Statistical Mechanics - 2

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1. Finding $\langle E \rangle$ & C_v:—

The energy of a single particle with position (x,y,z) and momentum (p_x, p_y, p_z) be given by:

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz .$$

Now here as the system is very weakly interacting: we can use the formula:

$$Z = \frac{1}{N!}(Z_1)^N$$

when Z_1 is single particle partition fn.

$$Z_1 = \int e^{-\beta E} \cdot \frac{\prod dp_i \, dx_i}{h^3}$$

$$= \frac{1}{h^3} \int_{-\infty}^{\infty} e^{-\frac{\beta(p_x^2+p_y^2+p_z^2)}{2m}} dp_x dp_y dp_z \int_{\text{over } A} dx\,dy \cdot \int_0^h e^{-mg\beta z} dz .$$

$$= \frac{A}{h^3} \int_0^{\infty} 4\pi p^2 e^{-\frac{\beta p^2}{2m}} dp \cdot \int_0^h e^{-mg\beta z} dz .$$

$$= \frac{4\pi A}{h^3} \times \frac{1}{2} \cdot \Gamma\left(\frac{3}{2}\right) \cdot \left(\frac{2m}{\beta}\right)^{3/2} \cdot \left[\frac{e^{-mg\beta z}}{mg\beta}\right]_h^0$$

$$= \frac{2\pi A}{h^3} \cdot \frac{\sqrt{\pi}}{2} \cdot \left(\frac{2m}{\beta}\right)^{3/2} \cdot \frac{1}{mg\beta}\left(1 - e^{-mg\beta h}\right)$$

$$= \frac{A}{h^3} \times (2m\pi kT)^{3/2} \frac{\left(1 - e^{-mgh/kT}\right) \cdot kT}{mg}$$

So the partition fn of the system:

$$Z = \frac{1}{N!} \times \left[\frac{A}{h^3} (2m\pi kT)^{3/2} . \frac{kT \left(1 - e^{-mgh/kT}\right)}{mg} \right]^N.$$

So the energy density be given by:

$$\langle \varepsilon \rangle = \frac{U}{V} = \frac{-1}{V} \left[\frac{\partial}{\partial \beta} (\ln Z) \right]_{N,V}.$$

Here I've used Mathematica for the derivative. I'm just writting the result and the mathematica file will be sent also.

$$\langle \varepsilon \rangle = \frac{5N \left(1 - e^{ghm\beta}\right) + 2ghmN\beta}{2Ah\beta \left(1 - e^{ghm\beta}\right)}$$

$$= \frac{5N \left(1 - e^{\frac{ghm}{kT}}\right) + \frac{2ghmN}{kT}}{\frac{2Ah}{kT} \left(1 - e^{\frac{ghm}{kT}}\right)}$$

And, the specific ~~heat~~ heat be given by:

$$C_v = \frac{\partial U}{\partial T} = \frac{\partial U}{\partial \beta} . \frac{\partial \beta}{\partial T} = -k\beta^2 Ah \frac{\partial \langle \varepsilon \rangle}{\partial \beta}.$$

Using mathematica we find:

$$C_v = C_v(\beta)$$

$$= -Ahk\beta^2 \left[\frac{ghmN\left(2 - 5e^{ghm\beta}\right)}{2Ah\beta\left(1 - e^{ghm\beta}\right)} - \frac{2Ah\left(1 - e^{ghm\beta} - ghm\beta e^{ghm\beta}\right) \times \left(5N - 5Ne^{ghm\beta} + 2ghmN\beta\right)}{\left\{2Ah\beta\left(1 - e^{ghm\beta}\right)\right\}^2} \right]$$

$$= -\frac{Ah}{kT^2}\left[\frac{ghmNkT\left(2-5e^{\frac{ghm}{kT}}\right)}{\left(1-e^{\frac{ghm}{kT}}\right)} - \frac{2Ah\left(1-e^{\frac{ghm}{kT}}-\frac{ghm}{kT}e^{\frac{ghm}{kT}}\right)\times\left(5N-5Ne^{\frac{ghm}{kT}}+\frac{2ghmN}{kT}\right)k^2T^2}{\left\{2Ah\left(1-e^{\frac{ghm}{kT}}\right)\right\}^2}\right]$$

Now for weak gravitational field i.e $g \to 0$

$$\langle \varepsilon \rangle_{free} = \lim_{g\to 0} \langle \varepsilon \rangle = \lim_{g\to 0} \frac{5N\left(1-e^{ghm\beta}\right)+2ghmN\beta}{2Ah\beta\left(1-e^{ghm\beta}\right)}$$

$$= \frac{3N}{2Ah\beta} = \frac{3NkT}{2V}.$$

which is the ideal gas law in kinetic theory in terms of energy: $pV = NkT = \frac{2}{3}E = \frac{2}{3}\varepsilon V$.

