Matrix Theory EE5609 - Assignment 11

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***Abstract*—Minimum Polynomial**

1 Problem

Let A $\in M_3(\mathbb{R})$ be such that $A^8 = I_{3\times3}$ Then

1) minimal polynomial of A can only be of degree 2.
2) minimal polynomial of A can only be of degree 3.
3) either $A = I_{3\times3}$ or $A = -I_{3\times3}$
4) there are uncountably many A satisfying the above.

2 Solution

Given	A $\in M_3(\mathbb{R})$ be such that $A^8 = I_{3\times3}$.
Option 1 : minimal polynomial of A can only be of degree 2	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ The Characteristic polynomial is $-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$ Minimum polynomial is of degree 1. Hence this option is not correct
Option 2 : minimal polynomial of A can only be of degree 3	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ as given in option 1, the minimum polynomial is of degree 1. Hence this option is not correct
Option 3 : either $A = I_{3\times3}$ or $A = -I_{3\times3}$	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ Here, $A^8 = I_{3\times3}$ and $A \neq I_{3\times3}$ or $A \neq -I_{3\times3}$. Hence this option is not correct

Option 4 : there are uncountably many A satisfying the above	Let A be any 3×3 involuntary matrix. **Involuntary matrix:** A matrix is said to be involuntary matrix if the matrix is its own inverse. Therefore, for an involuntary matrix, $A^2 = I$. For an involuntary matrix, A^n will be equal to A if n is odd and I if n is even. Cleary, $A^8 = I$ for all involuntary matrices. The set of involuntary matrices is uncountable. Hence there are uncountably many A which satify the above condition Hence, this option is the correct answer. Example: $$A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$ $$A^2 = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$ $$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $$\therefore A^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$