FFT and Convulutions

Special class

Nishchay Manwani • Jan 11, 2021

Nishchay Manwani

- EnEm at Codeforces
- EnEm at Codechef
- unacademy.com/@EnEm at Unacademy

Prerequisites

- Divide and conquer
- Complex numbers
- Number theory / modulo operations (inverse, ordoni))

Polynomial multipliction

$p_1(x) \rightarrow a_0 x^0 + a_1 x^1 \ldots\ldots a_{n-1} x^{n-1}$

$p_2(x) \rightarrow b_0 x^0 + b_1 x^1 \ldots\ldots b_{m-1} x^{m-1}$

$p(x) \rightarrow p_1(x) \times p_2(x) \rightarrow$

?

$c_0 x^0 + c_1 x^1 + c_2 x^2 \ldots\ldots c_{n+m-2} x^{n+m-2}$

Eg. $(1 + x^2) \times (1 + 3x)$

$1 + 0x + 1x^2$ $\qquad$ $1 + 3x$

$\hookrightarrow \ 1 \times (1 + x^2) + 3x(1 + x^2)$

$\hookrightarrow \ 1 + x^2 + 3x + 3x^3$

$\hookrightarrow \ 1 + 3x + 1x^2 + 3x^3$

$a_0 \quad b_0$

$a_1 \quad b_1$

$\vdots$

$a_{n-1} \quad b_{m-1}$

$0 \quad 0$

$0 \quad 0$

$0 \quad 0$

$0 \quad 0$

$$\boxed{c_i = \sum_{j=0}^{j=i} a_j \times b_{i-j}}$$

$$0 \le i \le n+m-2$$

Ex- $(1+x+0x^2)$ $\quad c_0 \to 1 \quad c_1 \to 3$

$(1+2x+0x^2)$ $\quad c_2 \to 2$

$$c_2 = \sum_{j=0}^{2} a_j \times b_{2-j}$$

$(1+3x+2x^2)$

$\Rightarrow a_0 b_2 + a_1 b_1 + a_2 b_0$

$\Rightarrow 1 \times 0 \quad 1 \times 2 \quad 0 \times 1$

$\hookrightarrow O\left((n+m)^2\right)$

$\hookrightarrow O\left(n^2\right)$

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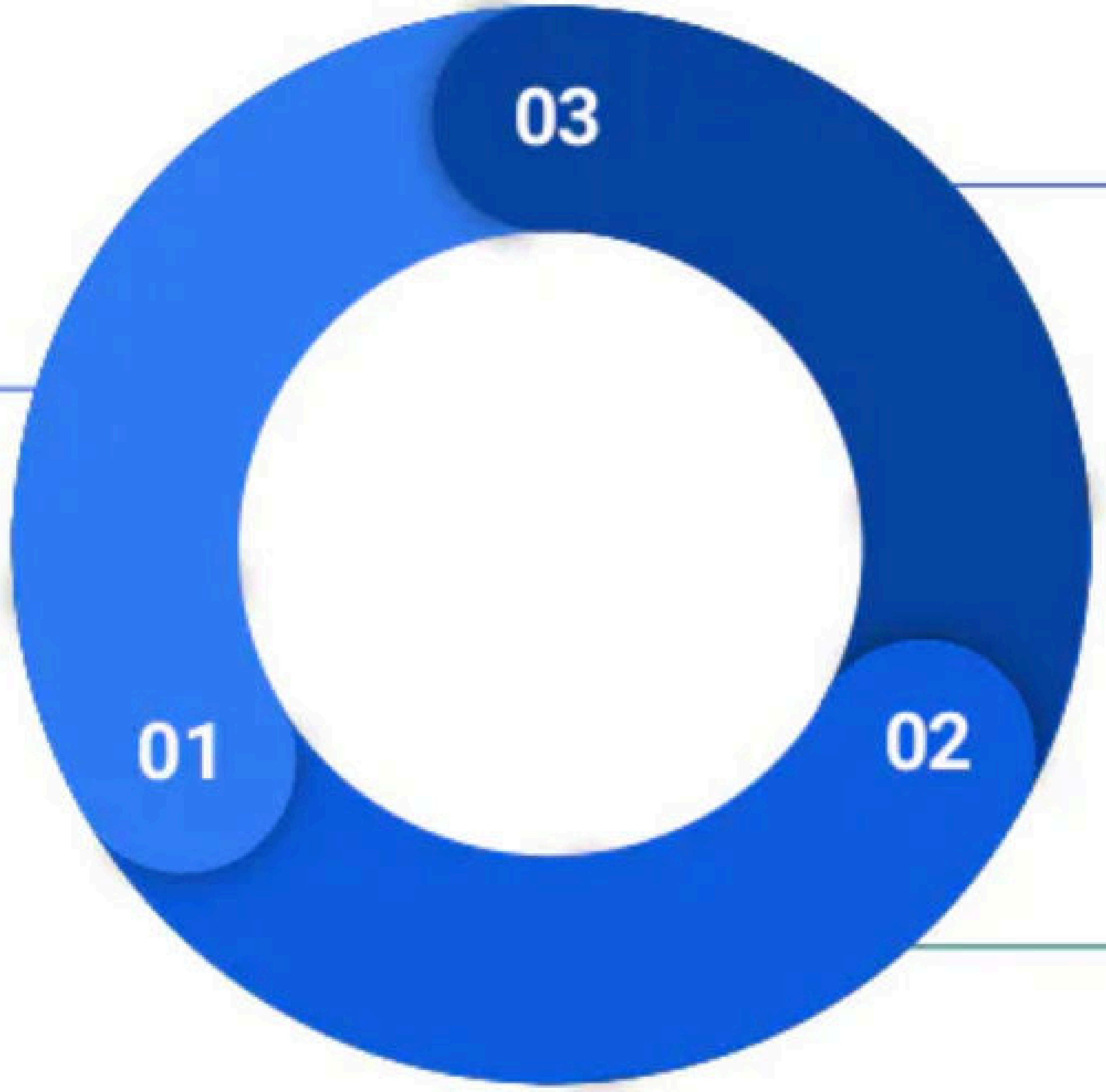

