The **Goranko Paper** on completeness of axioms for Game algebra given by Van Benthem

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In this article we shall go through the Goranko's Paper. The core ideas of this paper are "minimal cannonical game terms" and "the modal translation of game terms". He gives a, sort of, normal form for formula in the game language $\mathcal{GL}$, which he calls "minimal canonical terms". Then he uses this normal form to talk about formula in $\mathcal{GL}$, from the well established Modal Logic $\mathcal{ML}$. This we say 'purely syntactic' reasoning from $\mathcal{ML}$. This normal form also gives a better inductive structure on formula of $\mathcal{GL}$. Before diving into the meat of the paper let's do a little bit of excursion through Proposotinal Logic($\mathcal{PL}$). In $\mathcal{PL}$, think about the Disjunctive Normal Form(DNF):

$$\bigvee_{i \in I} \bigwedge_{k \in K_i} \varphi_{i,k} \tag{1}$$

Note here $\varphi_{i,k}$ is a atomic term or it's negation, if we were to embed $\mathcal{PL}$ in a first order logic, say, $\mathcal{L}_{\mathcal{PL}}$. Similar thought is expressed while defining "canonical terms". But obviously in our $\mathcal{GL}$ appropriate syntatic element will be used in place of $\mathcal{L}_{\mathcal{PL}}$'s atomic terms. But one question arises here why DNF? Why not CNF? We shall see the an explanation later in the article.

Now let's discuss a bit about the translation to Modal Logic($\mathcal{ML}$). Whenever we talk about playing a two player game, we use a phrase something on the line: "there is a play for player 1 such that for every play of the player 2, something holds". This thought is expresed verbosely in $\mathcal{ML}$ using something on the line of "$\Diamond\Box(\cdots)$". Also he uses $\mathcal{ML}$ to talk about "syntax" of $\mathcal{GL}$ inside $\mathcal{ML}$. These ideas will be more clearer as we proceed.

First, lets do some house keeping by defining, $\mathcal{GL}$, game terms, game algebra($\mathcal{GA}$), game boards(the models for these $\mathcal{GA}$s) and the axiomatic system for $\mathcal{GA}$.

1 Definitions

Definition 1. *The game language $\mathcal{GL}$ constitutes of set:*

- *Atomic game: $\mathcal{G}_a t = \{g_a | a \in A\}$. the 'idle' atomic game ι is also in $\mathcal{G}_a t$.*
- *Game operations: $\vee, d, \circ$.*

Definition 2. *Game terms:*

- *Every atomic game is a game term*
- *If G,H are game terms then G^d , $G \vee H$ and $G \circ H$ are game terms.*

We say $G \wedge H := (G^d \vee H^d)^d$. Atomic games and their duals will be called literals.

Intuitively, the operations ^{d}, $\vee$, $\wedge$, $\circ$ mean respectively dualization (swapping the two players' roles), choice of first player, choice of second player, and composition of games. The algebra of game terms will be denoted by $\mathcal{GA}$.

Definition 3. ***Game Boards*** *are models for $\mathcal{GL}$. It's defined as follows*

$$\langle S, \{\rho_a^i \mid a \in A, i \in \{1, 2\}\}\rangle$$

Where S is a set of states and $\rho_a^i \subseteq S \times \mathcal{P}(S)$ are atomic forcing relations satisfying the following forcing conditions:

- *upwards monotonicity (MON): for any $s \in S$ and $X \subseteq Y \subseteq S$, if $s\rho_a^i X$ then $s\rho_a^i Y$;*

- *consistency of the powers (CON): for any $s \in S$,$X \subseteq S$, if $s\rho_a^1 X$ then not $s\rho_a^2(S \setminus X)$ and (hence) likewise with 1 and 2 swapped.*

The forcing relations ρ_ι^i of the idle game ι have a fixed interpretation: $s\rho_\iota^i$ iff $s \in X$ .Compositions of idle literals (ι or ι^d) will be called idle game terms.

