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PART – 1

1. Here at equilibrium; the total no should be conserved as there is no further interactions between the particles.

i.e $\underbrace{n_\sigma^F(p) + n^B(p)}_{\text{for all possible } p} = N_{tot} = \text{const}.$

Now, $n^B(p) = f(E^B) = \dfrac{1}{z_1^{-1} e^{\frac{\beta p^2}{4m}} e^{-\beta E_B} - 1} = \dfrac{1}{z_1^{-1} e^{\beta E} - 1}$ $\quad \left(z_1^{-1} = e^{-\beta(\mu + E_B)} = z^{-1} e^{-\beta E_B}\right)$

when the degeneracy factor is given by:

$$g(E) = \frac{4 m' \sqrt{m'}\, V \pi \sqrt{2}}{h^3} \cdot \sqrt{E}. \qquad m' = 2m$$

in range E & $E + dE$.

∴ Total no of bose particle

$$= N^B(E) = f(E)\, g(E)\, dE$$

$$= \frac{16\, m\sqrt{m}\, \pi V}{h^3} \cdot \sqrt{E} \cdot \frac{dE}{z_1^{-1} e^{\beta E} - 1}.$$

Or if we write in terms of $\vec{n}$ then:

$$E = \frac{p^2}{4m} = \frac{4\pi^2 \hbar^2}{L^2 \times 4m}(n_x^2 + n_y^2 + n_z^2)$$

$$= \frac{\pi^2 \hbar^2}{m L^2}(n_x^2 + n_y^2 + n_z^2).$$

So in terms of $\vec{n}$ (here we dont have to include degeneracy factor $g(E)$ as we take sum for all possible n_x, n_y, n_z differently.)

$\left(z_1^{-1} = e^{-\alpha - \beta E_B} = e^{-\alpha'}\right)$

$$\therefore N^B = \sum_E \frac{1}{z_1^{-1} e^{\beta E} - 1}$$

$$= \sum_E \frac{e^{-(\alpha' + \beta E)}}{1 - e^{-(\alpha' + \beta E)}}$$

$$= \sum_E \sum_{j=1}^{\infty} e^{-j(\alpha' + \beta E)}$$

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$$= \sum_{j=1}^{\infty} e^{-j\alpha'} \left[\sum_{n_x} e^{-j\omega n_x^2} \cdot \sum_{n_y} e^{-j\omega n_y^2} \cdot \sum_{n_z} e^{-j\omega n_z^2} \right]$$

$$\left(\omega = \frac{\pi^2 \beta \hbar^2}{mL^2} \right)$$

So; $N_B = \sum_{j=1}^{\infty} e^{-j\alpha'} \left[\sum_{n_x} e^{-j\omega n_x^2} \cdot \sum_{n_y} e^{-j\omega n_y^2} \cdot \sum_{n_z} e^{-j\omega n_z^2} \right]$

Similarly for fermi system with up/down spin we get

$$E = \frac{4\pi^2\hbar^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$= \frac{2\pi^2\hbar^2}{mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$\therefore N^F = \sum_E \frac{2}{e^{\alpha+\beta E}+1}$$

(2 as there are ↑ & ↓ spins at eq α is same for Bose & Fermi)

$$= 2\sum_E \frac{e^{-(\alpha+\beta E)}}{1+e^{-(\alpha+\beta E)}}$$

$$= 2\sum_E \sum_{j=1}^{\infty} (-1)^{j-1} \cdot e^{-j(\alpha+\beta E)}$$

$$= 2\sum_{j=1}^{\infty} (-1)^{j-1} e^{-j\alpha} \cdot \left[\sum_{n_x} e^{-j\omega' n_x^2} \cdot \sum_{n_y} e^{-j\omega' n_y^2} \sum_{n_z} e^{-j\omega' n_z^2} \right]$$

$$\left(\omega' = \frac{2\pi^2\hbar^2\beta}{mL^2} = 2\omega \right)$$

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So the number conservation gives:

$$\sum_{j=1}^{\infty} e^{-j\alpha'}\left(\sum_{n=-\infty}^{\infty} e^{-j\omega n^2}\right)^3 + 2\sum_{j=1}^{\infty}(-1)^{j-1} e^{-j\alpha}\left(\sum_{m=-\infty}^{\infty} e^{-2j\omega m^2}\right)^3 = N_{tot}.$$

$$\Rightarrow N_{tot} = \sum_{j=1}^{\infty} e^{-j\alpha}\left\{\left(\sum_{n=-\infty}^{\infty} e^{-j\omega n^2}\right)^3 e^{-\beta j E_B} + 2\cdot(-1)^{j-1}\left(\sum_{n'=-\infty}^{\infty} e^{-2j\omega n'^2}\right)^3\right\} \quad \cdots (1)$$