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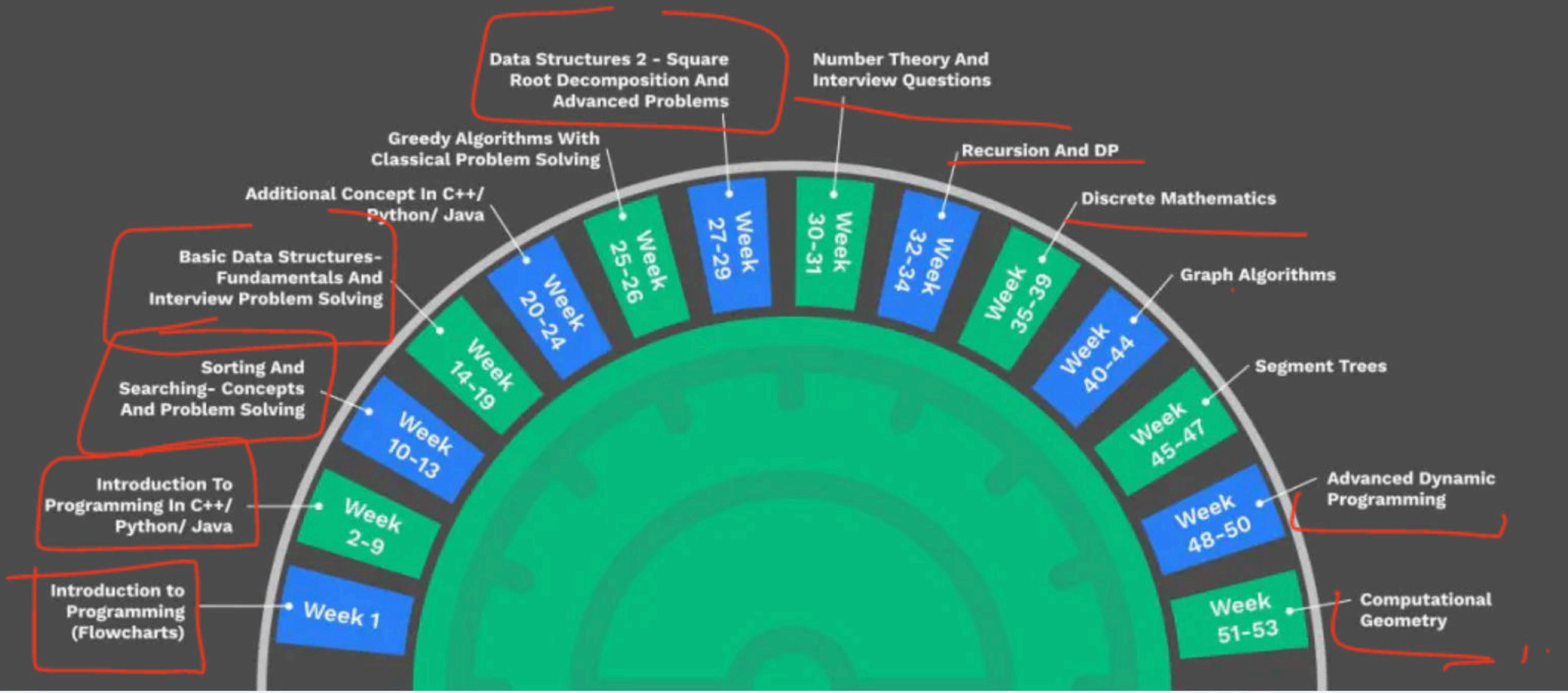
Conquest 2021



Educators

Tanuj Khattar

ACM ICPC World Finalist - 2017, 2018. Indian IOI Team Trainer 2016-2018. Worked @ Google, Facebook, HFT. Quantum Computing Enthusiast.

Sanket Singh

Software Development Engineer @ LinkedIn | Former SDE @ Interviewbit | Google Summer of Code 2019 @ Harvard University | Former Intern @ISRO

Pulkit Chhabra

Codeforces: 2246 | Codechef: 2416 | Former SDE Intern @CodeNation | Former Intern @HackerRank

Riya Bansal

Software Engineer at Flipkart | Former SDE and Instructor @ InterviewBit | Google Women TechMakers Scholar 2018

Triveni Mahatha

Qualified ICPC 2016 World Final. Won multiple Codechef Long Challenges (India). ICPC Onsite Regionals' Problem setter and Judge. IIT Kanpur.

Deepak Gour

ICPC World Finalist 2020 | Former Instructor @InterviewBit | Software Engineer at AppDynamics

Educators

Himanshu Singh

World Finalist ICPC 2020, Winner Techgig Code Gladiators 2020, Winner TCC '19, 2020 CSE Graduate from IIT BHU, Works at Nutanix

Murugappan S

Software engineer at Google. Have won many programming contests. Max Rating of 2192 in codeforces and 2201 in codechef.

Nishchay Manwani

Hey I am Nishchay Manwani from CSE, IIT Guwahati and I'm a Seven star on Codechef and International Grandmaster on Codeforces.

Vivek Chauhan

Codechef: 7 stars (2612) India Rank 6, Codeforces: MASTER (2279), Won Codechef Long Challenges(India), TCO20 Southern Asia Runner up