Given a game board, the atomic forcing relations are extended to forcing relations $\{\rho_G^i \mid G \in \mathcal{G}$ and $i \in \{1,2\}\}$for all game terms, following the recursive definition:

- $s\rho_{G^d}^1 X$ iff $s\rho_G^2 X$. And likewise with 1 and 2 swapped.
- $s\rho_{G_1 \vee G_2 x}^1$ iff $s\rho_{G_1}^1 X$ or $s\rho_{G_2}^1 X$.
- $s\rho_{G_1 \vee G_2 x}^2$ iff $s\rho_{G_1}^2 X$ and $s\rho_{G_2}^2 X$.
- $s\rho_{G_1 \circ G_2}^i X$ iff there exists Z such that $s\rho_{G_1}^i Z$ and $z\rho_{G_2}^2 X$ for each $z \in Z$, $\forall i \in \{1,2\}$.

1.1 Inclusions and identities of game terms

Let G_1 and G_2 be game terms and B a game board:

- G_1 is i-**included** in G_2 in B , if $\rho_{G_1}^i \subseteq \rho_{G_2}^i$ where $i \in \{1,2\}$. It's denoted by $G_1 \subseteq_i G_2$.
- G_1 is **included** in G_2 on B, if $G_1 \subseteq_1 G_2$ and $G_2 \subseteq_2 G_1$ on B. We denote it by $B \models G_1 \preccurlyeq G_2$.
- $B \models G_1 = G_2$ (equivalent on board B) if $B \models G_1 \preccurlyeq G_2$ and $B \models G_2 \preccurlyeq G_1$.
- G_1 is **included** in G_2 if $\models G_1 \preccurlyeq G_2$.
- $G_1 \sim G_2$ if $\models G_1 = G_2$. Note, $\models G_1 \sim G_2$ iff $\models G_1 \preccurlyeq G_2$ and $\models G_2 \preccurlyeq G_1$.

Another thing to note:

$$G_1 \preccurlyeq G_2 \leftrightarrow G_1 \vee G_2 \sim G2 \leftrightarrow G_1 \wedge G_2 \sim G_1$$

2 Axioms for $\mathcal{GA}$

1. $G \sim G^{dd}$
2. The usual identities for $\vee$ in distributive lattices: idempotency, commutativity, associativity.
3. $G_1 \vee (G_1 \wedge G_2) \sim G_2$
4. Associativity of $\circ$
5. $G_1^d \circ G_2^d \sim (G_1 \circ G_2)^d$
6. Left distribution for $\vee$ and $\circ$: $(G_1 \vee G_2) \circ G_3 \sim (G_1 \circ G_3) \vee (G_2 \circ G_3)$.
7. Right-distributive inclusion: $G_1 \circ G_2 \preccurlyeq G_1 \circ (G_2 \vee G_3)$
8. The extras for ι: multiplicative unit: $G \circ \iota \sim \iota \circ G \sim G$ and self-duality:$\iota \sim \iota^d$.

We denote the set of all these identities by $\mathbf{GA}^\iota$. Also we abreviate $G \vee H$ as $G \preccurlyeq H$. And it can be easily proven that $\mathbf{GA}^\iota \vdash G \vee H \sim H \leftrightarrow G \wedge H \sim G$. Note that the respective identities for $\wedge$, as well as the dual absorption, distributivity, left-distribution for $\wedge$ and $\circ$, right-distributive inclusion $G_1 \circ (G_2 \wedge G_3) \preccurlyeq G_1 \circ G_2$, and the two De Morgan's laws for $\vee$,$\wedge$ and ^{d} easily follow from the definition of $\wedge$ and $\mathbf{GA}^\iota$ in the equational logic for the algebra of games, which includes the standard set of derivation rules reflecting the fact that $\sim$ is a congruence in the algebra of games.

Remark 4. *In each declaration of theorem of this article, Pn or Cn or Ln or Tn is attached to refer to the respective Proposition, Corollary, Lemma or Theorem number from the Goranko's paper.*

Theorem 5 (Soundness(P5))**.** *All identities in $\boldsymbol{GA}^\iota$ are valid.*

Proof. It's a routine check for valididity of the axiom. □

The obvious next question is Completenenss. But before that we sould have an interlude explaning in details of "Cannonical Terms" and "Minimal Canonical Terms". After that we will observe, bird's eye point of view, how the proof of the completness theorem is laid down in Goranko's paper.

