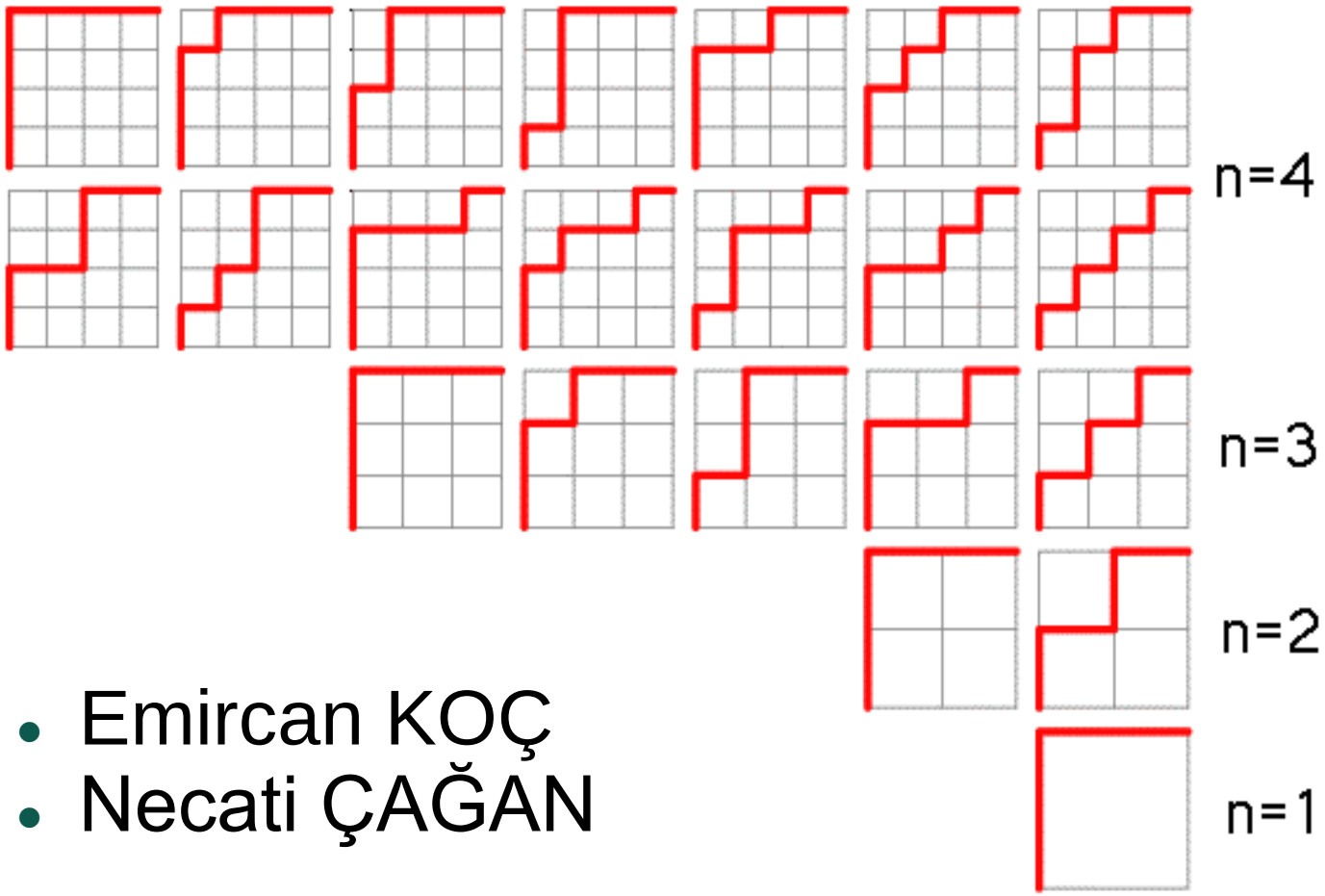
BBM 205 Discrete Mathematics
Hacettepe University

Lecture 13b: Catalan Numbers and the Art Gallery Problem

Lale Özkahya

Resources:
Presentations from BIL694 Class

Catalan Numbers



- Emircan KOÇ
- Necati ÇAĞAN

Catalan Numbers

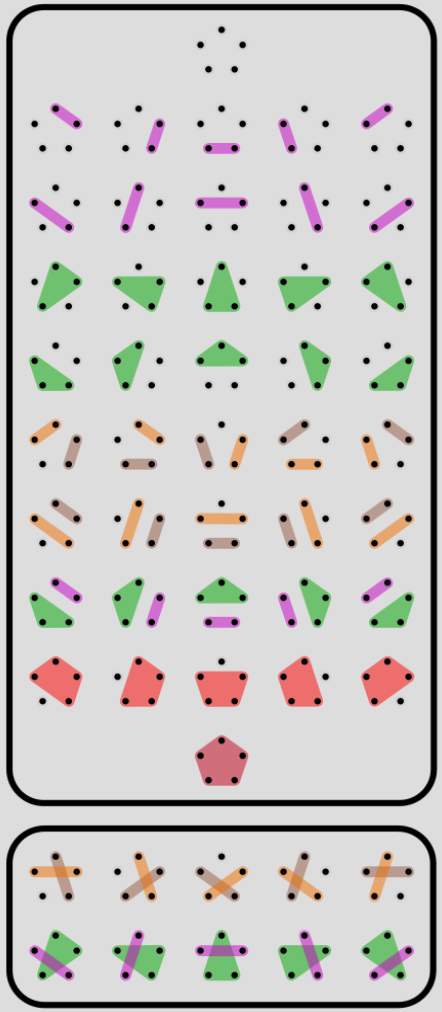
- The first Catalan numbers for n = 0, 1, 2, 3, … are
- 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452....
- Code: A000108 in oeis.org (The On-Line Encyclopedia of Integer Sequences)

