

# Matrix Theory EE5609 - Assignment 5

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**Abstract**—Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola

Download all python codes from

<https://github.com/SANDHYA-A/Assignment5/blob/master/Assignment5.py>

## 1 PROBLEM STATEMENT

Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola

$$19x^2 + 24xy + y^2 - 22x - 6y = 0 \quad (1.0.1)$$

## 2 SOLUTION

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.4)$$

Comparing equations 1.0.1 and 2.0.2 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \quad (2.0.5)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (2.0.6)$$

$$f = 0 \quad (2.0.7)$$

Expanding the Determinant of  $\mathbf{V}$ .

$$\Delta_V = \begin{vmatrix} 19 & 12 \\ 12 & 1 \end{vmatrix} < 0 \quad (2.0.8)$$

Hence from 2.0.8 given equation represents the hyperbola.

The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|V - \lambda \mathbf{I}| = 0 \quad (2.0.9)$$

$$\begin{vmatrix} 19 - \lambda & 12 \\ 12 & 1 - \lambda \end{vmatrix} = 0 \quad (2.0.10)$$

$$(19 - \lambda)(1 - \lambda) - 144 = 0 \quad (2.0.11)$$

$$\lambda_1 = -5, \lambda_2 = 25 \quad (2.0.12)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.13)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.0.14)$$

For  $\lambda_1 = -5$ ,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 19 + 5 & 12 \\ 12 & 1 + 5 \end{pmatrix} \quad (2.0.15)$$

By row reduction ,

$$\begin{pmatrix} 24 & 12 \\ 12 & 6 \end{pmatrix} \quad (2.0.16)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 24 & 12 \\ 0 & 0 \end{pmatrix} \quad (2.0.17)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{12}} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.18)$$

Substituting equation 2.0.18 in equation 2.0.14 we get

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.19)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_1 = t$

$$v_2 = -2t \quad (2.0.20)$$

Eigen vector  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} t \\ -2t \end{pmatrix} \quad (2.0.21)$$

Let  $t = 1$ , we get

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2.0.22)$$

For  $\lambda_2 = 25$ ,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} 19 - 25 & 12 \\ 12 & 1 - 25 \end{pmatrix} \quad (2.0.23)$$

By row reduction ,

$$\begin{pmatrix} -6 & 12 \\ 12 & -24 \end{pmatrix} \quad (2.0.24)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -6 & 12 \\ 0 & 0 \end{pmatrix} \quad (2.0.25)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \quad (2.0.26)$$

Substituting equation 2.0.26 in equation 2.0.14 we get

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.27)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_1 = t$

$$v_2 = \frac{t}{2} \quad (2.0.28)$$

Eigen vector  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} t \\ \frac{t}{2} \end{pmatrix} \quad (2.0.29)$$

Let  $t = 1$ , we get

$$\mathbf{p}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.30)$$

By eigen decomposition  $\mathbf{V}$  can be represented by

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (2.0.31)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.0.32)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.33)$$

Substituting equations 2.0.22, 2.0.30 in equation 2.0.32 we get

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \quad (2.0.34)$$

Substituting equation 2.0.12 in 2.0.33 we get

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \quad (2.0.35)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (2.0.36)$$

$$\Rightarrow \mathbf{c} = -\frac{1}{125} \begin{pmatrix} 1 & -12 \\ -12 & 19 \end{pmatrix} \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (2.0.37)$$

$$\Rightarrow \mathbf{c} = -\frac{1}{125} \begin{pmatrix} 25 \\ 75 \end{pmatrix} \quad (2.0.38)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} -\frac{1}{5} \\ \frac{3}{5} \end{pmatrix} \quad (2.0.39)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = -4 < 0 \quad (2.0.40)$$

we need to swap axes. In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (2.0.41)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{4}{5}} \quad (2.0.42)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{\frac{4}{25}} \quad (2.0.43)$$

Now we have,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.0.44)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.0.45)$$

To get  $\mathbf{y}$ ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (2.0.46)$$

$$\mathbf{y} = \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{5} \\ -\frac{3}{5} \end{pmatrix} \quad (2.0.47)$$

$$\mathbf{y} = \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{2} \end{pmatrix} \quad (2.0.48)$$

