(2) Find Condition Number based on Frobenius norm

(i) $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

$\det(A) = \frac{1}{2}(2) = 1$ $\quad A^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

Condition Number $K(A) = \|A\|_F \cdot \|A^{-1}\|_F$

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{ij}\|^2 \right)^{1/2}$$

$$= \left[\left(\tfrac{1}{2}\right)^2 + 2^2 \right]^{1/2} = \frac{\sqrt{17}}{2}$$

$$\|A^{-1}\|_F = \left(2^2 + 0^2 + 0^2 + \tfrac{1}{2}^2 \right)^{1/2} = \left(4 + \tfrac{1}{4}\right)^{1/2} = \left(\tfrac{17}{4}\right)^{1/2}$$

$$\|A^{-1}\|_F = \frac{\sqrt{17}}{2}$$

Condition Number $K(A) = \frac{\sqrt{17}}{2} \times \frac{\sqrt{17}}{2} = \frac{17}{4} = 4.25$

(ii) $B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \det(B) = 3(1) - 1 = 2$

$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$

$$\|B\|_F = \sqrt{3^2 + 1^2 + 1^2 + 1^2} = \sqrt{12}$$

$$\|B^{-1}\|_F = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \frac{\sqrt{12}}{2}$$

Condition Number $K(B) = \|B\|_F \cdot \|B^{-1}\|_F$

$$= \sqrt{12} \cdot \frac{\sqrt{12}}{2} = 6$$

(c)

$$C = \begin{bmatrix} 1.0001 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det(C) = 1.0001 - 1 = 0.0001$$

$$C^{-1} = \frac{1}{0.0001} \begin{bmatrix} 1 & -1 \\ -1 & 1.0001 \end{bmatrix} = \begin{bmatrix} 10000 & -10000 \\ -10000 & 10001 \end{bmatrix}$$

$$\|C\|_F = \sqrt{(1.0001)^2 + 1 + 1 + 1} = 2.000050002$$

$$\|C^{-1}\|_F = \sqrt{10000^2 + (-10000)^2 \times 2 + (10001)^2} = 20000.50002$$

Condition Number $K(C) = \|C\|_F \|C^{-1}\|_F$

$$= 40002.0001$$