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Problem: 1

1.a The spin singlet state for two e^- be given by:
$s=0$; $m=0$ i.e $\{|\frac{1}{2}\ \frac{1}{2}\rangle - |\frac{1}{2}\ -\frac{1}{2}\rangle\}/\sqrt{2}$

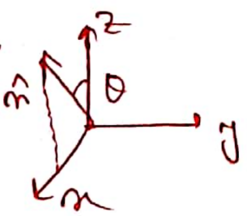
or; $|0\ 0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$.

Now for $S=0$; the spin ~~matrices~~ matrices are (1×1) & hence they are just numbers. So they will commute.

On other way we can say that for $S=0$:
$S^2|\chi\rangle = 0|\chi\rangle$; $S_z|\chi\rangle = 0|\chi\rangle$
($|\chi\rangle$ = spinor in 1 dimension)
And using that S^2 & S_z are both 1 dim square matrix; we get $S^2=0$; $S_z=0$
Similarly $S_+=0$; $S_-=0$; $S_x=0$; $S_y=0$
& $[S_+, S_-] = 2\hbar S_z = 0$

Now for the given problem; the state $|0\ 0\rangle$ can't be raised or lowered with $S_\pm$; makes the complete set in $S=0$ space.
So; $|0,0\rangle$ makes the complete set in $S=0$ space.
So; for rotation as 's' value is conserved and the coefficients for different 'm' (of the kets $|s\ m\rangle$) changes according to wigner D matrix; here the state will be kept unchanged as there is only one possible m value.
i.e $\mathcal{D}(R)|0\ 0\rangle = |0\ 0\rangle$

We can also proof this mathematically.
Here the rotation of axis is $(\theta, \phi = 0)$



$\therefore \hat{n} = \cos\theta \cdot \hat{z} + \sin\theta \cdot \hat{x}$

$\therefore \vec{S} \cdot \hat{n} = \sin\theta \, \hat{S}_x + \cos\theta \cdot \hat{S}_z$

If the rotation angle be β; then.

$$\mathcal{D}_{\beta}^{(0)}(R) = \exp\left\{-\frac{i\beta}{\hbar}\left(\sin\theta \cdot \hat{S}_x + \cos\theta \, \hat{S}_z\right)\right\} = e^{-\frac{i\beta \vec{S} \cdot \hat{n}}{\hbar}}$$

$$= \exp\left\{-\frac{i\beta}{\hbar} \sin\theta \, \hat{S}_x\right\} \cdot \exp\left\{-\frac{i\beta}{\hbar} \cos\theta \, \hat{S}_z\right\}$$

(Only for $S=0$ we can do this as $[S_z, S_x] = 0$)

So; $\mathcal{D}_{\beta}^{(0)}(R) \, |0 \; 0\rangle$

$$= \exp\left\{-\frac{i\beta}{\hbar} \sin\theta \, \hat{S}_x\right\} \cdot \exp\left\{-\frac{i\beta}{\hbar} \cos\theta \cdot \hat{S}_z\right\} |0 \; 0\rangle$$

but $|0 \; 0\rangle$ being eigenstate of $\hat{S}_z$ with eigenval 0:

$$\mathcal{D}_{\beta}^{(0)}(R) \, |0 \; 0\rangle = \exp\left\{-\frac{i\beta}{\hbar} (\sin\theta) \cdot \hat{S}_x\right\} \cdot \exp\left\{-\frac{i\beta \cos\theta \cdot 0}{\hbar}\right\} \cdot |0 \; 0\rangle$$

$$= \exp\left\{-\frac{i\beta}{\hbar} \sin\theta \, \hat{S}_x\right\} |0 \; 0\rangle$$

$$= \exp\left(-\frac{i\beta \sin\theta}{2\hbar} \left(S_+ + S_-\right)\right) |0 \; 0\rangle$$

$$= \exp\left(-\frac{i\beta \sin\theta}{2\hbar} S_+\right) \cdot \exp\left(-\frac{i\beta \sin\theta}{2\hbar} S_-\right) |0 \; 0\rangle$$

$(\because [S_+ \, S_-] = 2\hbar S_z = 0 \text{ for } S = 0)$

$$= \exp\left(-\frac{i\beta \sin\theta\, S_+}{2\hbar}\right)\cdot\left(1 + \left(\frac{-i\beta\sin\theta\, S_-}{2\hbar}\right) + \frac{1}{2!}\left(\frac{-i\beta\sin\theta\, S_-}{2\hbar}\right)^2 + \cdots\right)|0\ 0\rangle$$

$$= \exp\left(-\frac{i\beta\sin\theta\, S_+}{2\hbar}\right)|0\ 0\rangle \quad [\because S_-|0\ 0\rangle = 0]$$

$$= \left[1 + \left(\frac{-i\beta\sin\theta}{2\hbar} S_+\right) + \frac{1}{2!}\left(\frac{-i\beta\sin\theta}{2\hbar} S_+\right)^2 + \cdots\right]|0\ 0\rangle$$

$$[\because S_+|0\ 0\rangle = 0]$$

$$= |0\ 0\rangle$$

So; $\mathcal{D}^{(0)}_{\beta}(R)\,|0\ 0\rangle = |0\ 0\rangle$

i.e the state is invarient under rotation.

1.b.i) For the state $2p\,3p$:

$S_1 = S_2 = \frac{1}{2}$; $l_1 = l_2 = 1$

$\therefore$ $S = 0, 1$; $l = 0, 1, 2$.

For $S = 0$: $l = 0, 1, 2 \Rightarrow j = 0, 1, 2$

The states are ${}^1S_0$, ${}^1P_1$, ${}^1D_2$

$S = 1$: $l = 0 \Rightarrow j = 1$: state : ${}^3S_1$

$S = 1$: $l = 1 \Rightarrow j = 0, 1, 2$: states : ${}^3P_0$, ${}^3P_1$, ${}^3P_2$

$S = 1$; $l = 2 \Rightarrow j = 1, 2, 3$: states : ${}^3d_1$, ${}^3d_2$, ${}^3d_3$.

So there are 10 states,

${}^1S_0$, ${}^1P_1$, ${}^1d_2$, ${}^3S_1$, ${}^3P_0$, ${}^3P_1$, ${}^3P_2$, ${}^3d_1$, ${}^3d_2$, ${}^3d_3$.

Splitting:-

i) s-s Splitting: $S = 0, 1$.

The $\overline{\Delta\nu}$ for s-s splitting be given by:

$$\overline{\Delta\nu}_s = -a[s(s+1) - s_1(s_1+1) - s_2(s_2+1)]$$

for $s = 0$: $\overline{\Delta\nu}_s = -a\left[0 - \frac{3}{4} - \frac{3}{4}\right] = \frac{3a}{2}$

$s = 1$: $\overline{\Delta\nu}_s = -a\left[2 - \frac{3}{4}\times 2\right] = \frac{a}{2}$

ii) for l-l splitting, $\overline{\Delta\nu}_L = -m[L(L+1) - l_1(l_1+1) - l_2(l_2+1)]$

here $l_1 = l_2 = 1$; $L = 0, 1, 2$.

for $L = 0$: $\overline{\Delta\nu} = -m[0 - 2 - 2] = 4m$

$L = 1$: $\overline{\Delta\nu} = -m[2 - 2 - 2] = 2m$

$L = 2$: $\overline{\Delta\nu} = -m[6 - 2 - 2] = -2m$

iii) Splitting of sp orbital: $\overline{\Delta\nu} = x[J(J+1) - L(L+1) - S(S+1)]$

So splitting of 3p levels: $S = 1$; $L = 1$, $J = 0, 1, 2$

$\therefore \overline{\Delta\nu}\big|_{J=0} = \alpha[0 - 2 - 2] = -4x$

for $J = 1$: $\overline{\Delta\nu} = x[2 - 2 - 2] = -2x$

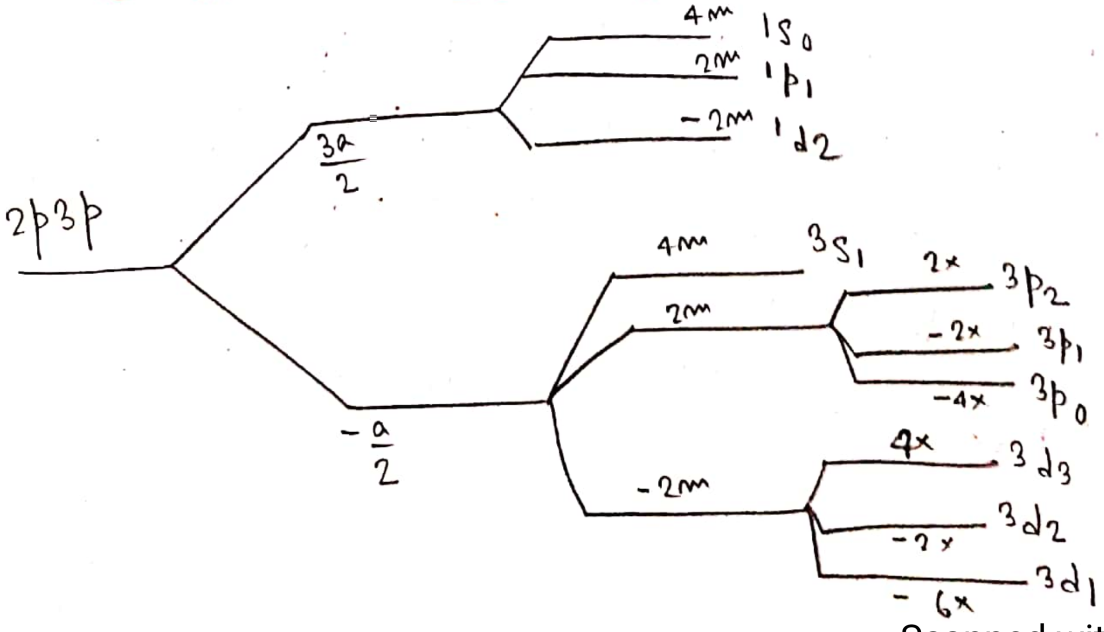
$J = 2$: $\overline{\Delta\nu} = \alpha[6 - 2 - 2] = +2x$

for 3d levels: $S = 1$; $L = 2$; $J = 1, 2, 3$.

$J = 1$: $\overline{\Delta\nu} = x[2 - 6 - 2] = -6x$

$J = 2$: $\overline{\Delta\nu} = x[6 - 6 - 2] = -2x$

$J = 3$: $\overline{\Delta\nu} = x[12 - 6 - 2] = 4x$



1.c $\hat{H} = -\gamma \vec{B}\cdot\hat{S} = -\frac{eB_x}{m} S_x$

Clearly $[\hat{H}, S_x] = 0$ and hence the eigenstates of $\hat{H}$ be: $|\uparrow x\rangle$: with energy $-\frac{e\hbar B}{2m}$

$|\downarrow x\rangle$: " " $+\frac{e\hbar B}{2m}$

given that $|\alpha(0)\rangle = |\uparrow\rangle = \frac{|\uparrow x\rangle + |\downarrow x\rangle}{\sqrt{2}}$

So $|\alpha(t)\rangle = |\alpha(0)\rangle \cdot e^{-\frac{iEt}{\hbar}}$

$$= \frac{1}{\sqrt{2}} \exp\left(-\frac{i\gamma Bt}{2}\right)|\uparrow\rangle_x + \frac{1}{\sqrt{2}} \exp\left(\frac{i\gamma Bt}{2}\right)|\downarrow\rangle_x$$

So the probability to find $|\downarrow\rangle$ at a later time t be given by:

$$P = |\langle\downarrow|\alpha(t)\rangle|^2$$

$$= \left|\left\{\frac{(\langle\uparrow x| - \langle\downarrow x|)}{\sqrt{2}}\right\}\cdot\left\{\frac{1}{\sqrt{2}}\exp\left(-\frac{i\gamma Bt}{2}\right)|\uparrow x\rangle + \frac{1}{\sqrt{2}}\exp\left(\frac{i\gamma Bt}{2}\right)|\downarrow x\rangle\right\}\right|^2$$

$$= \left|\frac{1}{2}\exp\left(-\frac{i\gamma Bt}{2}\right) - \frac{1}{2}\exp\left(\frac{i\gamma Bt}{2}\right)\right|^2$$

$$= \sin^2\left(\frac{\gamma Bt}{2}\right) = \sin^2\left(\frac{eBt}{2m}\right)$$

The spin will flip from $|\uparrow\rangle$ to $|\downarrow\rangle$ when the probability to find $|\downarrow\rangle$ will become 1.

i.e $\sin^2\left(\frac{eBt}{2m_e}\right) = 1 = \sin^2\left(\frac{\pi}{2}\right)$.

i.e $\frac{eBt}{2m} = \frac{\pi}{2}$ or $t = \frac{\pi m}{eB}$.

putting values:

$$t = \frac{\pi \times 9.11 \times 10^{-31}}{1.6 \times 10^{-19} \times 10^{-2}} \, s \quad \left(\because 100 \text{ gauss} = 10^{-2}\, T\right)$$

$= 17.89 \times 10^{-10}\, s$

$= 1.789 \times 10^{-9}\, s$.

1.d. $j_1 = 1$; $j_2 = \frac{1}{2}$

So; $J = \frac{3}{2}, \frac{1}{2}$.

$\boxed{I}$ Now for $j = \frac{3}{2}$: $m = \pm\frac{3}{2}$; $\pm\frac{1}{2}$.

$$|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = |1\ 1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle \quad \text{(obviously)}$$

$$\Rightarrow J^-|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = J_1^-|1\ 1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + J_2^-|1\ 1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$\Rightarrow \sqrt{\left(\tfrac{3}{2}+\tfrac{3}{2}\right)\left(\tfrac{3}{2}-\tfrac{3}{2}+1\right)}\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{(1+1)(1-1+1)}\ |1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)\left(\tfrac{1}{2}-\tfrac{1}{2}+1\right)}\ |1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\Rightarrow \sqrt{3}\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{2}\ |1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + |1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\Rightarrow |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\ |1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\ |1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

and obviously:

$$|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = |1\ -1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\Rightarrow J^+|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = J_1^+|1\ -1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + J_2^+|1\ -1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\Rightarrow \sqrt{\left(\tfrac{3}{2}+\tfrac{3}{2}\right)\left(\tfrac{3}{2}-\tfrac{3}{2}+1\right)}\ |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \sqrt{(1+1)(1+1-1)}\ |1\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)\left(\tfrac{1}{2}-\tfrac{1}{2}+1\right)}\ |1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$\Rightarrow \sqrt{3}\ |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \sqrt{2}\ |1\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + |1\ -1\rangle|\tfrac{1}{2}\ +\tfrac{1}{2}\rangle$$

$$\Rightarrow |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\ |1\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\ |1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle.$$

$\boxed{II}$ for $j = \frac{1}{2}$: $m = \pm\frac{1}{2}$.

let $|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = C_1\,|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + C_2\,|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$$\Rightarrow J^+|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = 0 = C_1 J_1^+|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \cancelto{0}{C_1 J_2^+|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle} + \cancelto{0}{C_2 J_1^+|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle} + C_2 J_2^+|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$\Rightarrow 0 = c_1\sqrt{(1-0)(1+1-0)}\,|1\ 1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + 0 + 0 +$
$c_2\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}+1-\tfrac{1}{2})}\,|1\,1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$\Rightarrow 0 = c_1\sqrt{2} + c_2 \quad \Rightarrow c_2 = -c_1\sqrt{2}.$

but $c_1^2 + c_2^2 = 1 \Rightarrow c_1^2 + 2c_1^2 = 1 \Rightarrow c_1 = \frac{1}{\sqrt{3}}$ i.e $c_2 = -\sqrt{\frac{2}{3}}.$

$\therefore |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1\,1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

And let:
$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = c_1|1\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + c_2|1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$\Rightarrow J^-|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = 0 = c_1\left(J_1^-|1\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + \cancel{J_2^-|1\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle}^{\,0}\right)$
$+ c_2\left(J_1^-|1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + J_2^-|1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle\right)$

$= c_1\sqrt{(1+0)(1+1-0)}\,|1\ -1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + c_2\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}+1-\tfrac{1}{2})}\,|1\ -1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$\Rightarrow \sqrt{2}c_1 + c_2 = 0 \quad \Rightarrow c_2 = -c_1\sqrt{2}$

but $c_1^2 + c_2^2 = 1 \Rightarrow c_1^2 + 2c_1^2 = 1 \Rightarrow c_1 = \frac{1}{\sqrt{3}}$ i.e $c_2 = -\sqrt{\frac{2}{3}}.$

So;
$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}|1\ 0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1\ -1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

Finding the CG coefficients for $J_1 = 1$; $J_2 = \frac{3}{2}$

(I use $|\,\rangle$ for uncoupled state & $|\,\rangle|\,\rangle$ for coupled state)

i.e. $J_1 = 1$; $J_2 = 3/2$.

$$J = \underbrace{\tfrac{5}{2}, \tfrac{3}{2}}, \tfrac{1}{2}.$$

For, $J = \frac{5}{2}$: $m = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}$.