Ans

$$\left(\alpha = -\frac{\mu}{KT}, \quad \omega = \frac{\pi^2\beta\hbar^2}{mL^2}\right)$$

However the sum is not simple to evaluate. Instead of that if we use the thermodynamic limit & let $L \gg$ microscopic limit; then we can directly do the integral to get:

$$N^B = \frac{V}{\lambda_B^3} g_{3/2}\left(e^{\beta(\mu+E_B)}\right); \quad \lambda_B = \frac{h}{(4\pi m KT)^{1/2}}.$$

$$N^F = \frac{2V}{\lambda_F^3} \cdot f_{3/2}\left(e^{\beta\mu}\right); \quad \lambda_F = \frac{h}{(2\pi m KT)^{1/2}}$$

(2 for ↑ & ↓)

$f_{3/2}$ & $g_{3/2}$ are Fermi & Bose functions of order 3/2.

So; $$\frac{V}{\lambda_B^3} g_{3/2}\left(e^{\beta(\mu+E_B)}\right) + \frac{2V}{\lambda_F^3} f_{3/2}\left(e^{\beta\mu}\right) = N_{tot} \quad \cdots (2)$$

Ans

(1) & (2) both are expressions of N_{tot} which are expressed as a function of a fn of the Z (or α) i.e. common chemical potential of equilibrium μ.

12. $$\underbrace{\int_p \epsilon^B(p)\cdot g^B(p)\, n^B(p)\, dp}_{E^B_{tot}} + \underbrace{2\int_p \epsilon^F_\sigma(p)\, g^F(p)\, n^F_\sigma(p)\, dp}_{E^F_{tot}} = E^{tot}$$

(for any one value of σ.)

here $g(p)\, dp = g_E(E)\, dE$

$$= \frac{4m\sqrt{m}\,\pi V\sqrt{2}}{h^3}\sqrt{E}\, dE$$

$$E = \frac{p^2}{2m} \Rightarrow dE = \frac{p\,dp}{m}$$

$$\therefore g(p) = \frac{4\cancel{m}\cancel{\sqrt{m}}\,\pi V \cancel{\sqrt{2}}}{h^3\, \cancel{m}} \cdot \frac{p^2}{\cancel{\sqrt{2}}\cdot\cancel{\sqrt{m}}} = \frac{4\pi V}{h^3} p^2$$

So; in thermodynamic limit $(\Sigma \to \int)$:

$$E^{tot} = \int_{p=-\infty}^{\infty} \frac{4\pi V p^2}{h^3}\left\{\epsilon^B(p)\, n^B(p) + 2\,\epsilon^F_\sigma(p)\, n^F_\sigma(p)\right\} dp$$

Ans

~~If we integrate using expressions of~~ $\cancel{\epsilon(p) \sim \frac{p^2}{2m}\ ;\ n(p) = \frac{1}{z^{-1}_{F/B}\, e^{\frac{\beta p^2}{2m}} \pm 1}}$ $\left(\begin{matrix} z_F = z = e^{\alpha} \\ z_B = e^{\alpha+\beta E_B}\end{matrix}\right)$

~~then~~ $\cancel{\left(e^{\beta(\mu+E_B)}\right)}$ ~~give just used the known~~

$\cancel{E^B = \frac{3}{} \frac{L^3}{} g}$

So; $E_{tot} = \frac{4\pi m'\sqrt{m'} V\sqrt{2}}{h^3} \int_0^\infty \frac{(E-E_B)\cdot\sqrt{E}}{Z_1^{-1} e^{-\beta E} - 1} dE$

$+ 2 \times \frac{4\pi m \sqrt{m} V\sqrt{2}}{h^3} \int_0^\infty \frac{E\sqrt{E}}{z^{-1} e^{-\beta E} - 1} dE$

$\left(m' = 2m; \; Z_1^{-1} = z^{-1} e^{-\beta E_B} = e^{-\alpha - \beta E_B} \right)$

The 2nd term is $\frac{2}{\lambda_F^3} f_{5/2}(e^{\beta\mu}) \times \frac{3V}{2\beta}$

The 1st integral is:

$\frac{4 m'\sqrt{m'} V\sqrt{2}\pi}{h^3} \left[\int_0^\infty \frac{E\sqrt{E}\, dE}{Z_1^{-1} e^{-\beta E} - 1} - \int_0^\infty \frac{E_B \sqrt{E}\, dE}{Z_1^{-1} e^{-\beta E} - 1} \right]$

$= \frac{3V}{2\beta\lambda_B^3} g_{5/2}(Z_1) - \frac{V E_B}{\lambda_B^3} g_{3/2}(Z_1)$

i.e

$E_{tot} = \frac{3V}{2\beta\lambda_B^3} g_{5/2}\left(e^{\beta(\mu+E_B)}\right) - \frac{V E_B}{\lambda_B^3} g_{3/2}\left(e^{\beta(\mu+E_B)}\right)$

$+ \frac{3V}{\beta\lambda_F^3} f_{5/2}\left(e^{\beta\mu}\right)$

clearly at $E_B \to 0$

$E_{tot} \to \frac{3}{2} PV\Big|_{Bose} + 3 PV\Big|_{Fermi}$

which is obvious.

