Sequences and Series

Definition: A sequence is a function $s: \mathbb{Z}^+ \to \mathbb{R}$.

We shall denote the sequence $s: \mathbb{Z}^+ \to \mathbb{R}$ by $\{s_n\}$ or $\langle s_n \rangle$

i.e $\{s_n\} = s_1, s_2, s_3, \dots, s_n, \dots$

The image $s(n)$ or s_n of a positive integer n is called the nth term of the sequence.

ex:- $1, 3, 5, 7, \dots$ is a sequence of odd numbers.

Constant sequence: A sequence $\{s_n\}$ defined by $s_n = k$, $\forall n \in \mathbb{Z}^+$ $(k \in \mathbb{R})$ is called a constant sequence. ex:- $2, 2, 2, \dots, 2, \dots$ is a constant sequence.

Boundedness of a sequence

Definition: A sequence $\{s_n\}$ is said to be bounded below if there exists $l \in \mathbb{R}$ such that $l \leq s_n$, $\forall n \in \mathbb{Z}^+$. Here l is called a lower bound of $\{s_n\}$.

If l is a lower bound of $\{s_n\}$ then any number less than l is also a lower bound of $\{s_n\}$

The greatest among all the lower bounds of $\{s_n\}$ is called the greatest lower bound (g.l.b) of $\{s_n\}$.

Def:- A sequence $\{s_n\}$ is said to be bounded above if there exists $u \in \mathbb{R}$ such that $s_n \le u, \forall n \in \mathbb{Z}^+$

The number u is called an upper bound of $\{s_n\}$

If u is an upper bound of $\{s_n\}$ then any number greater than u is also an upper bound of $\{s_n\}$.

If $\{s_n\}$ is bounded above then the least among all the upper bounds of $\{s_n\}$ is called least upper bound (l.u.b) of $\{s_n\}$.

Def:- A sequence $\{s_n\}$ is said to be bounded if it is both bounded above and bounded below.

ex:- $s_n = \frac{1}{n}, \forall n \in \mathbb{Z}^+$. Then $\{s_n\}$ is bounded since $0 < \frac{1}{n} \le 1$ i.e $0 < s_n \le 1, \forall n \in \mathbb{Z}^+$.

Here 0 is the g.l.b and 1 is the l.u.b of $\{s_n\}$.

ex:- Let $\{s_n\}$ be defined by $s_n = 1 + (-1)^n, \forall n \in \mathbb{Z}^+$

i.e $s_1 = 0, s_2 = 2, s_3 = 0, s_4 = 2, \ldots$

$\therefore$ Range of $\{s_n\} = \{0, 2\}$ is a finite set

Hence $\{s_n\}$ is bounded

The l.u.b of $\{s_n\} = 2$

The g.l.b of $\{s_n\} = 0$

Convergent Sequence

A Sequence $\{s_n\}$ is said to be convergent if there exists $s \in \mathbb{R}$ such that, for each $\epsilon > 0$, there exists $m \in \mathbb{Z}^+$ such that $|s_n - s| < \epsilon$, $\forall n \geq m$.

In this case 's' is called the limit of the sequence. we write $\lim_{n \to \infty} s_n = s$.

The limit of a sequence if exists, is unique.

Note:- Every convergent sequence is bounded, but the converse is not true.

For example, $\{\frac{1}{n}\}$ is convergent and hence it is bounded

But the sequence $\{(-1)^n\} = -1, 1, -1, 1, \ldots$ is bounded but not convergent.

Divergent Sequence

A sequence $\{s_n\}$ is said to be divergent if it is not convergent.

Divergent sequence can be classified as

1) diverging to $+\infty$
2) diverging to $-\infty$
3) oscillates finitely (i.e bounded but not convergent)
4) oscillates infinitely (i.e neither bounded nor convergent)

Sequences and Series

Infinite Series :

An expression of the form $u_1+u_2+u_3+\cdots$ is called an infinite series and is denoted by $\sum_{n=1}^{\infty} u_n$.

Let $S_n = u_1+u_2+\cdots+u_n$ be the nth partial sum. If the sequence $\{S_n\}$ converges to a number S, the we say that the series converges to S. The number S is called the sum of the series and we write $\sum_{n=1}^{\infty} u_n = S$.

* The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent for $|r|<1$ and divergent for $|r| \geq 1$.

* The auxiliary series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p>1$ and is divergent for $p \leq 1$.

Comparison Tests

1. If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that i) there is a positive integer m and $K \in \mathbb{R}^+$ such that $u_n \leq K v_n, \ \forall n \geq m$

ii) $\sum v_n$ is convergent, then $\sum u_n$ is convergent.

2. If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that i) there exists $m \in \mathbb{Z}^+$ and $K \in \mathbb{R}^+$ such that $u_n \geq K v_n, \ \forall n \geq m$ and

ii) $\sum v_n$ is divergent, then $\sum u_n$ is also divergent.

3. (Limit comparison Test):

If $\sum u_n$ and $\sum v_n$ are two series of positive terms and $\lim\limits_{n \to \infty} \frac{u_n}{v_n} \neq 0$, then both series $\sum u_n$ and $\sum v_n$ are convergent or divergent together.

Pb! Test the convergence of the series "

$$\sum \frac{n^3-5n^2+7}{n^5+4n^4-n}.$$

Sol. Here $u_n = \dfrac{n^3-5n^2+7}{n^5+4n^4-n}$

Take $v_n = \dfrac{1}{n^2}$ $\left(\text{i.e } \dfrac{n^3}{n^5}\right)$

Now
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^3-5n^2+7}{n^5+4n^4-n} \times n^2$$

$$= \lim_{n\to\infty} \frac{n^3\left(1-\frac{5}{n}+\frac{7}{n^3}\right)}{n^5\left(1+\frac{4}{n}-\frac{1}{n^4}\right)} \times n^2$$

$$= \lim_{n\to\infty} \frac{1-\frac{5}{n}+\frac{7}{n^3}}{1+\frac{4}{n}-\frac{1}{n^4}}$$

$$= \frac{1-0+0}{1+0-0}$$

$$= 1 \neq 0$$

Hence, by limit comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent

($\because \sum \frac{1}{n^p}$ is convergent for $p>1$)

Hence $\sum u_n$ is also convergent.