Ans, $C_{v_{free}} = \lim_{g\to 0} C_V = \frac{3Nk}{2} = \frac{\partial}{\partial T}\left\{\langle \varepsilon \rangle_{free}\right\}$

(Limits have been evaluated using Mathematica)

2. Calculating expectations at equilibrium :-

If the two systems are denoted by a & b and their respective energies be E_a & E_b then

$$E_a + E_b = E = \text{const}$$

If the no of microstate be denoted by $\Omega(E)$ then the probability to find system (1) with energy E_a is:

$$P(E_a) = \frac{\Omega_a(E_a) \cdot \Omega_b(E_b)}{\Omega(E)}.$$

Now if at equilibrium, their corresponding energies be given by E_a^0 & E_b^0 then; by condition of equilibrium i.e $\beta_a = \beta_b$ gives,

$$\left.\frac{\partial \ln \Omega_a(E_a)}{\partial E_a}\right|_{E_a^0} = \left.\frac{\partial \ln \Omega_b(E_b)}{\partial E_b}\right|_{E_b^0}.$$

And $E_a + E_b = E = \text{const}$ i.e $d(E_a + E_b) = 0$

hence $dE_a = -dE_b$

So if we deviate the system a little bit from equilibrium; i.e

$$E_a^0 \rightarrow \underbrace{E_a^0 + \delta E_a}_{= E_a} ; \quad E_b^0 \rightarrow \underbrace{E_b^0 + \delta E_b}_{= E_b}.$$

then;

$$\ln \Omega_a(E_a) = \ln \Omega_a(E_a^0) + \left.\frac{\partial \ln \Omega_a(E_a)}{\partial E_a}\right|_{E_a^0} \cdot \delta E_a$$

$$+ \frac{\partial^2 \ln \Omega_a(E_a)}{\partial E_a^2}\bigg|_{E_a^0} \cdot (\delta E_a)^2 + \cdots \qquad \cdots (1)$$

Any, $\ln \Omega_b(E_b) = \ln \Omega_b(E_b^0) +$

$$\frac{\partial \ln \Omega_b(E_b)}{\partial E_b}\bigg|_{E_b^0} \cdot (\delta E_b) + \frac{\partial^2 \ln \Omega_b(E_b)}{\partial E_b^2}\bigg|_{E_b^0} \cdot (\delta E_b)^2$$

$$+ \cdots \qquad \cdots (2)$$

(1) + (2) gives:

$$\ln \Omega_a(E_a) + \ln \Omega_b(E_b) = \ln\{\Omega_a(E_a) \cdot \Omega_b(E_b)\}$$

$$= \ln\{\Omega_a(E_a^0)\,\Omega_b(E_b^0)\} + \underbrace{(\beta_a - \beta_b)}_{0}\bigg|_{E_a = E_a^0,\, E_b = E_b^0} \times (\delta E_a)$$

$$+ (\eta_a + \eta_b)\bigg|_{\substack{E_a = E_a^0 \\ E_b = E_b^0}} \times (\delta E_a)^2$$

(using fact that $\delta E_a^2 - \delta E_b$; $\beta_a = \beta_b$ at equilibrium i.e $E_a = E_a^0$; $E_b = E_b^0$)

$$\Rightarrow \ln\left[\Omega(E) \times \underbrace{\frac{\Omega_a(E_a) \cdot \Omega_b(E_b)}{\Omega(E)}}_{= P(E_a)}\right]$$

$$= \ln\left[\Omega(E) \times \underbrace{\frac{\Omega_a(E_a^0)\,\Omega_b(E_b^0)}{\Omega(E)}}_{P(E_a^0)}\right]$$

$$+ (\eta_a + \eta_b) \cdot \delta E_a^2$$

$$\Rightarrow \ln(\Omega(E) \cdot P(E_a)) = \ln(\Omega(E) \cdot P(E_a^0)) + (\eta_a + \eta_b) \cdot (\delta E_a)^2$$

i.e $P(E_a) = P(E_a^0) \times \exp\left[\frac{(\eta_b+\eta_a)}{2} \cdot (\delta E_a)^2\right]$

as $P(E_a)$ is maximum at $E = E_a^0$ (as that is the equilibrium) So the maximum of R.H.S occurs iff $\delta E_a = 0$.

