Matrix Theory EE5609 - Assignment 7

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***Abstract*—Find foot of the perpendicular using SVD**

Download python code from

https://github.com/SANDHYA−A/Assignment7

1 Problem

Find the foot of the perpendicular for a point on the line of intersection of planes $9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0$ on to the plane containing the point $(-1, -4, 3)$.

2 Solution

Given the equation of two intersecting planes is

$$9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0 \tag{2.0.1}$$

1) **Second order equation in terms of matrices:** The general equation of a second order algebraic surface is given by

$$\begin{aligned} &ax^2 + by^2 + cz^2 + 2dxy + 2exz \\ &+ 2fyz + 2lx + 2my + 2nz + q = 0. \end{aligned} \tag{2.0.2}$$

This equation can be written as:-

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

$$\mathbf{V} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} \tag{2.0.4}$$

Substituting values from the given equation (2.0.1), we get,

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}^T \mathbf{x} - 1 = 0 \tag{2.0.5}$$

$$\mathbf{V} = \begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \quad f = -1 \tag{2.0.6}$$

a) The eigen values of matrix $\mathbf{V}$ can be calculated as:-

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{2.0.7}$$

$$\begin{vmatrix} 9-\lambda & 0 & -6 \\ 0 & -4-\lambda & 0 \\ -6 & 0 & 1-\lambda \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies -\lambda^3 + 6\lambda^2 + 67\lambda + 108 = 0 \tag{2.0.9}$$

$$(-\lambda - 4).(\lambda^2 - 10\lambda - 27) = 0 \tag{2.0.10}$$

From above, we get,

$$\lambda_1 = -4 \tag{2.0.11}$$

$$\lambda_2 = -2\sqrt{13} + 5 \tag{2.0.12}$$

$$\lambda_3 = 2\sqrt{13} + 5 \tag{2.0.13}$$

The corresponding eigen vectors for these eigen values are:-

$$\text{for } \lambda_1 = -4, \mathbf{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.0.14}$$

$$\text{for } \lambda_2 = -2\sqrt{13} + 5, \mathbf{v_2} = \begin{pmatrix} \frac{\sqrt{13}-2}{3} \\ 0 \\ 1 \end{pmatrix} \tag{2.0.15}$$

$$\text{for } \lambda_3 = 2\sqrt{13} + 5, \mathbf{v_3} = \begin{pmatrix} \frac{-\sqrt{13}-2}{3} \\ 0 \\ 1 \end{pmatrix} \tag{2.0.16}$$

Eigen vectors matrix is

$$\begin{pmatrix} 0 & \frac{\sqrt{13}-2}{3} & -\frac{\sqrt{13}-2}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.0.17}$$

The normalizing these values, we obtain

$$\begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \tag{2.0.18}$$

b) Affine transformation:
We can obtain the affine transformation for equation at (2.0.3) by taking the value of $\mathbf{x}$ as:

$$\mathbf{x} = \mathbf{P}\mathbf{y} + c \tag{2.0.19}$$

such that

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D} \tag{2.0.20}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{2.0.21}$$

and $\mathbf{P}$ is the transformation matrix and can be given by the normalized eigen vectors of matrix $\mathbf{V}$. Therefore,

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \tag{2.0.22}$$

Also, $\mathbf{P}^T = \mathbf{P}^{-1}$

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} \\ \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \tag{2.0.23}$$

$$= \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} = \mathbf{D} \tag{2.0.24}$$

(2.0.3) can be expressed as:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad \therefore |\mathbf{V}| \neq 0 \tag{2.0.25}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}$$

$$= \begin{pmatrix} 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{27} & 0 & \frac{-2}{9} \\ 0 & \frac{-1}{4} & 0 \\ \frac{-2}{9} & 0 & \frac{-1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = -1 \tag{2.0.26}$$

