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On topological solutions to a generalized Chern-Simons equation on lattice graphs

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Abstract

In this paper, we study a generalized self-dual Chern-Simons equation on the lattice graph $\mathbb{Z}^n$ for $n \geq 2$ as given by

$$\Delta u = \lambda e^u (e^u - 1)^{2p+1} + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j},$$

where Δ denotes the Laplacian operator, λ is a positive constant, p is a non-negative integer, $n_1, \ldots, n_M$ are positive integers, and $p_1, \ldots, p_M$ are distinct points with δ_{p_j} representing the Dirac delta function at p_j. We establish the existence of a topological solution that is maximal among all possible solutions. Our findings extend those of of Hua et al. [arXiv:2310.13905], Chao and Hou [J. Math. Anal. Appl. **519**(1), 126787(2023)], and Hou and Qiao [J. Math. Phys. **65**(8), 081503(2024)].

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1. Introduction

The Chern-Simons theory was indeed proposed by Shiing-Shen Chern and James Simons in 1974. This theory was initially developed within the field of mathematics to study the geometric structures on three-dimensional manifolds. Later, it found broad applications in physics, particularly in quantum physics and condensed matter physics. The theory quickly gained significance in understanding topological phase transitions, especially in the context of quantum Hall effects, topological insulators, and high-temperature superconductors [8, 44, 45].

Considerable research has been conducted in the area of field theory models influenced by Chern-Simons type dynamics, such as [34, 35, 41] and so on. From a mathematical perspective, the equations of motion for various Chern-Simons models pose significant analytical challenges, even in scenarios involving radial symmetry and static conditions [32]. The discovery of the

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self-dual structure in the Abelian Chern-Simons model in [21, 31] has inspired a lot of subsequent research. Many problems of existence can be transformed into studies of elliptic partial differential equations or systems of equations, with a particular focus on exploring topological and non-topological solutions [3, 4, 18, 19, 29, 49].

Partial differential equations on discrete graphs have recently garnered significant interest and are now widely applied in diverse fields such as image processing, social network analysis, bioinformatics, and machine learning. Recent advancements have extended some traditional methods for solving partial differential equations in Euclidean spaces to the context of graphs.

In seminal works by Grigor'yan et al. [15–17], a variational approach was developed to study Yamabe type equations, the Kazdan-Warner equation, and some nonlinear equations. These studies primarily focused on establishing the existence of solutions. Subsequent research has extensively explored various types of partial differential equations on graphs. Notably, the existence results for Yamabe type equations have been detailed in references such as [11, 13, 52]. Studies addressing Kazdan-Warner equations are found in [10, 12, 37, 47, 51], and the results concerning Schrödinger equations appear in [5, 40, 43, 50]. Additionally, significant findings related to the heat equations are documented in [36, 38].

In this paper, we study a generalized self-dual Chern-Simons equation given by

$$\Delta u = \lambda e^u (e^u - 1)^{2p+1} + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j}, \tag{1.1}$$

where Δ denotes the Laplacian operator, λ is a positive constant, p is a non-negative integer, $n_1, \ldots, n_M$ are positive integers , and $p_1, \ldots, p_M$ are distinct points with δ_{p_j} being the Dirac delta function at p_j. In the case of Euclidean spaces, a solution $u(x)$ to equation (1.1) is classified as topological if $u(x) \to 0$ as $|x| \to +\infty$, and non-topological if $u(x) \to -\infty$ as $|x| \to +\infty$.

When $p = 0$ and $n_i = 1$ (for $i = 1, 2, \ldots, M$), Eq.(1.1) reduces to

$$\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j}. \tag{1.2}$$

Caffarelli et al. [2] and Tarantello [48] investigated Eq.(1.2) in a doubly periodic region or the 2-torus in $\mathbb{R}^2$ and established the existence of solutions. Huang et al. [27] and Hou et al. [24] studied Eq.(1.2) on finite graphs and obtained the existence of solutions. When $p = 2$ and $n_i = 1$ (for $i = 1, 2, \ldots, M$), Eq.(1.1) is reduced to

$$\Delta u = \lambda e^u (e^u - 1)^5 + 4\pi \sum_{j=1}^{M} \delta_{p_j}. \tag{1.3}$$

Han [20] established the existence of multi-vortices for Eq.(1.3) over a doubly periodic region in $\mathbb{R}^2$. Chao et al. [6] and Hu [25] obtained multiple solutions to Eq.(1.3) on finite graphs. Further studies on the Chern-Simons models on graphs include [7, 9, 22, 23, 26, 30, 33, 39].

