Micro course → FFT and combinatorics

Stirling number of Second kind

Special class

Nishchay Manwani • Jan 15, 2021

Prerequisites

↳ Basic knowledge of permutations and combinations

Later part of the lecture

↳ FFT and convolutions

Nishchay Manwani

- EnEm at Codeforces
- EnEm at Codechef
- unacademy.com/@EnEm at Unacademy

Combinatorics

- section of mathematics
- deals with ways of arrangements
- countin things

PnC

What is the number of ways to arrange n distinguishable objects?

3 4 1 2

A. $1 \times 2 \times 3 \times \ldots (n-1)$ $\rightarrow (n-1)!$

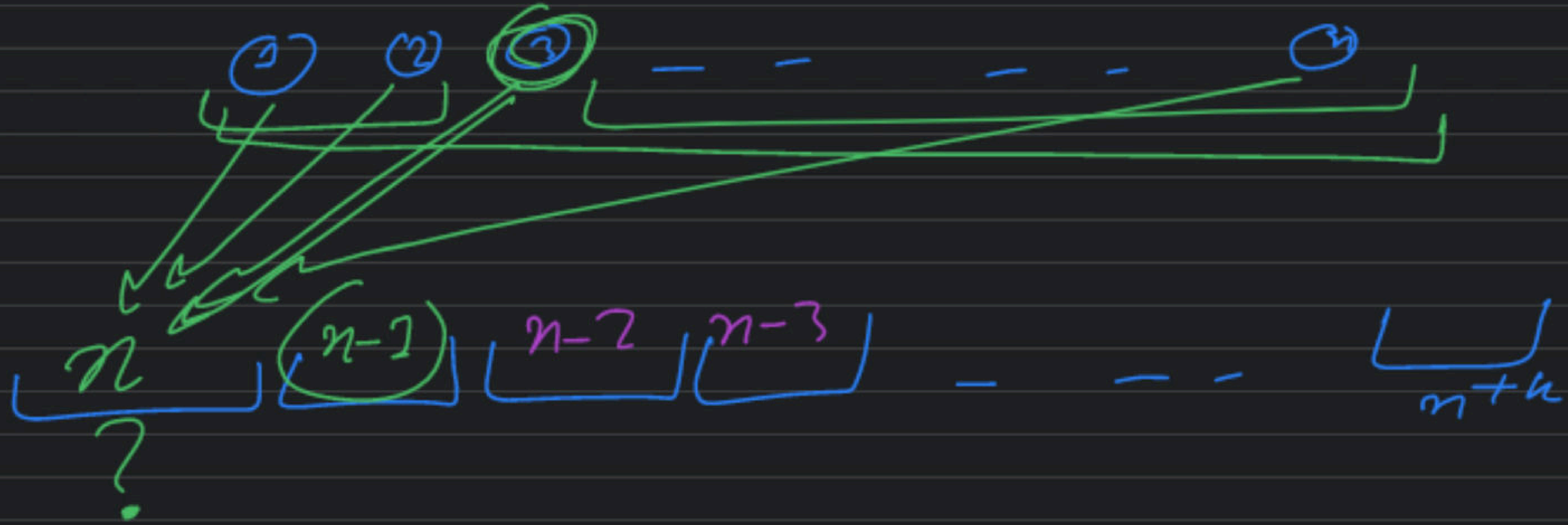
B. $n \times n \times n \times \ldots\ n$ (n times) $\rightarrow n^n$

C. $\boxed{1 \times 2 \times 3 \times \ldots\ n \rightarrow n!}$

D. $\underbrace{2 \times 2 \times 2 \times \ldots\ 2}_{2^n}$ (n times) $\rightarrow 2^n$

4 5

A B C D



$n \times (n-1) \; - \; - \; - \; 1 \Rightarrow n!$

Combinations

$$^{n}C_{k} \Rightarrow \frac{n!}{k!\,(n-k)!}$$

Ex. 1 2 3 4 5 → 2 $\Rightarrow$ $^{5}C_{2} \Rightarrow \frac{5!}{2! \times 3!} \Rightarrow \frac{5 \times 4}{2}$

$\Rightarrow 10$

1,2   1,3   1,4   1,5

2,3   2,4   2,5   3,4   3,5

4,5

Binomial coefficients

$$(1+x)^n \Rightarrow {}^nC_0 \times x^0 + {}^nC_1 \times x^1 + \dots$$

$$\dots\ {}^nC_i \times x^i + \dots$$

$$+\ {}^nC_n \times x^n$$

$${}^nC_k \Rightarrow \binom{n}{k}$$

What is the number of ways to ~~pick~~ put n distinguishable objects in m distinguishable sets when each set can have more than one object?