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Pb1 Test the convergence of $\sum \frac{1}{n2^n}$

Sol. we know that $n2^n \geq 2^n, \forall n \in \mathbb{Z}^+$

$$\Rightarrow \frac{1}{n2^n} \leq \frac{1}{2^n}, \forall n \quad \text{i.e } u_n \leq v_n$$

The series $\sum \frac{1}{2^n}$ is convergent, since it is a geometric series with common ratio $r = \frac{1}{2} < 1$.

Hence by comparison test, $\sum \frac{1}{n2^n}$ is also convergent.

D'Alembert's ratio test

If $\sum u_n$ is a series of positive terms such that $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$, then

i) $\sum u_n$ is convergent if $l < 1$

ii) $\sum u_n$ is divergent if $l > 1$

iii) The test fails to decide the nature if $l = 1$.

Pb) Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1.3.5....(2n+1)}{2.5.8....(3n+2)}$$

Sol. Here $u_n = \frac{1.3.5.....(2n+1)}{2.5.8.....(3n+2)}$

$$u_{n+1} = \frac{1.3.5......(2n+1)(2n+3)}{2.5.8......(3n+2)(3n+5)}$$

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\cancel{1.3.5.....(2n+1)}(2n+3)}{\cancel{2.5.8.....(3n+2)}(3n+5)} \times \frac{\cancel{2.5.8.....(3n+2)}}{\cancel{1.3.5.....(2n+1)}}$

$$= \lim_{n\to\infty} \frac{2n+3}{3n+5}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{5}{n}}$$

$$= \frac{2+0}{3+0}$$

$$= \frac{2}{3} < 1$$

Hence by D'Alembert's ratio test $\sum u_n$ is convergent i.e the given series is convergent.

6) Test the convergence of $\sum_{n=1}^{\infty}\left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]$

Sol: Here $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$= \frac{\sqrt{n^4+1}-\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}} \times \sqrt{n^4+1}+\sqrt{n^4-1}$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

Take $v_n = \frac{1}{\sqrt{n^4}} = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}} \times n^2$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2\left[\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right]}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{1+1} = 1 \neq 0.$$

Hence, by limit comparison test, both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

Hence $\sum u_n$ is also convergent.

Pb) Test the convergence of $\frac{1}{2}+\frac{\sqrt{2}}{3}+\frac{\sqrt{3}}{8}+\cdots+\frac{\sqrt{n}}{n^2-1}+\cdots$

Sol. Given Series is $\frac{1}{2}+\frac{\sqrt{2}}{3}+\frac{\sqrt{3}}{8}+\cdots+\frac{\sqrt{n}}{n^2-1}+\cdots$

Here $u_n=\frac{\sqrt{n}}{n^2-1}$ (ignoring the first term)

Take $v_n=\frac{1}{n^{3/2}}$

Now $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sqrt{n}}{n^2-1}\times n^{3/2}$

$=\lim_{n\to\infty}\frac{\sqrt{n}}{n^2(1-\frac{1}{n^2})}\times n^{3/2}$

$=\lim_{n\to\infty}\frac{n^2}{n^2(1-\frac{1}{n^2})}$

$=\frac{1}{1-0}=1\neq 0$

$\therefore$ By limit comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n=\sum\frac{1}{n^{3/2}}$ is convergent ($\because \sum\frac{1}{n^p}$ is convergent for $p>1$)

Hence $\sum u_n$ is also convergent

Pb) Test the convergence of $\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$

Sol. Given series is $\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$

Here $u_n=\tan\left(\frac{1}{n}\right)>0$

Take $v_n=\frac{1}{n}$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\tan(\frac{1}{n})}{\frac{1}{n}}$

$= \lim_{t\to 0} \frac{\tan t}{t}$ (put $t = \frac{1}{n}$

as $n \to \infty$, $t = \frac{1}{n} \to 0$

$= 1 \neq 0$

$\therefore$ By limit comparison test, both series $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \frac{1}{n}$ is divergent.

Hence, $\sum u_n$ is also divergent i.e the given series is divergent.

Pbl. Test the convergence of the series $\sum_{n=1}^{\infty} \{ \sqrt[3]{n^3+1} - n \}$

Sol. Here $u_n = \sqrt[3]{n^3+1} - n > 0$

i.e $u_n = (n^3+1)^{\frac{1}{3}} - n$

$= n\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2}\cdot\frac{1}{n^6} + \cdots\right] - n$

($\because$ By binomial theorem,

$(x+y)^n = x^n + nc_1 x^{n-1}y + nc_2 x^{n-2}y^2 + \cdots + y^n$.

$= n\left[1 + \frac{1}{3n^3} + \frac{1}{9n^6} + \cdots\right] - n$

$= \frac{1}{3n^2} - \frac{1}{9n^5} + \cdots$

Take $v_n = \frac{1}{n^2}$

we know that $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right] \times n^2$

$= \frac{1}{3} - 0$

$= \frac{1}{3} \neq 0$

$\therefore$ By limit comparison test both series

$\sum u_n$ and $\sum v_n$ converge or diverge together.

Hence $\sum u_n$ is also convergent.

Practice Problems : Test the convergence of

1. $\sum \left(\sqrt{n+1} - \sqrt{n-1}\right)$ 2. $\sum \left[\sqrt{n^3+1} - \sqrt{n^3}\right]$

3. $\sum \frac{1}{n^{3/2}+n+1}$ 4. $\frac{\sqrt{2}-1}{3^2-1} + \frac{\sqrt{3}-1}{4^2-1} + \frac{\sqrt{4}-1}{5^2-1} + \cdots$

5. $\frac{1}{4\cdot 7\cdot 10} + \frac{1}{7\cdot 10\cdot 13} + \frac{9}{10\cdot 13\cdot 16} + \cdots$ 6. $\sum \sqrt{\frac{n}{n^5+2}}$

7. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$ (or) $\sum \frac{n+1}{n^p}$

8. $\sum \frac{\log n}{2n^3-1}$ 9. $\sum \frac{2n-1}{n(n+1)(n+2)}$

Answers 1. convergent 2. convergent 3. convergent

4. convergent 5. divergent 6. convergent

7. converges for $p>2$, diverges for $p\leq 2$ 8. convergent

9. convergent.

