The Basic Algebra of Game Equivalences by Goranko

An overview

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Definition

Game Language and game terms

- The game language $\mathcal{GL}$ consists of the set:

    1. Atomic game: $\mathcal{G}_{at} = \{g_a \,|\, a \in A\}$. the 'idle' atomic game ι is also in $\mathcal{G}_{at}$.

    2. Game operations: $\vee$, d, $\circ$.

- Game terms:

    1. Every atomic game is a game term

    2. If G, H are game terms then G^d , $G \vee H$ and $G \circ H$ are game terms.

We say $G \wedge H := (G^d \vee H^d)^d$. Atomic games and their duals will be called literals.

Definition

Game boards

- **Game Boards** are models for $\mathscr{GL}$. It's defined as follows:

  1. $\langle S, \{\rho_a^i \mid a \in A, i \in \{1,2\}\}\rangle$

  2. Where S is a set of states and $\rho_a^i \subseteq S \times \mathscr{P}(S)$ are atomic forcing relations satisfying the following forcing conditions:

     A. upwards monotonicity (*MON*): For any $s \in S$ and $X \subseteq Y \subseteq S$, if $s\rho_a^i X$ then $s\rho_a^i Y$.

     B. consistency of the powers (*CON*): for any $s \in S$, $X \subseteq S$, if $s\rho_a^1 X$ then not $s\rho_a^2(S \backslash X)$ and (hence) likewise with 1 and 2 swapped.

Definition

Game Boards(Contd.)

- The forcing relations ρ_ι^i of the idle game ι have a fixed interpretation:$s\rho_\iota^i X$ iff $s \in X$. Compositions of idle literals (ι or ι^d) will be called idle game terms.
- Given a game board, the atomic forcing relations are extended to forcing relations $\{\rho_G^i \mid G \in \mathcal{G}$ and $i \in \{1,2\}\}$ for all game terms, following the recursive definition:

1. $s\rho_{G^d}^1 X$ iff $s\rho_G^2 X$. And likewise with 1 and 2 swapped.
2. $s\rho_{G_1 \vee G_2 x}^1$ iff $s\rho_{G_1}^1 X$ or $s\rho_{G_2}^1 X$.
3. $s\rho_{G_1 \vee G_2}^2 X$ iff $s\rho_{G_1}^2 X$ and $s\rho_{G_2}^2 X$.
4. $s\rho_{G_1 \circ G_2}^i X$ iff there exists Z such that $s\rho_{G_1}^i Z$ and $z\rho_{G_2}^2 X$ for each $z \in Z$, $\forall i \in \{1,2\}$.

Definitions

Inclusions and identities of game terms

- Let G_1 and G_2 be game terms and B a game board:

1. G_1 is i-**included** in G_2 in B , if $\rho^i_{G_1} \subseteq \rho^i_{G_2}$ where $i \in \{1,2\}$. It's denoted by $G_1 \subseteq_i G_2$.
2. G_1 is **included** in G_2on B, if $G_1 \subseteq_1 G_2$ and $G_2 \subseteq_2 G_1$ on B. We denote it by $B \vDash G_1 \preccurlyeq G_2$.
3. $B \vDash G_1 = G_2$ (equivalent on board B) if $B \vDash G_1 \preccurlyeq G_2$and $B \vDash G_2 \preccurlyeq G_1$.
4. G_1 is i**ncluded** in G_2 if $\vDash G_1 \preccurlyeq G_2$.
5. $G_1 \sim G_2$ if $\vDash G_1 = G_2$. Note, $\vDash G_1 \sim G_2$ iff $\vDash G_1 \preccurlyeq G_2$ and $\vDash G_2 \preccurlyeq G_1$.

Axioms for $\mathcal{GA}(GA^\iota)$

Proposed van Benthem.

1. $G \sim G^{dd}$

2. The usual identities for $\vee$ in distributive lattices: idempotency, commutativity, associativity.

3. $G_1 \vee (G_1 \wedge G_2) \sim G_2$

4. Associativity of $\circ$

5. $G_1^d \circ G_2^d \sim (G_1 \circ G_2)^d$

6. Left distribution for $\vee$ and $\circ$: $(G_1 \vee G_2) \circ G_3 \sim (G_1 \circ G_3) \vee (G_2 \circ G_3)$.

7. Right-distributive inclusion: $G_1 \circ G_2 \preccurlyeq G_1 \circ (G_2 \vee G_3)$

8. The extras for ι: multiplicative unit: $G \circ \iota \sim \iota \circ G \sim G$ and self-duality: $\iota \sim \iota^d$.