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} \quad \text{for } n \geq 0,$$

Applications in combinatorics

$n = 0$:	*	1 way
$n = 1$:	/\	1 way
$n = 2$:	/\ /\/\, / \	2 ways
$n = 3$:	/\ /\ /\ /\/\ / \ /\/\/\, /\/ \, / \/\, / \, / \	5 ways

Mountain Ranges



Applications in combinatorics

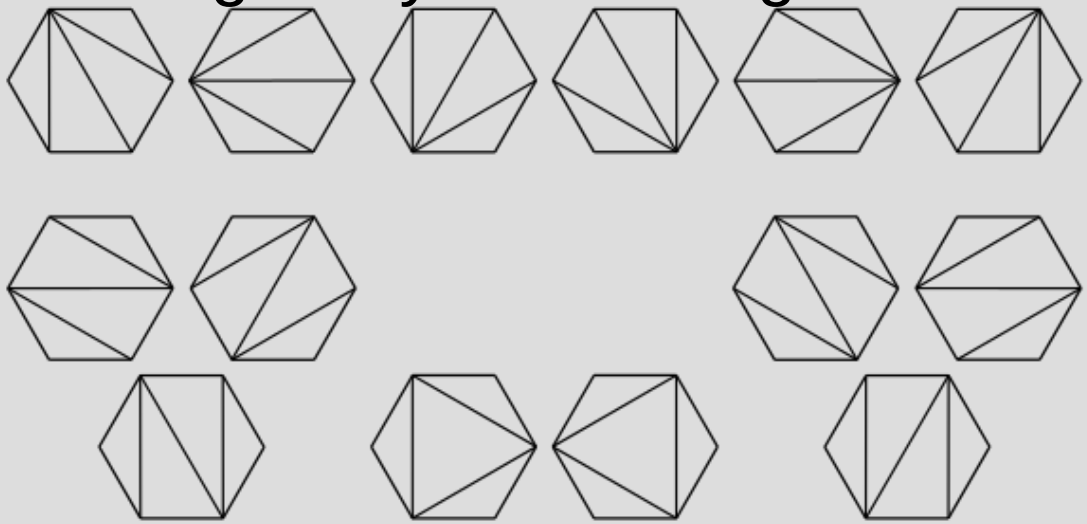
- Cn is the number of Dyck words of length 2n. A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.

XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY

Applications in combinatorics

- Sets consisting of 2n points on a circle joined with n non-intersecting chords, with each point on one chord.

- Or with the same principle;
- Number of ways a convex polygon of n+2 sides can split into triangles by connecting vertices.

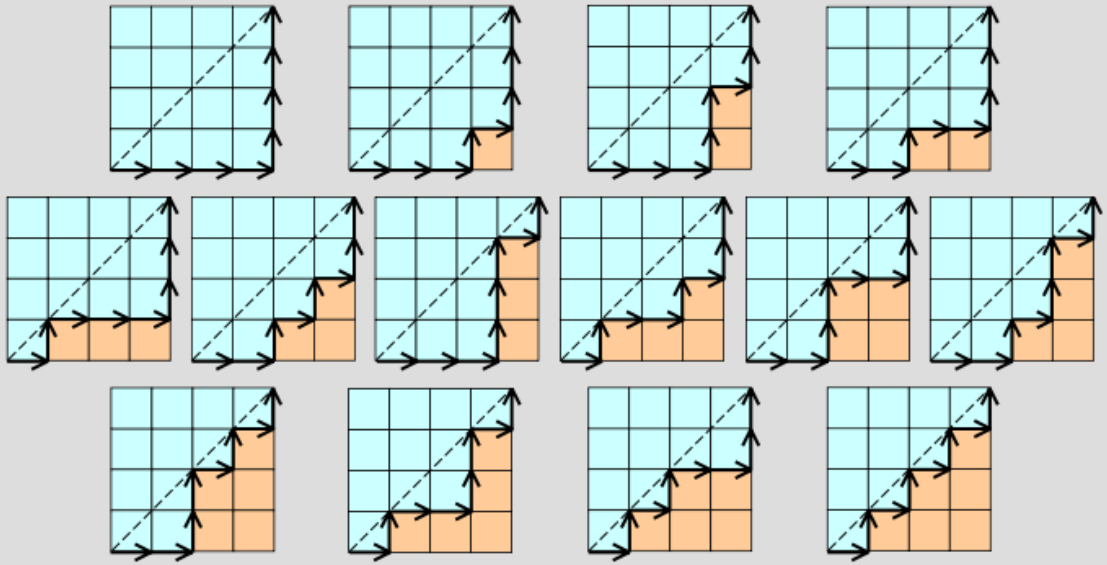


Applications in combinatorics

- Number of permutations of {1, …, n} that avoid the pattern 123 or 134.. (or any of the other patterns of length 3); that is, the number of permutations with no three-term increasing subsequence. For n = 3, these permutations are 132, 213, 231, 312 and 321. For n = 4, they are 1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312 and 4321