Pb) Test the convergence of the series

$$1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\cdots$$

Sol: Given Series is $1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\cdots$

Here $u_n=\frac{1}{\lfloor n}>0$ for all $n\in\mathbb{Z}^+$ and $u_0=1$

$$u_{n+1}=\frac{1}{\lfloor n+1}$$

Now $$\frac{u_{n+1}}{u_n}=\frac{1}{\lfloor n+1}\times\lfloor n=\frac{\lfloor n}{\lfloor n+1}=\frac{1}{n+1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1}{n+1}=0<1$$

Hence, by D'Alembert's ratio test $\sum u_n$ is convergent i.e given series is convergent.

Pb) Show that $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots\cdots$ is convergent for all values of p.

(or) Test the convergence of $\sum\frac{n^p}{n!}$

Sol. Given Series is $\sum\frac{n^p}{n!}$

Here $u_n=\frac{n^p}{n!}$. Then $u_{n+1}=\frac{(n+1)^p}{(n+1)!}$

Now $$\frac{u_{n+1}}{u_n}=\frac{(n+1)^p}{(n+1)!}\times\frac{n!}{n^p}=\frac{1}{n+1}\cdot\left(1+\frac{1}{n}\right)^p$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1}{n+1}\left(1+\frac{1}{n}\right)^p$$

$$=0<1$$

Hence, by D'Alembert's ratio test, $\sum u_n$ is convergent.

Pb) Test the convergence of the series
$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots\cdots \quad (x>0)$$

Sol. Given Series is $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots\cdots$ $(x>0)$

Here $u_n = \frac{x^n}{n^2+1} > 0, \ (\because x>0)$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{x^n} = x \cdot \frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x \cdot \frac{1+0}{(1+0)^2+0} = x$$

$(\because \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By D'Alembert's ratio test, $\sum u_n$ is convergent if $x<1$ and is divergent if $x>1$

If $x=1$ the test fails.

when $x=1$, $u_n = \frac{1}{n^2+1}$

Take $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^2+1} \times n^2 = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}}$$
$$= \frac{1}{1+0} = 1 \neq 0$$

By limit comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent ($\because \sum \frac{1}{n^p}$ is convergent for $p = 2 > 1$)

$\therefore \sum u_n$ is also convergent.

Hence $\sum u_n$ is convergent for $x \leq 1$ and is divergent for $x > 1$.

P10) * Test the convergence of the series whose nth term is $\dfrac{x^{2n-2}}{\sqrt{n}\,(n+1)}$

(or)

Test the convergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots\cdots$$

Sol. Here $u_n = \dfrac{x^{2n-2}}{\sqrt{n}\,(n+1)}$

$$u_{n+1} = \frac{x^{2n}}{\sqrt{n+1}\,(n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{\sqrt{n+1}\,(n+2)} \times \frac{\sqrt{n}\,(n+1)}{x^{2n-2}}$$

$$= x^2 \cdot \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}\left(1+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{x^2(1+0)}{\sqrt{1+0}\,(1+0)} \quad \left(\because \lim_{n\to\infty} \frac{1}{n} = 0\right)$$

$$= x^2$$

$\therefore$ By D'Alembert's ratio test, $\sum u_n$ is convergent if $x^2 < 1$ and is divergent if $x^2 > 1$.

If $x^2 = 1$, the test fails.

when $x^2=1$, $u_n = \dfrac{1}{\sqrt{n}\,(n+1)}$

Take $v_n = \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$

$\sum v_n = \sum \dfrac{1}{n^{3/2}}$ is convergent ($\because \sum \dfrac{1}{n^p}$ is convergent for $p>1$)

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}\,(n+1)} \times n\sqrt{n}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1 \neq 0$

$\therefore$ By limit comparison test, both $\sum u_n$ and $\sum v_n$ converge or diverge together.

Hence $\sum u_n$ is also convergent.

Thus, the given series is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

p6) Discuss the convergence of $\dfrac{x}{1\cdot 3} + \dfrac{x^2}{3\cdot 5} + \dfrac{x^3}{7\cdot 9} + \cdots$ $(x>0)$

Sol. Given Series is $\dfrac{x}{1\cdot 3} + \dfrac{x^2}{5\cdot 7} + \dfrac{x^3}{7\cdot 9} + \cdots$ $(x>0)$

Here $u_n = \dfrac{x^n}{(2n+1)(2n+3)}$ (ignoring 1st term)

$u_{n+1} = \dfrac{x^{n+1}}{(2n+3)(2n+5)}$

Now $\dfrac{u_{n+1}}{u_n} = \dfrac{x^{n+1}}{(2n+3)(2n+5)} \times \dfrac{(2n+1)(2n+3)}{x^n}$

$= \dfrac{2n+1}{2n+5} \cdot x$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+1}{2n+5} \cdot x$$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{2+\frac{5}{n}} \cdot x$$

$$= \frac{2+0}{2+0} \cdot x$$

$$= x$$

Hence, by D'Alembert's ratio test, $\sum u_n$ is convergent if $x<1$ and is divergent if $x>1$

The test fails if $x=1$

when $x=1$, $u_n = \frac{1}{(2n+1)(2n+3)}$

Take $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(2n+1)(2n+3)} \times n^2$$

$$= \lim_{n\to\infty} \frac{1}{(2+\frac{1}{n})(2+\frac{3}{n})}$$

$$= \frac{1}{(2+0)(2+0)} = \frac{1}{4} \neq 0 \quad \left(\because \lim_{n\to\infty} \frac{1}{n} = 0\right)$$

$\therefore$ By limit comparison test, both $\sum u_n$ and $\sum v_n$ converge or divergent together.

But $\sum v_n = \sum \frac{1}{n^2}$ is Convergent

Hence $\sum u_n$ is also convergent.

Thus, the given series is convergent if $x \le 1$ and is divergent if $x>1$.

Practice Problems

Test for convergence of

1. $\sum \frac{n}{n^n}$ 2. $\sum \frac{1.2.3.\cdots n}{3.5.7.\cdots(2n+1)}$ 3. $\sum \frac{n^2}{2^n}$

4. $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \cdots\cdots$

5. $\sum \frac{3.6.9\cdots 3n}{4.7.10\cdots(3n+1)} \cdot \frac{5^n}{(3n+2)}$

6. $\frac{2}{3.4}x + \frac{3}{4.5}x^2 + \frac{4}{5.6}x^3 + \cdots\cdots\infty$ $(x>0)$

7. $\sum \frac{x^{2n}}{(n+1)\sqrt{n}}$

8. Find the interval of convergence of the series

$$\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\cdots\infty$$

Answers

1. converges 2. converges 3. converges

4. converges for $x \le 1$ and diverges for $x > 1$.