Next, we introduce the lattice $\mathbb{Z}^n$ for $n \geq 2$. The lattice $\mathbb{Z}^n$ is a discrete graph which consists of the vertex set V and the edge set E:

$$V = \mathbb{Z}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \in \mathbb{Z} \text{ for all } i \in \{1, \ldots, n\}\},$$

$$E = \left\{ \{x, y\} : x, y \in V \text{ with } \sum_{i=1}^{n} |x_i - y_i| = 1 \right\}.$$

Write $x \sim y$ if $\{x, y\}$ is an edge in E. The distance on $\mathbb{Z}^n$ is defined by

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|\,, \quad x, y \in \mathbb{Z}^n,$$

with $d(x) = d(x, 0)$ representing the distance from the origin.

For a finite subset $\Omega \subset \mathbb{Z}^n$, the boundary of Ω is delineated by

$$\delta\Omega := \{y \in \mathbb{Z}^n \setminus \Omega : \exists x \in \Omega \text{ such that } y \sim x\}\,,$$

and the closure of Ω is designated as $\bar{\Omega} = \Omega \cup \delta\Omega$.

Now, we introduce function spaces and operators on graphs to prepare for the subsequent analysis. Denote $C(\mathbb{Z}^n)$ as the set of functions defined on $\mathbb{Z}^n$. For a finite subset $\Omega \subset \mathbb{Z}^n$, we define $C(\Omega)$ as the set of functions defined on Ω. The l^p-norm $(1 \leq p < \infty)$ of $u \in C(\mathbb{Z}^n)$ is defined as

$$\|u\|_{l^p(\mathbb{Z}^n)} = \left(\sum_{x\in\mathbb{Z}^n} |u(x)|^p\right)^{\frac{1}{p}},$$

and the l^∞-norm of $u \in C(\mathbb{Z}^n)$ is defined as

$$\|u\|_{l^\infty(\mathbb{Z}^n)} = \sup_{x\in\mathbb{Z}^n} |u(x)|.$$

We also define the following seminorm:

$$|u|_{1,p} := \left(\sum_{x\in\mathbb{Z}^n} \sum_{y\sim x} |u(y) - u(x)|^p\right)^{\frac{1}{p}}.$$

For any $u \in C(\mathbb{Z}^n)$, the Laplacian operator is expressed as

$$\Delta u(x) = \sum_{d(x,y)=1} (u(y) - u(x)).$$

Additionally, the difference operator is specified by

$$\nabla_{xy} u = u(y) - u(x), \quad \text{for any } u \in C(\mathbb{Z}^n) \text{ and } x, y \in \mathbb{Z}^n.$$

For functions $u, v \in C(\bar{\Omega})$, a bilinear form is introduced:

$$E_\Omega(u, v) := \frac{1}{2} \sum_{\substack{x,y\in\Omega\\ x\sim y}} \nabla_{xy} u \nabla_{xy} v + \sum_{\substack{x\in\Omega, y\in\delta\Omega\\ x\sim y}} \nabla_{xy} u \nabla_{xy} v, \tag{1.4}$$

and the Dirichlet energy of u on Ω is defined as $E_\Omega(u) = E_\Omega(u, u)$. The directional derivative operator $\frac{\partial u}{\partial \mathbf{n}}$ at $x \in \delta\Omega$ is

$$\frac{\partial u}{\partial \mathbf{n}}(x) := \sum_{\substack{y\in\Omega\\ x\sim y}} (u(x) - u(y)).$$

There also holds Green's identity [14]:

$$E_\Omega(u, v) = -\sum_{x\in\Omega} u(x)\Delta v(x) + \sum_{x\in\delta\Omega} u(x) \frac{\partial v}{\partial \mathbf{n}}(x). \tag{1.5}$$

In this study, we focus primarily on topological solutions to the generalized Chern-Simons equation defined on $\mathbb{Z}^n$:

$$\begin{cases} \Delta u = \lambda e^u (e^u - 1)^{2p+1} + 4\pi \sum_{j=1}^{M} n_j \delta_{p_j}, \\ \lim_{d(x)\to+\infty} u(x) = 0, \end{cases} \tag{1.6}$$

where we aim to construct a topological solution that is maximal among all possible solutions. Our main result is presented as follows.

Theorem 1.1. *The equation (1.6) admits a topological solution $u \in l^{2p+2}(\mathbb{Z}^n)$ on $\mathbb{Z}^n$ for $n \geq 2$, which is maximal among all possible solutions.*

In our proof, we employ methods from [26, 46]. First, we establish an iterative scheme, which yields a monotone sequence $\{u_k\}$ addressing the Dirichlet problem. Subsequently, we introduce the functional $J_\Omega(u)$ and demonstrate that $J_\Omega(u_k)$ is uniformly bounded. By using the discrete Gagliardo-Nirenberg-Sobolev inequality, we ensure that the sequence $\{\|u_k\|_{l^{2p+2}(\Omega)}\}$ is uniformly bounded. As we pass to the limit, we ascertain a solution on Ω. Utilizing the exhaustion method, we extend this solution to $\mathbb{Z}^n$. Additionally, we establish the maximality of the solution.

