Concepts of Mathematics EXCEL. Session 4

Topics: Review Induction, Relations
Session Date: Monday 4 March 2019
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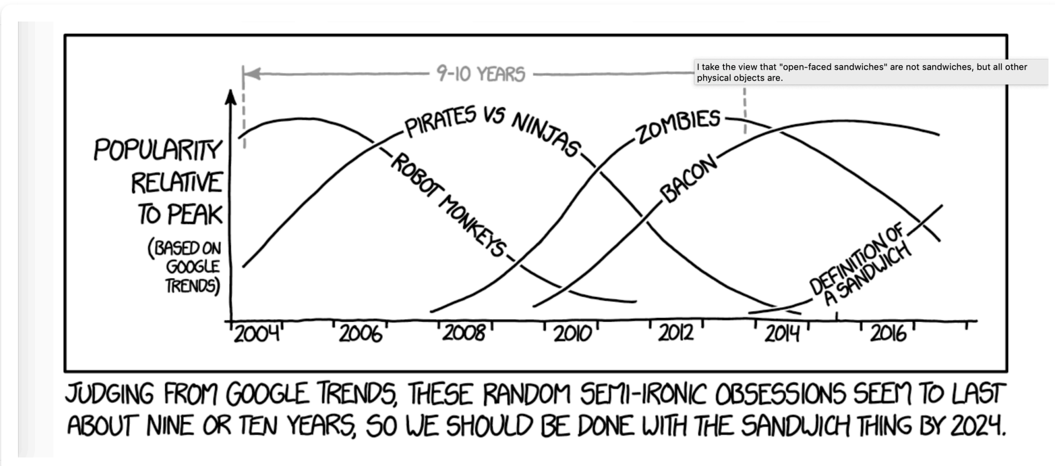

0 Mathematical Studies

0.1 Discuss the following questions with your peers.

- How do you prepare for a math lecture?
- What do you do when listening to a math lecture? What do you find easy to do? What do you find difficult?
- What do you do to help you understand a new math concept?
- What do you do to help you memorize a math definition/theorem/technique?
- How do you prepare for a math exam?
- What do you do before, during, and after working a math homework/problem set?
- Do you review graded homework and/or exams? If so, how?
- What do you find easy in studying the course? What do you find difficult? How do you tackle the difficulties?
- Do you have any good suggestion in studying the course/studying math?

1 Induction

1.1 Discuss the following questions.

- What are the differences between regular and strong induction?
- How might you identify when strong induction is *required*?
- Why is it that we might as well always use strong induction?

1.2 Let P be a mathematical statement on natural numbers. $P(n)$ states that the statement is true for natural number n. Suppose $P(1)$, $P(2)$, and $(P(n) \land P(n-1)) \Rightarrow P(n+1)$ for $n \geq 2$. Prove $\forall n \in \mathbb{N}.\, P(n)$.

1.3 Recall the definition of the Fibonacci sequence. $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Fill in the table below. Then find a formula for $F_{n-1}F_{n+1} - F_n^2$ and prove it.

	F_{n-1}	F_n	F_{n+1}	$F_{n-1}F_{n+1} - F_n^2$
$n = 2$				
$n = 3$				
$n = 4$				
$n = 5$				

1.4 Let Q_n be a $2 \times n$ grid and let a_n be the number of domino (2×1) tilings for Q_n. Find a recurrence for a_n in terms of a_{n-1} and a_{n-2} for $n \geq 3$. Find a general formula for a_n and prove it using induction.

2 Relations

Definition

> Let A, B be sets. A *relation* between A and B is a set of *ordered pairs* $R \subseteq A \times B$. Given elements $a \in A$ and $b \in B$ we say a and b are *related* if and only if $(a, b) \in R$.
>
> The set A is called the *domain* and the set B is called the *codomain*. The set R is called the *relation set*.
>
> If $A = B$ we say R is a relation *on* A.

Intuitively speaking, a *binary relation on set A* is some relation R where, for every $x, y \in A$, the statement xRy is either true or false.

For example, $<$ can be a binary relation on set $\mathbb{N}$. $1 < 2$ is true, so $(1, 2)$ is *in* the relation; $2 < 1$ is false, so $(2, 1)$ is *not* in the relation.

Check Your Understanding

2.1 For the following pairs of sets A, B write down all binary relations between A and B.

- $A = [2], B = \{x \in \mathbb{Z} : 2x^2 + x - 1\}$
- $A = B = \{x \in \mathbb{Z} : |x| = 1\}$
- $A = \{\text{Alice}, \text{Bob}\}, B = \{\{1, 2\}, \{2, 3\}\}$

Definitions

> Let A be a set and let R be a relation on A.
>
> - We say R is *reflexive* if $\forall x \in A.\, (x, x) \in R$.
> - We say R is *symmetric* if $\forall x, y \in A.\, (x, y) \in R \Rightarrow (y, x) \in R$.
> - We say R is *transitive* if $\forall x, y, z \in A.\, ((x, y) \in R \wedge (y, z) \in R) \Rightarrow (x, z) \in R$.
> - We say R is *anti-symmetric* if $\forall x, y \in A.\, ((x, y) \in R \wedge (y, x) \in R) \Rightarrow x = y$.

Check Your Understanding

2.2 For the following relations R on set $A = \{a, b, c\}$, discuss if R is reflexive? Symmetric? Transitive? Anti-symmetric?

- $R = \{(a, b), (b, a), (a, c), (c, a)\}$
- $R = \{(a, a), (a, b), (b, a)\}$
- $R = \{(a, b), (b, c), (a, c), (b, b)\}$

2.3 The following argument shows that symmetry and transitivity imply reflexivity. Find the flaw in the argument.

> Suppose xRy. By symmetry, yRx. By transitivity, $xRy \wedge yRx \Rightarrow xRx$.

3 Additional Content

3.1 From the previous questions we have seen that all $2 \times n$ grids can be tiled by dominos. Investigate the following questions.

- How many ways can we tile a $2 \times n$ grid with dominos and squares (1×1)?
- How many ways can we tile a $3 \times n$ grid with dominos?

3.2 You all know Pythagorean triples. $a, b, c \in \mathbb{N}$ such that $a^2 + b^2 = c^2$. List the primitive Pythagorean triples you can think of. A primitive Pythagorean triple consists of the three numbers a, b, c such that they do not share a common factor, i.e. $3, 4, 5$ but not $6, 8, 10$.

How many different primitive Pythagorean triples can you write down?

Now, let $a, b, a+b, a+2b$ be four consecutive Fibonacci numbers. Construct a Pythagorean triple from the four numbers. Find out if the triple is necessarily primitive. Hint: For $x, y \in \mathbb{Z}$, what is $(a^2 - b^2)^2 + (2ab)^2$?