15-151 Math Foundations CS – EXCEL

Topic: **Probability, Inequalities, Miscellaneous Review**
EXCEL Leader: Sam Yong
Email: myong@andrew.cmu.edu

Session Date: **Thu 6 Dec 2018**
Academic Development
Cyert Hall B5 | 412-268-6878

1. In a TV Game show, a contestant selects one of three doors; behind one of the doors there is a prize, and behind the other two there are no prizes. After the contestant selects a door, the game-show host opens one of the remaining doors, and reveals that there is no prize behind it. The host then asks the contestant whether they want to SWITCH their choice to the other unopened door, or STICK to their original choice. Is it probabilistically advantageous for the contestant to SWITCH doors, or is the probability of winning the prize the same whether they STICK or SWITCH? Explain your answer.

2. A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive when applied to a non-sufferer. It is estimated that 0.5% of the population are sufferers. Suppose that the test is now administered to a person about whom we have no relevant information relating to the disease. Calculate the following probabilities:
    a. That the test result will be positive;
    b. That, given a positive result, the person is a sufferer;
    c. That, given a negative result, the person is a non-sufferer;
    d. That the person will be misclassified.

3. For three events A, B, C we know that
    a. A and C are independent;
    b. B and C are independent;
    c. A and B are disjoint;
    d. $P(A \cup C) = 2/3$, $P(B \cup C) = 3/4$, $P(A \cup B \cup C) = 11/12$;

    Find $P(A), P(B), P(C)$.

4. A grab bag contains 12 packages worth 80 cents apiece, 15 packages worth 40 cents apiece, and 25 packages worth 30 cents apiece. Is it worthwhile to pay 50 cents for the privilege of picking one of the packages at random?

5. Given a fair dice with 6 faces, the dice is thrown n times, find the expected sum of all the results.

6. Consider random variables X, Y, Z. Assume the following:

   $E(X) = 2, E(Y) = 3, E(Z) = 6$

   $E(XY) = 2, E(YZ) = -3, E(ZX) = 4$

   $E(X^2) = 7, E(Y^2) = 10, E(Z^2) = 32$

   Define random variables U, V, W by

   $$U = X + Y + Z$$

   $$V = X + Y(1 + Z)$$

   $$W = (X + 2Y)(Y - 3Z) - X$$

   Find $E(U), E(V), E(W)$.

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7. Let u, v, w be non-negative real numbers. Prove

   $$uv + uw \geq u\sqrt{vw}$$

8. Prove that $|x| - |y| \leq |x - y|$.

9. Given non-negative real numbers x, y, z with $y + z \geq 2$ show that

   $$(x + y + z)^2 \geq 4yz + 4x$$

10. What is the maximum product of four positive real numbers which add to 20?

11. Let a, b, c be the lengths of sides of a triangle, and let $p = (a + b + c)/2$ be its semiperimeter. Then the area of the triangle can be found by Archimedes-Heron formula:

$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

Prove that out of all triangles with a given perimeter $2p$, the equilateral triangle has the greatest possible area.

Prove that out of all triangles with a given area, the equilateral triangle has the smallest possible perimeter.

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12. Prove that for every positive integer n,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

13. Find the remainder when $2(26!)$ is divided by 29.

14. Prove that

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$$

15. Prove that function $f: \mathbb{R}^2 \to \mathbb{R}^2$ via $f(x, y) = (x + y, x - y)$ is a bijection.

16. Prove or disprove $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

17. Given $g(x)$ is injective and $f(x)$ is surjective. Is $h = gof$ injective?

18. How many integers from 1 to 100 are multiples of 2 or 3?

Show that any party with at least 6 people will contain a group of three mutual friends or a group of three mutual non-friends. In fact, 6 is the minimal party size that guarantees this property.

[This is a famous example of a branch of mathematical studies called Ramsey theory.]