2. Proof of Theorem 1.1

Initially, we recall a foundational result: the maximum principle.

Lemma 2.1. *[26] Let Ω be a finite subset of $\mathbb{Z}^n$. Consider any positive function $f \in C(\overline{\Omega})$. Suppose that a function $v \in C(\overline{\Omega})$ satisfies the following conditions:*

$$\begin{cases} (\Delta - f)v \geqslant 0 & \text{on } \Omega, \\ v \leqslant 0 & \text{on } \partial\Omega. \end{cases}$$

Then, it follows that $v \leqslant 0$ on $\overline{\Omega}$.

Next, we use the maximum principle to establish the iterative sequence. Let Ω_0 be a finite subset of $\mathbb{Z}^n$ such that $\Omega_0 \supset \{p_j\}_{j=1}^M$. Additionally, let Ω be an arbitrary connected finite subset satisfying $\Omega_0 \subset \Omega \subset \mathbb{Z}^n$. Let $g = 4\pi \sum_{j=1}^M n_j \delta_{p_j}$ and $N = 4\pi \sum_{j=1}^M n_j$. Choose a constant $K > (2p+2)\lambda > 0$. Set $u_0 = 0$ and consider the following iterative equations:

$$\begin{cases} (\Delta - K)\, u_k = \lambda e^{u_{k-1}} (e^{u_{k-1}} - 1)^{2p+1} + g - K u_{k-1} \text{ on } \Omega, \\ u_k = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.2. *Let the sequence $\{u_k\}$ be defined as in (2.1). Then, for each k, u_k is uniquely defined and satisfies*

$$0 = u_0 \geqslant u_1 \geqslant u_2 \geqslant \dots .$$

Proof. First, we establish the following equation for u_1:

$$\begin{cases} (\Delta - K)\, u_1 = g & \text{on } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

By the variation method in Lemma 2.2 in [22], we get that (2.2) admits a unique solution. Utilizing Lemma 2.1, it follows that $u_1 \leqslant 0$.

Assuming that

$$0 = u_0 \geqslant u_1 \geqslant u_2 \geqslant \ldots \geqslant u_i,$$

and given that

$$\lambda e^{u_i} (e^{u_i} - 1)^{2p+1} + g - Ku_i \in l^2(\Omega),$$

we can also guarantee the existence and uniqueness of u_{i+1} by the variation method again.

Analyzing the iterative equation (2.1), we derive that

$$\begin{aligned}(\Delta - K)(u_{i+1} - u_i) &= \lambda e^{u_i} (e^{u_i} - 1)^{2p+1} - \lambda e^{u_{i-1}} (e^{u_{i-1}} - 1)^{2p+1} - K(u_i - u_{i-1}) \\ &\geqslant \lambda e^{\omega} (e^{\omega} - 1)^{2p} \left[(2p+2)e^{\omega} - 1\right](u_i - u_{i-1}) - K(u_i - u_{i-1}) \\ &\geqslant K(e^{2\omega} - 1)(u_i - u_{i-1}) \geqslant 0,\end{aligned}$$

where ω is a function satisfying $u_i \leqslant \omega \leqslant u_{i-1}$. This indicates that $u_{i+1} \leqslant u_i$, thereby affirming the lemma using Lemma 2.1. □

Next, we define the natural functional $J_\Omega(u)$ and demonstrate that $J_\Omega(u_k)$ decreases as k increases. Utilizing Green's identities, we introduce the following functional defined over Ω:

$$J_\Omega(u) = \frac{1}{2} E_\Omega(u) + \sum_{x\in\Omega} \left[\frac{\lambda}{2p+2}(e^{u(x)} - 1)^{2p+2} + g(x)u(x)\right]. \tag{2.3}$$

Lemma 2.3. *Let $\{u_k\}$ be the sequence defined in Eq.(2.1). Then, the following inequality holds:*

$$C \geqslant J_\Omega(u_1) \geqslant J_\Omega(u_2) \geqslant \ldots \geqslant J_\Omega(u_k) \geqslant \ldots,$$

where the constant C depends only on n, λ, p and N.

