Linear Algebra

Special class

Nishchay Manwani • Nov 20, 2020

INDIAN PROGRAMMING CAMP 2020

Introduction to Linear Algebra

Terminology:

Space $\rightarrow$ $\underline{R^3} \rightarrow \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

Vector $\rightarrow$ $\underline{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = x \in \underline{R^n}$

Linear combination of vectors

↳ $v_1, v_2, \ldots v_m \in R^n$

↳ $\underline{a_1 v_1 + a_2 v_2 + \ldots a_m v_m} = \underline{v}$

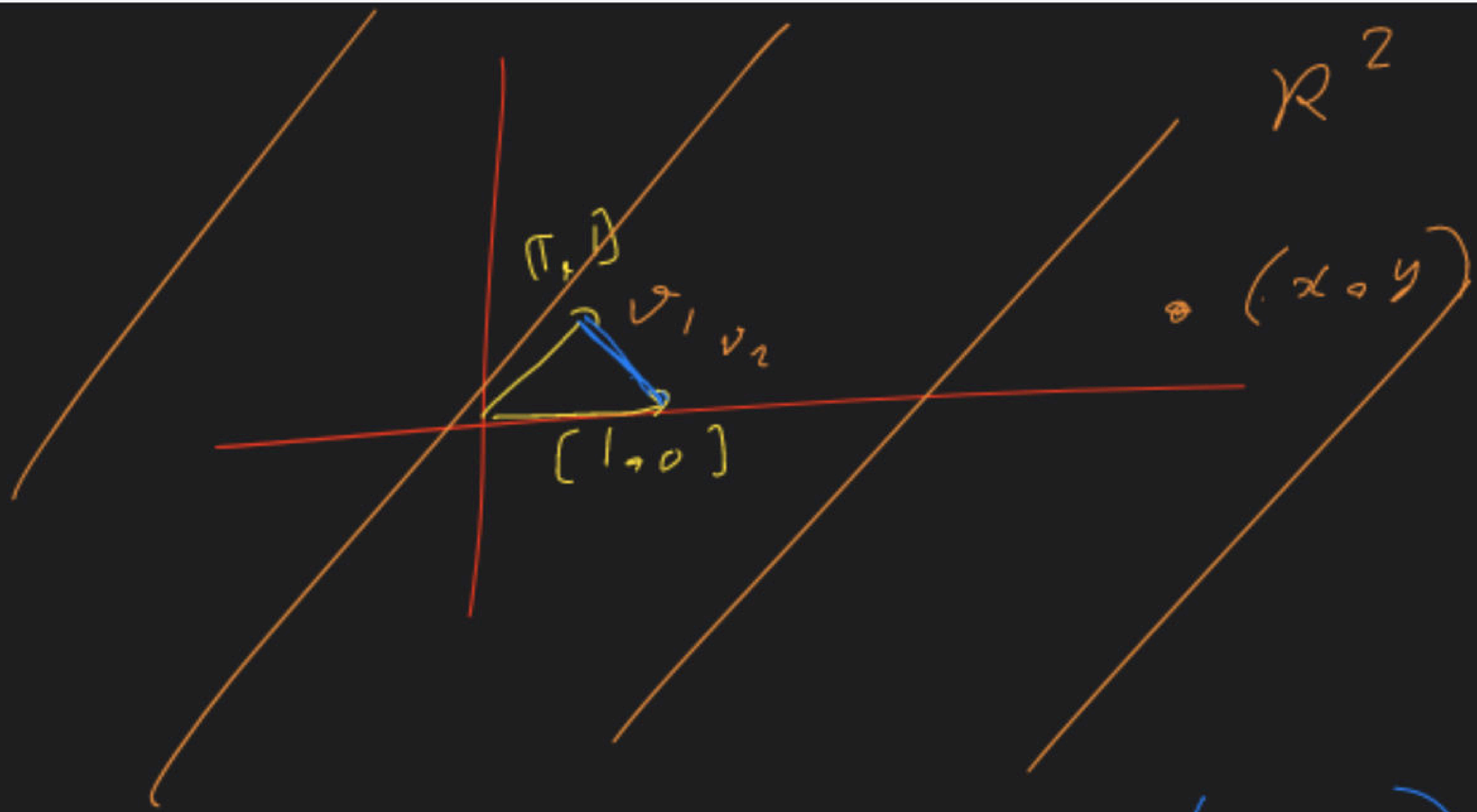
$a_i \in R$

Affined $\rightarrow$ $\sum a_i = 1$

Convex $\rightarrow$ $\sum a_i = 1$ & $a_i \geq 0$

Conical $\rightarrow$ $a_i \geq 0$

Ex.



$$\begin{bmatrix} x \\ y \end{bmatrix} = v_1 \times y + v_2 (x - y)$$

Linear indebendonce of vectors

$v_1 \quad v_2 \; - \; - \quad — \quad - \quad v_m \in R^n$

$\hookrightarrow \; \underline{a}_1 v_1 + a_2 v_2 \; - \quad - \quad a_m V_m = \underline{v} = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_n$$

$\hookrightarrow \underline{a_i = 0}$

Linearly dependent

$$\underbrace{a_1 v_1 + \cdots a_{i-1} v_{i-1} + a_{i+1} v_{i+1} \cdots = \underbrace{v_i}}$$

linearly independent

Is([5,3,2,7] , [9,-1,8 ,5], [7,1,5,6]) linearly independent?

false

$$\begin{bmatrix}5\\3\\2\\7\end{bmatrix} + \begin{bmatrix}9\\-1\\8\\5\end{bmatrix} - 2\times\begin{bmatrix}7\\1\\5\\6\end{bmatrix} = \vec{0}$$