3 Canonical Game Terms

Definition 6. ***Canonical Terms*** *are defined recursively:*

- ι *is a canonical term*
- *Let* $\{G_{ik} \mid i \in I,\, k \in K_i\}$ *be a finite non-emty family of canonical terms and* $\{g_{ik} \mid i \in I,\, k \in K_i\}$ *be a family of literals such that,* g_{ik} *is an idle literal* $\Longrightarrow$ G_{ik} *is an idle term. Then* $\bigvee_{i\in I} \bigwedge_{k\in K_i} g_{ik} \circ G_{ik}$ *is a canonical term.*

Example 7. *Examples will clear up things:*

1. $g_1 \circ \iota$
2. $g_2 \circ (g_1 \circ \iota)$
3. $\iota \circ (\iota \circ \iota^d)$
4. $(g_2 \circ (g_1 \circ \iota) \wedge g_1^d \circ (\iota^d \circ (\iota^d \circ \iota))) \vee (g_2^d \circ \iota^d \wedge g_2 \circ (g_1^d \circ (\iota \circ \iota^d)))$

Remark 8. *I hope now it's visible these canonical terms are very similar to DNFs in* $\mathcal{PL}$*. The only difference is here* $\varphi_{i,k}$ *is replaced by* $g_{ik} \circ G_{ik}$*. Here these are sort of "atomic games" rather than atomic terms in case of* $\mathcal{PL}$*.Also, Canonical terms impose a periodic structure on games: every game is a composition of one or several rounds, each consisting of:*

- *a choice of player I.(This is exactly why it looks like a DNF, rather than an CNF)*
- *followed by a choice of player II.(If this step came before the first, then we would've have a CNF style formulation)*
- *followed by an atomic game by one of the players(depending on the sign of literal).*

As in $\mathcal{PL}$ every formula can be written in a DNF, same is applicable for game terms in $\mathcal{GL}$. Next theorem describes that.

Theorem 9 (P8)**.** *Every game term* G *is equivalent to a canonical game term. This can be proved inside* $\boldsymbol{GA}^\iota$

Proof. The proof follows through the structural induction on game terms. For atomic case use the properties of ι from the axioms. For terms like $G = H^d$, use De'Morgans and distributivity properties. The case where $G = G_1 \vee G_2$ follows through trivially.

The remaining case is $G = G_1 \circ G_2$. By induction hypothesis both G_1 and G_2 are in canonical form. We can do another layer of induction on G_1 with respect to it's canonical form. So G_1, following the definiton of canonical term, is either an ι or of the form $\bigvee_{i\in I} \bigwedge_{k\in K_i} g_{ik} \circ G_{ik}$ where certain condition on G_{ik} and g_{ik} holds. For the $G_1 = \iota$ case, it's trivial. Use $\vdash \iota \circ G \sim G$. For another case, we can use the left distributivity for $\vee$ and $\wedge$. So we get $G = \bigvee_{i\in I} \bigwedge_{k\in K_i} g_{ik} \circ G_{ik} \circ G_2$. Now by second layer of induction hypothesis, $G_{ik} \circ G_2$ can be represented in a canonical term form. Hence whole G is in canonical form. □

Next we define a concept called **embedding**. This will capture the idea of **inclusion** from a 'purely syntactic' view point. This view does not depend on the '$\vdash$' at all. Rather it gives a way to talk about game inclusion from the language of $\mathcal{ML}$.

Definition 10. *We define recursively* ***embedding*** *of canonical terms, denoted by* $\rightarrowtail$ *as follows:*

- $\iota \rightarrowtail \iota$
- *Auxiliary notions: if* g, h *are literals and* G, H *are canonical terms,* $g \circ G$ *embeds into* $h \circ H$ *iff* $g = h$ *and* $G \rightarrowtail H$*; a conjunction* $\bigwedge_{k\in K} g_k \circ G_k$ *embeds into a conjunction* $\bigwedge_{m\in M} h_m \circ H_m$*if for every* $m \in M$ *there is some* $k \in K$ *such that* $g_k \circ G_k \rightarrowtail h_m \circ H_m$.
- *Let* $G = \bigvee_{i\in I} \bigwedge_{k\in K_i} g_{ik} \circ G_{ik}$ *and* $\bigvee_{j\in J} \bigwedge_{m\in M_j} h_{jm} \circ H_{jm}$*. Then* $G \rightarrowtail H$ *iff every disjunct of G embeds into some disjunct of H.*

Theorem 11 (P12)**.** *If* G *,*H *are canonical terms and* $G \rightarrowtail H$ *then* $G \preccurlyeq H$ *is provable in* $\boldsymbol{GA}^\iota$*. Hence* $\boldsymbol{GA}^\iota \vdash G \vee H \sim H$ *and* $\boldsymbol{GA}^\iota \vdash G \wedge H \sim G$.

