ELECTRE TRI-nB, pseudo-disjunctive: axiomatic and combinatorial results [a]

Denis Bouyssou [b] Thierry Marchant [c] Marc Pirlot [d]

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Abstract

ELECTRE TRI-nB is a method designed to sort alternatives evaluated on several attributes into ordered categories. It is an extension of ELECTRE TRI-B, using several limiting profiles, instead of just one, to delimit each category. ELECTRE TRI-nB comes in two flavours: pseudo-conjunctive and pseudo-disjunctive. In a previous paper we have characterized the ordered partitions that can be obtained with ELECTRE TRI-nB, pseudo-conjunctive, using a simple axiom called linearity. The present paper is dedicated to the axiomatic analysis of ELECTRE TRI-nB, pseudo-disjunctive. It also provides some combinatorial results.

Keywords: Multiple criteria analysis, Sorting models, ELECTRE TRI-nB.

1 Introduction

ELECTRE TRI (or ETRI for short) is a family of methods for sorting alternatives evaluated on several attributes into ordered categories. The first method in this family was ETRI-B (Roy and Bouyssou, 1993, Yu, 1992). Then came several variants, that we do not detail.[1] Recently, Fernández, Figueira, Navarro, and Roy (2017) proposed a new variant (named ELECTRE TRI-nB or ETRI-nB for short) that uses several limiting profiles instead of merely one as in the original ETRI-B.

[a]Authors are listed alphabetically. They have contributed equally.

[b]Former Senior Researcher, CNRS, Paris, France, e-mail: dbouyssou@gmail.com.

[c]Ghent University, Department of Data Analysis, H. Dunantlaan, 1, B-9000 Gent, Belgium, e-mail: thierry.marchant@UGent.be.

[d]Université de Mons, rue de Houdain 9, 7000 Mons, Belgium, e-mail: marc.pirlot@umons.ac.be.

[1]For an overview of ELECTRE methods, we refer to Roy and Bouyssou (1993, Ch. 5 & 6), Figueira, Greco, Roy, and Słowiński (2013), and Figueira, Mousseau, and Roy (2016).

Like ETRI-B, the new ETRI-nB has two versions: pseudo-conjunctive (pc) and pseudo-disjunctive (pd).

A simplified version of ETRI-B-pc received a detailed axiomatic analysis in Bouyssou and Marchant (2007a,b), Greco, Matarazzo, and Słowiński (2001), Słowiński, Greco, and (2002). Later, Bouyssou and Marchant (2015) have shown that ETRI-B-pd is much more difficult to analyze than ETRI-B-pc, although their definitions may seem dual to each other at first sight.

Bouyssou, Marchant, and Pirlot (2023)—hereafter referred to as BMP23—have characterized the pseudo-conjunctive version of ETRI-nB making auxiliary use of a simplified version thereof. This characterizations uses a single axiom—Linearity—that was first proposed by Goldstein (1991). Bouyssou, Marchant, and Pirlot (2022) have characterized the particular case of ETRI-nB using at most 2 limiting profiles.

The present paper intends to axiomatically analyze ETRI-nB-pd or a simplified version thereof. Our main findings are twofold. The first one is similar to that in Bouyssou and Marchant (2015): ETRI-nB-pd is much more difficult to analyze than ETRI-nB-pc, although their definitions may seem dual to each other. The second one is a characterization of a special case of ETRI-nB-pd, involving Linearity and a new condition, raising some interesting combinatorial questions about maximal antichains in direct products of chains.

2 Framework and notation

We use the framework of conjoint measurement (Krantz, Luce, Suppes, and Tversky, 1971). As in BMP23, we will restrict our attention to the case of two categories. This allows us to use a simple framework while not concealing any important difficulty.[2] For the same reasons, we suppose throughout that the set of objects to be sorted is finite.

The finite set of alternatives is $X = X_1 \times \ldots \times X_n$, with $n \geq 2$. The set of attributes is $N = \{1, \ldots, n\}$. For $x, y \in X, i \in N$ and $J \subseteq N$, we use $X_J, X_{-J}, X_i, X_{-i}, (x_J, y_{-J})$ and (x_i, y_{-i}) as usual. Our primitives consist of a twofold partition $\langle \mathcal{A}, \mathcal{U} \rangle$ of the set X, where $\mathcal{A}$ (resp. $\mathcal{U}$) contains the sAtisfactory (resp. Unsatisfactory) alternatives.

An attribute i is influential for $\langle \mathcal{A}, \mathcal{U} \rangle$ if there exist $x_i, y_i \in X_i$ and $a_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \in \mathcal{A}$ and $(y_i, a_{-i}) \in \mathcal{U}$. If an attribute is not influential, it does not play any role and can be suppressed. We therefore suppose without loss of generality that all attributes are influential.

[2]Bouyssou and Marchant (2007b) have shown how to extend the axiomatic analysis to the case of more than two categories, in the case of ETRI-B. Their technique applies mutatis mutandis to ETRI-nB.

3 Axiomatic analysis of ETRI-nB-pc: a digest

In this section, we recall some definitions and results presented in BMP23. All ETRI methods start with a preference modelling step during which a preference relation is built for each attribute. This valued preference relation depends on a number of parameters that we do not detail here. In a second step, these n valued preference relations are aggregated into a single valued preference relation that is afterwards cut to define a crisp outranking relation S. The assignment of alternatives to categories occurs in a third step. In order to save space, we do not present the exact definition of ETRI-nB-pc, but an idealization thereof: Model E. It mostly simplifies steps 1 and 2 and we will later see that this does not entail any loss of generality. See Fernández et al. (2017) for a complete description of ETRI-nB-pc and BMP23 for the relationship between Model E and ETRI-nB-pc.

