FFT and Variants

Special class

Nishchay Manwani • Nov 22, 2020

FFT and Variants

Nishchay Manwani

Course on Unacademy

Let's crack Competitive Programming together!

Nishchay Manwani

- zeus_orz at codeforces.com
- EnEm at codechef.com
- Educator at Unacademy : https://unacademy.com/@EnEm

combley numbers

D n C

number theory Z_b

What you will get

Live Interactive Classes

Attend live interactive classes with our top educators.

Doubt Support

Get your doubts resolved by our expert panel of teaching assistants and community members

Practice Relevant Problems @ CodeChef

Each class comes with a set of curated practice problems to help you apply the concepts in real time.

Educators

- **Curated faculty** with a strong **background in competitive programing** & hands on experience of educational training.
- Highly competent technical minds with **ICPC world finals, IOI medals,** IOI team training experience and Codeforces Grandmasters as accolades.
- Alumni of the most respected technology teams from around the world. (Google, Flipkart, Linkedin, Facebook, Amazon, Goldman Sachs, AppDynamics)
- Young & dynamic faculty to make each class as engaging as they are informative.

Educators

Deepak Gour

ICPC World Finalist 2020 | Former Instructor @InterviewBit | Software Engineer at AppDynamics

Himanshu Singh

World Finalist ICPC 2020, Winner Techgig Code Gladiators 2020, Winner TCC '19, 2020 CSE Graduate from IIT BHU, Works at Nutanix

Arjun P

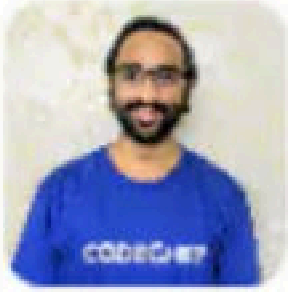
I am an IOI 2015 bronze medallist, and my team qualified for the upcoming ICPC 2020 World Finals to be held in Moscow, Russia.

Murugappan S

Software engineer at Google. Have won many programming contests. Max Rating of 2192 in codeforces and 2201 in codechef.

Triveni Mahatha

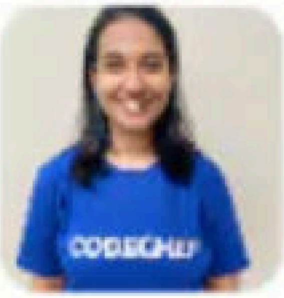
Qualified ICPC 2016 World Final. Won multiple Codechef Long Challenges (India). ICPC Onsite Regionals' Problem setter and Judge. IIT Kanpur.



Tanuj Khattar

ACM ICPC World Finalist - 2017, 2018. Indian IOI Team Trainer 2016-2018. Worked @ Google, Facebook, HFT. Quantum Computing Enthusiast.

Educators



Riya Bansal

Software Engineer at Flipkart | Former SDE and Instructor @ InterviewBit | Google Women TechMakers Scholar 2018

Sanket Singh

Software Development Engineer @ LinkedIn | Former SDE @ Interviewbit | Google Summer of Code 2019 @ Harvard University | Former Intern @ISRO

Nishchay Manwani

Hey I am Nishchay Manwani from CSE, IIT Guwahati and I'm a Seven star on Codechef and International Grandmaster on Codeforces.

Pulkit Chhabra

Codeforces: 2246 | Codechef: 2416 | Former SDE Intern @CodeNation | Former Intern @HackerRank

and many more joining soon…

Topic-wise structure

Beginner	• Introduction to programming • C++ Foundation	• Java Foundation • Python Foundation
Intermediate	• Basic Data Structures • STLs • Sorting and Searching • Greedy Algorithms	• Basic Data Structures 2 • Number Theory • Recursion and DP
Advanced	• Segment Trees • Trees and Graphs • Advanced Dynamic Programming	• Graphs 2 • Computational Geometry
Misc	• ICPC Regionals + World Finals problem solving	