(P5)GA^ι is a sound system.

Canonical Game terms

Definition

- ι is a canonical term
- Let $\{G_{ik} \mid i \in I, k \in K_i\}$ be a finite non-emty family of canonical terms and $\{g_{ik} \mid i \in I, k \in K_i\}$ be a family of literals such that, g_{ik} is an idle literal $\implies$ G_{ik} is an idle term. Then $\bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ is a canonical term.

Canonical Game terms

Examples

- $g_1 \circ \iota$
- $g_2 \circ (g_1 \circ \iota)$
- $\iota \circ (\iota \circ \iota^d)$
- $(g_2 \circ (g_1 \circ \iota) \wedge g_1^d \circ (\iota^d \circ (\iota^d \circ \iota))) \vee (g_2^d \circ \iota^d \wedge g_2 \circ (g_1^d \circ (\iota \circ \iota^d)))$

Canonical Terms

Equivalence Theorem

(P8)Every game term G is equivalent to a canonical game term. This can be proved inside $GA^{\prime}$

Canonical Game Terms

Embedding

- We define recursively **embedding** of canonical terms, denoted by $\rightarrowtail$ as follows:

  1. $\iota \rightarrowtail \iota$

  2. Auxiliary notions: if g, h are literals and G, H are canonical terms, $g \circ G$ embeds into $h \circ H$ iff $g = h$ and $G \rightarrowtail H$; a conjunction $\bigwedge_{k \in K} g_k \circ G_k$ embeds into a conjunction $\bigwedge_{m \in M} h_m \circ H_m$ if for every $m \in M$ there is some $k \in K$ such that $g_k \circ G_k \rightarrowtail h_m \circ H_m$.

  3. Let $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ and $H = \bigvee_{j \in J} \bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$. Then $G \rightarrowtail H$ iff every disjunct of G embeds into some disjunct of H.

(P12)If G, H are canonical terms and $G \rightarrowtail H$ then $G \preccurlyeq H$ is provable in GA^ι. Hence $GA^\iota \vdash G \vee H \sim H$ and $GA^\iota \vdash G \wedge H \sim G$

Minimal Canonical Terms

Minimal canonical term is defined recursively

- ι is a minimal canonical term
- Let $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ be a canonical term where all G_{ik} are minimal. Then $G$is minimal if:

  1. ι^d does not occur in G.
  2. g_{ik} is $\iota \Longrightarrow G_{ik}$ is ι.
  3. No conjunct occurring in a conjunction $\bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ embeds into another conjunct from the same conjunction.
  4. No disjunct in G embeds into another disjunct of$G$.

Minimal Canonical Term

Equivalence Theorem

(P14)Every term G can be reduced to an equivalent minimal canonical term $c(G)$ and this can be done provably in GA^{ι}.

Isomorphism

Between canonical terms

Two canonical terms G, H are isomorphic, denoted by $G \simeq H$, if one can be obtained from the other by means of successive permutations of conjuncts (resp. disjuncts) within the same $\bigwedge$'s (resp. $\bigvee$'s) in subterms.

(P10)Isomorphic terms are equivalent, provably in GA'

The Big picture…

Cogs and bolts…

- (L24)if G, H are minimal canonical terms then $\models G \preccurlyeq H$iff $G \rightarrowtail H$.
- (P25) if G, H are minimal canonical terms such that $G \rightarrowtail H$ and $H \rightarrowtail G$ then $G \simeq H$.
- (C26)The minimal canonical terms G and H are equivalent(in semantics sense) Iff they are isomorphic.

The big picture...

(T6)Every valid term identity of the game algebra can be derived from in the standard equational logic.

Given, $\vDash G \sim H$. Also we know from (P14),

$$\vdash G \sim c(G) \text{ and } \vdash H \sim c(H)$$

Since GA^ι is sound, $\vDash G \sim c(G)$ and $\vDash H \sim c(H)$.

So,

$\vDash c(G) \sim c(H)$

$c(G) \simeq c(H)$, By (C26),

$\vdash c(G) \sim c(H)$, By (P10),

$\vdash G \sim H$

$\blacksquare$

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

We consider the modal language $\mathscr{ML}$ comprising:

- A set of *atomic variable* $V = \{p_a \mid a \in A\} \cup \{q\}$. Here q is *auxilary variable*.
- the usual connectives: $\vee$, $\wedge$, $\neg$, $\Diamond$ and $\Box$.