5. divergent 6. convergent for $x<1$, divergent for $x \ge 1$

7. convergent if $x^2 \le 1$ and divergent if $x^2 > 1$

8. converges for $-1 \le x < 1$.

Cauchy's nth root test

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} u_n^{1/n} = l$, then

i) $\sum u_n$ is convergent if $l<1$

ii) $\sum u_n$ is divergent if $l>1$

iii) The test fails to decide the nature if $l=1$

pb) Test the convergence of the series

$$\sum \left(1+\frac{1}{n}\right)^{-n^2}$$

Sol. Here $u_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$u_n^{1/n} = \left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n} = \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

Now $\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$$= \frac{1}{e} < 1 \quad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right)$$

Hence, by Cauchy's nth root test, $\sum u_n$ is convergent.

Pb) Test the convergence of $\sum \frac{n^{n^2}}{(n+1)^{n^2}}$

Sol. Here $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$u_n^{1/n} = \left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{\frac{1}{n}} = \frac{n^n}{(n+1)^n}$$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

$\therefore$ By Cauchy's nth root test, $\sum u_n$ is convergent.

Pb) Test the convergence of $\sum \frac{1}{(\log\log n)^n}$

Sol. Given series is $\sum \frac{1}{(\log\log n)^n}$

Here $u_n = \frac{1}{(\log\log n)^n}$

$$u_n^{1/n} = \frac{1}{\log(\log n)}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log(\log n)} = 0 < 1$$

Hence, by Cauchy's nth root test $\sum u_n$ is convergent i.e the given series is convergent.

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(b) Test the convergence of $\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \cdots + \frac{(n+1)^n}{n^{n+1}}x^n + \cdots$ $(x>0)$

Sol: Here $u_n = \frac{(n+1)^n}{n^{n+1}}x^n$

$$u_n^{1/n} = \left[\frac{(n+1)^n}{n^{n+1}}\right]^{1/n} \cdot x = \frac{n+1}{n^{\frac{n+1}{n}}} \cdot x$$

i.e $$u_n^{1/n} = \frac{n+1}{n \cdot n^{1/n}}x = \frac{1+\frac{1}{n}}{n^{1/n}}x$$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = \frac{1+0}{1}x = x \quad (\because \lim_{n\to\infty} n^{1/n} = 1)$$

$\therefore$ By Cauchy's nth root test $\sum u_n$ is convergent if $x<1$ and is divergent if $x>1$

The test fails if $x=1$

when $x=1$, $$u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n(1+\frac{1}{n})^n}{n \cdot n^n} = \frac{(1+\frac{1}{n})^n}{n}$$

Take $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^n}{n} \times n$$

$$= \lim_{n\to\infty} (1+\frac{1}{n})^n = e \neq 0$$

$\therefore$ By limit comparison test both the series $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \frac{1}{n}$ is divergent.

Hence $\sum u_n$ is divergent.

Thus given series is convergent if $x<1$ and divergent if $x \geq 1$.

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Pb) Test the convergence of $\sum \frac{x^n}{n^{n-1}}$ $(x>0)$

Sol. Given series is $\sum \frac{x^n}{n^{n-1}}$

Here $u_n = \frac{x^n}{n^{n-1}} > 0$ $(\because x>0)$

$$u_n^{1/n} = \left(\frac{x^n}{n^{n-1}}\right)^{\frac{1}{n}} = \frac{x}{n^{\frac{n-1}{n}}} = \frac{x}{n} \cdot n^{\frac{1}{n}}$$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{x}{n} \cdot n^{1/n}$$

$$= \lim_{n\to\infty} \frac{x}{n} \cdot \lim_{n\to\infty} n^{1/n}$$

$$= 0 \times 1 = 0 < 1 \qquad \left(\because \lim_{n\to\infty} n^{1/n} = 1\right)$$

Hence by Cauchy's nth root test, $\sum u_n$ i.e the given series is convergent.

Practice Problems

Test the convergence of

1) $\sum n(e^{-n})^2$ 2) $\sum \left(\frac{n+1}{n+2}\right)^n x^n$ $(x>0)$

3) $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}$ 4) $\sum \frac{x^{2n}}{2^n}$ $(x>0)$ 5) $\sum n\left(\frac{2}{3}\right)^n$

6) $\sum \frac{2^n}{n^3}$

Answers : 1) convergent 2) converges if $x<1$, diverges if $x \geq 1$

3) divergent 4) converges if $x^2 < 2$ and diverges if $x^2 \geq 2$

5) convergent 6) divergent.

Cauchy's integral test

Let f be a non negative decreasing function defined on $[1,\infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converge or diverge according as the improper integral $\int_1^{\infty} f(x)\,dx$ converge (finite) or diverge (infinite).

pb) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Sol) Given series $\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \sum_{n=1}^{\infty} f(n)$ say

Then $f(x) = \frac{x}{x^2+1}$

Now $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{x}{x^2+1}\,dx$

$$= \left[\frac{1}{2}\log(x^2+1)\right]_1^{\infty}$$

$$= \frac{1}{2}\left[\log\infty - \log 2\right]$$

$$= \infty$$

$\therefore \int_1^{\infty} f(x)\,dx$ diverges

Hence, by Cauchy's integral test, given series is divergent.

p(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (harmonic series of order p) converges for $p>1$ and diverges for $p \le 1$.

Sol. Given series is $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Let $f(x) = \frac{1}{x^p}$ for $x \in [1, \infty)$

Clearly $f(x) > 0$ and f is decreasing on $[1, \infty)$

Now $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx$

$$= \begin{cases} \left(\frac{x^{1-p}}{1-p}\right)_1^{\infty} & ; \text{if } p \ne 1 \\ \left[\log x\right]_1^{\infty} & \text{if } p = 1 \end{cases}$$

Now, $x^{1-p} = \frac{1}{x^{p-1}} \to 0$ as $x \to \infty$, for $p > 1$

and $x^{1-p} \to \infty$ as $x \to \infty$ for $p < 1$

and $\log x \to \infty$ as $x \to \infty$

$\therefore \int_1^{\infty} f(x)\,dx$ converges to $\frac{1}{p-1}$ if $p > 1$

and $\int_1^{\infty} f(x)\,dx$ diverges (i.e does not exist)

for $p \le 1$. ($\because$ for $p \le 1$, $\int_1^{\infty} f(x)\,dx = \infty$

Hence, by Cauchy's integral test $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Pb) Use integral test, determine the convergence of the series $1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}+\cdots$

Sol. Let $f(x)=\frac{1}{2x-1}$ on $[1,\infty)$

Clearly $f(x)>0$ and f is decreasing on $[1,\infty)$

$$\int_1^\infty f(x)\,dx=\int_1^\infty \frac{1}{2x-1}\,dx=\left[\frac{1}{2}\log(2x-1)\right]_1^\infty=\infty$$

Hence, by Cauchy's integral test given series is divergent.

