ED ASSIGNMENT: 2

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D·N·A·P·

Prob: 3

Without loss of generality; let's take the Lorentz boost along $\vec{x}$ axis. i.e the coordinate transformation is given by:

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$ct' = \gamma\left(ct - \frac{vx}{c}\right)$$

i.e the $x^\mu \to x'^\mu$ transformation; the Jacobian determining the change of 4-volume element & that's given by,

$$J\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \end{pmatrix} = \begin{vmatrix} \frac{\partial x_0'}{\partial x_0} & \frac{\partial x_1'}{\partial x_0} & \cdots \\ \frac{\partial x_0'}{\partial x_1} & & \\ \frac{\partial x_0'}{\partial x_2} & \cdots & \\ \vdots & & \end{vmatrix}$$

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As $y' = y$; $z' = z$; so the only contribution comes from, ct & x.

i.e J reduces to:

$$J = \begin{vmatrix} \frac{\partial (ct')}{\partial (ct)} & \frac{\partial x'}{\partial (ct)} \\ \frac{\partial (ct')}{\partial x} & \frac{\partial x'}{\partial x} \end{vmatrix} = \begin{vmatrix} \frac{\partial t'}{\partial t} & \frac{\partial x'}{\partial t} \\ \frac{\partial t'}{\partial x} & \frac{\partial x'}{\partial x} \end{vmatrix}$$

$$= \begin{vmatrix} \gamma & -\gamma v \\ -\frac{vx}{c^2} & \gamma \end{vmatrix} = \gamma^2 - \frac{\gamma^2 v^2}{c^2}$$

$$= \gamma^2 (1-\beta^2) = 1.$$

i.e $dx_0\, dx_1\, dx_2\, dx_3 \longrightarrow \underbrace{J\left(x^\mu \to x^{\mu'}\right)}_{1} . dx_0'\, dx_1'\, dx_2'\, dx_3'$

i.e $\boxed{dx_0\, dx_1\, dx_2\, dx_3 = dx_0'\, dx_1'\, dx_2'\, dx_3'}$

$\longrightarrow$ ~~Under~~ Under L.T.

i.e 4 volume is conserved.

$\longrightarrow$ proved

Prob: 8

We have to show that Green's fn for a self adjoint operator is symmetric.

Now as told in class to show this for ∇^2; it can be shown for a more general operator with self adjoint property like Helmholtz operator.

$$\nabla^2 \psi(\vec{r}) + k^2 \psi(\vec{r}) = f(\vec{r}).$$

(k = 0 gives poisson eq.
k = 0; f = 0 gives Laplace eq.)

The green fn $G(\vec{r}, \vec{r}_0)$ satisfies:

$$\nabla^2 G(\vec{r}, \vec{r}_0) + k^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0)$$

i.e $\nabla^2 G(\vec{r}, \vec{r}_1) + k^2 G(\vec{r}, \vec{r}_1) = \delta(\vec{r} - \vec{r}_1) \quad \cdots (1)$

$\nabla^2 G(\vec{r}, \vec{r}_2) + k^2 G(\vec{r}, \vec{r}_2) = \delta(\vec{r} - \vec{r}_2) \quad \cdots (2).$

multiplying (1) by $G(\vec{r}, \vec{r}_2)$ & (2) by $G(\vec{r}, \vec{r}_1)$ and integrating over $d^3\vec{r}$ gives (using δ fn property)

$$G(\vec{r}_1, \vec{r}_2) = \int_V G(\vec{r}, \vec{r}_2) \left\{ \nabla^2 G(\vec{r}, \vec{r}_1) + k^2 G(\vec{r}, \vec{r}_1) \right\} d^3\vec{r}$$

$$G(\vec{r}_2, \vec{r}_1) = \int_V G(\vec{r}, \vec{r}_1) \left\{ \nabla^2 G(\vec{r}, \vec{r}_2) + k^2 G(\vec{r}, \vec{r}_2) \right\} d^3\vec{r}$$

