Matrix Theory EE5609 - Assignment 5

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***Abstract*—Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola**

Download all python codes from

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https://github.com/SANDHYA−A/Assignment5/
    blob/master/Assignment5.py
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1 Problem Statement

Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola

$$19x^2 + 24xy + y^2 - 22x - 6y = 0 \tag{1.0.1}$$

2 Solution

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

Comparing equations 1.0.1 and 2.0.2 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \tag{2.0.6}$$

$$f = 0 \tag{2.0.7}$$

Expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix} 19 & 12 \\ 12 & 1 \end{vmatrix} < 0 \tag{2.0.8}$$

Hence from 2.0.8 given equation represents the hyperbola.

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.0.9}$$

$$\begin{vmatrix} 19-\lambda & 12 \\ 12 & 1-\lambda \end{vmatrix} = 0 \tag{2.0.10}$$

$$(19-\lambda)(1-\lambda) - 144 = 0 \tag{2.0.11}$$

$$\lambda_1 = -5, \lambda_2 = 25 \tag{2.0.12}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.13}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.14}$$

For $\lambda_1 = -5$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 19+5 & 12 \\ 12 & 1+5 \end{pmatrix} \tag{2.0.15}$$

By row reduction ,

$$\begin{pmatrix} 24 & 12 \\ 12 & 6 \end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 24 & 12 \\ 0 & 0 \end{pmatrix} \tag{2.0.17}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{12}} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.18}$$

Subsituting equation 2.0.18 in equation 2.0.14 we get

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_1 = t$

$$v_2 = -2t \tag{2.0.20}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} t \\ -2t \end{pmatrix} \tag{2.0.21}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.0.22}$$

For $\lambda_2 = 25$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} 19-25 & 12 \\ 12 & 1-25 \end{pmatrix} \tag{2.0.23}$$

By row reduction ,

$$\begin{pmatrix} -6 & 12 \\ 12 & -24 \end{pmatrix} \tag{2.0.24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -6 & 12 \\ 0 & 0 \end{pmatrix} \tag{2.0.25}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \tag{2.0.26}$$

Subsituting equation 2.0.26 in equation 2.0.14 we get

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.27}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_1 = t$

$$v_2 = \frac{t}{2} \tag{2.0.28}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} t \\ \frac{t}{2} \end{pmatrix} \tag{2.0.29}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.30}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.31}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.33}$$

Substituting equations 2.0.22, 2.0.30 in equation 2.0.32 we get

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \tag{2.0.34}$$

Substituting equation 2.0.12 in 2.0.33 we get

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.0.35}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.36}$$

$$\implies \mathbf{c} = -\frac{1}{125} \begin{pmatrix} 1 & -12 \\ -12 & 19 \end{pmatrix} \begin{pmatrix} -11 \\ -3 \end{pmatrix} \tag{2.0.37}$$

$$\implies \mathbf{c} = -\frac{1}{125} \begin{pmatrix} 25 \\ 75 \end{pmatrix} \tag{2.0.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{5} \\ -\frac{3}{5} \end{pmatrix} \tag{2.0.39}$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = -4 < 0 \tag{2.0.40}$$

we need to swap axes. In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{2.0.41}$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{4}{5}} \tag{2.0.42}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{\frac{4}{25}} \tag{2.0.43}$$

Now we have,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{2.0.44}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.0.45}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.0.46}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{5} \\ -\frac{3}{5} \end{pmatrix} \tag{2.0.47}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{2} \end{pmatrix} \tag{2.0.48}$$

Subsituting the equation 2.0.35 in equation 2.0.44

$$\implies \mathbf{y}^T \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{y} = -4 \tag{2.0.49}$$

2.0.49 is the equation of the hyperbola. Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$19x^2 + 24xy + y^2 - 22x - 6y + K = 0 \quad (2.0.50)$$

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.0.51)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \quad (2.0.52)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (2.0.53)$$

$$f = K \quad (2.0.54)$$

$$\Delta = \begin{vmatrix} 19 & 12 & -11 \\ 12 & 1 & -3 \\ -11 & -3 & K \end{vmatrix} \quad (2.0.55)$$

