Matrix Theory EE5609 - Assignment 5

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***Abstract*—Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola**

Download all python codes from

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https://github.com/SANDHYA−A/Assignment5/
    blob/master/Assignment5.py
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1 Problem Statement

Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola

$$19x^2 + 24xy + y^2 - 22x - 6y = 0 \tag{1.0.1}$$

2 Solution

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

Comparing equations 1.0.1 and 2.0.2 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \tag{2.0.6}$$

$$f = 0 \tag{2.0.7}$$

Expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix} 19 & 12 \\ 12 & 1 \end{vmatrix} < 0 \tag{2.0.8}$$

Hence from 2.0.8 given equation represents the hyperbola.

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| \, V - \lambda\mathbf{I} \, \right| = 0 \tag{2.0.9}$$

$$\begin{vmatrix} 19-\lambda & 12 \\ 12 & 1-\lambda \end{vmatrix} = 0 \tag{2.0.10}$$

$$(19-\lambda)(1-\lambda) - 144 = 0 \tag{2.0.11}$$

$$\lambda_1 = -5, \lambda_2 = 25 \tag{2.0.12}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.13}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.14}$$

For $\lambda_1 = -5$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 19+5 & 12 \\ 12 & 1+5 \end{pmatrix} \tag{2.0.15}$$

By row reduction ,

$$\begin{pmatrix} 24 & 12 \\ 12 & 6 \end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 24 & 12 \\ 0 & 0 \end{pmatrix} \tag{2.0.17}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{12}} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.18}$$

Subsituting equation 2.0.18 in equation 2.0.14 we get

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_1 = t$

$$v_2 = -2t \tag{2.0.20}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} t \\ -2t \end{pmatrix} \tag{2.0.21}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.22}$$

For $\lambda_2 = 25$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} 19-25 & 12 \\ 12 & 1-25 \end{pmatrix} \tag{2.0.23}$$

By row reduction ,

$$\begin{pmatrix} -6 & 12 \\ 12 & -24 \end{pmatrix} \tag{2.0.24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -6 & 12 \\ 0 & 0 \end{pmatrix} \tag{2.0.25}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \tag{2.0.26}$$

Subsituting equation 2.0.26 in equation 2.0.14 we get

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.27}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_1 = t$

$$v_2 = \frac{t}{2} \tag{2.0.28}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} t \\ \frac{t}{2} \end{pmatrix} \tag{2.0.29}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.30}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.31}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.33}$$

Substituting equations 2.0.22, 2.0.30 in equation 2.0.32 we get

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix} \tag{2.0.34}$$

Substituting equation 2.0.12 in 2.0.33 we get

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.0.35}$$

Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$19x^2 + 24xy + y^2 - 22x - 6y + K = 0 \tag{2.0.36}$$

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.37}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \tag{2.0.38}$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \tag{2.0.39}$$

$$f = K \tag{2.0.40}$$

$$\Delta = \begin{vmatrix} 19 & 12 & -11 \\ 12 & 1 & -3 \\ -11 & -3 & K \end{vmatrix} \tag{2.0.41}$$

Since the equations represent pair of straight lines, equating the determinant to zero, we can get the value of K

$$\implies K = 4 \tag{2.0.42}$$

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \tag{2.0.43}$$

Substituting the values, we obtain,

$$\begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix} \tag{2.0.44}$$

$$\text{We get, } \alpha = \frac{1}{5}, \beta = \frac{3}{5} \tag{2.0.45}$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + c \tag{2.0.46}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2.0.47}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \tag{2.0.48}$$

under transformation 2.0.48 will become,

$$aX^2 + 2bXY + cY^2 = 0 \tag{2.0.49}$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.50}$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.51}$$

$$\begin{pmatrix} X' & Y' \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \tag{2.0.52}$$

$$\text{where } X' = Xu_1 + Yu_2 \tag{2.0.53}$$

$$Y' = Xv_1 + Yv_2 \tag{2.0.54}$$

$$\implies \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \tag{2.0.55}$$

2.0.55 is called *Spectral decomposition* of matrix

$$\implies X' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}Y' \tag{2.0.56}$$

$$u_1X + u_2Y = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1X + v_2Y) \tag{2.0.57}$$

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \tag{2.0.58}$$

Substituting values, we get

$$(x - \frac{1}{5}) - 2(y - \frac{3}{5}) = \pm\sqrt{\frac{25}{5}}(x - \frac{1}{5}) + \frac{1}{2}(y - \frac{3}{5}) \tag{2.0.59}$$

Simplifying above equation

$$8x + 9y - 7 = 0 \tag{2.0.60}$$

$$12x + y + 7 = 0 \tag{2.0.61}$$

$$\implies (8x + 9y - 7)(12x + y + 7) = 0 \tag{2.0.62}$$

Thus the equation of lines are

$$\begin{pmatrix} 8 & 9 \end{pmatrix} \mathbf{x} = 7 \tag{2.0.63}$$

$$\begin{pmatrix} 12 & 1 \end{pmatrix} \mathbf{x} = -7 \tag{2.0.64}$$

The Equation of Conjugate hyperbola is given by:

2(Equation of Asymptotes)- Equation of hyperbola.

From Eq 1.0.1 and 2.0.36, we obtain equation of Conjugate hyperbola as:-

$$19x^2 + 24xy + y^2 - 22x - 6y + 8 = 0 \tag{2.0.65}$$

Which can be expressed as,

$$\mathbf{x}^T \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -11 \\ -3 \end{pmatrix} \mathbf{x} + 8 = 0 \tag{2.0.66}$$

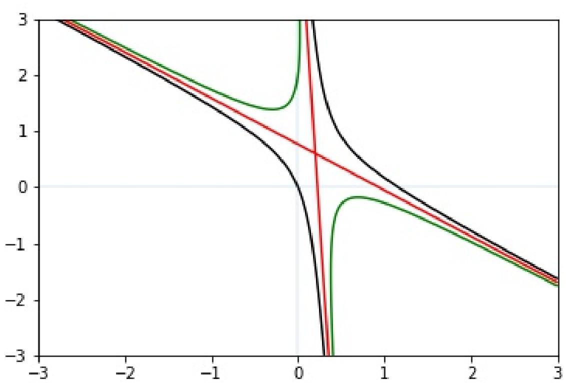

Fig. 1: Hyperbola, Conjugate Hyperbola and Asymptotes