Ans

3. As the two types of particles do not interact at equilibrium, we get

$$\mathcal{Z}_{grand} = \mathcal{Z}^{Fermi}_{grand} \cdot \mathcal{Z}^{Bose}_{grand}.$$

Now if we consider just one simple gas system (not for this problem) then for Fermi or Bose gas we get:

$$\mathcal{Z} = \sum_s \sum_r e^{-\beta(E_r - \mu N_s)}$$

$$= \sum_{N_s} e^{+\alpha N_s} \times \sum_{E_r} e^{-\beta E_r} \qquad (\alpha = \mu\beta)$$

This means N_s no of particles is populated in single particle state $\varepsilon_r = \frac{E_r}{N_s}$

$$\therefore \mathcal{Z} = \sum_{N_s} e^{+\alpha N_s} \cdot \sum_{\{N_s\}} e^{-\beta \varepsilon_r N_s}$$

But this is canonical grand fn of $N = \sum N_s$ particles i.e Q_{N_s}

$$\therefore \mathcal{Z} = \sum_N e^{+\alpha N} \prod_\varepsilon (e^{-\beta\varepsilon})^N = \prod_\varepsilon \sum_N e^{+\alpha N} (e^{-\beta\varepsilon})^N$$

$\therefore$ At energy level ε the ~~particul~~ partition fn be given by:

$$\mathcal{Z}_\varepsilon = \sum_N e^{+\alpha N} (e^{-\beta\varepsilon})^N = \sum_N (z e^{-\beta\varepsilon})^N.$$

$$(z = e^{\alpha})$$

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Now as we are dealing with one microstate of energy ε in Z_ε; we have to sum for $N = 0, 1, 2, \ldots$ for Bose
$N = 0, 1$ for Fermi

$$\therefore Z_\varepsilon^B = 1 + ze^{-\beta\varepsilon} + \left(ze^{-\beta\varepsilon}\right)^2 + \cdots$$

$$= \frac{1}{1 - ze^{-\beta\varepsilon}}$$

$$Z_\varepsilon^F = 1 + ze^{-\beta\varepsilon}$$

So; the whole partition fn when we vary energy we get:

$$Z^{Bose} = \prod_\varepsilon Z_\varepsilon^B = \prod_\varepsilon \frac{1}{1 - ze^{-\beta\varepsilon}}$$

$$Z^{Fermi} = \prod_\varepsilon Z_\varepsilon^F = \prod_\varepsilon \left(1 + ze^{-\beta\varepsilon}\right)$$

$\therefore$ The total grand partition fn of the system be given by:

$$Z = Z^{Bose} \cdot Z^{Fermi}$$

$$= \prod_{\varepsilon_B} \frac{1}{\left(1 - ze^{-\beta\varepsilon_B}\right)} \left\{ \prod_{\varepsilon_F} \left(1 + ze^{-\beta\varepsilon_F}\right) \right\}^2$$

In Fermi; the square comes for $\uparrow$ & $\downarrow$ contributions.

$\therefore \; Z_{grand} = \prod_{\varepsilon_F} \prod_{\varepsilon_B} \dfrac{(1+ze^{-\beta\varepsilon_F})^2}{(1-ze^{-\beta\varepsilon_B})}$ Ans

Now; $\varepsilon_F = \dfrac{p^2}{2m}$; $\varepsilon_B = \dfrac{p^2}{4m} - E_B$ & p^2 varies from 0 to ∞ in interval

$$p^2 = \frac{4\pi^2 m^2 \hbar^2}{L^2}$$

Here in the expression the product is over all microstates.

4. If we write Z_{grand} in terms of macrostates then we have to introduce the degeneracy factor $g(E)$

So; $Z_{grand} = \prod_{E_B} \prod_{E_F} \dfrac{(1+ze^{-\beta E_F})^{2g_F(E_F)}}{(1-ze^{-\beta E_B})^{g_B(E_B)}}$.

when in thermodynamic limit; and E_F & E_B changes from 0 to ∞; and $-E_{binding}$ to ∞.

So: $\ln(Z_{grand})$

$$= \sum_{E_F} 2g_F(E_F) \ln(1+ze^{-\beta E_F}) - \sum_{E_B} g_B(E_B) \ln(1-ze^{-\beta E_B})$$

This E_B is not the E_B of binding energy. Sorry for using same notation by mistake.