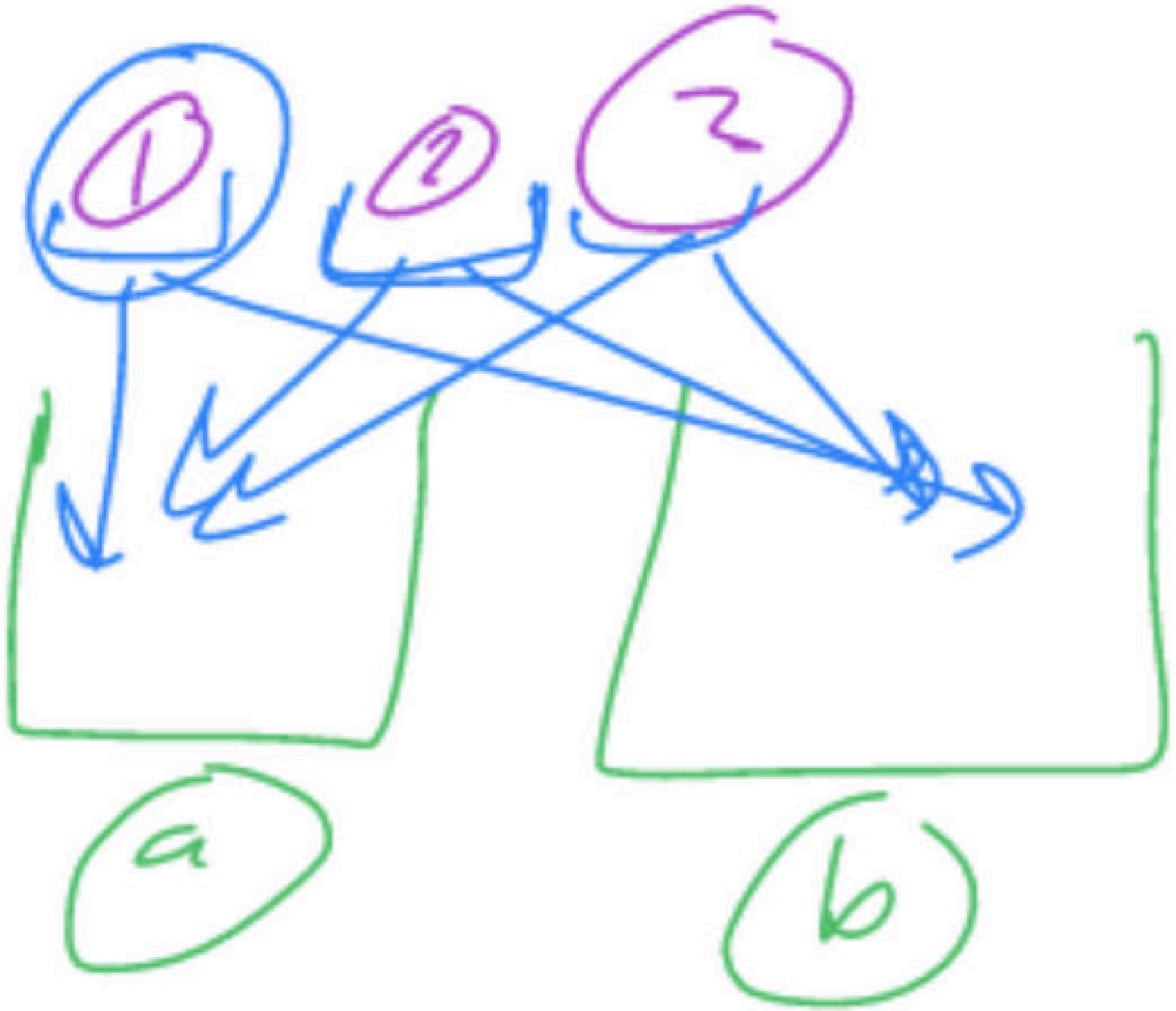
A. ${}^{n}C_{m}$

B. m^{n}

C. n^{m}

D. ${}^{n}C_{m} \times m!$

Sol.



A B C D

$$\underbrace{m} \times \underbrace{m} \cdots \cdots \underbrace{m}$$

$$m^n$$


1
2
3
1
3
2
1
2
3
2
3
1

What is the number of ways to pick n distinguishable objects in m distinguishable sets when each set has only one object?

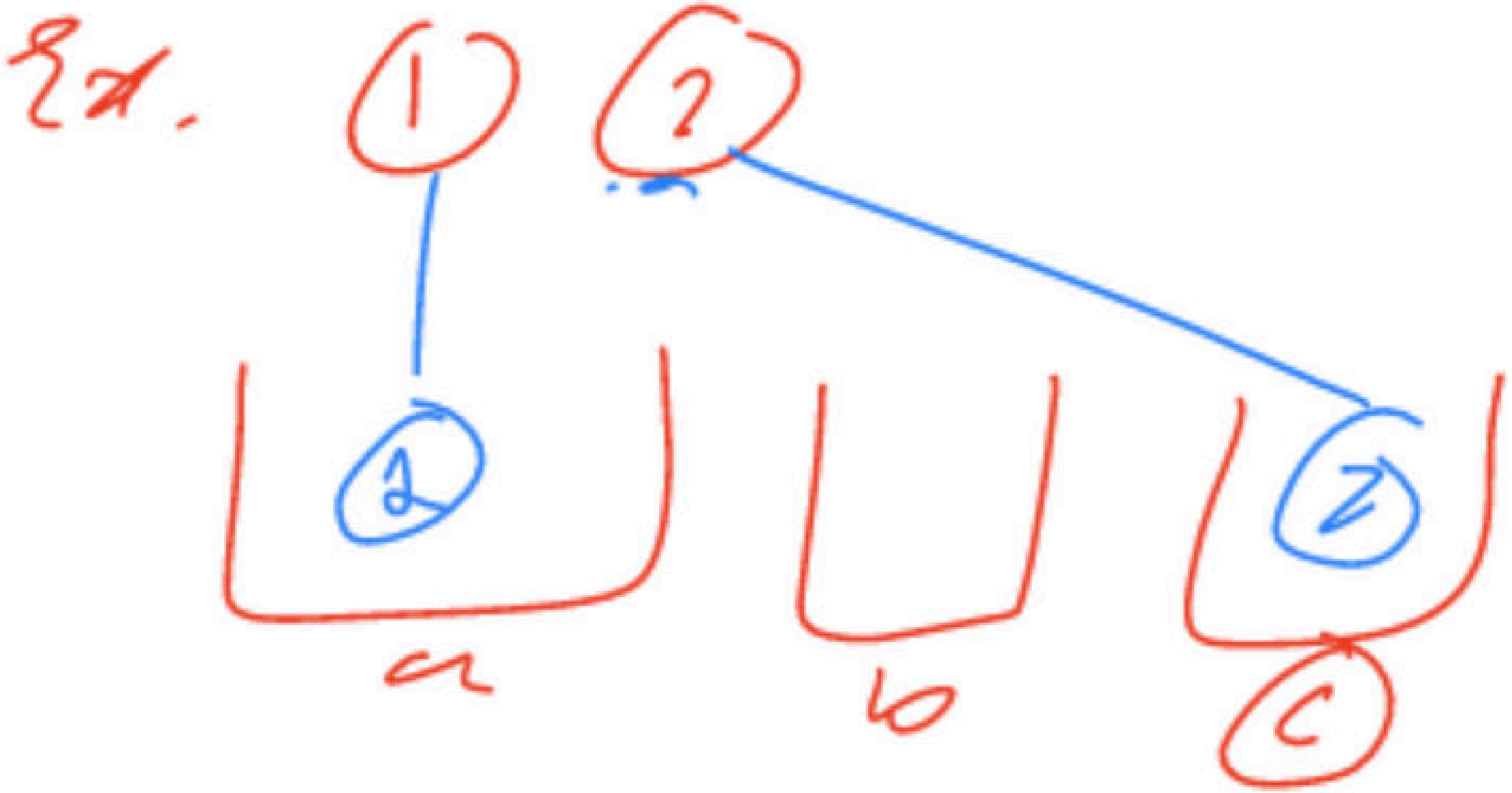
or zero object

A. ${}^{m\,n}C_{\cancel{m}\,n}$

B. m^n

C. n^m

D. $\boxed{{}^{m\,n}C_{\cancel{m}\,n} \times \cancel{m}\,n!}$



A B C D

$$n \times (n-1) \times (n-2)$$

$$\hookrightarrow n \times (n-1) \times (n-2) \; \text{- - - -} \; (n-(r-1))$$

$$0 \qquad 1 \qquad 2 \qquad \cdots \qquad r-1$$

$$\hookrightarrow \frac{n!}{(n-r)!} \qquad \frac{n!}{\cancel{r!}\,(n-r)!} \times \cancel{r!}$$

