Does Singular Value Decomposition (SVD) of a matrix always exist?

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***Abstract*—Let A be an $m \times n$ real matrix of rank r. Then there exist orthogonal matrices $\mathbf{U} \in \mathbb{R}^{m\times m}, \mathbf{V} \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with nonnegative diagonal entries $\sigma_1, \sigma_2, ..., ...$ such that**

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^t$$

1 Proof

Since $\mathbf{A}^T\mathbf{A}$ is symmetric and positive semi-definite, there exists an $n \times n$ orthogonal matrix $\mathbf{V}$ whose column vectors are the eigenvectors of $\mathbf{A}^T\mathbf{A}$ with non-negative eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

Hence $\mathbf{A}^T\mathbf{A}v_i = \lambda_i v_i$ for $i = 1, 2, ..., n$.
Let $r =$ rank $\mathbf{A}$. Assume that $\lambda_1 \geq \lambda_2 \geq \ \geq \lambda_r > 0$ and $\lambda_j = 0$ for $j = r+1, r+2, ..., n$.
Set $\sigma_i = \sqrt{\lambda_i}$ for all $i = 1, 2, ..., n$.
Then ${v^t}_i\mathbf{A}^t v\mathbf{A}v_i = \lambda_i {v^t}_i v_i = \lambda_i \geq 0$.
Then $\|\mathbf{A}v_i\| = \sigma_i$ for $i = 1, 2, ..., n$.

$$\text{Set } \frac{\mathbf{A}v_i}{\sigma_i} = u_i.$$

The set $u_1, u_2, ..., u_r$ is an orthonormal basis of $\mathbf{C}(\mathbf{A})$.
${u^t}_i u_j = \frac{(Av_i)^t Av_j}{\sigma_i\sigma_j} = \frac{(v_i)^t v_j\lambda_j}{\sigma_i\sigma_j} = \delta_{ij}$

We can add to it an orthonormal basis $\{u_{r+1}, ..., u_m\}$ of $\mathbf{N}(\mathbf{A^t})$ so that $\mathbf{U} = [u_1, u_2, ..., u_m]$ is an orthogonal matrix. Since $\mathbf{A}v_i = \sigma_i u_i$ for all i, we have the singular value decomposition

$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^t$ where $\Sigma = diag(\sigma_1, \sigma_2, ..., ...\sigma_r, 0, 0, ..., 0)$.