Proof. The proof pulls through induction on G and H. It primarily uses Right-distributive inclusion axiom. □

Remark 12. *Since* $\boldsymbol{GA}^\iota$ *is sound, the last theorem also connects 'purely syntactic' property such as embedding with inclusion in the semantic case. This and reverse of this, which shall be proved later, will be used to show the completeness. Before that let's define something which will give more control and structure on the game terms.*

3.1 Minimal Canonical Terms

Definition 13. *Again Minimal canonical term is defined recursively:*

- ι *is a minimal canonical term.*
- *Let* $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ *be a canonical term where all* G_{ik} *are minimal. Then* G *is minimal if:*
  1. ι^d *does not occur in* G
  2. g_{ik} *is* $\iota \implies G_{ik}$ *is* ι.
  3. *No conjunct occurring in a conjunction* $\bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ *embeds into another conjunct from the same conjunction.*
  4. *No disjunct in* G *embeds into another disjunct of* G.

Thus, minimal canonical terms are systematically 'minimized' canonical terms.

Theorem 14 (P14)**.** *Every term* G *can be reduced to an equivalent minimal canonical term* $c(G)$ *and this can be done provably in* $\boldsymbol{GA}^\iota$.

Proof. Use Theorem 9 to get a canonical form. Then use properties of ι such that no ι^d is there in the game term. Finally to assure point 3 and point 4 from above definiton 13 by using theorem 11. □

4 Proof sketch of the Completeness theorem

Before jumping into the main proof sketch, we will look into another purely syntactic notion, which will try to encapsulate term equivalency

4.1 Isomorphism

Definition 15 (Isomorphism)**.** *Two canonical terms* G, H *are isomorphic, denoted by* $G \simeq H$, *if one can be obtained from the other by means of successive permutations of conjuncts (resp. disjuncts) within the same* $\bigwedge$*'s (resp.* $\bigvee$*'s) in subterms.*

Theorem 16 (P10)**.** *Isomorphic terms are equivalent, provably in* $\boldsymbol{GA}^\iota$.

Proof. By Commutativity of $\bigvee$ and $\bigwedge$. □

4.2 The big picture...

First we look at two theorems, whose proof will follow later.

Theorem 17 (L24)**.** *if* G, H *are minimal canonical terms then* $\models G \preccurlyeq H$ *iff* $G \rightarrowtail H$.

Remark 18. *Theorem 17 connects 'semantics world' to a 'purely syntactic' world. Note this notion of syntactic connection does not depend on '*$\vdash$*', rather, we shall see later, this type of syntactic discussion shall purely be done via* $\mathcal{ML}$.

Theorem 19 (P25)**.** *if* G, H *are minimal canonical terms such that* $G \rightarrowtail H$ *and* $H \rightarrowtail G$ *then* $G \simeq H$.

Remark 20. *Theorem 19 talks about earlier discussed 'purely syntatic' notion. And just as above, these syntactic notion will be focused through* $\mathcal{ML}$.

From Theorem 17 and Theorem 19 we get the following

Theorem 21 (C26)**.** *The minimal canonical terms* G *and* H *are equivalent(in semantics sense) iff they are isomorphic.*

Proof. Proof easily follows from theorem 17 an 19. □

Now we shall prove main completeness theorem using theorem 21.

Theorem 22 (Completenes, T6)**.** *Every valid term identity of the game algebra can be derived from* $\boldsymbol{GA}^\iota$ *in the standard equational logic.*

Proof. We have,

$$\models G \sim H \tag{2}$$

Now by theorem 14,

$$\begin{aligned} &\mathbf{GA}^{\iota} \vdash G \sim c(G) \qquad \text{where } c(G) \text{ is the equivalent minimal canonical term for } G \\ &\mathbf{GA}^{\iota} \vdash H \sim c(H) \end{aligned} \tag{3}$$

Since our system is sound by Theorem 5,

$$\begin{aligned} &\models H \sim c(H) \\ &\models G \sim c(G) \end{aligned} \tag{4}$$

From 2 and 4 we get $\models c(G) \sim c(H)$. By Theorem 21, $c(G) \simeq c(H)$. But according to Theorem 16 we know that , we will get

$$\mathbf{GA}^{\iota} \vdash c(G) \sim c(H) \tag{5}$$

From 3 and 5:

$$\mathbf{GA}^{\iota} \vdash G \sim H$$

□

5 Translation of $\mathcal{GA}$ into $\mathcal{ML}$

Here we introduce a translation of GL into plain modal logic. we will use it in the next section to construct countermodels to invalid game equivalences, since Kripke models are rather more transparent, flexible and easier to deal with than game boards.

