2. $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

(a)

$$AA^T = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 \\ 4 & 5 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

$$\det(AA^T - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 4-\lambda & 4 & 0 \\ 4 & 5-\lambda & -2 \\ 0 & -2 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)[(5-\lambda)(4-\lambda)-4] - 4(4(4-\lambda)) = 0$$

$$= 80 + 4\lambda^2 - 36\lambda - 16 - 20\lambda - \lambda^3 + 9\lambda^2 + 4\lambda = 64 - 16\lambda$$

$$\Rightarrow \lambda^3 - 13\lambda^2 + 36\lambda = 0$$

$$\lambda = 0, \quad \lambda^2 - 13\lambda + 36 = 0$$

$$\lambda - 9\lambda - 4\lambda + 36 = 0$$

$$\boxed{\lambda = 9, 4, \phi}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{4} = 2$$

$$A^T A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $\lambda_1 = 9$

$$\text{ref}(A - \lambda_1 I) = \text{ref}\left(\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -9 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 = 0$

$x_3 = 0$ $\quad x_2 = 1 \Rightarrow x_1 = 2$

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad v_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

for v_2: $\lambda_2 = 4$

$$rref(A - \lambda_2 I) = rref\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 0.5x_2 = 0$

$x_3 = 0$ $\quad x_2 = 2$ $\quad X_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = -1$

$$v_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

for v_3 with $\lambda_3 = 0$

$$rref(A - \lambda_3 I) = rref\begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & +0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$ $\quad x_2 = 0$ $\quad x_3 = 1$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u_j = \frac{1}{\sigma_j} A v_j$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3}\begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$u_1 = \frac{1}{3}\begin{bmatrix} 4/\sqrt{5} \\ 5/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix}$$

$A = U \Sigma V^T$

$$= \begin{bmatrix} \frac{4}{3\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{5}{3\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}_{3\times 2} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

(b) 2-norm $= \sqrt{\lambda_{max}(A^T A)} = \sqrt{9} = 3$

Frobenius-norm $= \sqrt{4+4+1+4} = \sqrt{13} = 3.605$

(c) Rank-1 approximation:

$$A = \sum_{j=1}^{1} \sigma_j u_j v_j^T = \sigma_1 u_1 v_1^T$$

$$\Rightarrow 3 \times \frac{1}{3} \times \begin{bmatrix} 4/\sqrt{5} \\ 5/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 8 & 4 & 0 \\ 10 & 5 & 0 \\ -4 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1.6 & 0.8 & 0 \\ 2 & 1 & 0 \\ -0.8 & -0.4 & 0 \end{bmatrix}$$

Rank-2 approximation:

$$A = \sum_{j=1}^{2} \sigma_j u_j v_j^T = \frac{1}{5} \times 3 \times \frac{1}{3} \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$$

$$+ \; 2 \times \frac{1}{5} \times \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 10 & 0 & 0 \\ 10 & 5 & 0 \\ 0 & -10 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} = A$$