Substituting the equation 2.0.35 in equation 2.0.44

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{y} = -4 \quad (2.0.49)$$

2.0.49 is the equation of the hyperbola. Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$19x^2 + 24xy + y^2 - 22x - 6y + K = 0 \quad (2.0.50)$$

The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.51)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \quad (2.0.52)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (2.0.53)$$

$$f = K \quad (2.0.54)$$

$$\Delta = \begin{vmatrix} 19 & 12 & -11 \\ 12 & 1 & -3 \\ -11 & -3 & K \end{vmatrix} \quad (2.0.55)$$

Since the equations represent pair of straight lines, equating the determinant to zero, we can get the value of K

$$\Rightarrow K = 4 \quad (2.0.56)$$

Let the equations of lines be,

$$(\mathbf{n}_1^T \mathbf{x} - c_1) (\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.57)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1) (\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -11 & -3 \end{pmatrix} \mathbf{x} + 4 \quad (2.0.58)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 19 \\ 24 \\ 1 \end{pmatrix} \quad (2.0.59)$$

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.0.60)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (2.0.61)$$

$$c_1 c_2 = 4 \quad (2.0.62)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.0.63)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (2.0.64)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.0.65)$$

Substituting the given data in above equations 2.0.63 we get,

$$m^2 + 24m + 19 = 0 \quad (2.0.66)$$

$$m_1 = -12 + 5\sqrt{5}, m_2 = -12 - 5\sqrt{5} \quad (2.0.67)$$

$$\mathbf{n}_1 = \begin{pmatrix} 12 - 5\sqrt{5} \\ 1 \end{pmatrix} \quad (2.0.68)$$

$$\mathbf{n}_2 = \begin{pmatrix} 12 + 5\sqrt{5} \\ 1 \end{pmatrix} \quad (2.0.69)$$

Equation 2.0.68 and 2.0.69 satisfies equation 2.0.61, so  $c_1$  and  $c_2$  can be obtained as,

$$c_1 = 3 + \sqrt{5} \quad (2.0.70)$$

$$c_2 = 3 - \sqrt{5} \quad (2.0.71)$$

Equation 2.0.57 can be modified by using equation 2.0.68, 2.0.69, 2.0.70 and 2.0.71

$$(12 - 5\sqrt{5} \quad 1) \mathbf{x} = 3 + \sqrt{5} \quad (2.0.72)$$

$$(12 + 5\sqrt{5} \quad 1) \mathbf{x} = 3 - \sqrt{5} \quad (2.0.73)$$

$$\Rightarrow (12 - 5\sqrt{5})x + y - 3 - \sqrt{5} = 0 \quad (2.0.74)$$

$$\Rightarrow (12 + 5\sqrt{5})x + y - 3 + \sqrt{5} = 0 \quad (2.0.75)$$

$$((12 - 5\sqrt{5})x + y - 3 - \sqrt{5})$$

$$((12 + 5\sqrt{5})x + y - 3 + \sqrt{5}) = 0 \quad (2.0.76)$$

The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|V - \lambda \mathbf{I}| = 0 \quad (2.0.77)$$

$$\begin{vmatrix} 19 - \lambda & 12 \\ 12 & 1 - \lambda \end{vmatrix} = 0 \quad (2.0.78)$$

$$(19 - \lambda)(1 - \lambda) - 144 = 0 \quad (2.0.79)$$

$$\lambda_1 = -5, \lambda_2 = 25 \quad (2.0.80)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (2.0.81)$$

$$\Rightarrow (\mathbf{V} - \lambda \mathbf{I}) \mathbf{p} = 0 \quad (2.0.82)$$

For  $\lambda_1 = -5$ ,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} 19 + 5 & 12 \\ 12 & 1 + 5 \end{pmatrix} \quad (2.0.83)$$

By row reduction ,

$$\begin{pmatrix} 24 & 12 \\ 12 & 6 \end{pmatrix} \quad (2.0.84)$$

$$\xrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 24 & 12 \\ 0 & 0 \end{pmatrix} \quad (2.0.85)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{12}} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.86)$$

Substituting equation 2.0.86 in equation 2.0.82 we get

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.87)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_1 = t$

$$v_2 = -2t \quad (2.0.88)$$

Eigen vector  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} t \\ -2t \end{pmatrix} \quad (2.0.89)$$

