21-127 Concepts of Mathematics – EXCEL

Topic: **Functions, Image & Preimage, In-/Sur-/Bijection, Cardinality** Session Date: **Mon 25 Mar 19**

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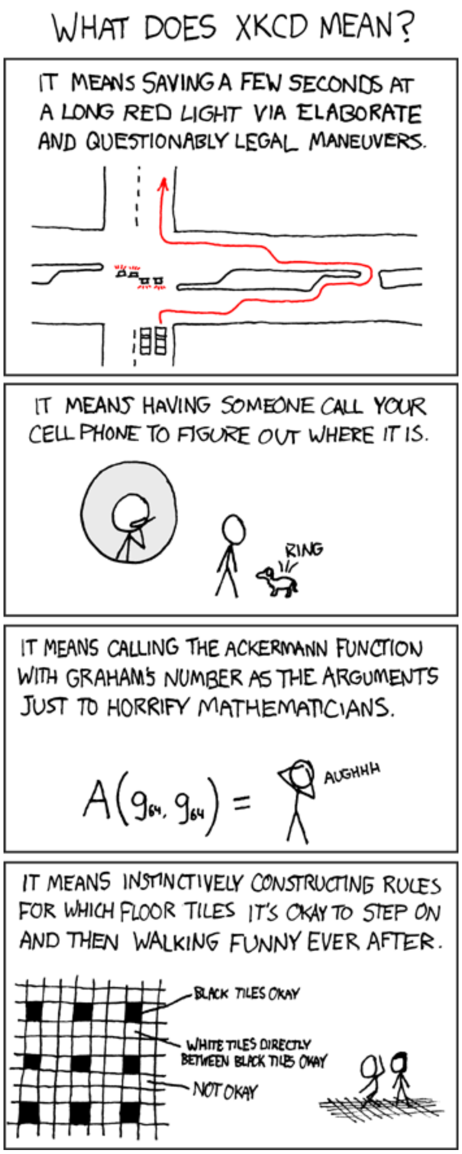


Image & Preimage

1.1

Let A, B be sets and let $f: A \to B$ be a function. Let $X \subseteq A$.

The *image* of X *under the function* f is defined as $\mathrm{Im}_f(X) =$ ___________________________. (Use set builder notation.)

1.2

Consider the function $p: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by

$$\forall (a, b) \in \mathbb{N}\times\mathbb{N}.\, p(a, b) = ab + a$$

What is the image of p?

1.3

Let A, B be sets. Let $f: A \to B$ be a function. Let $S, T \subseteq A$. Show that $\mathrm{Im}_f(S \cap T) \subseteq \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$.

1.4

Let A, B be sets and let $f: A \to B$ be a function. Let $Y \subseteq B$.

The *pre-image* of Y under the function f is defined as $\operatorname{PreIm}_f(Y) =$ ______________________________.

Let $X, Y \subseteq B$. Show that $\operatorname{PreIm}_f(X \cap Y) = \operatorname{PreIm}_f(X) \cap \operatorname{PreIm}_f(Y)$.

1.5

Let A, B be sets. Let $f: A \to B$ be a function. Let $Y \subseteq B$. Show that $\operatorname{Im}_f\left(\operatorname{PreIm}_f(Y)\right) \subseteq Y$.

Give an example such that the equality does not hold.

Hilbert's Hotel

Suppose you are a hotel manager and your hotel is full. In real life, if a new guest comes in, you will have to turn them away. But in the world of math, this is knot a problem – if your hotel has infinitely many rooms. Suppose they are numbered $1, 2, 3,$ and so on. Ask the guest in room 1 to move to room 2, ask the guest in room 2 to room $3\ldots$ Now, the new guest will happily take room 1.

Using this trick, you can accommodate any number of new guests coming in – simply ask each guest to move to the room with number n plus their original room number, if n new guests turn up. But there is more. Can you find a way to accommodate an infinite number of new guests?

Properties of Functions

2.1

Let A, B be sets and let $f: A \to B$ be a function.