Definition 1 (Models $\boldsymbol{E}, \boldsymbol{E^c}, \boldsymbol{E^u}$)
We say that a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model E if:

- for all $i \in N$, there is a semiorder S_i on X_i (with asymmetric part P_i and symmetric part I_i),
- for all $i \in N$, there is a strict semiorder V_i on X_i that is included in P_i and is the asymmetric part of a semiorder U_i,
- (S_i, U_i) is a homogeneous nested chain of semiorders and W_i is a weak order that is compatible with both S_i and U_i, [3]
- there is a set of subsets of attributes $\mathcal{F} \subseteq 2^N$ such that, for all $I, J \in 2^N$, $[I \in \mathcal{F} \text{ and } I \subseteq J] \Rightarrow J \in \mathcal{F}$,
- there is a binary relation S on X (with symmetric part I and asymmetric part P) defined by

  $$x \; S \; y \Leftrightarrow [S(x, y) \in \mathcal{F} \text{ and } V(y, x) = \varnothing],$$

  where $S(x, y) = \{i \in N : x_i \; S_i \; y_i\}$ and $V(x, y) = \{i \in N : x_i \; V_i \; y_i\}$,
- there is a set $\mathcal{P} = \{p^1, \ldots, p^k\} \subseteq X$ of k limiting profiles, such that for all $p, q \in \mathcal{P}$, $Not[p \; P \; q]$,

such that

$$x \in \mathcal{A} \Leftrightarrow \begin{cases} x \; S \; p & \text{for some } p \in \mathcal{P} \quad \text{and} \\ Not[q \; P \; x] & \text{for all } q \in \mathcal{P}. \end{cases} \tag{1}$$

[3] W_i is the intersection of the weak orders S_i^{wo} and U_i^{wo}, respectively induced by S_i and U_i. See Appendix A of the supplementary material of BMP23.

We then say that $\langle (S_i, V_i)_{i \in N}, \mathcal{F}, \mathcal{P} \rangle$ is a representation of $\langle \mathcal{A}, \mathcal{U} \rangle$ in Model E. Model E^c is the particular case of Model E, in which there is a representation that shows no discordance effects, i.e. in which all relations V_i are empty. Model E^u is the particular case of Model E^c, in which there is a representation that requires unanimity, i.e. such that $\mathcal{F} = \{N\}$. ⌟

In this definition, S_i is the idealization of the preference relation on attribute i, V_i represents all pairs of levels on attribute i for which a discordance could occur (step 1).[4] S is the idealization of the outranking relation (step 2). The third step (the assignment of alternatives to categories) is described by (1).

Goldstein (1991) has proposed a simple condition that may be satisfied by some partitions:

Definition 2 (Linearity)
The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is linear on attribute i if, for all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$\left.\begin{array}{c}(x_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathcal{A}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \in \mathcal{A} \\ \text{or} \\ (x_i, b_{-i}) \in \mathcal{A}.\end{array}\right. \tag{2}$$

The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is linear if it is linear on all attributes. If all partitions that can be represented in some Model M are linear, we say that Model M satisfies Linearity. ⌟

Replacing $\mathcal{A}$ by $\mathcal{U}$ in (2) yields an equivalent definition of Linearity. On each attribute X_i, we define the relation $\succsim_i$ letting, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i y_i \text{ if [for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathcal{A} \Rightarrow (x_i, a_{-i}) \in \mathcal{A}].$$

By construction, $\succsim_i$ is transitive and reflexive; it is complete if and only if the partition is linear on attribute i. The symmetric part of $\succsim_i$ is denoted by $\sim_i$. It is not useful to keep in X_i elements that are equivalent w.r.t. the equivalence relation $\sim_i$. Indeed, if $x_i \sim y_i$ then $(x_i, a_{-i}) \in \mathcal{A}$ iff $(y_i, a_{-i}) \in \mathcal{A}$. In order to simplify notation, we suppose throughout the paper that we are dealing with partitions on $\prod_{i=1}^n X_i$ for which all relations $\sim_i$ are trivial[5]. This non-restrictive convention implies that each relation $\succsim_i$ is antisymmetric.

Let $\succsim$ be the relation on X defined by $x \succsim y$ iff $x_i \succsim_i y_i$ for all $i \in N$. This relation is a partial order (reflexive, transitive and antisymmetric). Let $\mathcal{A}_* =$

[4] In Definition 1, S_i and V_i are supposed to be semiorders. The reason of this assumption is that the notion of semiorder is related to the existence of thresholds, as they appear in the modelling of preference and veto in the classical ELECTRE methods.

[5] If $\sim_i$ is not trivial, we can work without loss of generality with the quotient $X_i/\sim_i$.

$\min(\mathcal{A}, \succsim)$ be the set of minimal elements in $\mathcal{A}$ for $\succsim$. By construction, for any $x \in \mathcal{A}_*$ and $y_i \prec_i x_i$, we have $(y_i, x_{-i}) \in \mathcal{U}$.

We say Model M is nested in—or is a special case of—Model M' (denoted $M \subseteq M'$) if all partitions that can be represented in M can also be represented in M'. Models M and M' are equivalent (denoted $M \equiv M'$) if $M \subseteq M'$ and $M' \subseteq M$. We note $M \subsetneq M'$ if $M \subseteq M'$ and M is not equivalent to M'. By construction, we have $E^u \subseteq E^c \subseteq E$. The main results in BMP23 can now be summarized in the following theorem.

