 q E t}{\sqrt{E^2(0) + q^2 c^2 E^2 t^2}}.$$

i.e $$x = \int \frac{c^2 q E t}{\sqrt{E^2(0) + q^2 c^2 E^2 t^2}}\, dt$$

$$= \frac{1}{qE} \sqrt{E^2(0) + q^2 c^2 E^2 t^2} + x_0.$$

$$\frac{dy}{dt} = \frac{p_0 c^2}{\sqrt{E^2(0) + q^2 c^2 E^2 t^2}}$$ i.e $$y = \int \frac{p_0 c^2\, dt}{\sqrt{(\cdots)}}$$

i.e $$y = \frac{p_0 c^2}{qE} \sinh^{-1}\left(\frac{qcEt}{E(0)}\right) + y_0.$$

(integrals done in mathematica)

& $$\frac{dz}{dt} = 0$$ i.e $z = z_0$.

$$\therefore v = \frac{1}{\sqrt{E^2(0) + q^2 c^2 E^2 t^2}} \left(c^2 q E t,\ p_0 c^2,\ 0\right)$$