Pb) Show that $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$ converges if $p>1$ and diverges for $p\leq 1$

Sol. Let $f(x)=\frac{1}{x(\log x)^p}$, $\forall x\in[2,\infty)$

Clearly $f(x)>0$ and f is decreasing on $[2,\infty)$

$$\int_2^\infty f(x)\,dx=\int_2^\infty \frac{1}{x(\log x)^p}\,dx$$

$$=\int_{\log 2}^{\infty}\frac{1}{t^p}\,dt$$

(put $\log x=t$
$\Rightarrow \frac{1}{x}dx=dt$
when $x=2$, $t=\log 2$
when $x=\infty$, $t=\infty$)

$$=\left[\frac{t^{1-p}}{1-p}\right]_{\log 2}^{\infty}$$

when $p>1$, $1-p<0 \Rightarrow t^{1-p}\to 0$ as $t\to\infty$

$\therefore \int_2^\infty f(x)\,dx$ is convergent (exists) if $p>1$

when $p\leq 1$, $\int_2^\infty f(x)\,dx$ diverges ($\because t^{1-p}\to\infty$ as $t\to\infty$ for $p<1$)

Hence by Cauchy's integral test, given series is convergent for $p>1$, divergent for $p\leq 1$

Alternating Series

A series whose terms are alternatively positive and negative, is called an alternating series. An alternating series may be written as $u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$

It is denoted by $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Leibnitz's Test

If $\{u_n\}$ is a decreasing sequence of positive terms i.e $u_1 \geq u_2 \geq u_3 \geq \cdots$ such that $\lim_{n \to \infty} u_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent.

Pb) Test for convergence $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n(n+1)(n+2)}}$

Sol. Here $u_n = \frac{1}{\sqrt{n(n+1)(n+2)}}$

$$u_{n+1} = \frac{1}{\sqrt{(n+1)(n+2)(n+3)}}$$

we have $n(n+1)(n+2) < (n+1)(n+2)(n+3)$

$$\Rightarrow \sqrt{n(n+1)(n+2)} < \sqrt{(n+1)(n+2)(n+3)}$$

$$\Rightarrow \frac{1}{\sqrt{n(n+1)(n+2)}} > \frac{1}{\sqrt{(n+1)(n+2)(n+3)}}$$

i.e $u_n > u_{n+1}$, $\forall n$

Now $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

$= 0$

Hence, by Leibnitz's test, the alternating series $\sum(-1)^{n-1} u_n$ is convergent

i.e the given series is convergent.

pb) Examine the convergence of $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \cdots$

sol. Given series is $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \cdots$

which is an alternating series of the form

$\sum_{n=1}^{\infty}(-1)^{n-1} u_n$.

Here $u_n = \frac{1}{(2n-1)(2n)}$

Clearly $u_n > 0$ and $u_n > u_{n+1}$ for all n.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n-1)(2n)} = 0$

Hence, by Leibnitz's test given series is convergent.

pb) Examine the convergence of $\frac{1}{1\cdot3\cdot5}-\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}-\frac{1}{7\cdot9\cdot11}+\cdots$

Sol. Given series is an alternating series

Here $u_n = \frac{1}{(2n-1)(2n+1)(2n+3)} > 0$ for all

Clearly $u_n > u_{n+1}, \forall n$

Now $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n-1)(2n+1)(2n+3)} = 0$

Hence, by Leibnitz's test given series is convergent.

pb) Show that the series $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots$ converges.

Sol. Given series is $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots$

This is an alternating series

$u_n = \frac{1}{(2n-1)!} > 0, \forall n \in \mathbb{Z}^+$

and $u_n > u_{n+1}, \forall n \in \mathbb{Z}^+$ $\left(\because \frac{1}{(2n-1)!} > \frac{1}{(2n+1)!}\right)$

Now $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n-1)!} = 0$

Hence, by Leibnitz's test given series is convergent.

pb) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$

Sol. Given series is $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

$(\because \cos n\pi = (-1)^n)$

This is an alternating series.

$u_n = \frac{1}{n^2+1} > 0$ and $u_n > u_{n+1}, \forall n$

Now $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$

$\therefore$ By Leibnitz's test given series is convergent.

pb) Test for convergence of $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \cdots$ $(0<x<1)$

Sol. This is an alternating series

Here $u_n = \frac{x^n}{1+x^n} > 0, \forall n \quad (\because x > 0)$

$$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= \frac{x^n(1+x^{n+1}) - x^{n+1}(1+x^n)}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} > 0 \quad (\because 0<x<1$$

$\therefore u_n > u_{n+1}, \forall n \in \mathbb{Z}^+$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0 \quad (\because \lim_{n\to\infty} x^n = 0 \text{ for } 0<x<1$$

Hence by Leibnitz's test given series is convergent

pb) Test for convergence of $\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} + \cdots$

Sol. This is an alternating series

$$u_n = \frac{n}{5n+1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{5+\frac{1}{n}} = \frac{1}{5} \neq 0$$

Hence by Leibnitz's test given series is not convergent.

MEAN VALUE THEOREMS

Rolle's theorem

Let $f: [a,b] \to \mathbb{R}$ be a function such that

i) f is continuous on $[a,b]$

ii) f is differentiable on (a,b) and

iii) $f(a) = f(b)$

Then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

pbl Verify Rolle's theorem for $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$.

Sol. Given $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$

Since $\sin x$ and e^x are continuous and also differentiable, $\frac{\sin x}{e^x}$ is continuous and differentiable.

$\therefore$ f is continuous on $[0,\pi]$ and differentiable in (a,b) i.e $(0,\pi)$

$$f(0) = \frac{\sin 0}{e^0} = 0$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = 0 \qquad (\because \sin\pi = 0)$$

$\therefore$ f satisfies all the conditions of Rolle's theorem.

