Matrix Theory EE5609 - Assignment 11

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***Abstract*—Minimum Polynomial**

1 Problem

Let A $\in M_3(\mathbb{R})$ be such that $A^8 = I_{3\times3}$ Then

1) minimal polynomial of A can only be of degree 2.
2) minimal polynomial of A can only be of degree 3.
3) either $A = I_{3\times3}$ or $A = -I_{3\times3}$
4) there are uncountably many A satisfying the above.

2 Solution

Given	A $\in M_3(\mathbb{R})$ be such that $A^8 = I_{3\times3}$.
Option 1 : minimal polynomial of A can only be of degree 2	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ The Characteristic polynomial is $-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$ Minimum polynomial is of degree 1. Hence this option is not correct
Option 2 : minimal polynomial of A can only be of degree 3	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ as given in option 1, the minimum polynomial is of degree 1. Hence this option is not correct
Option 3 : either $A = I_{3\times3}$ or $A = -I_{3\times3}$	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ Here, $A^8 = I_{3\times3}$ and $A \neq I_{3\times3}$ or $A \neq -I_{3\times3}$. Hence this option is not correct

Option 4 : there are uncountably many A satisfying the above	Let M(θ) be 2×2 orthogonal matrix $$M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ Let $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then, $$M^{-1}(\theta) = M^T(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ $$\begin{aligned} M^T(\theta)PM(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \end{aligned}$$ This is clearly an uncountable family of 2×2 matrices. $$\begin{aligned} (M^T(\theta)PM(\theta))^2 &= M^T(\theta)PM(\theta)M^T(\theta)PM(\theta) \\ &= M^T(\theta)P^2M(\theta) = M^T(\theta)IM(\theta) = I \end{aligned}$$ Also, $$(M^T(\theta)PM(\theta))^8 = I$$ $$\text{Let } A(\theta) = \begin{pmatrix} M^T(\theta)PM(\theta) & 0 \\ 0 & -1 \end{pmatrix}$$ $$A^8(\theta) = \begin{pmatrix} (O^T(\theta)PO(\theta))^8 & 0 \\ 0 & (-1)^8 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} = I$$ and the family $\{A(\theta) \mid \theta \in \mathbb{R}\}$ is uncountable. Hence there are uncountably many A which satify the above condition Hence, this option is the correct answer.