So; $(\eta_a + \eta_b) = -ve. = -\eta$.

$$\therefore P(E_a) = P(E_a^0) \cdot \exp\left[-\frac{\eta}{2} \cdot (E_a - E_a^0)^2\right]$$

which is of the Gaussian form: $f(x) \sim e^{-\frac{(x-x_0)^2}{2\sigma^2}}$.

here the varience σ^2 be given by:

$$\sigma^2 = \frac{1}{\eta} = -\frac{1}{\eta_a + \eta_b}.$$

but $\sigma^2 = \langle (E_a - E_a^0)^2 \rangle = \langle (E_a - \langle E_a \rangle)^2 \rangle$

$$\text{So; } \langle (E_a - \langle E_a \rangle)^2 \rangle = -\frac{1}{\frac{\partial}{\partial E_a}\left(\frac{\partial \ln \Omega_a}{\partial E_a}\right)\Big|_{E_a^0} + \frac{\partial}{\partial E_b}\left(\frac{\partial \ln \Omega_b}{\partial E_b}\right)\Big|_{E_b^0}}$$

$$= -\frac{1}{\frac{\partial}{\partial E_a}\left(\frac{1}{T_a}\right)\Big|_{E_a^0} + \frac{\partial}{\partial E_b}\left(\frac{1}{T_b}\right)\Big|_{E_b^0}}$$

$\left(\because \frac{\partial \ln \Omega}{\partial E} = \frac{1}{T} \text{ by definition}\right)$

but $\frac{\partial}{\partial E}\left(\frac{1}{T}\right) = -\frac{1}{T^2}\frac{\partial T}{\partial E} = -\frac{1}{T^2} \cdot 1 \Big/ \left(\frac{\partial E}{\partial T}\right)$

$$= -\frac{1}{T^2} \times C^{-1}$$

$$\text{So: } \langle (E_a - \langle E_a \rangle)^2 \rangle = -\frac{1}{-\frac{1}{T_a^2} C_{1T_a}^{-1}\Big|_{E_a^0} + \left(-\frac{1}{T_b^2} C_{2T_b}^{-1}\right)_{E_b^0}}$$

but E_a^0 & E_b^0 being equilibrium energy: $T_a^0 = T_b^0 = T$

So; $\left\langle (E_a - \langle E_a\rangle)^2 \right\rangle_T = \dfrac{T^2}{(C_1^{-1} + C_2^{-1})_T}$

Similarly $\left\langle (E_b - \langle E_b\rangle)^2 \right\rangle_T = \dfrac{T^2}{(C_1^{-1} + C_2^{-1})_T}$

proved.

5. Interacting magnetic system:-

The Hamiltonian is given by:

$$H - \mu N = -U \sum_{i=1}^{N_{sites}} n_{i\uparrow} . n_{i\downarrow} - \mu \sum_{i=1}^{N_{sites}} (n_{i\uparrow} + n_{i\downarrow})$$

where; $n_i = 0, 1$

($\because$ for two e^- in a site both up & down have to present)

a) For one single site; the grand partition fn is given by:

$$Z_1 = \sum_{n_{i\uparrow}, n_{i\downarrow}} e^{-\beta(H - \mu N)}$$

$$= 1 + 2e^{\beta\mu} + e^{\beta(U + 2\mu)}$$

1: if $n_{i\uparrow}, n_{i\downarrow} = 0$

$2e^{\beta\mu}$: if $n_{i\uparrow}$ or $n_{i\downarrow}$ any one is 1 other is zero

$e^{\beta(U+2\mu)}$: if both of them are 1.

As the sites ~~themselves~~ themselves are non interacting; hence the Grand partition fn of the whole system is:

$$Z = Z_1^{N_{sites}} = \left[1 + 2e^{\beta\mu} + e^{\beta(U+2\mu)}\right]^{N_{sites}}$$

5.b. Here N_e = no of electrons in the system

As the chemical potential are due to the e^-

So; $N_e = \frac{1}{\beta} \cdot \frac{\partial}{\partial \mu}(\ln Z)$

$$= \frac{1}{\beta} \frac{\partial}{\partial \mu}\left[N_{sites} \ln\left(1 + 2e^{\beta\mu} + e^{\beta(U+2\mu)}\right)\right]$$

$$= \frac{N_{sites}}{\beta} \frac{2\beta e^{\beta\mu} + 2e^{\beta U} \cdot \beta e^{2\beta\mu}}{1+2e^{\beta\mu} + e^{\beta(U+2\mu)}}$$

