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(D.N.A.P)

Problem: B

Let the 4 States be given by: (possible for σ_i)

$$|\alpha\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ |\beta\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix};\ |\gamma\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ |\delta\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Now given $\hat{H} = -J\sum_{i=1}^{L} \delta_{\sigma_i,\sigma_{i+1}}$

$\therefore$ The partition fn be given by:

$$Z(\tilde{\beta}) = \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_L} \exp\left[+\tilde{\beta}J\cdot\sum_{i=1}^{L}\delta_{\sigma_i,\sigma_{i+1}}\right]$$

(I've used $\tilde{\beta}$ to distinguish from β of State vector given. i.e $\tilde{\beta} = \frac{1}{KT}$;

The condition be: $\sigma_{L+1} = \sigma_1$)

i.e $$Z(\tilde{\beta}) = \sum_{\sigma_1}\cdots\sum_{\sigma_L}\left\{\prod_{i=1}^{L}\left(\exp\left[\tilde{\beta}J\,\delta_{\sigma_i,\sigma_{i+1}}\right]\right)\right\}$$

Let's impose the transfer matrix here (P) So that we get:

$$\langle\sigma_i|P|\sigma_{i+1}\rangle = \exp\left(\tilde{\beta}J\,\delta_{\sigma_i\sigma_{i+1}}\right) \quad \left(\begin{matrix}P = 4\times4\\ \text{matrix}\end{matrix}\right)$$

for $\sigma_i = |\alpha\rangle$; $\sigma_{i+1} = |\alpha\rangle$; $P_{11} = \exp\left(\tilde{\beta}J\,\delta_{\alpha,\alpha}\right) = e^{\tilde{\beta}J}$

$\sigma_i = |\alpha\rangle$; $\sigma_{i+1} = |\beta\rangle$; $P_{12} = \exp\left(\tilde{\beta}J\,\delta_{\alpha\beta}\right) = 1$

$\sigma_i = |\alpha\rangle$; $\sigma_{i+1} = |\gamma\rangle$; $P_{13} = \exp\left(\tilde{\beta}J\,\delta_{\alpha\gamma}\right) = 1$

$\sigma_i = |\alpha\rangle$; $\sigma_{i+1} = |\delta\rangle$; $P_{14} = \exp\left(\tilde{\beta}J\,\delta_{\alpha\delta}\right) = 1$

In the similar way we construct the whole matrix P:

$$P = \begin{pmatrix} e^{\tilde{\beta}J} & 1 & 1 & 1 \\ 1 & e^{\tilde{\beta}J} & 1 & 1 \\ 1 & 1 & e^{\tilde{\beta}J} & 1 \\ 1 & 1 & 1 & e^{\tilde{\beta}J} \end{pmatrix}$$

$$\therefore Z = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \langle \sigma_1 | P | \sigma_2 \rangle \langle \sigma_2 | P | \sigma_3 \rangle \langle \sigma_3 | \cdots \cdots | P | \sigma_L \rangle \langle \sigma_L | P | \sigma_1 \rangle$$

Contructing the sum using $\sum_{\sigma_i} |\sigma_i\rangle\langle\sigma_i| = \mathbb{I}$

$$Z = \sum_{\sigma_1} \langle \sigma_1 | \underbrace{P P \cdots P}_{L} | \sigma_1 \rangle = \sum_{\sigma_1} \langle \sigma_1 | P^L | \sigma_1 \rangle$$

$$= \mathrm{Tr}(P^L) = \sum_{j=1}^{4} x_i^L$$

(x_i = eig-vals of P).

Now $\det(P - X\mathbb{I}) = 0$ gives: (with $x = e^{\tilde{\beta}J}$)

$$\begin{vmatrix} x-X & 1 & 1 & 1 \\ 1 & x-X & 1 & 1 \\ 1 & 1 & x-X & 1 \\ 1 & 1 & 1 & x-X \end{vmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2 + R_3 + R_4} \begin{vmatrix} 3+x-X & 3+x-X & 3+x-X & 3+x-X \\ 1 & x-X & 1 & 1 \\ 1 & 1 & x-X & 1 \\ 1 & 1 & 1 & x-X \end{vmatrix}$$

$$(3+x-X)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x-X & 1 & 1 \\ 1 & 1 & x-X & 1 \\ 1 & 1 & 1 & x-X \end{vmatrix} \xrightarrow[\substack{R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - R_1}]{R_2 \leftarrow R_2 - R_1} (3+x-X)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-X-1 & 0 & 0 \\ 0 & 0 & x-X-1 & 0 \\ 0 & 0 & 0 & x-X-1 \end{vmatrix}$$

$$= (3+\alpha - x)(\alpha - 1 - x)^3.$$

$\therefore \det(P\mathcal{g} - xI) = 0$ gives:

$$\left.\begin{aligned} x &= \alpha - 1 = e^{\tilde{\beta}J} - 1 \longrightarrow 3 \text{ fold} \\ x &= \alpha + 3 = e^{\tilde{\beta}J} + 3 \longrightarrow 1 \text{ fold.} \end{aligned}\right\}$$

So: $Z = (\alpha+3)^L + 3(\alpha-1)^L$

$$= (\alpha+3)^L \left(1 + 3\left(\frac{\alpha-1}{\alpha+3}\right)^L\right)$$

$$= \left(e^{\tilde{\beta}J}+3\right)^L \left(1 + 3\left(\frac{e^{\tilde{\beta}J}-1}{e^{\tilde{\beta}J}+3}\right)^L\right)$$

Taking $J > 0$ when $\tilde{\beta}$ is obviously > 0 we get

$|e^{\tilde{\beta}J} - 1| < |e^{\tilde{\beta}J} + 3|$ i.e $\dfrac{(e^{\tilde{\beta}J} - 1)}{(e^{\tilde{\beta}J} + 3)} < 1$

i.e $\lim\limits_{L \to \infty} \left(\dfrac{e^{\tilde{\beta}J} - 1}{e^{\tilde{\beta}J} + 3}\right)^L = 0.$

So at thermodynamic limit; the partition fn gives: $Z = \left(e^{\tilde{\beta}J} + 3\right)^L.$

$\therefore$ energy of the system in canonical ensemble:

$$\langle E \rangle = -\frac{\partial}{\partial \tilde{\beta}}(\ln Z) = -\frac{\partial}{\partial \tilde{\beta}}\left\{L \ln\left(e^{\tilde{\beta}J} + 3\right)\right\}$$

$$\Rightarrow \langle E \rangle = -\frac{LJ e^{\tilde{\beta} J}}{3 + e^{\tilde{\beta} J}}.$$

i.e the specific heat be given by:

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial \langle E \rangle}{\partial \tilde{\beta}} \cdot \frac{\partial \tilde{\beta}}{\partial T}$$

$$= \frac{-1}{KT^2} \times (-LJ) \cdot J \frac{\left(3 + e^{\tilde{\beta} J}\right) e^{\tilde{\beta} J} - e^{2\tilde{\beta} J}}{\left(3 + e^{\tilde{\beta} J}\right)^2}.$$

$$= \frac{3LJ^2}{KT^2} \cdot \frac{e^{J/KT}}{\left(3 + e^{J/KT}\right)^2} \Bigg\}$$ Answer

Problem: A

Given: $F = -h\phi + a\phi^2 + b\phi^4 + c\phi^6$

$a = \frac{T - T^*}{T^*}$; $T^* > 0$; $c > 0$ (fixed)

0.a: Clearly here h is the external field which tries to order the system (ordering field).