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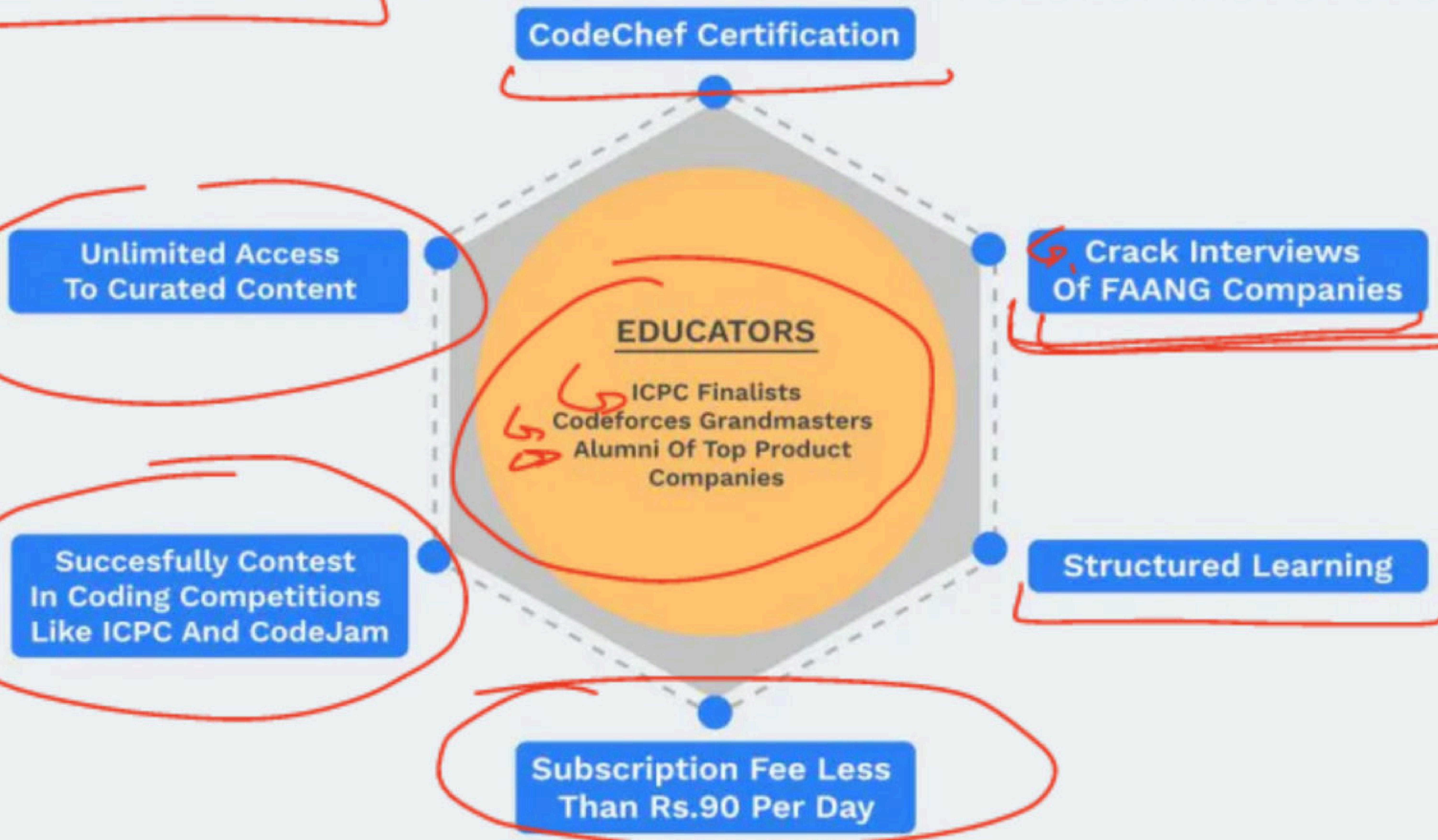

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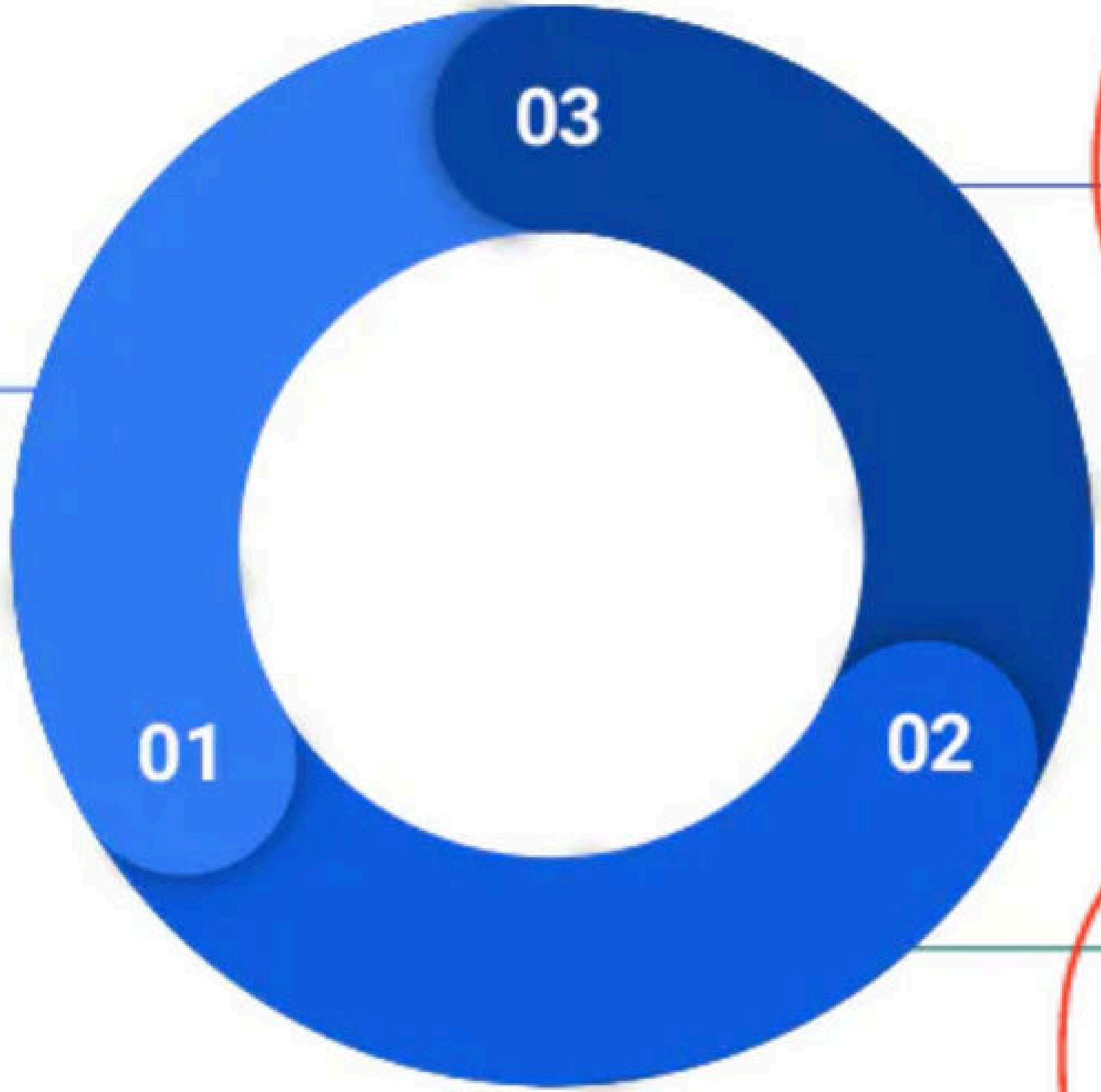