Proof. By multiplying Eq.(2.1) by the difference $u_k - u_{k-1}$ and performing a summation across the domain Ω, we derive:

$$\begin{aligned}&\sum_{x\in\Omega} (\Delta - K)\, u_k(x)\, [u_k(x) - u_{k-1}(x)] \\ =&\sum_{x\in\Omega} \left[\lambda e^{u_{k-1}} (e^{u_{k-1}} - 1)^{2p+1} (u_k - u_{k-1}) - Ku_{k-1} (u_k - u_{k-1}) + g\,(u_k - u_{k-1})\right](x).\end{aligned} \tag{2.4}$$

Utilizing the Green's identity as specified in (1.5), we have

$$\sum_{x\in\Omega} \Delta u_k(x)\,(u_k(x) - u_{k-1}(x)) = -E_\Omega(u_k - u_{k-1}, u_k) = -E_\Omega(u_k) + E_\Omega(u_{k-1}, u_k). \tag{2.5}$$

Integrating this with equation (2.4), we conclude

$$\begin{aligned}&E_\Omega(u_k) - E_\Omega(u_{k-1}, u_k) + \sum_{x\in\Omega} K\,(u_k(x) - u_{k-1}(x))^2 \\ =& -\sum_{x\in\Omega} \left[\lambda e^{u_{k-1}} (e^{u_{k-1}} - 1)^{2p+1} (u_k - u_{k-1}) + g(x)\,(u_k - u_{k-1})\right].\end{aligned} \tag{2.6}$$

Now we introduce a concave function for $x \leq 0$:

$$h(x) = \frac{\lambda}{2p+2}\left(e^{x}-1\right)^{2p+2} - \frac{K}{2}x^{2}.$$

It is easy to see that

$$h\left(u_{k-1}\right) - h\left(u_{k}\right) \geqslant h'\left(u_{k-1}\right)(u_{k-1}-u_{k}) = \left[\lambda\left(e^{u_{k-1}}-1\right)^{2p+1}e^{u_{k-1}} - Ku_{k-1}\right]\left(u_{k-1}-u_{k}\right),$$

which yields that

$$\begin{aligned}\frac{\lambda}{2p+2}\left(e^{u_{k}}-1\right)^{2p+2} \leqslant& \frac{\lambda}{2p+2}\left(e^{u_{k-1}}-1\right)^{2p+2} + \frac{K}{2}\left(u_{k}-u_{k-1}\right)^{2}\\ &+ \lambda e^{u_{k-1}}\left(e^{u_{k-1}}-1\right)^{2p+1}\left(u_{k}-u_{k-1}\right).\end{aligned} \tag{2.7}$$

It follows from (1.4) that

$$\begin{aligned}\left|E_{\Omega}\left(u_{k-1},u_{k}\right)\right| \leqslant& \frac{1}{2}\sum_{\substack{x,y\in\Omega\\ x\sim y}}\left|\nabla_{xy}u_{k-1}\nabla_{xy}u_{k}\right| + \sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}\left|\nabla_{xy}u_{k-1}\nabla_{xy}u_{k}\right|\\ \leqslant& \frac{1}{4}\sum_{\substack{x,y\in\Omega\\ x\sim y}}\left(\left|\nabla_{xy}u_{k-1}\right|^{2} + \left|\nabla_{xy}u_{k}\right|^{2}\right) + \frac{1}{2}\sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}\left(\left|\nabla_{xy}u_{k-1}\right|^{2} + \left|\nabla_{xy}u_{k}\right|^{2}\right)\\ =& \frac{1}{2}E_{\Omega}\left(u_{k-1}\right) + \frac{1}{2}E_{\Omega}\left(u_{k}\right).\end{aligned} \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we conclude that

$$J_{\Omega}\left(u_{k}\right) \leqslant J_{\Omega}\left(u_{k}\right) + \frac{K}{2}\left\|u_{k-1}-u_{k}\right\|_{l^{2}(\Omega)}^{2} \leqslant J_{\Omega}\left(u_{k-1}\right).$$

Next, we estimate the upper bound for $J_{\Omega}(u_1)$. Noting that

$$\begin{aligned}E_{\Omega}\left(u_{1}\right) =& \frac{1}{2}\sum_{\substack{x,y\in\Omega\\ x\sim y}}\left|\nabla_{xy}u_{1}\right|^{2} + \sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}\left|\nabla_{xy}u_{1}\right|^{2}\\ \leqslant& \sum_{\substack{x,y\in\Omega\\ x\sim y}}\left(u_{1}(x)^{2}+u_{1}(y)^{2}\right) + 2\sum_{\substack{x\in\Omega,y\in\delta\Omega\\ x\sim y}}\left(u_{1}(x)^{2}+u_{1}(y)^{2}\right)\\ \leqslant& 4n\left\|u_{1}\right\|_{l^{2}(\Omega)}^{2},\end{aligned}$$

