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Prob - 2

Particle in a ring :-

a. The given Hamiltonian : $\hat{H} = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\phi^2}$.

∴ T.I.S.E gives:

$$-\frac{\hbar^2}{2mR^2}\frac{d^2}{d\phi^2}\psi(\phi) = E\cdot\psi(\phi).$$

(The particle is effectively in 1-d with only generalized coordinate ϕ.)

$$\Rightarrow \frac{d^2\psi}{d\phi^2} = -\frac{2mR^2E}{\hbar^2}\psi = -k^2\psi. \quad (\because E \geq 0)$$

$$\therefore \psi = Ae^{\pm ik\phi}.$$

clearly B.C gives : $\psi(\phi) = \psi(\phi + 2\pi)$

$$\Rightarrow e^{\pm ik\cdot 2\pi} = 1 \quad \Rightarrow k = 0, \pm 1, \pm 2, \cdots$$

but $k = \sqrt{\frac{2mR^2E}{\hbar^2}}$

$$\Rightarrow \frac{2mR^2E}{\hbar^2} = m^2 \quad (m \text{ is an integer})$$

$$\Rightarrow E = \frac{m^2\hbar^2}{2mR^2}.$$

And $\int_0^{2\pi} |\psi|^2 d\phi = 1 \Rightarrow A^2 \times 2\pi = 1 \Rightarrow A = \frac{1}{\sqrt{2\pi}}$.

$$\cancel{\therefore \psi_m(\phi) = \frac{1}{\sqrt{2\pi}}\exp(im\phi)} \quad \cancel{(m = 0, 1, 2, \cdots, \pm 1, \pm 2, \cdots)}$$

~~with~~

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$$\therefore \quad \psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

with $E_m = \dfrac{m^2\hbar^2}{2mR^2}$.

clearly each energy state is doubly degenerate.

<u>b</u>. In presence of Magnetic field; the Hamiltonian goes as: $\hat{H} = \dfrac{1}{2m}\left(-i\hbar\vec{\nabla} - e\vec{A}\right)^2$

Here $\vec{A} = \dfrac{BR}{2}\hat{\phi}$ (given)

& As the particle is constrained to move only on the ring; so replacing

$$-i\hbar\vec{\nabla} \rightarrow -\frac{i\hbar\hat{\phi}}{R}\frac{\partial}{\partial\phi}.$$

$$\therefore \quad \hat{H} = \frac{1}{2m}\left[-\frac{i\hbar}{R}\frac{d}{d\phi} - \frac{eBR}{2}\right]^2$$

$\therefore$ T.I.S.E gives:

$$\frac{1}{2m}\left[-\frac{i\hbar}{R}\frac{d}{d\phi} - \frac{eBR}{2}\right]^2\psi(\phi) = E\cdot\psi(\phi).$$

$$\Rightarrow -\frac{\hbar^2}{R^2}\frac{d^2\psi}{d\phi^2} + \frac{e^2B^2R^2}{4}\psi + \frac{\cancel{2}i\hbar}{R}\cdot\frac{eB\cancel{R}}{\cancel{2}}\frac{d\psi}{d\phi} = 2mE\psi$$

$$\Rightarrow \frac{d^2\psi}{d\phi^2} - \frac{e^2B^2R^4}{4\hbar^2}\psi - \frac{i\hbar eB\cdot R^2}{\hbar^2}\frac{d\psi}{d\phi} = -\frac{2mER^2}{\hbar^2}\psi.$$

$$\Rightarrow \frac{d^2\psi}{d\phi^2} - \underbrace{\frac{i\cancel{2} eBR^2}{\hbar\cancel{2}}}_{\beta} \frac{d\psi}{d\phi} + \underbrace{\left(\frac{2mER^2}{\hbar^2} - \frac{e^2B^2R^4}{4\hbar^2}\right)}_{\gamma}\psi = 0$$

$$\Rightarrow \psi'' + \beta\psi' + \gamma\psi = 0.$$

The trial soln is of the form $\psi = e^{m\phi}$.

$$\therefore m^2 + \beta m + \gamma = 0.$$

$$\therefore m = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}.$$

Using values of β & γ we get:

$$m = \frac{1}{2}\left[\frac{ieBR^2}{\hbar} \pm \sqrt{-\frac{e^2B^2\cancel{R^4}}{\cancel{\hbar^2}} - \frac{8mER^2}{\hbar^2} + \frac{e^2B^2\cancel{R^4}}{\cancel{\hbar^2}}}\right]$$

$$= \frac{i}{2}\left[\frac{eBR^2}{\hbar} \pm \frac{2R}{\hbar}\sqrt{2mE}\right]$$

$$\therefore \psi(\phi) = A \cdot \exp\left[\frac{i}{2}\left[\frac{eBR^2}{\hbar} \pm \frac{2R}{\hbar}\sqrt{2mE}\right]\right].$$

Using $\psi(\phi + 2\pi) = \psi(\phi)$ gives:

$$e^{2\pi m i} = 1 \Rightarrow m = 0, \pm i, \pm 2i, \ldots$$

i.e, $\frac{eBR^2}{2\hbar} \pm \frac{R}{\hbar}\sqrt{2mE} = n$

$n = 0, \pm 1, \pm 2, \ldots$

i.e $\frac{R}{\hbar}\sqrt{2mE} = \left(n - \frac{eBR^2}{2\hbar}\right)$

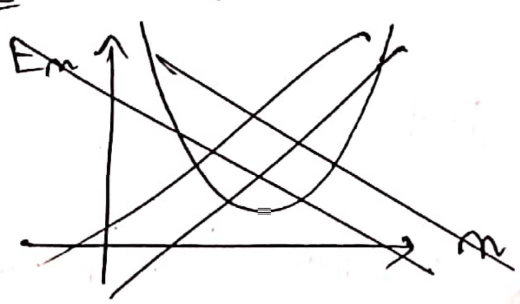
i.e $E = \frac{\hbar^2}{2mR^2}\left(m - \frac{eBR^2}{2\hbar}\right)^2 = E_m$

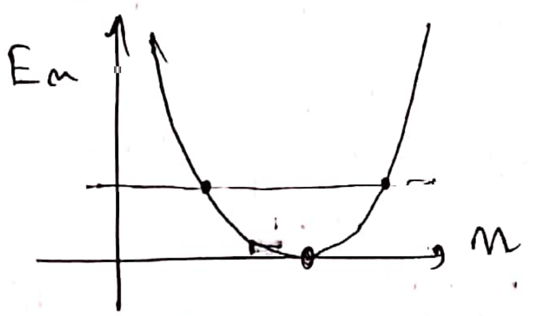
Clearly the magnetic field breaks the degeneracy & splits the m'th level into different energy states.

At $B \to 0$: $E_m = \frac{m^2\hbar^2}{2mR^2}$; which was the eigenstate of original problem.

Ans

c. The plot of E_m vs m be:





Clearly at a particular value of E_m there will be 2, m values iff m ≠ the ground state

when in the expression of

$$E_m = \frac{\hbar^2}{2mR^2}\left(m - \frac{eBR^2}{2\hbar}\right)^2$$

we get $\frac{eBR^2}{2\hbar}$ = an integer = m

The smallest value of B corresponds to m=1.

$\therefore B_{min} = \frac{1 \times 2\hbar}{eR^2} = \frac{e\hbar}{2R^2}$ **Ans**

Prob-9

Interacting Spin System :-

Instead of $|\uparrow\downarrow\rangle$; I use $|+-\rangle$ for notation.

Now here ; in absence of external magnetic field ; the conserved ~~quantities~~ quantity is neither S_1 nor S_2 but ; $S = (S_1 + S_2)$

$$S = S_1 + S_2 \Rightarrow S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2 .$$

$$\Rightarrow S_1 \cdot S_2 = \frac{S^2 - S_1^2 - S_2^2}{2} .$$

As $S = \frac{\hbar}{2}\sigma$ so; $\sigma_1 \cdot \sigma_2 = \frac{\sigma^2 - \sigma_2^2 - \sigma_1^2}{2}$.

$$\therefore \hat{H} = E_0 + \frac{A}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

$$= E_0 + \frac{A}{8}\left(\sigma^2 - \sigma_1^2 - \sigma_2^2\right)$$

a. we have to express $\hat{H}$ in the basis of $|++\rangle$ $|+-\rangle, |-+\rangle, |--\rangle$.

i.e we're to evaluate $\langle +-|H|-+\rangle$, ... etc quantities.

expressing $|+-\rangle$'s in the coupled basis we get ; the coupled basis has 4 vectors

$$|11\rangle = |++\rangle ; \; |10\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} ; \; |00\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

$$|1 -1\rangle = |--\rangle .$$

$\therefore\ |+-\rangle = \dfrac{|10\rangle + |00\rangle}{\sqrt{2}}$; $|-+\rangle = \dfrac{|10\rangle - |00\rangle}{\sqrt{2}}$.

Here we get:

$$\langle 11|11\rangle = \langle 10|10\rangle = \langle 1\,{-1}|1\,{-1}\rangle = \langle 00|00\rangle = 1$$

any, $\langle 10|11\rangle = \langle 1\,{-1}|00\rangle = \cdots = 0$.

$$\cancel{\therefore \langle ++|\hat{P}|++\rangle \quad (P = \sigma^2 - \sigma_1^2 - \sigma_2^2)}$$

$$\cancel{= \frac{1}{2}\left(\langle ++|\sigma^2 - \sigma_1^2 - \sigma_2^2|++\rangle\right)}$$

$$\cancel{= \frac{1}{2}\left(\langle 10| + \langle 00|\right)\left(\sigma^2 - \sigma_1^2 - \sigma_2^2\right)\left(|10\rangle + |00\rangle\right)}$$

$$\cancel{\text{but } \sigma^2|10\rangle = 1(1+1)|10\rangle =}$$

So: $\langle ++|\hat{P}|++\rangle \quad \left(P = \sigma^2 - \sigma_1^2 - \sigma_2^2\right)$

$$= \langle 11|\sigma^2 - \sigma_1^2 - \sigma_2^2|11\rangle$$

$$= \langle 11|\left\{2 - 2\times\frac{3}{4}\right\}|11\rangle$$

$$\left.\begin{aligned} \sigma^2|11\rangle &= 1(1+1)|11\rangle = 2|11\rangle \\ \sigma_1^2|11\rangle &= \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)|11\rangle = \tfrac{3}{4}|11\rangle \\ \sigma_2^2|11\rangle &= \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)|11\rangle = \tfrac{3}{4}|11\rangle \end{aligned}\right\} \text{ using these.}$$

$$= \langle 11|\frac{1}{2}|11\rangle = \frac{1}{2}.$$

$\langle ++|\sigma^2-\sigma_1^2-\sigma_2^2|+-\rangle$

$= \cancel{\langle ++|\sigma^2-\sigma_1^2-\sigma_2^2|}$

$= \langle +-|\sigma^2-\sigma_1^2-\sigma_2^2|++\rangle^*$

$= \left[\left\{\frac{\langle 10|+\langle 00|}{\sqrt{2}}\right\}\frac{1}{2}\cdot|11\rangle\right]^*$

$= 0$

Similarly $\langle ++|\hat{P}|-+\rangle = 0 = \langle ++|\hat{P}|--\rangle$

And in the same way:

$\langle --|\hat{P}|--\rangle = \frac{1}{2}$

$\langle --|\hat{P}|+-\rangle = 0 = \langle --|\hat{P}|-+\rangle$

$\langle --|\hat{P}|++\rangle = 0$

Now $\langle +-|\hat{P}|+-\rangle$

$= \left(\frac{\langle 10|+\langle 00|}{\sqrt{2}}\right)\cdot(\sigma^2-\sigma_1^2-\sigma_2^2)\left(\frac{|10\rangle+|00\rangle}{\sqrt{2}}\right)$

$= \frac{1}{2}\left(\langle 10|+\langle 00|\right)\Big\{1(1+1)|10\rangle - \frac{1}{2}\left(\frac{1}{2}+1\right)|10\rangle$

$- \frac{1}{2}\left(\frac{1}{2}+1\right)|10\rangle + 0(0+1)|00\rangle -$

$\frac{1}{2}\left(\frac{1}{2}+1\right)|00\rangle - \frac{1}{2}\left(\frac{1}{2}+1\right)|00\rangle\Big\}$