Since the equations represent pair of straight lines, equating the determinant to zero, we can get the value of K

$$\implies K = 4 \quad (2.0.56)$$

Let the equations of lines be,

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.0.57)$$

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -11 & -3 \end{pmatrix}\mathbf{x} + 4 \quad (2.0.58)$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 19 \\ 24 \\ 1 \end{pmatrix} \quad (2.0.59)$$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.0.60)$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (2.0.61)$$

$$c_1c_2 = 4 \quad (2.0.62)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.0.63)$$

$$\implies m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (2.0.64)$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.0.65)$$

Substituting the given data in above equations 2.0.63 we get,

$$m^2 + 24m + 19 = 0 \quad (2.0.66)$$

$$m_1 = -12 + 5\sqrt{5}, m_2 = -12 - 5\sqrt{5} \quad (2.0.67)$$

$$\mathbf{n_1} = \begin{pmatrix} 12 - 5\sqrt{5} \\ 1 \end{pmatrix} \quad (2.0.68)$$

$$\mathbf{n_2} = \begin{pmatrix} 12 + 5\sqrt{5} \\ 1 \end{pmatrix} \quad (2.0.69)$$

Equation 2.0.68 and 2.0.69 satisfies equation 2.0.61, so c_1 and c_2 can be obtained as,

$$c_1 = 3 + \sqrt{5} \quad (2.0.70)$$

$$c_2 = 3 - \sqrt{5} \quad (2.0.71)$$

Equation 2.0.57 can be modified by using equation 2.0.68, 2.0.69, 2.0.70 and 2.0.71

$$\begin{pmatrix} 12 - 5\sqrt{5} & 1 \end{pmatrix}\mathbf{x} = 3 + \sqrt{5} \quad (2.0.72)$$

$$\begin{pmatrix} 12 + 5\sqrt{5} & 1 \end{pmatrix}\mathbf{x} = 3 - \sqrt{5} \quad (2.0.73)$$

$$\implies (12 - 5\sqrt{5})x + y - 3 - \sqrt{5} = 0 \quad (2.0.74)$$

$$\implies (12 + 5\sqrt{5})x + y - 3 + \sqrt{5} = 0 \quad (2.0.75)$$

$$((12 - 5\sqrt{5})x + y - 3 - \sqrt{5}) ((12 + 5\sqrt{5})x + y - 3 + \sqrt{5}) = 0 \quad (2.0.76)$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \quad (2.0.77)$$

$$\begin{vmatrix} 19 - \lambda & 12 \\ 12 & 1 - \lambda \end{vmatrix} = 0 \quad (2.0.78)$$

$$(19 - \lambda)(1 - \lambda) - 144 = 0 \quad (2.0.79)$$

$$\lambda_1 = -5, \lambda_2 = 25 \quad (2.0.80)$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.81)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.0.82)$$

For $\lambda_1 = -5$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 19 + 5 & 12 \\ 12 & 1 + 5 \end{pmatrix} \quad (2.0.83)$$

By row reduction ,

$$\begin{pmatrix} 24 & 12 \\ 12 & 6 \end{pmatrix} \quad (2.0.84)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 24 & 12 \\ 0 & 0 \end{pmatrix} \quad (2.0.85)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{12}} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.86)$$

Subsituting equation 2.0.86 in equation 2.0.82 we get

$$\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{2.0.87}$$

Where, $\mathbf{p} = \begin{pmatrix}v_1\\ v_2\end{pmatrix}$ Let $v_1 = t$

$$v_2 = -2t \tag{2.0.88}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix}t\\ -2t\end{pmatrix} \tag{2.0.89}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix}1\\ -1\end{pmatrix} \tag{2.0.90}$$

For $\lambda_2 = 25$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix}19-25 & 12\\ 12 & 1-25\end{pmatrix} \tag{2.0.91}$$

By row reduction ,

$$\begin{pmatrix}-6 & 12\\ 12 & -24\end{pmatrix} \tag{2.0.92}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix}-6 & 12\\ 0 & 0\end{pmatrix} \tag{2.0.93}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix} \tag{2.0.94}$$