So; for $N_e = N_{sites}$ we get:

$$N_{sites} = \frac{2N_{sites}\left(e^{\beta\mu} + 2e^{\beta(U+2\mu)}\right)}{1+2e^{\beta\mu} + e^{\beta(U+2\mu)}}$$

$$\Rightarrow 1 + \cancel{2e^{\beta\mu}} + e^{\beta(U+2\mu)} = \cancel{2e^{\beta\mu}} + 2e^{\beta(U+2\mu)}$$

$$\Rightarrow e^{\beta(2\mu+U)} = 1 = e^0$$

as $T \neq \infty$ so $2\mu + U = 0$

$$\Rightarrow \mu = -\frac{U}{2}$$

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5.c After applying external field:

$$H - \mu N = -U \sum_{i=1}^{N_{sites}} n_{i\uparrow} \cdot n_{i\downarrow} - \mu \sum_{i=1}^{N_{sites}} (n_{i\uparrow} + n_{i\downarrow})$$
$$- B \sum_{i=1}^{N_{sites}} (n_{i\uparrow} - n_{i\downarrow})$$

Here the single site partition fn is:

$$Z_1 = \sum_{n_{i\uparrow}, n_{i\downarrow}} e^{-\beta(H - \mu N)}$$

$$= \underset{\substack{n_{i\uparrow}=0 \\ n_{i\downarrow}=0}}{1} + \underset{\substack{n_{i\uparrow}=0 \\ n_{i\downarrow}=1}}{e^{-\beta(0 - \mu + B)}} + \underset{\substack{n_{i\uparrow}=1 \\ n_{i\downarrow}=0}}{e^{-\beta(0 - \mu - B)}} + \underset{n_{i\uparrow}=1=n_{i\downarrow}}{e^{-\beta(U - 2\mu + 0)}}$$

$$= 1 + e^{\beta(\mu - B)} + e^{\beta(\mu + B)} + e^{\beta(U + 2\mu)}$$

$$\therefore Z_{total} = \left[1 + e^{\beta(\mu - B)} + e^{\beta(\mu+B)} + e^{\beta(U + 2\mu)}\right]^{N_{sites}}$$

Here like the previous case

$$N_e = + \frac{1}{\beta} \frac{\partial}{\partial \mu} (\ln Z)$$

$$= \frac{1}{\beta} \frac{\partial}{\partial \mu} \left[N_{sites} \cdot \ln \left\{ 1 + e^{\beta(\mu+B)} + e^{\beta(\mu - B)} + e^{\beta(U + 2\mu)} \right\} \right]$$

$$= \frac{N_{sites}}{\cancel{\beta}} \cdot \frac{0 + \cancel{\beta} e^{\beta(\mu+B)} + \cancel{\beta} e^{\beta(\mu - B)} + 2\cancel{\beta} e^{\beta(U+2\mu)}}{\left(1 + e^{\beta(\mu+B)} + e^{\beta(\mu - B)} + e^{\beta(U+2\mu)}\right)}$$

$\therefore N_{sites} = N_e$ gives,

$$1 + \cancel{e^{\beta(\mu+B)}} + \cancel{e^{\beta(\mu-B)}} + e^{\beta(U+2\mu)}$$

$$= \cancel{e^{\beta(\mu+B)}} + \cancel{e^{\beta(\mu-B)}} + 2e^{\beta(2\mu+U)}$$

i.e. $e^{\beta(U+2\mu)} = 1$

i.e. $2\mu + U = 0 \Rightarrow \mu = -\frac{U}{2}.$

So; $Z_1 = 1 + e^{\beta(\mu+B)} + e^{\beta(\mu-B)} + 1.$

$$= 2 + 2e^{\beta\mu}\cosh(\beta B)$$

Here for a single site

$$\langle M \rangle = \langle m_{i\uparrow} - m_{i\downarrow} \rangle = \frac{\sum_{m_{i\uparrow\downarrow}} M e^{-(H-\mu N)\cdot\beta}}{Z_1}$$

$$= \frac{1}{Z_1} \cdot \sum_{m_{i\uparrow}, m_{i\downarrow}} (m_{i\uparrow} - m_{i\downarrow}) \cdot e^{-\beta(H-\mu N)}$$

$$= \frac{1}{Z_1}\Big[\underbrace{0}_{\substack{m_{i\uparrow}=0 \\ m_{i\downarrow}=0}} + \underbrace{(-1)\cdot e^{\beta(\mu-B)}}_{\substack{m_{i\uparrow}=0 \\ m_{i\downarrow}=1}} + \underbrace{1\cdot e^{\beta(\mu+B)}}_{\substack{m_{i\uparrow}=1 \\ m_{i\downarrow}=0}} + \underbrace{0}_{m_{i\uparrow}=1=m_{i\downarrow}}\Big]$$

$$= \frac{\cancel{2}e^{\beta\mu}\sinh(\beta B)}{\cancel{2}(1+e^{\beta\mu}\cosh(\beta B))}$$

So for N_{sites}: no of sites:

$$\langle M_{tot} \rangle = N_{sites} \times \langle M_1 \rangle = \frac{N_{sites}\, e^{\beta\mu}\sinh(\beta B)}{1+e^{\beta\mu}\cosh(\beta B)}.$$