Courses

ENGLISH **INTERMEDIATE**

Course on Greedy Algorithms

Starts on Sep 21, 2020 • 8 lessons

Murugappan S

HINDI **ADVANCED**

Detailed Course on Graphs - I

Starts on Sep 21, 2020 • 9 lessons

Pulkit Chhabra

HINDI **INTERMEDIATE**

Course on Introduction to Number Theory

Starts on Sep 22, 2020 • 8 lessons

Nishchay Manwani

Courses

ENGLISH **BEGINNER**

Course on Recursion and Dynamic Programming

Starts on Sep 22, 2020 • 12 lessons

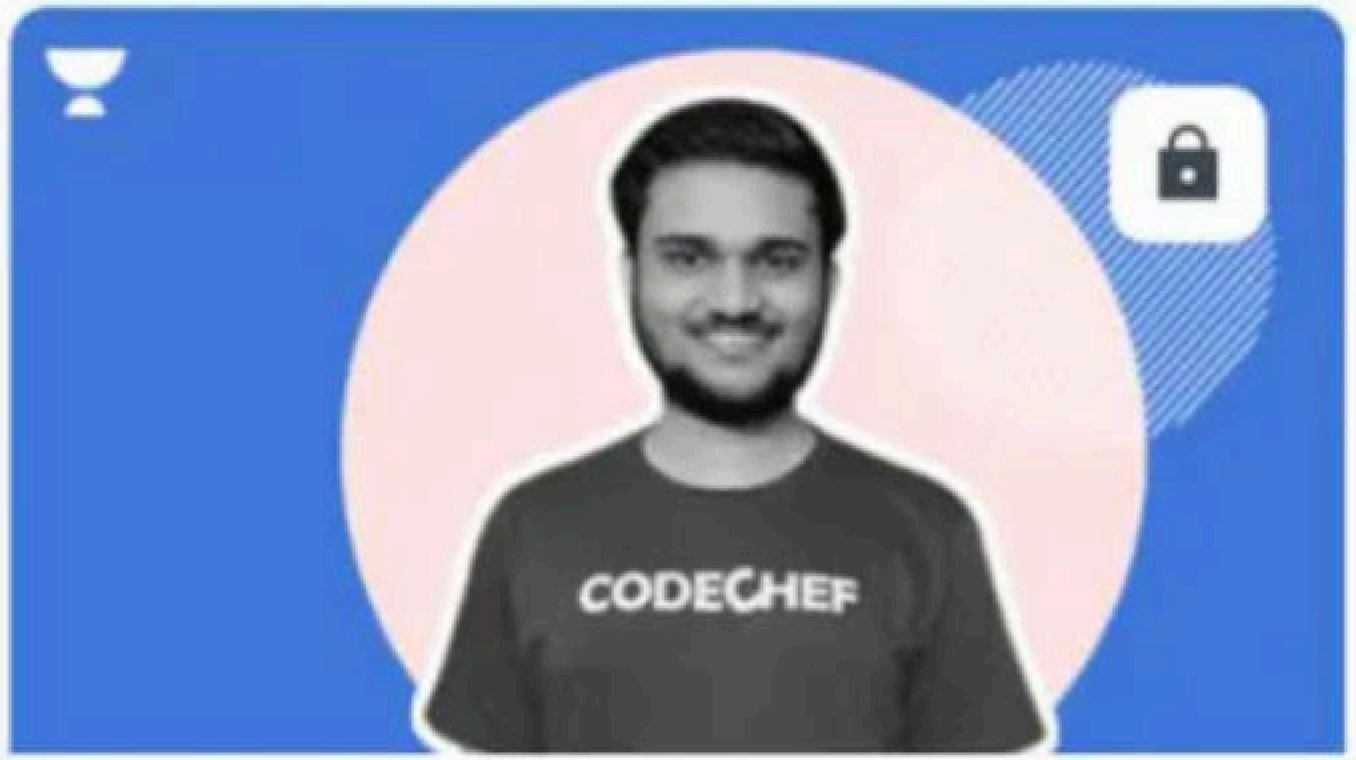
Arjun P

ENGLISH **INTERMEDIATE**

Course on Sorting and Searching

Starts on Sep 22, 2020 • 10 lessons

Riya Bansal



HINDI **INTERMEDIATE**

Course on Standard Template Library (STL) in C++

Starts on Sep 23, 2020 • 11 lessons

Sanket Singh

Courses

HINDI INTERMEDIATE

Course on Basic Data Structures - I

Starts on Sep 26, 2020 • 11 lessons

Deepak Gour

HINDI INTERMEDIATE

Course on Data Structures (Square Root Decomposition)

