

Matrix Theory EE5609 - Assignment 7

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Abstract—Find foot of the perpendicular using SVD

Download python code from

<https://github.com/SANDHYA-A/Assignment7>

1 PROBLEM

Find the foot of the perpendicular for a point on the line of intersection of planes $9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0$ on to the plane containing the point $(-1, -4, 3)$.

2 SOLUTION

Given the equation of two intersecting planes is

$$9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0 \quad (2.0.1)$$

1) Second order equation in terms of matrices:

The general equation of a second order algebraic surface is given by

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + 2lx + 2my + 2nz + q = 0. \quad (2.0.2)$$

This equation can be written as:-

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

$$\mathbf{V} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} \quad (2.0.4)$$

Substituting values from the given equation (2.0.1), we get,

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}^T \mathbf{x} - 1 = 0 \quad (2.0.5)$$

$$\mathbf{V} = \begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \quad f = -1 \quad (2.0.6)$$

a) The eigen values of matrix \mathbf{V} can be calculated as:-

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.0.7)$$

$$\begin{vmatrix} 9 - \lambda & 0 & -6 \\ 0 & -4 - \lambda & 0 \\ -6 & 0 & 1 - \lambda \end{vmatrix} = 0 \quad (2.0.8)$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 + 67\lambda + 108 = 0 \quad (2.0.9)$$

$$(-\lambda - 4)(\lambda^2 - 10\lambda - 27) = 0 \quad (2.0.10)$$

From above, we get,

$$\lambda_1 = -4 \quad (2.0.11)$$

$$\lambda_2 = -2\sqrt{13} + 5 \quad (2.0.12)$$

$$\lambda_3 = 2\sqrt{13} + 5 \quad (2.0.13)$$

The corresponding eigen vectors for these eigen values are:-

$$\text{for } \lambda_1 = -4, \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (2.0.14)$$

$$\text{for } \lambda_2 = -2\sqrt{13} + 5, \mathbf{v}_2 = \begin{pmatrix} \frac{\sqrt{13}-2}{3} \\ 0 \\ 1 \end{pmatrix} \quad (2.0.15)$$

$$\text{for } \lambda_3 = 2\sqrt{13} + 5, \mathbf{v}_3 = \begin{pmatrix} \frac{-\sqrt{13}-2}{3} \\ 0 \\ 1 \end{pmatrix} \quad (2.0.16)$$

Eigen vectors matrix is

$$\begin{pmatrix} 0 & \frac{\sqrt{13}-2}{3} & \frac{-\sqrt{13}-2}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (2.0.17)$$

The normalizing these values, we obtain

$$\begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \quad (2.0.18)$$

b) Affine transformation:

We can obtain the affine transformation for equation at (2.0.3) by taking the value of \mathbf{x} as:

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (2.0.19)$$

such that

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D} \quad (2.0.20)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad (2.0.21)$$

and \mathbf{P} is the transformation matrix and can be given by the normalized eigen vectors of matrix \mathbf{V} . Therefore,

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \quad (2.0.22)$$

Also, $\mathbf{P}^T = \mathbf{P}^{-1}$

$$\begin{aligned} \mathbf{P}^T \mathbf{V} \mathbf{P} &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} \\ \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \\ &\begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \\ &\quad (2.0.23) \end{aligned}$$

$$= \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} = \mathbf{D} \quad (2.0.24)$$

(2.0.3) can be expressed as:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad \therefore |\mathbf{V}| \neq 0 \quad (2.0.25)$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}$$

$$= \begin{pmatrix} 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{27} & 0 & \frac{-2}{9} \\ 0 & \frac{-1}{4} & 0 \\ \frac{-2}{9} & 0 & \frac{-1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = -1 \quad (2.0.26)$$

Therefore, we obtain

$$\mathbf{y}^T \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} \mathbf{y} = 0 \quad (2.0.27)$$