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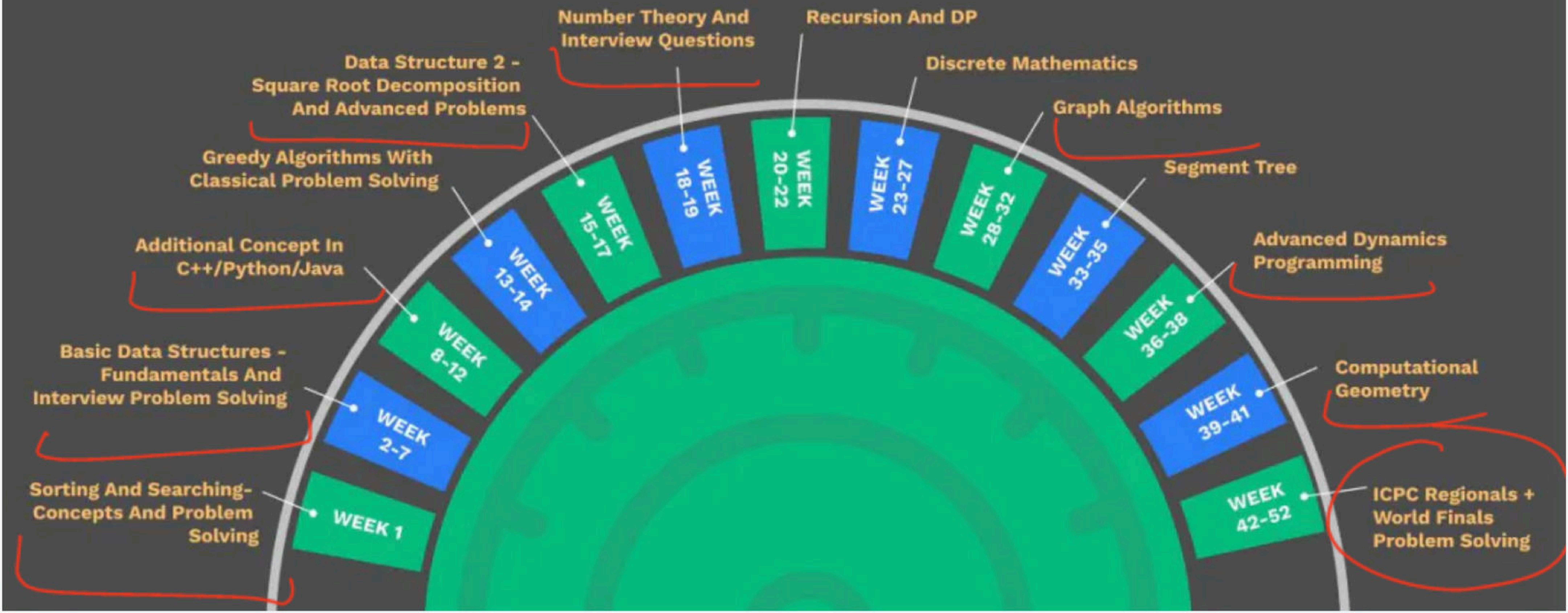

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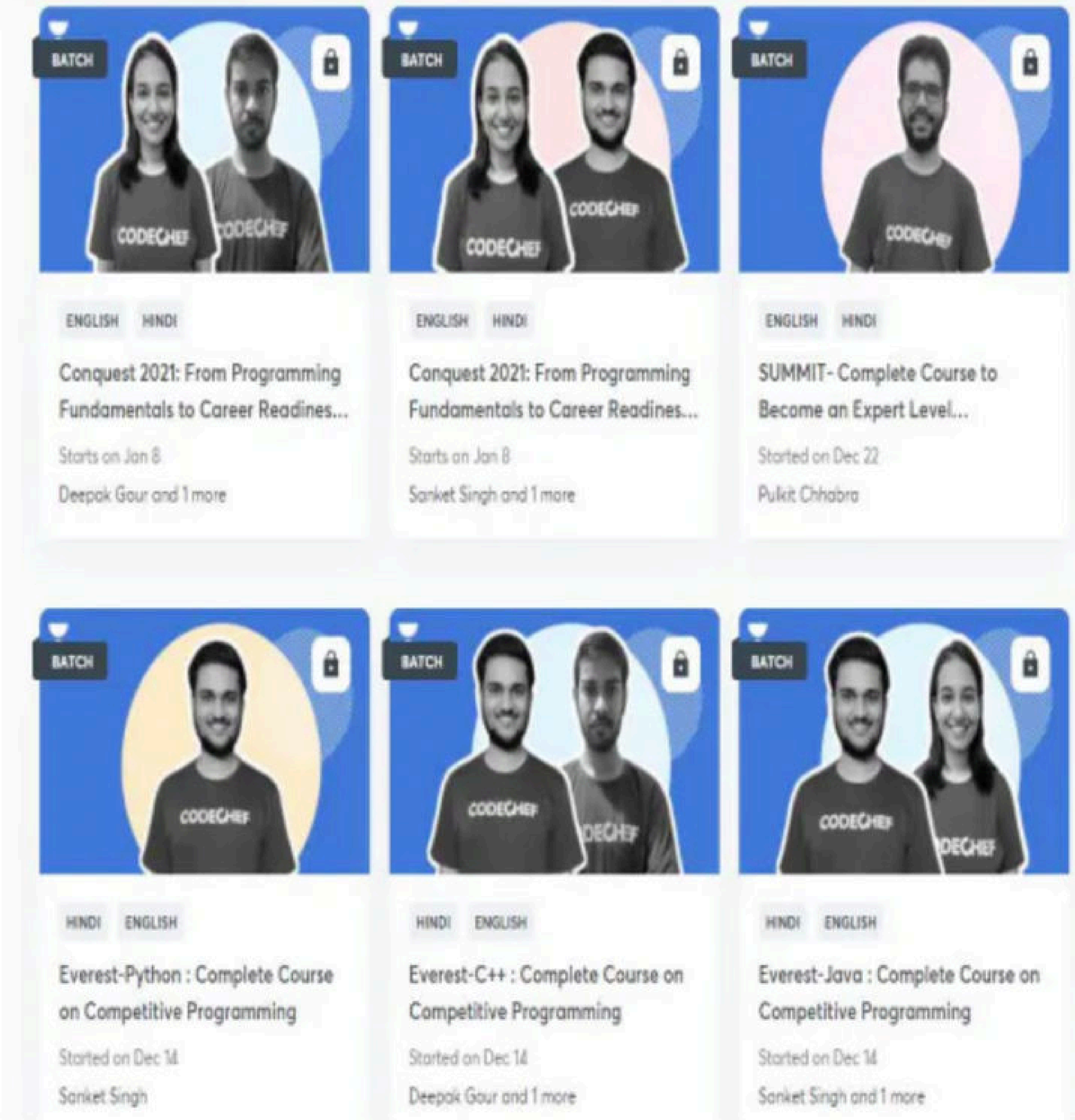
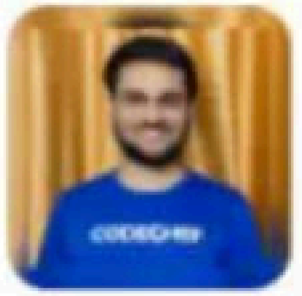


Educators

Tanuj Khattar

ACM ICPC World Finalist - 2017, 2018. Indian IOI Team Trainer 2016-2018. Worked @ Google, Facebook, HFT. Quantum Computing Enthusiast.



Sanket Singh

Software Development Engineer @ LinkedIn | Former SDE @ Interviewbit | Google Summer of Code 2019 @ Harvard University | Former Intern @ISRO

Pulkit Chhabra

Codeforces: 2246 | Codechef: 2416 | Former SDE Intern @CodeNation | Former Intern @HackerRank

Riya Bansal

Software Engineer at Flipkart | Former SDE and Instructor @ InterviewBit | Google Women TechMakers Scholar 2018

Triveni Mahatha

Qualified ICPC 2016 World Final. Won multiple Codechef Long Challenges (India). ICPC Onsite Regionals' Problem setter and Judge. IIT Kanpur.