Theorem 1

1. *ETRI-nB-pc* $\equiv E \equiv E^c \equiv E^u$.
2. *A partition* $\langle \mathcal{A}, \mathcal{U} \rangle$ *has a representation in any of these models iff it is linear.*
3. *This representation can always be taken to be* $\langle (\succsim_i, V_i = \varnothing)_{i \in N}, \mathcal{F} = \{N\}, \mathcal{P} = \mathcal{A}_* \rangle$*, that is a representation in Model* E^u*.*

We like to stress point 1: although model E and the nested models E^c and E^u seem to be simplifications of ETRI-nB-pc, they are not: all four models are fully equivalent.

4 ETRI-nB-pd: definition and difficulties

The pseudo-disjunctive version of ETRI-nB consists of three steps. The first and the second one are identical to steps 1 and 2 in ETRI-nB-pc. The only difference is the third step: the assignment of alternatives to categories. With ETRI-nB-pc, an alternative x is assigned to $\mathcal{A}$ iff it is weakly preferred (in terms of S) to a limiting profile and no limiting profile is strictly preferred to x (in terms of P), as in (1). With ETRI-nB-pd, an alternative x is assigned to $\mathcal{U}$ iff (i) there is a limiting profile strictly preferred (in terms of P) to x and (ii) x is not strictly preferred to any limiting profile. As in Section 3, in order to save space, we do not present the exact definition of ETRI-nB-pd, but an idealization thereof: Model F. We define Model F that is to ETRI-nB-pd what Model E is to ETRI-nB-pc.

Definition 3
Model F is defined exactly as Model E, except that we now replace (1) by:

$$x \in \mathcal{U} \Leftrightarrow \begin{cases} p \; P \; x & \text{for some } p \in \mathcal{P} \quad \text{and} \\ Not[x \; P \; q] & \text{for all } q \in \mathcal{P}. \end{cases} \tag{3}$$

The definition of Models F^c and F^u parallels that of E^c and E^u. ⌟

All pseudo-disjunctive models mentioned so far satisfy linearity.

Lemma 1
If $\langle \mathcal{A},\mathcal{U} \rangle$ has a representation in Model F, then it is linear. The same holds for ETRI-nB-pd, F^u and F^c.

Proof
Consider first Model F. Suppose that we have $(x_i, a_{-i}) \in \mathcal{U}$ and $(y_i, b_{-i}) \in \mathcal{U}$. We have either $x_i \; W_i \; y_i$ or $y_i \; W_i \; x_i$ since W_i is a weak order. Suppose wlog that $y_i \; W_i \; x_i$. Because $(y_i, b_{-i}) \in \mathcal{U}$, we know that $p \; P \; (y_i, b_{-i})$, for some $p \in \mathcal{P}$, and $Not[(y_i, b_{-i}) \; P \; q]$ for all $q \in \mathcal{P}$. Using Lemma 3 in BMP23, we obtain $p \; P \; (x_i, b_{-i})$ and $Not[(x_i, b_{-i}) \; P \; q]$ for all $q \in \mathcal{P}$.[6] Hence, $(x_i, b_{-i}) \in \mathcal{U}$ and linearity holds for Model F. By construction, $F^u \subseteq F^c \subseteq F$ and linearity thus also holds for these models.

Since we did not formally define ETRI-nB-pd, we cannot provide the proof that linearity holds for partitions generated by ETRI-nB-pd. For the interested reader, this proof closely follows that of Corollary 1 in BMP23. □

Hence, combining Lemma 1 with Theorem 1, we obtain the next proposition.

Proposition 1
$F^u \subseteq F^c \subseteq F \subseteq E$ and ETRI-nB-pd $\subseteq E \equiv$ ETRI-nB-pc.

At this stage, given the apparent duality between the definitions of the pseudo-conjunctive and pseudo-disjunctive models, we can suspect that $F^u \equiv F^c \equiv F \equiv$ ETRI-nB-pd $\equiv E$, but the next result shows that it does not hold.

Proposition 2
$F^u \subsetneq F^c$ and $F \subsetneq E$.

Proof
Part 1: $F^u \subsetneq F^c$

Let $N = \{1, 2, 3\}$ and $X_i = \{0, 1\}$ for all $i \in N$, so that X has $2^3 = 8$ elements. Consider the partition $\langle \mathcal{A},\mathcal{U} \rangle$ such that $\mathcal{A} = \{111, 101, 011\}$ and $\mathcal{U} = \{110, 100, 010, 001, 000\}$, abusing notation in an obvious way. It is simple to check that all attributes are influential for $\langle \mathcal{A},\mathcal{U} \rangle$ and that, for all $i \in N$, we have $1_i \succ_i 0_i$. Notice that we have $\mathcal{A}_* = \text{Min}(\succsim, \mathcal{A}) = \{101, 011\}$ and $\mathcal{U}^* = \text{Max}(\succsim, \mathcal{U}) = \{110, 010, 001\}$.

Let us show that this partition *cannot* be obtained with Model F^u. Observe first that, here, since all attributes are influential and can only take two values, we must have that $S_i = \succsim_i$, for all $i \in N$.

[6]Lemma 3 in BMP23 is established under the hypothesis that the partition $[\mathcal{A},\mathcal{U}]$ is representable in Model E. Since the proof only uses the properties of relation S, which are common to Models E and F, the result also holds for partitions representable in Model F.

