Monotonicity: A Good Samaritan

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October 15, 2025

What are we up to?

Recalling (my) set-up: Let $R, S \in \mathcal{P}(B \times \mathcal{P}(B) \times \{s_3 \mid s_3 \in S\})$.

Define

$$R \circ S = \{(p, T, s) \mid (p, U, s) \in R \text{ and } (U, T, s) \in S \text{ for all such } U\}.$$

Define $\circ$ as

$$(R_1, S_1) \circ (R_2, S_2) := (R_1 \circ R_2,\ S_1 \circ S_2).$$

Enquiry: What goes wrong if monotonicity is not imposed?

Let us try to examine here one of the game algebra axioms:

$$\big((R_1, S_1) \circ (R_2, S_2)\big) \circ (R_3, S_3) = (R_1, S_1) \circ \big((R_2, S_2) \circ (R_3, S_3)\big)?$$

Which is the same as trying to prove:

$$(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3),$$

and

$$(S_1 \circ S_2) \circ S_3 = S_1 \circ (S_2 \circ S_3).$$

Getting down to the business

We want to prove:

$$R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3.$$

Let

$$(p, T, s) \in R_1 \circ (R_2 \circ R_3).$$

Then, by definition,

$$(p, U, s) \in R_1 \text{ for some set } U,$$

and

$$(U, T, s) \in R_2 \circ R_3 \text{ for all } u \in U.$$

So, for each $u \in U$, there exists T_u such that

$$(U, T_u, s) \in R_2 \quad \text{and} \quad (t', T, s) \in R_3 \text{ for all } t' \in T_u.$$

We seek to show:

$$(p, T, s) \in (R_1 \circ R_2) \circ R_3.$$

Meaning, we must find a set U_1 such that

$$(p, U_1, s) \in R_1 \circ R_2 \quad \text{and} \quad (u_1, T, s) \in R_3 \text{ for all } u_1 \in U_1.$$

We must show that

$$(p, U_1, s) \in R_1 \text{ for some set } U_1 \quad \text{and} \quad (U_1, T, s) \in R_2 \text{ for all } u_1 \in U_1,$$

for some set U_1. $(*)$

Data we have so far:

$(p, U, s) \in R_1$ for some set U, and for each $u \in U$, there exists T_u such that

$$(U, T_u, s) \in R_2 \quad \text{and} \quad (t', T, s) \in R_3 \text{ for all } t' \in T_u.$$

Using this data alone, it is impossible to reach $(*)$, because T_u depends on u. It is not a universal T. *What to do, What to do??*

Monotonicity saves the day!

If we knew R_2 was monotonic, then from

$$(u, T_u, s) \in R_2 \text{ for each } u \in U,$$

we could infer

$$(u, \bigcup_{u \in U} T_u, s) \in R_2.$$

Then, since

$$(p, U, s) \in R_1 \quad \text{and} \quad (u, \bigcup_{u \in U} T_u, s) \in R_2 \text{ for all } u \in U,$$

we obtain

$$(p, T, s) \in (R_1 \circ R_2),$$

where $T = \bigcup_{u \in U} T_u$.

Thus, we are able to reach $(*)$.

The proof of the other inclusion goes through with or without monotonicity. No problem there.

Farewell for now....

Thank you, Monotonicity for having our backs in this expedition. Now we can't help but wonder if you are merely helpful or indispensable too in the grand scheme of things!