Subsituting equation 2.0.94 in equation 2.0.82 we get

$$\begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{2.0.95}$$

Where, $\mathbf{p} = \begin{pmatrix}v_1\\ v_2\end{pmatrix}$ Let $v_1 = t$

$$v_2 = \frac{t}{2} \tag{2.0.96}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix}t\\ \frac{t}{2}\end{pmatrix} \tag{2.0.97}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix}1\\ \frac{1}{2}\end{pmatrix} \tag{2.0.98}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.99}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} \tag{2.0.100}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \tag{2.0.101}$$

Substituting equations 2.0.90, 2.0.98 in equation 2.0.100 we get

$$\mathbf{P} = \begin{pmatrix}1 & 1\\ -1 & \frac{1}{2}\end{pmatrix} \tag{2.0.102}$$

Substituting equation 2.0.80 in 2.0.101 we get

$$\mathbf{D} = \begin{pmatrix}-5 & 0\\ 0 & 25\end{pmatrix} \tag{2.0.103}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.104}$$

$$\implies \mathbf{c} = -\frac{1}{125}\begin{pmatrix}1 & -12\\ -12 & 19\end{pmatrix}\begin{pmatrix}-11\\ -3\end{pmatrix} \tag{2.0.105}$$

$$\implies \mathbf{c} = -\frac{1}{125}\begin{pmatrix}25\\ 75\end{pmatrix} \tag{2.0.106}$$

$$\implies \mathbf{c} = \begin{pmatrix}\frac{-1}{5}\\ -\frac{3}{5}\end{pmatrix} \tag{2.0.107}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = -8 < 0 \tag{2.0.108}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.109}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.0.110}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.0.111}$$

$$\mathbf{y} = \begin{pmatrix}1 & -1\\ 1 & \frac{1}{2}\end{pmatrix}\mathbf{x} - \begin{pmatrix}1 & -1\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-\frac{1}{5}\\ -\frac{3}{5}\end{pmatrix} \tag{2.0.112}$$

$$\mathbf{y} = \begin{pmatrix}1 & -1\\ 1 & \frac{1}{2}\end{pmatrix}\mathbf{x} - \begin{pmatrix}\frac{2}{5}\\ -\frac{1}{2}\end{pmatrix} \tag{2.0.113}$$

Subsituting the equation 2.0.103 in equation 2.0.109

$$\implies \mathbf{y}^T\begin{pmatrix}-5 & 0\\ 0 & 25\end{pmatrix}\mathbf{y} = -8 \tag{2.0.114}$$

$$\implies \mathbf{y}^T\begin{pmatrix}-5 & 0\\ 0 & 25\end{pmatrix}\mathbf{y} + 8 = 0 \tag{2.0.115}$$

2.0.115 represents equation of asymptotes. The Equation of Conjugate hyperbola is given by:
2(Equation of Asymptotes)- Equation of hyperbola.

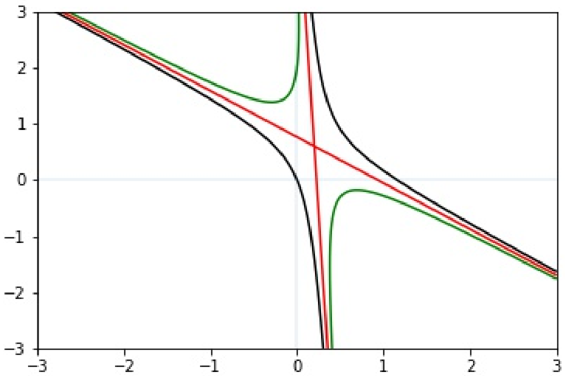

Fig. 1: Hyperbola, Asymptotes and Conjugate Hyberbola

So, from equation 2.0.49 and 2.0.115 , we get equation of conjugate hyperbola as:-

$$\implies \mathbf{y}^T \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{y} + 12 = 0 \qquad (2.0.116)$$