$$\left|\tfrac{5}{2}, \tfrac{5}{2}\right) = |1\ 1\rangle \cdot \left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle \quad \text{(obviously)}$$

$$J^-\left|\tfrac{5}{2}, \tfrac{5}{2}\right) = \left(J_1^- + J_2^-\right)\left(|1\ 1\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle\right)$$

$$\Rightarrow \sqrt{\left(\tfrac{5}{2}+\tfrac{5}{2}\right)\cdot\left(\tfrac{5}{2}-\tfrac{5}{2}+1\right)}\left|\tfrac{5}{2}\cdot\tfrac{3}{2}\right) = \sqrt{(1+1)(1-1+1)}\,|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + \sqrt{\left(\tfrac{3}{2}+\tfrac{3}{2}\right)\left(\tfrac{3}{2}-\tfrac{3}{2}+1\right)}\,|1\ 1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle$$

$$\Rightarrow \sqrt{5}\left|\tfrac{5}{2}\ \tfrac{3}{2}\right) = \sqrt{2}\,|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + \sqrt{3}\,|1\ 1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle$$

$$\Rightarrow \left|\tfrac{5}{2}\ \tfrac{3}{2}\right) = \frac{\sqrt{2}}{\sqrt{5}}|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + \frac{\sqrt{3}}{\sqrt{5}}|1\ 1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle \checkmark$$

$$J^-\left|\tfrac{5}{2}\ \tfrac{3}{2}\right) = \frac{\sqrt{2}}{\sqrt{5}}\left(J_1^-|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + J_2^-|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle\right) + \frac{\sqrt{3}}{\sqrt{5}}\left(J_1^-|1\ 1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + J_2^-|1\ 1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle\right)$$

$$\sqrt{\left(\tfrac{5}{2}+\tfrac{3}{2}\right)\left(\tfrac{5}{2}-\tfrac{3}{2}+1\right)}\left|\tfrac{5}{2}\ \tfrac{1}{2}\right) = \sqrt{\tfrac{2}{5}}\left(\sqrt{(1+0)(1-0+1)}\,|1\ -1\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + \sqrt{\left(\tfrac{3}{2}+\tfrac{3}{2}\right)\cdot\left(\tfrac{3}{2}-\tfrac{3}{2}+1\right)}\,|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle\right) +$$

$$\sqrt{\tfrac{3}{5}}\left(\sqrt{(1+1)(1-1+1)}\,|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{\left(\tfrac{3}{2}+\tfrac{1}{2}\right)\left(\tfrac{3}{2}-\tfrac{1}{2}+1\right)}\,|1\ 1\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle\right)$$

$$\sqrt{\cancel{2}\cdot\tfrac{4}{\cancel{2}}}\left|\tfrac{5}{2}\ \tfrac{1}{2}\right) = \sqrt{\tfrac{2}{5}}\cdot\sqrt{2}\,|1\ -1\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{5}}|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \frac{\sqrt{3}\cdot\sqrt{2}}{\sqrt{5}}|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{\tfrac{3}{5}}\cdot 2\,|1\ 1\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle$$

$$\Rightarrow \cancel{2}\sqrt{2}\left|\tfrac{5}{2}\ \tfrac{1}{2}\right) = \frac{\cancel{2}}{\sqrt{5}}|1\ -1\rangle\left|\tfrac{3}{2}\cdot\tfrac{3}{2}\right\rangle + \cancel{2}\frac{\sqrt{6}}{\sqrt{5}}|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \cancel{2}\cdot\sqrt{\tfrac{3}{5}}\,|1\ 1\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{5}{2}\ \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{10}}|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + \frac{\sqrt{6}}{\sqrt{10}}|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{\tfrac{3}{10}}|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle$$

$$J^-\left|\tfrac{5}{2}\ \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{10}}\left[J_1^-|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle + J_2^-|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle\right]$$

$$+\sqrt{\tfrac{6}{10}}\left[J_1^-|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + J_2^-|1\ 0\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle\right]$$

$$+\sqrt{\tfrac{3}{10}}\left[J_1^-|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle + J_2^-|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle\right]$$

$$\Rightarrow \sqrt{\left(\tfrac{5}{2}+\tfrac{1}{2}\right)\left(\tfrac{5}{2}-\tfrac{1}{2}+1\right)}\left|\tfrac{5}{2}\ {-\tfrac{1}{2}}\right\rangle = \frac{1}{\sqrt{10}}\left[0+\sqrt{\left(\tfrac{3}{2}+\tfrac{3}{2}\right)\left(\tfrac{3}{2}-\tfrac{3}{2}+1\right)}|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle\right]$$

$$+\sqrt{\tfrac{6}{10}}\left[\sqrt{(1+0)(1-0+1)}|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{\left(\tfrac{3}{2}+\tfrac{1}{2}\right)\left(\tfrac{3}{2}-\tfrac{1}{2}+1\right)}|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle\right]$$

$$+\sqrt{\tfrac{3}{10}}\left[\sqrt{(1+1)(1-1+1)}|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle + \sqrt{\left(\tfrac{3}{2}-\tfrac{1}{2}\right)\left(\tfrac{3}{2}+\tfrac{1}{2}+1\right)}|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{3}{2}}\right\rangle\right]$$

$$\Rightarrow \sqrt{3\cdot 3}\left|\tfrac{5}{2}\ {-\tfrac{1}{2}}\right\rangle = \frac{1}{\sqrt{10}}\cdot\sqrt{3}|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{\tfrac{6}{10}}\cdot\sqrt{2}|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \sqrt{\tfrac{3}{10}}\cdot\sqrt{2}|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle$$

$$+\sqrt{2\cdot 2}\cdot\frac{\sqrt{6}}{\sqrt{10}}|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle + \sqrt{\tfrac{3}{10}}\cdot\sqrt{3}|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{3}{2}}\right\rangle$$

$$\Rightarrow \left|\tfrac{5}{2}\ {-\tfrac{1}{2}}\right\rangle\cdot\sqrt{3} = \frac{\sqrt{3}\,|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle}{\sqrt{10}} + \frac{2}{\sqrt{10}}|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle$$

$$+|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle\left[\frac{2\sqrt{2}}{10}+\frac{\sqrt{2}}{10}\right] + \sqrt{\tfrac{3}{10}}|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{3}{2}}\right\rangle$$

$$\Rightarrow \left|\tfrac{5}{2}\ {-\tfrac{1}{2}}\right\rangle = \sqrt{\tfrac{3}{10}}|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + \frac{\sqrt{1}}{\sqrt{10}}|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle$$

$$+\frac{1}{\sqrt{10}}|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{3}{2}}\right\rangle$$

$$J^-\left|\tfrac{5}{2}\ {-\tfrac{1}{2}}\right\rangle = \sqrt{\tfrac{3}{10}}\left[J_1^-|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + J_2^-|1\ {-1}\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle\right]$$

$$+\sqrt{\tfrac{6}{10}}\left[J_1^-|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle + J_2^-|1\ 0\rangle\left|\tfrac{3}{2}\ {-\tfrac{1}{2}}\right\rangle\right]$$

$$+\sqrt{\tfrac{1}{10}}\left[J_1^-|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{3}{2}}\right\rangle + J_2^-|1\ 1\rangle\left|\tfrac{3}{2}\ {-\tfrac{3}{2}}\right\rangle\right]$$

$$\Rightarrow \sqrt{\left(\frac{5}{2}-\frac{1}{2}\right)\left(\frac{5}{2}+\frac{1}{2}+1\right)}\left|\frac{5}{2}\ \frac{-3}{2}\right\rangle = \sqrt{\frac{3}{10}}\left\{0+\sqrt{\left(\frac{3}{2}+\frac{1}{2}\right)\left(\frac{3}{2}-\frac{1}{2}+1\right)}\left|1\ 1\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle\right\}$$

$$+\sqrt{\frac{6}{10}}\left\{\sqrt{(1+0)(1-0+1)}\left|1\ 1\right\rangle\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle+\sqrt{\left(\frac{3}{2}-\frac{1}{2}\right)\left(\frac{3}{2}+\frac{1}{2}+1\right)}\left|1\ 0\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle\right\}$$

$$+\sqrt{\frac{1}{10}}\left\{\sqrt{(1+1)(1-1+1)}\left|1\ 0\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle+0\right\}$$

$$\Rightarrow \sqrt{2\cdot 4}\left|\frac{5}{2}\ \frac{-3}{2}\right\rangle = \sqrt{\frac{3}{10}}\cdot 2\left|1\ 1\right\rangle\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle+\sqrt{\frac{6}{10}}\cdot\sqrt{2}\left|1\ 1\right\rangle\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle$$

$$+\sqrt{\frac{6}{10}}\cdot\sqrt{3}\left|1\ 0\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle+\frac{1}{\sqrt{10}}\cdot\sqrt{2}\left|1\ 0\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle$$

$$\Rightarrow 2\left|\frac{5}{2}\ \frac{-3}{2}\right\rangle = \left(\sqrt{\frac{6}{10}}+\sqrt{\frac{6}{10}}\right)\left|1\ 1\right\rangle\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle$$

$$+\left(\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{10}}\left|1\ 0\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle\right)$$

$$\Rightarrow \left|\frac{5}{2}\ \frac{-3}{2}\right\rangle = \sqrt{\frac{6}{10}}\left|1\ 1\right\rangle\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle+\frac{2}{\sqrt{10}}\left|1\ 0\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle$$

And obviously.

$$\left|\frac{5}{2}\ \frac{-5}{2}\right\rangle = \left|1\ 1\right\rangle\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle$$

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$J = \frac{3}{2}$; $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$; Now let.

$$\left|\frac{3}{2}, \frac{3}{2}\right) = c_1 |1\ 1\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle + c_2 |1\ 0\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle$$

$$\cancel{c_1 = \left\langle 1\ 1 \left| \frac{3}{2}\ \frac{1}{2} \right| \frac{3}{2}\ \frac{3}{2}\right\rangle}$$

$$J_+ \left|\frac{3}{2}\ \frac{3}{2}\right) = 0 = c_1\left[\left(J_1^+ |1\ 1\rangle^{0} \left|\frac{3}{2}\ \frac{1}{2}\right\rangle + J_2^+ |1\ 1\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle\right)\right]$$
$$+ c_2 \left[J_1^+ |1\ 0\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle + J_2^+ |1\ 0\rangle^{0} \left|\frac{3}{2}\ \frac{3}{2}\right\rangle\right]$$

$$\Rightarrow 0 = c_1 \cdot \sqrt{\left(\frac{3}{2}-\frac{1}{2}\right)\left(\frac{3}{2}+\frac{1}{2}+1\right)}\ |1\,1\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle +$$
$$c_2 \sqrt{(1-0)(1+0+1)}\ |1\,1\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle$$

$$\Rightarrow c_1 \sqrt{1\cdot 3} + c_2\sqrt{2} = 0$$

$$\Rightarrow c_1 = -c_2 \cdot \sqrt{\frac{2}{3}} \quad ; \quad c_1^2 + c_2^2 = 1$$
$$\Rightarrow c_2^2 + \frac{2}{3}c_2^2 = 1$$
$$\Rightarrow \frac{5}{3} c_2^2 = 1 \Rightarrow c_2 = -\sqrt{\frac{3}{5}}$$
$$\Rightarrow c_1 = \sqrt{\frac{2}{5}}$$

$$\therefore \left|\frac{3}{2}\ \frac{3}{2}\right) = \sqrt{\frac{2}{5}}\ |1\ 1\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle - \sqrt{\frac{3}{5}}\ |1\ 0\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle$$

$$J^- \left|\frac{3}{2}\ \frac{3}{2}\right) = \sqrt{\frac{2}{5}}\left[J_1^- |1\,1\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle + J_2^- |1\,1\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle\right]$$
$$- \sqrt{\frac{3}{5}}\left[J_1^- |1\ 0\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle + J_2^- |1\ 0\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle\right]$$

$$\sqrt{\left(\frac{3}{2}+\frac{3}{2}\right)\left(\frac{3}{2}-\frac{3}{2}+1\right)} \left|\frac{3}{2}\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left[\sqrt{(1+1)(1-1+1)}\ |1\,0\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle\right.$$
$$\left. + \sqrt{\left(\frac{3}{2}+\frac{1}{2}\right)\left(\frac{3}{2}-\frac{1}{2}+1\right)}\ |1\ 1\rangle \left|\frac{3}{2}\ -\frac{1}{2}\right\rangle\right]$$
$$- \sqrt{\frac{3}{5}}\left[\sqrt{(1+0)(1-0+1)}\ |1\ -1\rangle \left|\frac{3}{2}\ \frac{3}{2}\right\rangle + \sqrt{\left(\frac{3}{2}+\frac{3}{2}\right)\left(\frac{3}{2}-\frac{3}{2}+1\right)}\ |1\ 0\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle\right.$$

$$\Rightarrow \sqrt{3}\left|\tfrac32\ \tfrac12\right\rangle = \sqrt{\tfrac25}\left[\sqrt2\,|1\ 0\rangle\left|\tfrac32\ \tfrac12\right\rangle + 2\,|1\ 1\rangle\left|\tfrac32\ -\tfrac12\right\rangle\right]$$

$$-\sqrt{\tfrac35}\left[\sqrt2\,|1\ -1\rangle\left|\tfrac32\ \tfrac32\right\rangle + \sqrt3\,|1\ 0\rangle\left|\tfrac32\ \tfrac12\right\rangle\right]$$

$$\Rightarrow \left|\tfrac32\ \tfrac12\right\rangle = \frac{2-3}{\sqrt{15}}|1\ 0\rangle\left|\tfrac32\ \tfrac12\right\rangle + \frac{2\sqrt2}{\sqrt{15}}|1\ 1\rangle\left|\tfrac32\ -\tfrac12\right\rangle$$

$$-\frac{\sqrt6}{\sqrt{15}}|1\ -1\rangle\left|\tfrac32\ \tfrac32\right\rangle$$

$$= -\frac{1}{\sqrt{15}}|1\ 0\rangle\left|\tfrac32\ \tfrac12\right\rangle + \frac{2\sqrt2}{\sqrt{15}}|1\ 1\rangle\left|\tfrac32\ -\tfrac12\right\rangle$$

$$-\sqrt{\tfrac{6}{15}}\,|1\ -1\rangle\left|\tfrac32\ \tfrac32\right\rangle \quad \checkmark$$

$$J_-\left|\tfrac32\ \tfrac12\right\rangle = -\frac{1}{\sqrt{15}}\left[J_1^-|1\ 0\rangle\left|\tfrac32\ \tfrac12\right\rangle + J_2^-|1\ 0\rangle\left|\tfrac32\ \tfrac12\right\rangle\right]$$

$$+\frac{2\sqrt2}{\sqrt{15}}\left[J_1^-|1\ 1\rangle\left|\tfrac32\ -\tfrac12\right\rangle + J_2^-|1\ 1\rangle\left|\tfrac32\ -\tfrac12\right\rangle\right]$$

$$-\sqrt{\tfrac{6}{15}}\left[J_1^-\cancel{|1\ -1\rangle}^{\,0}\left|\tfrac32\ \tfrac32\right\rangle + J_2^-|1\ -1\rangle\left|\tfrac32\ \tfrac32\right\rangle\right]$$

$$\Rightarrow \sqrt{\left(\tfrac32+\tfrac12\right)\left(\tfrac32-\tfrac12+1\right)}\left|\tfrac32\ -\tfrac12\right\rangle = -\frac{1}{\sqrt{15}}\left[\sqrt{(1+0)(1-0+1)}\,|1\ -1\rangle\left|\tfrac32\ \tfrac12\right\rangle\right.$$

$$\left.+\sqrt{\left(\tfrac32+\tfrac12\right)\left(\tfrac32-\tfrac12+1\right)}\,|1\ 0\rangle\left|\tfrac32\ -\tfrac12\right\rangle\right]$$

$$+\frac{2\sqrt2}{\sqrt{15}}\left[\sqrt{(1+1)(1-1+1)}\,|1\ 0\rangle\left|\tfrac32\ -\tfrac12\right\rangle \cancel{+\sqrt{\left(\tfrac32+\tfrac32\right)\left(\tfrac32-\tfrac32+1\right)}\,|1\ -1\rangle\left|\tfrac32\ \tfrac12\right\rangle}\right.$$

$$\left.\sqrt{\left(\tfrac32-\tfrac12\right)\left(\tfrac32+\tfrac12+1\right)}\,|1\ 1\rangle\left|\tfrac32\ -\tfrac32\right\rangle\right]$$

$$-\sqrt{\tfrac{6}{15}}\left[0+\sqrt{\left(\tfrac32+\tfrac32\right)\left(\tfrac32-\tfrac32+1\right)}\,|1\ -1\rangle\left|\tfrac32\ \tfrac12\right\rangle\right]$$

$$\Rightarrow 2\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = -\frac{1}{\sqrt{15}}\left[\sqrt{2}\,|1\,-1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle + 2\,|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle\right]$$

$$+\frac{2\sqrt{2}}{\sqrt{15}}\left[\sqrt{2}\,|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle + \sqrt{3}\,|1\,1\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle\right]$$

$$-\sqrt{\frac{6}{15}}\times\sqrt{3}\,|1\,-1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle.$$

$$= |1\,-1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle\left\{-\frac{\sqrt{2}}{\sqrt{15}}-\frac{\sqrt{18}}{\sqrt{15}}\right\}$$

$$+|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle\left\{\frac{4}{\sqrt{15}}-\frac{2}{\sqrt{15}}\right\}$$

$$+\frac{2\sqrt{6}}{\sqrt{15}}\,|1\,1\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$$

$$\Rightarrow\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = |1\,-1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle\left\{-\frac{1}{\sqrt{30}}-\frac{3}{\sqrt{30}}\right\}$$

$$+|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle\left\{\frac{\sqrt{2}}{\sqrt{30}}\right\}+\frac{2\sqrt{3}}{\sqrt{30}}\,|1\,1\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$$

$$= -\frac{4}{\sqrt{30}}\,|1\,-1\rangle\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle+\frac{\sqrt{2}}{\sqrt{30}}\,|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle$$

$$+\frac{2\sqrt{3}}{\sqrt{30}}\,|1\,1\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$$

$$\cancel{J_-\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = c_1\{J_1^-}$$