Hence, there exists a point $c \in (0, \pi)$

Such that $f'(c) = 0$

Now $f'(x) = \dfrac{e^x \cos x - \sin x \, e^x}{(e^x)^2}$

$= \dfrac{\cos x - \sin x}{e^x}$

Now $f'(c) = 0$

$\Rightarrow \dfrac{\cos c - \sin c}{e^c} = 0$

$\Rightarrow \cos c - \sin c = 0$

$\Rightarrow \cos c = \sin c$

$\Rightarrow \tan c = 1$

$\Rightarrow c = \dfrac{\pi}{4} \in (0, \pi)$

Hence Rolle's theorem is verified.

pb) verify Rolle's theorem for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$ where $a>0, b>0$

Sol. Given $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$

Clearly f is continuous in $[a,b]$ and derivable in (a,b).

$$f(a) = \log\left[\frac{a^2+ab}{a(a+b)}\right] = \log 1 = 0$$

$$f(b) = \log\left[\frac{b^2+ab}{b(a+b)}\right] = \log 1 = 0$$

$$\therefore f(a) = f(b)$$

Hence f satisfies all the conditions of Rolle's theorem.

$\therefore$ There exists a point $c \in (a,b)$ such that $f'(c) = 0$

Now $f(x) = \log(x^2+ab) - \log x - \log(a+b)$

$$\therefore f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} - 0$$

$$= \frac{x^2-ab}{x(x^2+ab)}$$

Now $f'(c) = 0$

$$\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0$$

$\Rightarrow c^2 - ab = 0$

$\Rightarrow c^2 = ab$

$\Rightarrow c = \pm\sqrt{ab}$

But $c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem is verified.

pbl Verify Rolle's theorem for i) $f(x) = \tan x$ in $[0,\pi]$

ii) $f(x) = \frac{1}{x^2}$ in $[-1,1]$

Sol. i) Given $f(x) = \tan x$ in $[0,\pi]$

f is discontinuous at $x = \frac{\pi}{2}$, as $f(x)$ is not defined there.

$\therefore$ Rolle's theorem is not applicable.

ii) Given $f(x) = \frac{1}{x^2}$ in $[-1,1]$

Here $f(x)$ is dis-continuous at $x=0$.

$\therefore$ Rolle's theorem is not applicable.

pbl Verify Rolle's theorem for $f(x) = x^2 - 2x - 3$ in the interval $(-1,3)$.

Sol. Given $f(x) = x^2 - 2x - 3$ in $[-1, 3]$

Since $f(x)$ is a polynomial, we have f is continuous in $[-1,3]$ and f is derivable in $(-1,3)$.

$f(-1) = (-1)^2 - 2(-1) - 3 = 0$

$f(3) = 3^2 - 2(3) - 3 = 0$

$\therefore f(-1) = f(3)$.

Hence f satisfies all the conditions of Rolle's theorem

$\therefore$ There exists a point $c \in (-1,3)$ such that

$f'(c) = 0$

$\Rightarrow 2c - 2 = 0 \qquad (\because f'(x) = 2x-2)$

$\Rightarrow c = 1 \in (-1, 3)$

Hence, Rolle's theorem is verified.

Practice Problems

1. verify Rolle's theorem for $f(x) = (x-a)^m (x-b)^n$ where m, n are positive integers

2. verify Rolle's theorem for $f(x) = |x|$ in $[-1, 1]$

3. Using Rolle's theorem, Show that $g(x) = 8x^3 - 6x^2 - 2x + 1$ has a zero between 0 and 1.

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Lagrange's Mean Value Theorem

If $f : [a,b] \to \mathbb{R}$ is a function such that

i) f is continuous on $[a,b]$

ii) f is differentiable on (a,b),

then there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Pb) Verify Lagrange's mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

Sol. Given $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

Since $f(x)$ is a polynomial function, f is continuous in $[0,4]$ and f is derivable in $(0,4)$.

$\therefore$ f satisfies all the conditions of Lagrange's mean value theorem.

Hence, there exists a point $c \in (0,4)$ such that

$$f'(c) = \frac{f(4)-f(0)}{4-0} \longrightarrow 1)$$

we have $f'(x) = 3x^2 - 2x - 5$

$$\therefore f'(c) = 3c^2 - 2c - 5$$

$$f(4) = 4^3 - 4^2 - 5(4) + 3$$
$$= 64 - 16 - 20 + 3 = 31$$

$$f(0) = 3$$

substitute these values in eq(1),

$$3c^2 - 2c - 5 = \frac{31-3}{4} = \frac{28}{4}$$
$$= 7$$

$$\Rightarrow 3c^2 - 2c - 12 = 0$$

$$\therefore c = \frac{2 \pm \sqrt{4+144}}{6}$$
$$= \frac{2 \pm \sqrt{148}}{6}$$
$$= \frac{2 \pm 2\sqrt{37}}{6}$$
$$= \frac{1 \pm \sqrt{37}}{3}$$

$$c = \frac{1+\sqrt{37}}{3} \in (0,4)$$

Hence Lagrange's mean value theorem is verified.

pb) verify Lagrange's mean value theorem for $f(x) = \log_e x$ in $[1, e]$.

Sol. Given $f(x) = \log_e x$ in $[1, e]$

clearly f is continuous in $[1, e]$ and

f is differentiable in $(1, e)$

Hence, by Lagrange's mean value theorem, there exists a point $c \in (1, e)$

such that $$f'(c) = \frac{f(e) - f(1)}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{\log_e e - \log_e 1}{e - 1} \quad \left(\because f'(x) = \frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{c} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}$$

$$\Rightarrow c = e - 1 \in (1, e)$$

Hence Lagrange's mean value theorem is verified.

Pb) If $a < b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ using Lagrange's mean value theorem.

Hence deduce the following

i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

ii) $\frac{5\pi + 4}{20} < \tan^{-1}2 < \frac{\pi + 2}{4}$.

Sol. Let $f(x) = \tan^{-1} x$ in $[a,b]$ for $0 < a < b$.