Starts on Sep 26, 2020 • 5 lessons

Tanuj Khattar

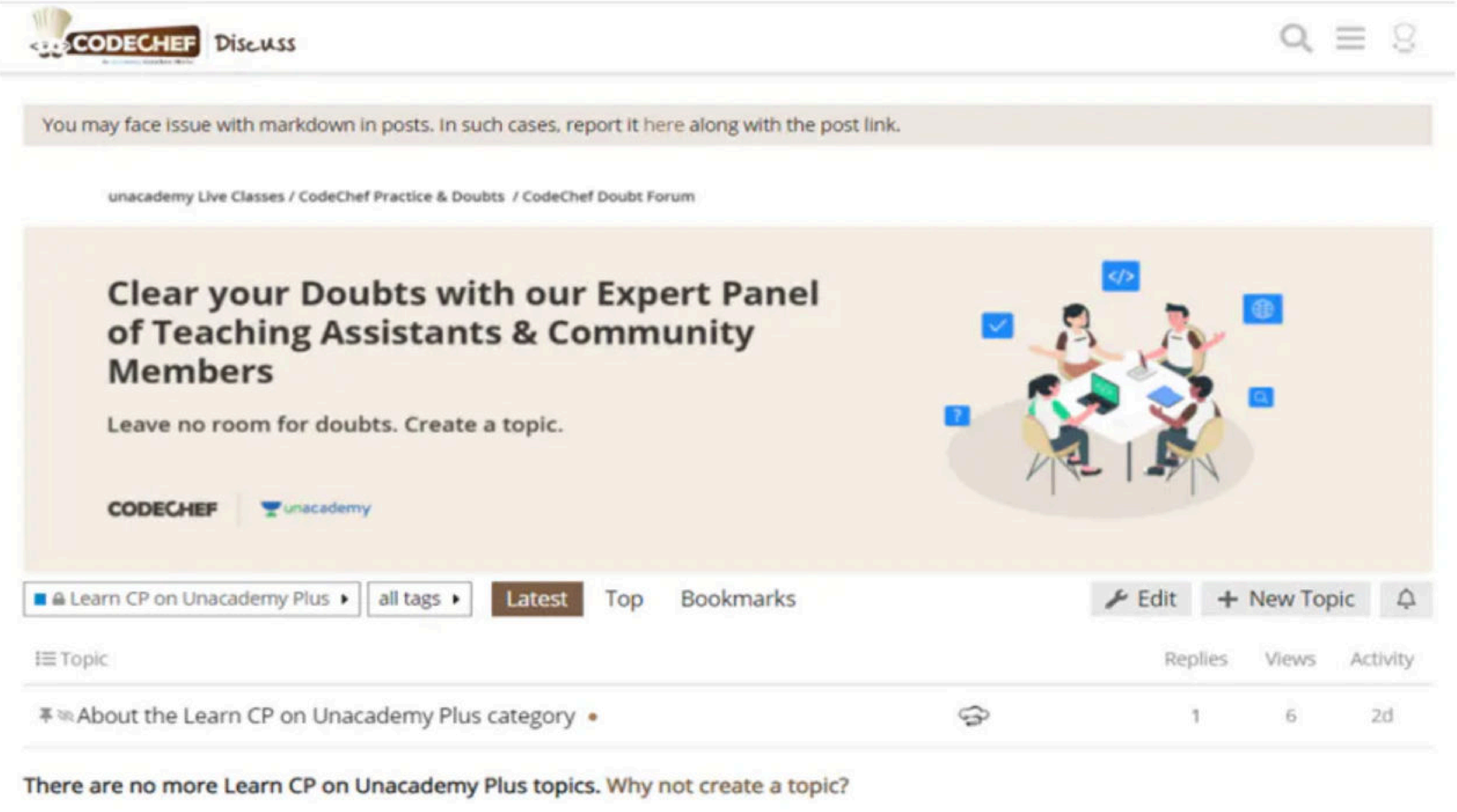
HINDI BEGINNER

Course on Introduction to Competitive Programming with C++

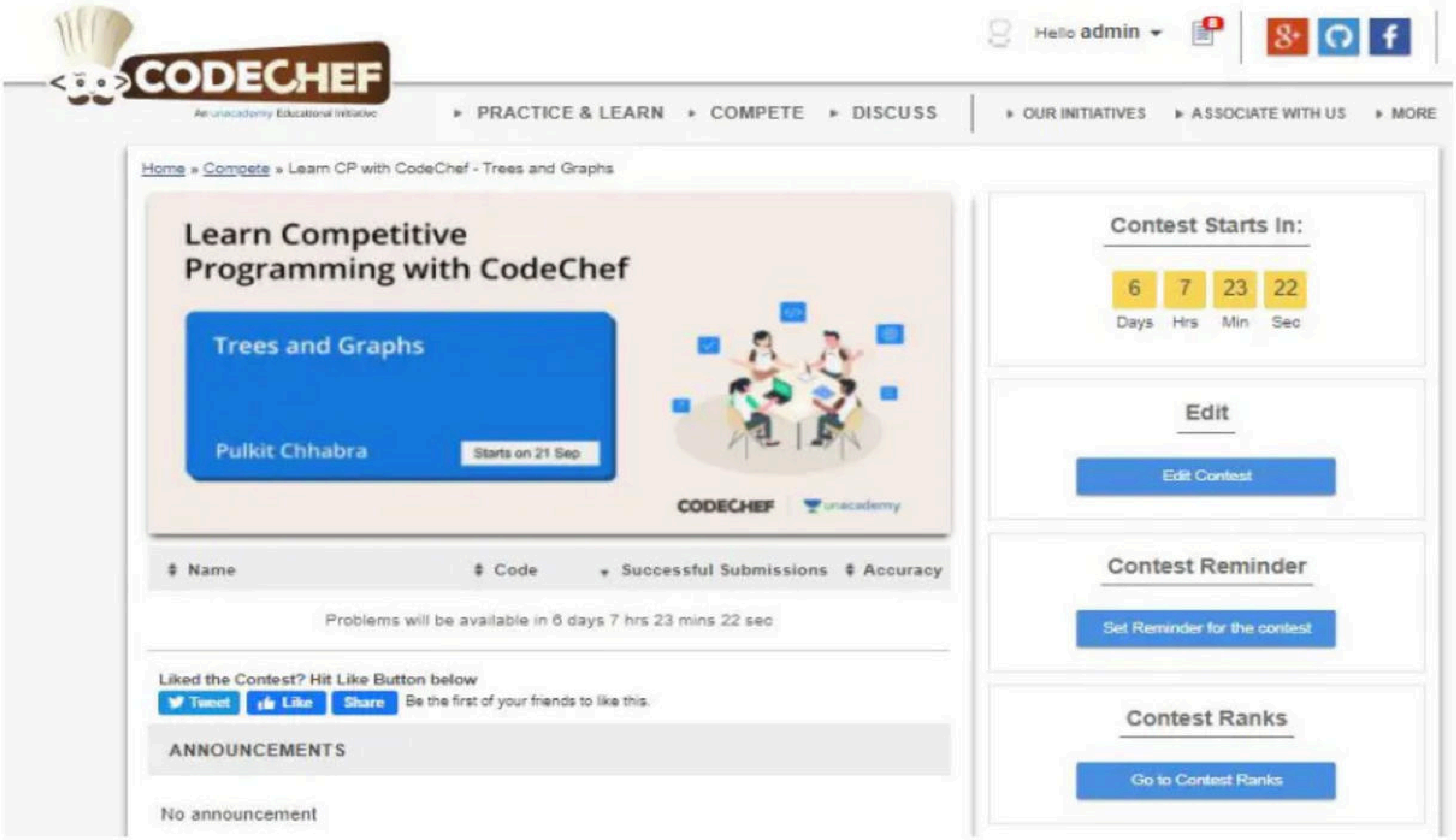
Starts on Sep 26, 2020 • 10 lessons

Triveni Mahatha

Teaching Assistants support on chat and Doubts Forum



Course-wise Practice Problems



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6 months	25% OFF	₹4,500 per month	₹27,000 Total (Incl. of all taxes)
12 months	54% OFF	₹2,750 per month	₹33,000 Total (Incl. of all taxes)

EnEm Apply

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Flexible Subscription Plans

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6 months	25% OFF	₹4,050 per month	₹24,300 Total (Incl. of all taxes)
12 months	54% OFF	₹2,475 per month	₹29,700 Total (Incl. of all taxes)

EnEm

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Awesome! You got 10% off

Upcoming Free Classes Schedule on website

FREE

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See all

Disjoint Set Union - II
Today, 7:00 PM
Pulkit Chhabra

Discussion on Merge Sort &...
Today, 9:00 PM
Riya Bansal

Headstart to Strings in STL
Today, 10:00 PM
Sanket Singh

ICPC past problems
Sep 19, 2020, 3:30 PM
Himanshu Singh

FFT

fast Fourier Transform

Polynomial multiplication

$A \to a_0 + a_1 x + a_2 x^2 \text{ --- } a_{n-1} x^{n-1}$

$B \to b_0 + b_1 x + b_2 x^2 \cdots b_{m-1} x^{m-1}$

$\deg_A \to n-1$

$\text{size}_A \to n$

$\underline{m-1}$

$\underline{m}$

$C = A \times B$

$size_c \rightarrow n+m-1$

$c \rightarrow c_0 + c_1 x + c_2 x^2 - - \quad c_{n+m-2} x^{n+m-2}$

$c_0 = a_0 \times b_0$

$c_1 = a_1 b_0 + a_0 b_1$

$c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$

$\vdots$

$c_{n+m-2} = a_{n-1} \times b_{m-1}$

$c_i = 0$

i from 0 to $n-1$

j from 0 to $m-1$

$c_{i+j} \mathrel{+}= a_i \times b_j$

$\hookrightarrow \underbrace{O(n \times m)}$

$A \rightarrow \underbrace{a_0 + \text{-- -- -- --} \quad a_{n-1} x^{n-1} + 0 \times x^{n} + 0 \times x^{n+1} \text{ --- } 0 \times x^{n+m-2}}_{\text{size} \rightarrow n+m-1}$

$B \rightarrow \underline{\hspace{10em}}$

$\text{size} \rightarrow n+m-1$

$A(x) \rightarrow$

$(0, A(0)) \rightarrow O(n+m)$

$(1, A(1))$

$(2, A(2))$

$\vdots$

$(n+m-2, A(n+m-2))$

point value

$\rightarrow O((n+m)^2)$

$n+m-1$

$$C(x) = A(x) \times B(x)$$

$$\boxed{C(0) = A(0) \times B(0)}$$

$$C(1) = A(1) \times B(1)$$

$$\left\{\begin{array}{l} 0,\ c(0) \\ 1,\ c(1) \\ 2,\ c(2) \\ \vdots \\ n+m-2,\ c(n+m-2) \end{array}\right)$$

$c \to$ deg $n+m-2$

t

$t+1$

$n+m-1$

$O(t^3)$

$O(t^2) \to O(n+m)^2$

Ex. $f(x) = 5 + 2x + 3x^2 +$

$$\boxed{\begin{aligned} f(0) &= 5 \\ f(1) &= 10 \\ f(2) &= 21 \end{aligned}}$$

$t + 1$

$\to a_0 + a_1 x + a_2 x^2$

$\hookrightarrow \boxed{a_0 = 5}$, $\boxed{a_0 + a_1 + a_2 = 10}$

, $\boxed{a_0 + 2a_1 + 4a_2 = 21}$

$A + B$ $(1, 2, 3 \ldots n+m-1)$ C

$O((n+m)^2)$

interpol $O((n+m)^2)$

point value

$c(0) = A(0) \times B(0)$

$c(1) = A(1) \times B(1)$

A B $O(n+m)$ C $O((n+m)\log(n+m))$

$c(n+m-1) = A(n+m-1) \times B$

$A(x)$ at $x=1$

$B(x)$ at $x=2$

t $t+1$

$O((n+m)\log(n+m))$

$$A(x) = a_0 + a_1 x - - - a_{n-1} x^{n-1}$$

5

$$\underbrace{a_0 + a_1 \times 5 + a_2 \times 5^2 + a_{\cdot} - - a_{n-1} \times 5^{n-1}}_{\cancel{O}(n)}$$