So: $\frac{\partial}{\partial B}\langle M_{tot}\rangle \Big|_{B=0}$

$$= \frac{\partial}{\partial B}\left[N_{sites}\frac{e^{\beta\mu}\sinh(\beta B)}{(1+\cosh(\beta B)\cdot e^{\beta\mu})}\right]\Bigg|_{B=0}$$

$\downarrow$ Using Mathematica

$$= \frac{N_{sites}\times\beta e^{\beta\mu}}{1+e^{\beta\mu}} = \eta \text{ (say)}$$

: for very high temp : $T\to\infty$; $\beta = 0$

$$\lim_{\beta\to 0}(\eta) = \lim_{\beta\to 0}\frac{N_{sites}\times\beta\left(1+\mu\beta+\frac{\beta^2\mu^2}{2}+\cdots\right)}{1+\left(1+\mu\beta+\frac{\beta^2\mu^2}{2}+\cdots\right)}$$

$$\left.\begin{aligned} &= 0, \text{ at zeroth order of } \beta \\ &= \frac{\beta N_{sites}}{2} \text{ at first " " } \beta \end{aligned}\right\}$$

for very low temp: $\beta\to\infty$.

$$\lim_{\beta\to\infty}\frac{N_{sites}\,\beta e^{\beta\mu}}{1+e^{\beta\mu}} = \lim_{\beta\to\infty}\frac{\beta N_{sites}}{e^{-\beta\mu}+1}$$

$$= \beta N_{sites}\left(1-e^{-\beta\mu}+e^{-2\beta\mu}-\cdots\right)$$

$$\simeq \beta N_{sites}$$

Ans

4. Bose gas in 2-d :-

For the given problem $E(\vec{k}) = \sqrt{k_x^2 + k_y^2}$;
which is the energy for a massless gas particle
($E \sim |\vec{p}|$ i.e in terms of k : $E \sim |k|$)
taking $\hbar, c = 1$.

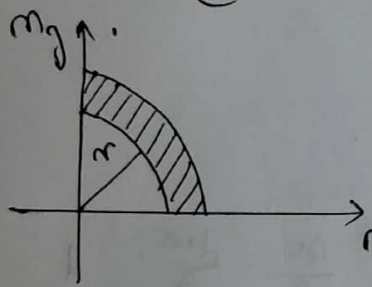
Now for a relativistic ~~Boson~~ or massless gas system

$$E = pc \Rightarrow \sum_i p_i^2 = \frac{E^2}{c^2} = \hbar^2 \sum k_i^2$$

Here the boundary condition gives $k_i = \frac{n_i \pi}{L}$

$$\therefore \frac{E^2}{c^2} = \frac{\hbar^2 \pi^2}{L^2} \sum n_i^2 = \frac{n^2 \pi^2 \hbar^2}{L^2} \quad (n^2 = \sum n_i^2)$$

$$\therefore \sum n_i^2 = \frac{L^2 E^2}{\pi^2 \hbar^2 c^2} \quad \ldots (1)$$



$\therefore$ The volume ~~...~~ of the section satisfying condition (1) with n_i value between n_i & $n + dn_i$ be given by:

$$d\tau = \frac{2\pi r \times dr}{4}$$

(4 in denominator to take the first quadrant as $n_i > 0$)

where r is given by condition (1) : $r = \frac{LE}{\pi \hbar c}$

$$\therefore dr = \frac{L\,dE}{\pi \hbar c}$$

$$\therefore d\tau = \frac{\cancel{2}\cancel{\pi}}{\cancel{4}_2} \times \frac{LE}{\cancel{\pi} \hbar c} \times \frac{L\,dE}{\pi \hbar c} = \frac{L^2 E\,dE}{2\pi \hbar^2 c^2}$$

As in n space the smallest cell volume be unity ($dn_i = 1$; as n_i is quantized.)
So total no of states for relativistic gas in energy range E & E+dE :

$$= \frac{dn}{dE} = \frac{L^2 E}{2\pi \hbar^2 c^2} = n(E)$$

taking $\hbar = 1$; $c = 1$ (in proper unit)

$$n(E) = \frac{L^2 E}{2\pi}.$$

∴ Total no of particles

$$N_{2D} = \int f(E)\, n(E)\, dE = \frac{L^2}{2\pi}\int_0^\infty \frac{E\, dE}{e^{\beta(E-\mu)}-1}$$

$$\cancel{= \frac{L^2}{2\pi}\int_0^\infty \frac{E}{e^{\beta(E-\mu)}}\frac{dE}{\left(1-e^{-\beta(E-\mu)}\right)}}$$