So at $h = 0$; we get:

$$F = a\phi^2 + b\phi^4 + c\phi^6 \quad (b > 0)$$

∴ condition for minima gives:

$$\frac{\partial F}{\partial \phi} = 2a\phi + 4b\phi^3 + 6c\phi^5 = 0.$$

the trivial soln is $\phi = 0$

for non trivial soln s: $2a + 4b\phi^2 + 6c\phi^4 = 0$

i.e $\phi^4 + \frac{2b}{3c}\phi^2 + \frac{a}{3c} = 0$.

i.e $\phi^2 = \frac{1}{2}\left(-\frac{2b}{3c} \pm \sqrt{\left(\frac{2b}{3c}\right)^2 - \frac{4a}{3c}}\right)$

Now as $b > 0$ so at $T \lesssim T^*$; the solution of ϕ would only exist. i.e at $T \lesssim T^*$ (or $T \leq T^*$) $a < 0$. So for the '+' ve ~~sing~~ sign in front of the root; it will give:

$$\phi^2 = \frac{1}{2}\left(-\eta + \sqrt{\eta^2 + \frac{4|a|}{3c}}\right) \quad (\eta > 0)$$

i.e $\phi^2 > 0$ ∴ ϕ = real.

but for $T > T^*$ we get

$$\phi^2 = \frac{1}{2}\left[-m \pm \sqrt{m - \frac{4|a|}{3c}}\right] < 0 \quad (m > 0)$$

i.e for any sign '$\pm$' we dont get any real nontrivial solution for ϕ.

Which means $T = T^*$ separates the two regions of getting / not getting the non trivial solution for $\frac{\partial F}{\partial \phi} = 0$.

i.e for $b > 0$: $T = T^*$ is transition temperature.

<u>proved</u>

~~Now~~ $\cancel{\frac{\partial^2 F}{\partial \phi^2} = 2a + 4b\phi^2 + 6c\phi^4}$

$\therefore$ $\cancel{\left.\frac{\partial^2 F}{\partial \phi^2}\right|_{T=T^*} = 2\times 0 + 4b\phi^2 + 16c\phi^4}$ $\left(\cancel{\because a = T - T^*}\right)$

~~at~~ =

<u>Q. a.</u> Now; $\phi^2 = \frac{1}{2}\left[-\frac{2b}{3c} + \sqrt{\left(\frac{2b}{3c}\right)^2 - \frac{4a}{3c}}\right]$

$$\Rightarrow \phi^2 = \frac{1}{2}\left[-\frac{2b}{3c} + \frac{2b}{3c}\left(1 - \frac{3ac}{b^2}\right)^{1/2}\right]$$

at $T \lesssim T^*$; $|a| \simeq 0$; so by expansion:

$$\therefore \phi^2 = \frac{1}{2}\left[-\frac{2b}{3c} + \frac{2b}{3c}\left(1 - \frac{3ac}{2b^2}\right)\right]$$

$$= -\frac{1}{4}\cdot\frac{2b}{3c}\times\frac{3ac}{b^2} = \frac{|a|}{2b} \quad (\because |a| = -a)$$

$\therefore \quad \bar{\phi} = \sqrt{\dfrac{|a|}{2b}} = \dfrac{1}{\sqrt{2b}} \cdot \left(\dfrac{T^* - T}{T^*}\right)^{1/2}$

<u>Answer</u>

As usual $\bar{\phi} \sim (T^* - T)^{1/2}$ for $T \lesssim T^*$

Now, $\left.\dfrac{\partial^2 F}{\partial \phi^2}\right|_{\substack{T = T_c \\ \phi = \bar{\phi}}} = 2a + 4b\phi^2 + 6c\phi^4$

$$= \left(2a + 4b\bar{\phi}^2 + 6c\bar{\phi}^4\right)_{T = T_c} = 0$$

So at $T = T^*$ we get simultaneous solution of $\dfrac{\partial F}{\partial \phi} = 0$ & $\dfrac{\partial^2 F}{\partial \phi^2} = 0$ at $\phi = \bar{\phi} = 0$.

This gives another ~~ps~~ proof that $T = T^*$ is indeed the true critical temp for $b > 0$.

<u>Q.b</u> In presence of a small ordering field:

$$F = -h\phi + a\phi^2 + b\phi^4 + c\phi^6.$$

$$\therefore \frac{\partial F}{\partial \phi} = -h + 2a\phi + 4b\phi^3 + 6c\phi^5$$

~~as $T \lesssim T^*$; So keeping the leading order of (T^*-T) we get: (i.e $T^* - T \approx 0$)~~

~~$$\frac{\partial F}{\partial \phi} = -h + \left(\frac{T^*-T}{T^*}\right)^{3/2}\left(\frac{2}{\sqrt{2b}}\right) -$$~~

Q-b In presence of the ordering field h:

$$\frac{\partial F}{\partial \phi} = -h + 2a\phi + 4b\phi^3 + 6c\phi^5$$

at $\phi = \bar{\phi}$; $\frac{\partial F}{\partial \phi} = 0$. So we can write:

$$2a\bar{\phi} + 4b\bar{\phi}^3 + 6c\bar{\phi}^5 - h = 0.$$

differentiating w.r.t h:

$$2a\frac{\partial \bar{\phi}}{\partial h} + 4b \times 3\bar{\phi}^2 \frac{\partial \bar{\phi}}{\partial h} + 30c\bar{\phi}^4 \frac{\partial \bar{\phi}}{\partial h} - 1 = 0.$$

i.e $$\frac{\partial \bar{\phi}}{\partial h} = \frac{1}{2a + 2 \times 6b\bar{\phi}^2 + 30c\bar{\phi}^4}$$

in vicinity of $T = T^c = T^*$ and $h \to 0$ we can replace $\bar{\phi}$ by the previously calculated value

$$\lim_{h\to 0} \bar{\phi} = \frac{1}{\sqrt{2b}}\left(\frac{T^*-T}{T^*}\right)^{1/2}.$$

$$\lim_{h\to 0} \frac{\partial \bar{\phi}}{\partial h} = \frac{1}{2\left(\frac{T-T^*}{T^*}\right) + \frac{2 \times 6b}{2b}\left(\frac{T^*-T}{T^*}\right) + O(T^*-T)^2}$$

Taking the leading order in denominator.
i.e dropping the term $\sim O(T^*-T)^2$ we get:

$$\lim_{h\to 0} \frac{\partial\bar{\phi}}{\partial h} \simeq \frac{1}{\frac{(6-2)(T^*-T)}{T^*}} = \frac{T^*}{4(T^*-T)}.$$

Clearly $\underbrace{\frac{\partial\bar{\phi}}{\partial h} \sim (T^*-T)^{-1}}$

The standard result that susceptibility diverges.