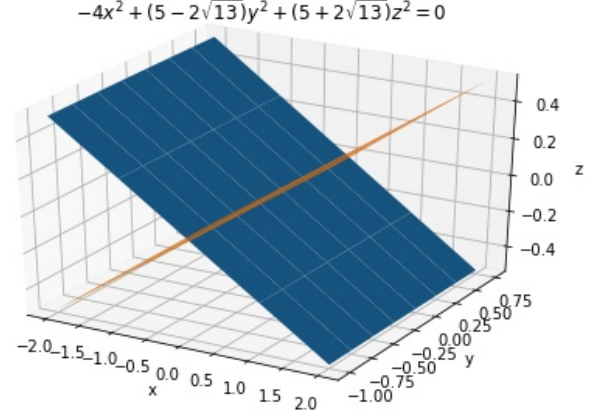


Fig. 1: Plot of intersecting planes

c) Obtaining equation of planes from transformation

From (2.0.19) we can write \mathbf{y} as

$$\mathbf{y} = \mathbf{P}^{-1}(\mathbf{x} - \mathbf{c}) \quad (2.0.28)$$

$$\mathbf{y} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} \\ \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \begin{pmatrix} x-0 \\ y-1/2 \\ z-0 \end{pmatrix} \quad (2.0.29)$$

$$\mathbf{y} = \begin{pmatrix} y - \frac{1}{2} \\ \frac{(\sqrt{13}-2)x+3z}{26-4\sqrt{13}} \\ \frac{(-\sqrt{13}-2)x+3z}{26+4\sqrt{13}} \end{pmatrix} \quad (2.0.30)$$

Substituting value of \mathbf{y} in (2.0.27), we obtain:

$$\begin{pmatrix} y - \frac{1}{2} & \frac{(\sqrt{13}-2)x+3z}{26-4\sqrt{13}} & \frac{(-\sqrt{13}-2)x+3z}{26+4\sqrt{13}} \end{pmatrix} \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} \begin{pmatrix} y - \frac{1}{2} \\ \frac{(\sqrt{13}-2)x+3z}{26-4\sqrt{13}} \\ \frac{(-\sqrt{13}-2)x+3z}{26+4\sqrt{13}} \end{pmatrix} = 0 \quad (2.0.31)$$

$$\Rightarrow 9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0 \quad (2.0.32)$$

Expressing in terms of individual plane equations, we get

$$(3x - 2y - z - 1)(3x + 2y - z + 1) = 0 \quad (2.0.33)$$

2) Finding equation of individual planes:

Let the two normals for these planes be \mathbf{n}_1 and \mathbf{n}_2

$$a_1x + b_1y + c_1z + d_1 = 0 \quad (2.0.34)$$

$$a_2x + b_2y + c_2z + d_2 = 0 \quad (2.0.35)$$

We have,

$$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0 \quad (2.0.36)$$

From equation (2.0.1) and (2.0.36) we get the equations of two planes as:

$$(3x - 2y - z - 1)(3x + 2y - z + 1) = 0 \quad (2.0.37)$$

$$3x - 2y - z - 1 = 0 \quad (2.0.38)$$

$$3x + 2y - z + 1 = 0 \quad (2.0.39)$$

3) Equation of normal to individual planes:

Let equation (2.0.38) and (2.0.39) be denoted as E_1 and E_2 . Let k_1 and k_2 be two arbitrary constants. Then,

$$k_1E_1 + k_2E_2 = 0 \quad (2.0.40)$$

represents a linear equation and this equation will be satisfied by coordinates of the points that lie on both these planes.

Let us assume the constants $k_1 = 1$ and $k_2 = 1$. Then the resultant equation from (2.0.40) represents a plane through the line of intersection of the planes E_1 and E_2 and is given by:

$$1.(3x - 2y - z - 1) + 1.(3x + 2y - z + 1) = 0 \quad (2.0.41)$$

$$3x = z \quad (2.0.42)$$

$$(3 \ 0 \ -1) \mathbf{x} = 0 \quad (2.0.43)$$

4) Equation of line of intersection of planes:

Equation (2.0.43) is the equation of the normal to E_1 and E_2 . Also, from equation (2.0.38) and (2.0.39), we obtain the equation of line of intersection of the given planes as

$$3x = z \text{ and } y = \frac{-1}{2} \quad (2.0.44)$$

5) Equation of plane passing through point (-1, -4, 3) and perpendicular to plane at (2.0.43):

Let, \mathbf{r} be $(x \ y \ z)$. Now, the equation of the plane passing through the point $Q(-1, -4, 3)$ can be obtained by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{Q}) = 0 \quad (2.0.45)$$

$$(3 \ 0 \ -1) \begin{pmatrix} x+1 \\ y+4 \\ z-3 \end{pmatrix} = 0 \quad (2.0.46)$$

Thus, equation of the plane containing the point $(-1, -4, 3)$ is obtained as

$$(3 \ 0 \ -1) \mathbf{x} = -6 \quad (2.0.47)$$

6) Finding normal vectors:

The equation of the plane at (2.0.47) can be expressed as:

$$\mathbf{n}^T \mathbf{x} = c \quad (2.0.48)$$

Rewriting given equation of plane in (2.0.48) form

$$(3 \ 0 \ -1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -6 \quad (2.0.49)$$

where the value of

$$\mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \quad (2.0.50)$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (2.0.51)$$

$$c = -6 \quad (2.0.52)$$

The two vectors \mathbf{q} and \mathbf{r} which are \perp to \mathbf{n} can be obtained by

$$(3 \ 0 \ -1) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (2.0.53)$$

$$\Rightarrow 3a - c = 0 \quad (2.0.54)$$

Put $a = 0$ and $b = 1$ in (2.0.54), $\implies c = 0$
 Put $a = 1$ and $b = 0$ in (2.0.54), $\implies c = 3$

Hence

$$\mathbf{q} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad (2.0.55)$$

An arbitrary point on line of intersection of two planes at equation (2.0.1) can be taken as:

$$\mathbf{b} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 3 \end{pmatrix} \quad (2.0.56)$$

7) Solution using Singular Value Decomposition:

So, now we have two orthogonal vectors \mathbf{q} and \mathbf{r} and \mathbf{M} is the matrix of these orthogonal vectors. We can solve the equation :

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.57)$$

Substituting the values of normal vectors and the point on the plane in (2.0.57), we get,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 3 \end{pmatrix} \quad (2.0.58)$$

To solve the above equation, we perform Singular Value Decomposition on \mathbf{M} as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.59)$$

Where the columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, the columns of \mathbf{U} are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \quad (2.0.60)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 9 \end{pmatrix} \quad (2.0.61)$$

As we know that,

$$\begin{aligned} \mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} &= \mathbf{b} \\ \implies \mathbf{x} &= \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \end{aligned} \quad (2.0.62)$$

Where \mathbf{S}_+ is Moore-Penrose Pseudo-Inverse of \mathbf{S} .

a) Calculating eigenvalues and eigen vectors of $\mathbf{M}\mathbf{M}^T$:

$$\begin{aligned} |\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| &= 0 \\ \implies \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 9-\lambda \end{vmatrix} &= 0 \\ \implies -\lambda^3 + 11\lambda^2 - 10\lambda &= 0 \end{aligned}$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = 10; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \quad (2.0.63)$$

And the corresponding eigenvectors are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u}_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \quad (2.0.64)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{\sqrt{10}} \\ 0 \\ \frac{1}{\sqrt{10}} \end{pmatrix} \quad (2.0.65)$$

\mathbf{U} is obtained as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & -\frac{3}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix} \quad (2.0.66)$$

Using values from (2.0.63),

$$\mathbf{S} = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.67)$$

b) Calculating eigen values and eigen vectors of $\mathbf{M}^T\mathbf{M}$:

$$\begin{aligned} |\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| &= 0 \\ \implies \begin{vmatrix} 1-\lambda & 0 \\ 0 & 10-\lambda \end{vmatrix} &= 0 \\ \implies \lambda^2 - 11\lambda + 10 &= 0 \end{aligned}$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 10; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.68)$$

From (2.0.68) we obtain \mathbf{V} as,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.69)$$

Hence we obtain SVD of \mathbf{M} as :

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & -\frac{3}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.70)$$

Moore-Penrose Pseudo inverse of \mathbf{S} is obtained as,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.71)$$

Using equation (2.0.56), (2.0.66), (2.0.69) and (2.0.71) in equation (2.0.62) we obtain value of \mathbf{x} as:

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \sqrt{10} \\ \frac{-1}{2} \\ 0 \end{pmatrix} \quad (2.0.72)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 1 \\ \frac{-1}{2} \end{pmatrix} \quad (2.0.73)$$

$$\mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-1}{2} \\ 1 \end{pmatrix} \quad (2.0.74)$$

8) **Solution verification:**

We can verify the obtained solution,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (2.0.75)$$

Computing the RHS values in equation 2.0.75, we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} \frac{-1}{2} \\ 10 \end{pmatrix} \quad (2.0.76)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-1}{2} \\ 10 \end{pmatrix} \quad (2.0.77)$$

The values of \mathbf{x} is obtained as:

$$\mathbf{x} = \begin{pmatrix} \frac{-1}{2} \\ 1 \end{pmatrix} \quad (2.0.78)$$

Comparing equation (2.0.74) and (2.0.78) solution is verified.