# Concepts of Mathematics EXCEL - Session 3

Topics: Induction, Strong Induction
Session Date: Monday 25 February 2019
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## 1 Induction Walkthrough

This exercise helps you review the structure of an induction proof. The following is a partial solution using strong induction. Complete the proof.

> Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by $7$ for all $n \geq 0$.

Proof.
We want to show that ________________________________ using _____________________.

Base Case 1: __________
We have $5^{2n+1} + 2^{2n+1} =$ _____________ $=$ ______. This is divisible by $7$.

Base Case 2: __________
We have $5^{2n+1} + 2^{2n+1} =$ _____________ $=$ ______. This is divisible by $7$.

Induction Step.
We assume ____________________________________________ (Induction Hypothesis).

We want to show ________________________________________.

By _________________________ we are done.

## 2 Everything You Need to Know About Leonardus Pisanus

Define the Fibonacci sequence $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove the following properties.

> Let $A$ be the $2 \times 2$ matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Let $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ denote this matrix multiplied by itself $n$ times. Find and prove a formula for the four entries $a_n, b_n, c_n, d_n$.

> Prove that the Fibonacci numbers follow the pattern odd, odd, even: that is, show that for any positive integer $m$, $F_{3m-2}, F_{3m-1}$ are odd and $F_{3m}$ is even.

> Prove that $F_{n+1} < \left(\frac{7}{4}\right)^n$ for all $n > 1$.

> Prove Cassini's identity: $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

> Prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$.

Following are some magical Fibonacci identities/properties/proofs.

- Prove Cassini's identity using the spiral diagram.
- Prove the square sum identity using the spiral diagram.
- Prove Cassini's identity using the matrix identity.
- Fibonacci numbers are diagonal sums of the Pascal triangle.
- Sum of ten numbers trick with recurrent additions.
- Generate Pythagorean triples with four Fibonacci numbers.

## 3 Omnipotent Induction

Following are a few exercises to use induction to prove all sorts of mathematical conclusions.

> Prove, by induction, that a set of $n$ elements has $2^n$ subsets.

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> Suppose the product of two odd numbers is odd. Show that if $x_1, x_2, \ldots, x_n$ are odd numbers, the product $x_1 x_2 \ldots x_n$ is odd.

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> Determine which postage amounts can be created using the stamps of 3 and 7 cents. In other words, determine the exact set of positive integers $n$ that can be written in the form $n = 3x + 7y$ with $x, y$ non-negative integers.
> (Hint: Check the first few values of $n$ directly, then use strong induction to show that, from a certain point $n_0$ onwards, all numbers $n$ have such a representation.)

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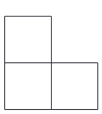
> Prove that for $n \in \mathbb{Z}^+$, a $2^n \times 2^n$ chessboard with any one square removed can be tiled by the 3-square "L" tiles.



L-tromino

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> What is the maximum number of regions into which a plane can be divided by $n$ straight lines?

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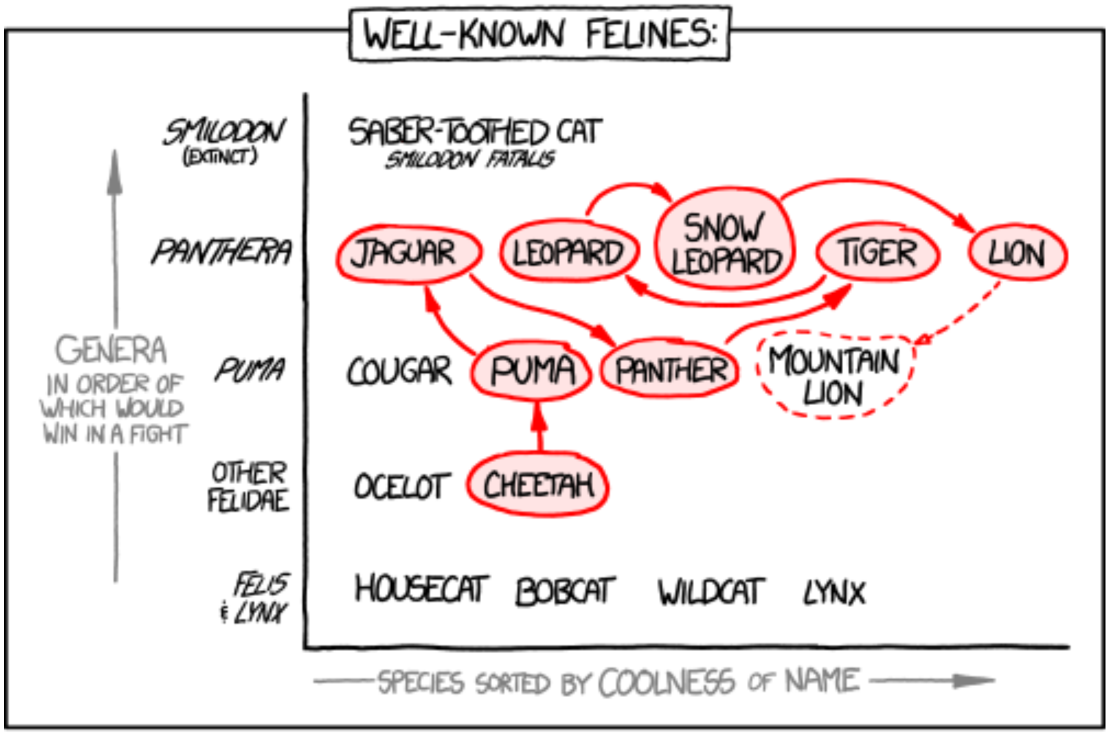


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