Answer

1.a

From the previous part calculation we get: for $\phi \neq 0$ solution of $\frac{\partial F}{\partial \phi} = 0$; the eq be given by:

$$\phi^2 = \frac{1}{2}\left[-\frac{2b}{3c} \pm \sqrt{\left(\frac{2b}{3c}\right)^2 - \frac{4a}{3c}}\right]$$

as here $b =$ '$-$ve' so $-b = |b|$ & we get:

$$\phi^2 = \frac{1}{2}\left[\frac{2|b|}{3c} \pm \frac{2|b|}{3c}\sqrt{1 - \frac{4a}{3c}\cdot\frac{9c^2}{4b^2}}\right]$$

$$= \frac{1}{2}\cdot\frac{2|b|}{3c}\left[1 \pm \sqrt{1 - \frac{3ac}{b^2}}\right] \quad \cdots (1)$$

So there is no non trivial real soln for ϕ if $1 - \frac{3ac}{b^2} < 0$.

i.e $\frac{3ac}{b^2} > 1$ i.e $\left(\frac{T-T^*}{T^*}\right) > \frac{b^2}{3c}$

i.e $\frac{T}{T^*} - 1 > \frac{b^2}{3c}$ i.e $T > T^*\left(1+\frac{b^2}{3c}\right)$

So for $T < T^*\left(1+\frac{b^2}{3c}\right)$ we get non trivial real solution & hence we call ~~it~~ it as T_c

$$\therefore \quad T_c = T^*\left(1+\frac{b^2}{3c}\right).$$

But there is one interesting phenomenon that I found out.

even for $T < T_c$ i.e $\frac{3ac}{b^2} < 1$; if a is ~~sufficiently large~~ less than 0 $(a<0)$; then for '$-$ve' sign in eq (1)

we get :

$$\phi^2 = \frac{|b|}{3c}\left(1 - \sqrt{1+\frac{3c|a|}{b^2}}\right)$$

i.e ϕ has no soln from '$-$ve' sign.

So we get 4 solutions (real) at $a \in \left(0, \frac{b^2}{3c}\right)$

2 " " $a < 0$

no solution for $a > \frac{b^2}{3c}$.

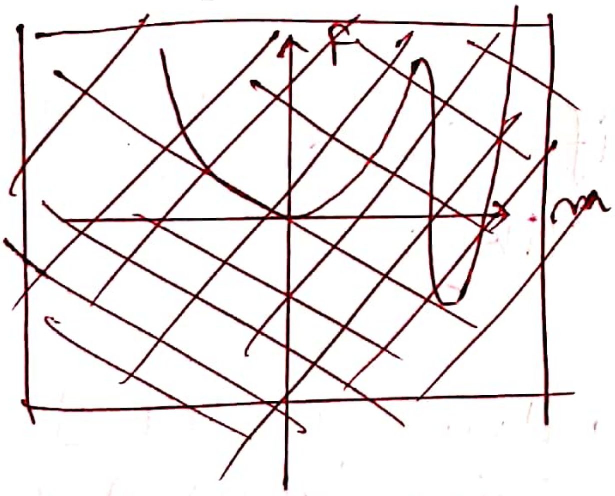
This situation is quite different from the normal case. Here I have solved out the problem with graphical method than algebric way.

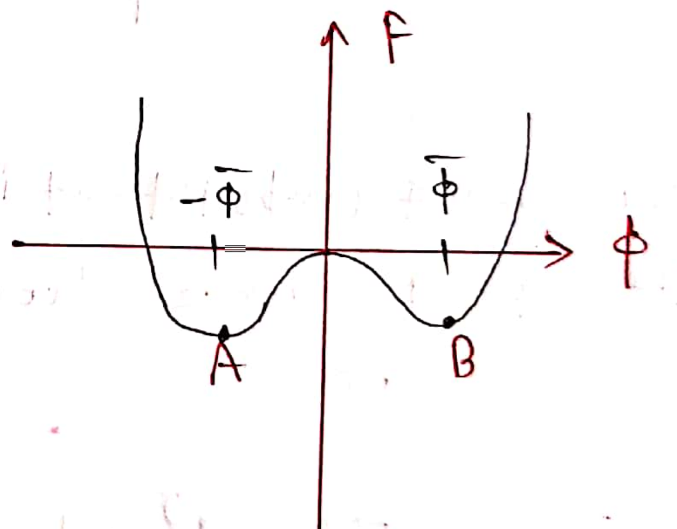
(iii) we know the true stable equilibrium is given by a minima of F.

For a magnetic system; $\chi = \dfrac{\partial M}{\partial h} > 0$ is a physical condition. So in the F, ~~m~~ ϕ plot; we must get for a possible region on F-~~m~~ ϕ curve so that

$$\frac{\partial^2 F}{\partial \phi^2} \; \cancel{\frac{\partial^2 \phi}{\partial m^2}} = \chi^{-1} = \text{curvature} > 0.$$

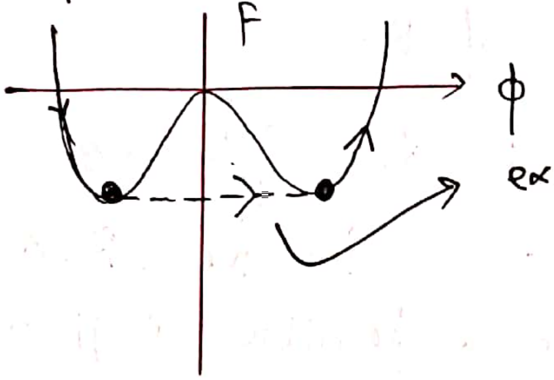
Now if $a < 0$ in the expression of $F(\phi)$ then as $b < 0$; the curve looks like:





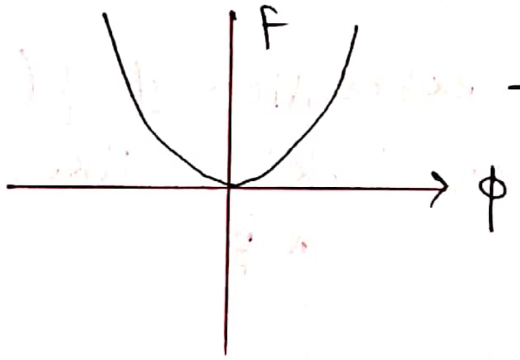
There is two stable minima symmetrically. If a system starts from extreme left then after reaching point A; it directly jumps to point B to avoid the middle region of negative curvature.

This type of jump is truely what we call as a phase transition.



excluding the middle region.

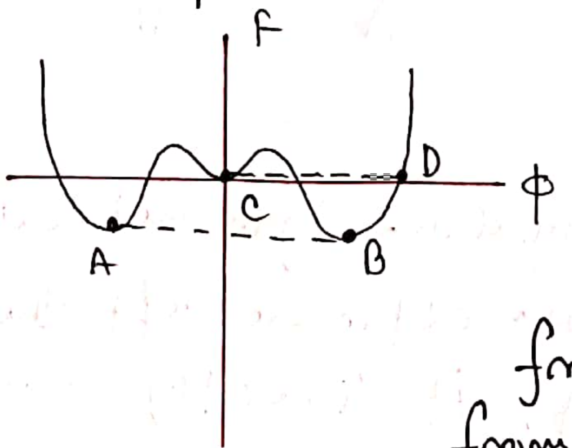
Now if $a > \frac{b^2}{3c}$; then we don't get any solution than $\phi = 0$.



→ Here we get no phase transition.

The most important is if $0 < a < \frac{b^2}{3c}$

The $F - \phi$ curve looks like:



→ Here we get 3 minima points. And if a system starts from left it first jumps from the left minima to a point on the middle minima region where the curvature is again '+ve' & then further jumps to the right wall.