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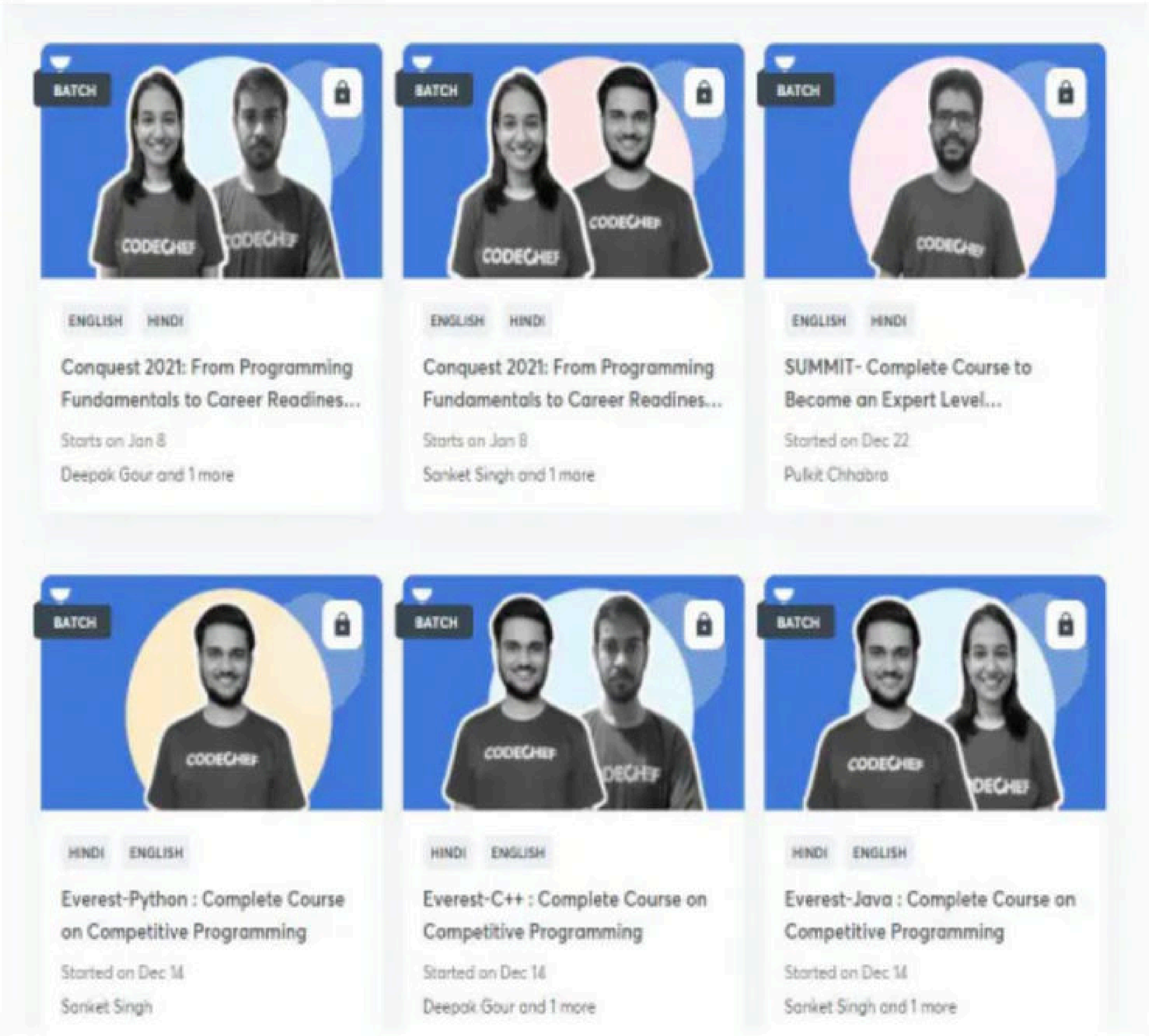
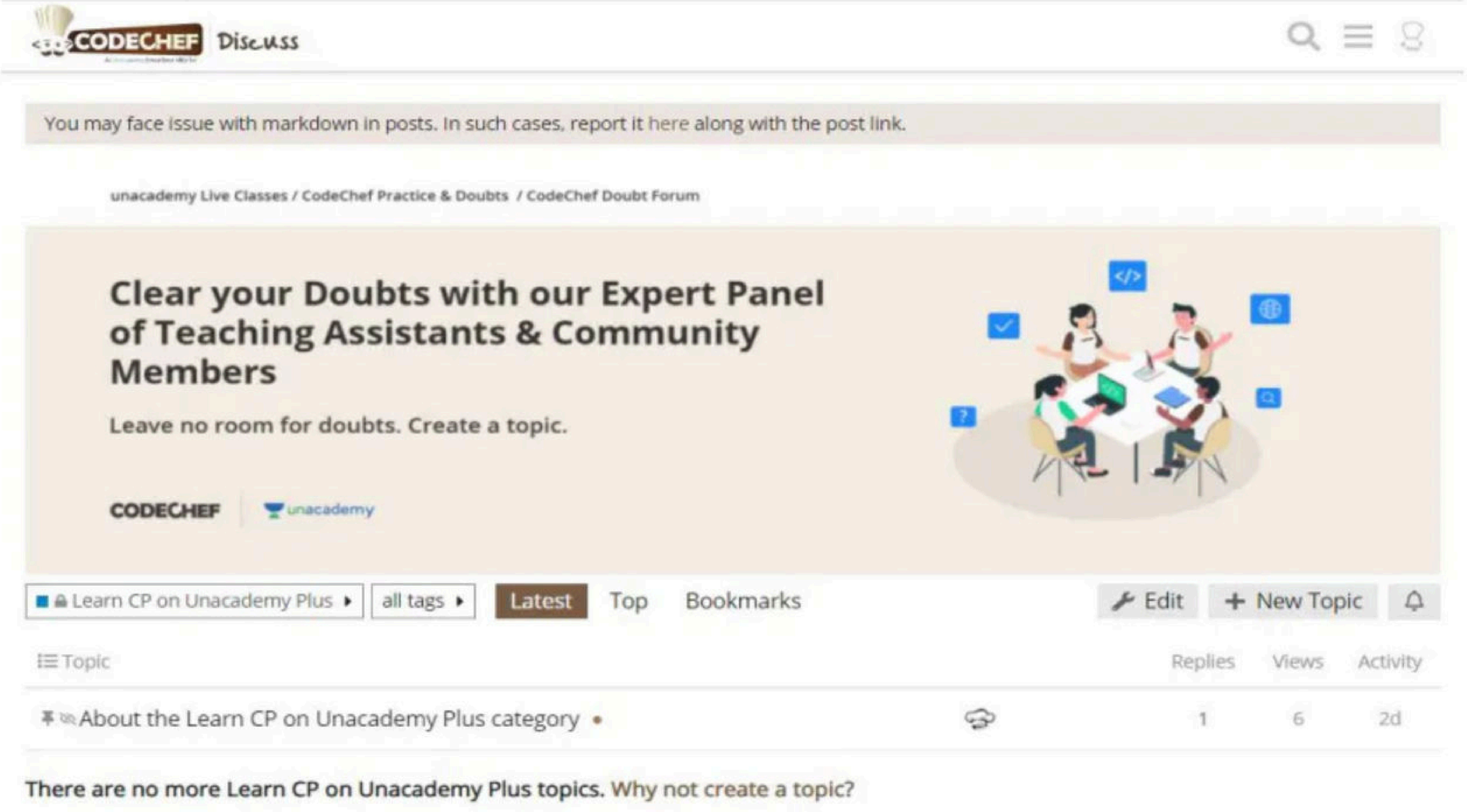
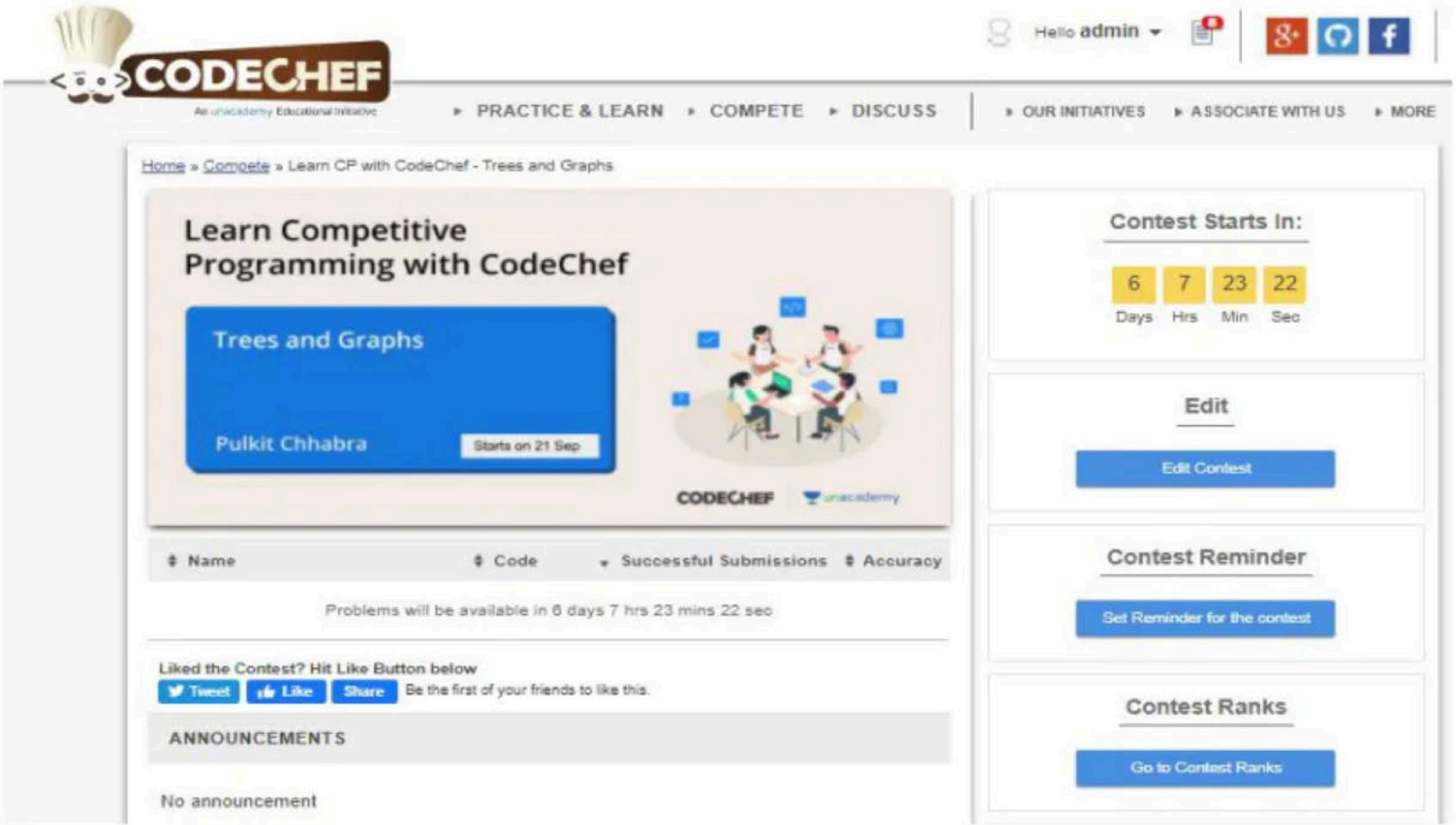