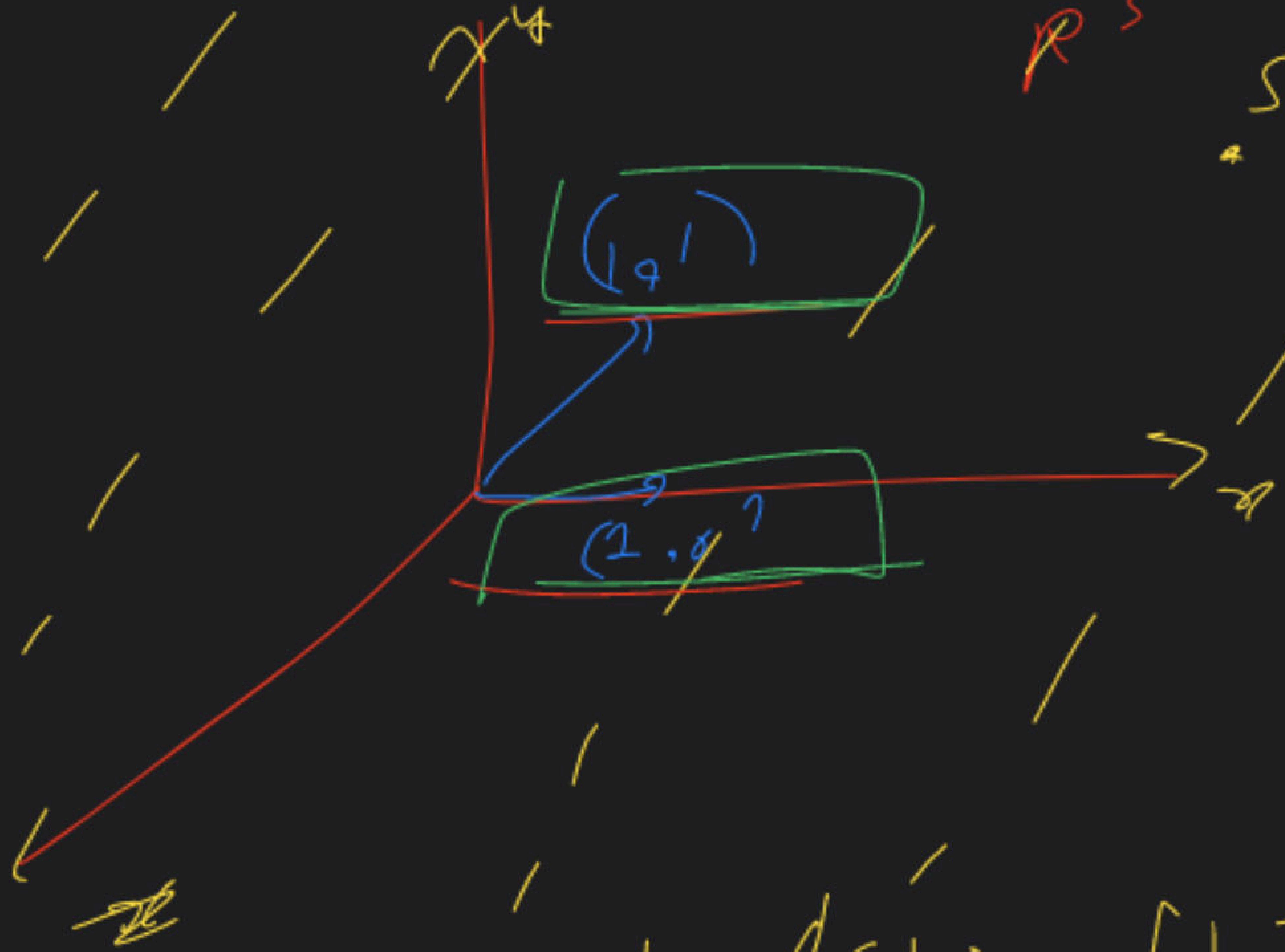
$v_1, v_2 \dots \dots v_m \in R^n$

$\hookrightarrow \underbrace{a_1 v_1 \dots \dots \dots a_m v_m}_{} = \underline{v}$

$span(v_1 \; v_2 \dots \dots \dots v_m) = SS$

Ex.

$\mathbb{R}^3$ SS



$$\text{span}\left(\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}\right)$$