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$$= \frac{1}{2}\left(\langle 10| + \langle 00|\right)\left\{\left(2 - \frac{3}{4} - \frac{3}{4}\right)|10\rangle - \left(\frac{3}{4} + \frac{3}{4}\right)|00\rangle\right\}$$

$$= \frac{1}{2}\left(\langle 10| + \langle 00|\right)\left\{\frac{1}{2}|10\rangle - \frac{3}{2}|00\rangle\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - 0 + 0 - \frac{3}{2}\right\}$$

$$= -1/2.$$

$$\langle +-|\hat{p}|-+\rangle = \langle -+|\hat{p}|+-\rangle^*$$

$$= \left[\frac{1}{2}\left(\langle 10| - \langle 00|\right)\cdot\underbrace{\left\{\frac{1}{2}|10\rangle - \frac{3}{2}|00\rangle\right\}}_{\text{from previous result.}}\right]^*$$

$$\cancel{= \frac{1}{2}\left(\langle 01| - \langle 00|\right)}$$

$$= \left[\frac{1}{2}\left\{\frac{1}{2} + 0 + 0 + \frac{3}{2}\right\}\right]^* = 1.$$

Ans $\langle -+|\hat{p}|-+\rangle$

$$= \frac{1}{2}\left(\langle 10| - \langle 00|\right)\left(\sigma^2 - \sigma_1^2 - \sigma_2^2\right)\left(|10\rangle - |00\rangle\right)$$

$$= \frac{1}{2}\left(\langle 10| - \langle 00|\right)\left\{2|10\rangle - \frac{3}{4}|10\rangle - \frac{3}{4}|10\rangle - 0|00\rangle + \frac{3}{4}|00\rangle + \frac{3}{4}|00\rangle\right\}$$

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$$= \frac{1}{2}\left(\langle 10| - \langle 00|\right)\left(\frac{1}{2}|10\rangle + \frac{3}{2}|00\rangle\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} + 0 + 0 - \frac{3}{2}\right) = -1/2.$$

As $\hat{p}$ is Hermitian; so these are sufficient to construct $\hat{p}$; clearly:

$$\hat{p} = \begin{array}{c} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -1/2 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{array}{c} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{array} \\ \begin{array}{cccc} \langle ++| & \langle +-| & \langle -+| & \langle --| \end{array} \end{array}$$

So: $\hat{H} = E_0 + \frac{A}{4}\sigma_1 \cdot \sigma_2$

$$= E_0 + \frac{A}{4} \cdot \hat{p} = E_0 \cdot I_4 + \frac{A}{4}\hat{p}$$

(As E_0 is constant; so that is some unit matrix)

$$\Rightarrow \hat{H} = \begin{pmatrix} E_0 + \frac{A}{16} & 0 & 0 & 0 \\ 0 & E_0 - \frac{A}{16} & \frac{A}{4} & 0 \\ 0 & \frac{A}{4} & E_0 - \frac{A}{16} & 0 \\ 0 & 0 & 0 & E_0 + \frac{A}{16} \end{pmatrix}$$

Ans

in the given basis.

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9.6 As $\hat{E}_0 = E_0 I_4$ = constant matrix

any ~~a~~ in the form of unit matrix; so all vectors is eigen vector to it.

$\sigma_1 \cdot \sigma_2 \sim \dfrac{\sigma^2 - \sigma_1^2 - \sigma_2^2}{2}$ So the eigen-basis of $\hat{H}$ will be those ~~basis~~ vectors which are eigenvector of $\sigma^2, \sigma_1^2, \sigma_2^2$ simultaneously.

Clearly those common eigenbasis are the coupled basis ket vectors of σ_1 & σ_2. (which I've already used in part a.)

The basis set is: ($|j, m_j\rangle$ form)

$$\left.\begin{aligned} |1\ 1\rangle &= |+\,+\rangle \\ |1\ 0\rangle &= \frac{|+\,-\rangle + |-\,+\rangle}{\sqrt{2}} \\ |1\ {-1}\rangle &= |-\,-\rangle \\ |0\ 0\rangle &= \frac{|+\,-\rangle - |-\,+\rangle}{\sqrt{2}} \end{aligned}\right\} \text{Ans}$$

$|j, m_j\rangle$

$\downarrow$

j is the added up angular momenta

m_j is the z component of total angular momentum

9.c Given $|\psi(0)\rangle = \alpha|++\rangle + \beta|+-\rangle$

Now, $|++\rangle = |11\rangle$

$|+-\rangle = \dfrac{|10\rangle + |00\rangle}{\sqrt{2}}$

$\Rightarrow |\psi(0)\rangle = \alpha|11\rangle + \dfrac{\beta}{\sqrt{2}}|10\rangle + \dfrac{\beta}{\sqrt{2}}|00\rangle$

↳ expressed in terms of eigenbasis of H.

9.d Let at time t: $|\psi(t)\rangle$ is given by:

$$|\psi(t)\rangle = a_1|11\rangle + a_2|10\rangle + a_3|1-1\rangle + a_4|00\rangle$$

$$\therefore \hat{H}\psi(t) = E_0|\psi(t)\rangle + \frac{A}{8}\left(\sigma^2 - \sigma_1^2 - \sigma_2^2\right)|\psi(t)\rangle$$

$$= E_0|\psi(t)\rangle + \frac{A}{8}\left\{a_1\left(2-\frac{3}{2}\right)|11\rangle + a_2\left(2-\frac{3}{2}\right)|10\rangle + a_3\left(2-\frac{3}{2}\right)|1-1\rangle + a_4\left(0-\frac{3}{2}\right)|00\rangle\right\}$$

$$= E_0|\psi(t)\rangle + \frac{A}{8}\left\{\frac{1}{2}a_1|11\rangle + \frac{a_2}{2}|10\rangle + \frac{a_3}{2}|1-1\rangle - \frac{3a_4}{2}|00\rangle\right\}$$

$$= a_1\left(E_0 + \frac{A}{16}\right)|11\rangle + a_2\left(E_0 + \frac{A}{16}\right)|10\rangle + a_3\left(E_0 + \frac{A}{16}\right)|1-1\rangle + a_4\left(E_0 - \frac{3A}{16}\right)|00\rangle$$

but $\hat{H}|\psi(t)\rangle = i\hbar \frac{\partial|\psi\rangle}{\partial t} = i\hbar \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \\ \dot{a}_4 \end{pmatrix}$

So by comparing we get:

$$i\hbar \dot{a}_1 = a_1\left(E_0 + \frac{A}{16}\right) \Rightarrow \int_{\alpha}^{a_1(t)} \frac{da_1}{a_1} = -\frac{i\left(E_0+\frac{A}{16}\right)}{\hbar}\int_0^t dt$$

$$, \quad a_1(t) = \alpha \exp\left[-\frac{i}{\hbar}\left(E_0 + \frac{A}{16}\right)t\right].$$

Similarly $a_2(t) = \frac{\beta}{\sqrt{2}} \cdot \exp\left[-\frac{i}{\hbar}\left(E_0 + \frac{A}{16}\right)t\right]$

$$a_3(t) = 0 \quad [\because a_3(t=0) = 0]$$

$$a_4(t) = \frac{\beta}{\sqrt{2}} \exp\left[-\frac{it}{\hbar}\left(E_0 - \frac{3A}{16}\right)\right]$$

So in the eigenbasis of $\hat{H}$; $|\psi(t)\rangle$ is given by:

$$|\psi(t)\rangle = \begin{pmatrix} \alpha \exp\left[-\frac{it}{\hbar}\left(E_0 + \frac{A}{16}\right)\right] \\ \frac{\beta}{\sqrt{2}} \exp\left[-\frac{it}{\hbar}\left(E_0 + \frac{A}{16}\right)\right] \\ 0 \\ \frac{\beta}{\sqrt{2}} \exp\left[-\frac{it}{\hbar}\left(E_0 - \frac{3A}{16}\right)\right] \end{pmatrix}$$

Ans

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Q7. $l=1$ Hydrogen State with variational method.

In the true wave fn of H atom; the Radial part varies as:

$$R_{nl}(r) \sim e^{-\frac{r}{na}} \cdot \left(\frac{2r}{na}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2r}{na}\right)$$

Now the associated ~~Leg~~ Laguerre poly be given by:

$$L^p_{q-p}(x) \sim (-1)^p \frac{d^p}{dx^p} L_q(x)$$

when; $L_q(x) = \sum_{j=0}^{q} c_j x^j$ (for some non zero c_j)

as in $L_q(x)$ all powers of x upto $j=q$ is present; hence in it's derivative; there will be at least one term which is constant any non zero. (otherwise the derivative is zero as a whole)

i.e $L_q(x) = \sum_{j=0}^{q} c_j x^j$

$\Rightarrow \frac{d^n}{dx^n} L_q(x) = 0$ if $n > q$

& $\frac{d^n}{dx^n} L_q(x)$ has one nonzero constant term in expansion if $n \leq q$ i.e it's not zero as whole.

So for any physical normalizable wave fn for an l State we get in Hydrogen

$$\lim_{r\to 0} R_{nl}(r) \sim \lim_{r\to 0} e^{-\frac{r}{na}} \cdot \left(\frac{2r}{na}\right)^l \left(c_0 + c_1 r + \cdots + c_k r^k\right)$$

for some $k \geq 0$ & we get

$c_0, c_1, \ldots, c_k \neq 0$

So; $\lim_{r \to 0} R_{nl}(r) \sim A e^0 \cdot r^l \cdot (c_0 + 0 + \cdots + 0)$

$\sim r^l$.

Here for the ~~given sta~~ trial state we use that information.

more over at $r \to \infty$; $e^{-r/na} \cdot r^\alpha \to 0$

for all α. So the good trial fn can be of the form:

$$R_{n1} = A r^1 e^{-\alpha r} = A r e^{-\alpha r} = \psi_\alpha \text{ (say)}$$

As the spherical Harmonics are also normalized so we get:

$$1 = \int_0^\infty r^2 R^2 \, dr = \int_0^\infty A^2 r^2 \cdot r^2 \cdot e^{-2\alpha r} \, dr \qquad (2\alpha r = z)$$

$$= A^2 \int_0^\infty \frac{z^4}{2^4 \alpha^4} \cdot e^{-z} \frac{dz}{2\alpha}$$

$$= \frac{A^2}{32\alpha^5} \Gamma(5) = \frac{A^2 \times 4 \times 3 \times 2}{32 \alpha^5} = \frac{3A^2}{4\alpha^5}.$$

$$\therefore A = \sqrt{\frac{4\alpha^5}{3}}$$

Now R_{nl} satisfies the eq:

$$-\frac{\hbar^2}{2m} \cdot \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[\frac{\hbar^2}{2m} \cdot \frac{l(l+1)}{r^2} + V(r)\right] R = ER$$

Here $V(r) = -\frac{e^2}{r}$; $l = 1$ gives:

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(\frac{\hbar^2}{m}\cdot\frac{1}{r^2}-\frac{e^2}{r}\right)R = ER.$$

i.e $$\hat{H}_R = \underbrace{-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)}_{\hat{T}_R}+\underbrace{\left(\frac{\hbar^2}{2m}\cdot\frac{1}{r^2}-\frac{e^2}{r}\right)}_{\hat{V}_R}$$

At the trial state ψ_α taken;

$$\langle H_R\rangle = \langle\hat{T}_R\rangle + \langle\hat{V}_R\rangle$$

when $$\langle\hat{A}_R\rangle = \int_0^\infty(\psi_\alpha\hat{A}_R\psi_\alpha)\cdot r^2\,dr.$$

∴ Here $\langle\hat{T}_R\rangle$

$$= -\int_0^\infty A^2 r e^{-\alpha r}\left(\frac{\hbar^2}{2m}\right)\cdot\frac{1}{\cancel{r^2}}\frac{d}{dr}\left(r^2\frac{d}{dr}(re^{-\alpha r})\right)\cdot\cancel{r^2}\,dr.$$

I've evaluated the integrals in mathematica which gives:

$$\langle\hat{T}_R\rangle_{\psi_\alpha} = \frac{\cancel{3}}{\cancel{8}}\frac{\alpha\hbar^2}{m\cancel{\alpha^4}}\cdot\frac{\cancel{8}\alpha^{\cancel{5}}}{\cancel{3}\times 2}$$ (using value of A)

$$= \frac{\alpha^2\hbar^2}{2m}$$

And $$\langle\hat{V}_R\rangle_{\psi_\alpha} = \int_0^\infty\left(\frac{\hbar^2}{2mr^2}-\frac{e^2}{r}\right)\cdot r^2 A^2 r^2 e^{-2\alpha r}\,dr$$

$$= -\frac{\cancel{3}}{\cancel{8}}\cdot\frac{\cancel{m}e^2}{\cancel{m}\cancel{\alpha^4}}\cdot\frac{\cancel{8}\alpha^{\cancel{5}}}{\cancel{3}\times 2} = -\frac{\alpha e^2}{2}.$$