Some terminology and notation:

- Substitution $\varphi(\psi/q)$: ψ is substituted for all occurrences of the variable q in φ.
- A dual of a modal formula φ with respect to variable q is $\varphi_q^d = \neg\varphi(\neg q)$.
- The notion of *set substitution*: Say $\mathscr{M} = \langle S, R, V\rangle$ is a Kripke Model. Let's define :

$$\varphi(q) : \mathscr{P}(S) \to \mathscr{P}(S)$$

$X \mapsto \{s \in S \mid \mathscr{M}', s \vDash \varphi$, where $\mathscr{M}' := \langle S, R, V' := V \backslash \{(q, V(q))\} \cup \{(q, X \cup V(q))\}\rangle\}$

So, $M, s \vDash \varphi(X)$, where $X \subseteq S$. It means $s \in \phi(X)$.

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

The Translation: With every game term $\boldsymbol{G}$ we associate a modal formula $\boldsymbol{m(G)}$as follows

- $m(\iota) = q$
- $m(g_a) = \Diamond\Box(p_a \to q)$ for any nono-idle atomic game g_a, $a \in A$.
- $m(G_1 \vee G_2) = m(G_1) \vee m(G_2)$
- $m(G^d) = (m(G))^d$, also denoted by $m^d(G)$.
- $m(G_1 \circ G_2) = m(G_1)(m(G_2))$.

A little detour:

Determined Game

$s\rho_a^2(S\backslash X)$ iff $\neg s\rho_a^1 X$. The class of determined game boards will be denoted by **DET**.

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

(T15)For any game terms G, H, if $\boldsymbol{DET} \vDash \boldsymbol{G} \preccurlyeq \boldsymbol{H}$, then $\vDash_{\mathscr{ML}} \boldsymbol{m(G) \to m(H)}$.

Proof: By contraposition, suppose $M, u \nvDash m(G) \to m(H)$ for some model $M := \langle S, R, V \rangle$ and $u \in S$. We define a game board, $B_M = \langle S, \{\rho_a^i \mid a \in A$ and $i \in \{1,2\}\rangle$ as follows. For $X \subseteq S$ and $s \in S$:

$$a\rho_a^1 X \iff M, s \vDash m(g_a)(X)$$

$$a\rho_a^2 X \iff M, s \vDash m^d(g_a)(X)$$

It can be shown that B_M is a determined game and it follows *CON* and *MON*. Next, we can prove by induction on structure of game terms, that $\forall s \in S$, $X \subseteq S$ and term D:

$$a\rho_D^1 X \iff M, s \vDash m(D)(X)$$

$$a\rho_D^2 X \iff M, s \vDash m^d(D)(X)$$

Now, we said earlier, $M, u \vDash m(G)$ and $M, u \nvDash m(H)$. Let $X = V(q)$ So $u\rho_G^1 X$ but, $\neg u\rho_H^1 X$. Hence $B_M \nvDash G \preccurlyeq H$.

∎

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

(L19)Let G and H be minimal canonical terms. The followings are equivalent:

1. $G \not\leq H$

2. There is an disjunct $\bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ such that every disjunct in H contains a conjunct $h_{jm_j} \circ H_{jm_j}$ not including any of the conjuncts $g_{ik} \circ G_{ik}$ for$k \in K_i$. *[Notice this carefully!!]*

3. There is a finite (tree-like) Kripke model M and a state $s \in M$ such that: $M, s \vDash m(G)$; $M, s \nvDash m(H)$; and s has no predecessors in M.