and $|e^{u_1}-1| = 1-e^{u_1} \leqslant -u_1$, we get

$$\begin{aligned}J_{\Omega}\left(u_{1}\right) \leqslant& \frac{1}{2}\cdot 4n\left\|u_{1}\right\|_{l^{2}(\Omega)}^{2} + \frac{\lambda}{2p+2}\sum_{x\in\Omega}u_{1}(x)^{2p+2} + \frac{1}{2}\sum_{x\in\Omega}\left[g(x)^{2}+u_{1}(x)^{2}\right]\\ =& c_{1} + c_{2}\left(\left\|u_{1}\right\|_{l^{2}(\Omega)}^{2} + \left\|u_{1}\right\|_{l^{2p+2}(\Omega)}^{2p+2}\right),\end{aligned}$$

where c_1, c_2 are constants that only depend on n, p, λ and N.

Multiplying (2.2) by u_1 and summing over Ω, we have

$$E_{\Omega}\left(u_{1}\right) + K\sum_{x\in\Omega}u_{1}(x)^{2} = -\sum_{x\in\Omega}g(x)u_{1}(x).$$

This yields

$$K\sum_{x\in\Omega}u_1(x)^2\leqslant\frac{1}{2K}\sum_{x\in\Omega}g(x)^2+\frac{K}{2}\sum_{x\in\Omega}u_1(x)^2.$$

Hence,

$$\sum_{x\in\Omega}u_1(x)^2\leqslant\frac{\|g\|^2_{l^2(\mathbb{Z}^n)}}{K^2},$$

which yields

$$\sum_{x\in\Omega}u_1(x)^{2p+2}\leqslant\left(\sum_{x\in\Omega}u_1(x)^2\right)^{p+1}\leqslant\left(\frac{\|g\|^2_{l^2(\mathbb{Z}^n)}}{K^2}\right)^{p+1}.$$

We derive that $J_\Omega(u_1)\leq C$, where C depends only on n, λ, p and N, and complete the proof. □

Next, utilizing the discrete Gagliardo-Nirenberg-Sobolev inequality along with Lemma 2.3, we establish the bound of $\|u_k\|_{l^{2p+2}(\Omega)}$. The proof of Theorem 4.1 in [42] provides the following discrete Gagliardo-Nirenberg-Sobolev inequality:

Lemma 2.4. *[42] Let $n\geqslant 2, q>1, \gamma\geqslant q$ and $q'=\frac{q}{q-1}$. Then for any $u\in l^q(\mathbb{Z}^n)$, we have*

$$\|u\|^{\gamma}_{l^{\frac{\gamma n}{n-1}}(\mathbb{Z}^n)}\leq C(q,n,\gamma)|u|_{1,q}\|u\|^{\gamma-1}_{l^{(\gamma-1)q'}(\mathbb{Z}^n)}.$$

Lemma 2.1 in [28] implies that

$$\|u\|_{l^{p''}(\mathbb{Z}^n)}\leqslant\|u\|_{l^{p'}(\mathbb{Z}^n)},$$

for any $p''\geqslant p'$. Letting $\gamma=2(p+1)$, $q=2$ and $q'=2$, we have for $u\in l^2(\mathbb{Z}^n)$,

$$\|u\|_{l^{4p+4}(\mathbb{Z}^n)}\leqslant\|u\|_{l^{\frac{2n(p+1)}{n-1}}(\mathbb{Z}^n)}\leqslant C(n,p)|u|^{\frac{1}{2p+2}}_{1,2}\|u\|^{\frac{2p+1}{2p+2}}_{l^{4p+2}(\mathbb{Z}^n)}. \tag{2.9}$$

Lemma 2.5. *Let $\{u_k\}$ be the sequence defined by Eq.(2.1). For any $k\geq 1$, we have*

$$\|u_k\|_{l^{2p+2}(\Omega)}\leq C_2\left(J_\Omega(u_k)+1\right)\leq C_1, \tag{2.10}$$

where C_2 and C_1depend only on n, λ, p and N.