Applications in combinatorics

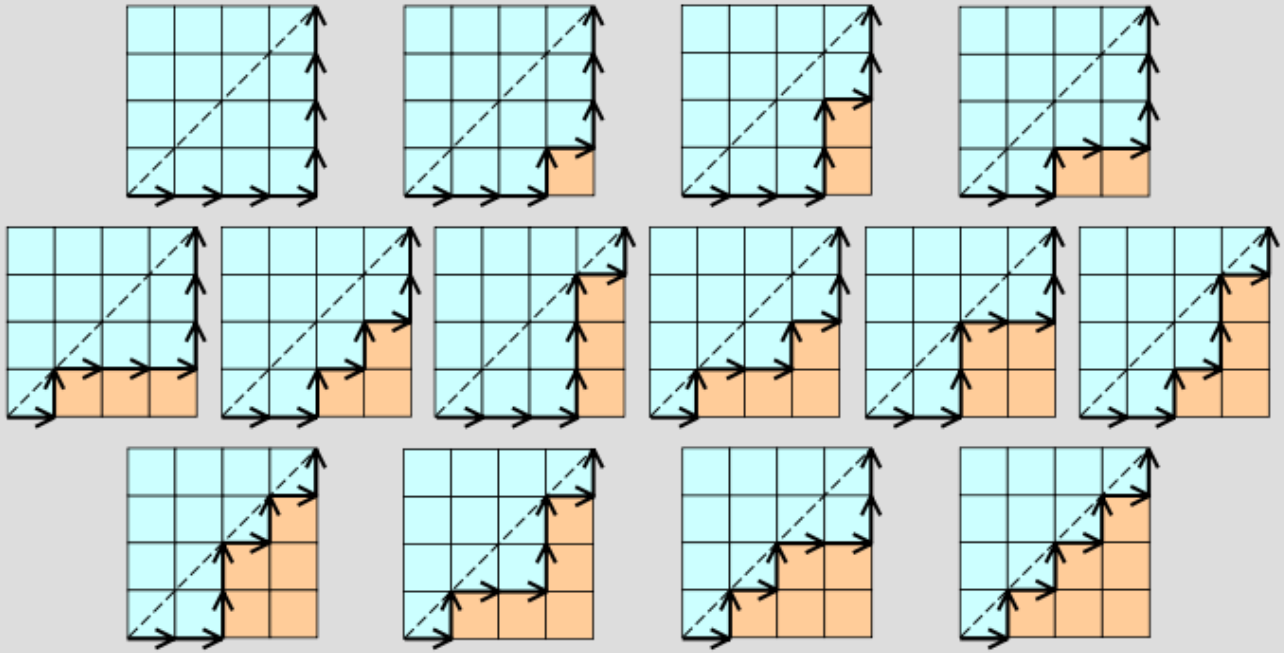
- Cn is the number of monotonic lattice paths along the edges of a grid with n × n square cells, which do not pass above the diagonal (Zero exceedance). A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. Analogy Dyck words: X stands for "move right" and Y stands for "move up".



Proof of Catalan Numbers

- **Proof 1 (by Rukavicka Josef (2011))**

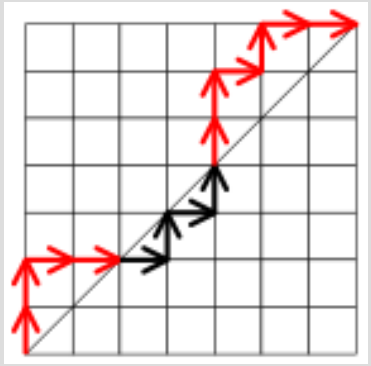
- Using monotonic lattice paths along the edges of a grid with n × n square cells



Proof of Catalan Numbers

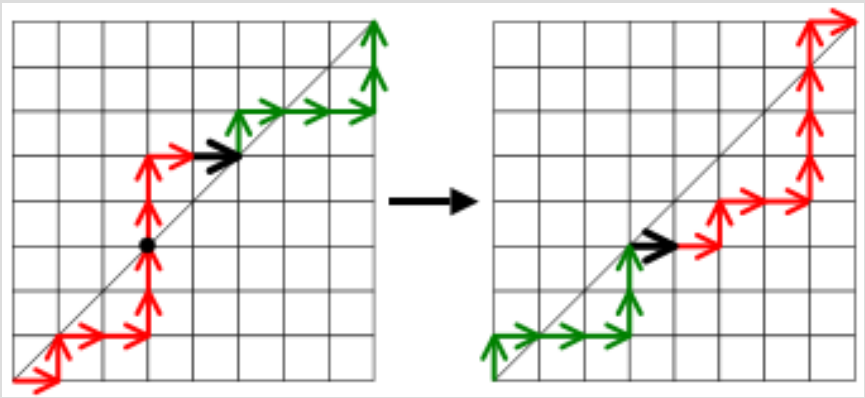
- **Proof 1;**

- **Exceedance:** Defined to be the number of vertical edges which lie above the diagonal.
- Exceedance is 5 in example below.