And let:

$$\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle = c_1\,|1\,-1\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle + c_2\,|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$$

$$\underbrace{J_-\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle}_{0} = c_1\left\{\underbrace{J_1^-|1\,-1\rangle}_{0}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle + J_2^-|1\,-1\rangle\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle\right\}$$

$$+c_2\left[J_1^-|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle + \underbrace{J_2^-|1\,0\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle}_{0}\right]$$

$$\Rightarrow 0 = c_1\cdot\sqrt{\left(\tfrac{3}{2}-\tfrac{1}{2}\right)\left(\tfrac{3}{2}+\tfrac{1}{2}+1\right)}\;|1\,-1\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$$

$$+c_2\sqrt{(1+0)(1-0+1)}\;|1\,-1\rangle\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$$

$$\cancel{2\, C_1\sqrt{\tfrac{3}{2}} = -C_2\sqrt{2} \Rightarrow C_2 = \tfrac{\sqrt{3}\,C_1}{4}}$$

$$\Rightarrow C_1\sqrt{3} = -C_2\sqrt{2} \Rightarrow C_1 = -C_2\sqrt{\tfrac{2}{3}}$$

$$C_1^2 + C_2^2 = 1 \Rightarrow \tfrac{2}{3}C_2^2 + C_2^2 = 1 \Rightarrow \tfrac{5}{3}C_2^2 = 1$$

$$\Rightarrow C_2 = \sqrt{\tfrac{3}{5}} \;;\; C_1 = -\sqrt{\tfrac{2}{5}}$$

$$\therefore \left|\tfrac{3}{2}\, -\tfrac{3}{2}\right\rangle = -\sqrt{\tfrac{2}{5}}\,|1\,-1\rangle\left|\tfrac{3}{2}\, -\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{3}{5}}\,|1\,0\rangle\left|\tfrac{3}{2}\, -\tfrac{3}{2}\right\rangle$$

Now for $j = \frac{1}{2}$; $m = \pm\frac{1}{2}$. Let.

$$\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = C_1\,|1\,0\rangle\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle + C_2\,|1\,-1\rangle\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle + C_3\,|1\,1\rangle\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle.$$

$$J_+\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = 0 = C_1\left[J_1^+|1\,0\rangle\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle + J_2^+|1\,0\rangle\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle\right]$$
$$+ C_2\left[J_1^+|1\,-1\rangle\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle + \cancel{J_2^+|1\,-1\rangle\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle}^{\,0}\right]$$
$$+ C_3\left[\cancel{J_1^+|1\,1\rangle}_{0}\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle + J_2^+|1\,1\rangle\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle\right]$$

$$= C_1\left[\sqrt{(1-0)(1+0+1)}\,|1\,1\rangle\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle + \sqrt{\left(\tfrac{3}{2}-\tfrac{1}{2}\right)\left(\tfrac{3}{2}+\tfrac{1}{2}+1\right)}\,|1\,0\rangle\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle\right]$$
$$C_2\left[\sqrt{(1+1)(1+1-1)}\,|1\,0\rangle\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle\right]$$
$$+ C_3\left[\sqrt{\left(\tfrac{3}{2}+\tfrac{1}{2}\right)\left(\tfrac{3}{2}-\tfrac{1}{2}+1\right)}\,|1\,1\rangle\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle\right]$$

$$= \left(\sqrt{2}\,C_1 + 2C_3\right)|1\,1\rangle\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle +$$
$$\left(\sqrt{2}\,C_2 + \sqrt{3}\,C_1\right)|1\,0\rangle\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle$$

So; $\sqrt{2}\,c_1 + 2c_3 = 0 \Rightarrow c_3 = -\frac{c_1}{\sqrt{2}}$.

$\sqrt{2}\,c_2 + \sqrt{3}\,c_1 = 0 \Rightarrow c_2 = -\frac{\sqrt{3}\,c_1}{\sqrt{2}}$.

$c_1^2 + c_2^2 + c_3^2 = 1$

$\Rightarrow c_1^2 + \frac{c_1^2}{2} + \frac{3c_1^2}{2} = 1 \Rightarrow 3c_1^2 = 1 \Rightarrow c_1 = \frac{1}{\sqrt{3}}$.

$\therefore c_2 = -\frac{1}{\sqrt{2}}\,;\ c_3 = -\frac{1}{\sqrt{6}}$.

$$\therefore \left|\tfrac12\ \tfrac12\right\rangle = \frac{1}{\sqrt{3}}|1\ 0\rangle\left|\tfrac32\ \tfrac12\right\rangle - \frac{1}{\sqrt{2}}|1\ {-1}\rangle\left|\tfrac32\ \tfrac32\right\rangle - \frac{1}{\sqrt{6}}|1\ 1\rangle\left|\tfrac32\ {-\tfrac12}\right\rangle \quad ✓$$

And finally let:

$$\left|\tfrac12\ {-\tfrac12}\right\rangle = c_1|1\ {-1}\rangle\left|\tfrac32\ \tfrac12\right\rangle + c_2|1\ 0\rangle\left|\tfrac32\ {-\tfrac12}\right\rangle + c_3|1\ 1\rangle\left|\tfrac32\ {-\tfrac32}\right\rangle$$

$$J^-\left|\tfrac12\ {-\tfrac12}\right\rangle = 0 = c_1\left[J_1^-|1\ {-1}\rangle\left|\tfrac32\ \tfrac12\right\rangle + J_2^-|1\ {-1}\rangle\left|\tfrac32\ \tfrac12\right\rangle\right] + J_2^-|1\ 0\rangle\left|\tfrac32\ {-\tfrac12}\right\rangle\Big]$$
$$+ c_2\left[J_1^-|1\ 0\rangle\left|\tfrac32\ {-\tfrac12}\right\rangle + J_2^-\left|\tfrac32\ {-\tfrac32}\right\rangle\right]$$
$$+ c_3\left[J_1^-|1\ 1\rangle\left|\tfrac32\ {-\tfrac32}\right\rangle + \cancel{|1\ {-1}\rangle\left|\tfrac32\ \tfrac12\right\rangle}\right]$$

$$\cancel{= c_1\left[0 + \sqrt{(1-1)(1+1-1)}\,|1\ {-1}\rangle\left|\tfrac32\ \tfrac12\right\rangle + |1\ {-1}\rangle\left|\tfrac32\ {-\tfrac12}\right\rangle\right.}$$
$$\cancel{+ c_2\left[\sqrt{(1-0)(1+0+1)}\ \sqrt{\left(\tfrac32-\tfrac12\right)\left(\tfrac32+\tfrac12+1\right)}\right.}$$

$$= C_1\left[0 + \sqrt{\left(\frac{3}{2}+\frac{1}{2}\right)\left(\frac{3}{2}+1-\frac{1}{2}\right)}\,|1\,-1\rangle\left|\frac{3}{2}\,-\frac{1}{2}\right\rangle\right] +$$

$$C_2\left[\sqrt{(1-0)(1+1+0)}\,|1\,-1\rangle\left|\frac{3}{2}\,-\frac{1}{2}\right\rangle + \sqrt{\left(\frac{3}{2}-\frac{1}{2}\right)\left(\frac{3}{2}+1+\frac{1}{2}\right)}\,|1\,0\rangle\left|\frac{3}{2}\,-\frac{3}{2}\right\rangle\right]$$

$$+ C_3\left[\sqrt{(1+1)(1+1-1)}\,|1\,0\rangle\left|\frac{3}{2}\,-\frac{3}{2}\right\rangle + 0\right]$$

$$\Rightarrow\quad 0 = |1\,-1\rangle\left|\frac{3}{2}\,-\frac{1}{2}\right\rangle\left[C_1\cdot 2 + \sqrt{2}\,C_2\right] + |1\,0\rangle\left|\frac{3}{2}\,-\frac{3}{2}\right\rangle\left[\sqrt{2}\,C_3 + \sqrt{3}\,C_2\right]$$

So; $\left.\begin{array}{l} 2C_1 + \sqrt{2}\,C_2 = 0 \\ \sqrt{2}\,C_3 + \sqrt{3}\,C_2 = 0 \end{array}\right\}$ i.e $\left.\begin{array}{l} C_1 = -\dfrac{C_2}{\sqrt{2}} \\ C_3 = -\dfrac{C_2\sqrt{3}}{\sqrt{2}} \end{array}\right\}$

but $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$

$$\Rightarrow \frac{C_2^2}{2} + \frac{3C_2^2}{2} + C_2^2 = 1 \Rightarrow 3C_2^2 = 1$$

$$\Rightarrow C_2 = \frac{1}{\sqrt{3}}\ ; \quad \text{i.e } C_1 = -\frac{1}{\sqrt{6}}\ ; \quad C_3 = \frac{1}{\sqrt{2}}.$$

So; $$\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle = -\frac{1}{\sqrt{6}}\,|1\,-1\rangle\left|\frac{3}{2}\,\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\,|1\,0\rangle\left|\frac{3}{2}\,-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\,|1\,1\rangle\left|\frac{3}{2}\,-\frac{3}{2}\right\rangle.$$

I was doing the problem informally in a rough work in my notebook. However as the calculations are so large & messy, I'm not going to copy them again on any paper. This is the rough I've done. However I'm finally rewritting all the C.G C's in a fresh way.

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1.e) Sterm Gerlash coefficients (finally) for $J_1 = 1; J_2 = \frac{3}{2}$

■ For $J = \frac{5}{2}$: $\left(m = \pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}\right)$

i) $\left|\frac{5}{2}\ \frac{5}{2}\right\rangle = |1\ 1\rangle\left|\frac{3}{2}\ \frac{3}{2}\right\rangle$

ii) $\left|\frac{5}{2}\ \frac{3}{2}\right\rangle = \sqrt{\frac{2}{5}}\,|1\ 0\rangle\left|\frac{3}{2}\ \frac{3}{2}\right\rangle + \sqrt{\frac{3}{5}}\,|1\ 1\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle$

iii) $\left|\frac{5}{2}\ \frac{1}{2}\right\rangle = \sqrt{\frac{1}{10}}\,|1\ {-1}\rangle\left|\frac{3}{2}\ \frac{3}{2}\right\rangle + \sqrt{\frac{6}{10}}\,|1\ 0\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle + \sqrt{\frac{3}{10}}\,|1\ 1\rangle\left|\frac{3}{2}\ {-\frac{1}{2}}\right\rangle$

iv) $\left|\frac{5}{2}\ {-\frac{1}{2}}\right\rangle = \sqrt{\frac{3}{10}}\,|1\ {-1}\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle + \sqrt{\frac{6}{10}}\,|1\ 0\rangle\left|\frac{3}{2}\ {-\frac{1}{2}}\right\rangle + \frac{1}{\sqrt{10}}\,|1\ 1\rangle\left|\frac{3}{2}\ {-\frac{3}{2}}\right\rangle$

v) $\left|\frac{5}{2}\ {-\frac{3}{2}}\right\rangle = \sqrt{\frac{6}{10}}\,|1\ {-1}\rangle\left|\frac{3}{2}\ {-\frac{1}{2}}\right\rangle + \frac{2}{\sqrt{10}}\,|1\ 0\rangle\left|\frac{3}{2}\ {-\frac{3}{2}}\right\rangle$

vi) $\left|\frac{5}{2}\ {-\frac{5}{2}}\right\rangle = |1\ {-1}\rangle\left|\frac{3}{2}\ {-\frac{3}{2}}\right\rangle$

■ For $J = \frac{3}{2}$: $\left(m = \pm\frac{3}{2}, \pm\frac{1}{2}\right)$:

i) $\left|\frac{3}{2}\ \frac{3}{2}\right\rangle = \sqrt{\frac{2}{5}}\,|1\ 1\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle - \sqrt{\frac{3}{5}}\,|1\ 0\rangle\left|\frac{3}{2}\ \frac{3}{2}\right\rangle$

ii) $\left|\frac{3}{2}\ \frac{1}{2}\right\rangle = -\frac{1}{\sqrt{15}}\,|1\ 0\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle + \frac{2\sqrt{2}}{\sqrt{15}}\,|1\ 1\rangle\left|\frac{3}{2}\ {-\frac{1}{2}}\right\rangle - \sqrt{\frac{6}{15}}\,|1\ {-1}\rangle\left|\frac{3}{2}\ \frac{3}{2}\right\rangle$

iii) $\left|\frac{3}{2}\ {-\frac{1}{2}}\right\rangle = \sqrt{\frac{2}{30}}\,|1\ 0\rangle\left|\frac{3}{2}\ {-\frac{1}{2}}\right\rangle - \frac{4}{\sqrt{30}}\,|1\ {-1}\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle + \frac{2\sqrt{3}}{\sqrt{30}}\,|1\ 1\rangle\left|\frac{3}{2}\ {-\frac{3}{2}}\right\rangle$

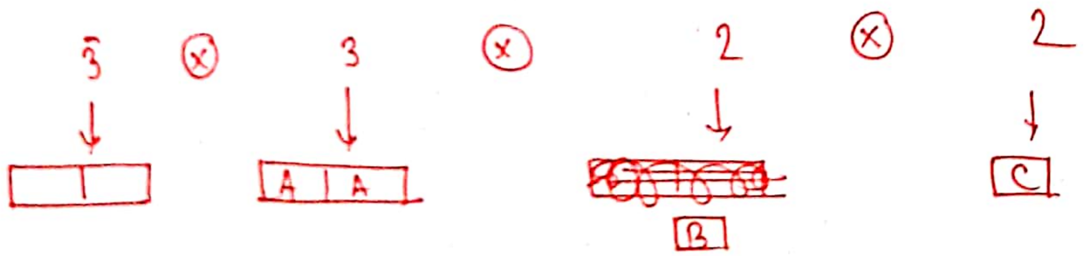
iv) $\left|\frac{3}{2}\ -\frac{3}{2}\right\rangle = -\sqrt{\frac{2}{5}}\,|1\ -1\rangle\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\,|1\ 0\rangle\left|\frac{3}{2}\ -\frac{3}{2}\right\rangle$

Ⅶ $J = \frac{1}{2}\ ;\ m = \pm\frac{1}{2}$

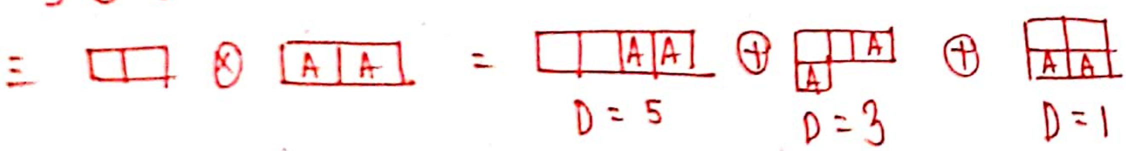
i) $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}|1\ 0\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}|1\ -1\rangle\left|\frac{3}{2}\ \frac{3}{2}\right\rangle - \frac{1}{\sqrt{6}}|1\ 1\rangle\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle$

ii) $\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = -\frac{1}{\sqrt{6}}|1\ -1\rangle\left|\frac{3}{2}\ \frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}|1\ 0\rangle\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}|1\ 1\rangle\left|\frac{3}{2}\ -\frac{3}{2}\right\rangle$

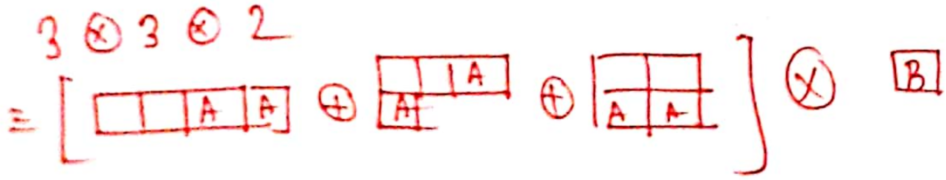
Ⅶ **1.f** Let the representations of the $3 \otimes 3 \otimes 2 \otimes 2$ blocks be given by:

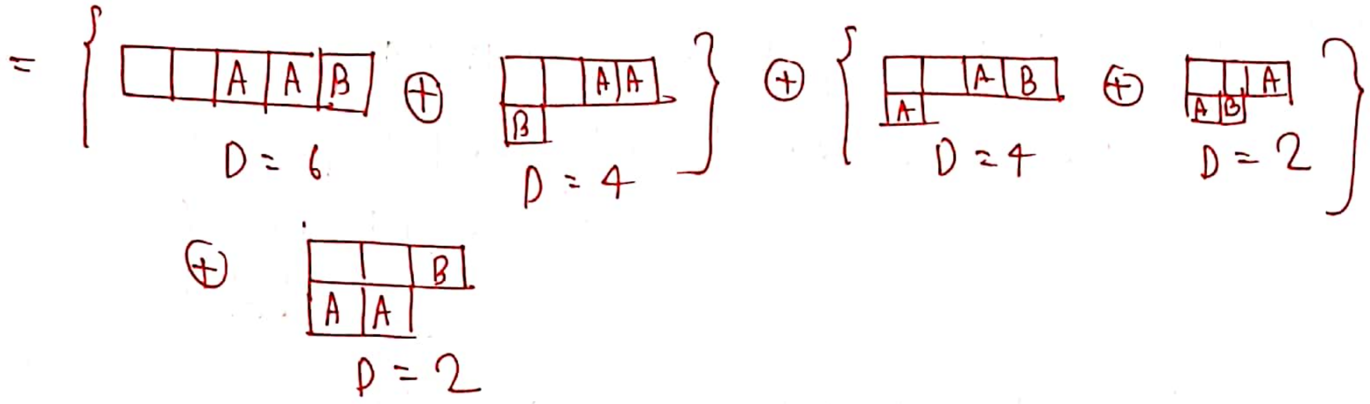


i) $3 \otimes 3$

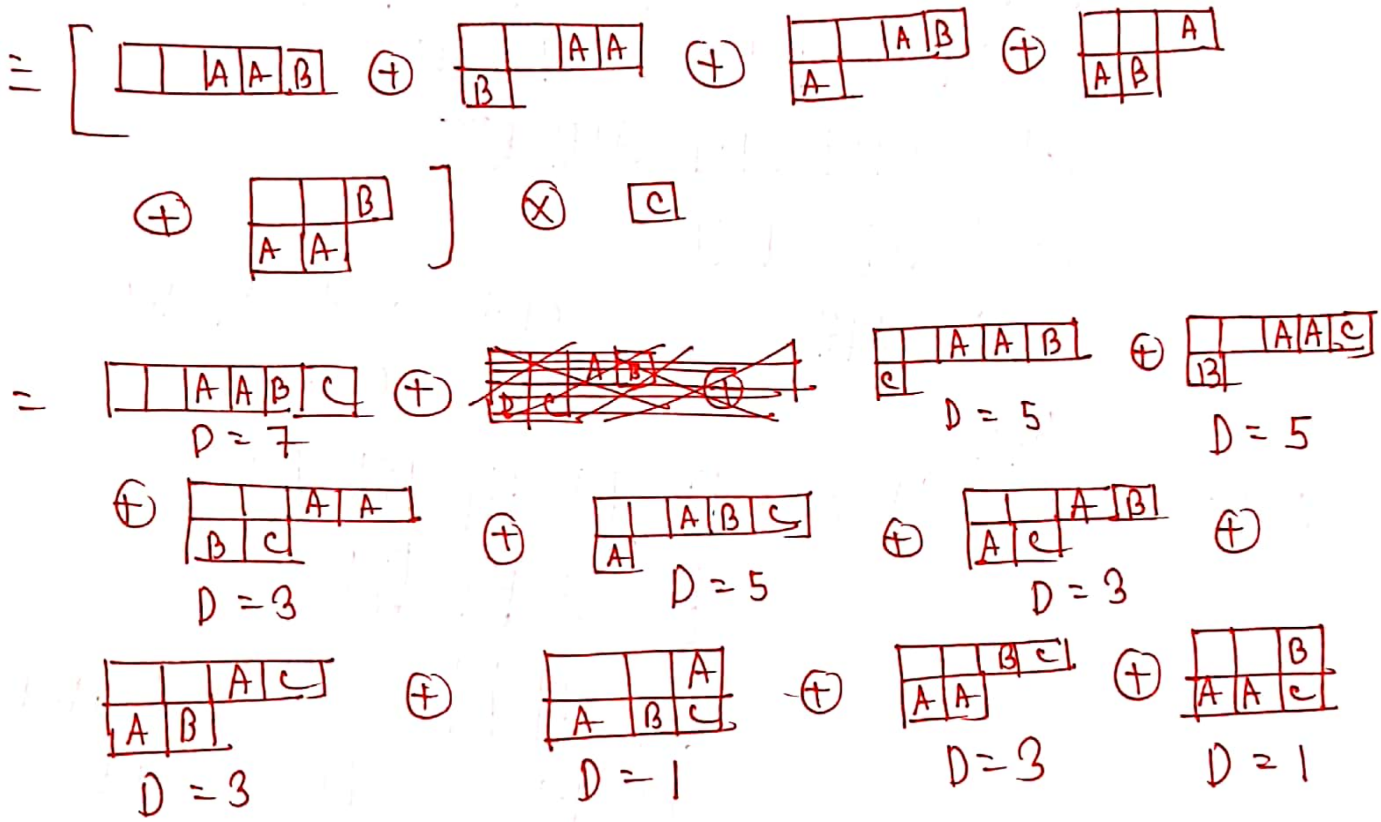


ii) $3 \otimes 3 \otimes 2$





III) $3 \otimes 3 \otimes 3 \otimes 2$



Ans

~~This~~ These are the irreducible combinations of $3 \otimes 3 \otimes 3 \otimes 2$ using Young tableau.

1.9 Two identical $\frac{5}{2}$ spins are combining.

$\therefore$ $\frac{5}{2}$ has 6 m values:

$$\left.\begin{array}{l} \pm\frac{5}{2} \rightarrow \boxed{a}, \boxed{d} \\ \pm\frac{3}{2} \rightarrow \boxed{b}, \boxed{e} \\ \pm\frac{1}{2} \rightarrow \boxed{c}, \boxed{f} \end{array}\right\}$$

The symmetrical states are given by:

$\boxed{a|a}$ $\boxed{b|b}$ $\boxed{c|c}$ $\boxed{d|d}$ $\boxed{e|e}$ $\boxed{f|f}$

$\boxed{a|b}$ $\boxed{b|c}$ $\boxed{c|d}$ $\boxed{d|e}$ $\boxed{e|f}$

$\boxed{a|c}$ $\boxed{b|d}$ $\boxed{c|e}$ $\boxed{d|f}$

$\boxed{a|d}$ $\boxed{b|e}$ $\boxed{c|f}$

$\boxed{a|e}$ $\boxed{b|f}$

$\boxed{a|f}$

Total no of states $1+2+\cdots+6 = 21.$

Antisymmetric states:

$\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}$ $\begin{array}{|c|}\hline b\\\hline c\\\hline\end{array}$ $\begin{array}{|c|}\hline c\\\hline d\\\hline\end{array}$ $\begin{array}{|c|}\hline d\\\hline e\\\hline\end{array}$ $\begin{array}{|c|}\hline e\\\hline f\\\hline\end{array}$

$\begin{array}{|c|}\hline a\\\hline c\\\hline\end{array}$ $\begin{array}{|c|}\hline b\\\hline d\\\hline\end{array}$ $\begin{array}{|c|}\hline c\\\hline e\\\hline\end{array}$ $\begin{array}{|c|}\hline d\\\hline f\\\hline\end{array}$

$\begin{array}{|c|}\hline a\\\hline d\\\hline\end{array}$ $\begin{array}{|c|}\hline b\\\hline e\\\hline\end{array}$ $\begin{array}{|c|}\hline c\\\hline f\\\hline\end{array}$

$\begin{array}{|c|}\hline a\\\hline e\\\hline\end{array}$ $\begin{array}{|c|}\hline b\\\hline f\\\hline\end{array}$

$\begin{array}{|c|}\hline a\\\hline f\\\hline\end{array}$

Total states $1+2+\cdots+5 = 15.$

for spin j : total possible no of values of $m = (2j+1)$

As symmetric states ~~are~~ gives ~~combin~~ combinations from $\boxed{-j|-j}$ $\cdots$ $\boxed{-j|j}$ So no of symmetric states: $1+2+\cdots+(2j+1) = \frac{(2j+1)(2j+2)}{2}$

$$= (j+1)(2j+1)$$

For antisymmetric states the sum goes from

$$1+2+3+\cdots+2j = \frac{2j(2j+1)}{2} = j(2j+1).$$

Ans

Question: 2

2.a $\hat{H}|n,l,j,m\rangle = E_n|n,l,j,m\rangle$

with $E_n = -\dfrac{m e^4}{2\hbar^2 \times 16\pi^2 \epsilon_0^2 n^2} = -\dfrac{m e^4}{32\hbar^2 \pi^2 \epsilon_0^2 n^2}$

but fine structure const $\alpha = \dfrac{e^2}{4\pi \epsilon_0 \hbar c}$

So, $\alpha^2 c^2 = \dfrac{e^4}{16\pi^2 \epsilon_0^2 \hbar^2}$

$\therefore E_n = -\dfrac{m}{2n^2} \times \dfrac{e^4}{16\pi^2 \hbar^2 \epsilon_0^2} = -\dfrac{m\alpha^2 c^2}{2n^2}$. Ans

$L^2|n,l,j,m\rangle = l(l+1)\hbar^2|n\,l\,j\,m\rangle$
Here the eigenvalue of $\hat{L}^2$ is already expressed in terms of those quantities. i.e $\boxed{l(l+1)\hbar^2}$

$J^2|n,l,j,m\rangle = j(j+1)\hbar^2|n\,l\,j\,m\rangle$
$\therefore$ eigenvalue of J^2 is: $\boxed{j(j+1)\hbar^2}$

$J_z|n\,l\,j\,m\rangle = m\hbar|n\,l\,j\,m\rangle$
$\therefore$ eigenvalue of J_z: $\boxed{m\hbar}$.

2.b Here n can take any value from 1 to ∞
for $n=1$; $E_n = -13.6$ ev; for $n \Rightarrow \infty$; $E_n \to 0$
i.e the electron becomes free at $n=\infty$. & makes the scattering continuum

For a particular value of n; the value of l can change of 0 to $n-1$. (total n values)
i.e $l = 0, 1, 2, \ldots, n-1$.

Here $J = L + S$; So according to addition of

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Angular momentum; J can take values as

$$J = L - \tfrac{1}{2} \ \& \ L + \tfrac{1}{2} \quad \text{if } L \neq 0$$
$$= \tfrac{1}{2} \quad \text{if } L = 0$$
$\left(\text{As } S = \frac{1}{2} \text{ for } e^- \right)$

For a particular value of J; m can take $(2j+1)$ values as:

$m = -j, -j+1, \dots, j-1, j$

These are the bounds on (n, l, j, m)

If an e^- stays at $2p$ ~~them:~~ state

$n = 2$; $l = 1$; $j = \frac{3}{2}, \frac{1}{2}$

for $j = \frac{3}{2}$: $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$

$j = \frac{1}{2}$: $m = -\frac{1}{2}, \frac{1}{2}$

2.c. We know $J_\pm$ lifts or lowers the m_j value by 1.

So: i) $\langle 3, 1, \frac{3}{2}, \frac{3}{2} | J_+ | 3, 1, \frac{3}{2}, -\frac{1}{2} \rangle$

$$= \hbar \sqrt{\left(\tfrac{3}{2} + \tfrac{1}{2}\right)\left(\tfrac{3}{2} - \tfrac{1}{2} + 1\right)} \underbrace{\langle 3, 1, \tfrac{3}{2}, \tfrac{3}{2} | 3, 1, \tfrac{3}{2}, \tfrac{1}{2} \rangle}_{0}$$

$= 0$

$\left(\because \langle n, l, j, m | n' \cdot l' \, j' \, m' \rangle = \delta_{nn'} \delta_{ll'} \delta_{jj'} \delta_{mm'} \right)$

ii) $\langle 3, 1, \frac{3}{2}, \frac{3}{2} | J_+ | 3, 1, \frac{3}{2}, \frac{1}{2} \rangle$

$$= \hbar \sqrt{\left(\tfrac{3}{2} - \tfrac{1}{2}\right)\left(\tfrac{3}{2} + \tfrac{1}{2} + 1\right)} \underbrace{\langle 3, 1, \tfrac{3}{2}, \tfrac{3}{2} | 3, 1, \tfrac{3}{2}, \tfrac{3}{2} \rangle}_{= 1}$$

$= \hbar \sqrt{1 \times 3}$

$= \hbar \sqrt{3}$.

iii) $\langle 2,1,\frac{3}{2},\frac{3}{2} | p_z | 2,1,\frac{3}{2},\frac{1}{2} \rangle =$

iv) $\langle 2,1,\frac{1}{2},-\frac{1}{2} | L^2 | 2\ 1\ \frac{1}{2}\ -\frac{1}{2} \rangle$

$= 1(1+1)\hbar^2 \langle 2,1,\frac{1}{2},-\frac{1}{2} | 2,1,\frac{1}{2},-\frac{1}{2} \rangle$

$= 2\hbar^2$

v) $\langle 3,2,\frac{3}{2},-\frac{1}{2} | J^2 | 3,2,\frac{3}{2},-\frac{1}{2} \rangle$

$= \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2 \langle 3,2,\frac{3}{2},-\frac{1}{2} | 3,2,\frac{3}{2},-\frac{1}{2} \rangle$

$= \frac{15}{4}\hbar^2$

vi) $\langle 3,1,\frac{3}{2},\frac{3}{2} | J_z | 3,1,\frac{3}{2},\frac{1}{2} \rangle$

$= \frac{1}{2}\hbar \langle 3,1,\frac{3}{2},\frac{3}{2} | 3,1,\frac{3}{2},\frac{3}{2} \rangle = \frac{\hbar}{2}.$

2.d. $\langle 1\,0\,\frac{1}{2}\,\frac{1}{2} | p_i p_j | 1\,0\,\frac{1}{2}\,\frac{1}{2} \rangle$

$p_i p_j$ can be written as sum of tensor operators of rank

$q = 0,1,2$

but for $q=2$; $j_1 = \frac{1}{2}$ $\Rightarrow$ C.G coefficient will be 0 since

$2 \otimes \frac{1}{2}$ can give $j = \frac{3}{2}, \frac{5}{2}$

$\therefore q = 0,1$

for $q=1$: $\langle 1\,0\,\frac{1}{2}\,\frac{1}{2} | T_0^1 | 1\,0\,\frac{1}{2}\,\frac{1}{2} \rangle = \langle 1\,0\,\frac{1}{2}\,\frac{1}{2} | 1\,\frac{1}{2}\,0\,\frac{1}{2} \rangle \times$ const

$= 0$

2.e $\langle n\, l'\, j'\, m' | \vec{S}\cdot\vec{r} | n\, l\, j\, m \rangle$

$\vec{S}\cdot\vec{r}$ is a rank 0 tensor. (Scaler)

$\therefore$ By Wigner Eckart theorem:

$\langle n\, l'\, j' m' | \vec{S}\cdot\vec{r} | n\, l\, j\, m \rangle \propto \langle \underset{j_1}{j}\ \underset{j_2}{0}\ \underset{j}{j'}\ \underset{m}{m'} | \underset{j_1}{j}\ \underset{j_2}{0}\ \underset{m_1}{m}\ \underset{m_2}{0} \rangle$

$\neq 0$ iff $j' = j$; $m' = m + 0 = m$

Also $\vec{S}\cdot\vec{r}$ under parity goes to $-\vec{S}\cdot\vec{r}$.

& under parity $|l\rangle \longrightarrow (-1)^l |l\rangle$

$\therefore$ The over all parity must be (+) for the matrix element to be non zero.

i.e $(-1)^{l+l'+1} = 1 \Rightarrow l+l'+1 =$ even.

$\therefore l + l' =$ odd.

but $\vec{J} = \vec{L} + \vec{S}$ & $j' = j$ $\Rightarrow$ either $l' = l$ or $l - l' = \pm 1$.

$s' = s = \frac{1}{2}$

But $l = l'$ is not possible as $l + l' =$ odd.

So the condition is:

$j' = j$, $m' = m$ & $l' - l = \pm 1$. Ans

Question : 3

3.a $\langle\psi|\psi\rangle = 1$

but $\psi(r,\theta,\phi) = N\left\{R_{21}(r)\left(2i\,Y_1^{-1} + (2+i)Y_1^0 + 3i\,Y_1^1\right)\right\}$

i.e $|\psi\rangle = N\left\{2i\,|2\,1\,{-1}\rangle + (2+i)|2\,1\,0\rangle + 3i\,|2\,1\,1\rangle\right\}$

so; $\langle\psi|\psi\rangle = N^2\left\{|2i|^2 + |2+i|^2 + |3i|^2\right\}$

$= N^2\left(4 + (2^2+1^2) + 9\right) = N^2(4+9+5) = 18N^2$

so; $18N^2 = 1$ i.e $N = \frac{1}{\sqrt{18}}$ (n being real +ve)

3.b. $\langle L_z\rangle = \langle\psi|L_z|\psi\rangle$

$= \frac{1}{18}\left[\left(-2i\langle 2\,1\,{-1}| + (2-i)\langle 2\,1\,0| - 3i\langle 2\,1\,1|\right)\cdot L_z\right.$
$\left.\left(2i\,|2\,1\,{-1}\rangle + (2+i)|2\,1\,0\rangle + 3i\,|2\,1\,1\rangle\right)\right]$

$= \frac{1}{18}\left[\left(-2i\langle 2\,1\,{-1}| + (2-i)\langle 2\,1\,0| - 3i\langle 2\,1\,1|\right)\cdot\right.$
$\left.\hbar\left(-2i\,|2\,1\,{-1}\rangle + 0 + 3i\,|2\,1\,1\rangle\right)\right]$

$= \frac{\hbar}{18}\left[(-2i)(-2i) + (-3i)(3i)\right]$

$= \frac{\hbar}{18}\left[9 - 4\right] = \frac{5\hbar}{18}.$

3.c $\langle L^2\rangle = \frac{1}{18}\left[\left(-2i\langle 2\,1\,{-1}| + (2-i)\langle 2\,1\,0| - 3i\langle 2\,1\,1|\right)L^2\right.$
$\left.\left(2i\,|2\,1\,{-1}\rangle + (2+i)|2\,1\,0\rangle + 3i\,|2\,1\,1\rangle\right)\right]$

$= \frac{\hbar^2}{18}\left[\left(-2i\langle 2\,1\,{-1}| + (2-i)\langle 2\,1\,0| - 3i\langle 2\,1\,1|\right)\times\right.$
$\left.1(1+1)\left(2i\,|2\,1\,{-1}\rangle + (2+i)|2\,1\,0\rangle + 3i\,|2\,1\,1\rangle\right)\right]$

[$\because$ for each state $l=1$ so I've taken $l(l+1) = 1(1+1) = 2$ as common factor.]