Teaching Assistants support on chat and Doubts Forum



Course-wise Practice Problems



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$\hookrightarrow O\left((n+m)^2\right)$

$$9 \quad 4\ 7\ 8\ 9\ 1\ 5\ 6\ 4 \ldots\ldots 987$$

n digits $\rightarrow$

$$1\ 2\ 3 \ldots\ldots 6$$

n digits $\rightarrow$

$$O(n)$$

$$\begin{array}{r} 1\quad 1\quad \\ 5\ 6\ 7 \\ \times\ 1\ 2\ 7 \\ \hline 1\ 7\ 0\ 1\ 9 \\ 1\ 1\ 3\ 4\ 0 \\ +\ 5\ 6\ 7\ 0\ 0 \\ \hline 6\ 9\ 7\ 4\ 1 \end{array}$$

ans += $d \times n$,

$$O(n)$$

$$O(n^2)$$

n_1 $\quad$ n_1 left $\quad$ n_1 right

n_2 $\quad$ n_2 left $\quad$ n_2 rigt

$$T(n) = 3\,T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = n^{\log_2 3} \approx \underbrace{n^{1.57\ldots}}$$

$\hookrightarrow d_{n-1}$ ______ _ $d_2 \; d_1 \; d_0$

$\hookrightarrow \boxed{d_0 + d_1 \times 10 + d_2 \times 10^2 + \dots \; d_{n-1} \times 10^{n-1}}$

$\hookrightarrow$

Convulutions

$$\boxed{h(x) = \sum_{i=0}^{x} f(i) \times g(x-i)}$$

$p_1(x) \rightarrow f(0)x^0 + f(1)x^1 - \cdots f \ldots$

$p_2(x) \rightarrow g(0)x^0 + g(1)x^1 + \cdots - \cancel{g}$

$p(x) \rightarrow p_1(x) \times p_2(x) \rightarrow h(0)x^0 + h(1)x^1 + h(2)x^2 \ldots$

Ex. $h(x) = \sum_{i=0}^{x} \binom{x}{i}$

$h(5) \rightarrow \binom{5}{0} + \binom{5}{1} - - - \binom{5}{5} \rightarrow 2^5$

