21-127 Concepts of Mathematics – EXCEL

Topic: **Relations, Functions**
EXCEL Leader: Sam Yong
Email: myong@andrew.cmu.edu

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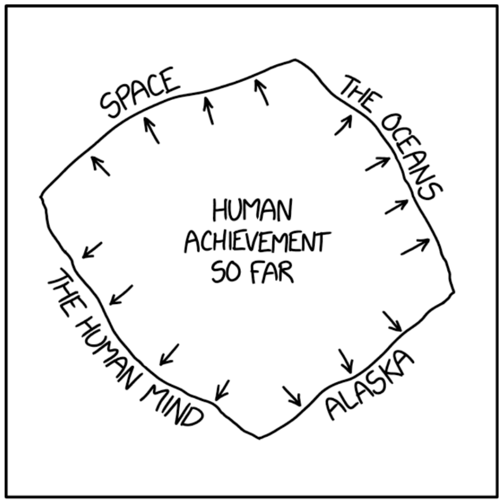
Academic Development
Cyert Hall B5 | 412-268-6878



FINAL REMAINING "FRONTIERS," ACCORDING TO POPULAR USAGE

1.1

- Let A be a set and let R be a relation on A, i.e. $R \subseteq A \times A$.

  We say R is reflexive if ____________________.

  We say R is symmetric if ____________________.

  We say R is transitive if ____________________.

  We say R is anti-symmetric if ____________________.

  We say R is a **partial order** on A if ____________________.

  We say R is a **total order** on A if ____________________.

  We say R is a **strict partial order** on A if ____________________.

  We say R is a **strict total order** on A if ____________________.

- What is a **poset**? Is a poset a set?
- Let A be a set and let R be a relation on A.

  We say R is an **equivalence relation** on A if ____________________.

- What is an **equivalence class**?
- What is a **partition** of a set?

"You have to spend some energy and effort to see the beauty of math."

"I don't have any particular recipe [for developing new proofs]. … It is like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks, and with some luck you might find a way out."

"You're torturing yourself along the way, but life isn't supposed to be easy."

– Maryam Mirzakhani

1.2

- Let A, B be sets. Let f be a relation between A and B, so $f \subseteq A \times B$.

  f is called a **function** from A to B if ______________________________________.

  We call _____ the **domain** of the function and _____ the **codomain** of the function.

  We write _______________ to mean f is a function from A to B.

  We write _______________ if $(a, b) \in f$.

- What is a **well-defined** function? What are some examples of non-well-defined functions?

1.3

For the following definitions of a relation R on a set A, discuss whether the relation is

Reflexive	Symmetric
Transitive	Anti-symmetric
An equivalence relation	A partial order relation
A total order relation	A well-defined function

(1) $A = \mathbb{N}, x\ R\ y$ iff $x \mid y$

(2) $A = \mathbb{N}, x\ R\ y$ iff $x < y + 1$

(3) $A = \mathbb{Z}, x\ R\ y$ iff $x - y \in \mathbb{Z}$

(4) $A = \mathbb{N}^2, (x_1, y_1)\ R\ (x_2, y_2)$ iff $x_1 y_2 = x_2 y_1$

(5) $A =$ a party of you and your friends, $a\ R\ b$ iff a and b have the same birthday

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The Birthday Problem

Consider a party of n random people. What is the probability that some pair(s) of them have the same birthday? It is perhaps more fun to consider when the probability is greater than 50% and when it is greater than 70%. (The answer is quite unintuitive.)

Here is a rather famous riddle. You have two buckets. One holds exactly five gallons of water. The other exactly three. How do you measure four gallons of water assuming you have unlimited water supply? Can you prove that you can measure any whole gallons of water from 1 to 8 gallons?

- A **permutation** of a set is an arrangement of the elements in the set in some particular order. For example, $\{1, 2, 3\}$ has six distinct permutations: $123, 132, 213, 231, 312, 321$
- A **superpermutation** is a string that contains all permutations as substrings. For example, a superpermutation of $\{1, 2, 3\}$ is 123121321 This is also a shortest superpermutation of the set of three elements.

Can you find a superpermutation of $\{1, 2, 3, 4\}$ and make it as short as possible?

(This problem is incredibly hard. We still do not know the length of the shortest superpermutation on 6.)

2.1

Suppose R is an equivalence relation on A. Then for all $a, b \in A$, show that the following are equivalent.

(1) $a\ R\ b$

(2) $[a] \cap [b] \neq \emptyset$

(3) $[a] = [b]$

2.2

Discuss if love is an equivalence relation. Is it reflexive? Symmetric? Transitive?

A taste of *infinity*

Suppose you have infinite number of candies labeled with all the natural numbers – candy 1, candy 2, etc. Now you would like to put some candies into a big jar. Following are two ways you may proceed.

(1) At first step put candies 1 to 10 into the jar, and remove candy 1; at second step put candies 11 to 20 into the jar, and remove candy 2; at third step put candies 21 to 30, remove candy 3…

(2) At first step put candies 1 to 9 into the jar, then write digit zero after candy 1 to make it candy 10; at second step put candies 11 to 19 into the jar, write digit zero after candy 2 to make it 20…

For each method, how many candies do you have in the jar after the infinite steps are complete?