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

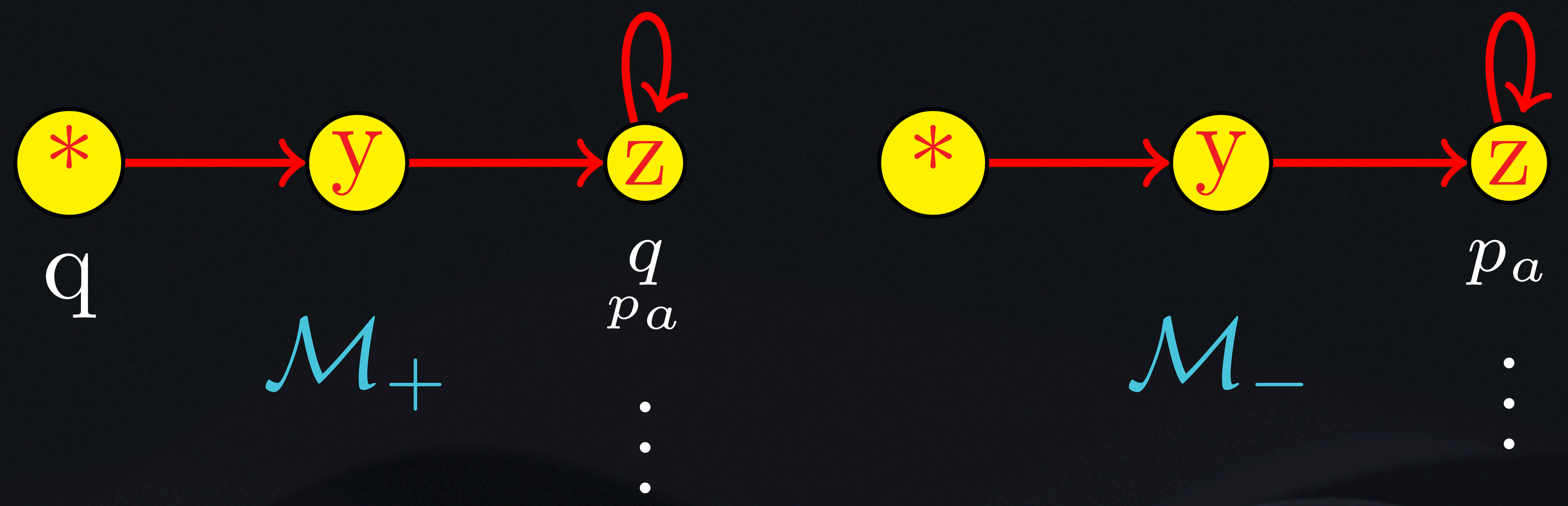
Proof of (L19): (1 $\Longrightarrow$2)Double Induction on structure of G and H. Routine.

(2$\Longrightarrow$3)

We will build a Kripke model $\mathscr{M}$ which will satisfy all $\{m(g_{ik} \circ G_{ik}) \mid k \in K_i\}$, and hence $m(G)$, while none of $\{m(h_{jm_i} \circ H_{jm_i}) \mid j \in J\}$, hence it will falsify $m(H)$.

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

Gadgets:



$\mathscr{M}_+$ satisfies all $m(G)$ at it's root $*$, $\mathscr{M}_-$ falsifies all $m(G)$ at it's root $*$

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

Proof of (L19):

Translation of $\mathscr{GA}$ into $\mathscr{ML}$

Corollaries

- (C20)For any game terms G,H:

$$\vDash_{\mathscr{ML}} m(G) \rightarrow m(H) \text{ iff } \vDash G \preccurlyeq H$$

$$\vDash_{\mathscr{ML}} m(G) \leftrightarrow m(H) \text{ iff } \vDash G \sim H$$

- (L22)Let G, H be any terms and g, h be non-idle literals. Then $g \circ G \preccurlyeq h \circ H$ iff $g = h$ and $G \preccurlyeq H$.
- (L23)Next we will take care of the case when either \$g\$ or \$h\$ is an idle term :
  - $g \circ G \preccurlyeq \imath \circ \imath$ iff g is an idle literal and $G \preccurlyeq \imath$.
  - $\imath \circ \imath \preccurlyeq g \circ G$ Iff g is an idle literal and $\imath \preccurlyeq G$.

Cogs and Bolts

(L24)if G, H are minimal canonical terms then $\vDash G \precsim H$ iff $G \rightarrowtail H$.

Proof: Double induction on structure of G and H. Primarily using (L22) and (L23) from the last slide.

Cogs and Bolts

(P25) if G, H are minimal canonical terms such that $G \rightarrowtail H$ and $H \rightarrowtail G$ then $G \simeq H$

Proof: We will be trying to prove two terms are ''isomorphic''. But isomorphism is defined as a `purely syntactic" notion. So how to prove it? Well we need to establish some sort of a positional bijection, from which it can be inferred which disjunct(conjunct) is equivalent to which disjucnt(conjunct).

We proceed by double induction. Suppose $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ and $H = \bigvee_{j \in J} \bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$ be minimal canonical terms such that $G \rightarrowtail H$ and $H \rightarrowtail G$ and the claim holds for all G_{ik}'s and H_{jm}'s.

Cogs and Bolts

(P25) if G, H are minimal canonical terms such that $G \rightarrowtail H$ and $H \rightarrowtail G$ then $G \simeq H$

Cogs and Bolts

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