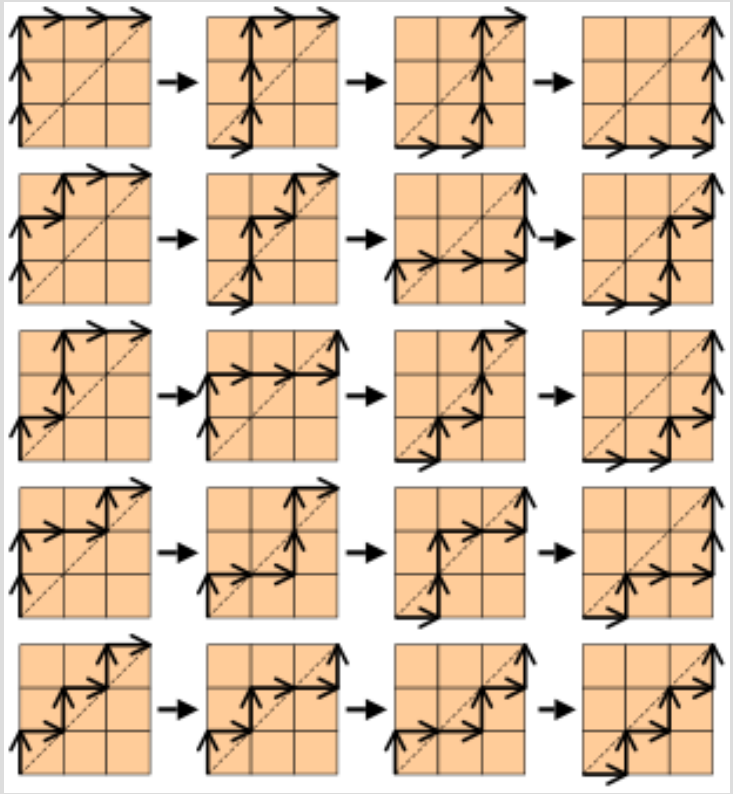
Proof of Catalan Numbers

- Proof 1;
- Algorithm to decrease **Exceedance** by one.
  - Start from bottom left. Follow the path until it first travels above the diogonal. (It is the black dot)
  - Follow path until it touches the diogonal again. (It is the black arrow)
  - Swap the portion before the black arrow and after the black arrow. (Swap Red portion and Green portion)



Proof of Catalan Numbers

- Proof 1;
- All monotonic paths in a 3×3 grid, illustrating the exceedance-decreasing algorithm.



Proof of Catalan Numbers

- Proof 1;
- This implies that the number of paths of exceedance n is equal to the number of paths of exceedance n − 1 and is equal to the number of paths of exceedance n − 2 and so on down to zero.

Proof of Catalan Numbers

- Proof 1;
- We know that the count of all monotonic paths equals to $\binom{2n}{n}$

- And we know number of exceedance zero path are equal to number of exceedance one path an so on. So we can just divide to n+1 to find only the exceedance zero paths. $\frac{1}{n+1}$

Proof of Catalan Numbers

- Proof 1;

And we can see that the formula for Catalan numbers is equal to.

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Catalan Numbers – Generating Functon

- We have;

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i \, C_{n-i} \quad \text{for } n \geq 0.$$

Known as ***Segner's recurrence relation***

Proof

Define a function f(z)

$$f(z) \triangleq C_0 + C_1 z + C_2 z^2 + C_3 z^3 \ldots = \sum_{i=0}^{\infty} C_i z^i$$

Multiply by itself

$$[f(z)]^2 = C_0 C_0 + (C_1 C_0 + C_0 C_1) z + (C_2 C_0 + C_1 C_1 + C_0 C_2) z^2 + \cdots$$

$$[f(z)]^2 = C_1 + C_2 z + C_3 z^2 + \cdots$$

$$f(z) = C_0 + z[f(z)]^2$$

Proof

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

$$\sqrt{1 - 4z} = (1 - 4z)^{1/2}$$

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{2 \cdot 1}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1}a^{n-3}b^3 + \cdots.$$

$$(1 - 4z)^{1/2} = 1 - \frac{1}{1!}2z - \frac{1}{2!}4z^2 - \frac{3 \cdot 1}{3!}8z^3 - \frac{5 \cdot 3 \cdot 1}{4!}16z^4 - \frac{7 \cdot 5 \cdot 3 \cdot 1}{5!}32z^5 - \cdots$$

$$f(z) = 1 + \frac{1}{2!}2z + \frac{3.1}{3!}4z^2 + \frac{5.3.1}{5!}8z^3 + \frac{7.5.3.1}{7!}16z^4 + \cdots$$

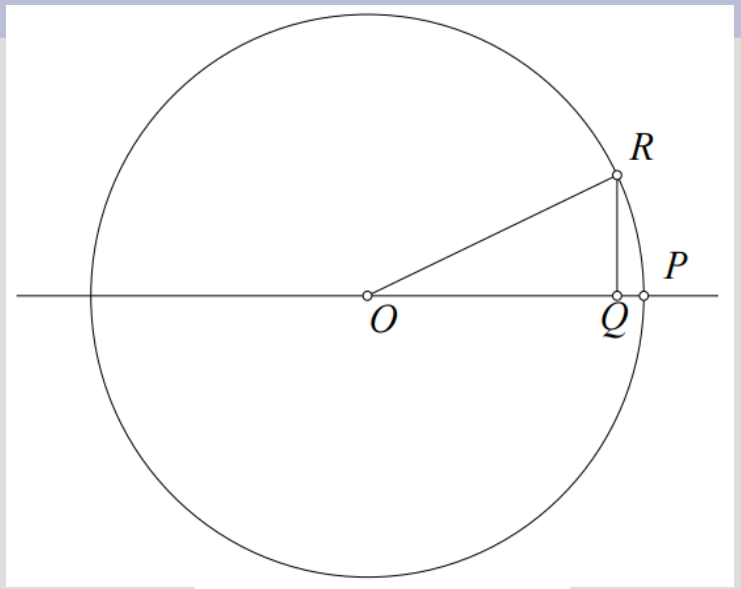
Proof

$$f(z) = 1 + \frac{1}{2}\left(\frac{2!}{1!1!}\right)z + \frac{1}{3}\left(\frac{4!}{2!2!}\right)z^2 + \frac{1}{4}\left(\frac{6!}{3!3!}\right)z^3 + \frac{1}{5}\left(\frac{8!}{4!4!}\right)z^4 + \dots$$

$$f(z) = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} z^i$$

$$C_i = \frac{1}{i+1}\binom{2i}{i}$$

A Strange Geometric Result



$$QP = r - \sqrt{r^2 - 1}$$

Let r = 5 and we obtain:
QP = 0.10102051443364380 3 . . .
Notice that the catalan numbers appear in the decimal expansion; we can see 1, 1, 2, 5, 14 and
“almost” 42.