To begin with, we consider the **modal language** $\mathcal{ML}$ comprising:

- a set of *atomic variable* $V = \{p_a \mid a \in A\} \cup \{q\}$. Here q is *auxilary variable.*
- the usual connectives: $\vee$, $\wedge$, $\neg$, $\Diamond$ and $\Box$.

Some terminology and notation:

- Substitution $\varphi(\psi/q)$: ψ is substituted for all occurrences of the variable q in φ.
- A dual of a modal formula φ with respect to variable q is $\varphi^d_q = \neg\varphi(\neg q)$.
- The notion of *set substitution*: Say $\mathcal{M} = \langle S, R, V\rangle$ is a Kripke Model. Let's define :

$$\begin{aligned} &\varphi(q) : \mathcal{P}(S) \to \mathcal{P}(S) \\ &X \mapsto \{s \in S \mid \mathcal{M}', s \models \varphi, \text{ where } \mathcal{M}' := \langle S, R, V' := V \setminus \{(q, V(q))\} \cup \{(q, X \cup V(q))\}\rangle\} \end{aligned} \tag{6}$$

  So $\varphi(q)$ acts as an set operator. Later we shall see something like $M, s \models \varphi(X)$, where $X \subseteq S$. It means $s \in \varphi(X)$.

- All duals of modal formulas used in the translation will be with respect to q, so we can safely omit the subscript. Likewise, all substitutions will be of the type $\varphi(\psi/q)$, which hereafter we will simply write as $\varphi(\psi)$.

5.1 The translation:

With every game term Gwe associate a modal formula $m(G)$ as follows:

- $m(\iota) = q$
- $m(g_a) = \Diamond\Box(p_a \to q)$ for any nono-idle atomic game g_a, $a \in A$.
- $m(G_1 \vee G_2) = m(G_1) \vee m(G_2)$
- $m(G^d) = (m(G))^d$, also denoted by $m^d(G)$.
- $m(G_1 \circ G_2) = m(G_1)(m(G_2))$.

Somethings to note before we move on:

- Every formula $m(G)$ is monotone in q. A proof sketch: Do an induction based proof on modal formula φ.
- $m(g_a^d) = \Box\Diamond(p_a \wedge q)$.
- $m(G_1 \wedge G_2) = m(G_1) \wedge m(G_2)$
- $m^d(G_1 \circ G_2) = m^d(G_1)(m^d(G_2))$

Definition 23 (Determined Game)**.** *$s\rho_a^2(S \setminus X)$ iff $\neg s\rho_a^1 X$. The class of determined game boards will be denoted by $\boldsymbol{DET}$.*

Theorem 24 (T15)**.** *For any game terms G, H, if $\boldsymbol{DET} \models G \preccurlyeq H$, then $\models_{\mathcal{ML}} m(G) \to m(H)$.*

Proof. By contraposition, suppose $M, u \not\models m(G) \to m(H)$ for some model $M := \langle S, R, V\rangle$ and $u \in S$. We define a game board, $B_M = \langle S, \{\rho_a^i \mid a \in A$ and $i \in \{1,2\}\rangle$ as follows. For $X \subseteq S$ and $s \in S$:

- $a\rho_a^1 X \iff M, s \models m(g_a)(X)$
- $a\rho_a^2 X \iff M, s \models m^d(g_a)(X)$

Note: B_M is a determined game and it follows CON and MON. All three of them can be proven by just following the definition. Next we need to prove the following:

For every $s \in S$, $X \subseteq S$ and term D:

- $a\rho_D^1 X \iff M, s \models m(D)(X)$
- $a\rho_D^2 X \iff M, s \models m^d(D)(X)$