So; $$\langle H\rangle_{\psi_\alpha} = \frac{\alpha^2\hbar^2}{2m} - \frac{\alpha e^2}{2}.$$

varying over α gives:

$$\frac{d\langle H\rangle_{\psi_\alpha}}{d\alpha} = 0 = \frac{2\alpha\hbar^2}{2m} - \frac{e^2}{2}$$

$$\Rightarrow \alpha = \frac{me^2}{2\hbar^2} = \alpha_0$$

$$\therefore \langle H\rangle_{\psi_\alpha}\Big|_{\alpha=\alpha_0} = \left(\frac{\hbar^2\alpha^2}{2m} - \frac{\alpha e^2}{2}\right)\Big|_{\alpha = \frac{me^2}{2\hbar^2}}$$

$$= \frac{\cancel{\hbar^2}}{2\cancel{m}} \cdot \frac{m^{\cancel{2}} e^4}{4\cancel{\hbar^4}\,\hbar^2} - \frac{e^2}{2} \cdot \frac{me^2}{2\hbar^2}$$

$$= -\frac{me^4}{8\hbar^2}.$$

So $E^{gs}\Big|_{l=1} \leq -\dfrac{me^4}{8\hbar^2}.$ Ans

(iii) The actual energy is given by:

$$E_n = -\frac{me^4}{2n^2\hbar^2} \quad \left(\text{taking } \frac{1}{4\pi\epsilon_0} = 1\right)$$

for $l = 1$ we have $n \geq 2$. (obvious for H)

$$\therefore E_n^{true} = -\frac{me^2}{2n^2\hbar^2} = -\frac{me^2}{8\hbar^2}.$$

$\therefore$ The result is true for $n = 2$.

i.e for all n values ($n = 2, 3, 4, \ldots$) possible for $l=1$; the ground state ($n=2$) exactly matching the energy. This is because of the form of ψ_α, which is equal to the form of $R_{n=2}(r)$ state.

Clearly the ~~value~~ value is indp of m.
m only takes part for determining 'Angular' part. ($Y_l^m(\theta,\phi)$)

Ans

Problem:- 11

Given: $H_0 = \frac{1}{2m_0}\left(\vec{p} - q\vec{A}\right)^2 I$.

Now using the traditional shorthand for

$\vec{p} - q\vec{A} = \vec{\pi}$

we get $[\pi_i, \pi_j] = [p_i - qA_i, p_j - qA_j]$

$= \underbrace{[p_i, p_j]}_{0} + q^2 \underbrace{[A_i, A_j]}_{0} - q[A_i, p_j] - q[p_i, A_j]$

$\left(\because A_i = A_i(x, y, z) \ \& \ [x_i, x_j] = 0\right)$

$= -i\hbar q \frac{\partial A_i}{\partial x_j} + i\hbar q \frac{\partial A_j}{\partial x_i}$

$= -i\hbar q \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) = -i\hbar q \epsilon_{ijk} B_k$.

Here $q = -e$ so:

$[\pi_i, \pi_j] = i\hbar e \epsilon_{ijk} B_k$.

Now using $(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B)$

we get:

$\cancel{\sigma \cdot (p + eA)}$ $\left[\sigma \cdot (\vec{p} + e\vec{A})\right]^2 = (\sigma \cdot \pi)^2 = (\sigma \cdot \pi)(\sigma \cdot \pi)$

$= \pi \cdot \pi + i\sigma \cdot (\pi \times \pi)$

$= I(p + eA)^2 + i\sigma \cdot -i\hbar e \vec{B}$

$\left(\because \vec{\pi} \times \vec{\pi}\Big|_i = \epsilon_{ijk} \pi_j \pi_k = [\pi_j, \pi_k] \times (\pm 1)\right.$

$\left.\text{according to circular order of } j \text{ or } k\right)$

$$= (p + eA)^2 \cdot I + 2m \cdot \mu_B \cdot \vec{B} \cdot \vec{\sigma}$$

$$\left(\text{when } \mu_B = \frac{e\hbar}{2m_0}\right)$$ proved

Problem :- 6

a. For spin 1 particle the states are ($|l, m\rangle$ form) be denoted by:

$$|1\,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |1\,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad |1\,{-1}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

when $\hat{S}_z |l\,m\rangle = m\hbar\,|l\,m\rangle$

$$\therefore (\hat{S}_z)_{ij} = \langle l\,m_i | S_z | l\,m_j\rangle = m_j \hbar\, \delta_{ij}$$

So; $$[\hat{S}_z] = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (l = 1)$$

Now; $$S_+ |l\,m\rangle = \hbar\sqrt{(l-m)(l+m+1)}\,|l\,m+1\rangle \quad (l=1)$$

$$\Rightarrow (S_+)_{ij} = \hbar\sqrt{(1-m_j)(2+m_j)}\;\delta_{i,j+1}$$

So; $$[\hat{S}_+] = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_- |1\,m\rangle = \hbar\sqrt{(1+m)(2-m)}\,|1\,m-1\rangle$$

$$\therefore (S_-)_{ij} = \hbar\sqrt{(1+m_j)(2-m_j)}\;\delta_{i,j-1}$$

$$\therefore \left[\hat{S}_-\right] = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\therefore \hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\hat{S}_- = \frac{S_+ - S_-}{2i} = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$\therefore S_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$\hat{S}_y^2 = -\frac{\hbar^2}{4} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{-\hbar^2}{4} \begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$\therefore \hat{H} = \hbar^2 \left[A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + B \left\{ \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \right\} \right]$$

$$= \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

Now we find eigenvalue of $\hat{H}$ $\left(\text{or } \frac{\hat{H}}{\hbar^2}\right)$:

$\det\left(\frac{H}{\hbar^2} - \lambda I\right) = 0$ gives

$$\det\begin{pmatrix} A-\lambda & 0 & B \\ 0 & -\lambda & 0 \\ B & 0 & A-\lambda \end{pmatrix} = 0$$

$\Rightarrow (A-\lambda)^2 \cdot (-\lambda) + B^2\lambda = 0$

i.e $\lambda = 0 = \lambda_0$

or; $(A-\lambda)^2 = B^2$ i.e $A - \lambda = \pm B$

i.e $\lambda = (A \pm B) = \lambda_\pm$.

Let the eig state be $X = (x\ y\ z)^T$.

for $\lambda = \lambda_0$

$$\begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} Ax + Bz &= 0 \\ Bx + Az &= 0 \end{aligned}\right\} \quad x = 0 = z \quad (A \neq B)$$

$\therefore X = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

for $\lambda = A + B$.

$$\begin{pmatrix} -B & 0 & B \\ 0 & -A-B & 0 \\ B & 0 & -B \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} Bz - Bx &= 0 \\ (A+B)y &= 0 \\ Bx - Bz &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x &= z \\ y &= 0 \end{aligned}\right\} \quad \begin{pmatrix} \text{assuming} \\ B \neq 0 \\ A \neq -B \end{pmatrix}$$

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$\therefore\ x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$

for $\lambda = A - B$:

$$\begin{pmatrix} +B & 0 & B \\ 0 & -A+B & 0 \\ B & 0 & B \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} B(x+z) = 0 \\ (B-A)y = 0 \\ B(x+z) = 0 \end{array}\right\} \left[\begin{array}{l} x = -z \\ y = 0 \end{array}\right]$$

$$\therefore\ x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$\therefore$ The normalized eig-state of $\hat{H}$ with eig value are given by:

$$\left.\begin{array}{l} |x_1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \longrightarrow \lambda_1 = 0 \\ |x_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \longrightarrow \lambda_2 = \hbar^2(A+B) \\ |x_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} \longrightarrow \lambda_3 = \hbar^2(A-B). \end{array}\right\}$$

Ans

b. $\Theta S_i^2 \Theta^{-1} = \Theta S_i \Theta^{-1} \Theta S_i \Theta$ $(i = x, y, z)$

(Θ = Time reversal operator.)

$= (-S_i)\cdot(-S_i) = S_i^2$.

Using $\Theta S_i \Theta = -S_i$ or $S_i \longrightarrow -S_i$ (under Θ)

$\therefore \; \Theta \hat{H} \Theta^{-1} = [A(\Theta S_z^2) + B(\Theta S_x^2 - \Theta S_y^2)]\Theta^{-1}$

$$= A \underbrace{\Theta S_z^2 \Theta^{-1}}_{S_z^2} + B(\underbrace{\Theta S_x^2 \Theta^{-1}}_{S_x^2} - \underbrace{\Theta S_y^2 \Theta^{-1}}_{S_y^2})$$

$$= A S_z^2 + B(S_x^2 - S_y^2)$$

$$= H.$$

Ans

$\therefore \hat{H}$ is invarient under time reversal.

c. $\Theta|\chi_1\rangle = \Theta\begin{pmatrix}0\\1\\0\end{pmatrix} = \Theta|1\;0\rangle = (-1)^0|1\;0\rangle$

(using $\Theta|jm\rangle = (-1)^m|j\;-m\rangle$)

$\therefore \; \Theta|\chi_1\rangle = |1\,0\rangle = |\chi_1\rangle$ Ans

$\Theta|\chi_2\rangle = \Theta \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\Theta\left\{\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$$= \frac{1}{\sqrt{2}}\Theta\{|1\,1\rangle + |1\,-1\rangle\} = \underbrace{\frac{-1}{\sqrt{2}}\{|1\,-1\rangle + |1\,1\rangle\}}_{-|\chi_2\rangle}$$

$\Rightarrow \Theta|\chi_2\rangle = -|\chi_2\rangle$ Ans

$\Theta|\chi_3\rangle = \frac{1}{\sqrt{2}}\Theta\{|1\,1\rangle - |1\,-1\rangle\} = \frac{1}{\sqrt{2}}\{|1\,-1\rangle - |1\,1\rangle\}$

$$= -|\chi_3\rangle$$

$\therefore \; \Theta|\chi_3\rangle = -|\chi_3\rangle$ Ans

Problem:- 10

Here for $n=2$ state there are (ignoring spin) n^2 degenerate states.

Now; $E(t) = \hat{k} E_0 e^{-t^2/\tau^2}$

$\therefore V(t) = \int E(t)\, dz = -z E_0 e^{-t^2/\tau^2}$

(That satisfies $E = -\frac{\partial V}{\partial z}\hat{z} = -\nabla V$)

Now $\langle n\,l\,m | z | n'\,l'\,m' \rangle$

$= \int r^3 R_{nl}(r) R_{n'l'}(r)\, dr \int Y_l^{m*}(\theta\phi) \cdot Y_{l'}^{m'}(\theta\phi) \cos\theta \sin\theta\, d\theta\, d\phi$

$(\because z = r\cos\theta)$

The ϕ integral gives:

$\langle n\,l\,m | z | n'\,l'\,m' \rangle = (r \text{ integral}) \times (\theta \text{ integral})$

$\times \underbrace{\int_0^{2\pi} e^{i(m-m')\phi}\, d\phi}_{\neq 0 \text{ iff } m=m'}$

for $n=2$; the l values are 0 & 1.