A Strange Geometric Result

if we let r = 5, 000, 000. We obtain:

QP =

.000000**1**00000000000000**1**00000000000000**2**00000000000000**5**00000000000000**14**00000
0000000**42**00000000000**132**00000000000**429**00000000000**1430**00000000000**4862**000
000000**16796**000000000**58786**00000000**208012**00000000**742900**0000000**26744400**
000000**9694845**000000**35357670**00000**129644790**00000**477638700**0000**176726319**
00000**6564120420**000**24466267020**000**91482563640**00**343059613650**01**289904147**
3240**4861946401452 18367353072152....**

Applications

(i) How many legal bracketings (paranthesizations) are there that use *n* pairs of brackets?

For example, for n=3 pairs, we have 5 different options.

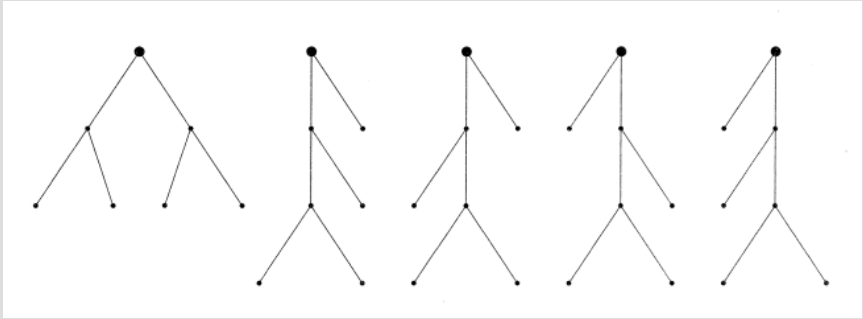
()()(), (())(), ()(()), (()()), ((()))

Applications

(ii) How many integer sequences $(a_i)_1^{n-1}$ are there such that $1 \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1}$ and $a_i \leq$ i for every *i?*

Applications

(iii) How many binary plane trees are there with $n + 1$ end-vertices? A *plane tree* is a rooted tree such that the subtrees at each vertex are linearly ordered; such a tree is *binary* if the root has degree 2 and every other vertex has degree either 1 or 3.



Five Binary Plane-Trees with Four End-Vertices

Solutions to Problems

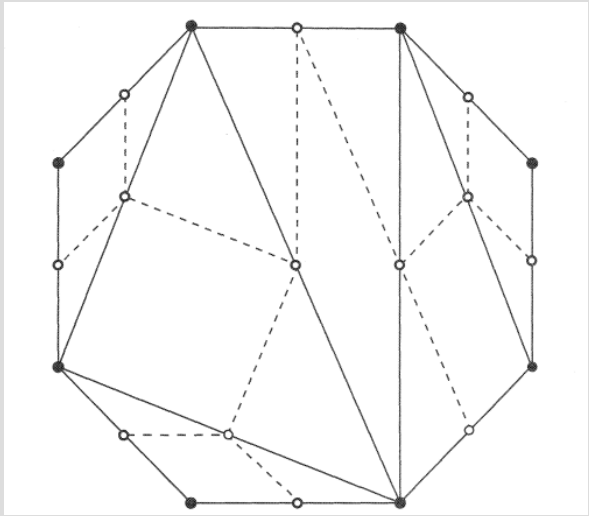
Proofs for (i) and (ii) are rather trivial since in each case we can identify the set in question with the set of increasing LATTICE PATHS that do not rise above the *y=x line.*

- An opening bracket is '(' is identified as a step to the right while a closing bracket ')' is identified as a step upwards.

- The increasing lattice path from (0,0) to (n,n) can be mapped into the sequence $(a_i)_1^{n-1}$ where a_i is the number of steps upwards before the (i+1)st step to the right.

Proofs of Problems

The set of binary trees with triangulations on (n+2)-gon is identified and TRIANGULATION OF POLYGONS concept is used. An edge of the polygon is kept fixed, which means it correponds to the root of the given tree.



Solid lines with black circles show the triangulation
Dashed lines with open circles show the binary plane tree

