EXCEL 21-127
2/3 March 2020
Relations

1. For the following relations R on set $A = \{a, b, c\}$, determine if R is reflexive, symmetric, transitive, and/or anti-symmetric?
    a. $R = \{(a, b), (b, a), (a, c), (c, a)\}$
    b. $R = \{(a, a), (a, b), (b, a)\}$
    c. $R = \{(a, b), (b, c), (a, c), (b, b)\}$

2. The following argument claims to show that symmetric and transitivity implies reflexivity. Find the flaw in the argument. [Hint. Pay attention to quantifiers.]
$$x{\sim}y \Rightarrow y{\sim}x \wedge \big((x{\sim}y \wedge y{\sim}x) \Rightarrow x{\sim}x\big)$$

3. For the following relations R on sets A, determine if R is reflexive, symmetric, transitive, and/or anti-symmetric?
    a. $A = \mathbb{N}, x{\sim}y$ iff $x \mid y$
    b. $A = \mathbb{N}, x{\sim}y$ iff $x < y + 1$
    c. $A = \mathbb{Z}, x{\sim}y$ iff $x - y \in \mathbb{Z}$
    d. $A = \mathbb{N}^2, (x_1, y_1){\sim}(x_2, y_2)$ iff $x_1 y_2 = x_2 y_1$
    e. $A =$ a group of friends, $a{\sim}b$ iff a and b have the same birthday

4. * (The birthday paradox) Consider a party of n people. What is the likelihood that some pair of two people share the same birthday? Alternatively, what is the smallest n such that the likelihood is greater than 50%?

5. Let R be an equivalence relation on A. Then for all $a, b \in A$, show that the following three statements are equivalent.
    a. $a{\sim}b$
    b. $[a] \cap [b] \neq \emptyset$
    c. $[a] = [b]$

6. Suppose R is a reflexive relation on A, and for all $a, b, c \in A$ we have
$$(a{\sim}b \wedge a{\sim}c) \Rightarrow b{\sim}c$$
Show that R is an equivalence relation.

7. [Review] Prove or disprove $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

8. [Review] Prove $\sum_{i=1}^{n} 1/i^2 \leq 2 - 1/n$ for all $n \in \mathbb{N}$.

9. [Review] Prove $9 \mid 4^n - 3n + 8$ for all $n \in \mathbb{N}$.

10. * Find all integer solutions to

$$\frac{x^6 - 3}{x^2 + 2}$$

11. * Are there more rational numbers than there are integers?

12. Construct a relation R on the set $[6]$ that is reflexive, transitive, and not symmetric.

13. Construct a relation R on the set $[6]$ that is reflexive, anti-symmetric, and not transitive.

14. Show that the lexicographical order relation $\leq$ on $\mathbb{Z}^2$ is a partial order.

$$(x_1, y_1) \leq (x_2, y_2) \text{ iff } x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 \leq y_2)$$

15. * Express the mathematical condition for the lexicographical $\leq$ on $\mathbb{Z}^n$.

		1		8	4	6		
	4		2				3	
		7				5		9
		6				1		7
						3		6
			1		6			
				7				5
	8							
3		4					2	