Deepak Gour

ICPC World Finalist 2020 | Former Instructor @InterviewBit | Software Engineer at AppDynamics

Educators

Himanshu Singh

World Finalist ICPC 2020, Winner Techgig Code Gladiators 2020, Winner TCC '19, 2020 CSE Graduate from IIT BHU, Works at Nutanix

Murugappan S

Software engineer at Google. Have won many programming contests. Max Rating of 2192 in codeforces and 2201 in codechef.

Nishchay Manwani

Hey I am Nishchay Manwani from CSE, IIT Guwahati and I'm a Seven star on Codechef and International Grandmaster on Codeforces.

Vivek Chauhan

Codechef: 7 stars (2612) India Rank 6, Codeforces: MASTER (2279), Won Codechef Long Challenges(India), TCO20 Southern Asia Runner up

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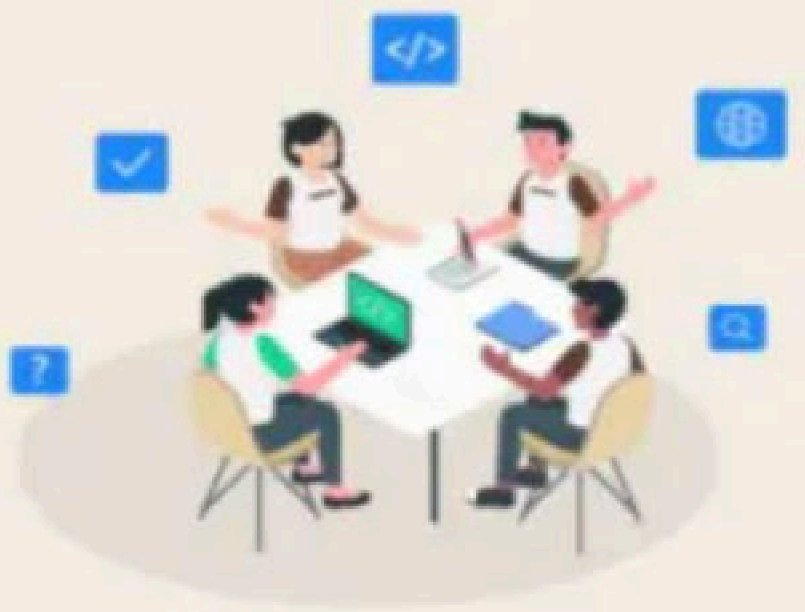

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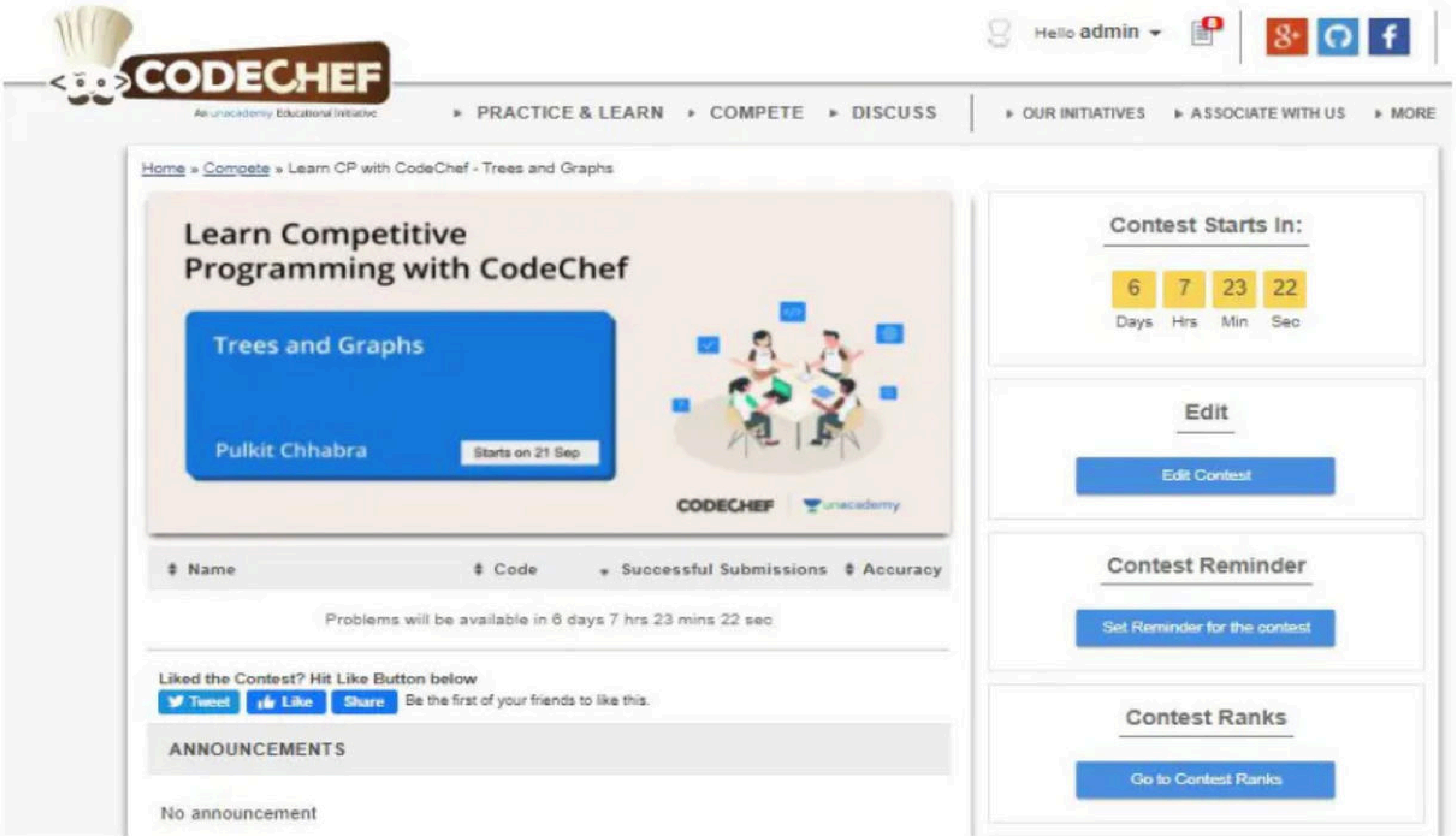
Course-wise Practice Problems



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What is the number of ways to arrange n distinguishable objects?

A. 1 x 2 x 3 x ... (n-1)

B. n x n x n x ... n (n times)

C. 1 x 2 x 3 x ... n ✓ $\rightarrow n!$

D. 2 x 2 x 2 x ... 2 (n times)

What is the number of ways to pick n distinguishable objects in m distinguishable sets when each set can have more than one object?

A. ${}^{n}C_{m}$

B. m^{n}

C. n^{m}

D. ${}^{n}C_{m} \times m!$

What is the number of ways to pick n distinguishable objects in m distinguishable sets when each set has only one object?