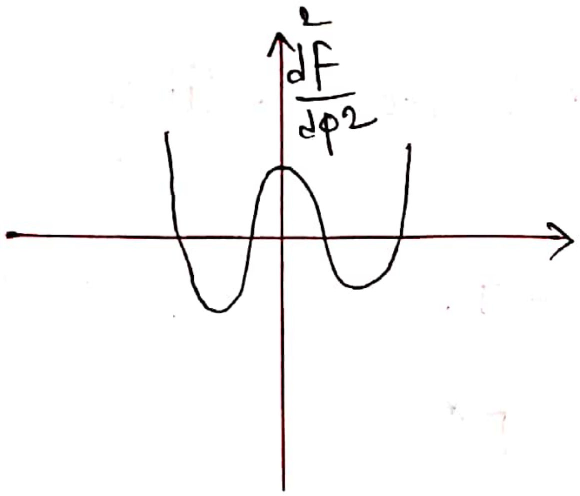
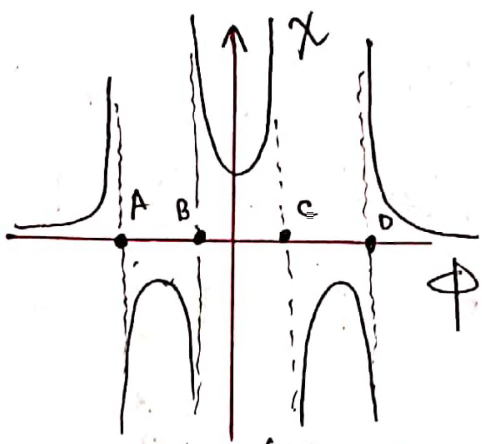


fig (1)

i.e $\longrightarrow$



$\chi = \left(\frac{d^2F}{d\phi^2}\right)^{-1}$

fig — (2)

So in fig (2); the system will reach at point A ~~A~~ then jump to B by one transition & then after continuous phase it will ~~..~~ again jump from point C to point D.

$$A \xrightarrow{\text{jump}} B \xrightarrow{\text{Continuous}} C \xrightarrow{\text{jump}} D.$$

Among all of them I call T_c as the last case to the 2nd case boundary; i.e the f^n $F(\phi)$ gets from $\phi = 0$ sol^n to 5 real solutions. (3 minima & ~~3 or~~ 2 maxima)

So finally I get $T_c = T^*\left(1 + \frac{b^2}{3c}\right)$

Ans

1.b Let the system starts from a little less temp than T_c.

i.e $T = T_c - \epsilon$; $\epsilon = T_c - T$.

$$\therefore a = \frac{T - T^*}{T^*} = \frac{(T_c - \epsilon) - T^*}{T^*}$$

$$= \frac{T^*\left(1 + \frac{b^2}{3c}\right) - \epsilon - T^*}{T^*} = \frac{\frac{b^2 T^*}{3c} - \epsilon}{T^*}$$

$$= \frac{b^2}{3c} - \frac{\epsilon}{T^*}$$

$$\therefore \frac{3ac}{b^2} = 1 - \frac{3c\epsilon}{T^* b^2}$$

So;
$$\phi^2 = \frac{|b|}{3c}\left(1 \pm \sqrt{1 - \frac{3ac}{b^2}}\right)$$

$$= \frac{|b|}{3c}\left(1 \pm \sqrt{\frac{3\epsilon c}{b^2 T^*}}\right)$$

$$\therefore \phi = \bar{\phi} = \pm \sqrt{\frac{|b|}{3c}}\left(1 \pm \sqrt{\frac{3\epsilon c}{b^2 T^*}}\right)^{1/2}.$$

$$\simeq \pm \sqrt{\frac{|b|}{3c}}\left(1 \pm \frac{1}{2} \cdot \sqrt{\frac{3\epsilon c}{b^2 T^*}}\right) \quad (\because \epsilon \simeq 0)$$

The minimas are located lower than maxima & they are symmetric & far away

So we obviously have to take the inner sign '+ve'

$$\therefore\ \bar{\phi} = \pm\sqrt{\frac{|b|}{3c}}\left(1+\frac{1}{2}\sqrt{\frac{3c(T_c-T)}{T^* b^2}}\right)$$

Ans

1.c

In previous calculation we got:

$$\frac{\partial\bar{\phi}}{\partial h} = \frac{1}{2a + 2\times 6b\bar{\phi}^2 + 30c\bar{\phi}^4}.$$

Now, for $h \to 0$: $\bar{\phi} = \pm\sqrt{\frac{|b|}{3c}}\left(1+\frac{1}{2}\sqrt{\frac{3c(T_c-T)}{T^* b^2}}\right)$

$$\left.\begin{aligned} \therefore\ \bar{\phi}^2 &\simeq \frac{|b|}{3c}\cdot\left(1+\sqrt{\frac{3c(T_c-T)}{T^* b^2}}\right) \\ \bar{\phi}^4 &\simeq \frac{b^2}{9c^2}\left(1+2\sqrt{\frac{3c(T_c-T)}{T^* b^2}}\right) \end{aligned}\right\} \text{upto leading order of } (T_c-T)$$

So: $\underset{h\to 0}{Lt}\left(\frac{\partial\bar{\phi}}{\partial h}\right)$

$$= \frac{1}{2\frac{(T_c-T)}{T^*} - \frac{2\times 6b^2}{3c}\left(1+\sqrt{(\ )}\right) + \frac{30b^2}{9c}\left(1+2\sqrt{(\ )}\right)}$$

$$= \frac{1}{\frac{b^2}{c}\left(\frac{10}{3}-2^2\right) + \frac{b^2}{c}\left(\frac{20}{3}-2^2\right)\cdot\sqrt{(\ )}}$$

upto leading order of (T_c-T)

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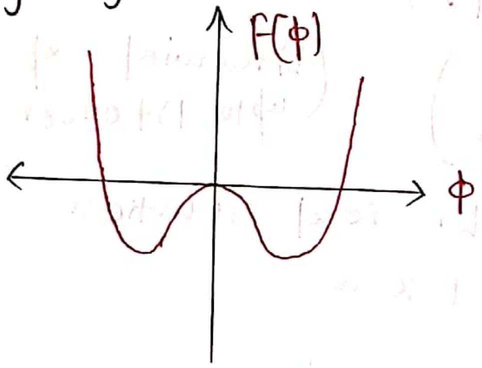
$$= \frac{c}{b^2} \cdot \left(\frac{1}{\left(\frac{10}{3} - 24\right) + \left(\frac{20}{3} - 24\right)\sqrt{\frac{3c(T_c - T)}{b^2 T^*}}} \right)$$

<u>Ans</u>

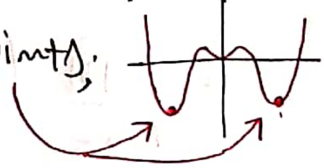
Remark

(This fact I ~~noticed~~ included later after computing the other questions and at time of revision. Hence I've written as remark.)

For $T < T^*$ i.e $a < 0$; the plot of F ~~changes~~ ~~for~~ ~~from~~ is given by: $(b<0)$



So as the both plots of $F(\phi)$ for ~~⇒~~ $T > T_c$ i.e $T > T^*\left(1 + \frac{b^2}{3c}\right)$ & $T^* > T > T_c$

Contains another minima at $\phi = 0$;
So without any perturbation (as that is not global minima) the system will sit there until temperature is lowered upto $T = T^*$.
(This is experimentally not possible; as there is always some external perturbation that will tend to move the system at lower (global) minima of F; which in case of $T^* < T < T_c$ occurs at non trivial points. )
i.e the transition occurs.