$h(x) = \sum_{i=0}^{x} \binom{x}{i}$

$\hookrightarrow h(x) = \sum_{i=0}^{x} \frac{x!}{i!(x-i)!}$

$$\rightarrow \frac{h(x)}{x!} = \sum_{i=0}^{x} \underbrace{\left(\frac{1}{i!}\right)} \times \underbrace{\left(\frac{1}{(x-i)!}\right)}$$

$$b_1 \rightarrow \frac{1}{0!} x^0 + \frac{1}{1!} x^1 - - - - \frac{1}{n!} x^n$$

$$b_2 \rightarrow \frac{1}{0!} x^0 + \frac{1}{1!} x^1 - - - \frac{1}{n!} x^n$$

$$\underline{c_i} \, x^i \qquad c_i = \frac{h(i)}{i!}$$

↳ $O((n+m)^2) \longrightarrow O(nm)$

$p(x) \rightarrow a_0 x^0 + a_1 x^1 - - - - a_{n-1} x^{n-1}$

$(c_1, p(c_1)) \rightarrow (c_2, p(c_2)) - - -$

$O(n)$

$\hookrightarrow O(n)$

$(c_n, p(c_n))$

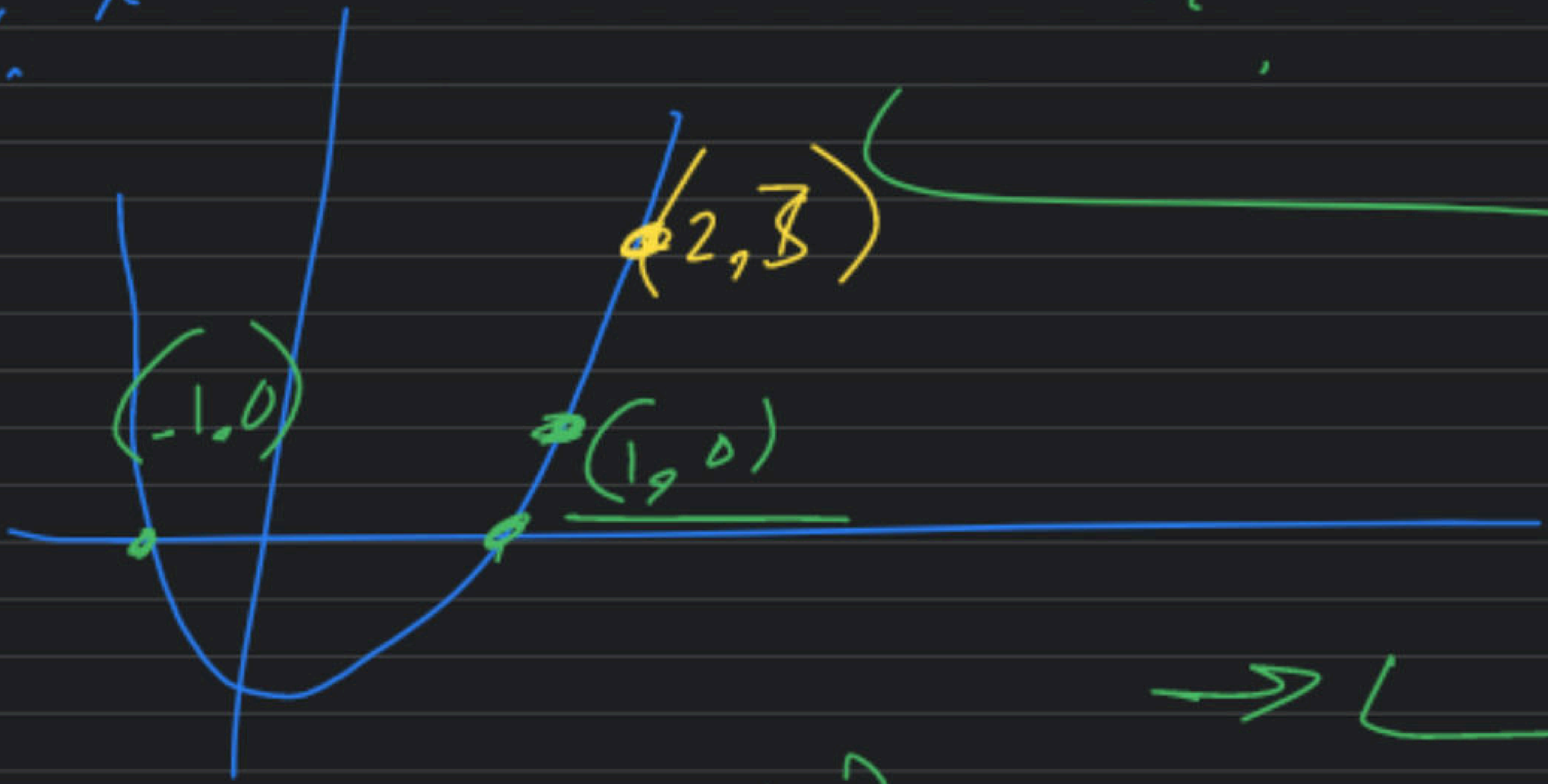
$O(n) \times n$

$\hookrightarrow O(n^2)$

$c_i \neq c_j \; \forall i \neq j$

→ We can find the orig poly of deg $n-1$ if we have n distinct points on it

예1. x^2-1



$x-1$

x

$(x-1)\times(ax+b)$

$(x-1)\times(x+1)\times a$

polynomial

convert $O(n^2)$

$O(n^2)$ <u>interpolation</u>

point-value form
(n points)

$$p_1(x) \rightarrow n-1$$

$$p_2(x) \rightarrow m-1$$

$$p(x) \rightarrow \frac{n+m-2}{n+m-1}$$

$p_1(x)$ $\qquad\qquad$ $p_2(x)$

$O((n+m-1)^2)$ $\qquad$ $\Leftrightarrow$ $O((n+m-1)^2)$

$\underline{O((n+m-1)\log(n+m-1))}$

point value form $\qquad$ multiply $\qquad$ point value form

$(n+m-1$ points$)$ $\qquad$ $O(n+m-1)$ $\qquad$ $(n+m-1$ points$)$

$c_1, p_1(c_1)$ $\qquad\qquad$ $c_1, p_2(c_1)$

$c_2, p_1(c_2)$ $\qquad$ interpolate $\qquad$ $c_2, p_2(c_2)$

$\vdots$ $\qquad\qquad$ $O((n+m-1)^2)$

$c_{n+m-1}, p_1(c_{n+m-1})$ $\qquad\qquad$ $c_{n+m-1}, p_2(c_{n+m-1})$

$p(x)$

$p_1(n) \rightarrow n$

$p_2(n) \rightarrow n$

$\hookrightarrow$ $\underline{O(n \log n)}$

$$p(x) = p_1(x) \times p_2(x)$$

$$p(c_1) = p_1(c_1) \times p_2(c_1)$$

FFT (fast fourier Transform)