Clearly f is continuous on $[a,b]$ and differentiable in (a,b)

Hence, by Lagrange's mean value theorem there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

we have $f'(x) = \frac{1}{1+x^2}$

$$\therefore \frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} \longrightarrow 1)$$

Now, $a < c < b$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1+a^2 < 1+c^2 < 1+b^2$$

$$\Rightarrow 1+b^2 > 1+c^2 > 1+a^2$$

$$\Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2} \longrightarrow 2)$$

from 1) and 2), we get

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

$$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \longrightarrow 3)$$

Deduction:

i) put $b = \frac{4}{3}$ and $a = 1$ in eq 3),

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1^2}$$

$$\Rightarrow \frac{1}{3} \times \frac{9}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\Rightarrow \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$$

ii) put $b = 2$, $a = 1$ in eq 3), we get

$$\frac{2-1}{1+2^2} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+1^2}$$

$$\Rightarrow \frac{1}{5} < \tan^{-1}(2) - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{\pi}{4} + \frac{1}{5} < \tan^{-1}(2) < \frac{\pi}{4} + \frac{1}{2}$$

$$\Rightarrow \frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}.$$

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Practice Problems

1. Verify Lagrange's mean value theorem for $f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$

2. Show that for any $x > 0$, $1 + x < e^x < 1 + xe^x$

3. Verify Lagrange's mean value theorem for $f(x) = 2x^2 - 7x + 10$ in $[0,5]$

4. Prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$.

Cauchy's Mean Value Theorem

Let $f:[a,b] \to \mathbb{R}$, $g:[a,b] \to \mathbb{R}$ be two functions such that

i) f, g are continuous on $[a,b]$

ii) f, g are differentiable on (a,b) and

iii) $g'(x) \neq 0, \ \forall x \in (a,b)$.

Then there exists a point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Problems

1. Verify cauchy's mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$ where $0 < a < b$.

Sol. Given that $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$

Clearly f, g are continuous on $[a, b]$

we have $f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{-1}{2x\sqrt{x}}$

$\therefore f, g$ are differentiable on (a,b)

Also $g'(x) = \frac{-1}{2x\sqrt{x}} \neq 0, \ \forall x \in (a,b)$

Hence, all the conditions of Cauchy's mean value theorem are satisfied

$\therefore$ There exists a point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\Rightarrow -c = \frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}-\sqrt{b}} \times \sqrt{ab}$$

$$\Rightarrow c = \sqrt{ab}$$

Since $a,b>0$, $\sqrt{ab}$ is the geometric mean of a,b and $a < \sqrt{ab} < b$

$$\therefore c = \sqrt{ab} \in (a,b)$$

Hence Cauchy's mean value theorem is verified.

2. Find c of Cauchy's mean value theorem on $[a,b]$ for $f(x) = e^x$ and $g(x) = e^{-x}$ $(a,b>0)$

Sol. Given that $f(x) = e^x$, $g(x) = e^{-x}$ on $[a,b]$

Clearly f, g are continuous on $[a,b]$

and also f,g are differentiable on (a,b)

we have $f'(x) = e^x$, $g'(x) = -e^{-x}$

Also, $g'(x) \neq 0, \ \forall x \in (a,b)$

Hence, the conditions of Cauchy's mean value theorem are satisfied.

$\therefore$ There exists a point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \quad \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\Rightarrow \quad -e^{2c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \frac{e^b - e^a}{e^a - e^b} \times e^a e^b$$

$$\Rightarrow \quad e^{2c} = e^{a+b}$$

$$\Rightarrow \quad 2c = a+b$$

$$\Rightarrow \quad c = \frac{a+b}{2} \in (a,b)$$

Hence Cauchy's mean value theorem is verified.

3. If $f(x) = \log x$ and $g(x) = x^2$ in $[a,b]$ with $b > a > 1$, using Cauchy's theorem prove that $\frac{\log b - \log a}{b-a} = \frac{a+b}{2c^2}$

Sol. Given that $f(x) = \log x$, $g(x) = x^2$

clearly f, g are continuous in $[a, b]$ and differentiable in (a, b)

we have $f'(x) = \frac{1}{x}$, $g'(x) = 2x$

Also $g'(x) = 2x \neq 0$, $\forall x \in (a, b)$

Hence, the conditions of Cauchy's mean value theorem are satisfied.

$\therefore$ There exists a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\Rightarrow \frac{\frac{1}{c}}{2c} = \frac{\log b - \log a}{b^2 - a^2}$$

$$\Rightarrow \frac{1}{2c^2} = \frac{\log b - \log a}{(b-a)(b+a)}$$

$$\Rightarrow \frac{\log b - \log a}{b - a} = \frac{a + b}{2c^2}.$$

practice problems

1. Verify Cauchy's mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in $[3, 7]$ and find c

2. Verify Cauchy's theorem for $f(x) = \sin x$, $g(x) = \cos x$ on $[0, \frac{\pi}{2}]$.

Generalized Mean Value Theorems

Taylor's Theorem

If $f:[a,b]\to\mathbb{R}$ is a function such that

i) $f^{(n-1)}$ is continuous on $[a,b]$

ii) $f^{(n-1)}$ is derivable on (a,b) or $f^{(n)}$ exists on (a,b) and $p\in\mathbb{Z}^+$, then there exists a point $c\in(a,b)$ such that

$$f(b)=f(a)+\frac{b-a}{1!}f'(a)+\frac{(b-a)^2}{2!}f''(a)+\cdots+\cdots+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n$$

where $R_n=\dfrac{(b-a)^p(b-c)^{n-p}}{(n-1)!\,p}f^{(n)}(c)$.

Note :- 1. Schlomilch - Roche's form of Remainder

$$R_n=\frac{(b-a)^p(b-c)^{n-p}}{(n-1)!\,p}f^{(n)}(c)$$

2. Lagrange's Form of Remainder

put $p=n$

$$R_n=\frac{(b-a)^n}{n!}f^{(n)}(c).$$

3. Cauchy's Form of Remainder:

put $p=1$

$$R_n = \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f^{(n)}(c).$$

Another Form of Taylor's Theorem

If $f:[a,a+h] \to \mathbb{R}$ is a function such that

i) $f^{(n-1)}$ is continuous on $[a,a+h]$

ii) $f^{(n-1)}$ is derivable on $(a,a+h)$ and $p \in \mathbb{Z}^+$

then there exists a real number $0<\theta<1$

such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where $R_n = \frac{h^n (1-\theta)^{n-p}}{(n-1)!\, p} f^{(n)}(a+\theta h)$

Note:- 1. Schlomilch-Roche's Form of Remainder

$$R_n = \frac{h^n (1-\theta)^{n-p}}{(n-1)!\, p} f^{(n)}(a+\theta h)$$

2. Lagrange's Form of Remainder.

put $p = n$

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

3. Cauchy's Form of Remainder

put $p = 1$

$$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

Maclaurin's Theorem

If $f: [0, x] \to \mathbb{R}$ is a function such that:

i) $f^{(n-1)}$ is continuous on $[0, x]$

ii) $f^{(n-1)}$ is derivable on $(0, x)$ and $p \in \mathbb{Z}^+$

then there exists a real number $\theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $R_n = \dfrac{x^n (1-\theta)^{n-p}}{(n-1)!\, p} f^n(\theta x)$.