Subtracting & using ~~Gradient~~ Green's theorem we gets

$$G(\vec{r}_1,\vec{r}_2) - G(\vec{r}_2,\vec{r}_1) = \oint_S \left\{ G(r,r_2)\vec{\nabla} G(r,r_1) - G(r,r_1)\vec{\nabla} G(r,r_2) \right\} \cdot d\vec{S}$$

Using Dirichlet or Neumann boundary condition; (evaluated on surface) we get the integral (a) vanishes.

i.e, that gives

$$G(\vec{r}_1,\vec{r}_2) - G(\vec{r}_2,\vec{r}_1) = 0$$

i.e $\boxed{G(\vec{r}_1,\vec{r}_2) = G(\vec{r}_2,\vec{r}_1)}$

Green fn is symmetric → proves
for self adj operator.

P: 1

The L.T is:

$$x' = \gamma(x - vt) \ ; \ y' = y ; \ z' = z$$

$$ct' = \gamma\left(ct - \frac{vx}{c}\right)$$

i.e $$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x' \\ y' \\ z' \end{pmatrix}$$

Now, $$e^{\phi} = \gamma(1+\beta)$$

$$\therefore e^{-\phi} = \frac{1}{\gamma(1+\beta)} = \frac{\gamma(1-\beta)}{\gamma^2(1-\beta^2)} = \gamma(1-\beta)$$

$$\therefore \left.\begin{aligned} \gamma &= \frac{e^{\phi} + e^{-\phi}}{2} = \cosh\phi \\ \gamma\beta &= \frac{e^{\phi} - e^{-\phi}}{2} = \sinh\phi \end{aligned}\right\}$$

i.e $$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

i.e L.T is a Rotation in hyperbolic spacetime coordinate with angle of rotation given by:

$$\phi = \cosh^{-1} \gamma \longrightarrow \underline{\text{proof}}$$

Prob: 4

The field tensor changes as:

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = \left(\Lambda^T F \Lambda\right)_{\mu\nu}.$$

i.e

$$F' = \Lambda^T F \Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -B_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

doing the matrix multiplication in Mathematica we get:

$$F' = \begin{pmatrix} 0 & E_x & \gamma(E_y - \beta B_z) & \gamma(E_z + \beta B_y) \\ -E_x & 0 & \gamma(-E_y\beta + B_z) & -\gamma(B_y + \beta E_z) \\ -\gamma(E_y - \beta B_z) & -\gamma(B_z - \beta E_y) & 0 & 0 \\ -\gamma(E_z + \beta B_y) & \gamma(B_y + \beta E_z) & -B_x & 0 \end{pmatrix}$$

But

$$F' = \begin{pmatrix} 0 & E_x' & E_y' & E_z' \\ & 0 & B_z' & -B_y' \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix} \text{ Symmetric}$$

And the comparison gives:

$$\left.\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma(E_y - \beta B_z) \\ E_z' &= \gamma(E_z + \beta B_y) \end{aligned}\right\} \qquad \left.\begin{aligned} B_x' &= B_x \\ B_y' &= \gamma(B_y + \beta E_z) \\ B_z' &= \gamma(B_z - \beta E_y) \end{aligned}\right\}$$

i.e for $\vec{v} \parallel \hat{x}$ this is true. Generalizing the result gives:

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel} \qquad \vec{B}_{\parallel}' = \vec{B}_{\parallel}$$

$$\vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} - \vec{\beta} \times \vec{B}) \qquad \vec{B}_{\perp}' = \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E})$$

$\longrightarrow$ proves

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Prob: 2

1) Under L.T along $\vec{x}$ direction; we get:

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}\right\} \Rightarrow \begin{aligned} dx' &= \gamma(dx - v\,dt) \\ dy' &= dy \\ dz' &= dz \\ dt' &= \gamma\left(dt - \frac{v\,dx}{c^2}\right) \end{aligned}$$

$\therefore$ v'_x ($= u'$ in question)

$$= \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v.dx}{c^2}} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

but $\frac{dx}{dt}$ is the velocity from unprimed frame i.e u_1

i.e $$\boxed{v'_x = u'_1 = \frac{u_1 - v}{1 - \frac{vu_1}{c^2}}} \longrightarrow \text{proved}$$

ii) The 4-velocity u^μ be given by:

$u^\mu = \gamma(c, \vec{v})$ i.e $u_\mu = \gamma(-c, \vec{v})$

$\therefore$ $u^\mu u_\mu = \gamma^2(-c^2 + v^2)$

i.e $\underrightarrow{u} \cdot \underrightarrow{u} = u^{\mu} u_{\mu} = -c^2 \gamma^2 \left(1 - \frac{v^2}{c^2}\right)$

$= -c^2 \underbrace{\gamma^2 (1-\beta^2)}_{1}$

i.e $\boxed{\underrightarrow{u} \cdot \underrightarrow{u} = -c^2} \longrightarrow$ proved

iii) $A^{\hat{\mu}} = \Lambda^{\mu}_{\nu} A^{\nu}$; i.e for L.T with Boost along x direction:

$$A' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix} = \begin{pmatrix} \gamma (A^0 - \beta A^1) \\ \gamma (A^1 - \beta A^0) \\ A^2 \\ A^3 \end{pmatrix}$$

$\Rightarrow A^{0'} = \gamma (A^0 - \beta A^1) = \frac{1}{c} (\gamma c A^0 - \gamma v A^1)$

$= \frac{1}{c} (U_0 A^0 - U_1 A^1) \longrightarrow$ proved

$\left(u_{\nu} = (\gamma c, -\gamma \vec{v})\right)$

iv) for rest frame: $\vec{v} = 0$

~~$\therefore A^{0\prime} = \ldots u^0 A_0 \ldots \gamma A_0$~~

but $A^{0\prime} = \frac{u_0 A^0}{c} = \frac{\gamma c A^0}{c}$.

but for $\vec{v} = 0$; $\gamma = 1$.

i.e $A^{0\prime} = A^0$

i.e $\boxed{u^{0\prime} = u^0 = c}$

$$x^{0\prime} = \int_0^t u^{0\prime} dt = \int_0^\tau u^0 d\tau \quad \left(\because \begin{matrix} dt = d\tau \\ \text{for } \gamma = 1 \end{matrix}\right)$$

$$= \int_0^\tau c \, d\tau$$

i.e $\boxed{x^{0\prime} = c\tau}$ $\longrightarrow$ proved

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Prob: 5.

The 4 momentum is given by: $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

calling $a^\mu = \frac{1}{m_0} \frac{\partial p^\mu}{\partial \tau}$ we get:

$$a^\mu = \left(\frac{1}{m_0 c} \frac{\partial E}{\partial \tau}, \frac{1}{m_0} \frac{\partial \vec{p}}{\partial \tau}\right)$$

And the 4 force be $m_0 a^\mu = \left(\frac{1}{c} \frac{dE}{d\tau}, \frac{d\vec{p}}{d\tau}\right)$

The Lorentz force law in relativistic form is:

$$\frac{dp^\mu}{d\tau} = \frac{q F^{\mu\nu} u_\nu}{c}$$

i.e $m_0 a^\mu = \frac{q F^{\mu\nu} u_\nu}{c}$

i.e $\boxed{a^\mu = \frac{q}{m_0 c} F^{\mu\nu} u_\nu} \rightarrow$ proved

for 0'th component it gives:

$$\frac{1}{m_0 c} \frac{dE}{d\tau} = \frac{q}{m_0 c} \left[F^{00} u_0 + F^{01} u_1 + F^{02} u_2 + F^{03} u_3\right]$$

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i.e $\gamma \frac{dE}{dt} = q \left\{ \gamma \left(E_x x + E_y y + E_z z \right) \right\}$

$\left(\vec{u} = 3 \text{ component vector} \right)$

i.e $$\boxed{\begin{aligned} \frac{dE}{dt} &= q \vec{E} \cdot \vec{u} \\ \text{or} \quad \frac{dE}{dt} &= \vec{E} \cdot \vec{J} \end{aligned}}$$

$\left(\begin{array}{l} \vec{E} = \text{electric field} \\ E = \text{energy} \\ \text{not confuse for notation} \end{array} \right)$