One Final Catalan Number Result

- Hankel's Matrix
  - Nxn matrix
  - Entry at location (i,j) is the Catalan number C_{i+j-1}
  - Determinant is always 1

$$\det \begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{bmatrix} = 1.$$

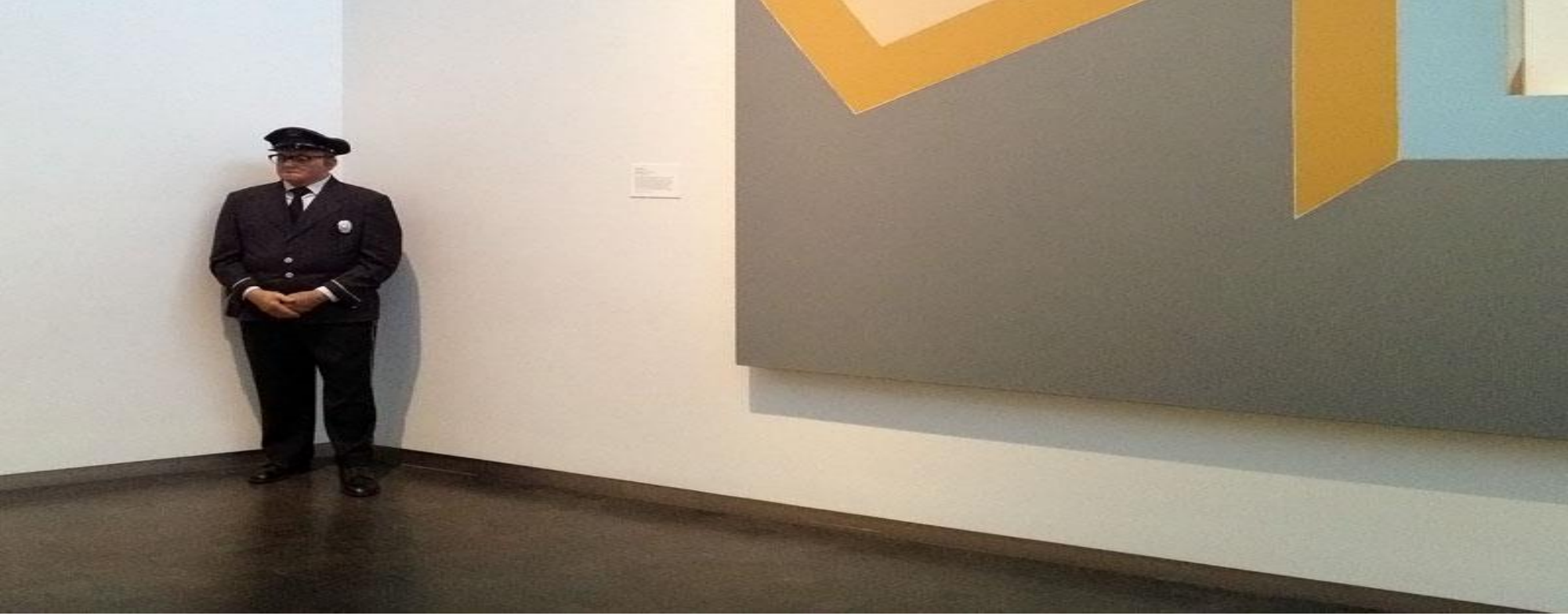
References

- **Catalan number. (2017, March 21). Retrieved April 01, 2017, from https://en.wikipedia.org/wiki/Catalan_number**

- **Bollobás, B. (2010). The art of mathematics: coffee time in Memphis. Cambridge: Cambridge University Press.**

- **Applications of Catalan Numbers. (2016, November 19). Retrieved April 02, 2017, from http://www.geeksforgeeks.org/applications-of-catalan-numbers/**

- **Davis, T. (2016), Catalan Numbers, http://www.geometer.org/mathcircles**



How to Guard a Museum?

Handan GÜRSOY, Ozan TEKDUR

Assoc. Prof. Dr. Lale ÖZKAHYA

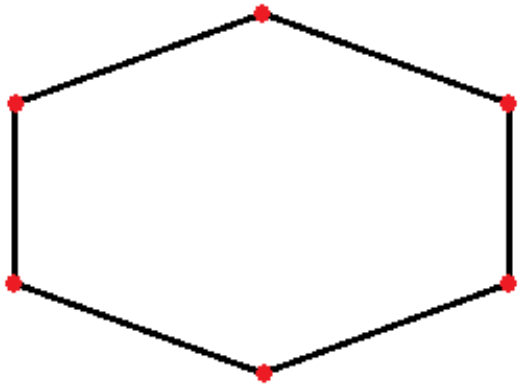
How to guard a museum?

Suppose that a museum is shaped like some sort of n-polygon, and we want to place cameras with 360-viewing angles along the vertices of museum in such a way that the entire gallery is under surveillance.

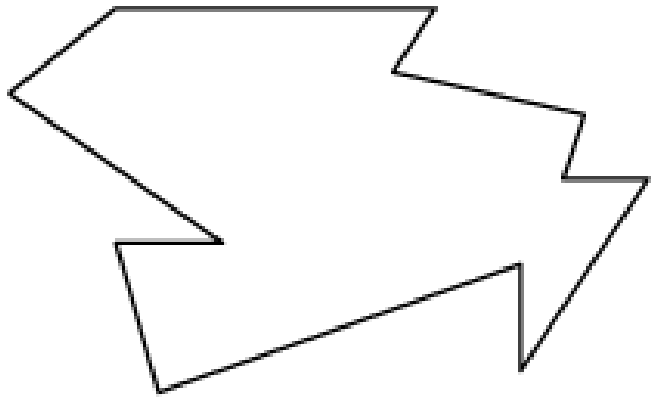
How many cameras are needed? (Victor Klee,1973)

Trivial upper bound you can come up with is n guards: just put one guard on each vertex of our polygon with n sides!

Can we do better?