$\langle 1\,0\,0 | z | 2\,0\,0 \rangle$

$\sim \underbrace{\int r^3 R_{10}(r) R_{20}(r)\, dr}_{\Downarrow} \int Y_0^{0*}(\theta\phi) \cdot Y_0^0(\theta\phi) \sin\theta \cos\theta\, d\theta\, d\phi.$

$\sim I(R) \cdot \int_0^{\pi} \sin\theta\cos\theta\, d\theta \cdot \int_0^{2\pi} d\phi$

(excluding normalization const)

but $\int_0^{\pi} \sin\theta \cos\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin\psi\, d\psi = 0$

So: $\langle 100|z|200\rangle = 0$

by $m \neq m'$ reason: $\langle 100|\hat{z}|21\pm1\rangle = 0$

And $\langle 100|z|210\rangle = \int r^3 R_{10}(r) R_{21}(r)\, dr \int Y_0^{0*} Y_1^0 \sin\theta \cos\theta\, d\theta\, d\phi$

$$= \int_0^\infty 2a^{-3/2} e^{-r/a}\, r^3 \cdot \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}\, dr$$

$$\times \int_0^\pi \frac{1}{\sqrt{4\pi}} \cdot \sqrt{\frac{3}{4\pi}} \sin\theta \cos^2\theta\, d\theta \cdot \int_0^{2\pi} d\phi.$$

$$= \frac{2\cancel{a^{-3}}}{\sqrt{24}\,\cancel{a}} \cdot \frac{256\, a^{\cancel{5}}}{81} \times \cancel{2\pi} \times \frac{1}{\cancel{2}\sqrt{3}\,\cancel{\pi}}.$$

(evaluated

$$= \frac{\cancel{2} \cdot 2^8\, a}{3^4 \cdot \sqrt{3} \cdot \cancel{8} \cdot \sqrt{3} \cdot \sqrt{2} \times \cancel{2}} = \frac{2^7 \sqrt{2}}{3^5} a \quad \left(a = \text{Bohr radius}\right)$$

$$H'(t) = q V(t) = -qzE_0 e^{-t^2/\tau^2}.$$

By previous calculation; we get: the charge can only go to $|210\rangle$ state.

The transition amplitude be (using formula for non degenerate time dependent perturbation theory which we can use as only one state with that particular energy with $n=2$ state is available here:)

$$c_b = \frac{-i}{\hbar} \int \langle 100|z|210\rangle \cdot \left(-qE_0 e^{-t^2/\tau^2}\right) e^{i\omega_0 t}\, dt$$

$$= \frac{i}{\hbar} qE_0 \frac{2^7 \sqrt{2}\, a}{3^5} \int_{-\infty}^{\infty} e^{i\omega_0 t} \cdot e^{-t^2/\tau^2}\, dt$$

$$\left(\omega_0 = \frac{E_{n=2} - E_{n=1}}{\hbar} = \frac{E_{21m} - E_{100}}{\hbar}\right)$$

$$\Rightarrow \quad c_b = \frac{i}{\hbar} \frac{2^7\sqrt{2}}{3^5} qE_0 a \cdot \tau \sqrt{\pi}\, e^{-\frac{\omega_0^2\tau^2}{4}}$$

So transition probability (required quantity)

$$|c_b|^2 = p = \left(\frac{eE_0}{\hbar}\right)^2 \cdot \left(\frac{2^{15} a_0^2}{3^{10}}\right) \pi \tau^2 e^{-\frac{\omega^2\tau^2}{2}}$$

Proved

(ii) The timely varying $\vec{E}$ field induces a $\vec{B}$ field given by Maxwell's eq. The spin of the e^- will interact with that $\vec{B}$ field and add another perturbation

$$H'' = -\frac{eB(t)}{m} \cdot \hat{S}_z$$

$\therefore$ The answer will change in presence of spin.

Problem: 5

Linear Stark effect

a. This problem is quite similar (by calculation) with problem 10 (which I've done earlier.)

The perturbing potential be given by,

$$H' = V = -ez \cdot |\vec{E}| \quad \left(\text{so that } e\vec{E} = -\nabla V\right)$$

Now for $n = 2$ states we get $\left.\begin{array}{l} l=0 \;;\; m=0 \\ l=1 \;;\; m=0,\pm 1 \end{array}\right\}$

i.e there are 4 th order degeneracy.

∴ The matrix elements of V (or H') be given by:

$$\left(H'_{4\times 4}\right)_{ij} = \langle i | H' | j \rangle$$

when $|i\rangle, |j\rangle = |2\,0\,0\rangle, |2\,1\,1\rangle, |2\,1\,0\rangle, |2\,1\,\bar{1}\rangle$

As; $H \sim z$ so $\langle 2\,l\,m | z | 2\,l'\,m' \rangle$

$$\sim \int_0^\infty R_{2l} R_{2l'} \cdot r^2 \cdot r\, dr \cdot \int_0^\pi Y_l^m (\theta \text{ part}) \cdot Y_{l'}^{m'} (\theta \text{ part}) \cdot \underset{\cos\theta \sin\theta\, d\theta}{}$$

$$\times \underbrace{\int_0^{2\pi} e^{i(m-m')\phi} d\phi}_{\neq 0 \text{ iff } m = m'}$$

So in H'; there are only two non zero elements $\langle 2\,0\,0 | z | 2\,1\,0 \rangle$ & h.c.

The diagonal terms are zero by the selection

rule on l; that tells $\langle n l m | z | n' l' m' \rangle \neq 0$
iff $l = l' \pm 1$.

As here we need only $\langle H' \rangle$ to evaluate 1st order perturbation; so by the previous reasoning the only two states connected via the given field are $|200\rangle$ & $|210\rangle$.

b. $\langle 200 | z | 210 \rangle$ is given by:

$$H'_{12} \sim \int_0^\infty R_{20}(r) R_{21}(r) \cdot r^3 \, dr \int_{\theta,\phi} Y_0^{0*} Y_1^0 \cdot \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= \int_0^\infty r^3 \cdot \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{r}{2a}\right) e^{-r/2a} \cdot \frac{1}{\sqrt{24}} a^{-3/2} \left(\frac{r}{a}\right) \cdot e^{-r/2a} \, dr.$$

$$\times \int_0^\pi \frac{1}{\sqrt{4\pi}} \cdot \sqrt{\frac{3}{4\pi}} \cos^2\theta \sin\theta \, d\theta \cdot \int_0^{2\pi} d\phi.$$

$$= \cancel{2}\cancel{\pi} \cdot \frac{1}{\cancel{2}\cancel{\sqrt{3}}\cancel{\pi}} \cdot (-3\cancel{\sqrt{3}}a) = -3a. = \langle 210 | z | 200 \rangle$$

$$\therefore H'_{12} = 3a|E|e$$

$$\therefore H = \begin{array}{c} \\ \left(\begin{array}{cccc} 0 & 3ae|E| & 0 & 0 \\ 3a|E|e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array} \begin{array}{l} |200\rangle \\ |210\rangle \\ |211\rangle \\ |21{-1}\rangle \end{array}$$

Here we only need to diagonalize the upper left block of $\hat{H}$; which contains the interaction term between $|200\rangle$ & $|210\rangle$.

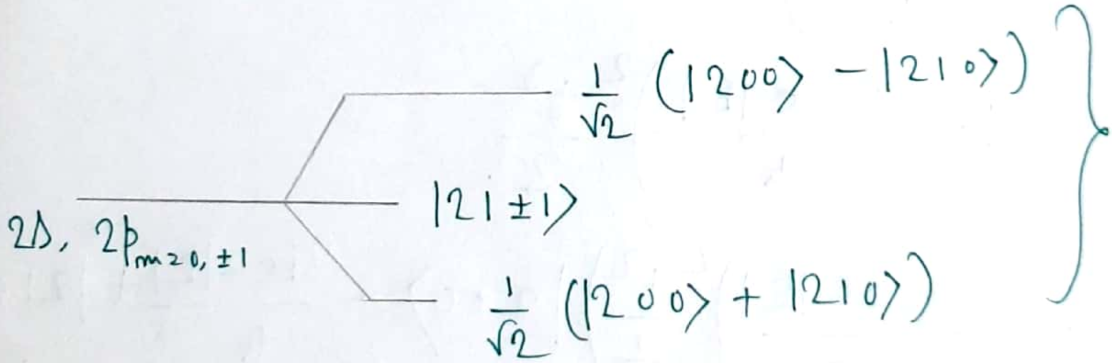
$$H_{\text{upper left}} \sim A \begin{array}{c} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} |200\rangle \\ |210\rangle \end{array}$$

As it is corresponding to σ_x; so the eigenvectors will be $\frac{1}{\sqrt{2}}\left(|200\rangle \pm |210\rangle\right)$ {like for σ_x: $\frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$}

So the good linear combinations are

$$\alpha_1 = \frac{1}{\sqrt{2}}\left(|200\rangle + |210\rangle\right)$$

$$\alpha_2 = \frac{1}{\sqrt{2}}\left(|200\rangle - |210\rangle\right)$$



c. Let $|\psi(t)\rangle = \cancel{c_a(t)}\ c_a(t)|200\rangle + c_b(t)|210\rangle$

for $t < \frac{L}{v}$ the ~~pert~~ perturbation is given the H' matrix.

$c_a(0) = 1$; $c_b(0) = 0$ (given).

first order perturbation gives:

$$\dot{c}_a^1(t) = \frac{-i}{\hbar} H'_{12}\, e^{-i\omega_0 t}\, c_b^0(t) = 0$$

$$\Rightarrow c_a^1(t) - c_a(0) = 0 \qquad \Rightarrow c_a^1(t) = 1 = c_a(0).$$

$$\dot{c}_b^1(t) = -\frac{i}{\hbar} \cdot 3ae|E|\, e^0 \cdot 1 \qquad \left(\because \underbrace{\omega_0 = \frac{E_b - E_a}{\hbar}}_{\text{here}} = 0\right)$$

$$\Rightarrow c_b^1(t) - \cancel{c_b^1(0)}^{\,0} = -\frac{3iae|E|t}{\hbar}$$

$$\Rightarrow c_b^1(t) = -\frac{3iae|E|t}{\hbar}.$$

2nd order for $c_a(t)$:

$$\dot{c}_a^2(t) = -\frac{i}{\hbar} \cdot 3ae|E| \cdot e^0 \cdot \frac{-3iae|E|}{\hbar} t$$

$$= -\left(\frac{3ae|E|}{\hbar}\right)^2 t.$$

$$\therefore c_a^2(t) = 1 - \left(\frac{3ae|E|}{\hbar}\right)^2 \cdot \frac{t^2}{2}.$$

$$\therefore |\psi(t)\rangle \simeq \left(1 - \left(\frac{3ae|E|}{\hbar}\right)^2 \cdot \frac{t^2}{2}\right)|200\rangle - \frac{3iea|E|t}{\hbar}|210\rangle$$

Answer

5. d.) prob to find the e^- in different $n=2$ states,

$$P_{|200\rangle} = |c_a|^2 = \left\{1 - \left(\frac{3ae|E|}{\hbar}\right)^2 \cdot \frac{t^2}{2}\right\}^2$$

$$P_{|210\rangle} = |c_b|^2 = \left(\frac{3ea|E|t}{\hbar}\right)^2.$$

$$P_{|21\pm1\rangle} = 0$$

Ans

Clearly $\sum_i P_i^2 = 1 + \underbrace{O(t^4)}_{\text{error of 2nd order theory.}}$

Problem:- 12

a.) $\vec{J} = \bar{L}_1 + \bar{L}_2$

So j varies from $(l_1+l_2) = 2$ to $|l_1 - l_2| = 0$.

clearly for $j = 2$; $m_j = \pm 2, \pm 1, 0$

$|2\ 2\rangle = |1\ 1\rangle|1\ 1\rangle$

$J_-|2\ 2\rangle = J_1^-|1\ 1\rangle|1\ 1\rangle + J_2^-|1\ 1\rangle|1\ 1\rangle$

$\Rightarrow 2|2\ 1\rangle = \sqrt{2}\,|1\ 0\rangle|1\ 1\rangle + \sqrt{2}|1\ 1\rangle|1\ 0\rangle$

$\Rightarrow |2\ 1\rangle = \dfrac{|1\ 0\rangle|1\ 1\rangle}{\sqrt{2}} + \dfrac{|1\ 1\rangle|1\ 0\rangle}{\sqrt{2}}$

So; $J^-|2\ 1\rangle = \dfrac{1}{\sqrt{2}}\left(J_1^-|1\ 0\rangle|1\ 1\rangle + J_2^-|1\ 0\rangle|1\ 1\rangle\right) + \dfrac{1}{\sqrt{2}}\left(J_1^-|1\ 1\rangle|1\ 0\rangle + J_2^-|1\ 0\rangle|1\ 1\rangle\right)$

$\Rightarrow \sqrt{3\times 2}\ |2\ 0\rangle = \dfrac{1}{\sqrt{2}}\Big(\sqrt{1\times 2}\ |1\ {-1}\rangle|1\ 1\rangle + \sqrt{2}\,|1\ 0\rangle|1\ 0\rangle + \sqrt{1\times 2}\ |1\ 1\rangle|1\ {-1}\rangle + \sqrt{2}\,|1\ 0\rangle|1\ 0\rangle\Big)$