In thermodynamic limit ; integration gives.

$$\ln(Z_{grand}) = 2\int_0^\infty g_F(E)\ln(1+ze^{-\beta E})\,dE - \int_0^\infty g_B(E)\ln(1-z_1e^{-\beta E})\,dE.$$

$$\left.\begin{aligned} g_F(E) &= \frac{4m\sqrt{m}\pi V\sqrt{2}\sqrt{E}}{h^3} \\ g_B(E) &= \frac{16\, m\sqrt{m}\, V\pi\sqrt{E}}{h^3} \end{aligned}\right\} \qquad \begin{aligned} z &= e^{\beta\mu} \\ z_1 &= e^{\beta(\mu+E_B)} \\ &= e^{\alpha'} \\ \text{or } \alpha' &= \alpha+\beta E_B. \end{aligned}$$

$$\therefore \langle N_{tot}\rangle = -\frac{\partial}{\partial\alpha}(\ln Z_{grand})$$

$$= \int_0^\infty \frac{4m'\sqrt{m'}\pi V\sqrt{2}}{h^3}\cdot\frac{\sqrt{E}\cdot e^{\alpha'}e^{-\beta E}}{1-e^{\alpha'}e^{-\beta E}}\,dE$$

$$+ 2\int_0^\infty \frac{4m\sqrt{m}\,V\pi\sqrt{2}}{h^3}\,\frac{\sqrt{E}\,e^{\alpha}e^{-\beta E}}{1+e^{\alpha}e^{-\beta E}}\,dE$$

(substituting $z = e^{\alpha}$ in first line of page) $\left(\begin{aligned} m' &= 2m \\ \alpha' &= \alpha+\beta E_B \end{aligned}\right)$

$$= \frac{4m'\sqrt{m'}\,V\pi\sqrt{2}}{h^3}\int_0^\infty \frac{\sqrt{E}}{z_1^{-1}e^{\beta E}-1}\,dE$$

$$+ 2\cdot\frac{4m\sqrt{m}\,V\pi\sqrt{2}}{h^3}\int_0^\infty \frac{\sqrt{E}}{z^{-1}e^{\beta E}+1}\,dE$$

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$$= \frac{4m'\sqrt{m'}\, V\pi\sqrt{2}}{h^3}\cdot\left(\frac{2}{\sqrt{\pi}}\right)^{-1}\cdot\frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^\infty \frac{\sqrt{x/\beta}\; dx/\beta}{z^{-1}e^{x}-1}$$

$$+ \frac{4m\sqrt{m}\, V\pi\sqrt{2}}{h^3}\left(\frac{\sqrt{\pi}}{2}\right)\cdot\frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^\infty \frac{\sqrt{x/\beta}\;\frac{dx}{\beta}}{z^{-1}e^{x}+1}$$

$$\left(x = \frac{E}{\beta} \Rightarrow dx = \frac{dE}{\beta} \right)$$

$$= \frac{4m'\sqrt{m'}\, V\pi\sqrt{2}}{\beta\sqrt{\beta}\cdot h^3}\cdot\frac{\sqrt{\pi}}{2}\cdot g_{3/2}(z_1)$$

$$+ 2\cdot\frac{4m\sqrt{m}\, V\pi\sqrt{2}}{\beta\sqrt{\beta}\, h^3}\cdot\frac{\sqrt{\pi}}{2}\; f_{3/2}(z).$$

$$= \frac{V(4m\pi KT)^{3/2}}{h^3}\, g_{3/2}(z_1)$$

$$+ \frac{2V(2m\pi KT)^{3/2}}{h^3}\, f_{3/2}(z)$$

$$= \frac{V(2\pi mKT)^{3/2}}{h^3}\left[2\sqrt{2}\, g_{3/2}\left(e^{\beta(\mu+E_B)}\right) + 2 f_{3/2}\left(e^{\beta\mu}\right)\right]$$

Ans

$= \langle N_{tot} \rangle$ as obtained in part 1

Which is same as of the question

5. Here I think, I don't have to do anything new as I've already taken thermodynamic limit in Q-4 and evaluated the answer. However I'm just rewritting the expressions once more:

$$P_{tot} = \frac{1}{L^3} \langle N_{tot} \rangle = \frac{1}{V} \langle N_{tot} \rangle$$

$$= \underbrace{\frac{4m\sqrt{m}\,\pi\sqrt{2}}{h^3} \int_0^\infty \frac{\sqrt{E}\, dE}{e^{-\alpha'} e^{\beta E} - 1}}_{P^B_{tot} = I_B} + \underbrace{\frac{\overset{\times 4}{2m\sqrt{m}\,\pi\sqrt{2}}}{h^3} \int_0^\infty \frac{\sqrt{E}\, dE}{e^{-\alpha} e^{\beta E} + 1}}_{P^F_{tot} = I_F}$$

including ↑ & ↓

$$= \frac{(2\pi m k T)^{3/2}}{h^3} \left[2\sqrt{2}\, g_{3/2}\left(e^{\beta(\mu + E_B)}\right) + 2 f_{3/2}\left(e^{\beta\mu}\right) \right]$$