Therefore, we obtain

$$\mathbf{y}^T \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} \mathbf{y} = 0 \tag{2.0.27}$$

c) Obtaining equation of planes from transformation
From (2.0.19) we can write $\mathbf{y}$ as

$$\mathbf{y} = \mathbf{P}^{-1}(\mathbf{x} - c) \tag{2.0.28}$$

$$\mathbf{y} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} \\ \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \begin{pmatrix} x-0 \\ y-1/2 \\ z-0 \end{pmatrix} \tag{2.0.29}$$

$$\mathbf{y} = \begin{pmatrix} y - \frac{1}{2} \\ \frac{(\sqrt{13}-2)x+3z}{26-4\sqrt{13}} \\ \frac{(-\sqrt{13}-2)x+3z}{26+4\sqrt{13}} \end{pmatrix} \tag{2.0.30}$$

Substituting value of $\mathbf{y}$ in (2.0.27), we obtain:

$$\begin{pmatrix} y - \frac{1}{2} & \frac{(\sqrt{13}-2)x+3z}{26-4\sqrt{13}} & \frac{(-\sqrt{13}-2)x+3z}{26+4\sqrt{13}} \end{pmatrix} \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} \begin{pmatrix} y - \frac{1}{2} \\ \frac{(\sqrt{13}-2)x+3z}{26-4\sqrt{13}} \\ \frac{(-\sqrt{13}-2)x+3z}{26+4\sqrt{13}} \end{pmatrix} = 0 \tag{2.0.31}$$

$$\implies 9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0 \tag{2.0.32}$$

Expressing in terms of individual plane equations, we get

$$(3x - 2y - z - 1)(3x + 2y - z + 1) = 0 \tag{2.0.33}$$

2) **Finding equation of individual planes:**
Let the two normals for these planes be $\mathbf{n_1}$ and $\mathbf{n_2}$

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{2.0.34}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \tag{2.0.35}$$

We have,

$$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0 \tag{2.0.36}$$

From equation (2.0.1) and (2.0.36) we get the equations of two planes as:

$$(3x - 2y - z - 1)(3x + 2y - z + 1) = 0 \tag{2.0.37}$$

$$3x - 2y - z - 1 = 0 \tag{2.0.38}$$

$$3x + 2y - z + 1 = 0 \tag{2.0.39}$$

3) **Equation of normal to individual planes:**
Let equation (2.0.38) and (2.0.39) be denoted as E_1 and E_2. Let k_1 and k_2 be two arbitrary constants. Then,

$$k_1E_1 + k_2E_2 = 0 \tag{2.0.40}$$

represents a linear equation and this equation will be satisifed by coordinates of the points that lie on both these planes.
Let us assume the constants $k_1 = 1$ and $k_2 = 1$. Then the resultant equation from (2.0.40) represents a plane through the line of intersection of the planes E_1 and E_2 and is given by:

$$1.(3x - 2y - z - 1) + 1.(3x + 2y - z + 1) = 0 \tag{2.0.41}$$

$$3x = z \tag{2.0.42}$$

$$\begin{pmatrix} 3 & 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.43}$$

4) **Equation of line of intersection of planes:**
Equation (2.0.43) is the equation of the normal to E_1 and E_2. Also, from equation (2.0.38) and (2.0.39), we obtain the equation of line of intersection of the given planes as

$$3x = z \text{ and } y = \frac{-1}{2} \tag{2.0.44}$$

5) **Equation of plane passing through point (-1, -4 ,3) and perpendicular to plane at** (2.0.43)**:**

Let, $\mathbf{r}$ be $\begin{pmatrix} x & y & z \end{pmatrix}$. Now, the equation of the plane passing through the point Q(-1, -4 ,3) can be obtained by

$$\mathbf{n}.(\mathbf{r} - \mathbf{Q}) = 0 \tag{2.0.45}$$

$$\begin{pmatrix} 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x+1 \\ y+4 \\ z-3 \end{pmatrix} = 0 \tag{2.0.46}$$

