Matrix Theory EE5609 - supporting Assignment 5

Sandhya Addetla
PhD Artificial Inteligence Department
AI20RESCH14001

***Abstract*—Relation between hyperbola and conjugate hyperbola in terms of $\mathbf{V}, \mathbf{u}$ and f.**

1 Solution

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.0.1)$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (1.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (1.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (1.0.4)$$

Equation 1.0.2 refers equation of a hyperbola if $|V| < 0$.
Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + (f + K) = 0 \quad (1.0.5)$$

The above equation 1.0.5 represents equation of asymtotes for equation 1.0.2 and can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f_1 = 0 \quad (1.0.6)$$

Where

$$f_1 = f + K \quad (1.0.7)$$

For a pair of straight lines,

$$\Delta = \begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f_1 \end{vmatrix} = 0 \quad (1.0.8)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f + K \end{vmatrix} = 0 \quad (1.0.9)$$

$$\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f + K \end{vmatrix} = 0 \quad (1.0.10)$$

expanding this determinant, we get,

$$ac(f + K) + 2bde - ae^2 - cd^2 - (f + K)b^2 = 0 \quad (1.0.11)$$

$$K = -\frac{(acf + 2bde - ae^2 - cd^2 - b^2 f)}{ac - b^2} \quad (1.0.12)$$

$$K = -\frac{\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}}{|\ \mathbf{V}\ |} \quad (1.0.13)$$

Let equation of the conjugate hyperbola be as given below

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f_2 = 0 \quad (1.0.14)$$

and it can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f_2 = 0 \quad (1.0.15)$$

The Equation of Conjugate hyperbola is also given by:

2(Equation of Asymptotes)- Equation of hyperbola.

$$2 \times (\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f_1) - (\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f) = 0 \quad (1.0.16)$$

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + (2f_1 - f) = 0 \quad (1.0.17)$$

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + (f + 2K) = 0 \quad (1.0.18)$$

comparing equations 1.0.15 and 1.0.18, we get

$$f_2 = f + 2K \quad (1.0.19)$$

$$f_2 = f - 2\frac{\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}}{|\ \mathbf{V}\ |} \quad (1.0.20)$$

This shows the relation between hyperbola and conjugate hyperbola in terms of $\mathbf{V}, \mathbf{u}$ and f.