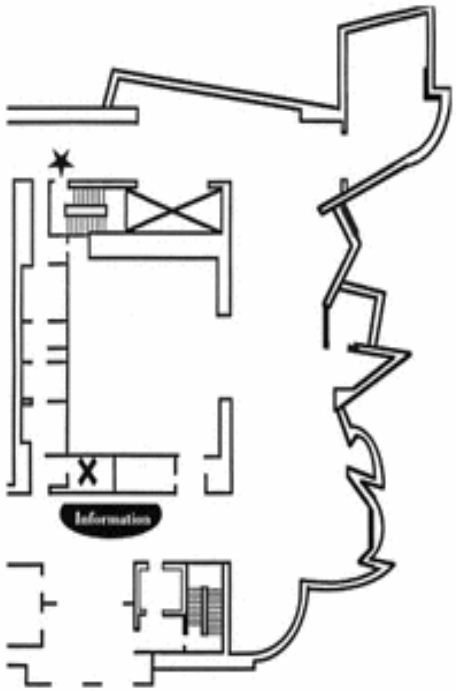


convex polygon



non-convex polygon

How to guard a museum?



A real life art gallery…

How to guard a museum?

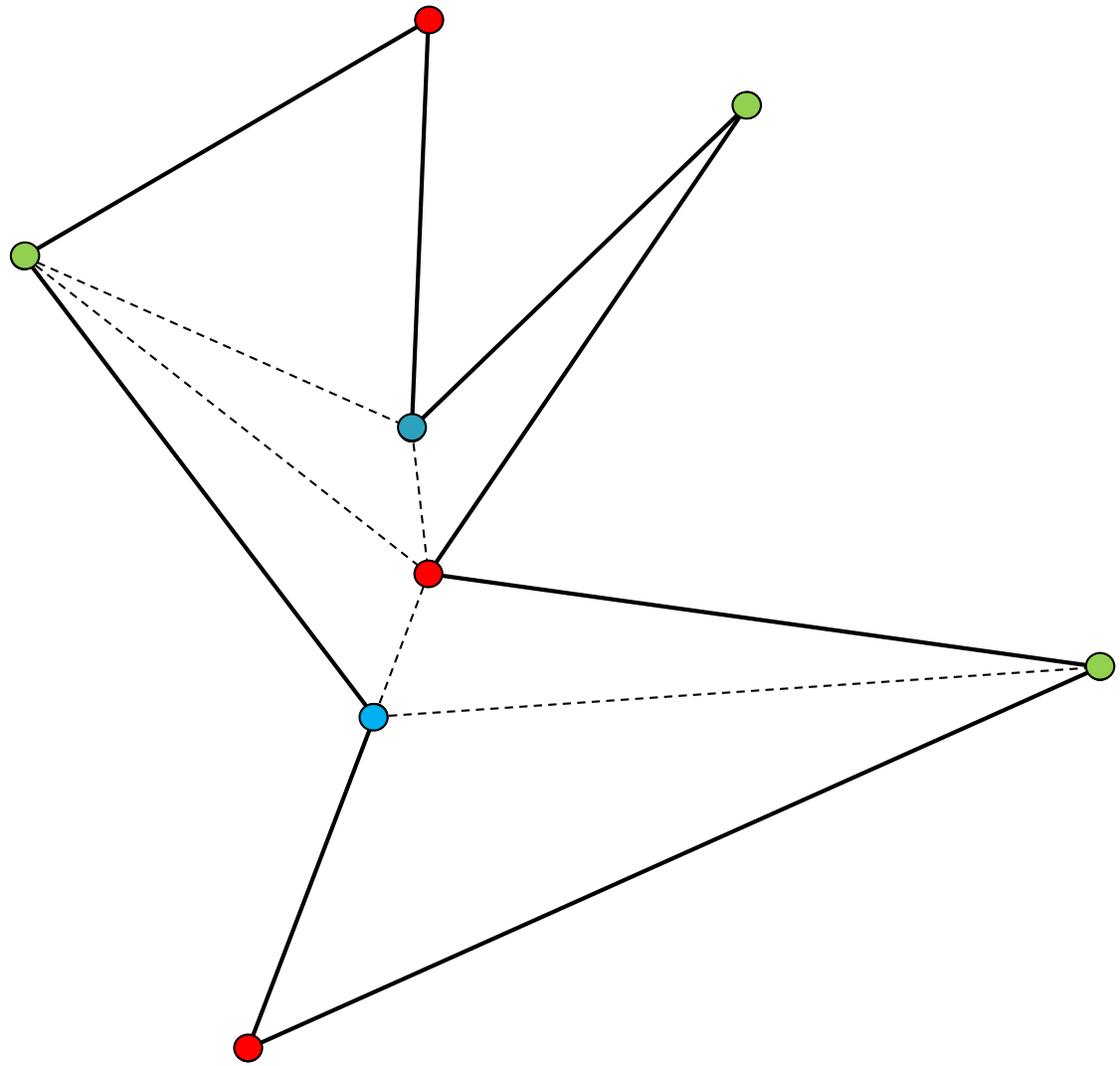
Theorem: For any museum with n walls, n/3 guards are sufficient (Chvatal, 1975).

Proof (Fisk 1978):

Claim: Any n-polygon can be triangulated.

Claim: Any triangulated graph is 3 colorable.

Once a 3-coloring is found, the vertices with any one color form a valid guard set, since each guard can see everything in their assigned triangle.



How to guard a museum?

Claim: Any n-polygon can be triangulated (and always with $n - 2$ triangles).