Ans

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6. $P_{tot} = I_B + I_F$

Now $I_B \sim \int_0^\infty \frac{\sqrt{x}\, dx}{e^{-\alpha'} \cdot e^{x} - 1}$

So the integral will not diverge if for all possible x (i.e $\beta p^2/4m$) the denominator is non zero.

i.e $e^{-\alpha'} e^{x} - 1 \geq 0$ $\left(\because \text{ at } x \to \infty \;\; e^{x-\alpha'} \to \infty\right)$

$\Rightarrow$ i.e $e^{-\alpha'} \cdot e^{x} \geq 1$

but the minimum value of e^x is 1 for $x = 0$.

So $e^{-\alpha'} \geq 1$ i.e $-\alpha' \geq 0$

i.e $-\beta(\mu + E_B) \geq 0$

i.e $\beta(\mu + E_B) \leq 0$

$$\boxed{\text{i.e } \mu \leq -E_B.}$$

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PART - 2

For all calculation from here I take $E_B = 0$

i.e. in previous results $\left.\begin{array}{l} z_1 \text{ is replaced by } z \\ \alpha' \quad " \quad " \quad " \quad \alpha \end{array}\right\}$

1. $E_B = 0$ i.e $\mu \leq -E_B$ gives $\mu \leq 0$.

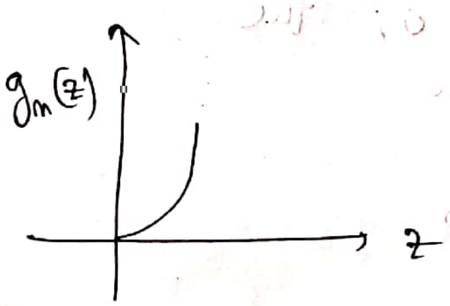
$\therefore$ $z = e^{\beta\mu} \leq e^0$ i.e $z \leq 1$.

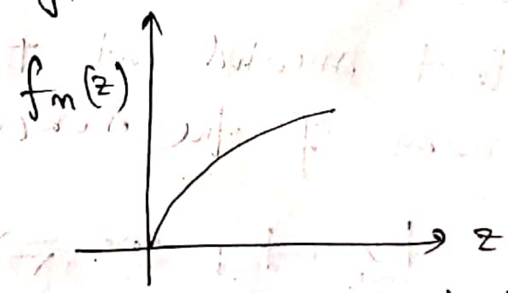
$\therefore$ For $E_B = 0$: $0 < z \leq 1$.

Now, $I_F + I_B$

$$= \frac{(2\pi m k T)^{3/2}}{h^3}\left[2\sqrt{2}\, g_{3/2}(z) + 2 f_{3/2}(z)\right]$$

Now g_n & f_n both are monotonic increasing function of z.





So here the maximum value of the left side is given by:

$$I_F + I_B \leq \frac{(2\pi m k T)^{3/2}}{h^3}\left[2\sqrt{2}\, \zeta\left(\tfrac{3}{2}\right) + 2 f_{3/2}(1)\right]$$

$$(\because g_n(1) = \zeta(n))$$

So for very low temp : $T \to 0$

we can make $I_F + I_B$ sufficiently small which is less than any non zero P_{tot}

for the range $\mu \leq 0$.

i.e by making the system colder & colder; we make the density of system arbitrarily small (and even → 0) which is inconsistent with physical point of view.

2. At $T \to 0$ we know that for Bosonic system $z \to 1$ i.e $\mu \to 0$.

So to keep equlibrium we must have

$$\mu_F = \mu_B$$

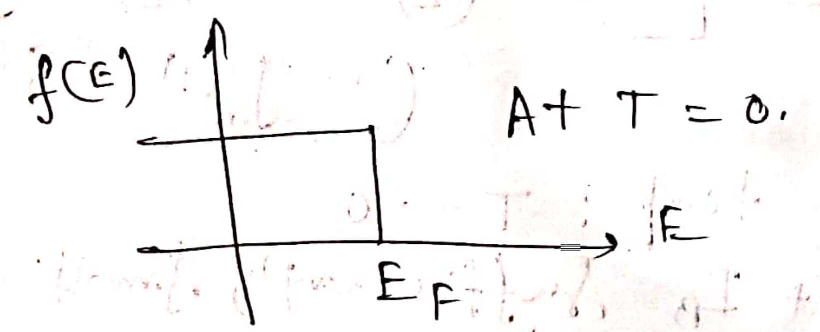
So at $T \to 0$; the Fermi energy E_F also goes to zero to keep mutual equilibrium.