$$\cancel{= \frac{L^2}{2\pi}\int_0^\infty \frac{E}{e^{\beta(E-\mu)}}\left(\sum_{j=0}^{\infty} e^{-\beta j(E-\mu)}\right) dE.}$$

$$\cancel{= \frac{L^2}{2\pi}\int_0^\infty E\left(\sum_{j=1}^{\infty} e^{-\beta j(E-\mu)}\right) dE.}$$

$$\cancel{= \frac{L^2}{2\pi}\int_0^\infty E\left(\sum_{j=1}^{\infty} e^{-\beta j E}\cdot e^{\beta\mu j}\right) dE.}$$

$$= \frac{L^2}{2\pi}\int_0^\infty \frac{E\, dE}{e^{-\beta\mu}\, e^{\beta E}-1} = \frac{L^2}{2\pi}\int_0^\infty \frac{\frac{x}{\beta}\cdot\frac{dx}{\beta}}{e^{-\beta\mu}\, e^{x}-1}$$

$$= \frac{L^2}{2\pi}\,\frac{1}{\beta^2}\int_0^\infty \frac{x\, dx}{Z^{-1}e^{x}-1} \qquad \left(Z = e^{+\beta\mu}\right)$$

$$= \frac{L^2 T^2}{2\pi} f_2(Z).$$

As $Z \in (0,1)$ for physical system & N_{2D} is max for $f_2(Z) = \max$; i.e. $Z = 1$ so if the temp at condensed state be T_c then.

$$N_{2D} = \frac{L^2 T_c^2}{2\pi} f_2(1) = \frac{L^2 T_c^2}{2\pi}\cdot \zeta(2) = \frac{L^2 T_c^2}{2\pi}\times\frac{\pi^2}{6}.$$

$$= \frac{L^2 T_c^2 \pi}{12}$$

$\Rightarrow T_c = \sqrt{\dfrac{12 \cdot N_{2D}}{L^2 \pi}} > 0$

So at $T \in (0, T_c)$ the Bose gas is in a condensed state for massless Bosons; i.e B.E.C is possible.

VII) If the Bosons were massive then the density of state is a constant $\left(= \dfrac{2mL^2\pi}{\hbar^2}\right)$

$\therefore$ In that case:

$$N_{2D} = \int f(E) N(E)\, dE = \frac{2\pi L^2 m}{\hbar^2} \int_0^\infty \frac{dE}{z^{-1} e^{\beta E} - 1}.$$

$$= \frac{2\pi L^2 m}{\hbar^2 \beta} \int_0^\infty \frac{dx}{z^{-1} e^{x} - 1} = \frac{2\pi L^2 m}{\hbar^2 \beta} \cdot f_1(z).$$

$\therefore$ for $z \to 1$:

$$N_{2D} = \frac{2\pi L^2 m K T_c}{\hbar^2} G(1)$$

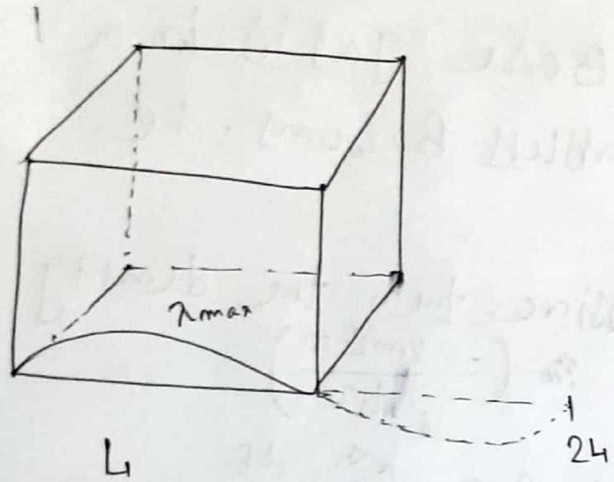
$$\therefore T_c = \frac{\hbar^2 N_{2D}}{2\pi L^2 m K} \times \frac{1}{G(1)}.$$

But $\lim_{z \to 1} G_1(z) = \infty$;

So $T_c = 0$

$\therefore$ In 2d massive boson the critical temp is OK. i.e Bose condensation is not possible in that situation.