But without any perturbation; this fact do not happen upto $T = T^*$ & the transition occurs there..

(iii) At $T \lesssim T^*$ for $b<0$:

$$\bar{\phi}^2 = \frac{|b|}{3c}\left(1 \pm \sqrt{1+\frac{3c(T^*-T)}{b^2 T^*}}\right) \quad \left(\because a = \frac{T-T^*}{T^*}\right)$$

let $T^* = T + \epsilon$ i.e $\epsilon = T^* - T$ & $|\epsilon| \ll 1$.

$$\therefore \bar{\phi}^2 = \frac{|b|}{3c}\left(1 \pm \sqrt{1+\frac{3c\epsilon}{b^2T^*}}\right)$$

$$\simeq \frac{|b|}{3c}\left(1 \pm 1 + \frac{3}{2}\frac{c\epsilon}{b^2T^*}\right) \quad \left(\text{binomial exp upto 1st order of } \epsilon\right)$$

as $\epsilon \gtrsim 0$ so only possible real solution will be iff '+' sign is taken.

i.e $$\bar{\phi}^2 = \frac{|b|}{3c}\left(2 + \frac{3}{2}\frac{c\epsilon}{b^2T^*}\right)$$

$$= \frac{2|b|}{3c}\left(1 + \frac{3}{4}\frac{c\epsilon}{b^2T^*}\right)$$

$$\therefore \bar{\phi} = \pm\sqrt{\frac{2|b|}{3c}}\left(1+\frac{3}{4}\frac{c\epsilon}{b^2T^*}\right)^{1/2}.$$

$$\simeq \pm\sqrt{\frac{2|b|}{3c}}\left(1+\frac{3}{8}\frac{c\epsilon}{b^2T^*}\right)$$

i.e $$\bar{\phi}_{T\lesssim T^*} = \pm\sqrt{\frac{2|b|}{3c}}\left(1+\frac{3}{8}\frac{c(T^*-T)}{b^2T^*}\right)$$

<u>Answer</u>

Clearly $\bar{\phi}^2 \simeq \frac{2b}{3c}\left(1+\frac{3}{4}\frac{\epsilon c}{b^2 T^*}\right)$

$\therefore \quad \phi^4 \simeq \left(\frac{2b}{3c}\right)^2\left(1+\frac{3}{2}\frac{\epsilon c}{b^2T^*}\right)$

$$\therefore \lim_{\substack{h\to 0\\ T\lesssim T^*}} \frac{\partial\bar{\phi}}{\partial h} = \frac{1}{2a+12b\bar{\phi}^2+30c\bar{\phi}^4}$$

$$= \frac{1}{\frac{2\epsilon}{T^*}+\frac{12b\times 2b}{3c}\left(1+\frac{3}{4}\frac{\epsilon c}{b^2T^*}\right)+\frac{30c}{9c^2}\cdot 4b^2\left(1+\frac{3}{2}\frac{\epsilon c}{b^2T^*}\right)}$$

$$= \frac{1}{\frac{2\epsilon}{T^*}+\frac{8b^2}{c}\left(1+\frac{3}{4}\frac{\epsilon c}{b^2T^*}\right)+\frac{40b^2}{c}\left(1+\frac{3}{2}\frac{\epsilon c}{b^2T^*}\right)}$$

$$= \frac{1}{\frac{2\epsilon}{T^*}+\frac{8b^2}{c}+\frac{6\epsilon}{T^*}+\frac{40b^2}{3c}+\frac{20\epsilon}{T^*}}$$

$$= \frac{1}{\frac{64b^2}{c}+\frac{28\epsilon}{T^*}} = \frac{1}{\frac{64b^2}{c}+\frac{28(T^*-T)}{T^*}}$$

i.e $$\lim_{\substack{h\to 0\\ T\lesssim T^*}}\left(\frac{\partial\bar{\phi}}{\partial h}\right) \sim \frac{1}{\alpha+\beta(T^*-T)}$$

Ans

2.a Here $b = 0$

So; Here $\frac{\partial F}{\partial \phi} = 0$ gives for $\phi \neq 0$:

$$2a + 6c\phi^4 = 0 \quad \text{i.e} \quad \phi^2 = \pm\sqrt{-\frac{a}{3c}}$$

we get soln for '+' sign

$$\bar{\phi} = \left(-\frac{a}{3c}\right)^{1/4}.$$

iff $-a > 0$ i.e $-\frac{T - T^*}{T^*} > 0$.

i.e $T < T^*$.

So here $T_c = T^*$.

2.b

$$\bar{\phi} = \left(\frac{1}{3c} \cdot \frac{T_c - T}{T^*}\right)^{1/4}.$$

$\therefore$ at the vicinity of $T = T_c = T^*$

order parameter changes as:

$$\tilde{\phi} \sim (T_c - T)^{1/4}$$

[This is not matching with $\phi \sim (T_c - T)^{1/2}$ i.e the exponent is different as there is no ϕ^4 term. Which is ~~[illegible]~~ there for normal magnetic system or van-der-walls gas]

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2C

In presence of field h:

$$\underset{\substack{h \to 0 \\ T \simeq T_c}}{Lt} \left(\frac{\partial \Phi}{\partial h}\right) = \frac{1}{2a + 30 c \Phi^4}.$$

$$= \frac{1}{-2\left(\frac{T_c - T}{T_c}\right) + \cancel{30}^{10} \cdot \cancel{\alpha} \cdot \left(\frac{T_c - T}{T_c}\right) \cdot \frac{1}{\cancel{3}\cancel{\alpha}}}$$

$$= \frac{T_c}{8 (T_c - T)}$$

Ans

Problem: C

1) Given true Hamiltonian of the System:

$$H = -J \sum_{\langle ij \rangle} Q_i Q_j \quad ; \quad Q_i = \left(S_i^{z^2} - \frac{2}{3}\right)$$

When S_i^z is a spin z matrix for s = 1 particle.

Now let's take the trial Hamiltonian be

$$H_0 = -q \sum_i Q_i \quad \text{(as given)}$$

i.e $H_0 = \sum_i H_i^0 \qquad (H_i^0 = -q Q_i)$

clearly in this model of trial Hamiltonian; the particles / sites are self non interacting.