Since $110 \in \mathcal{U}$, there must be $p \in \mathcal{P}$ such that $p\ P\ 110$. Since we are looking for a representation in Model F^u and we know that $S_i = \succsim_i$, for all $i \in N$, we must find a profile $p \in \mathcal{P}$ such that $p \succ 110$. The only candidate is 111. But taking $\mathcal{P} = \{111\}$ together with $\mathcal{F} = \{N\}$ does not lead to the desired partition. Indeed, we have $111 \succ 101$, so that 101 should be in $\mathcal{U}$.

This partition can be obtained with Model F^c, taking $S_i = \succsim_i$, for all $i \in N$, $\mathcal{P} = \{111\}$ and $\mathcal{F} = \{\{1,3\},\{2,3\}\}$.

Part 2: $F \subsetneq E$

Let $n = 4$ and $X_1 = X_2 = X_3 = \{2,1,0\}$ and $X_4 = \{0,1\}$, so that X has 54 elements. Consider the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ such that $\mathcal{A} = \{2221$, 2211, 2121, 1221, 2111, 1211, 1121, 1111, $2220\}$. Notice that $\mathcal{A}_* = \{1111, 2220\}$. It is easy to check that all attributes are influential for $\langle \mathcal{A}, \mathcal{U} \rangle$ and that, for all $i \in \{1,2,3\}$, we have $2_i \succ_i 1_i \succ_i 0_i$, while $1_4 \succ_4 0_4$. Hence, the partition is linear and, by Theorem 1, it can be represented in Model E.

In order to show that this partition cannot be obtained in Model F, we have to examine, all cases of indifference thresholds (associated with the strict semiorders S_i), combined with all cases of veto thresholds (associated with the strict semiorders V_i), and combined with all choices for $\mathcal{F}$.

Notice that if an attribute in $i \in \{1,2,3\}$ has thresholds (i.e. S_i is not a weak order), this means that $2_i\ I_i\ 1_i$ and $1_i\ I_i\ 0_i$. But veto effects can only occur among the elements that are strictly preferred. Hence, in this case, the only possibility is to take $2_i\ V_i\ 0_i$.

If $\{1,2,4\} \in \mathcal{F}$, then, without veto, $2201 \in \mathcal{U}$ outranks all elements in $\mathcal{A}$, a contradiction. This will remain true unless, there is a veto effect on attribute 3.

If $2_3\ V_3\ 0_3$, the only elements in $\mathcal{A}$ that are not strictly beaten by anther element in $\mathcal{A}$ are 2220, 1111, and 1121. It is easy to check that taking all of them or any subset of them as the set of profiles does not lead to the desired partition (consider $2201 \in \mathcal{U}$). If, furthermore, $2_3\ V_3\ 1_3$, the only elements in $\mathcal{A}$ that are not strictly beaten by anther element in $\mathcal{A}$ are 2220 and 1111. It is easy to check that taking all of them or any subset of them as the set of profiles does not lead to the desired partition (consider $2201 \in \mathcal{U}$).

The analysis of the cases $\{1,3,4\} \in \mathcal{F}$ and $\{2,3,4\} \in \mathcal{F}$ is entirely similar.

Suppose now that $\mathcal{F} = \{\{1,2,3\}, N\}$. Suppose that only attribute 1 has thresholds. Without veto, it is easy to check that $1220 \in \mathcal{U}$ outranks all elements in $\mathcal{A}$. This remains true, whatever the choice of veto thresholds on attributes 2 and 3. This also remains true if $1_4\ V_4\ 0_4$. But veto effects on attribute 1 are immaterial since 1_1 is indifferent to both 2_1 and 0_1.

The situation is entirely similar if 2 (resp. 3) is the only attribute to have thresholds.

Suppose that only attributes 1 and 2 have thresholds. Without veto, it is easy to check that $1120 \in \mathcal{U}$ outranks all elements in $\mathcal{A}$. This remains true, whatever the choice of veto thresholds on attributes 1 and 2 since 1_1 (resp. 1_2) is indifferent to 2_1 and 0_1 (resp. 2_2 and 0_2). This also remains true if $1_4\ V_4\ 0_4$. Clearly, the veto threshold on attribute 3 is immaterial.

The analysis of the cases in which 1 and 3 or 2 and 3 have thresholds is entirely similar.

It remains to tackle the case $\mathcal{F} = \{N\}$.

Suppose that only attribute 1 has thresholds. Without veto, there are only 3 elements in $\mathcal{A}$ that are not strictly beaten by another element in $\mathcal{A}$: 2220, 1111 and 2111. It is easy to check that taking all of them or any subset of them as the set of profiles does not lead to the desired partition. It is simple to check that whatever the choice of veto we make on attributes 2, 3 and 4, the situation remains the same.

There is only one possibility to put a veto on attribute 1, i.e. $2_1\ V_1\ 0_1$. In this case there are only 2 elements in $\mathcal{A}$ that are not strictly beaten by another element in $\mathcal{A}$: 2220 and 1111. In any case, it is impossible to recover the desired partition.

The situation is entirely symmetric in the case only attribute 2 or only attribute 3 has thresholds.

Suppose that both attributes 1 and 2 have thresholds. Without veto, there are only 5 elements in $\mathcal{A}$ that are not strictly beaten by another element in $\mathcal{A}$: 2220, 1111, 2211, 2111, and 1211. It is easy to check that taking all of them or any subset of them as the set of profiles does not lead to the desired partition. It is simple to check that whatever the choice of veto we make on attributes 3 and 4, the situation remains the same. There is only one possibility to put a veto on attribute 1 (resp. 2), i.e. $2_1\ V_1\ 0_1$ (resp. $2_2\ V_2\ 0_2$). It is simple to check that any of the three possible choices for the veto on these attributes does not alter the situation.