$= \frac{\hbar^2}{9}\left[|2i|^2 + |2+i|^2 + |3i|^2 \right] = \frac{18\hbar^2}{9} = 2\hbar^2.$

which is obvious as for each m; $l = 1$ so we get

$L^2 = 1(1+1)\hbar^2 = 2\hbar^2.$

3.d. $\left\langle \frac{\hat{p}^2}{2m} \right\rangle = \langle E - V \rangle \quad \left[\because \frac{\hat{p}^2}{2m} = E - \hat{V} \right]$

$= \langle K.E \rangle = \langle E \rangle - \langle V \rangle$

Now all the states (superposed) being $n = 2$;

we get:

$\langle E \rangle = \langle E \rangle \big|_{n=2} = E_2 = -\frac{13.6}{2^2}$ ev

$= -\frac{13.6}{4}$ ev

or in terms of α; $\hbar$; c; e : $E_2 = -\frac{m\alpha^2 c^2}{2n^2}\Big|_{n=2}$

$= -\frac{m\alpha^2 c^2}{8}$

Now; $\langle V \rangle = \left\langle \frac{-1}{4\pi\epsilon_0} \frac{e^2}{r} \right\rangle$

but for state with principal Q number n; we get

$\langle V \rangle_n = -\frac{e^2}{4\pi\epsilon_0} \left\langle \frac{1}{r} \right\rangle_n = 2E_n$

$\therefore$ for $n=2$: $\langle V \rangle_2 = \frac{2E_2}{2} = -\frac{4m\alpha^2c^2}{2\times 2\times 2^2} = -\frac{m\alpha^2c^2}{4}$

$\therefore \langle T \rangle = \frac{m\alpha^2c^2}{4} - \frac{m\alpha^2c^2}{8} = \frac{m\alpha^2c^2}{8}.$

Question :- 4

a) $\rho = \frac{1}{2}(AI + \sigma.B) = \frac{1}{2}\left[\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \underbrace{\begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix}}_{\sigma.B}\right]$

$$= \begin{pmatrix} \frac{A+B_3}{2} & \frac{B_1 - iB_2}{2} \\ \frac{B_1+iB_2}{2} & \frac{A-B_3}{2} \end{pmatrix}$$

Now $\hat{\rho}$ being a density matrix we get:

$Tr(\hat{\rho}) = 1$

i.e $\frac{A+B_3}{2} + \frac{A-B_3}{2} = 1$ i.e $A = 1$.

So; $\hat{\rho} = \begin{pmatrix} \frac{1+B_3}{2} & \frac{B_1-iB_2}{2} \\ \frac{B_1+iB_2}{2} & \frac{1-B_3}{2} \end{pmatrix}$

$\therefore \hat{\rho}^2 = \begin{pmatrix} \frac{1+B_3}{2} & \frac{B_1-iB_2}{2} \\ \frac{B_1+iB_2}{2} & \frac{1-B_3}{2} \end{pmatrix}^2 = \begin{pmatrix} \left(\frac{1+B_3}{2}\right)^2 + \frac{B_1^2+B_2^2}{4} & \frac{B_1-iB_2}{2} \\ \frac{B_1+iB_2}{2} & \left(\frac{1-B_3}{2}\right)^2 + \frac{B_1^2+B_2^2}{4} \end{pmatrix}$

But for a pure ensembel we get

$\hat{\rho}^2 = \hat{\rho}$ i.e $Tr(\rho^2) = Tr(\rho) = 1$

i.e $\frac{(1+B_3)^2 + (1-B_3)^2}{4} + \frac{B_1^2+B_2^2}{2} = 1$

So; $\frac{2(1+B_3^2)}{4} + \frac{B_1^2+B_2^2}{2} = 1$

i.e $1 + B_1^2 + B_2^2 + B_3^2 = 2$

$\Rightarrow B_1^2 + B_2^2 + B_3^2 = 1$. ... (1)

So; $-1 \leqslant B_1, B_2, B_3 \leqslant 1$ ... (2) (obviously)

as $B_i \in \mathbb{R}$.

So the vector $\vec{B}$ is a real unit vector in 3-d space (along any direction $\hat{m}$)

And the conditions on A & $\vec{B}$ be given by
$A = 1$; $\vec{B} = \hat{m}$ (in 3d space)
to make $\hat{\rho}$ a pure state density matrix. } **Ans**

4.b $\langle \hat{S}_y \rangle = \mathrm{Tr}(\hat{\rho}\cdot\hat{S}_y)$ (when both are given by same representations)

Here $\hat{\rho}$ is given in $|\uparrow\rangle_z, |\downarrow\rangle_z$ representation.
In the same basis; $\hat{S}_y$ be given by:

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\therefore \hat{\rho}\cdot\hat{S}_y = \frac{\hbar}{2\times 2}\begin{pmatrix} 1+B_3 & B_1-iB_2 \\ B_1+iB_2 & 1-B_2 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} B_2+iB_1 & -i(1+B_3) \\ i(1-B_2) & B_2-iB_1 \end{pmatrix}$$

~~So; $\langle \hat{S}_y \rangle = \mathrm{Tr}(\hat{\rho}\cdot\hat{S}_y) = \frac{\hbar}{4}\times 2B_2 = \frac{B_2\hbar}{2}$.~~
~~(as $-1 \leq B_2 \leq 1$)~~
~~Clearly $\langle S_y \rangle_{max} = \frac{\hbar}{2}$~~

~~**4.c** As for all (pure/mixed) density matrices the $\mathrm{Tr}(\hat{\rho}) = 1$ so we get essential condition be~~

~~$A = 1$~~

~~But for mixed state we don't have to get~~

So $\langle \hat{S}_y \rangle = \mathrm{Tr}(\hat{\rho}\hat{S}_y) = \frac{\hbar}{4}\times 2B_2 = \frac{B_2\hbar}{2}$.

Now $-1 \leq B_2 \leq 1$; So the max possible value of $\langle S_y \rangle$ be $\langle S_y \rangle_{max} = \frac{\hbar}{2}$ with $B_2 = 1$

But as $B_1^2 + B_2^2 + B_3^2 = 1$ So for $B_2 = 1$ we get
$B_1 = 0 = B_3$.

And hence the vector $\vec{B}$ be given as:

$$\vec{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ Ans

4.c. For all (pure/mixed) state we must have $Tr(\hat{\rho}) = 1$ which gives the condition $A = 1$.
But for mixed state we don't necessarily need $\rho^2 = \rho$ and hence the 2nd condition is been violated.

i.e $B_1^2 + B_2^2 + B_3^2 \neq 1$

i.e $\vec{B}$ is not an unit vector in 3d real space. Ans

Question no 5:-

5.a (As 3 marks is given; I'm doing this explicitly in a long way.)

Let the given state (i.e at $t=0$) be given by $|\alpha(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ (represented in basis of $|\uparrow_z\rangle$ & $|\downarrow_z\rangle$)

Now this is an eigenstate of S_x with eig-val $\frac{\hbar}{2}$.

The represntation of S_x in $|\uparrow_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be given by:

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So from given information: $S_x|\alpha(0)\rangle = \frac{\hbar}{2}|\alpha(0)\rangle$

i.e $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore$ by normalization $\langle\alpha(0)|\alpha(0)\rangle = 1$ we get

$a = b = 1/\sqrt{2}$. i.e $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So $|\alpha(0)\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

but $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ being $|\uparrow_z\rangle$ & $|\downarrow_z\rangle$:

$$|\alpha(0)\rangle = |\uparrow_x\rangle = \frac{|\uparrow_z\rangle + |\downarrow_z\rangle}{\sqrt{2}}.$$

5.b Here $\hat{H} = -\vec{\mu}\cdot\vec{B} = -\gamma B\hat{S}_z$

Clearly eigen state of $\hat{H}$ = eigenstate of $\hat{S}_z$

i.e $[H, S_z] = 0$.

$$\hat{H}|\uparrow\rangle = -\gamma B S_z|\uparrow\rangle = -\frac{\gamma B\hbar}{2}|\uparrow\rangle$$

$$\hat{H}|\downarrow\rangle = -\gamma B S_z|\downarrow\rangle = \frac{\gamma B\hbar}{2}|\downarrow\rangle$$

So; if $|\alpha(0)\rangle = c_+|\uparrow\rangle + c_-|\downarrow\rangle$

then $|\alpha(t)\rangle = c_+ e^{\frac{i\gamma B\hbar t}{2\hbar}}|\uparrow\rangle + c_- e^{-\frac{i\gamma B\hbar t}{2\hbar}}|\downarrow\rangle$

here $c_\pm = 1/\sqrt{2}$. So;

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}\left\{e^{\frac{i\gamma Bt}{2}}\begin{pmatrix}1\\0\end{pmatrix} + e^{-\frac{i\gamma Bt}{2}}\begin{pmatrix}0\\1\end{pmatrix}\right\}$$ Ans

5.c $$\langle S_y(t)\rangle = \langle\alpha(t)|S_y|\alpha(t)\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{-\frac{i\gamma Bt}{2}} \quad e^{+\frac{i\gamma Bt}{2}}\right)\cdot\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}e^{\frac{i\gamma Bt}{2}}\\ e^{-\frac{i\gamma Bt}{2}}\end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{-\frac{i\gamma Bt}{2}} \quad e^{+\frac{i\gamma Bt}{2}}\right)\begin{pmatrix}-i\,e^{-\frac{i\gamma Bt}{2}}\\ i\,e^{\frac{i\gamma Bt}{2}}\end{pmatrix}$$

$$= \frac{\hbar}{4}\left(i e^{i\gamma Bt} - i e^{-i\gamma Bt}\right)$$

$$= -\frac{\hbar}{2}\left(\frac{e^{i\gamma Bt} - e^{-i\gamma Bt}}{2i}\right) = -\frac{\hbar}{2}\sin(\gamma Bt)$$

So; $\langle S_y(t)\rangle = -\frac{\hbar}{2}\sin(\gamma Bt)$ Ans

5.d. The spin operator in the given direction is:

$$S_{\hat{n}} = \vec{S}\cdot\hat{n} = \sin\theta . S_x + \cos\theta . S_z$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

The given state is a eigenstate of $S_{\hat{n}}$. with eig-val $-\frac{\hbar}{2}$.

So $$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

i.e $$\begin{pmatrix} \alpha\cos\theta + \beta\sin\theta \\ \alpha\cdot\sin\theta - \beta\cos\theta \end{pmatrix} = -\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

which gives: $$\left.\begin{aligned} \alpha(1+\cos\theta) + \beta\sin\theta &= 0 \\ \alpha\sin\theta + \beta(1-\cos\theta) &= 0 \end{aligned}\right\}$$

$$\Rightarrow \alpha = -\frac{\beta\sin\theta}{(1+\cos\theta)} = \frac{-2\beta\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\beta\tan\frac{\theta}{2}$$

$$\therefore |\psi\rangle = \alpha\begin{pmatrix} 1 \\ -\cot\frac{\theta}{2} \end{pmatrix}$$

Normalizing, we get: $\alpha^2\operatorname{cosec}^2\frac{\theta}{2} = 1$ i.e $\alpha = \sin\frac{\theta}{2}$

$$\therefore |\psi\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = \sin\left(\frac{\theta}{2}\right)|\uparrow\rangle + \cos\left(\frac{\theta}{2}\right).|\downarrow\rangle$$

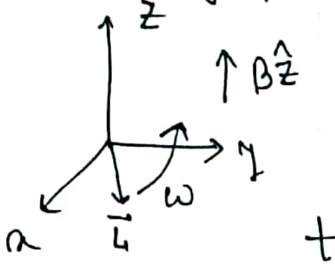
Ans

(e) Here from the ~~give~~ expression of $|\psi(t)\rangle$ calculated in previous part we get

$$\langle S_z\rangle = \langle\psi(t)|S_z|\psi(t)\rangle = \frac{\hbar}{4}\left(e^{\frac{-i\gamma Bt}{2}}\ e^{\frac{i\gamma Bt}{2}}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{\frac{i\gamma Bt}{2}} \\ e^{-\frac{i\gamma Bt}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2}\times\frac{1}{2}\left(e^{-\frac{i\gamma Bt}{2}}\ e^{\frac{i\gamma Bt}{2}}\right)\begin{pmatrix} e^{\frac{i\gamma Bt}{2}} \\ -e^{-\frac{i\gamma Bt}{2}} \end{pmatrix} = \frac{\hbar}{4}(1-1) = 0$$

And which is also clear from the physical view of the problem. The initial spin is in x direction (classical view) due to the $\hat{z}$ magnetic field; it precesses around $\hat{z}$ on xy plane. So the given direction will only give the spin-angular momentum value as $\left(-\frac{\hbar}{2}\right)$ is only possible iff $\theta = \frac{\pi}{2}$ (for all value of ϕ). However; if the eigenvalue of ~~spins~~ spin angular momentum along given direction be $-\frac{\hbar}{2}$, then the state will be the calculated one.



5. e. The required probability:

$$P = \left| \langle \psi(t) | \chi_{\hat{n}}^{-} \rangle \right|^2 \quad \text{when } |\chi_{\hat{n}}^{-}\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \begin{pmatrix} e^{-\frac{i\gamma B t}{2}} & e^{\frac{i\gamma B t}{2}} \end{pmatrix} \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \sin\frac{\theta}{2} \cdot e^{-\frac{i\gamma B t}{2}} + \cos\frac{\theta}{2}\, e^{\frac{i\gamma B t}{2}} \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \sin\frac{\theta}{2}\left(\cos\frac{\gamma B t}{2} - i\sin\frac{\gamma B t}{2}\right) + \cos\frac{\theta}{2}\left(\cos\frac{\gamma B t}{2} + i\sin\frac{\gamma B t}{2}\right) \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left| \cos\left(\frac{\gamma B t}{2}\right) \cdot \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) + i\sin\left(\frac{\gamma B t}{2}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) \right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left\{ \cos^2\left(\frac{\gamma B t}{2}\right) \cdot \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2 + \sin^2\left(\frac{\gamma B t}{2}\right) \cdot \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2 \right\}$$

if $\theta = 0$ then $P = \frac{1}{2}\left\{\cos^2\frac{\gamma B t}{2} + \sin^2\frac{\gamma B t}{2}\right\} = 1/2$

i.e the prob to find the spin value as $-\frac{\hbar}{2}$; when the spin was oriented along xy plane is $\frac{1}{2}$; as expected from

$$|\downarrow\rangle_z = \left(|\uparrow x\rangle - |\downarrow x\rangle\right)/\sqrt{2}.$$

Problem: 6

6.a. $B = B_0\hat{z}$; So; $H = -\frac{g\mu_0}{\hbar}.\vec{S}\cdot\vec{B} = -\frac{g\mu_0}{\hbar}BS_z$

i.e $[H, S_z] = 0$; So the eigenstates of H are eigenstates of S_z also, i.e $|\pm\rangle$ with eigenvalue $\mp\frac{g\mu_0 B}{\hbar}.\frac{\hbar}{2}$ i.e $\mp\frac{g\mu_0 B}{2}$

The energy difference between states: $\Delta E = g\mu_0 B$.

6.b. given $|\psi(0)\rangle = |+x\rangle$

In terms of $|\pm\rangle$: $|\psi(0)\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$.

clearly $|\psi(t)\rangle = |\psi(0)\rangle.\, e^{-\frac{iEt}{\hbar}}$

$$= \frac{1}{\sqrt{2}}|+\rangle.\, e^{\frac{ig\mu_0 Bt}{2\hbar}} + \frac{1}{\sqrt{2}}|-\rangle e^{-\frac{ig\mu_0 Bt}{2\hbar}}.$$

If $|\pm\rangle$ be given by $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ then:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\phi t}\\e^{-i\phi t}\end{pmatrix} \quad \left(\phi = \frac{g\mu_0 B}{2\hbar}\right)$$

6.c. $\langle S_x\rangle_t = \langle\psi(t)|S_x|\psi(t)\rangle = \frac{\hbar}{2\times 2}\left(e^{-i\phi t}\ e^{+i\phi t}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}e^{i\phi t}\\e^{-i\phi t}\end{pmatrix}$

$$= \frac{\hbar}{4}\left(e^{-i\phi t}\ e^{i\phi t}\right)\begin{pmatrix}e^{-i\phi t}\\e^{i\phi t}\end{pmatrix} = \frac{\hbar}{4}\left(e^{-2i\phi t} + e^{2i\phi t}\right)$$

$$= \frac{\hbar}{2}\cos(2\phi t) = \frac{\hbar}{2}.\cos\left(\frac{g\mu_0 Bt}{\hbar}\right)$$

$\langle S_z(t)\rangle = \langle\psi(t)|S_z|\psi(t)\rangle = \frac{\hbar}{4}\left(e^{-i\phi t}\ e^{i\phi t}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}e^{i\phi t}\\e^{-i\phi t}\end{pmatrix}$

$$= \frac{\hbar}{4}\left(e^{-i\phi t}\ e^{+i\phi t}\right)\begin{pmatrix}e^{i\phi t}\\-e^{-i\phi t}\end{pmatrix} = \frac{\hbar}{4}(1-1) = 0$$

$\langle S_y(t)\rangle = \langle\psi(t)|S_y|\psi(t)\rangle = \frac{\hbar}{4}\left(e^{-i\phi t}\ e^{i\phi t}\right)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}e^{i\phi t}\\e^{-i\phi t}\end{pmatrix}$

$$= \frac{\hbar}{4}\left(e^{-i\phi t}\ e^{i\phi t}\right)\begin{pmatrix}-ie^{-i\phi t}\\ie^{i\phi t}\end{pmatrix} = \frac{\hbar}{4}\left(ie^{2i\phi t} - ie^{-2i\phi t}\right)$$

$$= -\frac{\hbar}{2}\frac{\left(e^{2i\phi t} - e^{-2i\phi t}\right)}{2i} = -\frac{\hbar}{2}\sin(2\phi t) = -\frac{\hbar}{2}\sin\left(\frac{g\mu_0 Bt}{\hbar}\right)$$

7. We know $S_1^z + S_2^z = S_z$

& $S_1 \cdot S_2 = S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z$

Now; $H = J\left(S_1^x S_2^x + S_1^y S_2^y + K S_1^z S_2^z\right) + \mu\left(S_1^z + S_2^z\right)B$.