poly $\xrightarrow{O(n \log n)}$ point value

c_1 $\quad$ c_2 $\quad$ - - $\quad$ c_n

$$\boxed{x^n = 1}$$ solutions

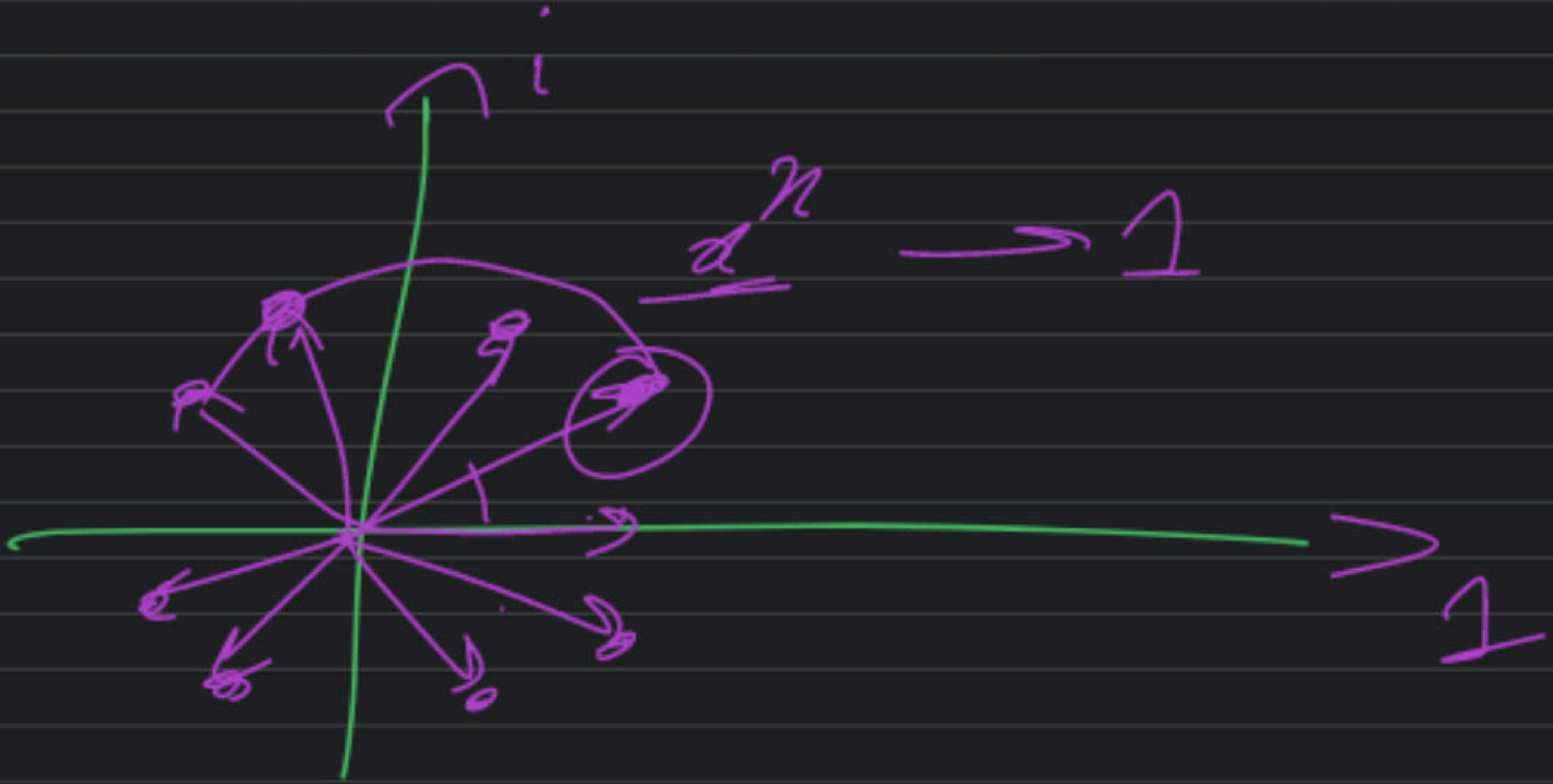
$\hookrightarrow \omega_n^0, \omega_n^1, \omega_n^2, \ldots \omega_n^{n-1}$

$\hookrightarrow \cancel{e^{i\,0\times\frac{1}{2\pi}}, e^{i\,1\times\frac{1}{2\pi}}}$

$$\cancel{W_n^j \Rightarrow e^{i 2\pi \frac{j}{n}}}$$

$$\boxed{W_n^k} \Rightarrow \boxed{e^{i \times 2\pi \frac{k}{n}}}$$

$$\sqrt{-1}$$


i
$\alpha^n \to 1$
1

$$p(x) = a_0 x^0 + a_1 x^1 \cdots \quad a_{n-1} x^{n-1}$$

$$\boxed{n = 2^k}$$

$$\begin{cases} w_n^0 , & p(w_n^0) \\ w_n^1 , & p(w_n^1) \\ \vdots & \\ w_n^{n-1} , & p(w_n^{n-1}) \end{cases}$$

$O(x) \Rightarrow a_1 + a_3 x + a_5 x^2 - - -$

$a_{n-1} x^{\frac{n}{2}-1}$

$E(x) \Rightarrow a_0 + a_2 x + a_4 x^2 - - -$

$a_{n-2} x^{\frac{n}{2}-1}$

$p(x) = E(x^2) + x\, O(x^2)$

$p(x) \to \deg \to n-1 \quad (n = 2^k)$

$O(x) \to \deg \to \frac{n}{2}-1 \quad (\frac{n}{2} = 2^{k-1})$

$E(x) \to \deg \to \frac{n}{2}-1$

$w_{\frac{n}{2}}^{0}, \; O(w_{\frac{n}{2}}^{0})$

$w_{\frac{n}{2}}^{1}, \; O(w_{\frac{n}{2}}^{1})$

$\vdots$

$w_{\frac{n}{2}}^{\frac{n}{2}-1}, \; O(w_{\frac{n}{2}}^{\frac{n}{2}-1})$

$w_{\frac{n}{2}}^{0}, \; E(w_{\frac{n}{2}}^{0})$

$w_{\frac{n}{2}}^{1}, \; E(w_{\frac{n}{2}}^{1})$

$\vdots$

$w_{\frac{n}{2}}^{\frac{n}{2}-1}, \; E(w_{\frac{n}{2}}^{\frac{n}{2}-1})$

$$w_n^0 \quad p(w_n^0) \; --- \; w_n^{n-1}, p(w_n^{n-1})$$

$$\hookrightarrow p(x) = E(a^2) + x\, O(x^2)$$

$$\hookrightarrow p(w_n^k) = E(w_n^{2k}) + w_n^k\, O(w_n^{2k})$$

$$\boxed{k < \frac{n}{2}}$$

$$\hookrightarrow \boxed{p(w_n^k) = E(w_{\frac{n}{2}}^k) + w_n^k\, O(w_{\frac{n}{2}}^k)}$$