The situation is entirely symmetric in the case only attributes 1 and 3 or only attributes 2 and 3 have thresholds.

Finally, if all attributes have thresholds, there is only one element in $\mathcal{A}$ that is not strictly beaten by another element in $\mathcal{A}$: 2220. It is easy to check that taking this element to be the unique profile, does not lead to the desired partition. Now, the choice of veto thresholds (they must be of the type $2_i\ V_i\ 0_i$) on attributes 1, 2, and 3 is immaterial. But it is also simple to check that adding a veto on attribute 4 does not change the situation. □

Given Propositions 1 and 2, it would be highly desirable to know whether ETRI-nB-pd $\equiv E$ or ETRI-nB-pd $\equiv F$. Unfortunately, we are presently unable to prove or disprove these equivalences. This shows that the relations between the pseudo-disjunctive models are more complex than between the corresponding pseudo-conjunctive models.

5 Two characterizations

In view of the above-metioned difficulties, we devote this section to two simpler problems: (1) the characterization of Models F^c, F, E and ETRI-nB-pc when all attributes are binary and (2) the characterization of a special case of Model F^u.

5.1 The case of binary attributes

Suppose the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is linear on attribute i. We say attribute i is binary if the weak order $\succsim_i$ has exactly two equivalence classes. Such attributes are common in many applications. The case in which all attributes are binary corresponds to the well-developed theory of monotone Boolean functions (see Crama and Hammer, 2011).

Proposition 3
$F^c \equiv F \equiv E \equiv$ ETRI-nB-pc whenever all attributes are binary.

Proof
When all attributes are binary, each X_i contains only two elements that we can denote by 1_i and 0_i with $1_i \succ_i 0_i$. Each element in X corresponds to a unique coalition $C(x) = \{i \in N : x_i = 1_i\} \subseteq 2^N$. Hence, all linear partitions have a representation in F^c with $S_i = \succsim_i$, for all $i \in N$, $\mathcal{P} = \{111\}$ and $\mathcal{F} = \{C(x) : x \in \mathcal{A}\}$. □

Since the proof uses a set $\mathcal{P}$ containing only one limiting profile, the reader may have the impression that Proposition 3 only applies to ETRI-B and not to ETRI-nB. What the result actually says is that any partition generated by a model F^c, F, E or ETRI-nB-pc (irrespective of the number of limiting profiles) can be represented in the other three models. The proof further shows that the representation can be chosen so that $\mathcal{P}$ is a singleton.

5.2 A special case of Model F^u

In order to reduce the complexity of the models, let us assume that the data are of good quality—in the sense of Roy (1996, Section 8.2)—meaning that there is no imprecision, uncertainty, or inaccurate determination. In that case, there is no need to use preference or indifference thresholds and the relation S_i is a weak order. Since $\succsim_i$ is also a weak order and we cannot have $x_i \succ_i y_i$ while $y_i \; S_i \; x_i$, it must be the case that S_i is a refinement of $\succsim_i$ (i.e. $S_i \subseteq \succsim_i$). But since we have assumed that the relation $\sim_i$ is trivial, the equality $S_i = \succsim_i$ must hold.

So, in this section, we restrict our attention to partitions having a representation in Model F^u such that $S_i = \succsim_i$ is a weak order for all $i \in N$. Model F^u together with this additional constraint will be denoted by $F^{\underline{u}}$. In such a model, $S = \succsim$, $P = \succ$ and Condition 3 reduces to $x \in \mathcal{U}$ iff $p \succ x$, for some $p \in \mathcal{P}$. Indeed, we may not have $p \succ x \succ q$ for $q \in \mathcal{P}$, otherwise $p \succ q$, a contradiction. Notice that $F^{\underline{u}} \subseteq F^u \subsetneq F^c \subseteq F \subsetneq E$.

By construction, the set $\mathcal{A}_* = \text{Min}(\succsim, \mathcal{A})$ is an antichain in the poset $(X, \succsim)$, remembering our convention that each relation $\sim_i$ is trivial. Observe that in the first part of the proof of Proposition 2, the antichain $\mathcal{A}_* = \{101, 011\}$ is not a maximal antichain, i.e. it is strictly included in the antichain $\{110, 101, 011\}$. As shown below, a characteristic feature of partitions that can be represented in Model $F^{\underline{u}}$ is that $\mathcal{A}_*$ is a maximal antichain in the poset $(X, \succsim)$.

Theorem 2
Let $X = \prod_{i=1}^n X_i$ be a finite set and $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold linear partition of X. The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model $F^{\underline{u}}$ iff the antichain $\mathcal{A}_*$, in the poset $(X, \succsim)$, is maximal.

Proof
Necessity. Suppose that $\mathcal{A}_*$ is not a maximal antichain. Hence there is $x \in X$ such that x is incomparable, using $\succsim$, w.r.t. all elements in $\mathcal{A}_*$. In view of the definition of $\mathcal{A}_*$, it is impossible that $x \in \mathcal{A}$ (since this would imply that $x \succsim z$, for some $z \in \mathcal{A}_*$). Hence, we must have $x \in \mathcal{U}$, so that there must be a profile $p \in \mathcal{P}$ such that $p \succ x$. This profile must be in $\mathcal{A}$. But, by hypothesis, this profile cannot belong to $\mathcal{A}_*$. Hence, by construction, we know that $p \succ y$, for some $y \in \mathcal{A}_* \subseteq \mathcal{A}$, which implies $y \in \mathcal{U}$, a contradiction.