Proof. Let $\tilde{u}_k$ denote the null extension of u_k to $\mathbb{Z}^n$ as follows:

$$\tilde{u}_k(x)=\begin{cases}u_k(x) & \text{on } \Omega,\\ 0 & \text{on } \Omega^c.\end{cases} \tag{2.11}$$

It is evident that $\tilde{u}_k\in l^2(\mathbb{Z}^n)$. By (2.9), the following inequality holds:

$$\|\tilde{u}_k\|^{2p+2}_{l^{4p+4}(\mathbb{Z}^n)}\leqslant(C(n,p))^{2p+2}|\tilde{u}_k|_{1,2}\|\tilde{u}_k\|^{2p+1}_{l^{4p+2}(\mathbb{Z}^n)}.$$

Referring to (2.11), we obtain:

$$\|\tilde{u}_k\|^{4p+4}_{l^{4p+4}(\mathbb{Z}^n)}=\sum_{x\in\Omega}u_k(x)^{4p+4},$$
$$\|\tilde{u}_k\|^{4p+2}_{l^{4p+2}(\mathbb{Z}^n)}=\sum_{x\in\Omega}u_k(x)^{4p+2},$$

and

$$|\tilde{u}_k|_{1,2} \leqslant (2E_\Omega(u_k))^{\frac{1}{2}}.$$

Thus, it follows that:

$$\sum_{x\in\Omega} u_k(x)^{4p+4} \leqslant C_3 E_\Omega(u_k) \sum_{x\in\Omega} u_k(x)^{4p+2}, \tag{2.12}$$

where $C_3 = 2(C(n,p))^{4p+4}$.

Using the identity $1 - e^{u_k} = 1 - e^{-|u_k|} = \frac{e^{|u_k|}-1}{e^{|u_k|}} \geq \frac{|u_k|}{1+|u_k|}$, and (2.3), we deduce:

$$\begin{aligned}
J_\Omega(u_k) &= \frac{1}{2}E_\Omega(u_k) + \sum_{x\in\Omega}\left[\frac{\lambda}{2p+2}\left(e^{u_k(x)}-1\right)^{2p+2} + g(x)u_k(x)\right] \\
&\geqslant \frac{1}{2}E_\Omega(u_k) + \frac{\lambda}{2p+2}\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2} - \|g\|_{l^{\frac{4p+4}{4p+3}}(\Omega)}\|u_k\|_{l^{4p+4}(\Omega)} \\
&\geqslant \frac{1}{2}E_\Omega(u_k) + \frac{\lambda}{2p+2}\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2} - C_4\left(E_\Omega(u_k)\right)^{\frac{1}{4p+4}}\left(\sum_{x\in\Omega}u_k(x)^{4p+2}\right)^{\frac{1}{4p+4}},
\end{aligned}$$

where C_4 is a uniform constant depending only on n, p and N.

Let $\epsilon > 0$ is a constant to be chosen later. By Yung's inequality, we have

$$\begin{aligned}
&C_4\left(E_\Omega(u_k)\right)^{\frac{1}{4p+4}}\left(\sum_{x\in\Omega}u_k(x)^{4p+2}\right)^{\frac{1}{4p+4}} \\
&= \left[C_4\epsilon^{-\frac{2p+1}{2p+2}}\left(\frac{2p+2}{2p+1}\right)^{-\frac{2p+1}{2p+2}}\left(E_\Omega(u_k)\right)^{\frac{1}{4p+4}}\right]\left[\epsilon^{\frac{2p+1}{2p+2}}\left(\frac{2p+2}{2p+1}\right)^{\frac{2p+1}{2p+2}}\left(\sum_{x\in\Omega}u_k(x)^{4p+2}\right)^{\frac{1}{4p+4}}\right] \\
&\leq \epsilon\|\tilde{u}_k\|_{l^{4p+2}(\mathbb{Z}^n)} + C_5 E_\Omega(u_k)^{\frac{1}{2}} \\
&\leq \epsilon\|\tilde{u}_k\|_{l^{2p+2}(\mathbb{Z}^n)} + \frac{1}{4}E_\Omega(u_k) + C_6,
\end{aligned}$$

where C_5 and C_6 depend only on ϵ, n, N and p.

Hence, we obtain

$$\begin{aligned}
J_\Omega(u_k) &\geqslant \frac{1}{2}E_\Omega(u_k) + \frac{\lambda}{2p+2}\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2} - \epsilon\|\tilde{u}_k\|_{l^{2p+2}(\mathbb{Z}^n)} - \frac{1}{4}E_\Omega(u_k) - C_6 \\
&= \frac{1}{4}E_\Omega(u_k) + \frac{\lambda}{2p+2}\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2} - \epsilon\|u_k\|_{l^{2p+2}(\Omega)} - C_6.
\end{aligned} \tag{2.13}$$