Which is the eq. for conservation of energy. $\longrightarrow$ proves

Problem: 6

Let's take the coordinate transformation

$x^\mu \longrightarrow x^\mu + \delta x^\mu$

i.e $A^\mu \longrightarrow A^\mu(x) + \underbrace{\frac{\partial A^\mu}{\partial x^\nu}\delta x^\nu}_{= \delta A^\mu}$

$\therefore \delta \mathcal{L} = \delta \mathcal{L}(A, \partial_\alpha A)$

$$= \frac{\partial \mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\delta(\partial_\nu A^\mu)$$

$$= \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\right)\delta A^\mu + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\right)\partial_\nu(\delta A^\mu)$$

(using E.L eq for field)

$$= \partial_\nu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\delta A^\mu\right]$$

Now if the coordinate & the field parameter both changes like:

$A^\mu(x) \longrightarrow A'^\mu(x')$

Then the total change of A^μ be given by:

$$\Delta A^\mu = A'^\mu(x') - A^\mu(x)$$

$$= A'^\mu(x') - A^\mu(x') + A^\mu(x') - A^\mu(x)$$

$$= \delta A^\mu(x') - \frac{\partial A^\mu}{\partial x^\nu}\delta x^\nu$$

$$= \delta\left\{A^\mu(x) + (\partial_\nu A^\mu)(\delta x^\nu)\right\} + \frac{\partial A^\mu}{\partial x^\nu}\delta x^\nu$$

$$= \delta A^\mu(x) + (\partial_\nu A^\mu)\,\delta x^\nu + O\left[(\delta x)^2\right]$$

Now the integral of a quantity $f(x)$ changes like (under $A \to A'$; $x \to x'$):

$$\int_a^b f(x)\,dx \longrightarrow \int_{a'}^{b'} f'(x')\,dx'$$

i.e for invarience of the integral; we get:
(later it will be generalized to 4 dimensional action integral with parameters $A^\mu, \partial_\nu A^\mu$.)

$$\int_{a'}^{b'} f'(x')\,dx' - \int_a^b f(x)\,dx = 0$$

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i.e $$\int_{a+\delta a}^{b+\delta b} f'(x)\,dx - \int_a^b f(x)\,dx = 0$$

but $$f'(x) = f(x) + \delta f$$

i.e $$\int_{a+\delta a}^{b+\delta b} \left(f(x) + \delta f\right) dx - \int_a^b f(x)\,dx = 0$$

i.e $$\underbrace{\int_{a+\delta a}^{b+\delta b} f(x)\,dx - \int_a^b f(x)\,dx}_{\text{this equals } = f(b)\,\delta b - f(a)\,\delta a} + \int_{a+\delta a}^{b+\delta b} \delta f(x)\,dx = 0$$

this equals $= \cancel{\delta}\; f(b)\,\delta b - f(a)\,\delta a$

(shown ~~as~~ in assignment (1). just a little bit calculation)

i.e $$\cancel{\int \delta f\, dx} \int_a^b \left[\delta f + \frac{d}{dx}\left(f(x)\,\delta x\right)\right] dx + O(\delta^2) = 0$$

$$\left(\because \int_{a+\delta a}^{b+\delta b} (\delta f)\,dx = \int_a^b \delta f\,dx + O\left(\delta^2\right)\right)$$

i.e the similar action in 4 dimentions gives:

$$\int d^4x \left\{ \delta\mathcal{L} + \partial_\mu\left(\mathcal{L}\,\delta x^\mu\right)\right\} = 0$$

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \delta A^\nu \right) \longrightarrow \text{previously shown.}$$

And $\quad \delta A^\nu = \Delta A^\nu - (\partial_\mu A^\nu)\, \delta x^\mu$

i.e the conservation law gives:

$$\int d^4x \left\{ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \cdot \left(\Delta A^\nu - (\partial_\rho A^\nu)\, \delta x^\rho \right) \right) + \partial_\mu \left(\mathcal{L}\, \delta x^\mu \right) \right\} = 0$$

comparison with the law of $\partial_\mu j^\mu = 0$ we get the conserved current j^μ is given by:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \Delta A^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} (\partial_\rho A^\nu)\, \delta x^\rho + \mathcal{L}\, \delta x^\mu$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \Delta A^\nu - \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\rho)} (\partial_\nu A^\rho) - \mathcal{L} \cdot \delta^\mu_\nu \right)} \delta x^\nu$$

$$= \frac{\delta \mathcal{L}}{\partial(\partial_\mu A^\nu)} \Delta A^\nu - T^\mu_\nu\, \delta x^\nu$$

