5) a) Eigen values and Eigen vectors of a $n \times n$ Identity Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \cdots & & & & 1 \end{bmatrix}_{n \times n}$$

$$A - \lambda I_{n\times n} = \begin{bmatrix} 1-\lambda & 0 & \cdots & 0 \\ 0 & 1-\lambda & \cdots & 0 \\ 0 & \cdots & & 1-\lambda \end{bmatrix}$$

$\det(A - \lambda I_{n\times n}) = 0$

$\Rightarrow (1-\lambda)^n = 0 \Rightarrow \lambda = 1$

$\Rightarrow \boxed{\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1} \rightarrow$ Eigen Values.

Eigen vectors:

$[A - \lambda_1 I]x_1 = 0$

$$\Rightarrow \left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow x_1, x_2, \ldots x_n \in \mathbb{R}$ can be any real value.

$\therefore$ Eigen vectors for an Identity matrix are infinite $X \in \mathbb{R}$

b) $Ax = \lambda x$

Show that $(A + I)x = (\lambda + 1)x$

$(A+I)x \Rightarrow \underbrace{Ax}_{\lambda x} + Ix$

$\Rightarrow \lambda x + I x$

$= (\lambda + \underset{1}{\cancel{I}})x = (\lambda+1)x$

$\Rightarrow (A+I)x = (\lambda+1)x \Rightarrow$ Thus eigen value changes to $\lambda+1$..

$(A + I - (\lambda+1)I)x = 0 \Rightarrow (A + \cancel{I} - \lambda + -\cancel{I})x = 0$

$= (A - \lambda I)x = 0$ $\rightarrow$ same as $[AX = \lambda X]$

Thus eigen vector does not change.