21-127 EXCEL

25 February 2020

Session 3 (Induction)

Section I. For this section, we are concerned with the following four sequence of numbers. Choose two of the problems below to explore. (More stars means more difficulty.)

Definitions.

A. The Fibonacci sequence

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}. \forall n > 1$$

B. The Lucas sequence

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}. \forall n > 1$$

C. The Pell sequence

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}. \forall n > 1$$

D. The Pell-Lucas sequence

$$Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}. \forall n > 1$$

Problems.

- * Prove Cassini's identity. $F_{n+1}F_{n-1} - F_n^2 = (-1)^n. \forall n \in \mathbb{N}$
- Prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}. \forall n \in \mathbb{N}$
- * Prove Cassini's identify. $P_{n+1}P_{n-1} - P_n^2 = (-1)^n. \forall n \in \mathbb{N}$
- Prove that $L_n = F_{n+1} + F_{n-1}. \forall n \in \mathbb{N}$
- ** Prove that $F_{2n} = L_n F_n. \forall n \in \mathbb{N} \cup \{0\}$
- Prove that $F_n = \frac{1}{5}(L_{n+1} + L_{n-1}). \forall n \in \mathbb{N}$
- *** Prove that $Q_n = P_{2n}/P_n. \forall n \in \mathbb{N}$

Section II. Explore the problems below <u>using induction</u> as well.

~~1.~~ Let T_n be a $2 \times n$ grid. Let a_n denote the number of different 2×1 domino tilings of T_n. Find a recurrence formula for a_n in terms of a_{n-1} and a_{n-2}. ~~Then, find a general formula for a_n and prove it.~~

2. Determine the set of postage amounts which can be created by combing only 3-cent stamps and 7-cent stamps. For example, $10 = 1 \times 3 + 1 \times 7$ can be created, but 11 cannot. [Hint. Check the first few natural numbers by hand then observe the pattern in large numbers.]

3. What is the <u>maximum</u> number of regions into which an infinite plane can be divided by n straight lines?

4. *** We shall attempt to prove the second-term self-similarity property of the Thue-Morse sequence with strong induction, or not.

Section III. The problems in this section are completely <u>arbitrary and fixed</u>.

For today, let's prove Fermat's Christmas theorem.

An odd prime p can be expressed as $p = x^2 + y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p \equiv 1 \bmod 4$.

There are several steps to proving this based on Zagier's.

a) determine the parity of x and y
b) find solutions to a relaxed equation with 3 variables instead of two
c) count the number of solutions and observe the pattern
d) find a visual representation of the solutions
e) observe the pattern in the visual representations