Now, $S_i^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (\hbar = 1)$

$\Rightarrow S_i^{z^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow S_i^{z^2} - \frac{2}{3} = Q_i = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore H_i = -q Q_i = -\frac{q}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

$\therefore Z_i^0 = \mathrm{Tr}\left(e^{-\beta H_i^0}\right) = \mathrm{Tr}\begin{pmatrix} e^{\beta q/3} & 0 & 0 \\ 0 & e^{-\frac{2\beta q}{3}} & 0 \\ 0 & 0 & e^{\beta q/3} \end{pmatrix}$

$$= \left(2e^{\frac{\beta q}{3}} + e^{-\frac{2\beta q}{3}}\right)$$

∴ Partition fn for the whole System:

$$Z_0 = (Z_i)^N = \left(2e^{\frac{q\beta}{3}} + e^{-\frac{2\beta q}{3}}\right)^N. \quad \left(\begin{matrix}N = \text{total no.} \\ \text{of sites}\end{matrix}\right)$$

$$\therefore F_0 = -KT \ln Z_0 = -NKT \ln\left(2e^{\frac{\beta q}{3}} + e^{-\frac{2\beta q}{3}}\right)$$

Now we deal with the 2nd quantity of variation:

$$H - H_0 = -J \sum_{\langle ij \rangle} Q_i Q_j + q \sum_i Q_i$$

$$\therefore \langle H - H_0 \rangle_0 = q \sum_i \langle Q_i \rangle_0 - J \sum_{\langle ij \rangle} \langle Q_i \cdot Q_j \rangle_0$$

As in the Hamiltonian H_0; all sites are identical & the sites are mutually indp; so we can ~~same if~~ safely assume,

$$\langle H - H_0 \rangle_0 = qN \langle Q_i \rangle_0 - JN_{\text{link}} \langle Q_i \rangle_0 \langle Q_j \rangle_0.$$

but $\langle Q_i \rangle_0 = \langle Q_j \rangle_0$. from symmetry of H

$$\therefore \langle H - H_0 \rangle_0 = qN \langle Q_i \rangle_0 - JN_{\text{link}} \langle Q_i \rangle_0^2.$$

$$\langle Q_i \rangle_0 = \frac{\mathrm{Tr}\left(e^{-\beta H_i} \cdot Q_i\right)}{\mathrm{Tr}\left(e^{-\beta H_i}\right)}.$$

here $e^{-\beta H_i} \cdot Q_i = \begin{pmatrix} e^{\frac{q\beta}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\beta q}{3}} & 0 \\ 0 & 0 & e^{\frac{\beta q}{3}} \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & -2/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$

i.e $e^{-\beta H_i} \cdot Q_i = \begin{pmatrix} \frac{1}{3} e^{\frac{\beta q}{3}} & 0 & 0 \\ 0 & -\frac{2}{3} e^{-\frac{2q\beta}{3}} & 0 \\ 0 & 0 & \frac{1}{3} e^{\frac{\beta q}{3}} \end{pmatrix}$

$\therefore Tr\left(e^{-\beta H_i} \cdot Q_i\right) = \frac{2}{3}\left(e^{\beta q} \cancel{*} - e^{-\frac{2\beta q}{3}}\right)$

$$\therefore \langle Q_i \rangle_0 = \frac{2}{3} \frac{e^{\beta q} - e^{-\frac{2\beta q}{3}}}{\left(2e^{\beta q} + e^{-\frac{2\beta q}{3}}\right)}$$

We got $F_0 = -NkT \ln\left(2e^{\frac{\beta q}{3}} + e^{-\frac{2\beta q}{3}}\right)$

$$\therefore \frac{\partial F_0}{\partial q} = -NkT \cdot \frac{1}{\left(2e^{\frac{\beta q}{3}} + e^{-\frac{2\beta q}{3}}\right)} \cdot \frac{2\beta}{3}\left(e^{\frac{\beta q}{3}} - e^{-\frac{2\beta q}{3}}\right)$$

$$= - \cancel{N} N \langle Q_i \rangle_0$$

Now we ~~minimi~~ take the tightest bound of the quantity $\langle F_0 + (H - H_0)\rangle_0$ w.r.t the variational parameter q.

$$\therefore \frac{\partial A}{\partial q} = \frac{\partial F_0}{\partial q} + \frac{\partial}{\partial q}\langle H - H_0 \rangle_0.$$

$$= -N\langle Q_i \rangle_0 + \frac{\partial}{\partial q}\left\{ Nq\langle Q_i \rangle_0 - N_{link} J \langle Q_i \rangle_0^2 \right\}$$

$$= -N\cancel{\langle Q_i \rangle_0} + N\cancel{\langle Q_i \rangle_0} + \frac{\partial \langle Q_i \rangle_0}{\partial q}\left\{ Nq - 2JN_{link}\langle Q_i \rangle_0 \right\}$$

$$= \frac{\partial \langle Q_i \rangle_0}{\partial q} \left\{ Nq - 2JN_{link} \langle Q_i \rangle_0 \right\}$$

but
$$\frac{\partial \langle Q_i \rangle_0}{\partial q} = \frac{2}{3} \cdot \frac{\left[\left(2e^{\frac{\beta q}{3}} + e^{-\frac{2\beta q}{3}}\right) \cdot \frac{\beta}{3} \left(e^{\frac{\beta q}{3}} - 2e^{-\frac{2\beta q}{3}}\right) - \frac{2\beta}{3}\left(e^{\frac{\beta q}{3}} - e^{-\frac{2\beta q}{3}}\right)\left(e^{\frac{\beta q}{3}} - e^{-\frac{2\beta q}{3}}\right)\right]}{\left(2e^{\frac{\beta q}{3}} + e^{-\frac{2\beta q}{3}}\right)^2}$$

$$= \frac{2}{3} \cdot \frac{\beta e^{\beta q}}{\left(1 + 2e^{\beta q}\right)^2}$$

as $\beta > 0$ so this term is never zero & we get for the tightest bound the eq that the variational parameter has to satisfy is:

$$Nq - 2JN_{link} \langle Q_i \rangle_0 = 0.$$

i.e
$$\boxed{q = \frac{2JN_{link}}{N} \cdot \frac{2}{3} \frac{\left(e^{\frac{\beta q}{3}} - e^{-\frac{2\beta q}{3}}\right)}{\left(2e^{\frac{\beta q}{3}} + e^{-\frac{2\beta q}{3}}\right)}} \rightarrow (1)$$

Solution of this transcendental eq is the optimal value of q i.e $\bar{q}$.

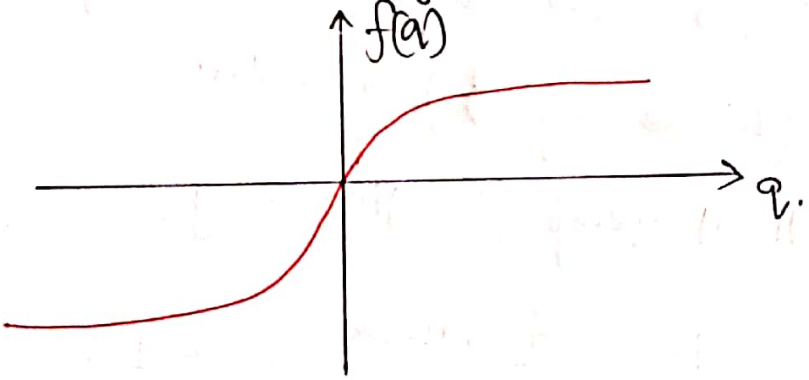
C.2

from eq (2) we get from the ~~RHS~~ R.H.S quantity (which is defined as $f(q)$) is given by at limiting cases:

$$\left.\begin{aligned} \underset{q\to\infty}{Lt} f(q) &= \frac{2JN\mu_{imk}}{3N} \\ \underset{q\to-\infty}{Lt} f(q) &= -\frac{4JN\mu_{imk}}{3J} \end{aligned}\right\}$$

And $\underset{q\to 0}{Lt} f(q) = 0$.