3. Photon gas System:-



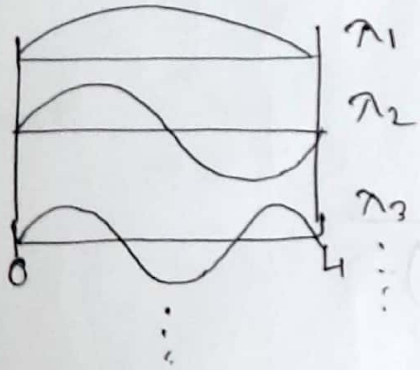
For the photon system;

$$E = pc = \hbar k c = \hbar \omega.$$

but from fig we get the max possible wavelength can be

$$\lambda_{max} = 2L.$$

moreover all possible λ is some integral division to fit a integral no of wave in the box. (i.e antinodes at the edges.)



$$\therefore \nu = \frac{c}{\lambda} = \frac{c}{2L/m} = \frac{mc}{2L}$$

$$\therefore E = h\nu = 2\pi\hbar\nu$$

$$= \hbar \frac{m\pi c}{L}$$

$$= m\hbar\omega_0 \quad \left(\omega_0 = \frac{\pi c}{L}\right)$$

(ω_0 = freq of zero mode)

Now for photon the chemical potential is zero ($\mu_{photon} = 0$) and the states are populated by no of photons.

So if m no of photons be populated at state E then $E = m\hbar\omega_0 = \hbar|\vec{k}_n|c$

with $|k_n| = \frac{n\omega_0}{c}$

So the partition fn is:

$$Z = \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega_0}{k_B T}} = \sum_{n=0}^{\infty} e^{-nx} \quad \left(x = \frac{\hbar\omega_0}{k_B T}\right)$$

i.e the mean no of photon at state $\langle n_E \rangle$

E_n or ~~[illegible]~~ or ~~[illegible]~~

or $\langle n_{\vec{k}} \rangle$ (anything can label the state)

is given by:

$$\langle n_{\vec{k}} \rangle_T = \frac{\sum_{n=0}^{\infty} n e^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}} = \frac{\sum_{n=0}^{\infty} n e^{-nx}}{1/(1-e^{-x})}$$

but $$\sum_{n=0}^{\infty} n e^{-nx} = -\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} e^{-nx}\right) = -\frac{\partial}{\partial x}\left(\frac{1}{1-e^{-x}}\right)$$

$$= -\frac{0 + e^{-x}}{-(1-e^{-x})^2} = +\frac{e^{-x}}{(1-e^{-x})^2}$$

$$\therefore \langle n_{\vec{k}} \rangle_T = \frac{e^{-x}}{(1-e^{-x})^2} \Big/ \left(1/(1-e^{-x})\right)$$

$$= \frac{e^{-x}}{1-e^{-x}}$$

And $$\langle n_k^2 \rangle = \frac{\sum n^2 e^{-nx}}{\sum e^{-nx}}$$

$$\sum n^2 e^{-nx} = \frac{\partial^2}{\partial x^2}\left(\sum e^{-nx}\right)$$

$$= -\frac{\partial}{\partial x}\left\{\frac{e^{-x}}{(1-e^{-x})^2}\right\}$$

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$$= -\left\{\frac{-e^{-x}}{(1-e^{-x})^2} + \frac{(-2)\,e^{-x}\cdot e^{-x}}{(1-e^{-x})^3}\right\}$$

$$= \frac{e^{-x}}{(1-e^{-x})^2}\left\{1 + \frac{2e^{-x}}{(1-e^{-x})}\right\}$$

So; $\langle n_{\vec{k}}^2\rangle = \dfrac{\partial^2 Z}{\partial x^2}\Big/ Z$

$$= \frac{e^{-x}}{(1-e^{-x})}\left\{1 + \frac{2e^{-x}}{(1-e^{-x})}\right\}$$

i.e $\langle n_{\vec{k}}^2\rangle - \langle n_{\vec{k}}\rangle^2 = \dfrac{e^{-x}}{1-e^{-x}}\left\{1 + \dfrac{2e^{-x}}{1-e^{-x}}\right\} - \dfrac{e^{-2x}}{(1-e^{-x})^2}$

$$= \frac{e^{-x}}{1-e^{-x}}\left\{1 + \frac{e^{-x}}{1-e^{-x}}\right\} = \frac{e^{-x}}{1-e^{-x}}\cdot\frac{1}{1-e^{-x}}$$

$$= \frac{e^{-x}}{(1-e^{-x})^2} = \frac{e^{x}}{(e^{x}-1)^2}$$

$$= \frac{e^{-\frac{\hbar\pi c}{L k_B T}}}{\left(1-e^{-\frac{\hbar\pi c}{L k_B T}}\right)^2} = \underbrace{\frac{e^{\frac{\hbar\pi c}{L k_B T}}}{\left(e^{\frac{\hbar\pi c}{L k_B T}}-1\right)^2}}_{\text{Ans}} \qquad \left(\because x = \frac{\hbar\omega_0}{k_B T} = \frac{\hbar\pi c}{L k_B T}\right)$$

Clearly $\sigma_{n_k}^2$ depends on length of the box (L).

