Matrix Theory EE5609 - Assignment 3 Find if a triangle is isosceles triangle.

Sandhya Addetla
PhD Artificial Inteligence Department
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AI20RESCH14001

***Abstract*—This document provides a solution for finding if a traingle is isosceles given two equal altitudes of the triangle**

I. Problem Statement

BE and CF are two equal altitudes of a triangle ABC. Prove that the triangle ABC is isosceles.

II. Solution

Given that BE and CF are two equal altitudes of a triangle ABC.

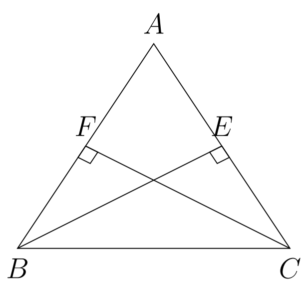

Fig. 1: Triangle with equal altitudes on two sides

So, we have :-

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{1}$$

Let $\mathbf{m}_{AB}$ and $\mathbf{m}_{CF}$ are the direction vectors of AB and CF respectively. Since AB $\perp$ CF hence,

$$\mathbf{m}_{AB}\mathbf{m}_{CF} = 0 \tag{2}$$

$$\left(\mathbf{B}-\mathbf{E}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{C}\right) = \mathbf{0} \tag{3}$$

$$\left(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{E}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{C}\right) = \mathbf{0} \tag{4}$$

$$\left(\mathbf{B}-\mathbf{A}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{C}\right) + \|\mathbf{A}-\mathbf{C}\|^2 + \left(\mathbf{C}-\mathbf{F}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{C}\right) = \mathbf{0} \tag{5}$$

Similarly, AC $\perp$ BE hence,

$$\mathbf{m}_{AC}\mathbf{m}_{BE} = 0 \tag{6}$$

$$\left(\mathbf{C}-\mathbf{F}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{B}\right) = \mathbf{0} \tag{7}$$

$$\left(\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{F}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{B}\right) = \mathbf{0} \tag{8}$$

$$\left(\mathbf{C}-\mathbf{A}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{B}\right) + \|\mathbf{A}-\mathbf{B}\|^2 + \left(\mathbf{B}-\mathbf{F}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{B}\right) = \mathbf{0} \tag{9}$$

In $\triangle ABC$, taking inner product of sides AB and AC we can write :

$$\implies \cos\angle BAC = \frac{\left(\mathbf{B}-\mathbf{A}\right)^T\left(\mathbf{A}-\mathbf{C}\right)}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{10}$$

Simlilarly,

$$\cos\angle CAB = \frac{\left(\mathbf{C}-\mathbf{A}\right)^T\left(\mathbf{A}-\mathbf{B}\right)}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|} \tag{11}$$

From equation 10, and 11, we have ,

$$\left(\mathbf{B}-\mathbf{A}\right)^T\left(\mathbf{A}-\mathbf{C}\right) = \left(\mathbf{C}-\mathbf{A}\right)^T\left(\mathbf{A}-\mathbf{B}\right) \tag{12}$$

using equation 12 in 5 and 9 we can write,

$$\|\mathbf{A}-\mathbf{C}\|^2 + \left(\mathbf{C}-\mathbf{E}\right)^T\left(\mathbf{A}-\mathbf{C}\right) = \|\mathbf{A}-\mathbf{B}\|^2 + \left(\mathbf{B}-\mathbf{F}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{B}\right) \tag{13}$$

$$\|\mathbf{A}-\mathbf{C}\|^2 + \left(\mathbf{C}-\mathbf{E}\right)^T\left(\mathbf{A}-\mathbf{C}\right) = \|\mathbf{A}-\mathbf{B}\|^2 + \left(\mathbf{B}-\mathbf{F}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{B}\right) \tag{14}$$

$$\|\mathbf{A}-\mathbf{C}\|^2 + \left(\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{E}\right)^T\left(\mathbf{A}-\mathbf{C}\right) = \|\mathbf{A}-\mathbf{B}\|^2 + \left(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{F}\right)^{\mathbf{T}}\left(\mathbf{A}-\mathbf{B}\right) \tag{15}$$

$$\|\mathbf{A}-\mathbf{C}\|^2+\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})+ \\ (\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C})=\|\mathbf{A}-\mathbf{B}\|^2+ \\ \|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B})+ \\ (\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{16}$$

since BE $\perp$ AC and CF $\perp$ AB, hence :

$$(\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C})=\mathbf{0} \tag{17}$$

$$(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B})=\mathbf{0} \tag{18}$$

Now equation 16 become :

$$2\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})= \\ 2\|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{19}$$

Substituting equation 12 in equation 19,

$$\|\mathbf{A}-\mathbf{C}\|=\|\mathbf{A}-\mathbf{B}\| \tag{20}$$

Therefore, the magnitude of sides AB and AC of $\triangle ABC$ are equal and hence they form the sides of an isosceles triangle.