But that means at $T \to 0$; the equilibrium of the reaction

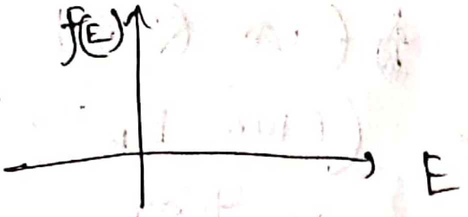
$$F_\uparrow + F_\downarrow \rightleftharpoons B$$

shifts towards right.

I have given the argument as the Fermi distⁿ is given by:



So $E_F \to 0$ means the graph will look like:



i.e $f(E) = 0$ for all possible $E \geq 0$.

That means there are no Fermi particle. That is only possible by increasing the reaction rate

$$F_{\uparrow} + F_{\downarrow} \longrightarrow B$$

So at $T \to 0$ all of the Fermi particles combine to make Bose particle and do Bose Einstein condensation.

The condensed Bose particles at ground state are not included in expression of $I_F + I_B$ as when we take density of state $\sim \sqrt{E}$ we take that at ground state i.e $E = 0$; $g(E) = 0$ and hence no particle can sit there.

So at $T \to 0$ the lost particles from Fermi gas makes n_0^B no of condensed molecules.

i.e in true expression of N_{tot} there should be one term $\sim 2n_0^B$ (as one Bose comes from two Fermi particles.)

i.e

$$P_{tot} = I_B + I_F \xrightarrow[\text{correction}]{\text{after}} I_B + I_F + 2n_0^B.$$

3. I have already used (in Q 2) that at $T \to 0$; $\mu \to 0$. (when $E_p = 0$) However here I'm going to prove that in two ways:

First method:-

As mentioned in Q-2 ; $g(E) \sim \sqrt{E}$
So truly saying that the density of Bose particles in excited state is given by:

$$\frac{N_{ex}^B}{V} = \rho_{ex}^B = \int_0^\infty g(E) \cdot f(E)\, dE.$$

$$= \frac{(2\pi m' kT)^{3/2}}{h^3} g_{3/2}(z) \quad (m' = 2m)$$

(integral has been evaluated in earlier question (prob 4 part 1))

Now as at $T \to 0$; maximum no of particles are pushed down to ground state so this will be done after the most possible fill up of excited state following previous eq.

$$\therefore \rho_{ex}^B \leqslant \underbrace{\frac{(2\pi m' kT)^{3/2}}{h^3} g_{3/2}(z)}$$

for all $z \in [0,1)$

That means the system tries to maximize the right side until it is enforced to fill up ground state abruptly.

but in the limit $\mu \leq 0$ i.e $z \in [0,1)$
the max value of $g_{3/2}(z)$ is given
at $z \to 1$ as $g_{3/2}(z)$ is monotonic
increasing fn of z.

i.e $P^{B}_{ex}\Big|_{T\to 0} \sim \frac{(2\pi m k T)^{3/2}}{h^3}\, \zeta\left(\frac{3}{2}\right)$

$\left(\because g_{3/2}(1) = \zeta\left(\frac{3}{2}\right)\right)$

However that means
$T \to 0$ corresponds to $z \to 1$
i.e $\mu \sim \ln z \to 0$

$\therefore$ at $T \to 0$; $\mu \to 0$. (the upper bound.)

Ans

Second method :-

Here we use the physical reasoning that at ~~thermo~~ classical limit or $T \to \infty$ ~~where the~~ we must get limit
$pV = NkT$
the law coming from kinetic theory.
Now I'm going to show ~~that~~ that at $z \to 0$ i.e $\mu \to -\infty$ that is the case.

We know $\frac{pV}{kT} = \ln \mathcal{Z} = \int_0^\infty g(E) \ln f(E)\, dE$

$= \frac{4\pi\sqrt{m}\, V\pi\sqrt{2}}{h^3} \int_0^\infty \sqrt{E} \ln\left(1 - ze^{-\beta E}\right) dE$

for $z = e^{\beta\mu} << 1$ i.e $e^{\beta\mu} \simeq 0$ we get

$$\frac{PV}{KT} = -\frac{4m\sqrt{m}\,V\pi\sqrt{2}}{h^3}\int_0^\infty \sqrt{E}\,\ln\left(1-ze^{-\beta E}\right)dE$$

$$\simeq -\frac{4m\sqrt{m}\,V\pi\sqrt{2}}{h^3}\int_0^\infty \sqrt{E}\times -ze^{-\beta E}\,dE.$$