A. $^{n}C_{m}$

B. m^{n}

C. n^{m}

D. $\cancel{^{n}C_{m} \times m!}$ $\quad$ $^{m}C_{n} \times n!$

What is the number of ways to pick n indistinguishable objects in m distinguishable sets when each set has only one object?

or zero objects

~~A. $^{n}C_{m}$~~ $^{m}C_{n}$

B. m^n

C. n^m

D. $^{n}C_{m} \times \cancel{m!}$ $^{m}C_{n} \times n!$

Eg: 2

$m \times (m-1) \times \ldots \, m-(n-1)$

$\frac{m!}{(m-n)!} \times \frac{1}{n!}$

$^{m}C_{n}$

$m < n$

$$\binom{m}{n} = 0 \qquad m < n$$

$1 \quad - \quad - \quad - \quad m$

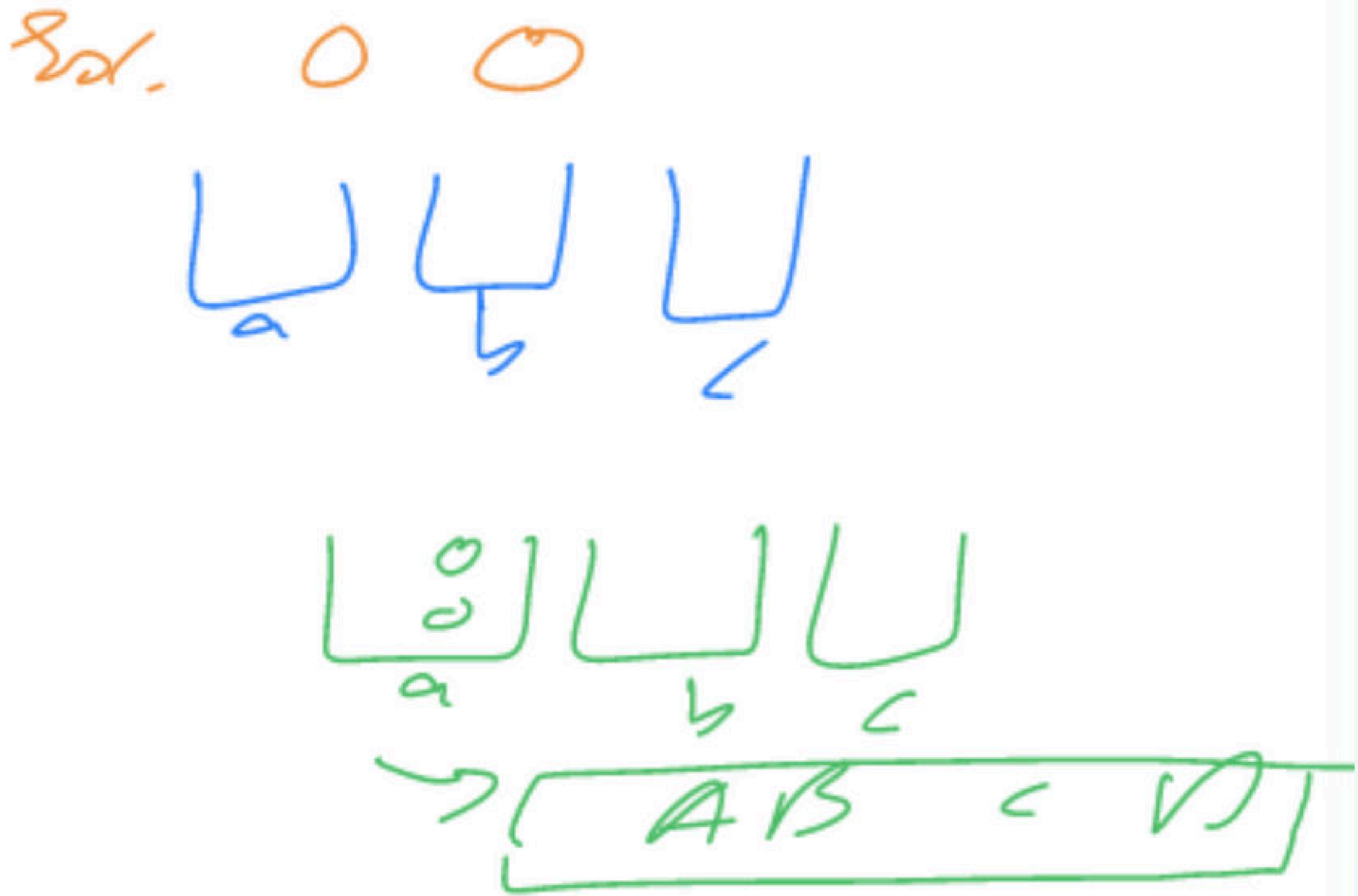
What is the number of ways to ~~pick~~ put n indistinguishable objects in m distinguishable sets when each set can have more than one object?

A. $^{n+m-1}C_m$

B. $(n+m)^n$

C. $(n+m)^m$

D. $^{n+m-1}C_m \times m!$



~~lines~~ n+1 lines

| 000 | 0 | x | 0000 | 00 | 0 |

$$\frac{(m+n-1)!}{m! \times (n-1)!}$$

$$\Rightarrow \boxed{{}^{m+n-1}C_m}$$

$$\underbrace{x_1 + x_2 - - - x_m = n}$$

$$x_i \geq 0$$

What is the number of ways to pick 4 distinguishable objects in 2 indistinguishable non-empty sets?