Thus, equation of the plane containing the point$(-1, -4, 3)$ is obtained as

$$\begin{pmatrix} 3 & 0 & -1 \end{pmatrix} \mathbf{x} = -6 \tag{2.0.47}$$

6) **Finding normal vectors:**
The equation of the plane at (2.0.47) can be expressed as:

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.48}$$

Rewriting given equation of plane in (2.0.48) form

$$\begin{pmatrix} 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -6 \tag{2.0.49}$$

where the value of

$$\mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \tag{2.0.50}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.0.51}$$

$$c = -6 \tag{2.0.52}$$

The two vectors $\mathbf{q}$ and $\mathbf{r}$ which are $\perp$ to $\mathbf{n}$ can be obtained by

$$\begin{pmatrix} 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.0.53}$$

$$\implies 3a - c = 0 \tag{2.0.54}$$

Put $a = 0$ and $b = 1$ in (2.0.54), $\implies c = 0$
Put $a = 1$ and $b = 0$ in (2.0.54), $\implies c = 3$

Hence

$$\mathbf{q} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \tag{2.0.55}$$

An arbitrary point on line of intersection of two planes at equation (2.0.1) can be taken as:

$$\mathbf{b} = \begin{pmatrix} 1 \\ \frac{-1}{2} \\ 3 \end{pmatrix} \tag{2.0.56}$$

7) **Solution using Singular Value Decomposition:**
So, now we have two orthogonal vectors $\mathbf{q}$

and $\mathbf{r}$ and $\mathbf{M}$ is the matrix of these orthogonal vectors. We can solve the equation :

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{2.0.57}$$

Substituting the values of normal vectors and the point on the plane in (2.0.57) , we get,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ \frac{-1}{2} \\ 3 \end{pmatrix} \tag{2.0.58}$$

To solve the above equation, we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.59}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \tag{2.0.60}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 9 \end{pmatrix} \tag{2.0.61}$$

As we know that,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$
$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U^T}\mathbf{b} \tag{2.0.62}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.

a) **Calculating eigenvalues and eigen vectors of $\mathbf{M}\mathbf{M}^T$:**

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 9-\lambda \end{vmatrix} = 0$$
$$\implies -\lambda^3 + 11\lambda^2 - 10\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = 10; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{2.0.63}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \tag{2.0.64}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{\sqrt{10}} \\ 0 \\ \frac{1}{\sqrt{10}} \end{pmatrix} \tag{2.0.65}$$

$\mathbf{U}$ is obtained as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & -\frac{3}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix} \tag{2.0.66}$$

Using values from (2.0.63),

$$\mathbf{S} = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.67}$$

b) **Calculating eigen values and eigen vectors of $\mathbf{M}^T\mathbf{M}$:**

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & 0 \\ 0 & 10-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - 11\lambda + 10 = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 10; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.68}$$

From (2.0.68) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.69}$$

Hence we obtain SVD of $\mathbf{M}$ as :

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & -\frac{3}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.70}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is obtained as,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.71}$$

Using equation (2.0.56),(2.0.66), (2.0.69) and (2.0.71) in equation (2.0.62) we obtain value of $\mathbf{x}$ as:

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \sqrt{10} \\ \frac{-1}{2} \\ 0 \end{pmatrix} \tag{2.0.72}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 1 \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.73}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-1}{2} \\ 1 \end{pmatrix} \tag{2.0.74}$$

8) **Solution verification:**
We can verify the obtained solution,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.75}$$

Computing the RHS values in equation 2.0.75, we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{-1}{2} \\ 10 \end{pmatrix} \tag{2.0.76}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-1}{2} \\ 10 \end{pmatrix} \tag{2.0.77}$$

The values of $\mathbf{x}$ is obtained as:

$$\mathbf{x} = \begin{pmatrix} \frac{-1}{2} \\ 1 \end{pmatrix} \tag{2.0.78}$$

Comparing equation (2.0.74) and (2.0.78) solution is verified.