EXCEL 21-127
13/14 April 2020

1. Show that the set of all functions $f: [n] \to \mathbb{N}$ is countable, for any $n \in \mathbb{N}$.

   We use the fact that a finite cartesian product of countable sets is countable. More specifically, for any n, the set $\mathbb{N}^n$ is countable.

   Now, we show a bijective function $F: \{f: [n] \to \mathbb{N}\} \to \mathbb{N}^n$. Define function F via $F(f) = \big(f(1), f(2), \dots, f(n)\big)$. Note that $\big(f(1), f(2), \dots, f(n)\big)$ for any f in the domain is a n-tuple with elements in $\mathbb{N}$, so F is well-defined. We will prove that function F is both injective and surjective.

   [injective] We want to show that for any two distinct functions f_1, f_2 in the domain, $F(f_1) \neq F(f_2)$. Since f_1 and f_2 are different, there exists some $x \in \mathbb{N}$ such that $f_1(x) \neq f_2(x)$. Now, note that the x-th position of $F(f_1)$ is $f_1(x)$, and the x-th position of $F(f_2)$ is $f_2(x)$. Since the same position of $F(f_1), F(f_2)$ exhibits different values, $F(f_1) \neq F(f_2)$.

   [surjective] We want to show that for any n-tuple $T = (x_1, x_2, \dots, x_n)$ in the codomain $\mathbb{N}^n$, there exists some function f in the domain such that $F(f) = T$. Consider $f: [n] \to \mathbb{N}$ via $f(k) = x_k$. Indeed $F(f) = T$.

2. Show that the set of all functions $f: \mathbb{N} \to \mathbb{N}$ is uncountable.

   We prove the statement using Cantor's diagonalization technique. Assume the set $\{f: \mathbb{N} \to \mathbb{N}\}$ is countable, then there is some bijective function $F: \mathbb{N} \to \{f: \mathbb{N} \to \mathbb{N}\}$. Suppose F is such a bijective function. We will now show that F in fact cannot be surjective.

   Consider function $g: \mathbb{N} \to \mathbb{N}$, which is in the codomain, via $g(x) = F(x)(x) + 1$. We claim that function g is not in the image of the domain of F, i.e. there does not exist any $n \in \mathbb{N}$ such that $F(n) = g$. For contradiction, assume that there is some $n \in \mathbb{N}$ such that $F(n) = g$. Note that $g(n) = F(n)(n) + 1 = g(n) + 1$, which is nonsense.

   [Intuitively, function g is defined, using the diagonalization technique, as such that it maps n to one plus whatever the function $F(n)$ maps n to. Note that $F(n)$ is a function that is an element of the codomain.]

3. Show that any non-finite subset of $\mathbb{N}$ is countable.

   We use the fact that countable infinity is the smallest infinity. Then, for any non-finite set $S \subseteq \mathbb{N}$, $|S| \geq |\mathbb{N}|$. It remains to show that S is not uncountable, i.e. $|S| \leq |\mathbb{N}|$. We show this with an injective function $f: S \to \mathbb{N}$ via $f(x) = x$. It is easy to verify that f is injective.

4. Find gcd(42,1001).

   Answer is 7.

5. Find gcd(273,754).

   Answer is 13.

6. Find the prime factorization of 10!.

   This one should be straightforward.

7. Let x, y be integers satisfying $x^4 + x^2 = 8y$. Show that $4 \mid x$.

   We may come back to this one next week.

8. Show that for any three <u>consecutive</u> integers, we can choose two of them x, y such that $10 \mid a^3b - ab^3$.

   We may come back to this one next week as well.

9. Prove or disprove the following for $a, b \in \mathbb{Z}$.
   a. $\gcd(a, b) = \gcd(a, a + b)$

      This is true.

   b. $\gcd(a, b) = \gcd(a, ab)$

      This is <u>not</u> true. We can show a counterexample.