T^μ_ν is the stress energy tensor.

$$\boxed{T^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\rho)}(\partial_\nu A^\rho) - \mathcal{L}\,\delta^{\mu}_{\nu}} \longrightarrow \underline{\text{Answer}}$$

Clearly the $T^{\mu}_{\ \nu}$ will be conserved iff $\delta A^\mu = 0, \forall \mu$

i.e the system is only gone through translation and hence the field parameter A^μ is unchanged.

Then the conservation gives: ($\partial_\mu j^\mu = 0$) $\boxed{\partial_\mu T^{\mu}_{\ \nu} = 0}$

Ans:

$$T^{\mu\nu} = T^{\mu}_{\ \alpha} g^{\alpha\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\rho)}(\partial^\nu A^\rho) - \mathcal{L}\,\delta^{\mu\nu}$$

if we define $p^{0\mu} = T^{0\mu}$ then we get:

$$p^0 = T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\rho)}(\partial^0 A^\rho) - \mathcal{L}$$

$= \mathcal{H} =$ Hamiltonian density (by defn)

$$p^i = T^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\rho)}(\partial^i A^\rho)$$

$=$ local momentum density.

Ans

And conservation law tells us that ~~a particular combination of~~ energy & momentum is conserved for this $\delta x \neq 0$; $\delta A^\mu = 0$ transform.

Prob: 7

a) This part is very similar to problem 6 which I have already shown for a general case ($i.e$ $A \to A'$; $x \to x'$)

However still I'm showing it for a scalar field $\phi(x)$ when $\phi(x) \to \phi'(x)$

i.e the field parameter changes.

The action is given by: $S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$

i.e for $\phi(x) \longrightarrow \phi'(x)$

$$\mathcal{L}(\phi, \partial_\mu \phi) \longrightarrow \mathcal{L}' = \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi)$$

(from E-L eq)

$$= \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right\}$$

clearly for the action invariance we must get: $S = S'$ i.e $\int \delta\mathcal{L} \, d^4x = 0$

i.e for arbitrary vol element τ this gives:

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right) = 0$$

i.e $\boxed{\partial_\mu j^\mu = 0}$ $\longrightarrow$ Some conservation law.

i.e $\frac{\partial j^0}{\partial t} = \vec{\nabla}\cdot\vec{j}$

i.e $\int\left(\frac{\partial j^0}{\partial t}\right) d^3x = \int(\nabla\cdot\vec{j})\,d^3x = \oint \vec{j}\cdot d\vec{s}$

if at boundary $\vec{j} \longrightarrow 0$ then the quantity

$$\int\left(\frac{\partial j^0}{\partial t}\right) d^3x = \frac{\partial}{\partial t}\left(\int j^0\, d^3x\right) = 0$$

i.e $\boxed{\int j^0 d^3x = Q = \text{A conserved quantity.}}$ proved

for general case $(\phi \to \phi' \,;\, x \to x')$ the conserved current is shown in prob(6) And given by:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\,\delta x^\nu + \mathcal{L}\,\delta x^\mu$$

And the conserved charge be,

$$Q = \int\left(\frac{\partial \mathcal{L}}{\partial\dot{\phi}}\,\Delta\phi - \frac{\partial \mathcal{L}}{\partial\dot{\phi}}(\partial_\nu\phi)\,\delta x^\nu + \mathcal{L}\,\delta x^0\right) d^3x$$

Ans

b. Given: $G = \vec{p} \cdot \delta\vec{r} - H\,\delta t$.

