21-127 Concepts of Mathematics – EXCEL

Topic: Infinite sets, countability, CSB Theorem

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Academic Development

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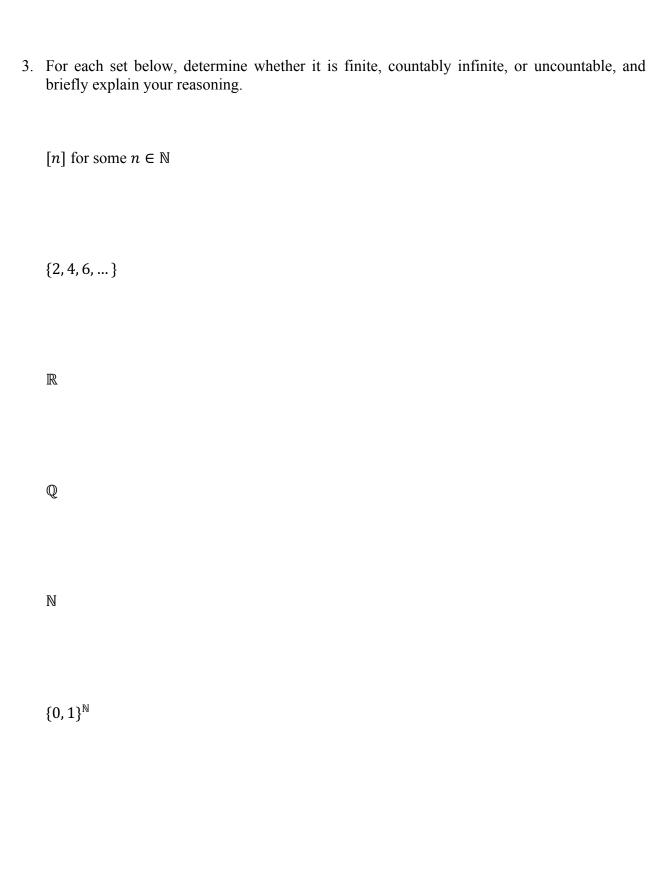
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Warm-up

1. For each question below, choose the strongest response.

A(n) union	of sets	is necessarily (finite, countable, uncountable, none of these)
Finite	Finite	
Finite	Countable	
Countable	Finite	
Countable	Countable	
Uncountable	Countable	

2. Consider the following statement: "A countable cartesian product of finite sets is necessarily countable." Is this statement true? Informally, explain why or why not. *You do not need to give a proof.* We will work on a proof (with a simplifying assumption) later in the session.



Board work

1. During our last EXCEL session, you showed that $\mathbb{N} \times \mathbb{N}$ was countable by providing an injection $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Today, we will consider an alternate, more informal proof method. Consider $\mathbb{N} \times \mathbb{N}$ as a two-dimensional grid of points, where the x coordinate represents the first coordinate in the tuple and the y coordinate represents the second. Show that $\mathbb{N} \times \mathbb{N}$ is countable by associating a unique n in \mathbb{N} to each lattice point. 2. Use the technique from (2) to show that the set \mathbb{Q}^+ is rational: that is, the set of all positive rational numbers. (Hint: use one axis to denote the numerator, and another to denote the denominator.) 3. We denote by C the set of complex numbers; that is, the set of numbers expressible in the form a + bi for some $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$. Is this set countable? Prove or disprove. 4. Prove that for any $n \in \mathbb{N}^+$, the set \mathbb{N}^n is countable. You may cite without proof the fact that $\mathbb{N} \times \mathbb{N}$ is countable. (*Hint: induction.*)

5.	Show that if $ A = B $, then $ P(A) = P(B) $. An informal argument is acceptable.
6.	Using the Cantor-Schroeder-Bernstein Theorem, show that there exists a bijection between the set of natural numbers and the set of integers with exactly two distinct prime factors. You may assume the existence of a function which, given a natural number i , outputs the i^{th} prime. (Hint: assume you know what the prime factors are, and create a function using them. The other direction is loosely similar to the way you proved $\mathbb{N} \times \mathbb{N}$ was countable last session.)
7.	(Challenge!) We say a number x is algebraic if it is the root of some polynomial with integer coefficients. Is the set of all algebraic numbers countable? Prove your answer. (Hint: countable union.)
8.	(Super challenge!) Show that the set of irrational numbers in $[0, 1]$ is uncountable. You don't need to provide a full proof for this, but instead think of a series of transformations which would allow you to achieve the desired result. In fact, using your argument, you will be able to show that for any $a, b \in \mathbb{R}$, $a \neq b$, the set of irrational numbers in $[a, b]$ is uncountable.