$n+m-1$

$$O((n+m)^2)$$

FFT / DFT

Discrete fourier tramform

$$\boxed{O((n+m)\log(n+m))}$$

$$\underbrace{\left(\omega_n^0,\ \omega_n^1,\ \omega_n^2,\ \cdots\ \omega_n^{n-1}\right)}$$

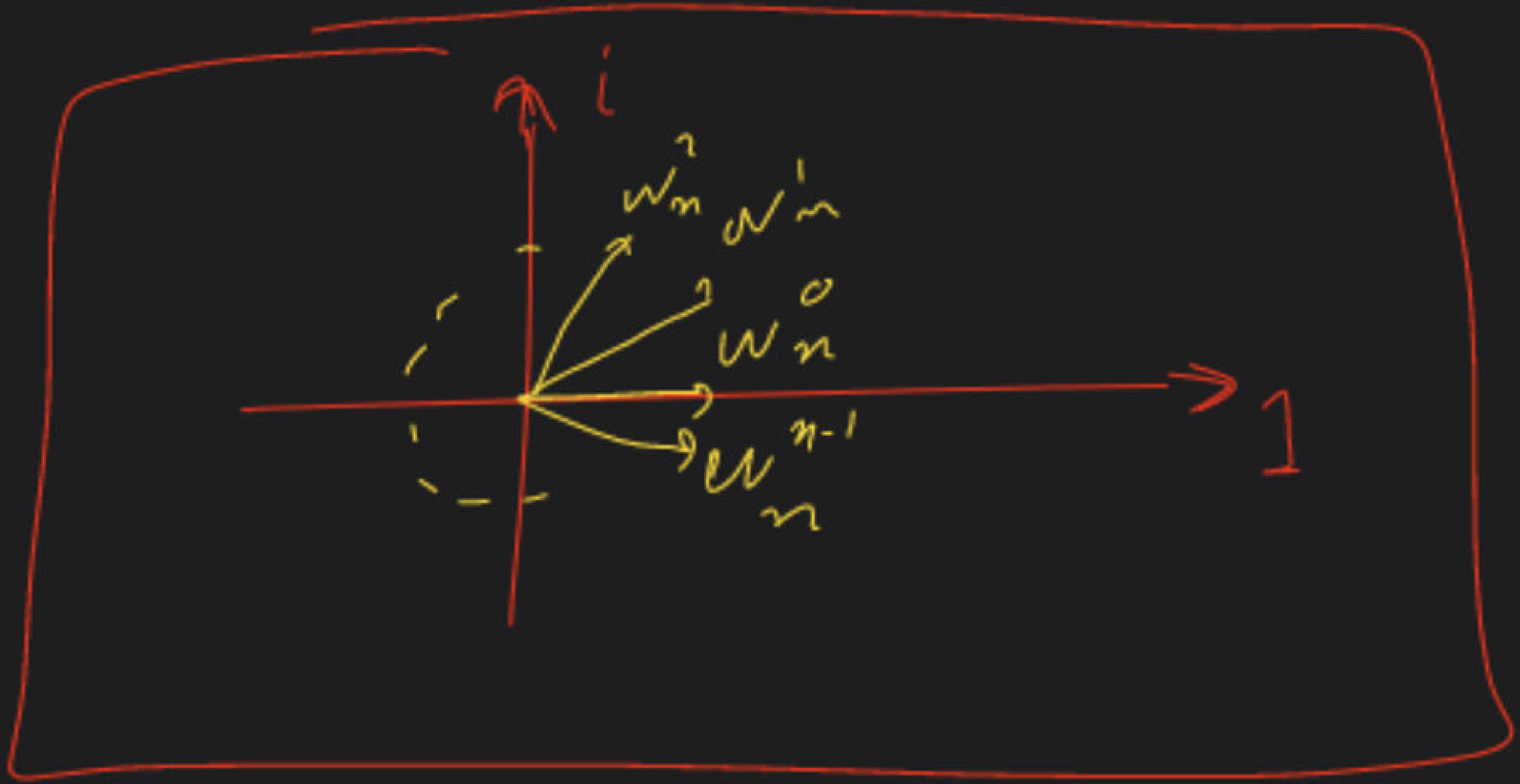
$\omega^0, \omega^1, \omega^2 \cdots \longrightarrow \boxed{x^n = 1}$

$x^2 = 1$

$\omega_2^0 = 1 \quad , \omega_2^1 = -1$

$n \to 4$

$$w_4^0 = 1 \qquad w_4^1 = i \;,\; w_4^2 = -1 \;,\; w_4^3 = -i$$



$$n \quad A(x) = a_0 + a_1 x + a_2 x^2 - - - - - a_{n-1} x^{n-1}$$

$$\frac{n}{2} \quad A_0(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{\frac{n}{2}-1}$$

$$\frac{n}{2} \quad A_1(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{\frac{n}{2}-1}$$

$$\boxed{n = 2^K}$$

$$\boxed{A(x) = A_0(x^2) + x \times A_1(x^2)}$$

$\hookrightarrow A_2(x) \rightarrow \frac{n}{2}$

$$\left(\omega_{\frac{n}{2}}^{0},\ A_0\left(\omega_{\frac{n}{2}}^{0}\right)\right)$$

$$\left(\omega_{\frac{n}{2}}^{1},\ A_0\left(\omega_{\frac{n}{2}}^{1}\right)\right)$$

$$\vdots$$

$$\left(\omega_{\frac{n}{2}}^{\frac{n}{2}-1},\ A_0\left(\omega_{\frac{n}{2}}^{\frac{n}{2}-1}\right)\right)$$

$\hookrightarrow A_1(x) \rightarrow \frac{n}{2}$

$$\left(\omega_{\frac{n}{2}}^{0},\ A_1\left(\omega_{\frac{n}{2}}^{0}\right)\right)$$

$$\left(\omega_{\frac{n}{2}}^{1},\ A_1\left(\omega_{\frac{n}{2}}^{1}\right)\right)$$

$$\vdots$$

$$\left(\omega_{\frac{n}{2}}^{\frac{n}{2}-1},\ A_1\left(\omega_{\frac{n}{2}}^{\frac{n}{2}-1}\right)\right)$$