Basis

↳ a linearly indebendent set of vectors is said to be a basis of its span

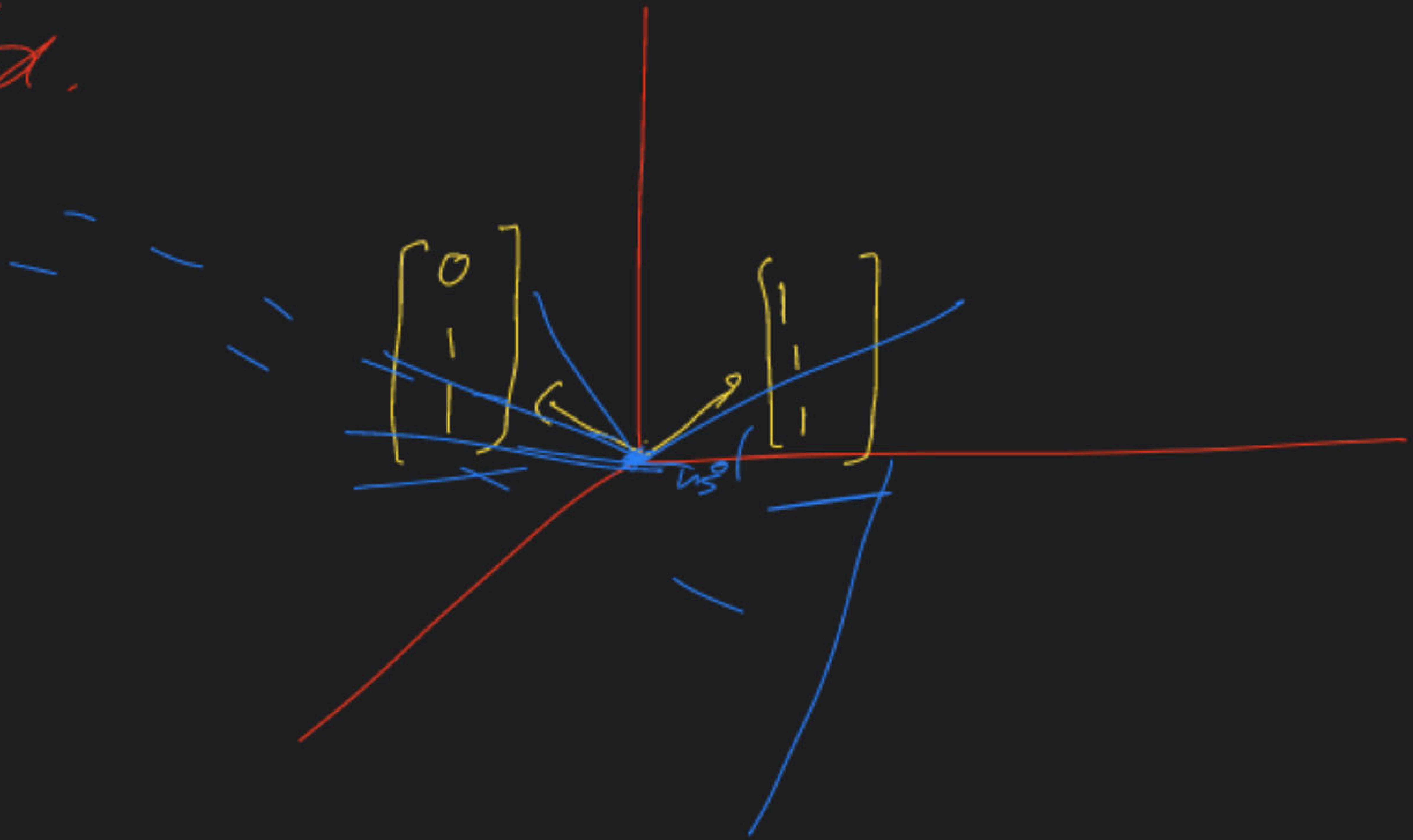
$$\text{Basis}(\underline{XY}) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\underset{\hookrightarrow 0}{a_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underset{\hookrightarrow 0}{a_2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\hookrightarrow \quad \begin{bmatrix} \underset{\hookrightarrow 0}{a_1} + a_2 \\ a_1 \end{bmatrix} \begin{matrix} \rightarrow 0 \\ \rightarrow 0 \end{matrix}$$

3d.



Affine SS

↳ $\underline{SS_0} + V_0$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$c_1 \times (a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n) = c_1 \times b_2) \; e/ob$$

$$\vdots$$

$$\cancel{R^n}$$

$$Z_p$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m$$

$$\hookrightarrow A = \begin{bmatrix} a_{11} & a_{12} & - & - & - \\ a_{21} & a_{22} & & & \\ & & & & \\ & & & & \bar{a}_{mn} \end{bmatrix}$$

$m \times n \leftarrow$

$$\hookrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leftarrow n \times 1$$

$\hookrightarrow$ $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \leftarrow m \times 1$

$$\boxed{\underbrace{A}_{m \times n}\ \underbrace{x}_{n \times 1} = \underbrace{b}_{m \times 1}}$$

Gaussian elim.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & & & \end{array}\right] \times \lambda$$

