EXCEL 127 Session 10 Notes
22 April 2019

I. Practice Problems on Number Theory

1. Prove or disprove the following statements [5].
   a. $\gcd(a, b) = \gcd(a, a + b)$
   b. $\gcd(a, b) = \gcd(a, ab)$
2. Find the greatest common divisors of 273 and 754. Express the positive greatest common divisor as a linear combination of 273 and 754 [6].
3. Find the canonical prime factorization of the following [5].
   a. 111
   b. 127
   c. 1001
   d. 2019
   e. 10!
4. Find all integer solutions *x* to the modular equation $2x + 9 \equiv 3x + 7 \bmod 5$ [4].
5. Fix a modulus *n*. Prove or disprove for all *a*, *b*, *q* integers $q \neq 0 \bmod n$ [5]
$$qa \equiv qb \to a \equiv b$$
6. Find all integer solutions *x* to the modular equation $25x - 4 \equiv 4x + 3 \bmod 13$ [7].
7. Find the remainder of 3^{10601} divided by 13 [5].
8. Find the remainder of $3^{45} \cdot 44!$ divided by 47 [7].
9. Prove that the gaps between consecutive primes can be made arbitrarily large [8].
10. Let *x*, *y* be integers. Prove that if 7 divides $x^2 + y^2$ then 7 divides *x* and *y* [6].
11. Let *p* be a prime. Find $\gcd\big((p-1)! + 1, p!\big)$ [7].
12. Let *x*, *y* be integers satisfying $x^4 + x^2 - 8y = 0$. Prove 4 divides *x* [4].
13. Find the last digit of $7^{7^{7^{7}}}$ [5].
14. Let *p* be a prime greater than 3. Prove $24 \mid p^2 - 1$ [6].
15. Show that for any three consecutive integers we can choose two *a*, *b* such that 10 divides $a^3b - ab^3$ [8].
16. Show that 30 divides $n^5 - n$ for all natural numbers *n* [6].

II. Practice Problems on Counting

1. Sophie is browsing new hats to buy. The brand offers *k* different colors of hats (each hat has a single color). There are 42 purchase options – some options include a single hat of one color, and other options include two hats of different colors (two of the same color is *not* an option). What is the least number of different colors the brand has [3]?
2. How many ways can you choose 4 distinct groups of 4 people from 16 people [3]?
3. How many ways can you pair up 8 boys and 8 girls [3]?
4. How many different ways can you arrange the letters in the word “repetition” [5]?
5. How many numbers can be expressed as a sum of four distinct members of the set $\{17, 21, 25, 29, 33, 27\}$ [6]?

III. Counting Review Questions

1. What is a permutation?
2. What is the addition principle?
3. What is the multiplication principle?
4. What is a partition?
5. What is the inclusion-exclusion principle?

IV. Counting in Two Ways

1. $\sum_{i=1}^{n} i = \binom{n+1}{2}$
2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$