$A(x)$

$\hookrightarrow$ $A(w_n^0)$, $A(w_n^1)$ ---- $A(w_n^{n-1})$

$\hookrightarrow$ $A(w_n^i)$ $\quad 0 \le i \le n-1$

$\hookrightarrow$ $$\boxed{A(w_n^i) = \underline{A_0}\left((w_n^i)^2\right) + w_n^i \times \underline{A_1}\left((w_n^i)^2\right)}$$

$$\rightarrow \underline{\left(w_n^i\right)^2} \longrightarrow \underline{w_{\underbrace{n}}^{2i}} \longrightarrow \underline{w_{\frac{n}{2}}^{i}}$$

$$w_n^i$$

$$\underline{e^{-i\frac{2\pi}{n}}} \longrightarrow \qquad \longrightarrow$$

$$\boxed{A(w_n^i) = A_0(w_{\frac{n}{2}}^i) + w_n^i A_1(w_{\frac{n}{2}}^i)}$$

$$\boxed{0 \le i \le \frac{n}{2} - 1}$$

$$w_{\frac{n}{2}}^{\frac{n}{2}} = 1 \quad \underline{-1}$$

$$A(w_n^i) = A_0(w_{\frac{n}{2}}^{i-\frac{n}{2}}) + w_n^{\frac{n}{2}} w_n^{(i-\frac{n}{2})} A_1(w_{\frac{n}{2}}^{i-\frac{n}{2}})$$

$$\frac{n}{2} \le i \le n-1$$

$$A(w_n^i) = A_0\left(w_{\frac{n}{2}}^{i-\frac{n}{2}}\right) - w_n^{i-\frac{n}{2}} A_1\left(w_{\frac{n}{2}}^{i-\frac{n}{2}}\right)$$

$$T(n) = 2 \times T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = \underline{O(n \log n)}$$

Inverse FFT

$A = \underline{a_0} + \underbrace{a_1 x} + \underbrace{a_2 x^2} \cdots a_{n-1} x^{n-1}$

$$\left[w_n^0 \; w_n^1 \; w_n^2 \cdots w_n^{n-1} \right]$$

$$\left[w_n^0 \; w_n^{1i} \; w_n^{2i} \cdots w_n^{i(n-1)} \right]$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

$$A(w_n^i) = \boxed{A(w_n^i)}$$

$$\boxed{x^n = 1}$$

$$(w_n^{n-1})^2 = w_n^{2n-2}$$

$$w_n^{n-2}$$

$$w_n^n = 1$$

$$
\underbrace{
\begin{array}{c c}
 & j \\
i &
\begin{bmatrix}
w_n^0 & w_n^0 & \cdots & w_n^0 \\
w_n^0 & w_n^1 & w_n^2 \cdots & w_n^{n-1} \\
w_n^0 & w_n^2 & w_n^4 \cdots & w_n^{n-2} \\
\vdots & & & \\
w_n^0 & w_n^{n-1} & w_n^{n-2} \cdots & w_n^1
\end{bmatrix}
\end{array}
}_{B \quad \text{Vandermonde matrix}}
\underbrace{
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1}
\end{bmatrix}
}_{A}
=
\underbrace{
\begin{bmatrix}
A(w_n^0) \\ A(w_n^1) \\ \vdots \\ A(w_n^{n-1})
\end{bmatrix}
}_{C}
$$

↳ $\underbrace{BA}_{} = \underbrace{C}_{}$ $\underbrace{O(n \log n)}_{}$

↳ $\boxed{\underbrace{A}_{} = \underbrace{B^{-1} C}_{}}$

↳ $\underbrace{O(n \log n)}_{}$

$$\boxed{\frac{1}{n}}\begin{bmatrix} w_n^0 & w_2^0 & - & - & \cdots & - & - \\ w_n^0 & w_n^{-1} & w_n^{-2} & - & - & \cdots & w_n^{-(n-1)} \\ w_n^0 & \underline{\overline{w_n^{-2}}} & w_n^{-4} & - & - & \cdots & w_n^{-(n-2)} \\ \vdots & & & & & & \\ w_n^0 & \underline{w_n^{-(n-1)}} & \cdots & - & - & - & - \end{bmatrix}$$

Poly mult

$\hookrightarrow \boxed{O\left((n+m)\log(n+m)\right)}$

NTT

$A, B \rightarrow C$

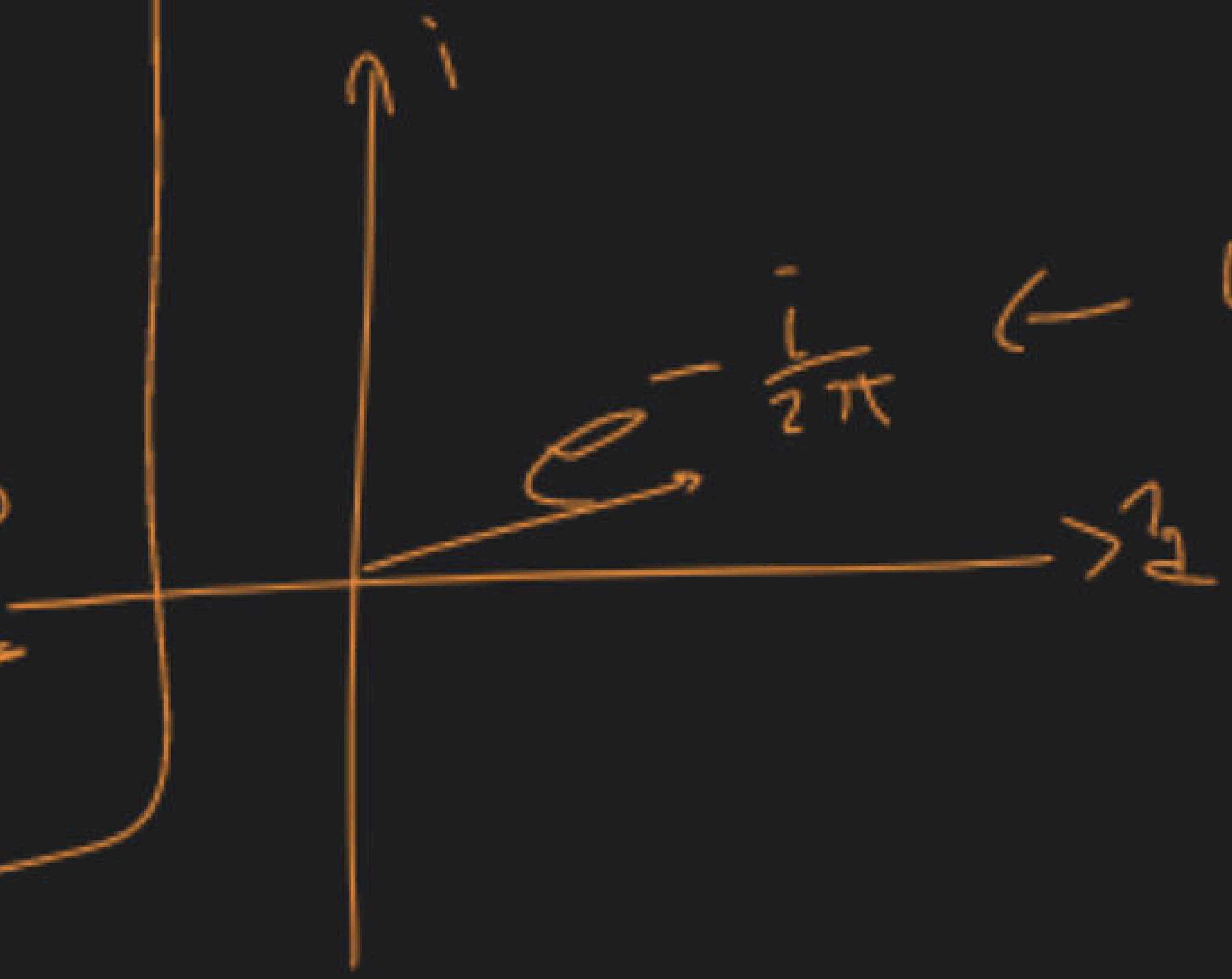
$\underbrace{\% \; p}$ ← prime no.