↳ $R \times= \lambda$

↳ $\underbrace{R_i \mathrel{+}= \lambda \times R_j}$

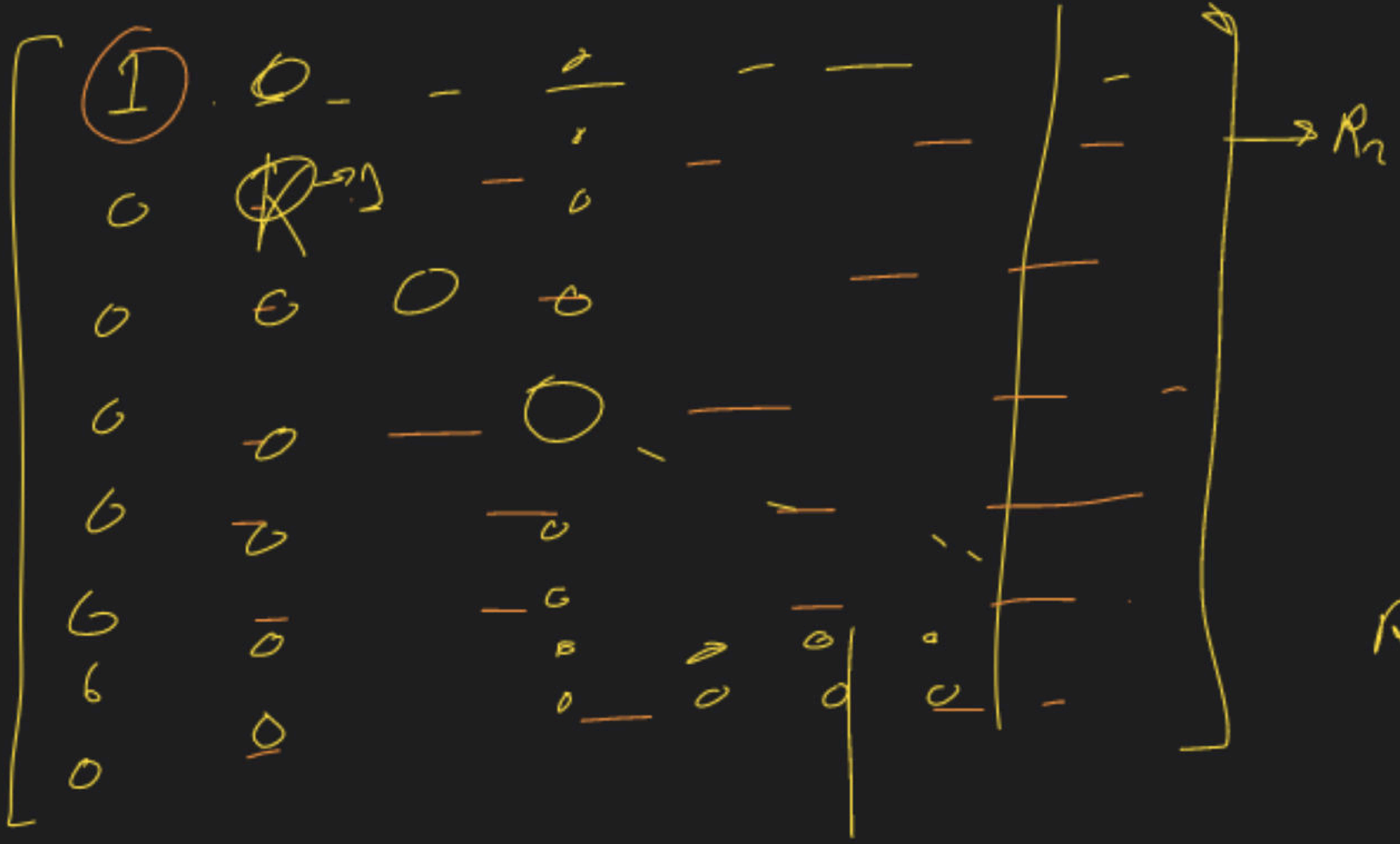
↳ $\text{swap}(R_i, R_j)$

Row operations

↳ Z_b^n

$$\left[\begin{array}{ccccc|c} \frac{a_{11}}{a_{11}} \rightarrow 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & --- & \frac{a_{1n}}{a_{11}} & \frac{b_1}{a_{11}} \\ R_2 \; a_{21} & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{array}\right]$$

$$R_2 = R_2 - a_{21} \times R_1$$



$$R_2 \times = \frac{1}{K}$$

$x_1 = b'_1$

$x_2 = b'_2$

$x_3 = b'_3$

$$\left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & b'_1 \\
0 & 1 & 0 & 0 & 0 & b'_2 \\
0 & 0 & 1 & 0 &   & b'_3 \\
0 & 0 & 0 & 1 & 0 & \\
0 & 0 & 0 & 0 & 1 & ' \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
  &   &   &   &   & b'_m
\end{array}\right]$$

$0x_1 + 0x_2 - - - - = 1$

Does $x + 2y + 3z = 2$, $2x+6y+4z=4$, $z-x=4$, $3x+5y+6z=2$ have a unique solution?