$K \geq \frac{n}{2}$

$$p(w_n^k) = E\left((w_n^k)^2\right) + w_n^k \, O\left((w_n^k)^2\right)$$

$$w_n^{\frac{n}{2}} \Rightarrow e^{i 2\pi \frac{n/2}{n}}$$

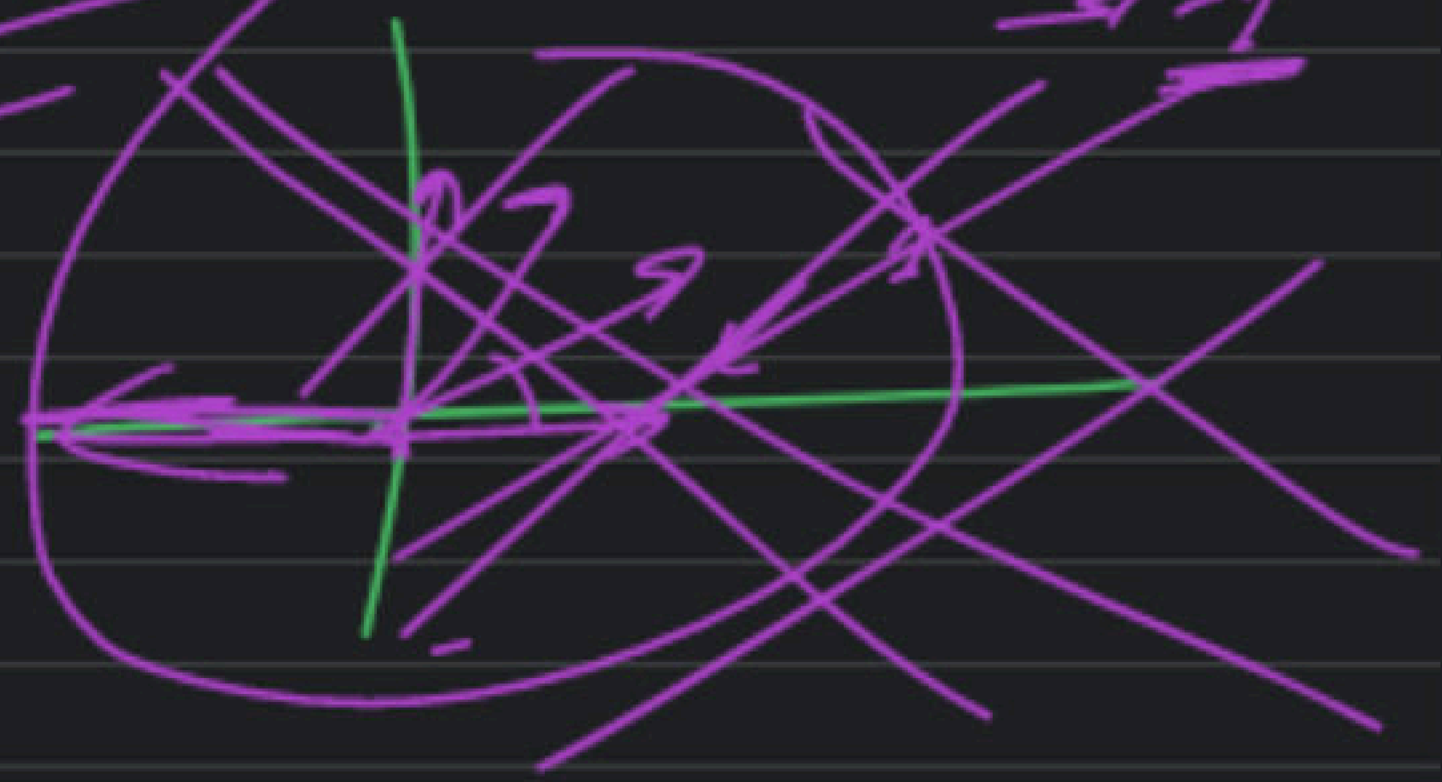
$$\Rightarrow e^{i\pi}$$

$$\Rightarrow \boxed{-1}$$

$\cos(\pi) + i\sin(\pi)$

$\rightarrow -1 + i \times 0$

$\rightarrow -1$



$$\hookrightarrow b(W_n^k) = E\left(\left(W_n^{k-\frac{n}{2}} \times (-1)\right)^2\right) + \left(W_n^{k-\frac{n}{2}} \times (-1)\right)$$

$$+\, O\left(\left(W_n^{k-\frac{n}{2}} \times (-1)\right)^2\right)$$

$$\hookrightarrow W_n^k \rightarrow W_n^{k-\frac{n}{2}} \times (-1)$$

$$\hookrightarrow W_n^{k-\frac{n}{2}} \times W_n^{\frac{n}{2}}$$

$$\boxed{k \geq \frac{n}{2}}$$

$$P(w_n^k) = E(w_{n/2}^{k-\frac{n}{2}}) - w_n^{k-\frac{n}{2}} O(w_{n/2}^{r-\frac{n}{2}})$$

$$T(n) = \underbrace{T\left(\frac{n}{2}\right)}_{E(n)} + \underbrace{T\left(\frac{n}{2}\right)}_{O(x)} + O(n)$$

$$\hookrightarrow T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$\boxed{T(n) = O(n\log n)}$$

$$W_n^k \Rightarrow e^{i 2\pi \frac{k}{n}}$$

$$\left(W_n^k\right)^2 \Rightarrow e^{i 2\pi \frac{2k}{n}} \Rightarrow e^{i 2\pi \frac{k}{\left(\frac{n}{2}\right)}}$$