$$\left(\ln(1+x) \simeq x + \cdots\right)$$

$$\simeq \frac{4m\sqrt{m}\,V\pi\sqrt{2}\,z}{h^3}\int_0^\infty \sqrt{E}\,e^{-\beta E}\,dE.$$

$$= \frac{4m\sqrt{m}\,\pi V\sqrt{2}\,z}{h^3}\int_0^\infty \frac{\sqrt{x}}{\sqrt{\beta}}\cdot e^{-x}\cdot\frac{dE}{\beta}$$

$$= \frac{4m\sqrt{m}\,V\pi\sqrt{2}}{h^3}\times\frac{z}{\beta\sqrt{\beta}}\cdot\Gamma\left(\frac{3}{2}\right)$$

Now $N/V = \rho$ be given by:

$$\frac{N}{V} = \frac{4\pi m\sqrt{m}\,\pi\sqrt{2}}{h^3}\int_0^\infty \frac{\sqrt{E}}{z^{-1}e^{\beta E}-1}\,dE$$

$$= \frac{4\pi m\sqrt{m}\,\pi\sqrt{2}}{h^3}\int_0^\infty \frac{z e^{-\beta E}\cdot\sqrt{E}}{1-ze^{-\beta E}}\,dE$$

$$\simeq \frac{4\pi m\sqrt{m}\,\pi\sqrt{2}}{h^3}\, z\int_0^\infty \sqrt{E}\,e^{-\beta E}\times\left(1+ze^{-\beta E}+\dots\right)dE$$

taking first order:

$$\frac{N}{V} \simeq \frac{4\pi m\sqrt{m}\,\pi\sqrt{2}}{h^3}\times z\int_0^\infty \sqrt{E}\,e^{-\beta E}\,dE + O(z^2)$$

$$\simeq \frac{4\pi m\sqrt{m}\,\pi\sqrt{2}\,z}{h^3}\times\frac{1}{\beta\sqrt{\beta}}\cdot\Gamma\left(\frac{3}{2}\right) + O(z^2)$$

$$\Rightarrow N \simeq \frac{4\pi m\sqrt{m}\,\pi V\sqrt{2}\,z}{h^3}\times\frac{1}{\beta\sqrt{\beta}}\times\Gamma\left(\frac{3}{2}\right) + O(z^2)$$

So at small z: $\frac{pV}{KT} \simeq N$.

i.e $pV \simeq NKT$.

which is the classical limit.

So $e^{\beta\mu} \ll 1 \Rightarrow T \to \infty$

Now as $e^{\beta\mu} = e^{\frac{\mu}{KT}}$ so at $T\to\infty$ the only way to make $e^{\mu/KT} \to 0$ by taking $\mu = -ve$ and; $\mu \sim T^\alpha$ when $\alpha > 1$

As the $T \to \infty$ extremum is given by $\mu \to -\infty$ limit. So by physical point of view it is safe to assume that the other limit of temperature ~~of the~~ corresponds to the other limit of μ; (as all physical formulas in physics changes continuously in general.)

So; $T \to \infty \Rightarrow \mu \to \mu_{min}$ i.e $-\infty$

$$\downarrow$$

$T \to 0 \Rightarrow \mu \to \mu_{max}$ i.e $\mu \to 0$

Ans

4 From part 2 ; Q-2 I get :

$$P_{tot} = I_B + I_F + 2n_0^B$$

But $I_B + I_F = (\text{Some term}) \cdot T^{3/2} \left[\alpha g_{3/2}(z) + \beta f_{3/2}(z)\right]$

$\therefore \lim_{T \to 0} (I_B + I_F) = 0.$

So $2n_0^B = P_{tot}$

i.e. $n_0^B = \frac{P_{tot}}{2} = \frac{N_{tot}}{2L^3}.$ Ans

However in proper limit

$$n_0^B \simeq \frac{P_{tot}}{2} - \frac{V(2\pi k m_B)^{3/2}}{h^3} \left[2\sqrt{2}\, G\left(\tfrac{3}{2}\right) + 2 f_{\frac{3}{2}}(1)\right] \propto T^{3/2}$$

This is however a very naive result. The actual change of n_0^B with T is done in Q-8 as asked.