By the inequality (2.12), we have the following estimate:

$$
\begin{aligned}
&\left(\sum_{x\in\Omega} u_k(x)^{2p+2}\right)^2 = \left[\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{p+1}(1+|u_k(x)|)^{p+1}|u_k(x)|^{p+1}\right]^2\\
&\leqslant \sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\sum_{x\in\Omega}(1+|u_k(x)|)^{2p+2}u_k(x)^{2p+2}\\
&\leqslant 2^{2p+2}\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\sum_{x\in\Omega}\left(u_k(x)^{2p+2}+u_k(x)^{4p+4}\right)\\
&\leqslant 2^{2p+2}\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\sum_{x\in\Omega}u_k(x)^{2p+2}+2^{2p+2}C_3\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}E_\Omega(u_k)\sum_{x\in\Omega}u_k(x)^{4p+2}\\
&\leq 2^{2p+2}\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\sum_{x\in\Omega}u_k(x)^{2p+2}+2^{2p+2}C_3\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}E_\Omega(u_k)\left(\sum_{x\in\Omega}u_k(x)^{2p+2}\right)^{\frac{2p+1}{p+1}}\\
&\leq \frac{1}{4}\left(\sum_{x\in\Omega}u_k(x)^{2p+2}\right)^2+C_7\left[\left(\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\right)^2+\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}E_\Omega(u_k)\left(\sum_{x\in\Omega}u_k(x)^{2p+2}\right)^{\frac{2p+1}{p+1}}\right]\\
&\leq \frac{1}{2}\left(\sum_{x\in\Omega}u_k(x)^{2p+2}\right)^2+C_8\left[\left(\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\right)^2+\left(\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\right)^{2p+2}E_\Omega(u_k)^{2p+2}\right]\\
&\leqslant \frac{1}{2}\left(\sum_{x\in\Omega}u_k(x)^{2p+2}\right)^2+C_9\left[1+\left(\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}\right)^{4p+4}+E_\Omega(u_k)^{4p+4}\right],
\end{aligned}
$$

which yields that

$$
\|u_k\|_{l^{2p+2}(\Omega)} \leqslant C_{10}\left[1+\sum_{x\in\Omega}\left(\frac{|u_k(x)|}{1+|u_k(x)|}\right)^{2p+2}+E_\Omega(u_k)\right], \tag{2.14}
$$

where C_7-C_{10} are constants depending only on n and p.

Choosing $\epsilon = \frac{\min\{\frac{1}{8},\frac{\lambda}{4p+4}\}}{C_{10}}$ and combining (2.13) and (2.14), we get

$$
\|u_k\|_{l^{2p+2}(\Omega)} \leqslant C_2(J_\Omega(u_k)+1).
$$

Furthermore, by Lemma 2.3, we conclude

$$
\|u_k\|_{l^{2p+2}(\Omega)} \leqslant C_2(J_\Omega(u_k)+1) \leq C_1,
$$

where C_2 and C_1 depend only on n, λ, p and N. □

The boundedness of $\|u_k\|_{l^{2p+2}(\Omega)}$ ensures the existence of a solution to the Chern-Simons equation on Ω.

Lemma 2.6. *Let Ω be a finite subset of $\mathbb{Z}^n$ containing the distinct points $\{p_j\}_{j=1}^M$. Consider the boundary value problem*

$$
\begin{cases}\Delta u = \lambda e^u(e^u-1)^{2p+1}+g & on\ \Omega,\\ u(x)=0 & on\ \delta\Omega.\end{cases}
$$

A solution $u_\Omega : \bar{\Omega} \to \mathbb{R}$ exists for this problem. This solution is maximal among all possible solutions and satisfies the condition $\|u\|_{l^{2p+2}(\Omega)} \leq C_1$, where C_1 depends only on n, λ, p and N.

Proof. By Lemmas 2.5 and 2.2, we have

$$u_k \to u_\Omega \text{ in } l^{2p+2}(\Omega),$$

and

$$\|u_\Omega\|_{l^{2p+2}(\Omega)} \leqslant C_0.$$

Since Δ is a local operator, and given the pointwise convergence, the function u_Ω in $l^{2p+2}(\Omega)$ is the solution to the equation

$$\begin{cases} \Delta u = \lambda e^u (e^u - 1)^{2p+1} + g \text{ on } \Omega, \\ u(x) = 0 \text{ on } \delta\Omega. \end{cases}$$

This completes the main proof.

The remaining task is to demonstrate that this solution is maximal. We claim that for any function $V \in C(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta V \geqslant \lambda e^V \left(e^V - 1\right)^{2p+1} + g \text{ on } \Omega, \\ V(x) \leqslant 0 \text{ on } \delta\Omega, \end{cases}$$

it holds that

$$u_0 \geqslant u_1 \geqslant \ldots \geqslant u_k \geqslant \ldots \geqslant u_\Omega \geqslant V. \tag{2.15}$$

Initially, it is noted that

$$\Delta V \geqslant \lambda e^V \left(e^V - 1\right)^{2p+1} + g \geqslant \lambda e^V \left(e^V - 1\right)^{2p+1}.$$

We assert that $\sup_{x\in\Omega} V(x) \leqslant 0$. Should this not hold, and $V(x_0) = \sup_{x\in\Omega} V(x) > 0$ for some $x_0 \in \Omega$, then

$$0 \geqslant \Delta V(x_0) \geqslant \lambda e^{V(x_0)} \left(e^{V(x_0)} - 1\right)^{2p+1} > 0,$$

resulting in a contradiction and thus establishing the claim.