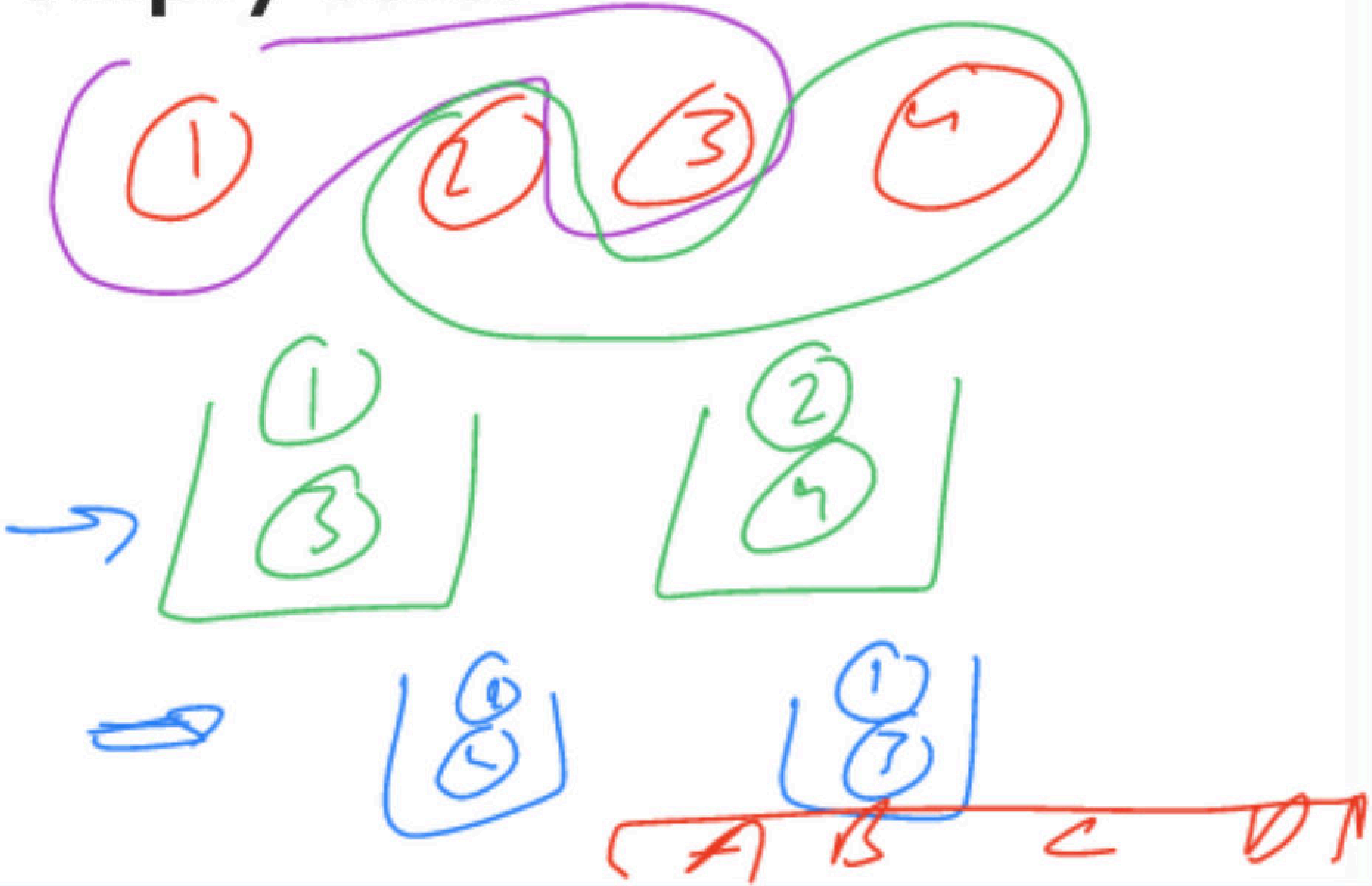
A. 4

B. 5

C. 6

D. 7

Sol.



Ls 1, 2, 3, 4 $\to$ $\{1, 2\}$ $\{3, 4\}$

$\to$ $\{1, 3\}$ $\{2, 4\}$

$\to$ $\{1, 4\}$ $\{2, 3\}$

$\to$ $\{1\}$ $\{2, 3, 4\}$

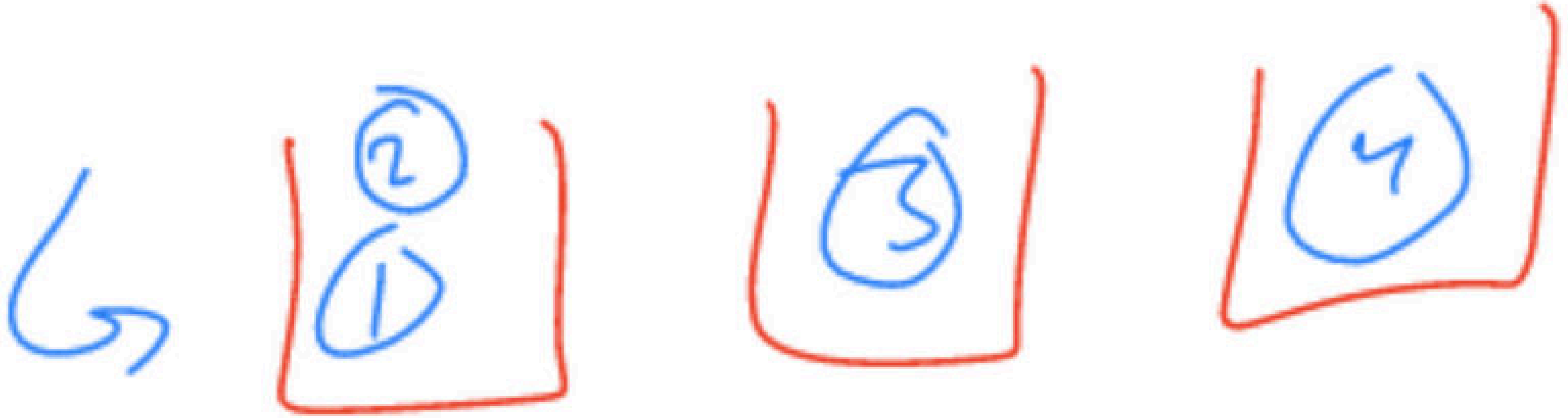
$\to$ $\{2\}$ $\{1, 3, 4\}$

$\to$ $\{3\}$ $\{1, 2, 4\}$

$\to$ $\{4\}$ $\{1, 2, 3\}$

What is the number of ways to pick 4 distinguishable objects in 3 indistinguishable non-empty sets?

1 2 3 4



A. 4

B. 5

C. 6

D. 7

$\binom{4}{2}$

2 + 1 + 1

$\frac{4!}{2!\,2!} \Rightarrow 6$

A B C D

→ partition n distiguishable objects into k in — ॥ — non-empty sets

↳ stirling no. of second kind

$$S(n, k) = \left\{ {n \atop k} \right\}$$

if $n<k$: $\left\{ {n \atop k} \right\} \rightarrow 0$

if $k<0$: $\left\{ {n \atop k} \right\} \rightarrow 0$

Recurrence of stirling no. of second kind

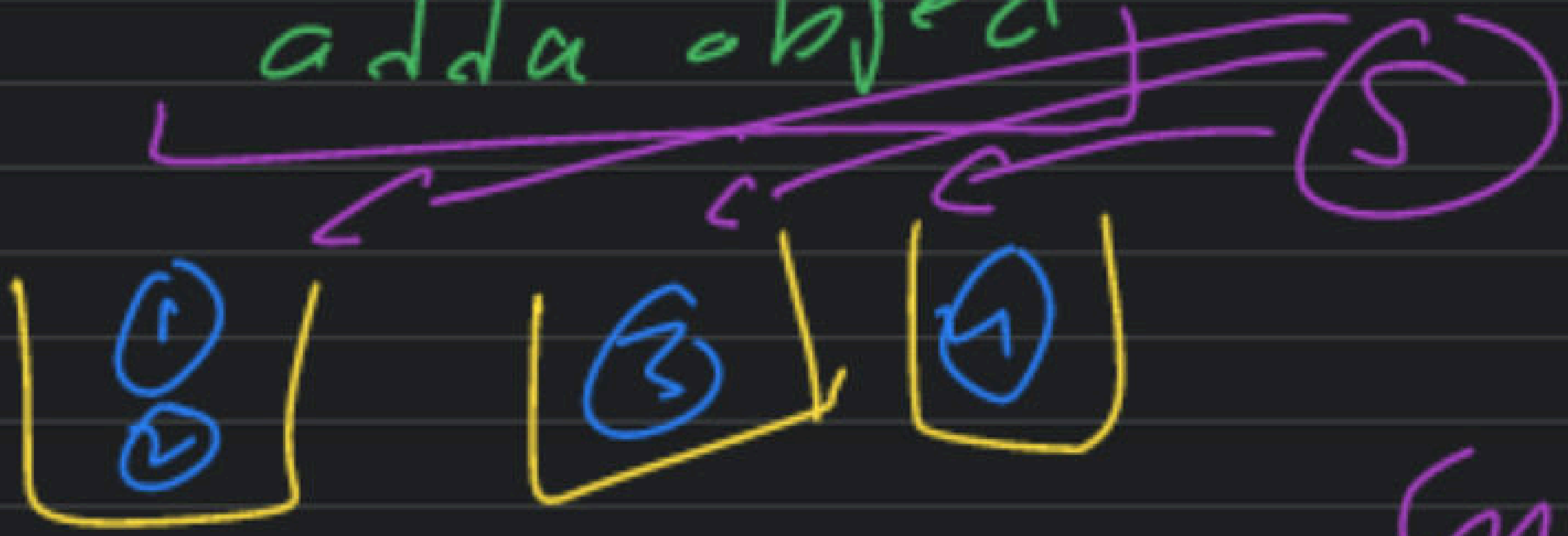
$$\boxed{k \times \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} = \left\{ {n+1 \atop k} \right\}}$$