5. As I argued in Q-2; in $T \to 0$ limit; the reaction $f_\uparrow + f_\downarrow \to B$ increases rapidly. Hence the Fermions goes to Bosons (to keep equilibrium at $\mu|_{T=0} = E_F = 0$)

So at $T \to 0$ limit
$\langle N_F \rangle \to 0$. (if equilibrium exists)

However
$$\langle N_F \rangle = 2 \frac{V(2\pi k m T)^{3/2}}{h^3} \cdot f_{3/2}(e^{\beta\mu})$$

So taking proper limit: $T \to 0$; $\mu \to 0$

$$\underset{T\to 0}{Lt} \langle N_F \rangle = \frac{2V}{h^3} \cdot f_{3/2}(1)(2\pi m k)^{3/2} \cdot T^{3/2}.$$

i.e $\langle N_F \rangle_{T\to 0} \sim T^{3/2}$. Ans

$$\left[\text{here } f_{3/2}(1) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^{3/2}} \approx 0.763 \quad \text{(using mobile calculator)} \right]$$

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6. At $T \to 0$ i.e all population goes to ground state; due to $\sqrt{E}$ factor; we have to evaluate ground state population differently.

Using B-E distn:

no of particles in $E=0$:

$$N_0^B = \left.\frac{1}{z^{-1}e^{\beta E}-1}\right|_{E=0} = \frac{1}{z^{-1}-1}.$$

$$\therefore N^B = \frac{1}{z^{-1}-1} + \frac{V(2\pi m' kT)^{3/2}}{h^3} g_{3/2}(z).$$

but as $z \approx 1$ so $g_{3/2}(z) \approx \zeta(3/2)$

$$\therefore N^B = \underbrace{\frac{1}{z^{-1}-1}}_{\text{ground state population}} + \underbrace{\frac{V(2\pi m' kT)^{3/2}}{h^3}\zeta(3/2)}_{\text{excited state population}}.$$

Ans

8. From Q-6:

$$N_0^B + N_{ex}^B = N_{tot}$$

i.e $N_0^B = N_{tot} - N_{ex}^B$

but $N_{ex}^B \simeq \frac{V(2\pi m'KT)^{3/2}}{h^3} G(3/2)$ at $T \to 0$.

as $z \sim 1 - \frac{1}{N}$ at $T \to 0$ and N is huge.

So,

$$N_0^B \simeq N_{tot} - \frac{V.(2\pi m'K)^{3/2}}{h^3} G\left(\frac{3}{2}\right). T^{3/2}$$

i.e $\rho_0^B = \frac{N_0^B}{V} = \frac{N_{tot}}{V} - \frac{(2\pi m'K)^{3/2}}{h^3} G\left(\frac{3}{2}\right) T^{3/2}$.

$$= A - BT^{p}.$$

$\therefore A = \frac{N_{tot}}{V}$; $B = \frac{(2\pi m'K)^{3/2}}{h^3} G\left(\frac{3}{2}\right)$; $p = \frac{3}{2}$.

Ans.

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9. At $T = T_c$ we get $\rho_0^B \simeq 0$.

i.e $P_{tot} - \frac{(2\pi m' k)^{3/2}}{h^3} G\left(\frac{3}{2}\right) T_c^{3/2} \simeq 0$.

i.e $T_c^{3/2} \simeq \frac{h^3 P_{tot}}{(2\pi m k)^{3/2} \, G\left(\frac{3}{2}\right)}$.

i.e $T_c \simeq \frac{h^2}{(2\pi m k)} \cdot \left[\frac{P_{tot}}{G\left(\frac{3}{2}\right)}\right]^{2/3}$. Ans

here $G\left(\frac{3}{2}\right) \simeq 2.612 \ldots$

7. It is given to prove $\left.\frac{d\mu}{dT}\right|_{T=0} = 0$.
I've shown this in a naive way using results of Q-8 and Q-6. But as it was written to prove this and then use the result in Q-8; I was unable to do so after a long trial.

At $T \to 0$; we get $z \to 1$
(as the no of particles in ground state of Bose gas diverges and I've shown $N_0^{\beta} \sim \frac{1}{z^{-1}-1}$ so $z \to 1$ is an essential condition for that.)

but $z = e^{\beta\mu}$

$\Rightarrow \lim_{T\to 0} e^{\beta\mu} = 1$ i.e. $\lim_{T\to 0} \beta\mu = 0$

i.e. $\lim_{T\to 0} \frac{\mu}{KT} = 0$ ... (1).

So if μ has a power expansion of T; then to satisfy (1) we must have the first term of $\mu(T)$ expansion has a power of T higher than 1.

i.e. let $\mu(T) = \alpha T^a + \beta T^b + \gamma T^c + \cdots$
(when $a < b < c < \cdots$)
then we must have $a > 1$ to satisfy (1).

$\therefore \frac{d\mu}{dT} = \alpha a T^{a-1} + \beta b T^{b-1} + \cdots$

i.e. $\left.\frac{d\mu}{dT}\right|_{T=0} = 0 + 0 + \cdots = 0$ Ans

(as $a-1>0$ & $b>a$ ... etc).

So μ remains unchanged at a slightly higher temperature than absolute zero.

proved