$= J\, S_1 \cdot S_2 + (K-1)\, J\, S_1^z S_2^z + \mu B S_z$.

But $S = S_1 + S_2 \Rightarrow S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$

$\Rightarrow S_1 \cdot S_2 = \dfrac{\left(S^2 - S_1^2 - S_2^2\right)}{2}$

$\therefore H = \dfrac{J}{2}\left(S^2 - S_1^2 - S_2^2\right) + (K-1) J S_1^z S_2^z + \mu B S_z$.

Now for two spin $\frac{1}{2}$ particles: (in $|j\, m\rangle$ form).

The symmetric states are: $|1\, 0\rangle, |1\, 1\rangle, |1\, {-1}\rangle$

and the antisymmetric state: $|0\, 0\rangle$

with $|1\, 0\rangle = \dfrac{|+-\rangle + |-+\rangle}{2}$; $|0\, 0\rangle = \dfrac{|+-\rangle - |-+\rangle}{\sqrt{2}}$

So the energy measurement on different states give as following:

- $\hat{H}|1\, 1\rangle = \dfrac{J}{2}\left(S^2 - S_1^2 - S_2^2\right)|1\, 1\rangle + (K-1) J\, S_1^z S_2^z\, |++\rangle$
$+ \mu B S^z |1\, 1\rangle$

(using $|1\, 1\rangle \equiv |++\rangle$ in $|j\, m\rangle$ & $|m_1\, m_2\rangle$ form)

$= \dfrac{J\hbar^2}{2}\left(2 - \frac{3}{4} - \frac{3}{4}\right)|1\, 1\rangle + (K-1) J \cdot \frac{\hbar}{2} \cdot \frac{\hbar}{2}\, |++\rangle$
$+ \mu B \hbar\, |1\, 1\rangle$

$= \left(\dfrac{J\hbar^2}{4} + (K-1)\dfrac{J\hbar^2}{4}\right)|1\, 1\rangle + \mu B \hbar\, |1\, 1\rangle$

$= \left(\dfrac{K J \hbar^2}{4} + \mu B \hbar\right)|1\, 1\rangle$

- $\hat{H}|1\, 0\rangle = \dfrac{J}{2}\left(S^2 - S_1^2 - S_2^2\right)|1\, 0\rangle + (K-1) J\, S_1^z S_2^z\, \dfrac{|+-\rangle + |-+\rangle}{\sqrt{2}}$
$+ \mu B S^z |1\, 0\rangle$.

$$= \frac{J\hbar^2}{2}\left(2-\frac{3}{4}-\frac{3}{4}\right)|10\rangle + \frac{(K-1)J}{\sqrt{2}}\left(\frac{\hbar}{2}\cdot\left(-\frac{\hbar}{2}\right)|+-\rangle + \frac{\hbar}{2}\left(-\frac{\hbar}{2}\right)|-+\rangle\right) + 0$$

$$= \frac{J\hbar^2}{2}\times\frac{1}{2}|10\rangle - \frac{(K-1)J\hbar^2}{4}\frac{|+-\rangle+|-+\rangle}{\sqrt{2}}$$

$$= \left(\frac{J\hbar^2}{4} - \frac{(K-1)J\hbar^2}{4}\right)|10\rangle = \frac{J\hbar^2}{4}(2-K)|10\rangle$$

$$\hat{H}|1-1\rangle = \frac{J}{2}(S^2-S_1^2-S_2^2)|1-1\rangle + (K-1)J S_1^z S_2^z|--\rangle + \mu B S^z|1-1\rangle$$

$$= \frac{J\hbar^2}{2}\left(2-\frac{3}{4}-\frac{3}{4}\right) + (K-1)J\cdot\left(-\frac{\hbar}{2}\right)^2|--\rangle - \mu B\hbar|1-1\rangle$$

$$= \left(\frac{KJ\hbar^2}{4} - \mu B\hbar\right)|1-1\rangle$$

$$\hat{H}|00\rangle = \cancel{\frac{J}{2}\left(0-\frac{3}{4}-\frac{3}{4}\right)\hbar^2}$$

$$= \frac{J}{2}(S^2-S_1^2-S_2^2)|00\rangle + (K-1)J S_1^z S_2^z\frac{|+-\rangle-|-+\rangle}{\sqrt{2}} + \mu B S^z|00\rangle$$

$$= \frac{J\hbar^2}{2}\left(0-\frac{3}{4}-\frac{3}{4}\right) + \frac{(K-1)J}{\sqrt{2}}\left\{-\frac{\hbar^2}{4}|+-\rangle - \left(-\frac{\hbar^2}{4}\right)|-+\rangle\right\} + 0$$

(like $|10\rangle$)

$$= \frac{J\hbar^2}{2}\times\left(-\frac{3}{2}\right) + (K-1)J\cdot\left(-\frac{\hbar^2}{4}\right)\cdot\frac{|+-\rangle-|-+\rangle}{\sqrt{2}}.$$

$$= \left(-\frac{3J\hbar^2}{4} - (K-1)\frac{J\hbar^2}{4}\right)|00\rangle$$

$$= -\left(\frac{J\hbar^2}{2} + \frac{KJ\hbar^2}{4}\right)|00\rangle = -\frac{J\hbar^2}{4}(2+K)|00\rangle$$

So the energy measurement on different states give,

$$\left.\begin{aligned} |1\pm1\rangle &\rightarrow \frac{KJ\hbar^2}{4} \pm \mu B\hbar \\ |10\rangle &\rightarrow \frac{J\hbar^2}{4}(2-K) \end{aligned}\right\} \text{symmetric state}$$

$$|00\rangle \longrightarrow -\frac{J\hbar^2}{4}(2+K) \rightarrow \text{antisymmetric state.}$$

8. The given Hamiltonian:

$$\hat{H} = -t\Big\{\big(|1\uparrow,1\downarrow\rangle\langle 1\uparrow,2\downarrow| + |1\uparrow,2\downarrow\rangle\langle 1\uparrow,1\downarrow|\big) + \big(|2\uparrow,2\downarrow\rangle\langle 1\uparrow 2\downarrow| + |1\uparrow 2\downarrow\rangle\langle 2\uparrow 2\downarrow|\big) + \big(|1\uparrow,1\downarrow\rangle\langle 1\downarrow,2\uparrow| + |1\downarrow,2\uparrow\rangle\langle 1\uparrow,1\downarrow|\big) + \big(|2\uparrow,2\downarrow\rangle\langle 1\downarrow,2\uparrow| + |1\downarrow,2\uparrow\rangle\langle 2\uparrow,2\downarrow|\big)\Big\}$$

$$+ U\Big\{|1\uparrow,1\downarrow\rangle\langle 1\uparrow,1\downarrow| + |2\uparrow,2\downarrow\rangle\langle 2\uparrow,2\downarrow|\Big\}$$

The only states present in the $\hat{H}$ are:
$|1\uparrow,1\downarrow\rangle$, $|1\uparrow,2\downarrow\rangle$, $|2\uparrow,2\downarrow\rangle$, $|1\downarrow,2\uparrow\rangle$
(while $|1\uparrow 2\uparrow\rangle$, $|2\uparrow 2\uparrow\rangle$, ... etc are not)
So those 4 vectors forming the eigenbasis of the $\hat{H}$.
So the Hamiltonian be given by:

$$\hat{H} = \begin{array}{c} \begin{pmatrix} U & 0 & -t & -t \\ 0 & U & -t & -t \\ -t & -t & 0 & 0 \\ -t & -t & 0 & 0 \end{pmatrix} \begin{array}{l} |1\uparrow 1\downarrow\rangle \\ |2\uparrow 2\downarrow\rangle \\ |1\uparrow 2\downarrow\rangle \\ |1\downarrow 2\uparrow\rangle \end{array} \\ \langle 1\uparrow 1\downarrow| \;\; \langle 2\uparrow 2\downarrow| \;\; \langle 1\uparrow 2\downarrow| \;\; \langle 1\downarrow 2\uparrow| \end{array}$$

$$= U\begin{pmatrix} 1 & 0 & -t/u & -t/u \\ 0 & 1 & -t/u & -t/u \\ -t/u & -t/u & 0 & 0 \\ -t/u & -t/u & 0 & 0 \end{pmatrix}$$

(forgive for the small u in ( ) in place of U by mistake.)

but $U \gg t$ so $t/U \simeq 0$.

so; $$\hat{H} \simeq U\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Being in the diagonal form; we get
$\hat{H}$ have two eigen values $\{U, 0\}$ with two fold degeneracy.
with eigen vectors $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \rightarrow U$; $\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \rightarrow U$; $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \rightarrow 0$.

So the ground states are

$$\left.\begin{aligned} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} &= |1\uparrow, 2\downarrow\rangle \\ \begin{pmatrix}0\\0\\0\\1\end{pmatrix} &= |1\downarrow 2\uparrow\rangle \end{aligned}\right\} \text{ with energy } 0.$$

And the first excited states:

$$\left.\begin{aligned} \begin{pmatrix}1\\0\\0\\0\end{pmatrix} &= |1\uparrow 1\downarrow\rangle \\ \begin{pmatrix}0\\1\\0\\0\end{pmatrix} &= |2\uparrow 2\downarrow\rangle \end{aligned}\right\} \text{ with energy } U.$$

Now instead of doing; this if I'd take $t/u \neq 0$ & then find out the eigen values & ~~would~~ I'd get some eigenvectors of $\hat{H}$ then different result. Obviously $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; .... are not eigenvectors of $\hat{H}$. But due to the 4x4 matrix it is very hard to do by hand.
Using python I've found out the eigen values for $t/U = 0.01$. The eig vals are
$\lambda = U(1.0004, 1.0, -3.999 \times 10^{-4}, -2.040 \times 10^{-33})$
The coupling terms splits the degeneracy & we get 4 different energy levels.

9. Two spin $\frac{1}{2}$ particle will stay in spin singlet state iff the composite state be given by:

$$|\psi\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

Now for calculation; I assume that $\hat{m}_a$ & $\hat{m}_b$ are directions along (θ_1, ϕ_1) & (θ_2, ϕ_2).

The spin operator along $\hat{m}$ be given by:

$$\hat{S}_{\hat{m}} = \sin\theta \cos\phi \cdot \hat{S}_x + \sin\theta \sin\phi \, \hat{S}_y + \cos\theta \, \hat{S}_z$$

$$= \frac{\sin\theta \cos\phi}{2}(S_+ + S_-) + \frac{\sin\theta \sin\phi}{2i}(S_+ - S_-)$$

$$+ \cos\theta \cdot S_z$$

~~And ...~~

$$= \frac{S_+}{2}\sin\theta \, e^{-i\phi} + \frac{S_-}{2}\sin\theta \cdot e^{i\phi} + \cos\theta \cdot S_z .$$

Now we get:

$$S_{\hat{m}}|+\rangle = S_+ \frac{\sin\theta \cdot e^{-i\phi}}{2}|+\rangle + \frac{\sin\theta \, e^{i\phi}}{2} S_-|+\rangle$$

$$+ \cos\theta \cdot S_z|+\rangle$$

$$= 0 + \frac{\hbar}{2}\sin\theta \, e^{i\phi}|-\rangle + \frac{\hbar}{2}\cos\theta|+\rangle$$

Similarly,

$$S_{\hat{m}}|-\rangle = \frac{\hbar}{2}\sin\theta \cdot e^{-i\phi}|+\rangle - \frac{\hbar}{2}\cos\theta|-\rangle$$

Now for the given problem:

$$(S^a \cdot \hat{m}_a)(S^b \cdot \hat{m}_b)|\psi\rangle = (S_{\hat{m}_a} S_{\hat{m}_b}) \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

$$= \frac{S_{\hat{m}_a}|+\rangle_a \, S_{\hat{m}_b}|-\rangle - S_{\hat{m}_a}|-\rangle \, S_{\hat{m}_b}|+\rangle}{\sqrt{2}}.$$

$$= \frac{1}{\sqrt{2}}\left[\left\{\frac{\hbar \sin\theta_1 e^{i\phi_1}}{2}|-\rangle_a + \frac{\hbar}{2}\cos\theta_1|+\rangle_a\right\}\cdot\left\{\frac{\hbar}{2}\sin\theta_2 e^{-i\phi_2}|+\rangle_b - \frac{\hbar}{2}\cos\theta_2|-\rangle_b\right\} - \left\{\frac{\hbar}{2}\sin\theta_1 e^{-i\phi_1}|+\rangle_a - \frac{\hbar\cos\theta_1}{2}|-\rangle_a\right\}\left\{\frac{\hbar}{2}\sin\theta_2 e^{i\phi_2}|-\rangle_b + \frac{\hbar}{2}\cos\theta_2|+\rangle_b\right\}\right]$$

$$\begin{aligned} = \frac{\hbar^2}{4\sqrt{2}}\Big[& \sin\theta_1 \sin\theta_2\, e^{i(\phi_1-\phi_2)}\, |-+\rangle - \\ & \sin\theta_1 \cos\theta_2\, e^{i\phi_1}\, |--\rangle + \\ & \cos\theta_1 \sin\theta_2\, e^{-i\phi_2}\, |++\rangle - \\ & \cos\theta_1 \cos\theta_2\, |+-\rangle - \\ & \sin\theta_1 \sin\theta_2\, e^{i(\phi_2-\phi_1)}\, |+-\rangle - \\ & \sin\theta_1 \cos\theta_2\, e^{-i\phi_1}\, |++\rangle + \\ & \cos\theta_1 \sin\theta_2\, e^{i\phi_2}\, |--\rangle + \\ & \cos\theta_1 \cos\theta_2\, |-+\rangle \Big] \end{aligned}$$

So the given inner product be:

$$\langle S\hat{m}_a S\hat{m}_b\rangle = \langle\psi| S\hat{m}_a S\hat{m}_b |\psi\rangle$$

$$= \frac{\hbar^2}{4\times 2}\left(\langle +-| - \langle -+|\right)\left[\text{This term}\right]$$

$$= \frac{\hbar^2}{8}\left[-\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\, e^{i(\phi_2-\phi_1)} - \sin\theta_1\sin\theta_2\, e^{i(\phi_1-\phi_2)} - \cos\theta_1\cos\theta_2\right]$$

$-$

(using $\langle ++|++\rangle = 1$; $\langle +-|+-\rangle = 1$
$\langle +-|--\rangle = 0$; $\langle +-|-+\rangle = 0$ ... etc)

$$= -\frac{\hbar^2}{8}\left[2\cos\theta_1\cos\theta_2 + 2\sin\theta_1\sin\theta_2\,\frac{e^{i(\phi_1-\phi_2)}+e^{-i(\phi_1-\phi_2)}}{2}\right]$$

$$= -\frac{\hbar^2}{8}\left[2\cos\theta_1\cos\theta_2 + 2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\right]$$

So; in terms of $(\theta_1,\phi_1) \equiv \hat{m}_a$ and $(\theta_2,\phi_2) \equiv \hat{m}_b$; the value of the given expectation be:

$$\langle S_{\hat{m}_a} S_{\hat{m}_b}\rangle = -\frac{\hbar^2}{4}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 \cdot \cos(\phi_1-\phi_2)\right).$$

Ans

10. The spherical tensor of rank K be defined as:

$$u^{\dagger}(R)\, T_q^K\, u(R) = \sum_{q'=-K}^{K} D_{qq'}^{K*}\, T_{q'}^K$$

Here the compoments of the Tensor T changes according to this definition.