$\Rightarrow |2\ 0\rangle = \dfrac{1}{\sqrt{6}}\left(|1\ {-1}\rangle|1\ 1\rangle + |1\ 1\rangle|1\ {-1}\rangle + 2|1\ 0\rangle|1\ 0\rangle\right)$

And; $|2\ {-2}\rangle = |1\ {-1}\rangle|1\ {-1}\rangle$

$\Rightarrow J^+|2\ {-2}\rangle = J_1^+|1\ {-1}\rangle|1\ {-1}\rangle + J_2^+|1\ {-1}\rangle|1\ {-1}\rangle$

$\Rightarrow |2\ {-1}\rangle = \dfrac{1}{\sqrt{2}}\left(|1\ {-1}\rangle|1\ 0\rangle + |1\ 0\rangle|1\ {-1}\rangle\right)$

Using property that for $m \neq m_1 + m_2$ the C.G ~~cof~~ coefficients; we let: (for $j = 1$)

$$|1\ 1\rangle = a\,|1\ 0\rangle|1\ 1\rangle + b\,|1\ 1\rangle|1\ 0\rangle$$

$$\Rightarrow J^+|1\,1\rangle = a\left(J_1^+|1\,0\rangle|1\,1\rangle + J_2^+|1\,0\rangle|1\,1\rangle\right) + b\left(J_1^+|1\,1\rangle|1\,0\rangle + J_2^+|1\,1\rangle|1\,0\rangle\right)$$

using $J^+|j\,j\rangle = 0$:

$$\Rightarrow 0 = a\sqrt{1\cdot 2}\;|1\,1\rangle|1\,1\rangle + 0 + 0 + b\sqrt{2}\;|1\,1\rangle|1\,1\rangle$$

$\therefore a = -b$; now $a^2 + b^2 = 1 \Rightarrow 2a^2 = 1$

So; $a = \frac{1}{\sqrt{2}}$; $b = -\frac{1}{\sqrt{2}}$.

$$\therefore |1\,1\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\rangle|1\,1\rangle - |1\,1\rangle|1\,0\rangle\right)$$

Similarly for $|1\,-1\rangle$ gives:

$$|1\,-1\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\rangle|1\,-1\rangle - |1\,-1\rangle|1\,0\rangle\right)$$

Applying J^+ on both sides:

$$J^+|1\,-1\rangle = \frac{1}{\sqrt{2}}\left(J_1^+ + J_2^+\right)\left(|1\,0\rangle|1\,-1\rangle - |1\,-1\rangle|1\,0\rangle\right)$$

$$\Rightarrow \sqrt{2}\,|1\,0\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}|1\,1\rangle|1\,-1\rangle - \sqrt{2}|1\,-1\rangle|1\,1\rangle\right)$$

$$\Rightarrow |1\,0\rangle = \frac{1}{\sqrt{2}}|1\,1\rangle|1\,-1\rangle - \frac{1}{\sqrt{2}}|1\,-1\rangle|1\,1\rangle$$

Now let: (for $j=0$)

$$|0\ 0\rangle = a|1\ -1\rangle|1\ 1\rangle + b|1\ 1\rangle|1\ -1\rangle + c|1\ 0\rangle|1\ 0\rangle$$

applying J_+ on ~~both~~ both sides:

$$0 = a\, J_1^+ |1\ -1\rangle|1\ 1\rangle + b\, J_2^+ |1\ 1\rangle|1\ -1\rangle + 0 + 0$$
$$+ c\, J_1^+ |1\ 0\rangle|1\ 0\rangle + c\, J_2^+ |1\ 0\rangle|1\ 0\rangle$$

$$= a\sqrt{2}\,|1\ 0\rangle|1\ 1\rangle + b\sqrt{2}\,|1\ 1\rangle|1\ 0\rangle$$
$$+ c\sqrt{1\cdot 2}\,|1\ 1\rangle|1\ 0\rangle + c\sqrt{2}\,|1\ 0\rangle|1\ 1\rangle$$

$$\Rightarrow (a+c)|1\ 0\rangle|1\ 1\rangle + (b+c)|1\ 1\rangle|1\ 0\rangle = 0$$

$$\Rightarrow a = -c\ ; \quad b = -c$$

but $a^2+b^2+c^2 = 1 \Rightarrow c = \frac{-1}{\sqrt{3}}$.

$$\therefore a = b = \frac{1}{\sqrt{3}}.$$

$$\therefore |0\ 0\rangle = \frac{1}{\sqrt{3}}\left(|1\ -1\rangle|1\ 1\rangle + |1\ 1\rangle|1\ -1\rangle - |1\ 0\rangle|1\ 0\rangle\right)$$

These are the expansion of coupled state in the eigenstate of J_1, J_2

Now here the two particle state is given by

$l_1 = l_2 = 1\ ;\ m_1 = m_2 = 0$

i.e $|1\ 0\rangle|1\ 0\rangle$.

So from the previous eq we have to expand $|1\,0\rangle|1\,0\rangle$ in eigenstate of J^2; i.e in coupled state term.

Let $|1\,0\rangle|1\,0\rangle = a|2\,0\rangle + b|1\,0\rangle + c|0\,0\rangle$

from the previous relations; multiplying $|j\,0\rangle$ & the right side expansion by ~~$\langle 0\,0|$~~ $\langle 1\,0|\langle 1\,0|$ we get:

$$a = \left(\langle 1\,0|\langle 1\,0|2\,0\rangle\right)^* = \langle 2\,0|1\,0\rangle|1\,0\rangle$$

$$= \frac{2}{\sqrt{6}}.$$

$$b = \left(\langle 1\,0|\langle 1\,0|1\,0\rangle\right)^* = 0$$

$$c = \left(\langle 1\,0|\langle 1\,0|0\,0\rangle\right)^* = -\frac{1}{\sqrt{3}}.$$

$$\text{So; } \underbrace{|1\,0\rangle|1\,0\rangle}_{\text{given 2 particle state}} = \underbrace{\frac{2}{\sqrt{6}}|2\,0\rangle - \frac{1}{\sqrt{3}}|0\,0\rangle}_{\text{expansion in basis of } J^2.}$$

$\therefore$ j values 0 & 2 have non zero amplitude in the expansion.

The probabilities of finding those j^2 be:

$$\left.\begin{aligned} P(j=0) &= \frac{1}{3} \\ P(j=2) &= \frac{2}{3} \end{aligned}\right\} \text{ on a measurement of } J^2.$$

Ans

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12.b Here there is no external field. So the conserved quantities are L_1^2, L_2^2 & J^2, J_z but not (m_{l_1}, m_{l_2}).

$$\vec{J} = \vec{L}_1 + \vec{L}_2 \Rightarrow J^2 = L_1^2 + L_2^2 + 2\vec{L}_1 \cdot \vec{L}_2$$

$$\Rightarrow \vec{L}_1 \cdot \vec{L}_2 = \frac{1}{2}(J^2 - L_1^2 - L_2^2)$$

∴ Eigstate of H are simultaneously eigenstates of J^2, L_1^2, L_2^2. But those are the coupled basis: $|2\,2\rangle, |2\,1\rangle, \ldots, |0\,0\rangle$

(evaluated in part a.)

Let in terms of the eigenbasis of $\hat{H}$; the state vector at time t be given by:

$$|\psi(t)\rangle = \underbrace{a_1|2\,0\rangle + a_2|0\,0\rangle}_{\text{which were present in } |\psi\rangle \text{ at } t=0} + \underbrace{b_1|2\,2\rangle + b_2|2\,1\rangle + b_3|2\,{-1}\rangle + \cdots \text{ etc}}_{\text{which were not present in } |\psi(t=0)\rangle}$$

Now
$$\hat{H}|j\,m_j\rangle = \frac{r}{2}(J^2 - L_1^2 - L_2^2)|j\,m_j\rangle$$
$$= \frac{r\hbar^2}{2}\left(j(j+1) - 1(1+1) - 1(1+1)\right)|j\,m_j\rangle$$
$$= \frac{r\hbar^2}{2}\left(j(j+1) - 4\right)|j\,m_j\rangle$$

So here we get:

$$\hat{H}|\psi(t)\rangle = \frac{r\hbar^2}{2}\Big[(2\times 3-4)|2\,0\rangle a_1 + (0-4)|0\,0\rangle a_2$$

$$+ b_1'|2\,2\rangle + b_2'|2\,1\rangle + \cdots \text{etc}\Big]$$

$$\big[b_i' = b_i\,(j(j+1)-4)\big]$$

$$= \frac{r\hbar^2}{2}\Big[2|2\,0\rangle a_1 - 4|0\,0\rangle a_2 + b_1'|2\,2\rangle + b_2'|2\,1\rangle + \cdots \text{etc}\Big]$$

but $\hat{H}|\psi(t)\rangle = i\hbar\dfrac{\partial\psi}{\partial t}$

$$= i\hbar\left(\dot{a}_1|2\,0\rangle + \dot{a}_2|0\,0\rangle + \dot{b}_1|2\,2\rangle + \dot{b}_2|2\,1\rangle + \cdots \text{etc}\right)$$

So comparing coefficients gives:

$$i\hbar\dot{a}_1 = \frac{r\hbar^2}{2}\cdot 2a_1 = a_1 r\hbar^2.$$

$$\Rightarrow \int_{a_{1_0}}^{a_1(t)} \frac{da_1}{a_1} = -ir\hbar\int_0^t dt \qquad \text{but } a_1(0) = \sqrt{\tfrac{2}{3}}.$$

$$\Rightarrow a_1(t) = \sqrt{\tfrac{2}{3}}\exp\left(-ir t\hbar\right)$$

~~Similarly~~ $\cancel{a_2(t) = a_2(0)\exp}$

Similarly: $i\hbar\dot{a}_2 = -2r\hbar^2 a_2$

$$\Rightarrow \int_{a_2(0)}^{a_2(t)} \frac{da_2}{a_2} = \int_0^t 2i\hbar r\,dt \qquad a_2(0) = \frac{1}{\sqrt{3}}.$$

$$\Rightarrow a_2(t) = \frac{1}{\sqrt{3}}\exp\left(2i\hbar r t\right)$$

for $b_1, b_2, \dots$ etc

we get:

$$b_i(t) = b_i(0)\exp(\text{some complex factor})$$

but $b_i(0) = 0$ for all i

$\Rightarrow b_i(t) = 0$ for all i

i.e $|\psi(t)\rangle = \frac{\sqrt{2}}{\sqrt{3}}\exp(-i\gamma t\hbar)|20\rangle + \frac{1}{\sqrt{3}}\exp(2i\gamma\hbar t)|00\rangle$

So; $|\langle\psi(t)|\psi(0)\rangle|^2$

$$= \left|\left(\sqrt{\frac{2}{3}}\exp(+i\gamma t\hbar), \frac{1}{\sqrt{3}}\exp(-2i\gamma t\hbar)\right)\cdot\begin{pmatrix}\sqrt{\frac{2}{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}\right|^2$$

$$= \left|\frac{2}{3}\exp(+i\gamma t\hbar) + \left(\frac{1}{\sqrt{3}}\right)^2\exp(-2i\gamma t\hbar)\right|^2$$

$$= \left|\left\{\frac{2}{3}\cos(\gamma t\hbar) + \frac{1}{3}\cos(2\gamma t\hbar)\right\} + \left\{\frac{2i}{3}\sin(\gamma t\hbar) - \frac{i}{3}\sin(2\gamma t\hbar)\right\}\right|^2$$

$$= \left(\frac{2}{3}\cos(K) + \frac{1}{3}\cos(2K)\right)^2 + \left(\frac{2}{3}\sin K - \frac{1}{3}\sin 2K\right)^2$$

$(K = \gamma t\hbar)$

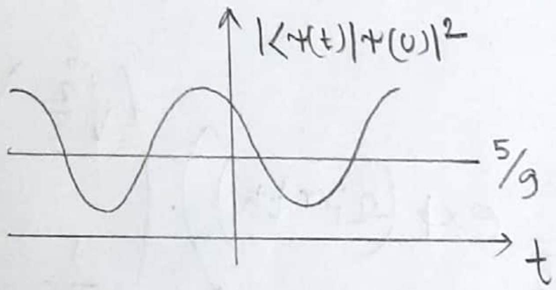
$$= \frac{4}{9}\left(\cos^2 k + \sin^2 k\right) + \frac{1}{9}\left(\cos^2(2k) + \sin^2(2k)\right)$$
$$+ 2 \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \left(\cos 2k \cos k - \sin 2k \sin k\right)$$

$$= \frac{5}{9} + \frac{4}{9} \cdot \cos(3k).$$

~~Clearly~~ $= \frac{5}{9} + \frac{4}{9} \cos(3\gamma t\hbar)$

clearly $\max\left(|\langle\psi(t)|\psi(0)\rangle|^2\right) = 1$; which is obvious.