$\underbrace{\qquad\qquad\qquad}_{n}$ ' $\underbrace{\qquad\qquad\qquad}_{n}$

$\swarrow$

k sets

add a object



$$\left\{ {n+1 \atop k} \right\} \longleftarrow k \times \left\{ {n \atop k} \right\}$$

$\swarrow$

$k-1$ sets

$$\underline{\left\{ {n \atop k-1} \right\}}$$

$$\left\{ {n+1 \atop k} \right\} \longleftarrow \left\{ {n \atop k-1} \right\}$$

n

k \ n	1	2	3	4	5	6	-	-	-	-
1	1	1	1	1	1	1	1	1	1	1
2	0	1	3	7	15	31				
3	0	0	1	6	25	90				
4	0	0	0	1	10	65				
5	0	0	0	0	1	15				
6	0	0	0	0	0	1				

$\times k$

$1 \times 3 = 3$

$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$

How

preprocessing $\underbrace{O(n \times k)}$

query time $\underbrace{O(1)}$

↳ $$\left\{ n \atop k \right\} = \frac{1}{k!} \times \sum_{i=0}^{k} \left((-1)^i \times \binom{k}{i} \times (k-i)^n \right)$$

↳ ~~combinatorial proof~~

↳ ~~inclusion - exclusion~~

↳ induction

$K \rightarrow 1$, $\left\{ {n \atop 2} \right\} \rightarrow 1$

$$\left\{ {n \atop 1} \right\} = \frac{1}{1!} \times \sum_{i=0}^{1} (-1)^i \times \binom{1}{i} \times (1-i)^n$$

$$\hookrightarrow \quad \underline{(-1)^0} \times \underline{\binom{1}{0}} \times \underline{1^n} + \underline{(-1)^1} \times \underline{\binom{1}{1}} \times \underline{0^n}$$

$$\hookrightarrow \quad 1 + 0$$

$$\rightarrow \quad \underline{1}$$

$$k \times \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \Rightarrow \left\{ {n+1 \atop k} \right\}$$

$$k \times \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \Rightarrow k \times \frac{1}{k!} \times \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

$$+ \frac{1}{(k-1)!} \times \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-1-i)^n$$

$$\Rightarrow \frac{1}{(k-1)!} \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n + \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-1-i)^n \right)$$

$$\Rightarrow \frac{1}{(k-1)!} \times \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n + \sum_{i=1}^{k} (-1)^{i-1} \binom{k-1}{i-1} (k-i)^n \right)$$

$$\Rightarrow \frac{1}{(k-1)!} \times \left((-1)^0 \times \binom{k}{0} \times k^n + \sum_{i=1}^{k} \left((-1)^i \binom{k}{i} (k-i)^n + (-1)^{i-1} \binom{k-1}{i-1} (k-i)^n \right) \right)$$

$$\Rightarrow \frac{1}{(k-1)!} \times \left(k^n + \sum_{i=1}^{k} \left((k-i)^n \times \left(\binom{k}{i} - \binom{k-1}{i-1} \right) \times (-1)^i \right) \right)$$

$$\hookrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\hookrightarrow \binom{n+1}{k} - \binom{n}{k-1} = \binom{n}{k}$$

$$\Rightarrow \frac{1}{(k-1)!} \times \left(k^n + \sum_{i=1}^{k} \left((-1)^i \times (k-i)^n \times \binom{k-1}{i} \right) \right)$$

$$\binom{k-1}{i} = \frac{(k-1)!}{i!\,(k-1-i)!} \times \frac{k}{k-i} \times \frac{k-i}{k}$$

$$\binom{k-1}{i} = \binom{k}{i} \times \frac{k-i}{k}$$

$$\Rightarrow \frac{k!}{i!\,(k-i)!} \Rightarrow \binom{k}{i} \times \frac{k-i}{k}$$

$$\Rightarrow \frac{1}{(k-1)!} \times \left(\frac{k^{n+1}}{k} + \sum_{i=1}^{k} \left((-1)^i \times \binom{k}{i} \times \frac{k-i}{k} \times (k-i)^n \right) \right)$$

$$\Rightarrow \frac{1}{k!} \times \left(k^{n+1} + \sum_{i=1}^{k} \left((-1)^i \times \binom{k}{i} \times (k-i)^{n+1} \right) \right)$$

$$\Rightarrow \left\{ {n+1 \atop k} \right\} \Rightarrow \frac{1}{k!} \times \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^{n+1}$$

$$\frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \; \% \; b$$

Time complexity

$\begin{Bmatrix} n \\ k \end{Bmatrix} \rightarrow O(K)$ per query

$$n! \qquad \frac{1}{n!}$$

$\leftarrow$ preprocess

Q. What are the values for

arr$_{ab}$ all i

n $\quad$ $0 \le i \le n$

$\boxed{O(n^2)}$

$\left\{ \begin{matrix} n \\ i \end{matrix} \right\}$ $\quad$ $O(n \log n)$

$\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}$ $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\}$ $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\}$ $\left\{ \begin{matrix} n \\ 3 \end{matrix} \right\}$ — — $\left\{ \begin{matrix} n \\ n \end{matrix} \right\}$

$O(1)$ $\quad$ $O(2)$ $\quad$ $O(3)$ $\quad$ $O(n)$

$\boxed{O(n^2)}$

prequisites

↳ polynomial multiplication using FFT

Convolutions

$$f(x) = \sum_{i=0}^{x} g(i) \times h(x-i)$$

$0 \le x \le n \rightarrow O(n^2)$

$\rightarrow O(n \log n)$

$$\left(g(0)x^0 + g(1)x^1 - - - g(n)x^n\right)$$

FFT $\times$

$$\left(h(0)x^0 + h(1)x^1 - - - h(n)x^n\right)$$

$$f(0)x^0 + f(1)x^1 + f(2)x^2 - - - -$$

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \times (k-i)^n$$

$$\Rightarrow \frac{1}{\cancel{k!}} \sum_{i=0}^{k} (-1)^i \times \frac{\cancel{k!}}{i! \times (k-i)!} (k-i)^n$$

$$\Rightarrow \left\{ {n \atop k} \right\} = \sum_{i=0}^{k} \frac{(-1)^i}{i!} \times \frac{(k-i)^n}{(k-i)}$$

$$g(i) \rightarrow \frac{(-1)^i}{i!} \qquad h(i) \rightarrow \frac{i^n}{i!}$$

$$g(0)x^0 + g(1)x^1 + \cdots g(n)x^n$$

$$\times \quad h(0)x^0 + h(1)x^1 - - \quad h(n)x^n$$

$O(n\log n)$

$$\underbrace{\left\{ {n \atop 0} \right\} x^0 + \left\{ {n \atop 2} \right\} x^1 + \cdots \left\{ {n \atop n} \right\} x^n + \cdots}$$

$O(n\log n)$

preprocessing

$O(1)$ per query

k

n

Stirling no. of first kind

$O(n^2)$ $O(n\log^2 n)$ $\rightarrow O(n\log n)$

easier

Factorial of a number in less than $\boxed{O(n)}$

$n! \to \underbrace{1 \times 2 \times 3 - - - \cdot n}$

<1e6

1e9

1e12

EnEm