Exercise Sheet 1

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Probabilistic Inference & Learning

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Probabilities & Densities

1 Inference in a directed graphical model

$$
\begin{aligned}
p(E|A) &= \frac{p(A|E)\cdot(E)}{p(A)} \\
&= \frac{p(A|E,B)p(B)p(E) + p(A|E,\bar{B})p(\bar{B})p(E)}{p(A|E,B)p(B)p(E) + p(A|E,\bar{B})p(\bar{B})p(E) + p(A|\bar{E},B)p(B)p(\bar{E}) + p(A|\bar{E},\bar{B})p(\bar{B})p(\bar{E})} \\
&= \frac{(0.9901099 \cdot 10^{-3} + 0.01099(1-10^{-3}))\cdot 10^{-3}}{(0.9901099 \cdot 10^{-3} + 0.01099(1-10^{-3}))\cdot 10^{-3} + (0.99001 \cdot 10^{-3} + 0.001(1-10^{-3}))(1-10^{-3})} \\
&= 5.98758 \cdot 10^{-3}
\end{aligned}
$$

2 Independencies in three-node directed graphs

Proof (a).

$$
\begin{aligned}
p(A,B,C) &= p(C|B)\cdot p(B|A)\cdot p(A) && \text{(factorization of the joint)}\\
p(A,B,C) &= p(C|B)\cdot \frac{p(A|B)\cdot p(B)}{p(A)}\cdot p(A)\\
p(A,B,C) &= p(C|B)\cdot p(A|B)\cdot p(B)\\
\frac{p(A,B,C)}{p(B)} &= p(C|B)\cdot p(A|B)\\
\frac{p(A,C|B)\cdot p(B)}{p(B)} &= p(C|B)\cdot p(A|B)\\
p(A,C|B) &= p(C|B)\cdot p(A|B)
\end{aligned}
$$

□

Proof (b).

$$
\begin{aligned}
p(A,B,C) &= p(A|B)\cdot p(B|C) * p(C) && \text{(factorization of the joint)}\\
p(A,B,C) &= p(A|B)\cdot \frac{p(C|B)\cdot p(B)}{p(C)}\cdot p(C)\\
p(A,B,C) &= p(A|B)\cdot p(C|B)\cdot p(B)\\
\frac{p(A,B,C)}{p(B)} &= p(A|B)\cdot p(C|B)\\
\frac{p(A,C|B)\cdot p(B)}{p(B)} &= p(A|B)\cdot p(C|B)\\
p(A,C|B) &= p(C|B)\cdot p(A|B)
\end{aligned}
$$

□

Proof (c).

$$p(A,B,C) = p(B) \cdot p(C|B) \cdot p(A|B) \qquad \text{(factorization of the joint)}$$

$$\frac{p(A,B,C)}{p(B)} = p(C|B) \cdot p(A|B)$$

$$\frac{p(A,C|B) \cdot p(B)}{p(B)} = p(C|B) \cdot p(A|B)$$

$$p(A,C|B) = p(C|B) \cdot p(A|B)$$

□

Proof (d).

$$p(A,B,C) = p(A) \cdot p(C) \cdot p(B|A,C) \qquad \text{(factorization of the joint)}$$

$$\frac{p(A,B,C)}{p(B|A,C)} = p(A) \cdot p(C)$$

$$\frac{p(B|A,C) \cdot p(A,C)}{p(B|A,C)} = p(A) \cdot p(C)$$

$$p(A,C) = p(A) \cdot p(C)$$

□

3 Inferring probabilities

Using the equation stated in the lecture for inferring a Bernoulli Probability:

$$p(\pi|n,m) = \frac{\pi^{n+a-1} \cdot (1-\pi)^{m+b-1}}{B(a+n, b+m)} \tag{1}$$

Also given an uniform distribution ($a = b = 1$) leads to the following expressions for the probabilities θ_1 and θ_2 for a negative review:

$$p(\theta_1|90,10) = \frac{\theta_1^{90} \cdot (1-\theta_1)^{10}}{B(91,11)} = \frac{\theta_1^{90} \cdot (1-\theta_1)^{10}}{\int_0^1 \theta_1^{90} \cdot (1-\theta_1)^{10} d\theta_1}$$

$$p(\theta_2|2,0) = \frac{\theta_2^{2} \cdot (1-\theta_2)^{0}}{B(3,1)} = \frac{\theta_2^{2} \cdot (1-\theta_2)^{0}}{\int_0^1 \theta_2^{2} \cdot (1-\theta_2)^{0} d\theta_2}$$

Using the solution for Beta distributed integrals $B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$ with $\Gamma(x) = (x-1)!$ given by Euler, for $m, n \in \mathbb{N}$ leads to the following expressions:

$$p(\theta_1|90,10) = \frac{\theta_1^{90}\cdot(1-\theta_1)^{10}}{\frac{90!\cdot 10!}{101!}} = \frac{101!}{90!\cdot 10!}\cdot\theta_1^{90}\cdot(1-\theta_1)^{10}$$
$$p(\theta_2|2,0) = \frac{\theta_2^2\cdot(1-\theta_2)^0}{\frac{2!\cdot 0!}{3!}} = \frac{\theta_2^2}{6}$$

4 The Poisson Distribution

Proof. The binomial distribution converges to the poisson distribution for a large number of trials with small probability.

$$p_b(r|f,n) = \binom{n}{r}\cdot f^r\cdot(1-f)^{n-r}$$
$$\lim_{n\to\infty} f_n = \frac{\lambda}{n}$$
$$p_b(r|f,n) = \lim_{n\to\infty}\frac{n\cdot(n-1)...(n-r-1)}{r!}\cdot(\frac{\lambda}{n})^r\cdot(1-\frac{\lambda}{n})^{n-r}$$
$$p_b(r|f,n) = \frac{\lambda^r}{r!}\lim_{n\to\infty}(n\cdot(n-1)...(n-r-1))\cdot\frac{1}{n^r}\cdot(1-\frac{\lambda}{n})^n\cdot(1-\frac{\lambda}{n})^{-r}$$

Now we divide this equation into three parts and look at each of the parts separately. The first part of the equation converges to 1:

$$\lim_{n\to\infty}(n\cdot(n-1)...(n-r-1))\cdot\frac{1}{n^r} \tag{2}$$

$$= \frac{n}{n}\cdot\frac{n-1}{n}...\frac{n-r-1}{n}$$
$$= 1$$

For the second part we need to recall that $e = \lim_{x\to\infty}(1+\frac{1}{x})^x$ and we introduce the variable x with $x = -\frac{n}{\lambda}$. If we plug these things in the second part of the equation we get:

$$\lim_{n\to\infty}(1-\frac{\lambda}{n})^n \tag{3}$$

$$\lim_{n\to\infty}(1-\frac{\lambda}{n})^n = \lim_{n\to\infty}(1+\frac{1}{x})^{-\lambda x} = e^{-\lambda}$$

The third and last part converges to 1 like the following:

$$\lim_{n\to\infty}(1-\frac{\lambda}{n})^{-r} \tag{4}$$

$$\lim_{n\to\infty}(1-\frac{\lambda}{n})^{-r}=1^{-r}=1$$

If we plug all three parts together one can see that the binomial distribution converges to the poisson distribution and it shows that $\lim_{n\to\infty} p_b(r|f,n) = p_p(r|\lambda)$.

$$p_b(r|f,n)=\frac{\lambda^r}{r!}\lim_{n\to\infty}(n\cdot(n-1)...(n-r-1))\cdot\frac{1}{n^r}\cdot(1-\frac{\lambda}{n})^n\cdot(1-\frac{\lambda}{n})^{-r}=\frac{\lambda^r}{r!}e^{-\lambda}$$

$\square$