At $t = 0$: $|\langle\psi(t)|\psi(0)\rangle|^2 = 1$; again obvious.



The quantity is periodic with a period

$$T = \frac{2\pi}{3\gamma\hbar}$$

Ans

At $t = T/2$:

$$|\langle\psi(t)|\psi(0)\rangle|^2 = \frac{5}{9} + \frac{4}{9}\cos\pi$$

$$= \frac{1}{9} =$$ minimum of the quantity.

Ans

2: Quantum H.O in E-M field:—

The original Hamiltonian: $H_0 = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} m\omega_0^2 r^2.$

In presence of magnetic field $\vec{B} = B_0 \hat{z} \sin(\omega t)$

I choose the Gauge field:

$$\vec{A} = (y B_0 \sin(\omega t), 0, 0)$$

$\therefore$ The new $\hat{H}$ gives:

$$\hat{H} = \frac{(p_x - q y B_0 \sin\omega t)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2} m\omega_0^2 r^2$$

$$= \underbrace{\frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega_0^2 r^2}{2}}_{H_0} + \underbrace{\frac{q^2 B_0^2 y^2 \sin^2\omega t}{2m} - \frac{p_x q B_0 y \sin(\omega t)}{2m}}_{H'}$$

$(\because [y, p_x] = 0)$

$$\therefore H' = \frac{q^2 B_0^2 \sin^2 \omega t}{2m} \cdot y^2 - \frac{q B_0 \sin\omega t}{2m} \cdot (y p_x)$$

Now, $\langle n_1, n_2, n_3 | H' | 0\,0\,0 \rangle$

($|n_1, n_2, n_3\rangle$ be quantum state of the actual oscillator with energy $(n_1 + n_2 + n_3 + \frac{3}{2})\hbar\omega$)

$= \langle n_1 n_2 n_3 | \alpha y^2 - \beta y p_x | 0\,0\,0 \rangle$

(α, β are those coefficients)

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Now; $\hat{y} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(a_y^+ + a_y^-\right)$; $p_x = i\sqrt{\frac{\hbar m\omega}{2}}\left(a_x^+ - a_x^-\right)$

$$\therefore \hat{y}^2 = \frac{\hbar}{2m\omega}\left(a_y^{+2} + a_y^{-2} + a_y^+ a_y^- + a_y^- a_y^+\right)$$

So; $\langle m_1 m_2 m_3 | H' | 0\,0\,0\rangle$

$$= \langle m_1 m_2 m_3 | \alpha' \left(a_y^{+2} + a_y^{-2} + a_y^+ a_y^- + a_y^- a_y^+ | 0\,0\,0\right\rangle$$

$$- \langle m_1 m_2 m_3 | \beta' \left(a_y^+ a_x^+ + a_y^- a_x^- - a_y^- a_x^+ - a_y^+ a_x^- | 0\,0\,0\right\rangle$$

$$\left(\hat{y} p_x = \frac{i\hbar}{2}\left(a_y^+ a_x^+ - a_y^+ a_x^- - a_y^- a_x^+ + a_y^- a_x^-\right)\right.$$

$$= \alpha' \left\{\langle m_1 m_2 m_3 | \sqrt{1\cdot 2} | 0\,2\,0\rangle + 0 + 0 + \langle m_1 m_2 m_3 | 0\,0\,0\rangle\right\}$$

$$- \beta' \left\{\langle m_1 m_2 m_3 | 1\,1\,0\rangle + 0 + 0 + 0\right\}$$

Using $\left\{\begin{array}{l} a_i^+ |m_i\, m_j\, m_k\rangle = \sqrt{m_i+1}\,|m_i+1\;\; m_j\, m_k\rangle \\ a_i^- |\cdots\rangle = \sqrt{m_i}\;|m_i-1 \cdots\rangle \end{array}\right\}$

So except $\langle m_1\, m_2\, m_3| = \langle 0\,0\,0|$; the only two non-zero terms present are given by $\langle 1\,1\,0|$ & $\langle 0\,2\,0|$

So: the allowed transitions from $|000\rangle$ (ground state) be:

$$\left.\begin{array}{l} |000\rangle \longrightarrow |020\rangle \\ |000\rangle \longrightarrow |110\rangle \end{array}\right\}$$ due to perturbation of 1st order.

ii) transition amplitude

$$P_{\substack{|000\rangle \\ |020\rangle}} = -\frac{i}{\hbar}\int_0^t \langle 020|H'|000\rangle e^{i\omega_c t}\,dt$$

$$\left(\omega_c = \frac{E_2 - E_1}{\hbar} = \frac{2\hbar\omega_0}{\hbar} = 2\omega_0\right)$$

$$= -\frac{i}{\cancel{\hbar}}\int_0^t \sqrt{2}\cdot\frac{\cancel{\hbar}}{2m\omega_0}\cdot\frac{q^2B_0^2\sin^2\omega t}{2m}\,e^{2i\omega_0 t}\,dt$$

$$= -\frac{\sqrt{2}\,q^2B_0^2}{4m^2\omega_0}\,\frac{i}{2}\int_0^t (1-\cos 2\omega t)e^{2i\omega_0 t}\,dt.$$

$$= -\frac{\sqrt{2}\,q^2B_0^2 i}{8m^2\omega_0}\left\{\int_0^t e^{2i\omega_0 t}\,dt - \frac{1}{2}\int_0^t e^{2i(\omega+\omega_0)t}\,dt - \frac{1}{2}\int_0^t e^{2i(\omega-\omega_0)t}\,dt\right\}$$

$$= -\frac{\sqrt{2}\,q^2B_0^2 i}{8m^2\omega_0}\left\{\frac{e^{2i\omega_0 t}-1}{\omega_0} - \frac{1}{2}\,\frac{e^{2i(\omega+\omega_0)t}-1}{\omega+\omega_0} - \frac{1}{2}\,\frac{e^{2i(\omega-\omega_0)t}-1}{\omega-\omega_0}\right\}$$

So, the transition rate be given by:

$$\tilde{R}_{|020\rangle} = \frac{P_{|020\rangle}}{t} = \frac{|b_{|020\rangle}|^2}{t}$$

(This is very long algebra; I've found out the value using Mathematica.)

$$\tilde{R}_{|020\rangle} = \frac{P_{|020\rangle}(t)}{t}$$

(evaluated avg $\tilde{R}$ instead of $\tilde{R}_0 = \frac{dP(t)}{dt}$)

$$= \frac{B^2 q^2}{16\sqrt{2}\, m^2 \omega_0^3 t}\Bigg[\left(\frac{2}{\omega_0} + \frac{1}{\omega - \omega_0} - \frac{1}{\omega+\omega_0} - \frac{2\cos(2t\omega_0)}{\omega_0} + \frac{\cos(2t(\omega-\omega_0))}{\omega_0-\omega} + \frac{\cos(2t(\omega_0+\omega))}{\omega_0+\omega}\right)^2 + \left(\frac{\sin(2t(\omega_0-\omega))}{\omega_0-\omega} + \frac{\sin(2t(\omega_0+\omega))}{\omega_0+\omega} - \frac{2\sin(2\omega_0 t)}{\omega_0}\right)^2\Bigg]^{1/2}$$

The actual (instantaneous) transition rate at time t be

~~Prob~~ probability to find the system in $|020\rangle$ at time t be given by:

$$P_{|020\rangle}(t) = |p_{|020\rangle}|^2$$

And the transition rate at time t:

$$\tilde{R}_{|020\rangle} = \frac{\partial}{\partial t} P_{|020\rangle}(t)$$

using Mathematica (as it is very complicated by hand.) the expression of $\tilde{R}_{|020\rangle}$ be given by:

$$\tilde{R}_{|020\rangle} = \frac{B_0^4 q^4}{512 m^4 \omega_0^2}\Bigg[2\Bigg\{\Bigg(\frac{2}{\omega_0} + \frac{1}{\omega-\omega_0} + \frac{1}{\omega+\omega_0} - \frac{\cos(2\omega_0 t)}{\omega_0} + \frac{\cos(2t(\omega_0-\omega))}{\omega_0-\omega} + \frac{\cos(2t(\omega_0+\omega))}{\omega_0+\omega}\Bigg) \times \Big(4\sin(2\omega_0 t) - 2\sin(2t(\omega_0-\omega)) - 2\sin(2t(\omega_0+\omega))\Big)\Bigg\} +$$

$$\Bigg\{2\Big(2\cos(2t(\omega_0-\omega)) + 2\cos(2t(\omega_0+\omega)) - 4\cos(2\omega_0 t)\Big) \times \Bigg(\frac{\sin(2t(\omega_0+\omega))}{\omega_0+\omega} + \frac{\sin(2t(\omega_0-\omega))}{\omega_0-\omega} - \frac{2\sin(2\omega_0 t)}{\omega_0}\Bigg)\Bigg\}\Bigg]$$

Ans

Similarly for $|110\rangle$

transition amplitude:

$$p_{|110\rangle} = -\frac{i}{\hbar}\int_0^t \langle 110|H'|000\rangle e^{i\omega_c' t}\, dt$$

$$\omega_c = \frac{E(110) - E(000)}{\hbar} = 2\omega_0 \text{ (again)}$$

$$= \frac{i\cancel{\hbar}}{2} \cdot \frac{qB_0}{2m} \cdot \frac{i}{\cancel{\hbar}} \int_0^t \sin(\omega t)\, e^{2i\omega_0 t}\, dt.$$

Again using mathematica I found out:

$$P_{|110\rangle} = |p_{|110\rangle}|^2 \quad \text{And} \quad \tilde{R}_{|110\rangle}(t) = \frac{\partial P_{|110\rangle}(t)}{\partial t}.$$

$$\tilde{R}_{|110\rangle}(t) =$$

$$\frac{B_0^2 q^2}{4m^2(\omega^2 - 4\omega_0^2)^2} \Big[\Big(\omega^2 \cos(2\omega_0 t) \sin(\omega t) - 4\omega_0^2 \cos(2\omega_0 t) \sin(\omega t) \Big) \times \Big(\omega - \omega \cos(2\omega_0 t) \cos(\omega t) - 2\omega_0 \sin(2\omega_0 t) \sin(\omega t) \Big) \Big]$$

Ans

Problem:- 3

a) Due to L-S coupling; the coupling Hamiltonian gives: $H \sim \bar{L} \cdot \bar{S}$

but $J = L + S \Rightarrow J^2 = L^2 + S^2 + 2L \cdot S$

$$\Rightarrow L \cdot S = \frac{J^2 - L^2 - S^2}{2}$$

Here for $n = 3$: $l = 0, 1, 2$; s is always $3/2$.

Now
$$H' = \underbrace{\frac{e^2}{4\pi\epsilon_0} \frac{1}{m^2 c^2 r^3}}_{= x \,=\, \text{const (let)}} \bar{S} \cdot \bar{L} = \frac{e^2 (J^2 - L^2 - S^2)}{4\pi\epsilon_0 m^2 c^2 r^3 \times 2}.$$

$$\Rightarrow \langle H' \rangle_{lmj} = x \cdot \left\langle \frac{1}{r^3} \right\rangle \cdot \langle S \cdot L \rangle$$

$$= x_0 \frac{j(j+1) - l(l+1) - s(s+1)}{l(l+1)(l+1/2)} \quad (s = 3/2)$$

$$= x_0 \frac{j(j+1) - l(l+1) - 15/4}{l(l+1/2)(l+1)} = (E_1) \text{ so.}$$

$\therefore$ For $l = 0$:

$j = 3/2$; $E_1' = 0$

for $l = 1$: $j = 1/2, 3/2, 5/2$

$j = 1/2 \Rightarrow E_1' = -\frac{10}{6} x_0$

$j = 3/2 \Rightarrow E_1' = -\frac{2}{3} x_0.$

$j = \frac{5}{2} \Rightarrow E_1' = \chi_0.$

for $l = 2$: $j = 7/2, \ldots, 1/2$.