$\rightarrow \boxed{x^n = 1} \leftarrow \underbrace{w_n^0 \; w_n^1 \; --- \; w_n^{n-1}}_{\text{distinct}}$

$w_n^1 \Rightarrow x$

$\underbrace{x^0, x^1, x^2 \cdots x^{n-1}}_{\text{distinct}}$

$(x^n = 1) \% p$



$e^{-\frac{i}{2\pi}} \leftarrow w_n^1$

$\hookrightarrow \mathbb{Z}_p \longrightarrow \% p$

$\hookrightarrow t^0, t, t^2, \ldots t^{p-2}$

$\hookrightarrow$ distinct

$\boxed{p-1}$

$\hookrightarrow$ primitive root

$(t^{p-1} = 1) \% p$

$$\underbrace{x^0 \quad x^1 \quad - \; - \; - \; - \quad x^{n-1}}_{\hookrightarrow} \qquad x^n = 1$$

$$\rightarrow \boxed{t^{\left(\frac{k-1}{n}\right)} = x} \longrightarrow$$

$$\hookrightarrow \underbrace{t^0}_{w_n^0} \qquad \underbrace{t^{\left(\frac{k-1}{n}\right)}}_{w_n^1} \qquad \underbrace{t^{2\left(\frac{k-1}{n}\right)}}_{W_n^1} \qquad \underbrace{t^{3\left(\frac{k-1}{n}\right)}} \quad - \quad \underbrace{t^{(n-1)\left(\frac{k-1}{n}\right)}} \; w_n^{n-1}$$

$\frac{p-1}{n}$ is an integer

$n = 2^{k \le 23}$

2^{23}

2^{k}

↳ $\boxed{(p-1)\%n \; != 0}$

↳ $\boxed{998244353} = \boxed{7 \times 17 \times 2^{23} + 1}$

$p-1 = 7 \times 17 \times 2^{23} \% 2^{k} = 0$

$k \le 23$

$p-1 = 2^{k_1} \times 7 \times 19 \times 23$

$\boxed{p-1}$

2^{17} 2^{18} ✗ — —

p_1

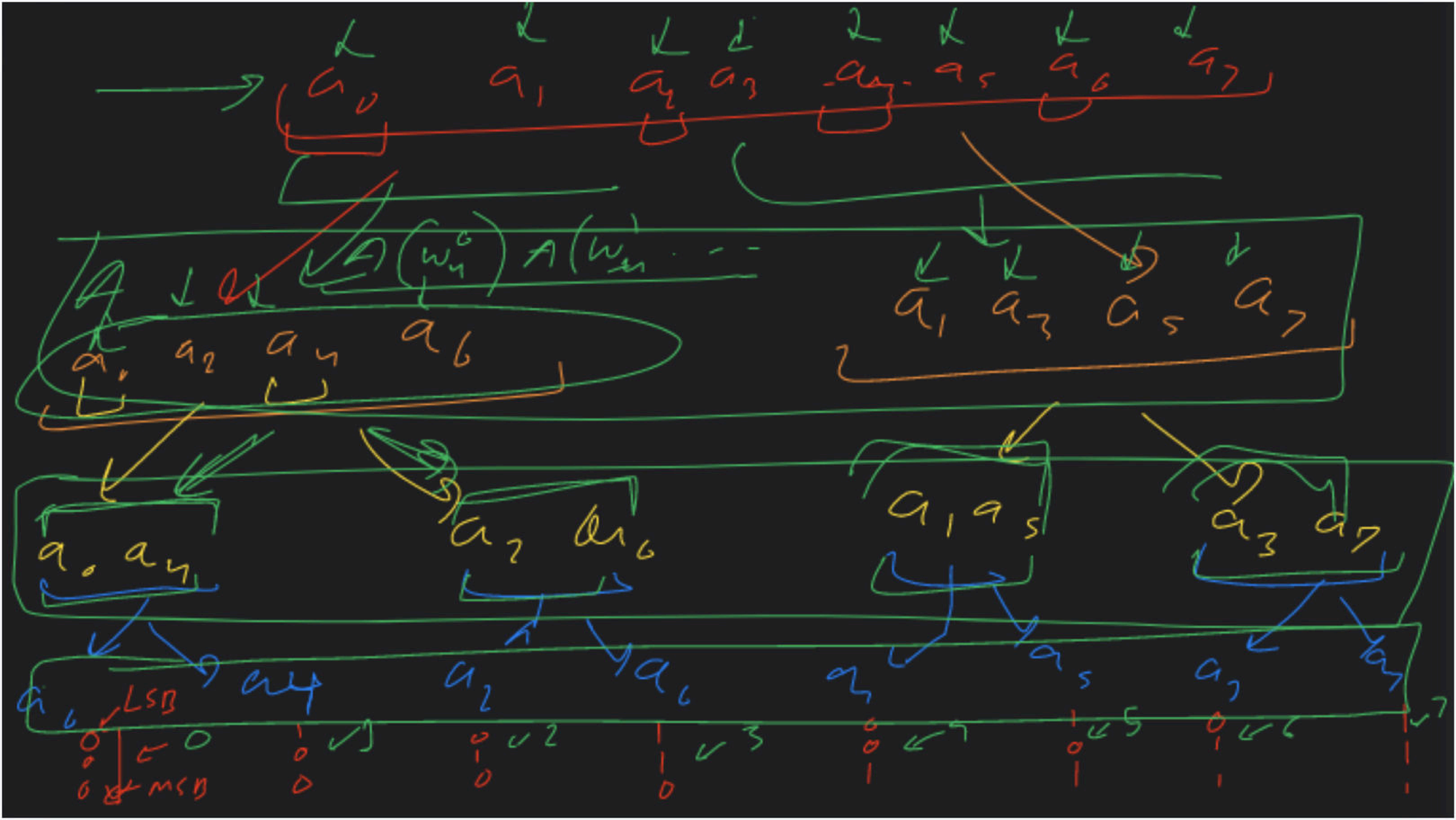
2^{k}

NTT

↳ $2^{23} \approx 10^7$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{10^7}$ %99824