$\delta\vec{r} = t\,\delta\vec{v}$; $\delta t = \frac{1}{c^2}(\delta\vec{v} \cdot \vec{r})$

i.e $G = \left(\vec{p}\,t - \frac{H\vec{r}}{c^2}\right) \cdot \delta\vec{v}$

Now, $\frac{dG}{dt} = 0$ i.e $\vec{p} \cdot \delta\vec{v} - \frac{H}{c^2}\,\delta\vec{v} \cdot \left(\frac{\partial\vec{r}}{\partial t}\right) = 0$

i.e $\left(\vec{p} - \frac{H\vec{v}}{c^2}\right) \cdot \delta\vec{v} = 0$

but $\delta\vec{v}$ is arbitrary; i.e $\vec{p} - \frac{H\vec{v}}{c^2} = 0$

i.e $\vec{p} = m_0\gamma\vec{v} = \frac{H\vec{v}}{c^2}$.

i.e $\boxed{H = m_0\gamma c^2 = mc^2} \rightarrow$ proves

Problem: 9

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\vec{r}\,',\ t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right)}{\text{ᴧ}}\, d^3\vec{r}\,'$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}\left(\vec{r}\,',\ t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right)}{\text{ᴧ}}\, d^3\vec{r}\,'$$

Now for $\left.\begin{aligned} \rho &= q\,\delta^3(\vec{r} - \vec{r}_0(t)) \\ \vec{J} &= q\,\vec{v}(t)\cdot\delta^3(\vec{r} - \vec{r}_0(t)) \end{aligned}\right\}$

$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int \frac{\delta^3(\vec{r}\,' - \vec{r}_0(t))}{\mathcal{R}}\, d^3\vec{r}\,' \qquad (\mathcal{R} = \vec{r} - \vec{r}\,')$$

$$= \frac{q}{4\pi\epsilon_0}\int d^3\vec{r}\,' \int dt' \,\frac{\delta^3(\vec{r}\,' - \vec{r}_0(t))}{\mathcal{R}}\,\delta\left(t' - t + \frac{|\vec{r} - \vec{r}\,'|}{c}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\int dt' \int d^3\vec{r}\,' \left\{\delta\left(t' - t + \frac{|\vec{r} - \vec{r}\,'|}{c}\right)\cdot \frac{\delta^3(\vec{r}\,' - \vec{r}_0(t'))}{\mathcal{R}}\right.$$

$$= \frac{q}{4\pi\epsilon_0}\int \frac{dt'\,\delta\left(t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c}\right)}{|\vec{r} - \vec{r}_0(t')|}$$

$$= \frac{q}{4\pi\epsilon_0}\int dt' \,\frac{\delta(f(t'))}{R(t')}$$

$$f(t') = t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c} \;;\; R(t') = |\vec{r} - \vec{r}_0(t')|$$

$$= \frac{q}{4\pi\epsilon_0} \int \left(\frac{dt'}{df}\right) \cdot df \, \frac{\delta(f(t'))}{R(t')}$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \left[\frac{1}{R(t')} \cdot \frac{1}{\left(\frac{df(t')}{dt'}\right)} \right]_{f(t')=0.}$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R(t')} \cdot \frac{1}{1 + \frac{1}{c}\frac{dR(t')}{dt'}} \right)_{f(t')=0}$$

$$\left(\because f(t') = t' - t + \frac{R(t')}{c} \right)$$

but $f(t') = 0 \Rightarrow t' - t + \frac{R(t')}{c} = 0$

i.e $R(t') = (t - t') \cdot c$

$$\& \quad \frac{dR(t')}{dt'} = -\hat{R}(t') \frac{d\vec{r}_0(t')}{dt'} = -\hat{R}(t') \vec{v}(t')$$

And $t' - t + \frac{R(t')}{c} = 0 \Rightarrow t' = \tau = t - \frac{|\vec{r} - \vec{r}\,'|}{c}$

$$\therefore \phi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R(\tau)} \cdot \frac{1}{1 - \frac{\vec{v}(\tau)\hat{R}(\tau)}{c}} \right)$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{|\vec{r} - \vec{r}_0(\tau)| - [\vec{r} - \vec{r}_0(\tau)] \cdot \vec{\beta}(\tau)}$$

Ans

Similarly we got:

$$A(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{q\vec{v}(\tau)}{|\vec{r}-\vec{r}_0(\tau)| - [\vec{r}-\vec{r}_0(\tau)]\cdot\vec{\beta}(\tau)}$$

proved

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