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MTH 317: Linear Algebra
Professor Sussan
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Homework #9 - 1.13

✓ 1.13 For the map $f\colon \mathcal{P}_1 \to \mathbb{R}^2$ given by

$$a + bx \overset{f}{\longmapsto} \begin{pmatrix} a-b \\ b \end{pmatrix}$$

Find the image of each of these elements of the domain.
(a) $3-2x$ (b) $2+2x$ (c) x
Show that this map is an isomorphism.

a) $3-2x$: $\boxed{a=3}$ $\boxed{b=-2}$

b) $2+2x$: $\boxed{a=2}$ $\boxed{b=2}$

c) x: $\boxed{a=0}$ $\boxed{b=1}$

$f(3-2) = \begin{pmatrix} 3-(-2) \\ -2 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ -2 \end{pmatrix}}$

$f(x) = \begin{pmatrix} 0-1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

$f(2+2) = \begin{pmatrix} 2-2 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}$

~~Exam pg 175: Definition~~
An Isomorphism between two vector space V and W is a map $f: V \to W$ that:
(1) Is a correspondence: f is one-to-one and onto
(2) preserves structure → Addiction
→ Scalar Multiplication

(A) preserves structure

(a) Addictivly

$$f((a_1+a_2)+(b_1+b_2)x) = \begin{bmatrix} (a_1+a_2)-(b_1-b_2) \\ b_1+b_2 \end{bmatrix} = \begin{bmatrix} a_1-b_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2-b_2 \\ b_2 \end{bmatrix} = f(a_1+b_1x)+f(a_2+b_2x) \checkmark$$

(b) Scalar multiplication

$$f(ra+rbx) = \begin{bmatrix} ra-rb \\ rb \end{bmatrix} = r\cdot\begin{bmatrix} a-b \\ b \end{bmatrix} = rf(a+bx) \checkmark$$

(B) correspondence

(a) one-to-one?

$$f(a+bx) = \vec{0} = \begin{bmatrix} a-b \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} b=0 \checkmark \\ a-b=0 \\ a=0 \checkmark \end{matrix}$$

map f is one-to-one. ✓

(b) onto?

$$f(a+bx) = \begin{bmatrix} a-b \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{matrix} b=y_2 \\ a-b=y_1 \\ a=y_1+y_2 \end{matrix}$$

map f is onto. ✓

$a+bx \in \mathcal{P}_2$ for any $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$