Sufficiency. Since $\langle \mathcal{A}, \mathcal{U} \rangle$ is linear, we know that it has a representation in Model E^u using the representation $\langle (\succsim_i, V_i = \varnothing)_{i \in N}, \mathcal{F} = \{N\}, \mathcal{P} = \mathcal{A}_* \rangle$. Since $\mathcal{A}_*$ is a maximal antichain, it is easy to see that this representation is also a representation in Model $F^{\underline{u}}$. Indeed, by construction, it is impossible that $x \in \mathcal{U}$ is incomparable, using $\succsim$, to all $p \in \mathcal{P} = \mathcal{A}_*$. Let $q \in \mathcal{P}$ be such that x and q are comparable using $\succsim$. It is impossible that $x \succsim q$ since this would imply that $x \in \mathcal{A}$, in view of the definition of $\mathcal{A}_* = \mathcal{P}$. Hence, we must have that $q \succ x$. $\square$

The next result shows that $F^{\underline{u}} \subsetneq F^u$, thereby showing that the hypothesis that the representation is such that $S_i = \succsim_i$, for all $i \in N$, is *not* innocuous.

Proposition 4
$F^{\underline{u}} \subsetneq F^u$.

Proof

Let $N = \{1, 2, 3, 4\}$ and $X_i = \{0, 1, 2\}$ for all $i \in N$, so that X has $3^4 = 81$ elements. Consider the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ such that $\mathcal{A} = \{$2222, 2221, 2220, 2212, 2211, 2210, 2202, 2201, 2200, 2122, 2121, 2120, 2112, 2111, 2110, 2102, 2101, 2022, 2021, 2020, 2012, 2011, 2010, 2002, 2001, 1222, 1221, 1220, 1212, 1211, 1210, 1202, 1201, 1200, 1122, 1121, 1120, 1112, 1111, 1110, 1102, 1101, 1022, 0222, 0221, 0220, 0212, 0211, 0210, 0202, 0201, 0122, 0121, 0120, 0112, 0111, 0110, 0102, 0101, 0022$\}$. The set $\mathcal{A}$ has 60 elements. It is easy to check that we have $\mathcal{A}_* = \{$2010, 2001, 1200, 0110, 0101, 0022$\}$.

We have $2012 \in \mathcal{A}$, $1012 \in \mathcal{U}$, $1200 \in \mathcal{A}$, $0200 \in \mathcal{U}$, so that $2_1 \succ_1 1_1 \succ_1 0_1$. Similarly, we have: $2200 \in \mathcal{A}$, $2100 \in \mathcal{U}$, $1101 \in \mathcal{A}$, $1001 \in \mathcal{U}$, so that $2_2 \succ_2 1_2 \succ_2 0_2$. We also have: $0022 \in \mathcal{A}$, $0012 \in \mathcal{U}$, $2110 \in \mathcal{A}$, $2100 \in \mathcal{U}$, so that $2_3 \succ_3 1_3 \succ_3 0_3$. Finally, we have: $0022 \in \mathcal{A}$, $0021 \in \mathcal{U}$, $2101 \in \mathcal{A}$, $2100 \in \mathcal{U}$, so that $2_4 \succ_4 1_4 \succ_4 0_4$ (notice that the role of attributes 3 and 4 is entirely symmetric, in this example).

Hence, using Theorem 2, this partition *cannot* be represented in Model $F^{\underline{u}}$. Indeed, the antichain $\mathcal{A}_*$ is not maximal: the element 2100 is incomparable, using $\succsim$, to all elements in $\mathcal{A}_*$.

Yet it is cumbersome but easy to check that this partition can be obtained in Model F^u, taking $\mathcal{P} = \{2200, 0022\}$, $\mathcal{F} = \{N\}$, $S_i = \succsim_i$, for $i = 2, 3, 4$, and $2_1\ P_1\ 0_1$, $2_1\ I_1\ 1_1$, and $1_1\ I_1\ 0_1$. $\square$

Let us define $E^{\underline{u}}$ in the same way as $F^{\underline{u}}$. By Theorem 1, $E^{\underline{u}}$ is equivalent to E^u. Summarizing Proposition 4 and previous results, we have that $F^{\underline{u}} \subsetneq F^u \subsetneq F^c \subseteq F \subsetneq E \equiv E^c \equiv E^u \equiv E^{\underline{u}}$. This long chain of inclusions and equivalences illustrates the strong asymmetry between the families of pseudo-conjunctive and pseudo-disjunctive models. In order to explore the gap between both families, we devote the rest of the paper to comparing the numbers of partitions that can be represented in models $F^{\underline{u}}$ and $E^{\underline{u}}$ (or any of the pseudo-conjunctive models discussed in this paper). This will help us quantify how restrictive $F^{\underline{u}}$ is compared to $E^{\underline{u}}$.

6 Counting maximal antichains

The number of partitions that can be represented in model $F^{\underline{u}}$ (resp. model $E^{\underline{u}}$) is the number of maximal antichains (resp. antichains) in the poset $(X, \succsim)$. This poset can be seen as a direct product of n chains, where n is the number of attributes and the ith chain ($i \in \{1, \dots, n\}$) is the set $[m_i] = \{1, \dots, m_i\}$ ordered by $\geq$ (the natural order on the integer interval $[m_i]$), with m_i being the number of equivalence classes of the weak order $\succsim_i$. Notice that antichains in the direct product of n chains also plays an important role in the analysis of multichoice cooperative games, as shown by Grabisch (2016a). More generally, the importance of studying discrete mathematics structures in decision theory was powerfully stressed in Grabisch (2016b).