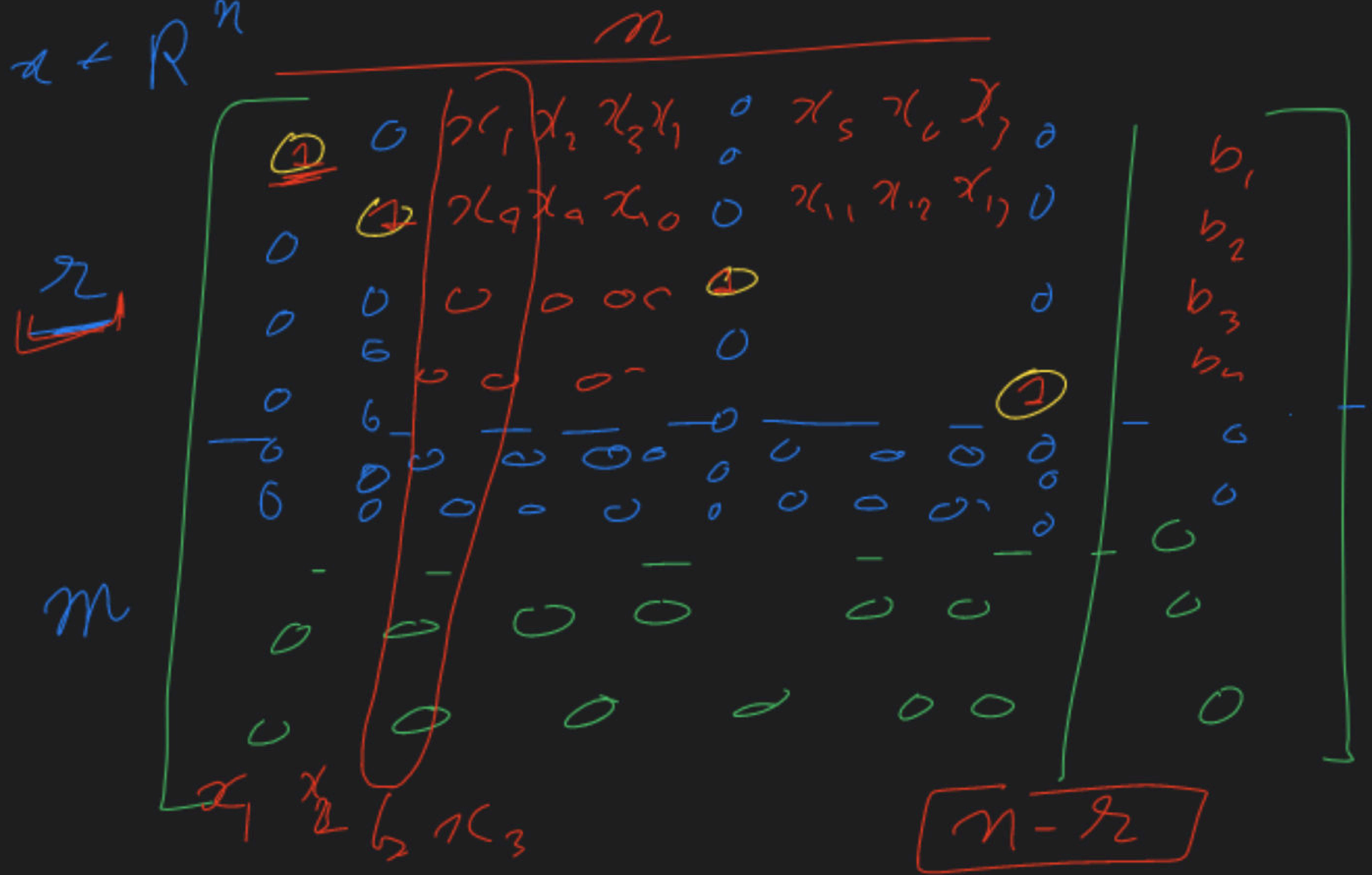
$\hookrightarrow$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ -1 & 0 & 1 \\ 3 & 5 & 6 \end{bmatrix}$ $\qquad$ $b = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix}$