So the rough sketch of $f(q)$ be:



And $$\left.\frac{\partial f}{\partial q}\right|_{q=0} = 2J\frac{N\mu_{imk}}{N}\cdot\left.\frac{\partial\langle Q_i\rangle_0}{\partial q}\right|_{q=0}$$

$$= \frac{4}{3}\cdot\frac{JN\mu_{imk}}{N}\left.\frac{\beta e^{\beta q}}{(1+2e^{\beta q})^2}\right|_{q=0}$$

As the L.H.S of eq (1) be q & the right side derivative at $q=0$ be $\frac{4}{3}\frac{JN\mu_{imk}}{N}\cdot\frac{\beta}{9}$.

and $q=0$ is a solution of eq (1).

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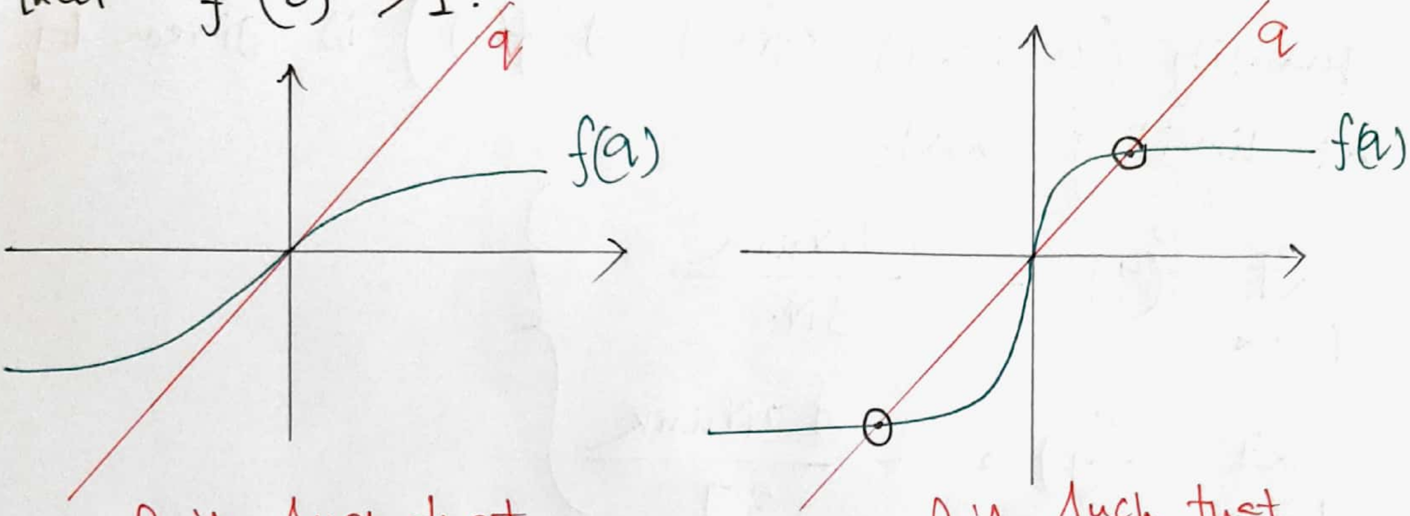
there will exist non trivial soln except $q=0$ iff the slope of $f(q)|_0$ for some β be such that $f'(0) > 1$.



β is such that $f'(0) < 1$

β is such that $f'(0) > 1$

And in 2nd case we get 2 non trivial solns.

as $f'(0) = \frac{4}{3} \frac{J N_{umn}}{9N} \times \beta$ so this is obviously possible by changing (highering β) β value.

$\therefore$ the ~~transit~~ critical temp is given by:

$$f'(0)\Big|_{\beta=\beta_c} = 1. \quad \text{i.e} \quad \frac{4 J N_{umn}}{27N} \beta_c = 1.$$

$$\text{i.e } T_c = \frac{4 J N_{umn}}{27 N k_B}.$$ Answer

C.3

The eq of ~~finding~~ finding the optimal value of $q = \bar{q}$ be given by:

$$q = \frac{2JN_{\text{link}}}{N} \langle Q_i \rangle_0$$

$$\Rightarrow \frac{2JN_{\text{link}}}{N} \cdot \frac{\partial \langle Q_i \rangle}{\partial q} = \frac{4}{3} \frac{JN_{\text{link}}}{N} \frac{\beta e^{\beta q}}{(1+2e^{\beta q})^2} = 1.$$

assuming $T \lesssim T_c$ we get $|q|$ is a first order small quantity & hence we can ~~binomial~~ Taylor expand the eq as:

$$\frac{\beta e^{\beta q}}{(1+2e^{\beta q})^2} \simeq \frac{\beta(1+\beta q)}{\{1+2(1+\beta q)\}^2} = \frac{\beta(1+\beta q)}{(3+2\beta q)^2}$$

(keeping terms upto 1st order of β).

$$= \frac{\beta}{9}(1+\beta q) \cdot \left(1+\frac{2}{3}\beta q\right)^{-2}$$

$$\simeq \frac{\beta}{9}(1+\beta q)\left(1-\frac{4}{3}\beta q\right)$$

$$\simeq \frac{\beta}{9}\left(1+\beta q - \frac{4}{3}\beta q - \frac{4}{3}\beta^2 q^2\right)$$

$$= \frac{\beta}{9}\left(1 - \frac{\beta q}{3} - \frac{4}{3}\beta^2 q^2\right)$$

i.e the condition gives: $\frac{4}{27}\frac{JN_{\text{link}}}{k_B N T}\left(1 - \frac{\beta q}{3} - \frac{4}{3}\beta^2 q^2\right) = 1$

Using $T_c = \frac{9JN_{ii}mK}{27NK_B}$,

$$1 - \frac{\beta q}{3} - \frac{4}{3}\beta^2 q^2 = \frac{T}{T_c}.$$

$$\Rightarrow \frac{4}{3}\beta^2 q^2 + \frac{\beta q}{3} - \left(1 - \frac{T}{T_c}\right) = 0.$$

i.e $$q = \frac{3}{8\beta^2}\left[-\beta/3 \pm \sqrt{\frac{\beta^2}{9} + \frac{16}{3}\beta^2\left(1 - \frac{T}{T_c}\right)}\right]$$

$$= \frac{3}{8\beta^2}\left[-\frac{\beta}{3} \pm \frac{\beta}{3}\sqrt{1 + 48\left(1 - \frac{T}{T_c}\right)}\right]$$

$$= \frac{1}{8\beta}\left[-1 \pm \sqrt{1 + 48\left(1 - \frac{T}{T_c}\right)}\right]$$

So there are two points on the solution line

① for '+' sign.

$$\bar{q}_+ = \frac{1}{8\beta}\left[-1 + \sqrt{1 + 48\left(1 - \frac{T}{T_c}\right)}\right]$$

which is a ~~first order~~ small quantity. $(\because T \approx T_c)$

② $\bar{q}_-$ (using '−' sign)

$$= \frac{1}{8\beta}\left[-1 - \sqrt{1 + 48\left(1 - \frac{T}{T_c}\right)}\right]$$

which is not a small quantity and lies on negative part of the curve.

Remark :-

Now this is a strange fact that just below $T = T_c$ the two solutions not only behaves differently but also one of them is a order small quantity & another is not. For normal longitudinal Ising model that is not the case when $\bar{m}_{\pm} \sim \left(1 - \frac{T}{T_c}\right)^{1/2}$ near $T = T_c$ & they form a parabola shape.