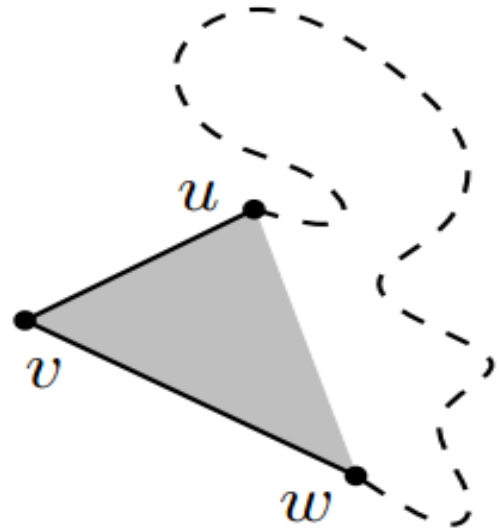
Proof by induction:

- for $n = 3$, polygon is a triangle, nothing to proove.
- for $n \geq 4$,

Consider the leftmost vertex v and its two neighbors u and w.

Case 1: uw is a diagonal ,
or part of the boundary of P is in Δuvw (case 2)

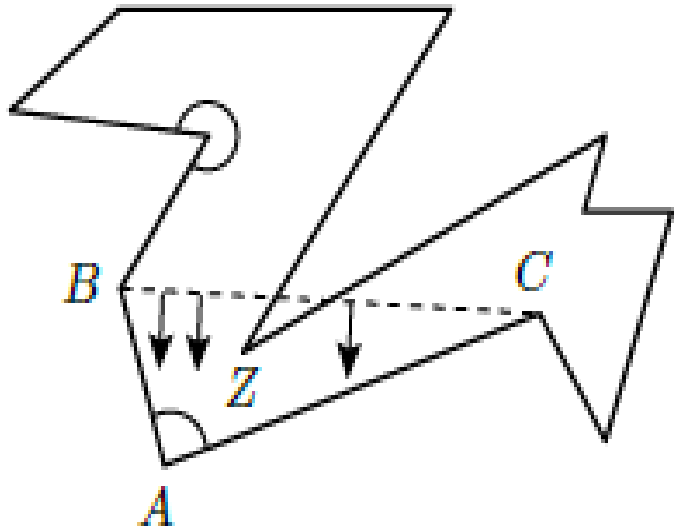
case 1:



How to guard a museum?

Case 2: The triangle ABC contains other vertices. Slide BC towards A until it hits the last vertex Z in ABC. Now AZ is within P, and we have a diagonal.

case 2:



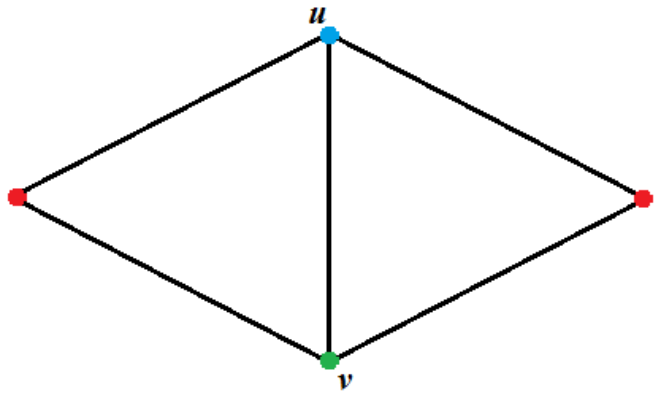
Claim: Any triangulated graph is 3 colorable..

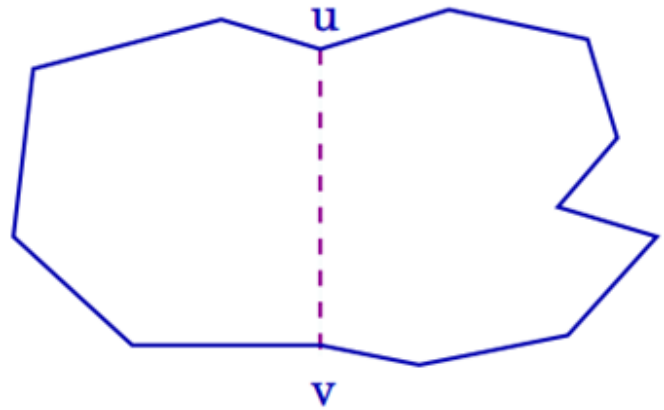
Proof by induction:

- for n = 3, there is nothing to prove.

- for n > 3, pick any two vertices u and v which are connected by a diagonal. This diagonal will split the graph into two smaller triangulated graphs both containing the edge uv.

By induction we may color each part with 3 colors where we may choose color 1 for u and color 2 for v in each coloring. Pasting the colorings together yields a 3-coloring of the whole graph.





Conclusion

Theorem: For any museum with n walls, n/3 guards are sufficient.

Proof (Fisk 1978):
Any n-polygon graph can be triangulated and any triangulated graph is 3 colorable.

Once a 3-coloring is found, clearly, every triangle contains all three colors. The vertices with any one color form a valid guard set, since each guard can see everything in their assigned triangle, as there are no walls of our museum (i.e. lines of our polygon) that intersect these triangles, by construction. Therefore, the entire museum is guarded.

Since the three colors partition the *n* vertices of the polygon, the color with the fewest vertices forms a valid guard set with at most n/3 guards.

https://youtu.be/OERYaFGBPbM

References

1. https://en.wikipedia.org/wiki/Art_gallery_problem
2. Proofs from the book, M. Aigner, G. M. Ziegler
3. http://www.cs.uu.nl/docs/vakken/ga/slides3.pdf
4. https://youtu.be/OERYaFGBPbM

Now you know quite a bit about discrete math!

Leonhard Euler