Let  $t = 1$ , we get

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2.0.90)$$

For  $\lambda_2 = 25$ ,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} 19 - 25 & 12 \\ 12 & 1 - 25 \end{pmatrix} \quad (2.0.91)$$

By row reduction ,

$$\begin{pmatrix} -6 & 12 \\ 12 & -24 \end{pmatrix} \quad (2.0.92)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -6 & 12 \\ 0 & 0 \end{pmatrix} \quad (2.0.93)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \quad (2.0.94)$$

Substituting equation 2.0.94 in equation 2.0.82 we get

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.95)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_1 = t$

$$v_2 = \frac{t}{2} \quad (2.0.96)$$

Eigen vector  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} t \\ \frac{t}{2} \end{pmatrix} \quad (2.0.97)$$

Let  $t = 1$ , we get

$$\mathbf{p}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.98)$$

By eigen decomposition  $\mathbf{V}$  can be represented by

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (2.0.99)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.0.100)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.101)$$

Substituting equations 2.0.90, 2.0.98 in equation 2.0.100 we get

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \quad (2.0.102)$$

Substituting equation 2.0.80 in 2.0.101 we get

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \quad (2.0.103)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (2.0.104)$$

$$\Rightarrow \mathbf{c} = -\frac{1}{125} \begin{pmatrix} 1 & -12 \\ -12 & 19 \end{pmatrix} \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (2.0.105)$$

$$\Rightarrow \mathbf{c} = -\frac{1}{125} \begin{pmatrix} 25 \\ 75 \end{pmatrix} \quad (2.0.106)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{3}{5} \end{pmatrix} \quad (2.0.107)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = -8 < 0 \quad (2.0.108)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.0.109)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.0.110)$$

To get  $\mathbf{y}$ ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (2.0.111)$$

$$\mathbf{y} = \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{5} \\ -\frac{3}{5} \end{pmatrix} \quad (2.0.112)$$

$$\mathbf{y} = \begin{pmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{2} \end{pmatrix} \quad (2.0.113)$$

Substituting the equation 2.0.103 in equation 2.0.109

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{y} = -8 \quad (2.0.114)$$

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{y} + 8 = 0 \quad (2.0.115)$$

2.0.115 represents equation of asymptotes. The Equation of Conjugate hyperbola is given by:  
2(Equation of Asymptotes)- Equation of hyperbola.

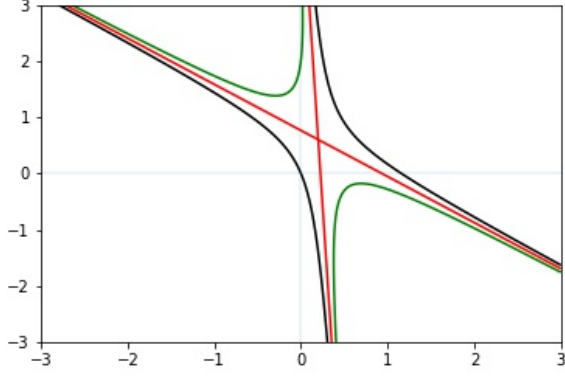


Fig. 1: Hyperbola, Asymptotes and Conjugate Hyperbola

So, from equation 2.0.49 and 2.0.115 , we get equation of conjugate hyperbola as:-

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{y} + 12 = 0 \quad (2.0.116)$$

Form equation 1.0.1, 2.0.1 and 2.0.2, we can write equation of the hyperbola as:-

$$\mathbf{x}^T \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -11 \\ -3 \end{pmatrix} \mathbf{x} = 0 \quad (2.0.117)$$

Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term. We obtained the constant term  $K=4$  for the equation of asymptotes from 2.0.54 and 2.0.56 Hence equation of asymptotes is obtained as:-

$$\mathbf{x}^T \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -11 \\ -3 \end{pmatrix} \mathbf{x} + 4 = 0 \quad (2.0.118)$$

The Equation of Conjugate hyperbola is given by:

2(Equation of Asymptotes)- Equation of hyperbola.

Applying this on equation 2.0.117 and 2.0.118 we obtain equation of Conjugate hyperbola as:-

$$\mathbf{x}^T \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -11 \\ -3 \end{pmatrix} \mathbf{x} + 8 = 0 \quad (2.0.119)$$