21-127 Concepts of Mathematics – EXCEL

Topic: **Infinite sets, countability, CSB Theorem**
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Warm-up

1. For each question below, choose the strongest response.

A(n) ___ union	of ___ sets	is necessarily ___ (finite, countable, uncountable, none of these)
Finite	Finite	
Finite	Countable	
Countable	Finite	
Countable	Countable	
Uncountable	Countable	

2. Consider the following statement: "A countable cartesian product of finite sets is necessarily countable." Is this statement true? Informally, explain why or why not. *You do not need to give a proof*. We will work on a proof (with a simplifying assumption) later in the session.

3. For each set below, determine whether it is finite, countably infinite, or uncountable, and briefly explain your reasoning.

$[n]$ for some $n \in \mathbb{N}$

$\{2, 4, 6, \ldots\}$

$\mathbb{R}$

$\mathbb{Q}$

$\mathbb{N}$

$\{0, 1\}^{\mathbb{N}}$

Board work

1. During our last EXCEL session, you showed that $\mathbb{N} \times \mathbb{N}$ was countable by providing an injection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Today, we will consider an alternate, more informal proof method. Consider $\mathbb{N} \times \mathbb{N}$ as a two-dimensional grid of points, where the x coordinate represents the first coordinate in the tuple and the y coordinate represents the second. Show that $\mathbb{N} \times \mathbb{N}$ is countable by associating a unique n in $\mathbb{N}$ to each lattice point.

2. Use the technique from (2) to show that the set $\mathbb{Q}^+$ is rational: that is, the set of all positive rational numbers. *(Hint: use one axis to denote the numerator, and another to denote the denominator.)*

3. We denote by $\mathbb{C}$ the set of complex numbers; that is, the set of numbers expressible in the form $a + bi$ for some $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$. Is this set countable? Prove or disprove.

4. Prove that for any $n \in \mathbb{N}^+$, the set $\mathbb{N}^n$ is countable. You may cite without proof the fact that $\mathbb{N} \times \mathbb{N}$ is countable. *(Hint: induction.)*

5. Show that if $|A| = |B|$, then $|P(A)| = |P(B)|$. An informal argument is acceptable.

6. Using the Cantor-Schroeder-Bernstein Theorem, show that there exists a bijection between the set of natural numbers and the set of integers with exactly two distinct prime factors. You may assume the existence of a function which, given a natural number i, outputs the i^{th} prime. *(Hint: assume you know what the prime factors are, and create a function using them. The other direction is loosely similar to the way you proved* $\mathbb{N} \times \mathbb{N}$ *was countable last session.)*

7. *(Challenge!)* We say a number x is *algebraic* if it is the root of some polynomial with integer coefficients. Is the set of all algebraic numbers countable? Prove your answer. *(Hint: countable union.)*

8. *(Super challenge!)* Show that the set of irrational numbers in $[0, 1]$ is uncountable. You don't need to provide a full proof for this, but instead think of a series of transformations which would allow you to achieve the desired result. In fact, using your argument, you will be able to show that for any $a, b \in \mathbb{R}$, $a \neq b$, the set of irrational numbers in $[a, b]$ is uncountable.