EXCEL 21-127
27, 28 April 2020
Number theory & counting

1. How many ways are there to choose 4 distinct groups of 4 people from 16 people?

$$\binom{16}{4}\cdot\binom{12}{4}\cdot\binom{8}{4}\cdot\binom{4}{4}$$

We complete the selection in 4 steps.
Step 1: choose 4 people from 16 people for group A.
Step 2: choose 4 people from the remaining 12 people for group B.
…

2. How many ways are there to pair up 8 boys and 8 girls?

$$8!$$

First boy has 8 choices; second boy has the remaining 7; …

3. How many ways are there to arrange the letters in the word "repetition" and obtain different character sequences?

$$\frac{10!}{(2!)^3}$$

10 letters, permutate all of them we have 10!, but there are three pairs of identical letters, the two letters in each pair are the same, so the relative positions of the two letters in a pair is non-relevant; the position of each pair of two letters is 2!

4. How many numbers can be expressed as the sum of a four-element subset of the set {17,21,25,29,33,37}?

All these numbers are $4k+1$ and consecutive, so the sums are multiples of 4. Smallest sum we can get is 17+21+25+29 which is 92, and the largest sum we can get is 25+29+33+37 which is 124. There are 9 multiples of 4 between 92 and 124, inclusive. So the answer is 9, given that all these multiples of 4 can actually be a sum.

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5. Prove by counting in two ways,

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

Suppose we want to select k people to join a committee from n people. The left-hand side counts how many ways we can choose such k people. The right-hand side counts the same number in two cases. Case 1, we want Alice on the committee. Case 2, we don't want Alice.

6. Prove by counting in two ways,

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

Suppose we want to select 3 people from 3 different groups of n people, i.e. in total $3n$ people. The left-hand side counts exactly that. The right-hand side counts the same number by 3 cases.
Case 1, all 3 people come from the same group.
Case 2, two people come from the same group, and the other come from another group.
Case 3, they come from different groups.

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7. How many ways are there to arrange 7 people in a line such that Alice and Bob are next to each other?

$$6! \cdot 2!$$

8. Given a 100×100 grid, a lattice path from $(0,0)$ to $(100,100)$ is a path that only moves up or right one step at a time.
   a. How many lattice paths are there from $(0,0)$ to $(100,100)$?

$$\binom{200}{100}$$

We need in total 200 steps on the path, among which 100 are up and 100 are right.
For example, a path from $(0,0)$ to $(3,3)$ can be Up-Right-Up-Up-Right-Right

   b. How many lattice paths are there from $(0,0)$ to $(100,100)$, which do <u>not</u> pass through $(10,10)$?

$$\binom{200}{100} - \binom{20}{10} \cdot \binom{180}{90}$$

   c. How many lattice paths are there from $(0,0)$ to $(100,100)$, which do <u>not</u> pass through $(20,50)$?

$$\binom{200}{100} - \binom{70}{20} \cdot \binom{130}{50}$$

9. How many ways are there to put 8 <u>numbered</u> marbles into 3 different boxes?
10. How many ways are there to put 8 <u>identical</u> marbles into 3 different boxes?
11. How many ways are there to put 8 <u>identical</u> marbles into 3 different boxes such that each box has at least 2 marbles?

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12. Let $a, b, c \in \mathbb{Z}$, find all $x \in \mathbb{Z}$ such that
    a. $x \equiv a \bmod 4$, and
    b. $x \equiv b \bmod 5$, and
    c. $x \equiv c \bmod 9$