Here this fact made me astonished. However I asked professor Kedar if we can use any plotter if needed; As he said to mention that as an appendix; I'm going to show 3 plots I got from plotter. All the results has also been submitted. But truly this subtle fact that I found out was not possible to derive without numerical process or plotting. So I had to use that & according to the choice of grader he may/may not include that in the exam part.

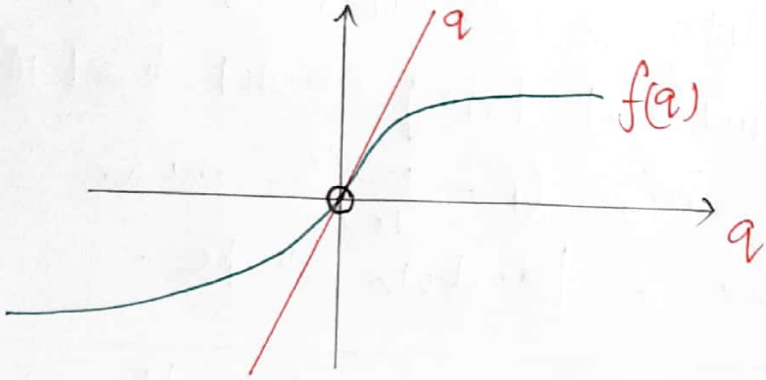
The condition of finding q be:

$$q = \frac{4}{3}\frac{J N_{link}}{N}\left\{\frac{e^{\beta q/3} - e^{-2\beta q/3}}{2e^{\beta q/3} + e^{-2\beta q/3}}\right\}$$

Taking $\frac{4}{3} \frac{J N_{link}}{N} = 1$ (which is just a scale and compensated by choice of J)

If I plot ~~LHS~~ L·H·S & R·H·S then I get:

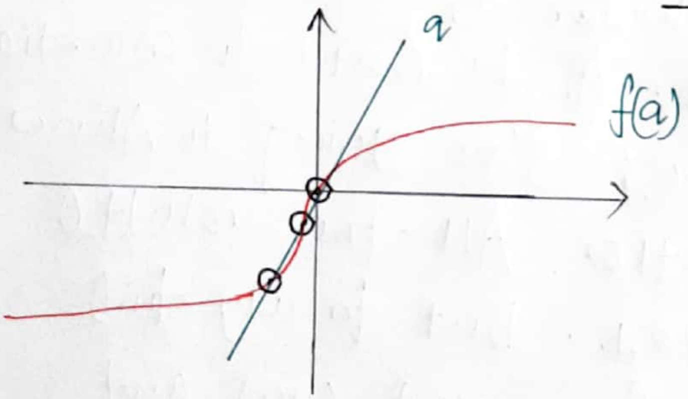
① for $\beta = 1.25$:



f(q) = R·H·S

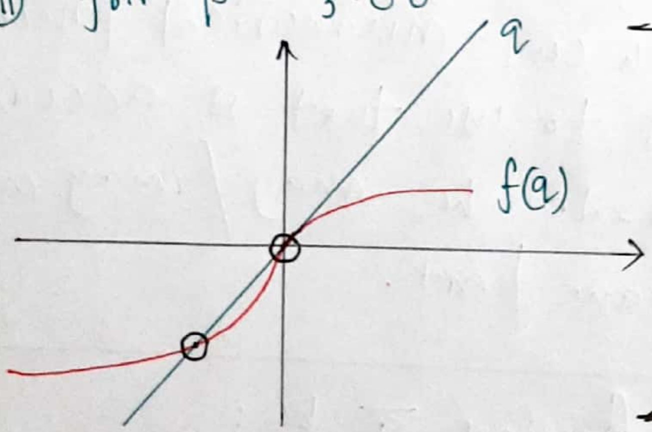
→ only the trivial zero as solution

(ii) for $\beta = 2.8$:



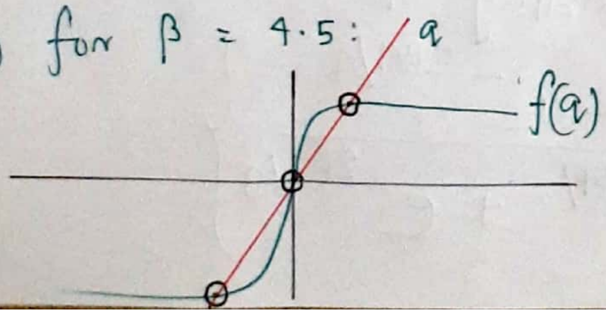
→ 3 solutions
one trivial zero.
two non-trivial and both are '-ve'

(iii) for $\beta = 3.00$



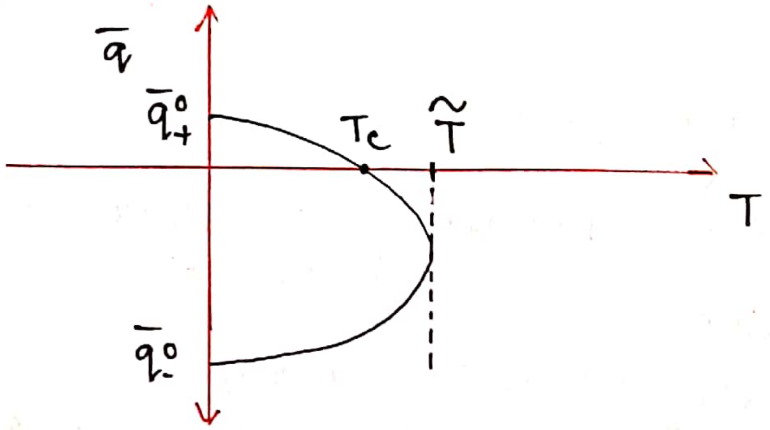
→ 2 solutions
one non-trivial zero
one " " '-ve'

(iv) for $\beta = 4.5$:



→ 3 solutions
one trivial zero.
one non trivial '+ve'
one " " '-ve'

That means the non trivial solutions changes (one of them) in a crazy manner like this.



And the condition of eq (1) of finding $\bar{q}$ is not going to give us $\tilde{T}$ but T_c as shown. At T_c we first get the situation when $f'(0) = 1$ and just below $T = T_c$

i.e $T \lesssim T_c$
one solution 'q_+' is a small quantity $(\to 0_+)$
other " (q_-) " not a small " . It's value $|q_-|$ is sufficienty large compared to a order small quantity.

And this is what I've achieved in part C.3

[Ⅳ] The reason is not because there is any mathematical error; but the previous ~~institution~~ intution that ~~the~~ when the non trival zero solution comes; the comes pairwise and with opposite sign (one +ve & one -ve) like the normal Ising case. Which is satisfied at $T < T_c$ & ~~&~~ not $T < \tilde{T}$. But this fact was not possible to derive analytically.

☒ I've also sent the true plot.

link: www.desmos.com/Calculator/~~03homkLb9k~~ qjvvqeazlo

(please let me know later if there is truely any analytical way to find $\tilde{T}$ & the nature of ~~q~~ $\bar{q}$ near $T \lesssim \tilde{T}$).

↳ which I think may follow $\tilde{q}_{T \lesssim \tilde{T}} \sim \left(1 - \frac{T}{\tilde{T}}\right)^{1/2}$

I'm just guessing. Not sure!!