15-151 Math Foundations CS – EXCEL

Topic: **Counting Principles, Counting in Two Ways, Infinite Sets**
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"The greatest enemy of knowledge is not ignorance; it is the illusion of knowledge."

– **Daniel J. Boorstin**

Puzzle 1. Prove the following math identity visually. (Hint: counting squares.)

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3, n \in \mathbb{N}$$

Of course, we can also prove this with good old induction.

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1. **A very challenging counting in two ways argument**

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3, n \geq 3$$

2. **An easier counting in two ways proof**

$$\sum_{i=1}^{n} i = \binom{n+1}{2}, n \in \mathbb{N}$$

Puzzle 2. (Putnam 1965) In a tennis tournament, every player plays every other player. Suppose that player i won w_i of his games and lost l_i of them. Prove $\sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} l_i^2$.

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3. Your very own counting in two ways recipe

When we use the technique of counting in two ways to prove a math identity, we often dissect the equation into pieces of counting identities, and then assign counting justification to these pieces.

For example, the following identity counts the total number of permutations of four distinct objects.

$4!$

Now, devise your own glossary of counting identities and their justification.

$n!$

$\binom{n}{k}$

n^k

$$\sum_{i=1}^{n} i$$

$k(n-k)$

Puzzle 3. Sudoku!

Hard

6	2	7	3	5				
		3		6				
	9						1	
	7		9	4	6			
3						7	4	
		8			3	1		
4								8
				1		4		9
							6	

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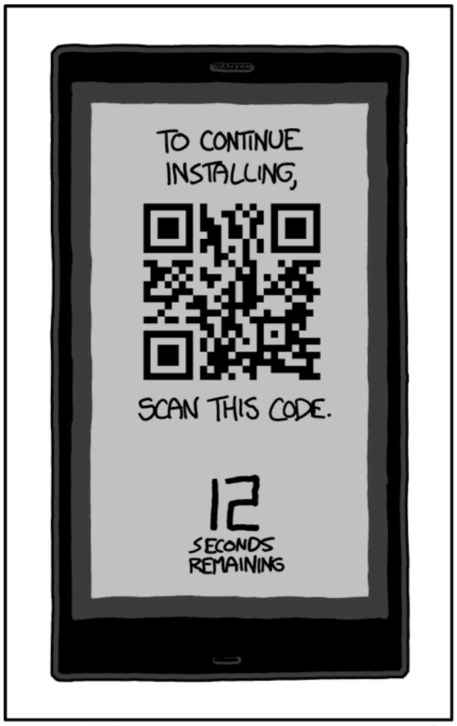
4. All of the Bijections

Construct an explicit bijection from the open interval $(0,1)$ to the interval $(0,1]$.

Construct three different explicit bijections from the open interval $(0,1)$ to the interval $(0,1]$.



HOW TO FREAK OUT A MOBILE APP USER

Title: QR Code

Alt: Remember, the installer is watching the camera for the checksum it generated, so you have to scan it using your own phone.

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5. A more challenging construction

Let $A \subseteq \mathbb{N}$ and A is infinite. Construct a bijection to show that $|A| = |\mathbb{N}|$.

6. Counting functions, and many sets

Problem 6 of the most recent pset shows that the set of all *periodic* functions $\mathbb{Z} \to \mathbb{Z}$ is countable.

Now, show that the set of all functions $\mathbb{N} \to \mathbb{N}$ is uncountable.

Show that the set of all functions $f: [n] \to \mathbb{N}$ is countable.

Summarize from your findings above and previous knowledge:

- A finite union of countable sets is _______________ (countable/uncountable/unknown)
- A finite intersection of countable sets is _______________ (countable/uncountable/unknown)
- A finite Cartesian product of countable sets is _______________
- A countable union of countable sets is _______________
- A countable intersection of countable sets is _______________
- A countable Cartesian product of countable sets is _______________

7. A more challenging countable set

A real number x is called *algebraic* if x is the root of a polynomial equation $c_0 + c_1 x + \cdots + c_n x^n$ where all the coefficients c_i are integers. A real number is called *transcendental* if it is not an algebraic number.

Show that the set of all algebraic numbers is countable.

Show that the set of all transcendental numbers is uncountable.

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