HW2

Tuesday, October 8, 2024 7:03 PM

Q.1) Solution:

we have,

$$\Sigma = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

a) Solution:

To find the population principal components Y_1 and Y_2

$\det(A - \lambda I) = 0$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 2(2) = 0$$

or, $10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

or, $\lambda^2 - 7\lambda + 6 = 0$

$\therefore \lambda^2 - 7\lambda + 6 = 0$

$\therefore \lambda_1 = 6, \lambda_2 = 1$

Next, we have to compute the eigen vectors corresponding to $\lambda_1 = 6$ and $\lambda_2 = 1$

when, $\lambda_1 = 6$

$AV = \lambda V$

$AV - \lambda V = 0$

$$\begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-v_1 + 2v_2 = 0$$
$$-v_1 = -2v_2$$
$$v_1 = 2v_2, \text{ let } v_2 = t, \; t = 0$$

So,

$$v_1 = 2t$$
$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$$

Without loss of generality, we assume $t = 1$.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

when $\lambda = 1$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$4v_1 + 2v_2 = 0$$
$$\text{or, } 4v_1 = -2v_2$$
$$\text{or, } 2v_1 = -v_2$$
$$\text{let } v_2 = 2t, \; t \neq 0$$
$$2v_1 = -2t$$
$$v_1 = -t$$
$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -t \\ 2t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} t$$

Again, without loss of generality we assume $t = 1$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So,

$$u = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vartheta = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\hat{u} = \frac{u}{\|u\|}, \quad \omega = \frac{\omega}{\|\omega\|}$$

$$= \frac{1}{\sqrt{(2)^2+1^2}}\begin{bmatrix}2\\1\end{bmatrix}, \quad = \frac{1}{\sqrt{(-1)^2+(2)^2}}\begin{bmatrix}-1\\2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{bmatrix} \qquad = \begin{bmatrix}\frac{-1}{\sqrt{3}}\\ \frac{2}{\sqrt{3}}\end{bmatrix}$$

So, the principal components are:

$$\boxed{Y_1 = \frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2} \qquad \boxed{Y_2 = -\frac{1}{\sqrt{3}}x_1 + \frac{2}{\sqrt{3}}x_2}$$

b) Proportion of population variance is explained by Y_1 is:

$$PCA = \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{6}{6+1} = \frac{6}{7} = \boxed{85.7\%}$$