The number of antichains (maximal antichains) in $[m_1] \times \ldots \times [m_n]$ will be denoted by $d_E(m_1, \ldots, m_n)$ (resp. $d_F(m_1, \ldots, m_n)$). When $m_1 = \ldots = m_n = m$, the numbers $d_E(m_1, \ldots, m_n)$ and $d_F(m_1, \ldots, m_n)$ are respectively denoted by $D_E(m, n)$ and $D_F(m, n)$. We first tackle two special cases ($n = 2$ and $m = 2$) and then the general case, for which we have few results.

6.1 The case $n = 2$

Let $\mathbb{N}$ denote the set of positive integers. The next result, due to Covington (2004), presents a recurrence relation for $d_F(m_1, m_2)$.

Theorem 3
For all $m_1, m_2 \in \mathbb{N}$, $d_F(m_1, m_2)$ is equal to

$$d_F(m_1 - 1, m_2 - 1) + \sum_{i=0}^{m_1-2} d_F(i, m_2 - 1) + \sum_{i=0}^{m_2-2} d_F(m_1 - 1, i). \tag{4}$$

A detailed proof of this result can be found in Bouyssou, Marchant, and Pirlot (2024). For $d_E(m_1, m_2)$, the following result easily follows from Berman and Köhler (1976).

Corollary 1
For all $m_1, m_2 \in \mathbb{N}$, we have

$$d_E(m_1, m_2) = \binom{m_1 + m_2}{m_1}.$$

Proof
According to Berman and Köhler (1976), the number of antichains in $[m_1] \times [m_2] \times [m_3]$ is equal to

$$\prod_{i=0}^{m_3-1} \frac{\binom{m_1+m_2+i}{m_1}}{\binom{m_1+i}{m_1}}. \tag{5}$$

Setting $m_3 = 1$ in this expression yields the desired result. □

For illustration purpose, we computed some numerical results under the constraint that $m_1 = m_2$ (to save space). Some terms of the sequences $D_E(m, 2)$ and $D_F(m, 2)$ can be found in Table 1, with the corresponding ratios $D_F(m, 2)/D_E(m, 2)$.

m	$D_F(m,2)$	$D_E(m,2)$	$D_F(m,2)/D_E(m,2)$
1	1	2	0.5
2	3	6	0.5
3	9	20	0.45
4	27	70	0.385714286
5	83	252	0.329365079
6	259	924	0.28030303
7	817	3432	0.238053613
8	2599	12870	0.201942502
9	8323	48620	0.171184698
10	26797	184756	0.145039945
11	86659	705432	0.122845292
12	281287	2704156	0.104020256
13	915907	10400600	0.088062900
14	2990383	40116600	0.074542284
15	9786369	155117520	0.06309003
100	3.76527E+51	9.05485E+58	4.15829E-08

Table 1: Number $D_F(m,2)$ of maximal antichains, number $D_E(m,2)$ of antichains and ratio of these numbers in $[m]^2$ for $m \in [15]$ and $m = 100$. Values of $D_F(m,2)$ are computed by means of (4).

For small values of m, the difference of expressivity between models $F^{\underline{u}}$ and $E^{\underline{u}}$ is not very large, but it grows for large values of m, since the ratio seems to converge to 0.

$D_F(m,2)$ and $D_E(m,2)$ are respectively sequences A171155 and A000984 in the On-line Encyclopedia of Integer Sequences OEIS (2023). A recurrence relation is mentioned by Alois P. Heinz (without proof) for $D_F(m,2)$ in OEIS (2023): $D_F(m,2)$ is equal to

$$\frac{(4m-3)D_F(m-1,2)-(2m-5)D_F(m-2,2)+D_F(m-3,2)-(m-3)D_F(m-4,2)}{m}.$$

Some other results (old and new) about the case $n=2$ are presented in Bouyssou et al. (2024). Therein, in addition to enumeration results, correspondences (bijections) between (maximal) antichains in products of chains and other mathematical structures are established.

6.2 The case $m=2$

$D_F(2,n)$ is sequence A326358 in OEIS (2023). No expression seems to be known for this sequence and the highest known value corresponds to $n=7$. Some terms can be found in Table 2.

$D_E(2,n)$ corresponds to the Dedekind numbers (sequence A000372 in OEIS (2023)), for which no expression is known. The highest known value corresponds to $n=9$. Some terms can be found in Table 2 with the corresponding ratios $D_F(2,n)/D_E(2,n)$. Here again, for small values of n, the difference of expressivity between models $F^{\underline{u}}$ and $E^{\underline{u}}$ is not very large, but for large values of n, the ratio seems to converge to 0.

n	$D_F(2,n)$	$D_E(2,n)$	$D_F(2,n)/D_E(2,n)$
1	2	3	0.6666667
2	3	6	0.5
3	7	20	0.35
4	29	168	0.172619
5	376	7581	0.04959768
6	31746	7828354	0.004055259
7	123805914	2414682040998	0.00005127214

Table 2: Number $D_F(2,n)$ of maximal antichains, number $D_E(2,n)$ of antichains and ratio of these numbers in $[2]^n$ for $n \in [7]$.

6.3 The general case

In the general case, analytic expressions for $D_F(m,n)$ and $D_E(m,n)$ are difficult to obtain and we therefore only provide a lower bound for $D_F(m,n)$ and some numerical results.

6.3.1 A lower bound for $D_F(m,n)$

Proposition 5
The number of maximal antichains in $[m]^n$ is at least the number of antichains of $[m]^{n-1}$, that is $D_F(m,n) \geq D_E(m,n-1)$.