We use the spherical tensor T as general form of spherical harmonics Y_l^m while replacing $\hat{n}$ by arbitrary vector $\vec{V}$ with compoments (V_x, V_y, V_z)

So: $T_q^K = Y_K^q(\vec{V})$

Now expressions given in form of S.T:

i) $xy \longrightarrow Y_2^{\pm 2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, \sin^2\theta \cdot e^{\mp 2i\phi}$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, \frac{(x \pm iy)^2}{r^2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, \frac{x^2 - y^2 \pm 2ixy}{r^2}$$

So; $T_{\pm 2}^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\left(V_x^2 - V_y^2 \pm 2iV_xV_y\right)$

$\left(V_i = \frac{x_i}{r}\right)$

i.e $T_2^2 - T_{-2}^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,(4iV_xV_y)$

or, $V_xV_y = i\sqrt{\frac{2\pi}{15}}\cdot\left(T_{-2}^2 - T_2^2\right)$

To get xy we have to choose $\vec{V} = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

So; $xy = i\sqrt{\frac{2\pi}{15}}\left(T_{-2}^2 - T_2^2\right)$

ii) From the previous one:

$T_{-2}^2 + T_2^2 = \frac{1}{4}\cdot\sqrt{\left[\frac{2\pi}{15}\right]^{-1}}\, 2\left(V_x^2 - V_y^2\right)$

$\Rightarrow V_x^2 - V_y^2 = \sqrt{\frac{8\pi}{15}}\left(T_{-2}^2 + T_2^2\right)$

Choosing $\vec{V} = \vec{r}$ we get:

$x^2 - y^2 = \sqrt{\frac{8\pi}{15}}\left(T_2^2 + T_{-2}^2\right)$

iii) $xz \rightarrow$

$$Y_2^{\pm 1} = \mp \frac{1}{2}\sqrt{\frac{15}{2\pi}}\, e^{\pm i\phi}(\sin\theta \cos\theta)$$

$$= \mp \frac{1}{2}\sqrt{\frac{15}{2\pi}}\, \frac{(x \pm iy)z}{r^2}$$

$$= \mp \frac{1}{2}\sqrt{\frac{15}{2\pi}}\, (V_x V_z \pm i V_y V_z)$$

So;

$$T_1^2 - T_{-1}^2 = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\, 2 V_x V_z$$

$$\Rightarrow V_x V_z = \sqrt{\frac{2\pi}{15}}\,(T_{-1}^2 - T_1^2)$$

Taking $\vec{V} = \vec{r}$ we get:

$$xz = \sqrt{\frac{2\pi}{15}}\,(T_{-1}^2 - T_1^2)$$

[b] We have to calculate

$$\langle \alpha j m' | x^2 - y^2 | \alpha j j \rangle$$

Now from Wigner-Eckart theorem:

$$\langle \alpha' j' m' | T_q^k | \alpha j m \rangle = \langle j k\, m q | j k\, j' m' \rangle \frac{\langle \alpha' j' \| T^k \| \alpha j \rangle}{\sqrt{2j+1}}$$

Now; the evaluating term.

$$Q = \langle \alpha j m' | x^2 - y^2 | \alpha j j \rangle = \sqrt{\frac{8\pi}{15}}\, \langle \alpha j m' | T_{-2}^2 + T_2^2 | \alpha j j \rangle$$

$$= \sqrt{\frac{8\pi}{15}} \cdot \frac{\langle \alpha j \| T^2 \| \alpha j \rangle}{\sqrt{2j+1}} \Big[\langle j 2, j\, {-2} | j 2, j m' \rangle + \langle j 2, j 2 | j 2, j m' \rangle \Big]$$

But $\langle j 2, j 2 | j 2, j m' \rangle = 0$ since $m' = j+2$ is not possible.

$$\therefore Q = \sqrt{\frac{8\pi}{15}} \cdot \frac{\langle \alpha j \| T^2 \| \alpha j \rangle}{\sqrt{2j+1}} \langle j 2, j\, j{-2} | j 2, j m' \rangle$$

Now

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} \left(\frac{3z^2 - r^2}{r^2}\right) \rightarrow T_0^2 = \sqrt{\frac{5}{16\pi}}\,(3z^2 - r^2)$$

$$\therefore (3z^2 - r^2) = \sqrt{\frac{16\pi}{5}}\, T_0^2$$

$\therefore$ $\langle \alpha\, j\, m | 3z^2 - r^2 | \alpha\, j\, m \rangle$

$= \sqrt{\frac{16\pi}{5}} \langle \alpha\, j\, m | T_0^2 | \alpha\, j\, m \rangle$

$= \sqrt{\frac{16\pi}{5}} \frac{\langle \alpha j \| T^2 \| \alpha j \rangle}{\sqrt{2j+1}} \langle j2, j0 | j2, jj \rangle$

So, $\langle \alpha\, j\, m' | x^2 - y^2 | \alpha\, j\, j \rangle$

$= \frac{Q}{\sqrt{2}} \frac{\langle j2, j-2 | j2, jm' \rangle}{\langle j2, j0 | j2, jj \rangle}$ Ans.

11) Instead of $|\uparrow\downarrow\rangle$, I prefer to write $|+-\rangle$ for writting.

Now, $S = S_1 + S_2$ = total spin. $\Rightarrow S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$

$$\Rightarrow S_1 \cdot S_2 = \frac{S^2 - S_1^2 - S_2^2}{2}$$

So for $t > 0$; the Hamiltonian be given by:

$$H = \frac{4\delta}{\hbar^2} S_1 \cdot S_2 = \frac{2\delta}{\hbar^2}\left(S^2 - S_1^2 - S_2^2\right)$$

Clearly the eigenstates of H are simultaneous eigenstates of S^2, S_1^2, S_2^2. But for two spin $\frac{1}{2}$ particles; we get 4 those type of states. i.e

$$|11\rangle = |++\rangle; \quad |10\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}; \quad |1-1\rangle = |--\rangle; \quad |00\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}.$$

which form a complete orthonormal set.

Let at a later time $t \geq 0$; the state be given by:

$$|\psi(t)\rangle = a_1|11\rangle + a_2|10\rangle + a_3|1-1\rangle + a_4|00\rangle \quad (a_i = a_i(t))$$

$$\therefore H|\psi(t)\rangle = \frac{2\delta}{\hbar^2}\left(S^2 - S_1^2 - S_2^2\right)\left[a_1|11\rangle + a_2|10\rangle + a_3|1-1\rangle + a_4|00\rangle\right]$$

$$= \frac{2\delta}{\hbar^2}\cdot\hbar^2\Big[a_1\left(1(1+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|11\rangle +$$

$$a_2\left(1(1+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|10\rangle + \left(1(1+1) - \tfrac{1}{2}(1+\tfrac{1}{2}) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|1-1\rangle \cdot a_3$$

$$+ a_4\left(0(0+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|00\rangle\Big]$$

$$= 2\delta\left[\frac{1}{2}a_1|11\rangle + \frac{a_2}{2}|10\rangle + \frac{a_3}{2}|1-1\rangle - \frac{3}{2}a_4|00\rangle\right]$$

$$= \delta\left[a_1|11\rangle + a_2|10\rangle + a_3|1-1\rangle - 3a_4|00\rangle\right] \quad \cdots (1)$$

Now; $$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = i\hbar\left[\dot{a}_1|11\rangle + \dot{a}_2|10\rangle + \dot{a}_3|1-1\rangle + \dot{a}_4|00\rangle\right] \cdots (2)$$

As; $H|\psi\rangle = i\hbar\frac{\partial|\psi\rangle}{\partial t}$; So comparing (1) & (2) we get for coefficients of orthonormal states gives:

$$i\hbar\dot{a}_1 = \delta a_1 \Rightarrow \frac{\dot{a}_1}{a_1} = -\frac{i\delta}{\hbar}, \quad \int_{a_1^0}^{a_1}\frac{\dot{a}_1\,dt}{a_1} = -\frac{i\delta t}{\hbar}$$

$$\Rightarrow a_1 = a_1^0 e^{-\frac{i\delta t}{\hbar}}$$

Similarly $a_2 = a_2^0 e^{-\frac{i\delta t}{\hbar}}$; $a_3 = a_3^0 e^{-\frac{i\delta t}{\hbar}}$.

$a_4 = a_4^0 e^{\frac{3i\delta t}{\hbar}}$.

So; $|\psi(t)\rangle = \left(a_1^0 |11\rangle + a_2^0 |10\rangle + a_3^0 |1-1\rangle\right) e^{-\frac{i\delta t}{\hbar}}$

$+ a_4^0 e^{\frac{3i\delta t}{\hbar}} |00\rangle$

The initial state $(t=0)$ be given by $|\psi(0)\rangle = |+-\rangle = \frac{|10\rangle + |00\rangle}{\sqrt{2}}$.

plugging in: $a_1^0 = 0$; $a_2^0 = \frac{1}{\sqrt{2}}$; $a_3^0 = 0$; $a_4^0 = \frac{1}{\sqrt{2}}$.

So; $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i\delta t}{\hbar}} |10\rangle + e^{\frac{3i\delta t}{\hbar}} |00\rangle\right)$

In terms of $|++\rangle, |+-\rangle \cdots$ etc we get:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \left(e^{-\frac{i\delta t}{\hbar}}\left(|+-\rangle + |-+\rangle\right) + e^{\frac{3i\delta t}{\hbar}}\left(|+-\rangle - |-+\rangle\right)\right)$$

$$= \frac{1}{2}\left[|+-\rangle \cdot \left\{e^{\frac{3i\delta t}{\hbar}} + e^{-\frac{i\delta t}{\hbar}}\right\} + |-+\rangle \left\{e^{-\frac{i\delta t}{\hbar}} - e^{\frac{3i\delta t}{\hbar}}\right\}\right].$$

$$= \frac{1}{2}\left[2|+-\rangle \cdot e^{\frac{i\delta t}{\hbar}} \frac{\left(e^{\frac{2i\delta t}{\hbar}} + e^{-\frac{2i\delta t}{\hbar}}\right)}{2} - 2i|-+\rangle e^{\frac{i\delta t}{\hbar}} \frac{\left(e^{\frac{2i\delta t}{\hbar}} - e^{-\frac{2i\delta t}{\hbar}}\right)}{2i}\right]$$

$$= e^{\frac{i\delta t}{\hbar}}\left[\cos\left(\frac{2\delta t}{\hbar}\right) \cdot |+-\rangle - i \sin\left(\frac{2\delta t}{\hbar}\right) \cdot |-+\rangle\right]$$

So the probabilities to get $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ as a fn of time be given by respectively:

$0, \cos^2\left(\frac{2\delta t}{\hbar}\right), \sin^2\left(\frac{2\delta t}{\hbar}\right), 0$.

Ans

12.a. If $A \in SU(N)$ then ~~the~~ the conditions on A be:-

$\det(A) = 1$; $A^{+}A = I$.

Now; if $A \in SU(N)$ then obviously $A^{+} \in SU(N)$.

$I \in SU(N)$

$I^{+} = I = e$ (in $SU(N)$)

$\therefore$ For any element x in $SU(N)$ $\exists$ some $y : y = x^{-1}$

here $y = A^{+}$ for $x = A$.

If $A, B \in SU(N)$ then let $x \equiv A$; $y \equiv B$

$\therefore x \circ y = A \cdot B = z$ (let).

$\therefore z^{-1} = (AB)^{+} = B^{+}A^{+}$

And ~~δ~~ $z^{-1}z = (B^{+}A^{+}) \cdot (AB) = B^{+} I B = B^{+}B = I$.

$\therefore x \circ y \in$ the given set.

And $\det(AB) = \det(A)\det(B) = 1 \times 1 = 1$

So it follows the closure ~~pro~~ property.

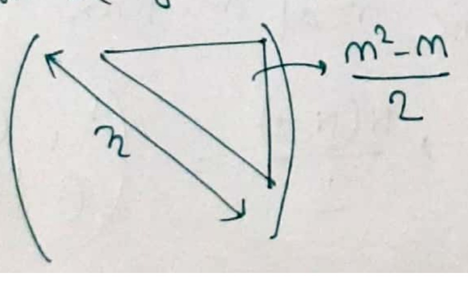
$\therefore$ The matrices in $SU(N)$ form a group.

b. 1) $SU(N)$: The n dimensional unitary matrix comes from exponentiating ~~A~~ a Hermitian matrix H by the form $\exp(iH)$

As if: $U = e^{iH}$

then $U^{+}U = e^{-iH} \cdot e^{iH} = I$.

Now for a general Hermitian matrix H:

H = 

There are n real no along the diagonal.
for the upper triangular part there are $\frac{n^2-n}{2}$ complex no. i.e (n^2-n) indp parameters.
The lower ~~tr~~ triangular part is just ~~a~~ complex conjugate of upper triangle as $H^+ = H$.

$\therefore$ Total no of ~~indp~~ parameters:

$$\frac{n^2-n}{2} \times 2 + n = n^2.$$

but for $SU(N)$; there is Another constrain of determinant.

~~trace.~~

$$\cancel{Tr(e^{iH}) = Tr(SU) = e^{Tr(iH)} = 1}$$

$$\det(e^{iH}) = (\det(SU(N)) = e^{Tr(iH)} = 1.$$

So; $Tr(iH) = 0$

This gives another constrain.

So no of indp parameters of $SU(N)$
= no of generators in $SU(N)$

$= N^2 - 1 \longrightarrow$ from constrain $Tr(iH) = 0$
$\hookrightarrow$ from $U(N)$

$= (N^2 - 1)$ Ans

ii) $O(N)$: If $R \in O(N)$; then R follows the rule $R^T R = R R^T = I$. ... (1)

So RR^T is a symmetric real matrix with N^2 ~~ent~~ items.

$RR^T = \left(N \right) \rightarrow \frac{N(N-1)}{2} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

As the lower part is equal to the upper triangular part of a symmetric real matrix; so no of constrains from $RR^T = I$ is

$$N + \frac{N(N-1)}{2} = \frac{N^2+N}{2}$$

$\therefore$ No of indep parameters = No of generators

$$= N^2 - \frac{N^2+N}{2} = \frac{N^2-N}{2} = \frac{N(N-1)}{2} \quad \text{Ans}$$

iii) SO(N) is special type of orthogonal matrix with the constrain $\det(R) = 1$.

Now it was actually in the definition of O(N) too.

As: $RR^T = I \Rightarrow \det(RR^T) = 1$

$$\Rightarrow \det(R) \cdot \det(R^T) = \det(R) \cdot \det(R) = (\det(R))^2 = 1$$

$$\Rightarrow \det(R) = \pm 1.$$

This is a disjoint condition; so doesn't bring a constrain to O(N) [As from U(N) to SU(N) one comes from $e^{i\gamma} = 1$; here we can't get like that.]

So No of generators in SO(N)

= " " " O(N)

$$= \frac{N(N-1)}{2} \quad \text{Ans}$$

iv) As SO(N) represents rotation in N dimension; and the rotation (in a plane) can be chosen in $^NC_2$ ways in N dimension (as one plane needs two axes from N i.e in $^NC_2$ ways); so no of generators (geometrically) in $SO(N) = {}^NC_2 = \frac{N(N-1)}{2}$ Ans

12.C. ▨ Previously I only know that the Gell-Mann matrices are the generators of SU(3). Obviously there will be $3^2-1=8$ matricy. But I don't know how to derive the generators of SU(N) generally for any N. (except that I have to use the basic rules like tracy will be zero; they will be Hermitian etc ....). My friend Souparna know this in a long ~~way~~ rigorous way. Generally other time we try to discuss among each other and then solve in own way. But now as we are at home so that type of discussion was not possible and I just have taken the problem from him. The complete credit goes ~~for~~ for Souporma and the evaluation completely depends on the ~~grader~~ grader.

▨ Using the Hermitian property, traceless property and the knowledge of SU(2); the 1st matrix be

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Another choice of the diagonal form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$ will be possible iff $a=-2$.

$$\therefore T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

After this I can't do from my own. I was going through his calculations. ~~[illegible]~~ However the final results (collected from wiki) are the

Sagar

Gell-Mann matrices are given by:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The assumed one

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};\quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix};\quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix};\quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Ans

13.a) $\psi(\vec{r}) = (x+y+3z)\cdot f(r)$

$= r\cdot f(r)\cdot[\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta]$.

Now from table

$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \Rightarrow 3\cos\theta = \sqrt{12\pi}\,Y_1^0(\theta,\phi)$

And, $Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi + i\sin\phi)$

$Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi - i\sin\phi)$.

$\therefore Y_1^{-1} - Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin\theta\cdot\times 2\cos\phi$

$\Rightarrow \sin\theta\cos\phi = \sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1)$

Similarly $\sin\theta\sin\phi = \sqrt{\frac{2\pi}{3}}\frac{(Y_1^{-1} + Y_1^1)}{i}$

$\therefore \psi(\vec{r}) = r\cdot f(r)\cdot\left[\sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1) - i\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1)\right.$

$\left. + \sqrt{12\pi}\,Y_1^0\right]$

$= r\cdot f(r)\cdot\left[Y_1^{-1}(1-i)\cdot\sqrt{\frac{2\pi}{3}} - Y_1^1(1+i)\sqrt{\frac{2\pi}{3}} + \sqrt{12\pi}\,Y_1^0\right]$

So the $l=1$ terms are present in the wavefn and hence the possible ~~values~~ of l only present is $l=1$.

So the state is an eigenstate of L^2 with

$L^2|\psi(\vec{r})\rangle = 1(1+1)\hbar^2|\psi(\vec{r})\rangle = 2\hbar^2|\psi(\vec{r})\rangle$.

