due 10:00 on Monday, 28 January 2019

Probabilistic Inference & Learning

Exercise Sheet #12

Monte Carlo Methods

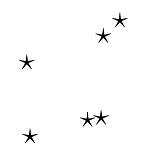
1. **Inverse Transform Sampling:** The standard Cauchy distibution is given by

$$p(x) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

Given a random variable u distributed uniformly over $(0, 1)$, find a transformation $x = h(u)$, such that $x \sim p$. Hint: This requires the CDF $P(x) = \int_{-\infty}^{x} p(\tilde{x})\, d\tilde{x}$, an integral that can be found in standard tables.[1] *20 points*

2. **Analysing Rejection Sampling:** The lecture introduced rejection sampling, but only provided a visual proof that it is a valid method. In this exercise you will formally show that it is. Consider samples x_i drawn i.i.d. from the distribution $q(x)$. In rejection sampling, such samples are accepted with probability $p(\text{accept} \mid x_i) = \frac{\tilde{p}(x_i)}{cq(x_i)}$, where $\tilde{p}(x) = Z \cdot p(x)$ with constants c, Z, such that $cq(x) \geq \tilde{p}(x)\, \forall x$. Note that the probability of rejection sampling producing the sample $x_i = x$ is given by the probability of drawing x_i from q times the probability of accepting that value, *given* that it was drawn. Make use of this, with the rules of probability (sum and product rule, Bayes theorem) to write down the probability of drawing x given that it was accepted, and show that this probability is equal to $p(x)$. *20 points*

3. **Implementing Rejection Sampling:**[2] *Two men looked through prison bars; one saw stars, the other tried to infer where the window frame was.*[3] It is dark night, you are sitting in a pitch-black room. In the darkness, you can see six stars twinkling.



You know that these stars are shining through a square window (a square with center (x_c, y_c) and width w. But where in the wall is this window (what are x_c, y_c), and how large is it (what is w)? For your computations: The locations of the stars are

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -0.2335 & -0.2123 & 0.3803 & 0.2564 & 0.1670 & 0.2709 \\ 0.0586 & -0.4310 & 0.3358 & -0.3078 & -0.3221 & 0.2334 \end{pmatrix}$$

and we will assume that stars are distributed uniformly at random across the night sky, which is assumed to be approximately a flat 2d-surface (that is actually how this dataset was generated).

   (a) What is the likelihood function $p(X, Y \mid x_c, y_c, w)$? *10 points*

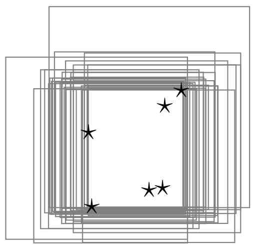
[1] e.g. `https://en.wikipedia.org/wiki/List_of_integrals_of_rational_functions`

[2] You can enter your answers in the prepared ipython notebook `PrisonStars.ipynb` available on ilias. The notebook also contains a scaffold for plotting.

[3] This is an extended form of Exercise 22.10 in David MacKay's book "Information Theory, Inference and Learning Algorithms", available at `http://www.inference.phy.cam.ac.uk/itprnn/book.html`. It is the two-dimensional (and more romantic) version of the famous "German Tank Problem" of estimating the outer bounds of a (uniform) distribution from samples.

(b) What is the *maximum likelihood* estimate for (x_c, y_c, w)? *10 points*

(c) Using a broad uniform prior $p(x_c, y_c, w) = U(x_c; -2, 2)U(y_c; -2, 2)U(w; 0, 2)$ (that is, values for $x_c, y_c \in [-2, 2]$ and $w \in [0, 2]$, $U(x; a, b)$ is the uniform distribution over $[a, b]$), implement a rejection sampler for windows that can produce samples from the posterior $p(x_c, y_c, w \mid X, Y)$, such as these: *30 points*



(d) What are the average (i.e. expected) values for (x_c, y_c, w) found by your sampler? *10 points*