$$
\begin{array}{r}
G \\
-2R_1 \\
+R_1 \\
-3R_1
\end{array}
\left[\begin{array}{ccc|c}
1 & 2 & 3 & 2 \\
2-2 & 6-4 & 4-6 & 4-4 \\
-1+1 & 0+2 & 1+3 & 4+2 \\
3-3 & 5-6 & 6-9 & 2-6
\end{array}\right]
$$

$$\begin{array}{r} \curvearrowright -2R_2 \\ \\ -2R_2 \\ +R_2 \end{array}\left[\begin{array}{ccc|c} 1+0 & 2-2 & 3+2 & 2+0 \\ 0 & 1 & -1 & 0 \\ 0 & 2-2 & 4+2 & 6+0 \\ 0 & -1+1 & -3-1 & -4 \end{array}\right]$$

$$
\begin{array}{r}
\hookrightarrow -R_3 \times 5 \\
+R_3 \\
\\
+4R_3
\end{array}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1 \\
\hline
\boxed{0} & 0 & 0 & 0
\end{array}\right]
\Rightarrow
\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}
$$

$$\begin{bmatrix} 1 & x & 0 & & & \\ 0 & 0 & K\neq 0 & 1 & -\cancel{x} & - \\ 0 & 0 & -K_2 & 0 & & \\ 0 & a & 0 & & & \\ 0 & b_1 & 0 & & & \\ 0 & b & 0 & & & \\ 0 & b & 0 & & & \\ 0 & b_1 & 0 & & & \end{bmatrix}$$



x ∈ R^n
n
r
m
b1
b2
b3
b4
n - r

Ex:

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad 1$$

$\hookrightarrow \underline{x} + 4y = 1 \quad \& \quad z = 2$

$y \rightarrow -10$

Q. $\rightarrow$ a_1 . - - - - a_n

1. p, x $\rightarrow$ $a'_b = x$

2. l, r, x

$$\left(a_l \left(a_{l+1} - - - \right) - a_r \right)$$

$\hookrightarrow$ a_{l+1} a_{l+5} - - - a_{l+9}

4 2 5 6 7 9

$l = 2 \quad r = 4 \quad x = 8$ **False**

$4 \; 0 \qquad 5 \to 5 \qquad 6 \to 6 \qquad 7 \to 7$

$5 \wedge 6 \to 3 \qquad 5 \wedge 7 \to 2$

↳ Segment tree ?

basis →

$a_1 \quad - \quad a_n$

XOR is add in $\mathbb{Z}_2$

$0 + 0 \ \%\ 2 \quad 0 \rightarrow$

$0 + 1 \ \%\ 2 \rightarrow 1$

$1 + 0 \ \%\ \rightarrow 1$

$1 + 1 \ \%\ 2 \rightarrow 0$

$\hookrightarrow x \wedge y \rightarrow (x+y) \% 2$

$a_i \leq 10^9$

$a_i \leq 2^{30}$

$$a_1 \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{30} \end{bmatrix} \begin{array}{l} \rightarrow z_2 \in [0,1] \\ \rightarrow \\ \rightarrow \\ \\ \\ \rightarrow z_2 \end{array}$$

$$a_1 \wedge a_2 \longrightarrow (\vec{a}_1 + \vec{a}_2)$$

$$a_1 \quad a_2 \quad a_3 \qquad\qquad a_x$$

Eg. $0a_1 + 1a_2 + 0a_3 + 1a_4 + 0a_5$

↳ Any bitwise xor value of a subset of this array is going to be a linear comb of the vectors.

2^{30}

↳ $\underline{< 30}$

↳ $x \notin$ span(basis)

$O(n^3)$