Proof
The set $\{x \in [m]^n : x_i = m\}$ is the set of elements $x \in X$ having their ith coordinate equal to m. We shall prove that any antichain, not necessarily maximal, in $\{x \in [m]^n : x_i = m\}$ can be extended into a maximal antichain of X, which has no other element with its ith coordinate x_i equal to m. This will establish Proposition 5 since any antichain of X_{-i} is in one-to-one correspondence with an antichain of $\{x \in [m]^n : x_i = m\}$.

We take wlog $i = 1$. If the antichain in $\{x \in [m]^n : x_1 = m\}$ is maximal, the result is obvious. Otherwise, let A be any non-maximal antichain in $\{x \in [m]^n : x_1 = m\}$. Since A is not maximal in $\{x \in [m]^n : x_1 = m\}$, there is at least one element $x = (m, x_2, \ldots, x_n)$ that is incomparable to all elements in A. Let $x' = (m-1, x_2, \ldots, x_n)$. We have that $x \succ x'$ and x' is incomparable to any element in A. Indeed, for no $y \in A$, we have $x' \succsim y$ (otherwise $x \succsim y$ would hold too) and, for no $y \in A$, we have $y \succsim x'$ (otherwise $y \succsim x$ would also hold). Consider the set of all elements in $\{x \in [m]^n : x_1 = m\}$ that are incomparable to all elements in A. Select the minimal elements from this set. Change the first coordinate of each minimal element x into $x_1 = m - 1$, yielding an element x'. Let A' be the set obtained by adding all such elements x' to the antichain A. These elements are incomparable to all elements in A and incomparable to one another. Therefore, A' is an antichain. It is easy to see that it is maximal in X. Furthermore, the intersection of A' with the set $\{x \in [m]^n : x_1 = m\}$ is exactly A. □

Since $D_F(m,n)$ is nondecreasing with m and n, we may conclude in particular that the number of maximal antichains in X is at least the number of antichains in $[2]^{n-1}$, which is Dedekind number $D_E(2, n-1)$. Table 2 suggests that this bound is very weak. It also suggests that $D_F(m,n)$ grows extremely fast with n even for $m = 2$.

6.3.2 Some numerical results

Table 3 presents some values of $D_F(m,n)$ for small values of m and n, computed with the help of the software system Macaulay2 (Grayson and Stillman, 2021). For $[3]^3$, we used the function `maximalAntichains` provided by the package `Posets` in the software system Macaulay2 (Grayson and Stillman, 2021) and manually checked the result. For $[4]^3$ and $[3]^4$, we also used the function `maximalAntichains`, but without manual check. For larger values (except when $m = 2$ or $n = 2$), the calculations are prohibitively long (indicated by question marks in Table 3).

$D_F(m,n)$	$n = 1$	2	3	4
$m = 1$	1	1	1	1
2	2	3	7	29
3	3	9	**144**	**116547**
4	4	27	**10631**	?
5	5	83	?	?

Table 3: Number of maximal antichains ($D_F(m,n)$) for small values of m and n. Boldface entries are new.

For $[3]^3$, using (5), we find $D_E(3,3) = 980$ so that the ratio $D_F(3,3)/D_E(3,3)$ is equal to 0.14693878. Similarly, for $[4]^3$, we obtain $D_E(4,3) = 232848$ so that the ratio $D_F(4,3)/D_E(4,3)$ is equal to 0.04565639, which implies a huge difference of expressivity between $F^{\underline{u}}$ and $E^{\underline{u}}$.

7 Conclusion

Although our results about ETRI-nB-pd and its special cases are very partial, we have axiomatic and combinatorial results showing that

1. the analysis of the pseudo-disjunctive models is far more complex than that of the pseudo-conjunctive models;

2. there is a whole variety of pseudo-disjunctive models that are not all equivalent, contrary to what we observed for pseudo-conjunctive models;

3. most pseudo-disjunctive models are strict special cases of the corresponding pseudo-conjunctive models;

4. the pseudo-disjunctive model $F^{\underline{u}}$ is much more restrictive than the corresponding pseudo-conjunctive model.

The strong asymmetry between the pseudo-conjunctive and pseudo-disjunctive models can be ascribed to the central role played by the relation P in the definition of ETRI-nB-pd while S is central in ETRI-nB-pc. Indeed, Bouyssou and Pirlot (2015a,b) have shown that the nature of the relation P is rather different from that of the relation S in the ELECTRE methods.

Hence, paralleling Bouyssou and Marchant (2015), we suggest to define the *dual* of ETRI-nB-pc not by means of (3), but rather by

$$x \in \mathcal{U} \Leftrightarrow \begin{cases} p \; S \; x & \text{for some } p \in \mathcal{P} \quad \text{and} \\ Not[x \; P \; q] & \text{for all } q \in \mathcal{P}. \end{cases} \tag{6}$$

It is easy to see that ETRI-nB-pc and its dual now correspond via the transposition operation consisting in inverting the direction of preference on all criteria and permuting $\mathcal{A}$ and $\mathcal{U}$ (see Almeida-Dias, Figueira, and Roy, 2010, Bouyssou and Marchant, 2015, Roy, 2002).

Mimicking Bouyssou et al. (2023, Th. 15), it is clear this dual model is characterized by Linearity. Instead of taking $\mathcal{A}_*$ as the set of profiles to delimit $\mathcal{A}$, we now take $\mathcal{U}^* = \text{Max}(\succsim, \mathcal{U})$ to delimit the category $\mathcal{U}$, still using $S_i = \succsim_i$ and $\mathcal{F} = \{N\}$.

If we replace (3) by (6) in the definition of Models F, F^c and F^u, it is also simple to see that they are all equivalent to the dual of ETRI-nB-pc.

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