Subproof. Proof shall be carried through induction on the structure of D. The only tricky part is when $D = D_1 \circ D_2$. Say $s\rho_D^1 X$, then $s\rho_{D_1}^1 Z$ for some $Z \subseteq S$ and $\forall z \in Z$, $z\rho_{D_2}^1 X$. By induction hypothesis, $M, s \models m(D_1)(Z)$ and $\forall z\ inZ$, $M, z \models m(D_2)(X)$. But this implies,

$$Z \subseteq m(D_2)(X).$$

By monotonnicity, $M, s \models m(D_1)(m(D_2(X)))$. Following holds,

$$m(D_1)(m(D_2(X))) = m(D_1)(m(D_2))(X)$$

A rudimentary proof: Say,

$m(D_1) = (\cdots q \cdots q \cdots)$
$m(D_2) = (\cdots q \cdots q \cdots q \cdots)$
$m(D_2)(X) = (\cdots q \cdots q \cdots q \cdots)$ $\qquad V(q) = X$
Then, $m(D_1)(m(D_2)(X))$ would look something like below
$(\cdots(\cdots q \cdots q \cdots q \cdots)\cdots(\cdots q \cdots q \cdots q \cdots)\cdots)$ $\qquad V(q) = X$
Now we can write $m(D_1)(m(D_2))$
$(\cdots(\cdots q \cdots q \cdots q \cdots)\cdots(\cdots q \cdots q \cdots q \cdots)\cdots)$
Hence, $m(D_1)(m(D_2))(X) =$
$(\cdots(\cdots q \cdots q \cdots q \cdots)\cdots(\cdots q \cdots q \cdots q \cdots)\cdots)$ $\qquad V(q) = X$

Hence we get $M, s \models m(D_1 \circ D_2)(X)$. The other direction follows on similar kind of rewriting rule and definitions. $\dashv$

$\square$

Theorem 25 (L19)**.** *Let G and H be minimal canonical terms. The followings are equivalent:*

1. *$G \not\preccurlyeq H$*
2. *($\odot$) There is an disjunct $\bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ such that every disjunct in H contains a conjunct $h_{jm_j} \circ H_{jm_j}$ not including any of the conjuncts $g_{ik} \circ G_{ik}$ for $k \in K_i$.*
3. *There is a finite (tree-like) Kripke model M and a state $s \in M$ such that: $M, s \models m(G)$; $M, s \not\models m(H)$; and s has no predecessors in M.*

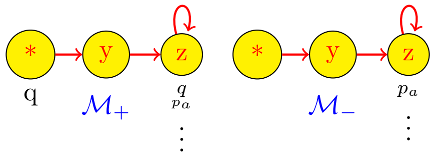

Figure 1: The M_+ and M_- models.

Proof. 1 $\implies$ 2 is straight forward. We do an induction on the structure of G and H, keeping in mind that G and H are in minimal canonical form. Please follow the paper's proof. The interesting part is 2 $\implies$ 3.

Proof of 2 $\implies$ 3. Before that let's take an interlude and define some Kripke models. which shall be used multiple times in this and upcomimg theorems. Refer to Fig.1. M_+ satisfies all $m(G)$ and M_- falsifies all $m(G)$.

We will build a Kripke model M which will satisfy all $\{m(g_{ik} \circ G_{ik}) \mid k \in K_i\}$, and hence m(G), while none of $\{m(h_{jm_i} \circ H_{jm_i}) \mid j \in J\}$, hence it will falsify m(H).M will be rooted at some state swith no predecessors, which is needed for the inductive hypothesis because models like this will be grafted at their roots on larger models as the induction goes on.

Depending on the signs of the literals g_{ik}, $k \in K_i$ and $h_{jm_i}, j \in J$,the set of all these terms splits into the following subsets:

1. $T_A = \{t_\alpha \circ D_\alpha \mid \alpha \in \mathbf{A}\}$, whose translation is true at s.
2. $T_B = \{t^d_\beta \circ D_\beta \mid \beta \in \mathbf{B}\}$, whose translation is true at s.
3. $T_B = \{t_\gamma \circ D_\gamma \mid \gamma \in \mathbf{\Gamma}\}$, whose translation is false at s.
4. $T_\Delta = \{t^d_\delta \circ D_\delta \mid \delta \in \mathbf{\Delta}\}$, whose translation is false at s.
5. Since both G and H are in minimal canonical form there might be an **single** occurence of idle term: $T_\iota = \{\iota \circ \iota\}$. Also both sets $\{m(g_{ik} \circ G_{ik}) \mid k \in K_i\}$ and $\{m(h_{jm_i} \circ H_{jm_i}) \mid j \in J\}$ simultenously can not have idle terms inside them, since the hypothese assumes, $h_{jm_j} \circ H_{jm_j}$ not including any of the conjuncts $g_{ik} \circ G_{ik}$ for $k \in K_i$.