Assuming $V \leqslant u_k$, then

$$\begin{aligned} (\Delta - K)(u_{k+1} - V) &\leqslant \lambda e^{u_k} (e^{u_k} - 1)^{2p+1} - \lambda e^V \left(e^V - 1\right)^{2p+1} - K(u_k - V) \\ &\leqslant \lambda e^\omega (e^\omega - 1)^{2p} \left[(2p+2)e^\omega - 1\right](u_k - V) - K(u_k - V) \\ &\leqslant K\left(e^{2\omega} - 1\right)(u_k - V) \leqslant 0, \end{aligned}$$

where the function ω satisfies $V \leqslant \omega \leqslant u_k \leqslant 0$. This ensures that $V \leqslant u_{k+1}$ by Lemma 2.1. Hence $V \leq u_\Omega$. If u'_Ω is a another solution, we conclude that $u'_\Omega \leq u_\Omega$, which means that u_Ω is maximal. □

We are now in a position to prove Theorem 1.1. Let Ω_i be finite and connected subsets satisfying

$$\Omega_0 \subset \Omega_1 \subset \ldots \subset \Omega_k \subset \ldots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \mathbb{Z}^n.$$

We will use these lemmas to establish Theorem 1.1. For any integers $1 \leqslant j \leqslant k$, since $\Omega_j \subset \Omega_k$, and noting that $u_{\Omega_k} \leqslant 0$ on $\overline{\Omega_j}$, by (2.15), we conclude

$$u_{\Omega_k} \leqslant u_{\Omega_j} \text{ on } \overline{\Omega_j}.$$

Let $\tilde{u}_{\Omega_k}$ be the null extension of u_{Ω_k} to $\mathbb{Z}^n$. Then,

$$0 \geqslant \tilde{u}_{\Omega_1} \geqslant \tilde{u}_{\Omega_2} \geqslant \ldots \geqslant \tilde{u}_{\Omega_k} \geqslant \ldots$$

on $\mathbb{Z}^n$. Noting that $\left\|\tilde{u}_{\Omega_k}\right\|_{l^{2p+2}(\mathbb{Z}^n)} \leqslant C_1$ for any $k \geqslant 1$, we have the pointwise convergence

$$\tilde{u}_{\Omega_k}(x) \to u(x), \quad \forall x \in \mathbb{Z}^n,$$

and $u \in l^{2p+2}(\mathbb{Z}^n)$. Hence, u satisfies the equations

$$\begin{cases} \Delta u = \lambda e^u \left(e^u - 1\right)^{2p+1} + 4\pi \sum_{j=1}^{M} \delta_{p_j} \text{ on } \mathbb{Z}^n, \\ \lim_{d(x)\to+\infty} u(x) = 0, \end{cases}$$

which constitutes a topological solution.

Assume there exists another topological solution $\hat{u}$ to (1.6). We now prove that $\hat{u}(x) \leqslant 0$ for all $x \in \mathbb{Z}^n$. If not, there exists $x_0 \in \mathbb{Z}^n$ such that $\hat{u}(x_0) > 0$. Noting that $\lim_{d(x)\to+\infty} \hat{u}(x) = 0$, we conclude there exists a domain Ω_i such that $\hat{u}(x_1) = \sup_{x\in\Omega_i} \hat{u}(x) > 0$ for some $x_1 \in \Omega_i$. Then, we obtain

$$0 \geq \Delta\hat{u}(x_1) \geqslant \lambda e^{\hat{u}(x_1)} \left(e^{\hat{u}(x_1)} - 1\right)^{2p+1} > 0,$$

which yields a contradiction.

Applying (2.15) on Ω_j, we have

$$\hat{u} \leqslant \tilde{u}_{\Omega_j}.$$

For a fixed integer $k \geqslant 1$, and for $j \geqslant k$, we have

$$\hat{u}(x) \leqslant \underline{\lim}_{j\to\infty} \tilde{u}_{\Omega_j}(x) = u(x) \text{ on } \Omega_k.$$

Thus, we obtain $\hat{u} \leqslant u$ on $\mathbb{Z}^n$, and the solution u is maximal among all possible solutions.

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