int t = ____ ;



a0 a1 a2 a3 a4 a5 a6 a7
A(w)
a0 a2 a4 a6
a1 a3 a5 a7
a0 a4
a2 a6
a1 a5
a3 a7
LSB
MSB

Bit reversal sort

b n

$$A(x) = a_0$$

$$A(\omega_1^0) = A(1) = \underline{a_0}$$

$\hookrightarrow \quad A^{-1} = \sum_{i=0}^{\infty} a_i x^i$

k coeff

$\underline{\underline{a}}$

$A^{-1} = \underline{B_k} \pmod{x^{c}} \Rightarrow \underline{\underline{B_{k+1}}} = A^{-1} \pmod{x^{2c}}$

$$B_K = A^{-1} \pmod{x^a}$$

$$2^k > n$$

$$B_{K+1} = A^{-1} \pmod{x^{2a}}$$

$$O(n \log n)$$

$$\Rightarrow \underbrace{B_{K+1} = \underline{B_K}\,(2 - A B_K) \pmod{x^{2a}}}$$

$$\underbrace{T(2a) = T(a) + O(a \log a)}$$

$$\hookrightarrow \underbrace{T(2a) = O(2a \log 2a)}$$

$$B_0 = A^{-1} \pmod{x^1}$$

$$B_0 = a_0^{-1}$$

$$2^k > n$$

$$B_1 = A^{-1} \pmod{x^2}$$

$$B_2 = A^{-1} \pmod{x^4}$$

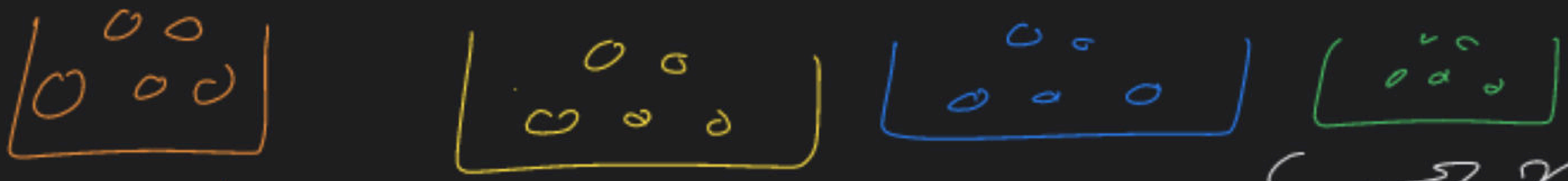
$$B_3 = A^{-1} \pmod{x^8}$$

$$\underbrace{A(x) = B(x) \times Q(x) + \underline{R(x)}}$$

$$\underbrace{\deg_R < \deg_B}$$

Q. n colors $\rightarrow K$ balls of each color

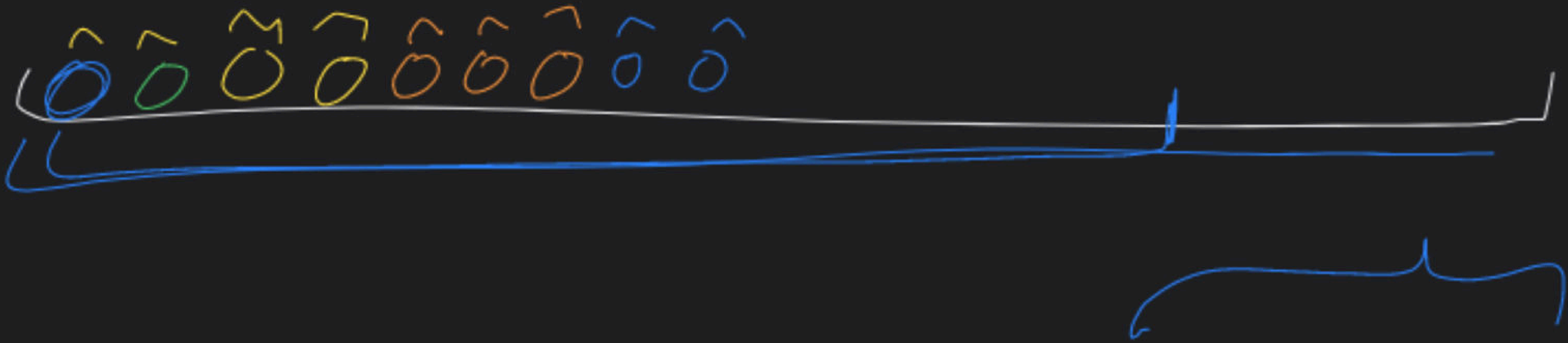
$K \leq 10$, $n \leq 10^4$



$\rightarrow n$

m rounds $\rightarrow \leq 10^{-4}$

$\% 998244353$



$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots\frac{x^k}{k!}\right)^n$$

$\hookrightarrow a_0 + a_1 x + a_2 x^2 \cdots \text{---} a_m x^m \quad a_{kn} x^{kn}$

$\hookrightarrow \quad m! \times a_m \quad \% \; 998244353$

$$\boxed{k_1 + k_2 + k_3 + k_4 + \cdots k_n = m}$$

$$\underbrace{0\ 0\ 0\ \ 0\ \ 0\ \ 0}_{m}$$

$k_i \geq 0$

$$\boxed{\frac{m!}{k_1!\, k_2!\, k_3! \cdots k_n!}}$$

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!} \cdots \frac{x^i}{i!} \cdots \frac{x^k}{k!}\right) \times \left(1+\frac{x}{1!}+\frac{x^2}{2!} \cdots \frac{x^j}{j!} \cdots \frac{x^k}{k!}\right)$$