* Taylor's Series expansion of $f(x)$ about the point $x=a$ is given by

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \cdots$$

* put $a=0$, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

which is called Maclaurin's Series expansion of $f(x)$.

Problems

1. Obtain the Taylor's Series expansion of $\sin x$ in powers of $x - \frac{\pi}{4}$.

Sol. Taylor's Series expansion of $f(x)$ about $x=a$ is given by

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

Here $f(x) = \sin x$, $a = \frac{\pi}{4}$

we have $f'(x) = \cos x$, $f''(x) = -\sin x$

$f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$, ...

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

$$f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad f^{iv}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ldots\ldots$$

$\therefore$ Required Taylor's series is

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x-\frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \ldots\ldots$$

$$\sin x = \frac{1}{\sqrt{2}} + \left(x-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{1}{2}\left(x-\frac{\pi}{4}\right)^2\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4\left(\frac{1}{\sqrt{2}}\right) + \ldots\ldots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{1}{4}\left(x-\frac{\pi}{4}\right)^4 + \ldots\ldots\right]$$

2. Obtain the Maclaurin's series expansion of the following functions

i) e^x ii) $\sin x$ iii) $\cos x$.

Sol. i) Let $f(x) = e^x$

Then $f'(x) = f''(x) = f'''(x) = \cdots\cdots = e^x$

$\therefore f(0) = f'(0) = f''(0) = f'''(0) = \cdots\cdots = e^0 = 1$

The Maclaurin's Series expansion of $f(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \cdots\cdots$$

i.e $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\cdots$

ii) Let $f(x) = \sin x$

Then $f'(x) = \cos x$, $f''(x) = -\sin x$

$f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$ etc.

$f(0) = f''(0) = f^{iv}(0) = 0$.

$f'(0) = 1$, $f'''(0) = -1$, $f^{v}(0) = 1$ etc.

Hence, the Maclaurin's Series expansion of $f(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots$$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots\cdots$$

i.e $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

iii) Let $f(x) = \cos x$

Then $f'(x) = -\sin x$, $f''(x) = -\cos x$

$f'''(x) = \sin x$, $f^{iv}(x) = \cos x$ etc.

$\therefore f(0) = 1$, $f'(0) = f'''(0) = f^{v}(0) = \cdots = 0$

$f''(0) = -1$, $f^{iv}(0) = 1$, $f^{(6)}(0) = -1$, etc.

$\therefore$ Maclaurin's series expansion of $f(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots$$

i.e $\cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots\cdots$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$

3. Show that $\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \cdots\cdots$

(or) Expand $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$ in powers of x.

Sol. Let $f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$f(0) = 0$

$\Rightarrow \sqrt{1-x^2}\, f(x) = \sin^{-1}x \longrightarrow 1)$

differentiating w.r.t x, we get

$$\sqrt{1-x^2}\, f'(x) + f(x) \cdot \frac{-2x}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow (1-x^2) f'(x) - x f(x) = 1 \longrightarrow 2)$ and $f'(0) = 1$

diff. eq 2) w.r.t x, we get

$(1-x^2) f''(x) - 2x f'(x) - f(x) - x f'(x) = 0$

$\Rightarrow (1-x^2) f''(x) - 3x f'(x) - f(x) = 0 \longrightarrow 3)$

$\therefore f''(0) - f(0) = 0$

$\Rightarrow f''(0) = f(0) = 0$

differentiating eq 3), w.r.t x, we get

$(1-x^2) f'''(x) - 2x f''(x) - 3x f''(x) - 3f'(x) - f'(x) = 0$

$\Rightarrow (1-x^2) f'''(x) - 5x f''(x) - 4 f'(x) = 0$

$\therefore f'''(0) - 5(0) - 4 f'(0) = 0$

$\Rightarrow f'''(0) = 4 \quad (\because f'(0) = 1$

Similarly $f^4(0) = 0$

Maclaurin's series expansion is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

i.e $$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\cdot 4 + \frac{x^4}{4!}(0) + \cdots$$

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \cdots$$

*4. Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in the interval.

Sol. Given $f(x) = (1-x)^{5/2}$ in $[0,1]$.

Clearly f, f' are continuous in $[0,1]$ and f' is differentiable in $(0,1)$

Hence f satisfies the conditions of Taylor's theorem.

Consider Taylor's theorem with Lagrange's form of remainder

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x) \longrightarrow 1)$$

where $0 < \theta < 1$

Here $n = p = 2$

$a = 0, \quad x = 1$

$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = -\frac{5}{2}(1-x)^{3/2} \Rightarrow f'(0) = -\frac{5}{2}$

$f''(x) = -\frac{5}{2} \cdot \frac{3}{2}(1-x)^{1/2}(-1) = \frac{15}{4}(1-x)^{1/2}$

$\therefore f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$

$\Rightarrow f''(\theta) = \frac{15}{4}(1-\theta)^{1/2} \quad (\because x = 1$

From eq 1), we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta)$$

$$\Rightarrow f(1) = 1 + 1\left(-\frac{5}{2}\right) + \frac{1}{2} \frac{15}{4}(1-\theta)^{1/2}$$

$$\Rightarrow 0 = -\frac{3}{2} + \frac{15}{8}(1-\theta)^{1/2}$$

$$\Rightarrow (1-\theta)^{1/2} = \frac{3}{2} \times \frac{8}{15} = \frac{4}{5}$$

$$\Rightarrow 1 - \theta = \frac{16}{25}$$

$$\Rightarrow \theta = 1 - \frac{16}{25} = \frac{9}{25} = 0.36$$

θ lies between 0 and 1.

Hence Taylor's theorem is verified.

Practice problems

1. Obtain the Maclaurin's Series expansion of i) $\sinh x$ ii) $\cosh x$

2. Obtain the Maclaurin's Series expansion of $\log_e(1+x)$

3. Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 3 terms in the interval $[0, 1]$.

4. Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$ and hence deduce that

$$\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$