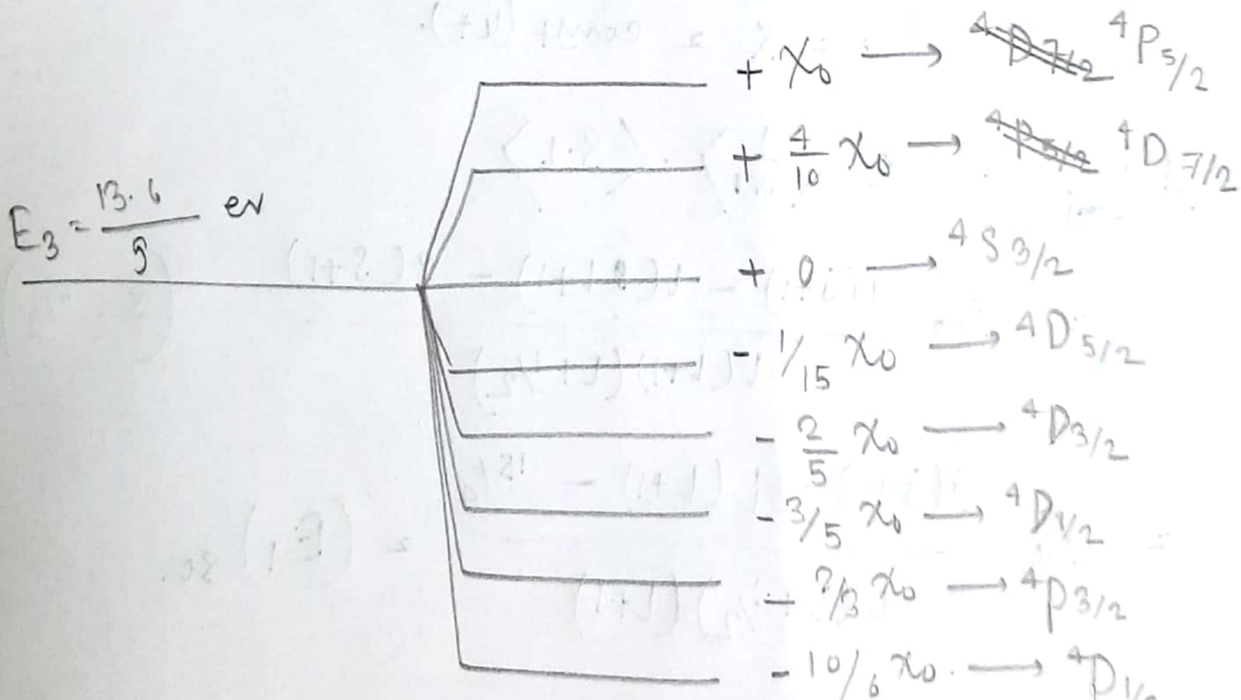
for $j = 1/2$: $E_1' = -\frac{6}{10}\chi_0$

$j = 3/2$: $E_1' = -4/_{10}\,\chi_0$

$j = 5/2$: $E_1' = -\frac{1}{15}\chi_0$

$j = 7/2$: $E_1' = \frac{4}{10}\chi_0.$

So the level E_3 $(n=3)$ splits into 8 lines.



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3.6 For multi e^- atom; there are L·S coupling between different e^- present in the atom.

In our universe S is $\cancel{3}$ $1/2$.

$\therefore$ j changes from $\left(l+\frac{1}{2}\right)$ to $\left|l-\frac{1}{2}\right|$

In that universe S is $3/2$.

$\therefore$ j changes from $\left|l+3/2\right|$ to $\left|l-3/2\right|$

So depending on value of l; there will be some extra terms.

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Here total $S = S_1 + S_2 = 0, 1, \cancel{2}$

in that universe $S = 0, 1, 2, 3.$

So for the extra 2 S values $(2,3)$; some more states will appear in parallel universe.

Ans

3.c $S = 3/2 \Rightarrow (2S+1) = 4.$

So for given n, l, there are 4 spin state.

$\therefore$ for given n there are $4n^2$ 'electronic configuration. As ~~full ext~~ filled shell gives inert gas; so the gases will have e^- no:

$\sum_i 4n_i^2$ (~~and~~ $n = 1, 2, \cdots$) (All shells upto n_{max} are filled.)

$\therefore$ 1st inert gas: $4n^2 = 4 \times 1^2 = 4 \rightarrow$ Be

~~2nd " " : $4n^2 = 4 \times 2^2 = 16 \rightarrow$~~

2nd inert gas: $4n_1^2 + 4n_2^2 \Big|_{(1,2)}$

$= 4 \times 1^2 + 4 \times 2^2$

$= 4 + 16 = 20\ e^- \rightarrow$ Ca

Ans

Problem: 4

4.a.i) For the state $2p3p$:

$S_1 = S_2 = \frac{1}{2}$; $l_1 = l_2 = 1$

$\therefore$ $S = 0, 1$; $l = 0, 1, 2.$

For $S = 0$: $l = 0, 1, 2 \Rightarrow j = 0, 1, 2$

The states are ${}^1S_0$, ${}^1P_1$, ${}^1D_2$

$S = 1$: $l = 0 \Rightarrow j = 1$: state: ${}^3S_1$

$S = 1$: $l = 1 \Rightarrow j = 0, 1, 2$: states: ${}^3P_0$, ${}^3P_1$, ${}^3P_2$

$S = 1$; $l = 2 \Rightarrow j = 1, 2, 3$: states: ${}^3d_1$, ${}^3d_2$, ${}^3d_3$.

So there are 10 states:

${}^1S_0$, ${}^1P_1$, ${}^1d_2$, ${}^3S_1$, ${}^3P_0$, ${}^3P_1$, ${}^3P_2$, ${}^3d_1$, ${}^3d_2$, ${}^3d_3$.

Splitting:-

i) s-s Splitting: $S = 0, 1$.

The $\overline{\Delta\nu}$ for s-s splitting be given by:

$$\overline{\Delta\nu}_s = -a[s(s+1) - s_1(s_1+1) - s_2(s_2+1)]$$

for $s = 0$: $\overline{\Delta\nu}_s = -a[0 - \frac{3}{4} - \frac{3}{4}] = \frac{3a}{2}$

$s = 1$: $\overline{\Delta\nu}_s = -a[2 - \frac{3}{4}\times 2] = -\frac{a}{2}$

ii) for L-L splitting: $\overline{\Delta\nu}_L = -m[L(L+1) - l_1(l_1+1) - l_2(l_2+1)]$

here $l_1 = l_2 = 1$; $L = 0, 1, 2$.

for $L = 0$: $\overline{\Delta\nu} = -m[0 - 2 - 2] = 4m$

$L = 1$: $\overline{\Delta\nu} = -m[2 - 2 - 2] = 2m$

$L = 2$: $\overline{\Delta\nu} = -m[6 - 2 - 2] = -2m$

iii) Splitting of sp orbital: $\overline{\Delta\nu} = x[J(J+1) - L(L+1) - S(S+1)]$

So splitting of 3p levels: $S = 1$; $L = 1$, $J = 0, 1, 2$

$\therefore \overline{\Delta\nu}\big|_{J=0} = x[0 - 2 - 2] = -4x$

for $J = 1$: $\overline{\Delta\nu} = x[2 - 2 - 2] = -2x$

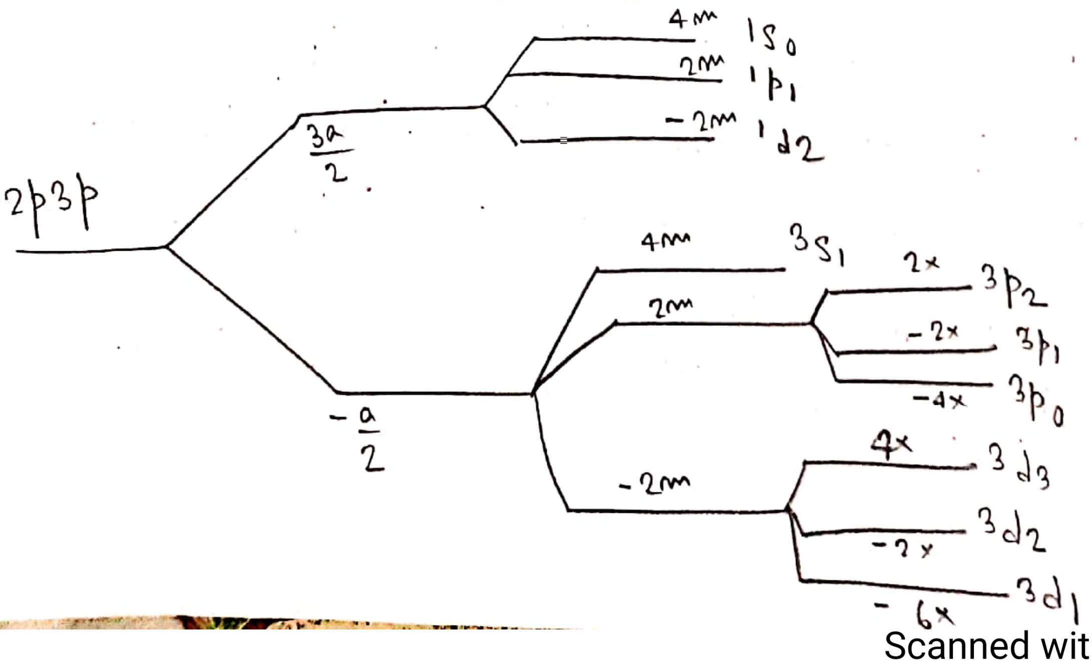
$J = 2$: $\overline{\Delta\nu} = x[6 - 2 - 2] = +2x$

for 3d levels: $S = 1$; $L = 2$; $J = 1, 2, 3$.

$J = 1$: $\overline{\Delta\nu} = x[2 - 6 - 2] = -6x$

$J = 2$: $\overline{\Delta\nu} = x[6 - 6 - 2] = -2x$

$J = 3$: $\overline{\Delta\nu} = x[12 - 6 - 2] = 4x$



4.b.

Electric dipole selection rule here given by:

$\Delta l = \pm 1$; $\Delta m = 0, \pm 1$, $\Delta S = 0$

for $j = 0$ no transition is possible here.

4.c. According to selection rule the transition from $^3p_1$ possible are:

$^3S_1$ & $^3d_{1,2,3}$.

And the forbiddens are:

$^1S_0$, $^1d_2$, $^3p_{0,1,2}$, $^1p_1$. Ans

Problem :- 1

a. Here parity is being violated.
When we apply π on s ; it does not change sign. But $\vec{p}$ changes sign.

$$\pi(\vec{s}\cdot\vec{p}) = \vec{s}\cdot\pi(\vec{p}) = -\vec{s}\cdot\vec{p}.$$

So in the reverse world; $\vec{s}$ is antiparallel to it's momentum; i.e violates parity.

b: i) $\langle 2 1 0 | x | 2 0 0\rangle$

$$= \int_0^\infty r^3 R_{21}(r)\cdot R_{20}(r)\, dr \cdot \int_{\theta,\phi} Y_{10}^* \cdot Y_{00} \cdot \sin\theta \cos\phi \, d\phi.$$

The ϕ integral gives: $\int_0^{2\pi} \cos\phi \, d\phi = 0.$

So; $\langle 210|x|200\rangle = 0.$

$|210\rangle$ doesn't have ϕ term in it. So it gives $Y_{10}(\theta,\phi)$. But as x contains $\cos\phi$

so; this ~~cos~~ can be written as in terms of $Y_1^{\pm 1}(\theta,\phi)$.

As spherical Harmonics are orthogonal f^n:
So; the whole space integral gives 0.