3.b $N = \int g(k) \cdot n(k)\, dk.$

$= \int g(E) f(E)\, dE$. (obviously)

Now for 3d relativistic particle

$E = pc \Rightarrow \sum p_i^2 = \frac{E^2}{c^2}$

$\therefore$ Vol of the region in hypersphere of momentum space of dim n (in n-dim case) between E value of E & E+dE

$= d\tau = d\left\{\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \cdot \left(\frac{E}{c}\right)^n\right\} = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \left(\frac{E}{c}\right)^{n-1} \cdot \frac{dE}{c}$

$\therefore$ phase space volume

$d\Omega = V d\tau = \frac{n V \pi^{n/2} E^{n-1}\, dE}{\Gamma\left(\frac{n}{2}+1\right) c^n}$

$\therefore g(E) = \frac{1}{h^n} \frac{d\Omega}{dE} = \frac{n V \pi^{n/2} E^{n-1}}{\Gamma\left(\frac{n}{2}+1\right) c^n h^n}$

for 3d case like here.

$g(E) = \frac{3 V \pi^{3/2} E^2}{\frac{3\sqrt{\pi}}{4} \times (hc)^3} = \frac{4 V \pi E^2}{h^3 c^3}$

but $E = \hbar\omega = \hbar c k$

$\therefore g(k) = \frac{4 V \pi \hbar^2 c^2 k^2}{h^3 c^3} = \frac{4 V \pi k^2}{hc}$

$\therefore N = \frac{4 V \pi}{hc} \int_{k=0}^{\infty} k^2 n_k\, dk \times (\hbar c)$

$(\because dE = \hbar c\, dk)$

$= 4 V \pi \int k^2 n_k\, dk.$

$k \neq k'$

However this $g(k)$ calculation was done for free particle. Here the box is of finite size & hence there is a little modification. Although still $g(k) \sim k^2$.

$$\therefore \quad N^2 = \int 4\pi V k^2 n_k dk \cdot \int 4\pi V k_1^2 n_{k_1} dk_1$$

$$= 16\pi^2 V^2 \int k^2 n_k dk \cdot \int k_1^2 n_{k_1} dk_1$$

$$= 16\pi^2 V^2 \left(\int k^2 n_k dk\right)^2$$

$$= 16\pi^2 V^2 \left[\iint k^2 k_1^2 n_k n_{k_1} \delta(k-k_1)\, dk\, dk_1 + \iint_{k \neq k_1} k^2 k_1^2 n_k n_{k_1} dk\, dk_1\right]$$

$$= 16\pi^2 V^2 \iint \left(\frac{K_B T}{\hbar c}\right)^6 n_x n_{x'} dx\, dx' \cdot \delta(x-x') x^2 x'^2$$

$$+ 16\pi^2 V^2 \iint_{k \neq k_1} k^2 k_1^2 n_k n_{k_1} dk\, dk_1 \qquad x = \frac{\hbar k c}{K_B T}$$

$$= 16\pi^2 V^2 \left(\frac{K_B T}{\hbar c}\right)^6 \int_x x^4 n_x^2 dx$$

$$+ 16\pi^2 V^2 \iint_{k_1 \neq k} (\cdots)\, dk\, dk_1$$

Similarly $\langle N \rangle^2 = \int g(k) g(k') \langle n_k \rangle \langle n_{k'} \rangle dk\, dk'$

$$= 16\pi^2 V^2 \left(\frac{K_B T}{\hbar c}\right)^6 \int x^4 \langle n_x \rangle^2 dx$$

$$+ 16\pi^2 V^2 \iint_{k \neq k'} (\cdots)\, dk\, dk'$$

So; $\langle N^2 \rangle - \langle N \rangle^2$

$$= 16 V^2 \pi^2 \left(\frac{K_B T}{\hbar c}\right)^6 \cdot \int x^4 \left(\langle n_x^2 \rangle - \langle n_x \rangle^2\right) dx$$

$$= 16 V^2 \pi^2 \left(\frac{K_B T}{\hbar c}\right)^6 \cdot \int_0^\infty x^4 \cdot \frac{e^x}{(e^x - 1)^2} dx$$

(last result)

↓ using mathematica

$$= 16 V^2 \pi^2 \left(\frac{K_B T}{\hbar c}\right)^6 \times 25.976.$$

1) Reference:- i) Pathria - Statistical mechanics
ii) dfcd.net (for B.E.C problem)
iii) Quantum mechanics - Bransden

ii) Acknowledgement:-
I'm very much thankful to my friend Akashdeep, who has helped me for the tricky integral in 3.b part. (use of δ fn and substitution etc)

— Sagar Dam