Matrix Theory EE5609 - Assignment 7

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***Abstract*—Find foot of the perpendicular using SVD**

Download python code from

https://github.com/SANDHYA−A/Assignment7

1 Problem

Find the foot of the perpendicular for a point on the line of intersection of planes $9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0$ on to the plane containing the point $(-1, -4, 3)$.

2 Solution

Given the equation of two intersecting planes is

$$9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0 \tag{2.0.1}$$

Let the two normals for these planes be $\mathbf{n_1}$ and $\mathbf{n_2}$

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{2.0.2}$$
$$a_2x + b_2y + c_2z + d_2 = 0 \tag{2.0.3}$$
$$\text{and, } \mathbf{n_1}.\mathbf{n_2} = 0 \tag{2.0.4}$$

We have,

$$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0 \tag{2.0.5}$$

From equation 2.0.1 and 2.0.5 we get the equations of two planes as:

$$(3x - 2y - z - 1)(3x + 2y - z + 1) = 0 \tag{2.0.6}$$

The vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ are

$$\mathbf{n_1} = \begin{pmatrix} 3 & 2 & -1 \end{pmatrix} \tag{2.0.7}$$
$$\mathbf{n_2} = \begin{pmatrix} 3 & -2 & -1 \end{pmatrix} \tag{2.0.8}$$

From equation 2.0.6, we obtain the equation of line of intersection as

$$3x = z \text{ and } y = \frac{-1}{2} \tag{2.0.9}$$

$\therefore$ The normal perpendicular to the intersection of two planes will be

$$\mathbf{n} = \mathbf{n_1} \times \mathbf{n_2} \tag{2.0.10}$$

Substituting eq 2.0.7 and 2.0.8 in 2.0.10 we obtain the normal vector to the intersection of the planes as

$$\mathbf{n} = \begin{pmatrix} -4 & 0 & -12 \end{pmatrix} \tag{2.0.11}$$

Let, $\mathbf{r}$ be $\begin{pmatrix} x & y & z \end{pmatrix}$. Now, the equation of the plane passing through the point Q(-1, -4 ,3) can be obtained by

$$\mathbf{n}.(\mathbf{r} - \mathbf{Q}) = 0 \tag{2.0.12}$$

$$\begin{pmatrix} -4 & 0 & -12 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 4 \\ z - 3 \end{pmatrix} = 0 \tag{2.0.13}$$

Equation of the plane containing the point$(-1, -4, 3)$ is

$$\begin{pmatrix} 1 & 0 & 3 \end{pmatrix} \mathbf{x} = -8 \tag{2.0.14}$$

Consider a point on the line of intersection at eq 2.0.9 as $\mathbf{b} = \begin{pmatrix} 1 & \frac{-1}{2} & 3 \end{pmatrix}$. To find the foot of the perpendicular on to the plane at eq 2.0.14, we can use SVD. We have the two orthogonal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ and $\mathbf{M}$ is the matrix of these orthogonal vectors

We solve the equation

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{2.0.15}$$

Substituting values of normal vectors and the point on the plane, in 2.0.15, We get,

$$\begin{pmatrix} 3 & 3 \\ 2 & -2 \\ -1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ \frac{-1}{2} \\ 3 \end{pmatrix} \tag{2.0.16}$$

To solve the above equation, we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.17}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of

$\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 14 & 6 \\ 6 & 14 \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{MM}^T = \begin{pmatrix} 18 & 0 & -6 \\ 0 & 8 & 0 \\ -6 & 0 & 2 \end{pmatrix} \tag{2.0.19}$$

Substituting eq 2.0.17 in 2.0.15, we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.20}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.21}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. The eigen values of $\mathbf{MM}^T$ are obtained as below,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.22}$$

$$\implies \begin{vmatrix} 18-\lambda & 0 & -6 \\ 0 & 8-\lambda & 0 \\ -6 & 0 & 2-\lambda \end{vmatrix} = 0 \tag{2.0.23}$$

$$\implies -\lambda^3 + 28\lambda^2 - 160\lambda = 0 \tag{2.0.24}$$

The eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = 20 \tag{2.0.25}$$

$$\lambda_2 = 8 \tag{2.0.26}$$

$$\lambda_3 = 0 \tag{2.0.27}$$

Eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \tag{2.0.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-3}{\sqrt{10}} \\ 0 \\ \frac{1}{\sqrt{10}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix} \tag{2.0.29}$$

$\mathbf{U}$ is obtained as follows,

$$\begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \tag{2.0.30}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{pmatrix} \tag{2.0.31}$$

The eigen values of $\mathbf{M}^T\mathbf{M}$ are obtained as below,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.32}$$

$$\implies \begin{vmatrix} 14-\lambda & 6 \\ 6 & 14-\lambda \end{vmatrix} = 0 \tag{2.0.33}$$

$$\implies \lambda^2 - 28\lambda + 160 = 0 \tag{2.0.34}$$

The eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 20 \tag{2.0.35}$$

$$\lambda_5 = 8 \tag{2.0.36}$$

Eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.0.37}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.0.38}$$

$\mathbf{V}$ is obtained as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.0.39}$$

From eq 2.0.30, 2.0.31 and 2.0.39, $\mathbf{M}$ as Singular Value Decomposition can be written as,

$$\mathbf{M} = \begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T \tag{2.0.40}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is obtained as,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{20}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{8}} & 0 \end{pmatrix} \tag{2.0.41}$$

Using equation 2.0.15, 2.0.30 and 2.0.39 in equation 2.0.21 we obtain value of $\mathbf{x}$ as:

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \frac{-1}{2} \\ \sqrt{10} \end{pmatrix} \tag{2.0.42}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \frac{-1}{4\sqrt{2}} \end{pmatrix} \tag{2.0.43}$$

$$\mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{1}{8} \\ \frac{-1}{8} \end{pmatrix} \tag{2.0.44}$$

We can verify the obtained solution,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.45}$$

Computing the RHS values in equation 2.0.45, we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.0.46}$$

$$\implies \begin{pmatrix} 14 & 6 \\ 6 & 14 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.0.47}$$

Solving the augmented matrix from eq 2.0.47 we get,

$$\begin{pmatrix} 14 & 6 & -1 \\ 6 & 14 & 1 \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{14}R_1} \begin{pmatrix} 1 & \frac{3}{7} & \frac{-1}{14} \\ 6 & 14 & 1 \end{pmatrix} \tag{2.0.48}$$

$$\xleftrightarrow{R_2=R_2-6R_1} \begin{pmatrix} 1 & \frac{3}{7} & -\frac{1}{14} \\ 0 & \frac{80}{7} & -\frac{10}{7} \end{pmatrix} \tag{2.0.49}$$

$$\xleftrightarrow{R_2=\frac{7}{80}R_2} \begin{pmatrix} 1 & \frac{3}{7} & -\frac{1}{14} \\ 0 & 1 & \frac{1}{8} \end{pmatrix} \tag{2.0.50}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{7}R_2} \begin{pmatrix} 1 & 0 & \frac{-1}{8} \\ 0 & 1 & \frac{1}{8} \end{pmatrix} \tag{2.0.51}$$

The value of $\mathbf{x}$ is obtained from eq 2.0.51 as

$$\mathbf{x} = \begin{pmatrix} \frac{1}{8} \\ \frac{-1}{8} \end{pmatrix} \tag{2.0.52}$$

From equations 2.0.44 and 2.0.52 the solution is verified.