We say f is a **surjective** function if and only if __________________________.

We say f is an **injective** function if and only if __________________________.

We say f is a **bijective** function is and only if _____________________.

2.2

Let A, B, C be sets. Let $f: A \to B$ and $g: B \to C$ be functions.

The composition of g with f is defined as $h: A \to C$ via ________________.

2.3

Let A, B, C be sets. Let $f: A \to B$ and $g: B \to C$ be functions.

Suppose $g \circ f$ is injective, show that f is injective.

2.4 *[won't do in session]

Given A, B, C, f, g defined above. Consider the following claims. Prove or disprove them.

- If $g \circ f$ is surjective, then f is surjective.
- If $g \circ f$ is surjective, then g is surjective.
- If f is injective, then $g \circ f$ is injective.
- If g is injective, then $g \circ f$ is injective.
- If f is surjective, then $g \circ f$ is surjective.
- If g is surjective, then $g \circ f$ is surjective.

2.5

For each of the following functions, prove or disprove it is bijective.

- $f: \mathbb{N} \to \mathbb{N}. f(x) = 2x$
- $f: \mathbb{R} \to \mathbb{R}. f(x) = 2x$
- $f: \mathbb{R} \to \mathbb{N}. f(x) = \lfloor x \rfloor$
- $f: \mathbb{Z}^- \to \mathbb{Z}^+. f(x) = |x|$

Iterated Functions

> In mathematics, an *iterated function* is a function $X \to X$ obtained by composing another function $f: X \to X$ with itself a certain number of times. The process of repeatedly applying the same function is called *iteration*.

Suppose you have a rope of length one meter. It is easy to find exactly $1/2$ meter of the rope – simply fold it in half. If you have no tools of measurement, how do you find exactly $1/3$ meter of the rope?

Here is a method.

> Make a guess – divide the rope into what you think are $1/3$ and $2/3$ meter sections. Now, fold in half the part that you think is about $2/3$ meter. If this halved portion is the same as your initial guess of the $1/3$ section, you are done! If not, use the halved portion as the new guess and repeat.

Your challenge: Prove the method is good enough – if you repeat the process many times, you will end up with a guess close enough to the actual $1/3$. Unsurprisingly, the mathematical representation of this method turns out to be an iterated function (with an attractive fixed point at $1/3$).

Cardinality

3.1

Put the following sets into correct places in the diagram below.

$\emptyset$

$\{1, 2, 3\}$

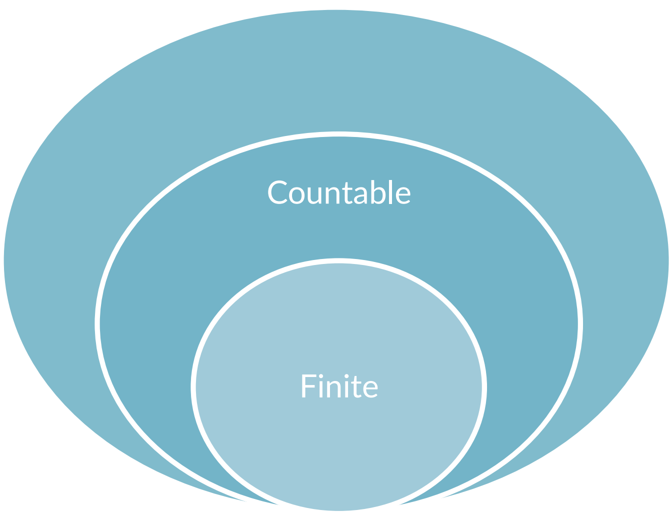
$[7]$

$\mathbb{N}$

$\mathbb{R}$

$\mathbb{Z}$

$\mathbb{Q}$



3.2

Show that $|(0,1)| = |(0,1]|$.

8	4		9			5		
	3		4	8			7	1
					1	6		
5				1			4	
		2		4		1		
	6			2				7
		6	1					
7	1			9	8		5	
		8			3		1	2