EXCEL 21-127
6, 7 April 2020
Countability

Note: We assume $0 \notin \mathbb{N}$ for every mention of $\mathbb{N}$ in the handout.

- What is a countable set?
  A set that is either finite or countably infinite.

- What is a countably infinite set?
  A set that is the same size as the naturals, i.e. there exists a bijection from the set to the set of natural numbers (or from the set of natural numbers to the set).

- Note: Countable infinity is a single, exact size – that is, there is a unique size/amount named "countable infinity". This means that any two sets, regardless of their definitions, have the exact same size if they are both countability infinite.
  This is not the case with uncountable infinity. Two sets who are both uncountable are not necessarily the same size. There are different uncountably infinite sizes.

- What are some common countably infinite sets?
  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Z} \times \mathbb{Z} \ldots$
  The naturals, the integers, the rational numbers, 2-D integer lattice points…

- What are some common uncountable sets?
  $\mathbb{R}, (0,1) \ldots$
  The real numbers, real numbers between 0 and 1 exclusive…

- Note: Any non-empty range of real numbers is uncountable.
  That is, (a, b) where $a < b$.

- How to establish the size relation of two sets A, B?
  - If there is a bijective function $f: A \to B$ or $f: B \to A$, then $|A| = |B|$.
  - If there is an injective function $f: A \to B$, then $|A| \leq |B|$.
  - If there is a surjective function $f: A \to B$, then $|A| \geq |B|$.

- Note: Given two sets A, B if there is an injective function $f_1: A \to B$ and there is a surjective function $f_2: A \to B$, then $|A| = |B|$. Think why?

Construct explicit bijections to show the following.

1. $|\mathbb{N}| = |\mathbb{Z}|$
   Consider $f: \mathbb{N} \to \mathbb{Z}$ via $f(x) = (x+1)/2$ if x is odd, and $f(x) = 1 - x/2$ otherwise.

2. $|\mathbb{N}| = |\{x \in \mathbb{N}: x \text{ is even}\}|$
   Consider $f: \mathbb{N} \to \{x \in \mathbb{N}: x \text{ is even}\}$ via $f(x) = 2x$.

3. $|(0,1)| = |(0,1]|$
   That is, the reals between 0 and 1 exclusive, and the reals between 0 exclusive and 1 inclusive. Consider $f: (0,1) \to (0,1]$ via $f(x) = 2x$ if $x = 1/2^n$ for some $n \in \mathbb{N}$, and $f(x) = x$ otherwise.
   Also consider $g: (0,1) \to (0,1]$ via $g(x) = 3x$ if $x = 1/3^n$ for some $n \in \mathbb{N}$, and $g(x) = x$ otherwise. Both functions are bijections. Prove it?
   In addition, consider $h: (0,1) \to (0,1]$ via $h(x) = 1/n$ if $x = 1/(n+1)$ for some $n \in \mathbb{N}$, and $h(x) = x$ otherwise. It is also a bijection. Prove it? (See page 3.)

Note: In other contexts, we often denote infinity as ∞. Here, infinity is <u>not</u> a number. Rather, ∞ is merely the concept of an infinitely large quantity. However, countable infinity is a very specific quantity, or number. But it is <u>not</u> a natural number, <u>nor</u> a real number.

Countable infinity is what we call a “cardinal number” in mathematics. Cardinal numbers are simply the numbers to be used in counting things. That is, cardinal numbers are sizes of sets. So, all the natural numbers are cardinal numbers. So is countable infinity.

Now, we should not use the symbol ∞ to denote countable infinity. Usually, we just write $|\mathbb{N}|$, which means the size of the set of natural numbers, which is exactly countable infinity. If you are curious, there is a special symbol to denote this quantity. $\aleph_0$ reads “Aleph null”.

We now prove function $h: (0,1) \to (0,1]$ from question 3 above is a bijection.
We prove that it is both an injection and a surjection.

To show it is an injection, take arbitrary $a, b \in (0,1)$ such that $a \neq b$. We claim $h(a) \neq h(b)$.
Case 1 – if $a = 1/(m+1)$ and $b = 1/(n+1)$ for some $m, n \in \mathbb{N}$. It must be that $m \neq n$. Then $h(a) = 1/m \neq 1/n = h(b)$.
Case 2 – without loss of generality suppose $a = 1/(m+1)$ but there does not exist n such that $b = 1/(n+1)$. Then $h(b) = b$. We now show that $h(a) = 1/m \neq b$. For contradiction assume $b = 1/m$. Since $b \in (0,1)$ we must have $m \geq 2$, which means $b = 1/(1+1)$, contradiction.
Case 3 – if there do not exist m, n such that $a = 1/(m+1)$ and $b = 1/(n+1)$, then $h(a) = a \neq b = h(b)$.

Now, we prove the function is also surjective, i.e. for an arbitrary $y \in (0,1]$ we claim that there exists $x \in (0,1)$ such that $h(x) = y$.
Case 1 – if there exists some $k \in \mathbb{N}$ such that $y = 1/k$, then $h\big(1/(k+1)\big) = y$. Note that $0 < 1/(k+1) < 1$.
Case 2 – if there does not exist such k, then we claim that $h(y) = y$. This is the case only if $y \in (0,1)$ and there does not exist $m \in \mathbb{N}$ such that $y = 1/(m+1)$. Since $y \in (0,1]$, we claim that $y \neq 1$. Note that if $y = 1$ or $y = 1/(m+1)$ for some $m \in \mathbb{N}$, then it would fall under case 1 above.

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