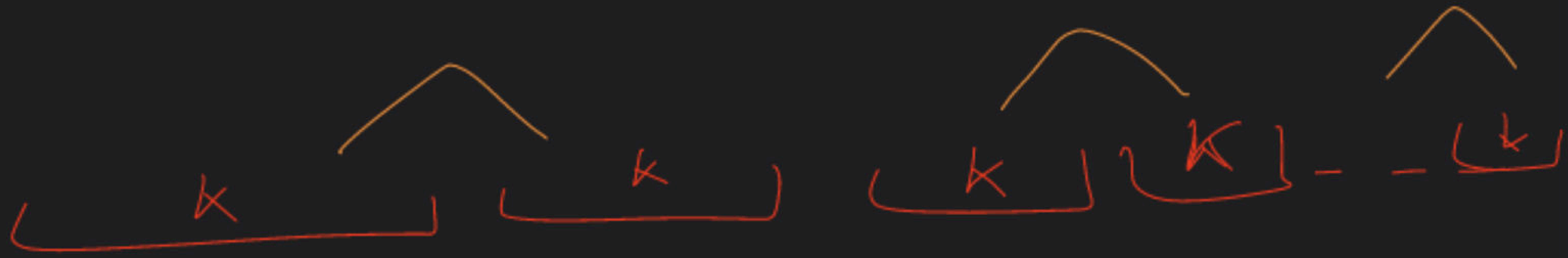
$$\hookrightarrow \frac{x^{i+j}}{i!\,j!} + \frac{x^{i+j}}{(i+1)!\,(j-1)!}$$

Ex. 2 ba 15 $\quad k=7$

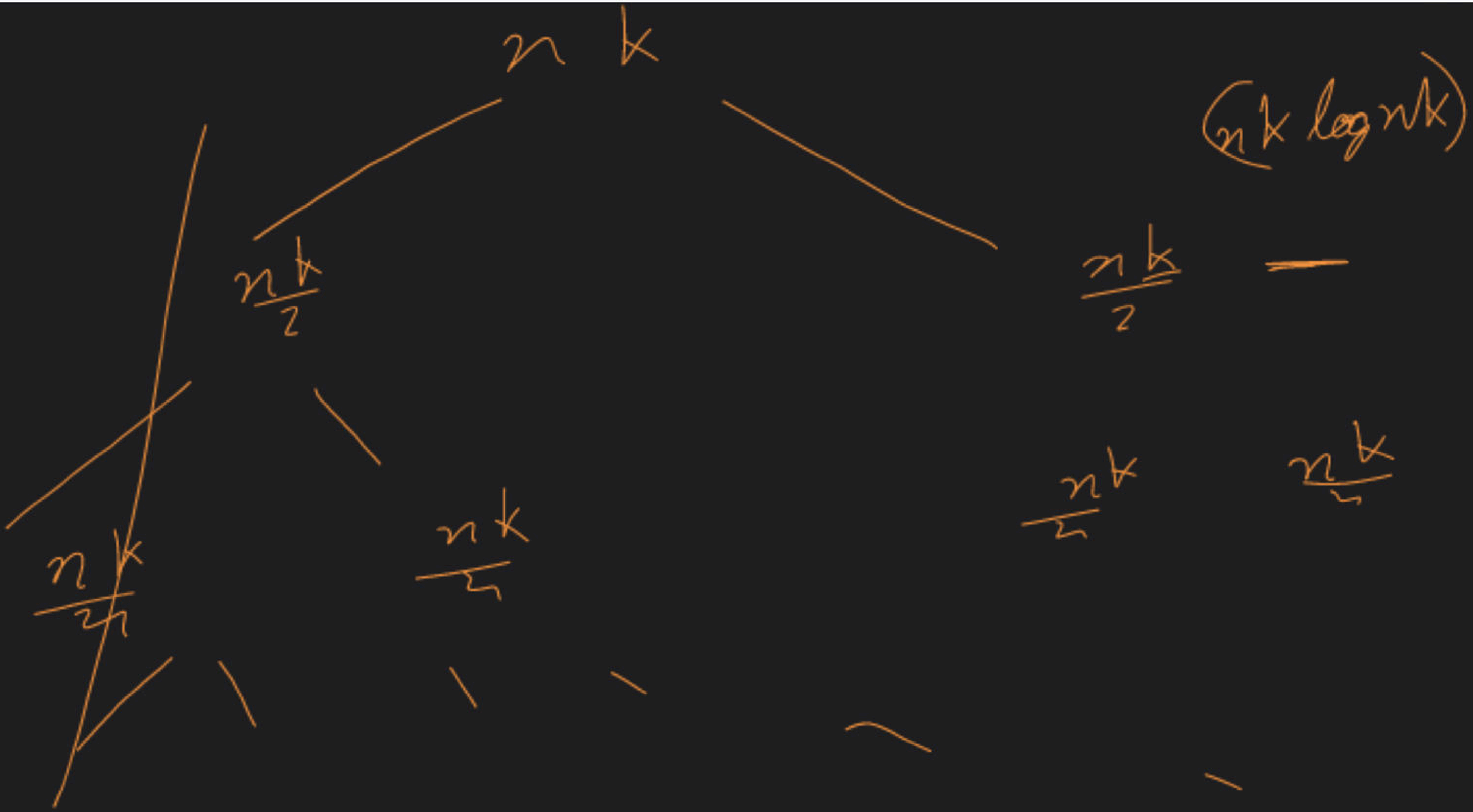
$$\left(1+x+\frac{x^2}{2}\right) \times \left(1+x+\frac{x^2}{2}\right)$$

$$\hookrightarrow 1 + \underbrace{2x}_{\downarrow} + \underbrace{3x^2}_{\downarrow} + \underbrace{\frac{x^4}{4}}_{\downarrow}$$

$\downarrow$ 0 length 1 $\quad$ 2 $\quad$ $3 \times 2!$ $\hookrightarrow 6$ $\quad$ $4! \times \frac{1}{4} \rightarrow 6$



k
k
k
k
k



nk
(nk log nk)
nk/2
nk/2
nk/4
nk/4
nk/4
nk/4

$\boxed{O(nk \log^2(nk))}$

A

K

FFT

$O(nk \log(nk))$

$+$

$nk \log(\cancel{k}n)$

↓

exponentiation

inverse FFT

A^n

O

O

nk

A^n.

$\boxed{O(nk \log(nk))}$

FFT $\longrightarrow$ $\underbrace{\text{FWHT}}$ $\leftarrow$ fast Walsh Hadamard Transform

XOR convulations

$$\left[\underbrace{f_0\ f_1\ f_2\ f_3\ \ldots\ \ldots\ f_{2^k-1}}\right]$$

$$\longrightarrow \left[f'_0\ f'_1\ f'_2\ f'_3\ \ldots\ \ldots\ f'_{2^k-1}\right]$$

$f_0\ f_1\ \ldots\ \ldots\ \ldots$

Ex. [1, 2] [2,3]

[0 1 2 0] ~~[0 1 0 1]~~

[0, 2, 3, 1]

[1, 1, 1, 1]