$$\Rightarrow \boxed{W_{\frac{n}{2}}^k}$$

Ex. $p(x) = 1 + 2\times x + 3\times x^2 + 0\,x^3$

$W_4^0, W_4^1, W_4^2, W_4^3$

$\to 1, i, -1, -i$

$E(x)$ $O(x)$

$1 + 3x$ $\quad$ $4, 2$

$2 + 0x$ $\quad$ $2, 2$

W_2^0, W_2^1

$\to 1, -1$

$E(x)$ $O(x)$ $\quad$ $E(x)$ $O(x)$

1 $\quad$ 3 $\quad$ 2 $\quad$ 0

W_1^0

$\to 1$

$$1 + 3W_2^0 = E(W_1^0) + W_2^0 \times O(W_1^0)$$

$$\to 1 + 3W_2^0 = 1 + 1 \cdot 3 \Rightarrow 4$$

$$p(w_4^0) = E(w_2^0) + w_4^0 \, O(w_2^0)$$

$$\rightarrow 4 + 1 \times 2 \Rightarrow 6$$

$$p(w_4^1) = E(w_2^1) + w_4^1 \, O(w_2^1)$$

$$\hookrightarrow -2 + i \times 2 \Rightarrow -2+2i$$

$$\hookrightarrow p(w_4^2) = \underline{E(w_2^0)} - \underbrace{w_4^0} \, \underline{O(w_2^0)}$$

$$\hookrightarrow p(w_4^3) = \underline{E(w_2^1)} - \underbrace{w_4^1} \, \underline{O(w_2^1)}$$

$$1 + 3 w_2^1 = f(w_1^0) - w_2^0 \, O(w_1^0)$$

$$\Rightarrow \quad 1 - 3$$

$$\Rightarrow -2$$

Interpolation

poly $\xrightarrow{FFT}$ point value

point value $\xrightarrow{\text{inv. FFT}}$ poly

$$
\begin{bmatrix}
w_n^0 & w_n^0 & \cdots & w_n^0 \\
w_n^0 & w_n^1 & \cdots & w_n^{n-1} \\
w_n^0 & w_n^2 \; w_n^4 & \cdots & w_n^{2(n-1)} \\
\vdots & & & \\
w_n^{n-1} & w_n^{2(n-1)} & \cdots & w_n^{(n-1)(n-1)}
\end{bmatrix}
\times
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
p(w_n^0) \\ p(w_n^1) \\ \vdots \\ p(w_n^{n-1})
\end{bmatrix}
$$

o FFT

Wandermonde matrix

$$\begin{bmatrix} w_n^0 & w_n^1 & & \\ & & & \\ & & & \end{bmatrix}_{n \times n}$$

$$\hookrightarrow \underbrace{w_n^{i \times j}}$$

$$\underline{M} \times \underline{a} = \underline{p(a)}$$

$$\hookrightarrow \underbrace{a =} \; \underbrace{M^{-1}} \times \underbrace{p(a)}$$

inverse vandermonde matrix

$$\frac{1}{n} \times$$

$$\begin{bmatrix} w_n^0 & w_n^{-0} & w_n^{-0} & \cdots & w_n^{-(n-1)} \\ w_n^0 & w_n^{-1} & w_n^{-2} & \cdots & w_n^{-(n-1)} \\ w_n^0 & w_n^{-2} & w_n^{-4} & \cdots & \\ & & w_n^{-i \times j} & & \end{bmatrix}$$

$$
\begin{bmatrix}
1 & 0 & 0 & & & & & & 0 \\
0 & 1 & 0 & & & - & & & 0 \\
0 & 0 & 1 & . & - & & - & & 0 \\
 & & & 1 & & & 0 & 0 & 0 \\
 & & & & 1 & & & & \\
 & & & & & & 1 & & \\
0 & & & 0 & & & & 1 & \\
0 & & & 0 & & & & & 1
\end{bmatrix}
$$

$$W_n^0 \times a_0 + w_n^k \times a_1 + W_n^{2k} \times a_2 \cdots$$

$$\cdots w_n^{(n-1)k} \times a_{n-1}$$

$$\hookrightarrow p(w_n^k)$$

$O(n \log(n))$

$\Rightarrow a_0 + a_1 x \cdots\cdots \quad a_{n-1} x^{n-1}$

$<$

$\rightarrow b_1 + b_1 x \cdots\cdots \quad b_{n-1} x^{n-1}$

$\% \; p$

$(x^n = 1)$

$$\underbrace{c_1 \quad c_2 \cdots\cdots \quad c_n}$$

$c_1^n = 1 \quad \% \; p$

$c_2^n = 1 \quad \% \; p$

$\vdots$

$c_n^n = 1$

$\rightarrow p$

$p \nmid g$

$G \quad \underbrace{g^0 \quad g^1 - - - - \quad g^{p-2}}_{\text{distinct}} \quad \underline{g^{p-1} = 1}$

$n = 2^k$

$\hookrightarrow x^n = 1$

$\hookrightarrow x^{2^k} = 1$

$\hookrightarrow g^{p-1} = 1$

$C_1 \Rightarrow g^{\left(\frac{p-1}{2^k}\right)}$

$C_2 \Rightarrow g^{\left(\frac{p-1}{2^k}\right)\times 2}$

$C_3 \Rightarrow g^{\left(\frac{p-1}{2^k}\right)\times 3}$

$\vdots$

$C_n \Rightarrow g^{\left(\frac{p-1}{2^k}\right)\times 2^k} \Rightarrow 1$

$$G\ \underbrace{g^{0},\ g^{1\times\left(\frac{p-1}{n}\right)},\ g^{2\times\left(\frac{p-1}{n}\right)},\ g^{3\left(\frac{p-1}{n}\right)}}_{G}$$

$$\underline{n} \mid p-1$$

$$\frac{p-1}{\underline{2^{k}}} \in Z$$

$\% \; \underbrace{998244353}$ $\swarrow p$

$\% \; (7 \times 17 \times \underline{2}^{23} + 1)$

$p-1 \Rightarrow 7 \times 17 \times \underline{2^{23}}$

$\dfrac{p-1}{\underline{2^{23}}}$ $t = z$

$2^{23}-1$ degree are

$\underbrace{\% \; 10^9 + 7}$

$k-1 \rightarrow \dfrac{10^9 + 6}{2}$

$\hookrightarrow 5 \times 10^8 + 3$

$\dfrac{k-1}{2} \in \mathbb{Z}$ $\qquad$ $\dfrac{k-1}{2l} \notin \mathbb{Z}$

↳ polynomial division

↳ polynomial inverse

↳ multipoint evaluation

↳ lagrange interpolation