13.b. From the calculated form of the wavefn in terms of $Y_l^m(\theta,\phi)$ we get ~~different~~ probabilities of different m values:

$m=0$: $P(m=0) = \frac{12\pi}{12\pi + \frac{2\pi}{3}\{|1+i|^2 + |1-i|^2\}}$

$$= \frac{12}{12 + \frac{2}{3}\{2+2\}} = \frac{9}{11}$$

$$P(m=1) = \frac{|1+i|^2 \cdot \frac{2\pi}{3}}{12 + \frac{2\pi}{3}\{|1+i|^2 + |1-i|^2\}} = \frac{\frac{2}{3}\times 2}{12 + \frac{2}{3}\times 4} = \frac{1}{11}$$

$$P(m=-1) = \frac{\frac{2\pi}{3}|1-i|^2}{12 + \frac{2\pi}{3}\{|1+i|^2 + |1-i|^2\}} = \frac{\frac{2}{3}\times 2}{12 + 8/3} = \frac{1}{11}.$$

Ans

Clearly $P(m=0) + P(m=1) + P(m=-1) = \frac{9+1+1}{11} = 1.$

13. iii) To find the energy eigenvalue we've to follow the steps:

i) For a given $f(r)$ we have to normalize $\psi(\bar{r})$.

ii) The radial part will be $\sim$ const $\times\, r f(r)$.
(normalization is not essential)

iii) $u = \frac{A r f(r)}{r} = A \cdot f(r)$ is an eigen ~~state~~ function of the radial eq:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu.$$

with $l = 1$

iv) every term except V is known.

So; $Vu = \left[E - \frac{\hbar^2}{2m}\cdot\frac{1(1+1)}{r^2}\right]u + \frac{\hbar^2}{2m}u''(r)$

$$\Rightarrow V = \frac{\left[E - \frac{\hbar^2}{mr^2}\right]\cdot u(r) + \frac{\hbar^2}{2m}u''(r)}{u(r)} = V(r).$$

Ans we get the form of $V = V(r)$.

Ans

14. In $|j, m_j\rangle$ notation the given state is $|2\ 0\rangle$ and the rotated state be $\mathscr{D}(R)|2\ 0\rangle$.
In the usual Eular ~~angle~~ angels; (θ, ϕ, ψ) which are taken w.r.t z, y', z' axes; the values for this problem be given by $\theta, \psi = 0$; $\phi = \alpha$.

Now let; $\mathscr{D}(R)|2\ 0\rangle = \sum_m \mathscr{D}(R)|2\ 0\rangle \cdot |2\ m\rangle\langle 2\ m|$

$$= \sum_m \underbrace{\langle 2\ m|\mathscr{D}(R)|2\ 0\rangle}_{C_m} \cdot |2\ m\rangle \quad \cdots (1)$$

However $\langle 2m|2\ 0\rangle$ be given by $Y_2^{m*}(\theta = 0, \phi)$
(I'm not going to derive the whole thing as the complete proof is not necessary. I'm just using the result from Sakurai.)

but $Y_2^{m*}(\theta = 0, \phi)$ be given by

$$: \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)\Big|_0^{l=2} \cdot \delta_m^0 \quad \left(\because \text{It gives only non-zero value for } m = 0\right)$$

$$= \sqrt{\frac{2l+1}{4\pi}}\Big|_{l=2} \delta_m^0 \quad (\because P_l(1) = 1)$$

So we get:

$\cancel{\mathscr{D}(R)|2\ 0\rangle = \sum_m \mathscr{D}_{m'}}$

$$\langle 2\ m'|\mathscr{D}(R)|2\ 0\rangle = \sum_m \langle 2m'|\mathscr{D}(R)|2m\rangle \langle 2m|2\ 0\rangle$$

$$= \sum_m \mathscr{D}_{m'm}(R) \cdot \sqrt{\frac{2l+1}{4\pi}}\, \delta_m^0 \Big|_{l=2}$$

$$= \mathscr{D}_{m'0}(R) \cdot \sqrt{\frac{2l+1}{4\pi}}\Big|_{l=2}$$

So from (1) we get:

$C_m = \langle 2m|\mathcal{D}(R)|20\rangle = \mathcal{D}_{m0}(R)\sqrt{\frac{2l+1}{4\pi}}\Big|_{l=2} = \sqrt{\frac{5}{4\pi}}\,\mathcal{D}_{m0}(R)$

Now $\langle 2m|\mathcal{D}(R)|20\rangle = Y_2^{m*}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\,\mathcal{D}_{m0}(R)$

So, $\mathcal{D}_{m0}(R) = \sqrt{\frac{4\pi}{5}}\,Y_2^{m*}(\theta,\phi)$

$\therefore \mathcal{D}(0,\alpha,0)|20\rangle = \sum_{m=-2}^{2}\mathcal{D}_{m0}(0,\alpha,0)\cdot|2m\rangle$

$\therefore$ The probability to find new state in $|2m\rangle$ state be: $|\mathcal{D}_{m0}(0,\alpha,0)|^2$

$= \frac{4\pi}{5}\left|Y_2^m(\theta=\alpha,\phi=0)\right|^2$

So; $\langle 20|\mathcal{D}(R)|20\rangle = \frac{4\pi}{5}\cdot\left|Y_2^0(\theta=\alpha,\phi=0)\right|^2$

$= \frac{4\pi}{5}\times\frac{5}{16\pi}\cdot(3\cos^2\alpha-1)^2 = \frac{1}{4}(3\cos^2\alpha-1)^2$

$\langle 2\pm1|\mathcal{D}(R)|20\rangle = \frac{4\pi}{5}\left|Y_2^{\pm1}(\alpha,0)\right|^2$

$= \frac{4\pi}{5}\cdot\frac{15}{8\pi}\sin^2\theta\cos^2\theta = \frac{3}{2}\sin^2\theta\cos^2\theta$

$\langle 2\pm2|\mathcal{D}(R)|20\rangle = \frac{4\pi}{5}\left|\langle Y_2^{\pm2}(\alpha,0)\rangle\right|^2$

$= \frac{4\pi}{5}\cdot\frac{15}{32\pi}\sin^4\alpha = \frac{3}{8}\sin^4\alpha$

Ans

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15. a) $\vec{\mu} = \gamma_1 \vec{J}_1 + \gamma_2 \vec{J}_2$ ($\vec{A}$ = Quantum vector operator)

So; $\mu_x = \gamma_1 J_1^x + \gamma_2 J_2^x$

$$= \gamma_1 \left(\frac{J_1^+ + J_1^-}{2} \right) + \gamma_2 \left(\frac{J_2^+ + J_2^-}{2} \right)$$

Now the given state $|j m j_1 j_2\rangle$ is a linear combination of the states $|j_1 m_1 j_2 m_2\rangle$

i.e $|j m j_1 j_2\rangle = \sum C^{j m j_1 j_2}_{m_1 m_2} |j_1 m_1 j_2 m_2\rangle$

$$\therefore \langle \mu_x \rangle = \langle j m j_1 j_2 | \mu_x | j m j_1 j_2 \rangle$$

$$= \left(\sum C^{j m j_1 j_2}_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | \right) \left(\gamma_1 \frac{J_1^+ + J_1^-}{2} + \gamma_2 \frac{J_2^+ + J_2^-}{2} \right) \left(\sum C^{j m j_1 j_2}_{m_1' m_2'} |j_1 m_1' j_2 m_2'\rangle \right)$$

(Such that $m_1 + m_2 = m = m_1' + m_2'$)

$$= \left(\sum C^{j m j_1 j_2}_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | \right) \times \sum C^{j m j_1 j_2}_{m_1' m_2'} \left[\frac{\gamma_1}{2} \alpha_1' |j_1, m_1'+1, j_2 m_2'\rangle + \frac{\gamma_1}{2} \alpha_2' |j_1, m_1'-1, j_2, m_2'\rangle + \frac{\gamma_2}{2} \alpha_3' |j_1 m_1' j_2 m_2'+1\rangle + \frac{\gamma_2}{2} \alpha_4' |j_1 m_1' j_2 m_2'-1\rangle \right]$$

($\alpha = \sqrt{(j \pm m)(j \mp m + 1)}$ = const)

Now for all states in the bra we get
$m_1 + m_2 = m$
And for the large ket part: $m_1' + m_2' \pm 1 = m \pm 1$
As the bra is the clebsch gordon expansion of $|j\, m\, j_1 j_2\rangle$ and the ket composed of such states for which $m_1' + m_2' \neq m$; So; each states in the bra is orthogonal to each states in the ket.

$$\text{e.g}\quad \langle \overset{j_1\, m_1\, j_2\, m_2}{1\ 0\ 1\ {-1}} | J_+ | \overset{j_1\, m_1'\, j_2\, m_2'}{1\ 0\ {-1}\ 1\ 0} \rangle$$

$$= \langle 1\,0\,1\,{-1} | (J_1^+ + j_2^+) | 1\,{-1}\,1\,0 \rangle$$

$$= \alpha_1 \underbrace{\langle 1\,0\,1\,{-1} | \overset{m_1'+1}{1\,0}\ \overset{m_2'}{1\,0} \rangle}_{0} + \alpha_2 \underbrace{\langle 1\,0\,1\,{-1} | \overset{m_1'}{1\,{-1}}\ \overset{m_2'+1}{1\,1} \rangle}_{0}$$

$$= 0$$

This situation happens for every term in the summation present in the expansion of $\langle M_x \rangle$.

So; by orthogonality we get,

$$\langle M_x \rangle = 0.$$

VIII Now for $\langle \mu_y \rangle$ the operator be given by:

$$\langle \mu_y \rangle = \frac{\gamma_1}{2i}\left(J_1^+ + J_1^-\right) + \frac{\gamma_2}{2i}\left(J_1^+ - J_2^-\right)$$

The calculation is totally similar like $\langle M_x \rangle$ with just 2 in denominator replaced by $2i$.

So by similar calculation we get

$$\langle \mu_y \rangle = \langle j\, m\, j_1 j_2 | f(J_1^\pm, J_2^\pm) | j\, m\, j_1 j_2 \rangle$$

$$= 0$$

(iii) $\mu_z = \gamma_1 J_1^z + \gamma_2 J_2^z = \frac{\gamma_1+\gamma_2}{2}(J_1^z + J_2^z) + \frac{\gamma_1-\gamma_2}{2}(J_1^z - J_2^z)$

but $J_1^z + J_2^z = J_z$.

So; $\mu_z = \frac{(\gamma_1+\gamma_2)}{2} \cdot J_z + \frac{(\gamma_1-\gamma_2)}{2}(J_1^z - J_2^z)$

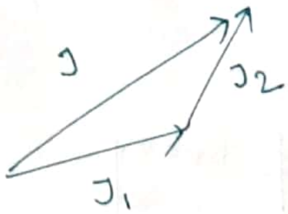
$|j\, m\, j_1 j_2\rangle$ are simultaneous ~~eigenstates~~ eigenstates of J^2, J_z, J_1^2, J_2^2; So.

$$\langle \mu_z \rangle = \frac{\gamma_1+\gamma_2}{2}\langle J_z\rangle + \frac{\gamma_1-\gamma_2}{2}\left(\langle J_1^z\rangle - \langle J_2^z\rangle\right)$$

but $\langle J_z \rangle = m\hbar$; So:

$$\langle \mu_z \rangle = \frac{\gamma_1+\gamma_2}{2} \cdot m\hbar + \frac{\gamma_1-\gamma_2}{2}\left(\langle J_1^z\rangle - \langle J_2^z\rangle\right) \quad \cdots (A)$$

Here comes the crucial part.



As $\langle J_1^x\rangle, \langle J_1^y\rangle = 0$; So we get

$$\langle J_1 \rangle = \underbrace{\langle J_1^x\rangle}_{=0} + \underbrace{\langle J_1^y\rangle}_{=0} + \langle J_1^z\rangle = \langle J_1^z\rangle$$

~~but from fig.~~ $\cancel{\langle J_1^{\text{along } J}\rangle = \left\langle \frac{(J_1 \cdot J)}{J^2} \cdot J \right\rangle = \langle J_1^z\rangle}$

~~but by comparing coefficient~~ $\cancel{\langle J_1^z\rangle = \frac{\langle (J_1 \cdot J)\rangle}{\langle J^2\rangle}\langle J_z\rangle}$.

$$\cancel{\langle J_1^z\rangle = \left\langle \frac{(J_1 \cdot J)}{J^2} \cdot J^z \right\rangle}$$

As J^z is conserved or better to say $|j\, m\, j_1 j_2\rangle$ is an eigenstate of J^z; So; the avg of J_1 is the avg of J_1 along 'J i.e. $\langle J_1 \rangle = \left\langle \frac{J_1 \cdot J}{J^2} \cdot J \right\rangle \quad \cdots (1)$

but $\langle J_1 \rangle = \langle J_1^z \rangle$: So comparing the z components from (1) we get:

$$\langle J_1^z \rangle = \left\langle \frac{J_1 \cdot J}{J^2} \right\rangle \langle J_z \rangle$$

Now, $J_2 = J - J_1 \Rightarrow J_2^2 = J^2 + J_1^2 - 2 J_1 \cdot J$

$$\Rightarrow J_1 \cdot J = \frac{J^2 + J_1^2 - J_2^2}{2} \Rightarrow \langle J_1 \cdot J \rangle = \left\langle \frac{J^2 + J_1^2 - J_2^2}{2} \right\rangle$$

Similarly $\langle J_2 \cdot J \rangle = \left\langle \frac{J^2 - J_1^2 + J_2^2}{2} \right\rangle$

So; $\langle J_1^z \rangle = \left\langle \frac{(J^2 + J_1^2 - J_2^2)}{2J^2} \cdot J_z \right\rangle$

Now $|j\, m\, j_1\, j_2\rangle$ is an eigenstate of every operator in the right paranthesis.

$$\therefore \langle J_1^z \rangle \Big|_{|j m j_1 j_2\rangle} = \left\langle \frac{(J^2 + J_1^2 - J_2^2)\, J_z}{2J^2} \right\rangle \Big|_{|j m j_1 j_2\rangle}$$

$$= \frac{\{ j(j+1) + j_1(j_1+1) - j_2(j_2+1) \} \hbar^2 \cdot m\hbar}{2j(j+1)\hbar^2}$$

$$= \frac{\{ j(j+1) + j_1(j_1+1) - j_2(j_2+1) \} \cdot m\hbar}{2j(j+1)}$$

Similarly:

$$\langle J_2^z \rangle = \frac{\{ j(j+1) - j_1(j_1+1) + j_2(j_2+1) \}}{2j(j+1)} m\hbar.$$

So from eq (A) we get:

$$\langle\mu_z\rangle = \frac{\gamma_1+\gamma_2}{2}\cdot m\hbar + \frac{\gamma_1-\gamma_2}{2}\cdot\frac{m\hbar}{2j(j+1)}\Big\{\cancel{j(j+1)} + j_1(j_1+1) - j_2(j_2+1) - \cancel{j(j+1)} + j_1(j_1+1) - j_2(j_2+1)\Big\}$$

$$= \frac{(\gamma_1+\gamma_2)}{2}\cdot m\hbar + \frac{(\gamma_1-\gamma_2)}{\cancel{2}}\cdot\frac{m\hbar\times\cancel{2}\left(j_1(j_1+1)-j_2(j_2+1)\right)}{2j(j+1)}$$

$$= m\hbar\left[\frac{\gamma_1+\gamma_2}{2} + \frac{(\gamma_1-\gamma_2)}{2}\cdot\frac{j_1(j_1+1)-j_2(j_2+1)}{j(j+1)}\right]$$

Proved

15.b excluding the factor of $\hbar$ in the previous expression, the term gives the magnetic moment factor of how many times the magnetic moment comes of the fundamental unit. (i.e Bohr magneton or nuclear magneton)

i.e if $\mu = \alpha\,\mu_{\text{fundamental}}$ then.

$$\alpha = m\left[\frac{\gamma_1+\gamma_2}{2} + \frac{\gamma_1-\gamma_2}{2}\cdot\frac{j_1(j_1+1)-j_2(j_2+1)}{j(j+1)}\right]$$

The given state: ${}^{2S+1}L_J = {}^2P_{1/2} \equiv |j\ m\ j_1\ j_2\rangle$

So; $S=\frac{1}{2}=J_1$; $L=1=J_2$; $J=\frac{1}{2}$. $\Rightarrow m=\pm\frac{1}{2}$.

And; $\gamma_1=\gamma_s=5.6$ (for proton); $\gamma_2=1$.

$$\text{So } \alpha_p = \pm\frac{1}{2}\left[\frac{6.6}{2} + \frac{4.6}{2}\cdot\frac{\frac{1}{2}(\frac{1}{2}+1)-1(1+1)}{\frac{1}{2}(\frac{1}{2}+1)}\right]$$

$$= \pm \frac{1}{2}\left[3.3 - 2.3 \times \frac{2-\frac{3}{4}}{3/4}\right] = \pm \frac{1}{2}\left[3.3 - 2.3 \times \frac{5}{3}\right]$$

$$= \pm 0.2666\cdots$$

$$\therefore \langle \mu_z \rangle_p = \pm 0.266\, \mu_{\text{nuclear magneton}}$$

c.) for e^- the only difference is: $\gamma_1 = 2$.

$$\text{So } \alpha_e = \pm \frac{1}{2}\left[\frac{3}{2} - \frac{1}{2} \times \frac{5}{3}\right]$$

$$= \pm \frac{1}{2} \times 0.666 = \pm 0.33$$

$$\therefore \langle \mu_z \rangle_e = \pm 0.33\, \mu_{\text{Bohr}}$$

$$\simeq \pm \frac{1}{3} \mu_{\text{Bohr}}.$$

proved