The terms $t_\alpha, t_\beta, t_\gamma, t_\delta$ above are non-idle atoms. Let p_α, p_β, p_γ, p_δ be their corresponding variables in the modal translation. Thus, we have to satisfy at ssimultaneously the following sets of formulae:

1. $F_{\mathbf{A}} = \{\Diamond\Box(p_\alpha \to m(D_\alpha)) \mid \alpha \in \mathbf{A}\}$
2. $F_{\mathbf{B}} = \{\Box\Diamond(p_\beta \wedge m(D_\beta)) \mid \beta \in \mathbf{B}\}$
3. $F_{\mathbf{\Gamma}} = \{\Box\Diamond(p_\delta \wedge \neg m(D_\gamma)) \mid \gamma \in \mathbf{\Gamma}\}$
4. $F_{\mathbf{\Delta}} = \{\Diamond\Box(p_\delta \to \neg m(D_\delta)) \mid \delta \in \mathbf{\Delta}\}$
5. Possibly, $F_\iota = \{q\}$ or $F_\iota = \{\neg q\}$, depending on whether there is an idle term in $\{g_{ik} \circ G_{ik} \mid k \in K_i\}$ or $\{h_{jm_i} \circ H_{jm_i} \mid j \in J\}$ respectively.

Next we shall build a model $M = \langle W, R, V\rangle$ which will obey the $F_{\mathbf{A}}$, $F_{\mathbf{B}}$, $F_{\mathbf{\Gamma}}$ and $F_{\mathbf{\Delta}}$ at a root vertex s. To save ourselves of redundancies, it's best to follow the proof from the paper itself. ☐

Theorem 26 (C20)**.** *For any game terms G,H:*

- $\models_{\mathcal{ML}} m(G) \to m(H)$ *iff* $\models G \preccurlyeq H$.
- $\models_{\mathcal{ML}} m(G) \leftrightarrow m(H)$ *iff* $\models G \sim H$.

Proof. Use the above theorem 25. ☐

Remark 27. *Theorem 26 solidifies that fact that we don't need **DET** in Theorem 24.*

Theorem 28 (L22)**.** *Let G, H be any terms and g, h be non-idle literals. Then $g \circ G \preccurlyeq h \circ H$ iff $g = h$ and $G \preccurlyeq H$.*

Proof. It's best to follow the paper's proof. ☐

Remark 29. *Theorem 28 is proving a property which holds for embedding of canonical terms. So in a way, the modal translation is helping us to establish a connection with the semantics and the 'purely syntactic' notions. Similar things can be noticed for the next theorem.*

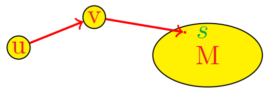

Figure 2: $T(M, s, u)$.

Given a Kripke model M and a state s, $T(M, s, u)$ will denote the model obtained from M by adding two new states u, v such that uRv and vRs. See 2. This will be used in the proof of theorem 30.

Theorem 30 (L23)**.** *Next we will take care of the case when either g or h is an idle term :*

- $g \circ G \preccurlyeq \iota \circ \iota$ *iff g is an idle literal and* $G \preccurlyeq \iota$.
- $\iota \circ \iota \preccurlyeq g \circ G$ *iff g is an idle literal and* $\iota \preccurlyeq G$.

Proof. Best to follow the proof from the paper. □

Proof of theorem 17. I've nothing to add to this proof. The proof uses induction on the structure of G and H. It uses Theorem 28 and Theorem 30. □

Proof of theorem 19. The proof in the paper is well written. Just a few heads up. Notice we will be trying to prove two terms are "isomorphic". But isomorphism is defined as a 'purely syntactic" notion. So how to prove it? Well we need to establish some sort of a positional bijection, from which it can be inferred which disjunct(conjunct) is equivalent to which disjucnt(conjunct). So for every disjunct D in G we need to map it to a disjunct D' in H. We do this by estabilishing a bijections(based in embedding and game inclusion) between the conjuncts in D with the conjuncts in D'. Obviously the backbone of the proof is supported by structural induction, Theorem 28, Theorem 30 and Theorem 17. □