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b ii) $p_z = -i\hbar \frac{\partial}{\partial z}$

in terms of (r, θ, ϕ) this can be written as:

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}.$$

$$\therefore \langle 210 | p_z | 200 \rangle = -i\hbar \langle 210 | \cos\theta \frac{\partial}{\partial r} | 200 \rangle$$

$$+ i\hbar \langle 210 | \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} | 200 \rangle$$

for the 1st $\langle \dots \rangle$ i.e $\langle 210 | \cos\theta \frac{\partial}{\partial r} | 200 \rangle$

the θ integral gives:

$$\sim \int_0^\pi P_{10}(\cos\theta) \cdot P_{00}(\cos\theta) \cdot \cos\theta \sin\theta \, d\theta$$

the r integral gives:

$$\sim \int_0^\infty R_{21}(r) \left(\frac{\partial}{\partial r} R_{20}(r) \right) r^2 \, dr$$

$$= \int_0^\infty r^3 \exp\left(-\frac{r}{2a}\right) \cdot \frac{\partial}{\partial r}\left(\left(1 - \frac{r}{2a}\right) e^{-r/2a} \right) dr$$

Using mathematica, I find the integral to be zero.

$$\therefore \langle 210 | \cos\theta \frac{\partial}{\partial r} | 200 \rangle = 0.$$

the 2nd integral gives.

$$\langle 210 | \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} | 200 \rangle$$

The θ integral gives $\sim \int_0^\pi P_{10}(\cos\theta) \cdot \sin^2\theta \frac{\partial}{\partial \theta} P_{00}(\cos\theta) \, d\theta$

But $P_{00}(\cos\theta) = 1$

$\therefore \frac{\partial}{\partial\theta}(P_{00}) = 0$ i.e $\langle 210| \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}|200\rangle = 0.$

So; $\langle 210|p_z|200\rangle = 0.$ Ans

iii) $j = 9/2$; $l = 4$; $s = 1/2$ (obvious)

$\therefore j = (4 + \frac{1}{2}) \longrightarrow m_j = 9/2, 7/2, \ldots, -9/2.$

clearly $|\frac{9}{2}\ \frac{9}{2}\rangle = |4\ 4\rangle|\frac{1}{2}\ \frac{1}{2}\rangle$

$\Rightarrow J^-|\frac{9}{2}\ \frac{9}{2}\rangle = L^-|4\ 4\rangle|\frac{1}{2}\ \frac{1}{2}\rangle$
$+ S^-|4\ 4\rangle|\frac{1}{2}\ \frac{1}{2}\rangle$

$\Rightarrow \sqrt{9\times(\frac{9}{2}+1-\frac{9}{2})}\,|\frac{9}{2}\ \frac{7}{2}\rangle = \sqrt{(4+4)(4-4+1)}\,|4\ 3\rangle|\frac{1}{2}\ \frac{1}{2}\rangle$
$+ \sqrt{1\times(\frac{1}{2}+1-\frac{1}{2})}\,|4\ 4\rangle|\frac{1}{2}\ -\frac{1}{2}\rangle$

$\Rightarrow 3|\frac{9}{2}\ \frac{7}{2}\rangle = \sqrt{8}\,|4\ 3\rangle|\frac{1}{2}\ \frac{1}{2}\rangle + |4\ 4\rangle|\frac{1}{2}\ -\frac{1}{2}\rangle$

$\Rightarrow |\frac{9}{2}\ \frac{7}{2}\rangle = \frac{2\sqrt{2}}{3}|4\ 3\rangle|\frac{1}{2}\ \frac{1}{2}\rangle + \frac{1}{3}|4\ 4\rangle|\frac{1}{2}\ -\frac{1}{2}\rangle$

This state is $j = 9/2$, $m_j = 7/2$.

$\therefore L_z|\frac{9}{2}\ \frac{7}{2}\rangle = \frac{2\sqrt{2}}{\cancel{3}}\times\cancel{3}\hbar\,|4\ 3\rangle|\frac{1}{2}\ \frac{1}{2}\rangle + \frac{4\hbar}{3}|4\ 4\rangle|\frac{1}{2}\ -\frac{1}{2}\rangle$

$\therefore \langle L_z\rangle = \langle\frac{9}{2}\ \frac{7}{2}|L_z|\frac{9}{2}\ \frac{7}{2}\rangle \cancel{= \left(2\sqrt{2} + \frac{4}{3}\right)\hbar}$

$= \frac{2\sqrt{2}\times 2}{3}\hbar + \frac{4\hbar}{3}\times\frac{1}{3} = \hbar\left(\frac{4\sqrt{2}}{3} + \frac{4}{3}\right)$ Ans

iv. Singlet $= \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$; triplet $(m=0) = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}$.

Now, if singlet & triplet be represented by:

$$\left.\begin{aligned} |S_{singlet}\rangle = |S_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} ; \\ |S_{triplet}\rangle = |S_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \end{aligned}\right\}$$

$$\left(|++\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} ; \ |+-\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} ; \ |-+\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} ; \ |--\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right)$$

$$\therefore \ S_z^{e^-} = S_z^{e^-} \otimes \mathbb{I}^{e^+} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{pmatrix}$$

$$S_z^{e^+} = \mathbb{I}^{e^+} \otimes S_z^{e^+} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}1 & & & 0\\ & -1 & & \\ & & 1 & \\0 & & & -1\end{pmatrix}$$

$$\therefore \ \underbrace{S_z^{e^-} - S_z^{e^+}}_{\Downarrow} = \frac{\hbar}{2}\begin{pmatrix}0 & & & 0\\ & 2 & & \\ & & -2 & \\0 & & & 0\end{pmatrix}$$

$$\langle S_0 | \hat{A} | S_1\rangle = \frac{\hbar}{4}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}\begin{pmatrix}0 & & & 0\\ & 2 & & \\ & & -2 & \\0 & & & 0\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}\begin{pmatrix}0\\2\\-2\\0\end{pmatrix} = \frac{\hbar}{4} \times (2+2)$$

$$= \hbar$$ <u>Ans</u>

V. $\vec{S}_1 \otimes \vec{S}_2 = S_{x_1} \otimes S_{x2} + S_{y_1} \otimes S_{y2} + S_{z_1} \otimes S_{z2}$.

$$S_{x_1} \otimes S_{x2} = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{4\times4}$$

$$S_{y_1} \otimes S_{y2} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$S_{z_1} \otimes S_{z2} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So; $\vec{S}_1 \otimes \vec{S}_2 = \sum_i S_{x_i1} \otimes S_{x_i2}$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In ground state (singlet configuration $\frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$)

i.e $|\chi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$: $\langle S_1 \otimes S_2 \rangle$

$$= \frac{\hbar^2}{4} \times \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{8} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 3 \\ 0 \end{pmatrix} = \frac{\hbar^2}{8} \times (-6) = -\frac{3\hbar^2}{4}$$ Ans

c. i) $(3z^2 - r^2) \sim T_0^2$ $\quad (\because z^2, r^2 \rightarrow \text{power } 2)$

$\therefore \langle n'l'm_l'm_s' | 3z^2 - r^2 | n l m_l m_s \rangle$

$\sim \langle n'l'm_l'm_s' | T_0^2 | n l m_l m_s \rangle$

$\neq 0$ for $l' = l+2$; $m_l' = m_l$; $m_s' = m_s$.

$\therefore$ Selection rule: $\Delta l = 2$; $\Delta m_l = \Delta m_s = 0$.

ii) $xy \sim T_2^2 - T_{-2}^2$

$\therefore \langle n'l'm_l'm_s' | xy | n l m_l m_s \rangle$

$\sim \langle n'l'm_l'm_s' | T_2^2 - T_{-2}^2 | n l m_l m_s \rangle$

$\neq 0$ if $l' = l+2$; $m_l' = m_l \pm 2$; $m_s' = m_s$.

$\therefore$ Selection rule $\Delta l = 2$; $\Delta m_l = \pm 2$; $\Delta m_s = 0$.

d. $xy = \frac{1}{2i}(T_2^2 - T_{-2}^2)$

$\therefore \langle 322 | xy | 300 \rangle = \frac{1}{2i} \langle 322 | T_2^2 - T_{-2}^2 | 300 \rangle$

$$= \frac{1}{2i} \left\{ \langle 02; 02 | 02; 22 \rangle \cdot \frac{\langle 32 \| T^2 \| 30 \rangle}{\sqrt{1}} - \langle 02; 02 | 02; 22 \rangle \frac{\langle 32 \| T^2 \| 30 \rangle}{\sqrt{1}} \right\}$$

$$\therefore \langle 32 \| T^2 \| 30 \rangle = \frac{2Ai}{\langle 02\, 02 | 02\, 22 \rangle - \langle 02\, 02 | 02\, 22 \rangle}$$

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$$= XA(\text{Say})$$

$$X = \frac{2i}{\langle 02\, 02 | 02\, 22\rangle - \langle 02\, 0{-2} | 02\, 22\rangle}$$

So; i) $\cancel{\langle 32\, m | 300\rangle}$ $\langle 32m | xy | 300\rangle$

$$= \frac{1}{2i} \langle 32m | T^2_2 - T^2_{-2} | 300\rangle$$

$$= \frac{1}{2i}\left[\langle 02\, 02 | 02\, 2m\rangle \frac{\langle 32 \| T^2 \| 30\rangle}{1} - \langle 02\, 0\bar{2} | 02\, 2m\rangle \langle 32 \| T^2 \| 30\rangle\right]$$

$$= \left(\langle 02\, 02 | 02\, 2m\rangle - \langle 02\, 0{-2} | 02\, 2m\rangle\right) \cdot XA.$$

ii) $\langle 32m | xz | 300\rangle = \frac{1}{2} \langle 32m | T^2_{-1} - T^2_1 | 300\rangle$

$$= \frac{1}{2}\left[\langle 02\, 0{-1} | 02\, 2m\rangle \frac{\langle 32 \| T^2 \| 30\rangle}{1} - \langle 02\, 01 | 02\, 2m\rangle \langle 32 \| T^2 \| 30\rangle\right]$$

$$= \left(\langle 02\, 0{-1} | 02\, 2m\rangle - \langle 02\, 01 | 02\, 2m\rangle\right) XAi.$$

iii) $\langle 32m | yz | 300\rangle = -\frac{1}{2i} \langle 32m | T^2_1 + T^2_{-1} | 300\rangle$

$$= -\frac{1}{2i}\left[\langle 02\, 01 | 02\, 2m\rangle \langle 32 \| T^2 \| 30\rangle + \langle 02\, 0{-1} | 02\, 2m\rangle \langle 32 \| T^2 \| 30\rangle\right]$$

$$= -XA\left[\langle 02\, 01 | 02\, 2m\rangle + \langle 02\, 0{-1} | 02\, 2m\rangle\right]$$

iv) $\langle 3\,2\,m | x^2 | 3\,0\,0 \rangle = \sqrt{\frac{2\pi}{15}} \langle 3\,2\,m | T^2_2 + T^2_{-2} | 3\,0\,0 \rangle$

$= \sqrt{\frac{2\pi}{15}} \cdot 2i \cdot \frac{1}{2i} \Big[\langle 0\,2\;0\,2 | 0\,2\;2\,m \rangle + \langle 0\,2\;0\,{-2} | 0\,2\;2\,m \rangle \Big] \langle 3\,2 \| T^2 \| 3\,0 \rangle$

$= 2i \sqrt{\frac{2\pi}{15}}\, XA \Big[\langle 0\,2\;0\,2 | 0\,2\;2\,m \rangle + \langle 0\,2\;0\,{-2} | 0\,2\;2\,m \rangle \Big]$

~~$\langle 3\,0\,m | z^2 | 3\,0\,0 \rangle = \frac{2\sqrt{\pi}}{3} T^0_0$~~

v) $\langle 3\,2\,m | z^2 | 3\,0\,0 \rangle \rightarrow \left\langle \frac{2\sqrt{\pi}}{3} T^0_0 + \frac{4\sqrt{\pi}}{3\sqrt{5}} T^2_0 \right\rangle$

$\therefore \langle 3\,2\,m | z^2 | 3\,0\,0 \rangle$

$= \frac{2\sqrt{\pi}}{3} \langle 3\,2\,m | T^0_0 | 3\,0\,0 \rangle + \frac{4\sqrt{\pi}}{3\sqrt{5}} \langle 3\,2\,m | T^2_0 | 3\,0\,0 \rangle$

$= \frac{2\sqrt{\pi}}{3} \cdot 2i \cdot XA \cdot \Big(\langle 0\,0\;0\,0 | 0\,0\;0\,m \rangle + \langle 0\,0\;0\,0 | 0\,0\;0\,m \rangle \Big)$

$+ \frac{4\sqrt{\pi}}{3\sqrt{5}} \cdot 2i \cdot XA \cdot \Big(\langle 0\,2\;2\,0 | 0\,2\;2\,m \rangle + \langle 0\,2\;2\,0 | 0\,2\;2\,m \rangle \Big)$

$$= \left. \begin{aligned} &\frac{4\sqrt{\pi}\, i AX}{3} \langle 0\,0\;0\,0 | 0\,0\;0\,m \rangle \\ &+ \frac{16\sqrt{\pi}\, i XA}{3\sqrt{5}} \langle 0\,2\;2\,0 | 0\,2\;2\,m \rangle \end{aligned} \right\}$$