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MECHANICS OF MATERIALS

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CONTENTS

Preface

Mechanics of Materials (or **Strength of Materials**) is an important engineering subject to students whose majors are civil engineering, mechanical engineering, architect, environmental engineering, etc. Mechanics of Materials provides essential knowledge on the mechanical behaviour of materials and structural members under loadings. The knowledge obtained from this subject builds a firm step for studying other follow-on engineering subjects. Also, it is useful for engineers in their higher study and future career. The role of this subject can be compared with a 'foundation' of a 'building'.

This book has been written closely based on the course syllabus of the subject to meet the increasing demand on training high-quality and international students at Ho Chi Minh City University of Technology (HCMUT) – Vietnam National University (VNU). In addition, Mechanics of Materials have been one of difficult subjects that students struggle and this book can therefore be a material for students to be successful in studying this subject.

This book is organized into 11 chapters:

Chapter 1. *Introduction* presents basic concepts, loads and support reactions, equilibrium equations and assumptions.

Chapter 2. *Internal forces* deals with many aspects of internal forces. The relationships between internal force and loading.

Chapter 3. *Axial loading* analyses of normal stress in members subjected to axial load and discusses on mechanical properties of materials are presented.

Chapter 4. *Stress state* transforms stress state in different coordinate systems.

Chapter 5. *Failure theories* briefly presents a summary of theories of failure.

Chapter 6. *Geometric properties of cross sections* expresses the concept of geometric properties of sections, followed by several formulas of geometric properties of commonly used sections.

Chapter 7. *Bending and shearing* provides the normal and shear stress in plane bending and shear.

Chapter 8. *Deflection of beams* presents methods to calculate the deflection of beams.

Chapter 9. *Pure torsion (torque)* deals with the shear stress in members under torsion.

Chapter 10. *Combined loadings* analyses members under asymmetric bending or combined bending and/or axial loading.

Chapter 11. *Stability of columns* provides concepts of buckling and formulas to determine critical loads of columns.

This is the first publication; therefore, errors may not be avoided. The author looks forward to the readers' suggestions to make the book more complete.

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Chapter 1

INTRODUCTION

§1. ABOUT MECHANICS OF MATERIALS

Mechanics of Materials is a branch of mechanics that studies the behaviour of solid bodies subjected to external loads. Solid bodies can be members or structures of buildings, bridges, airplane wings, ships, etc. The behaviour includes displacement, strain, stress, fracture, damage, vibration, stability, etc. while loading can be classified into static, variable, dynamic loads. Regarding loading, Mechanics of Materials mainly focuses on solid bodies or structures that are in rest or uniform (no-accelerated) motions. It is worth mentioning herein that the name Mechanics of Materials is sometimes called **Strength of Materials**.

The main objective of this subject is to check the strength (durability), rigidity (stiffness) and the stability conditions of solid bodies subjected to external mechanical and thermal loads. Strength condition satisfies when the solid bodies experience no failure, which is called strength limit state. Rigidity condition satisfies when the deformation has to be less than or equal to an allowable deformation, which is called serviceability limit state. Stability condition is the condition that solid bodies can retain the initial deformation state.

Understanding of the fundamental aspects of this subject is of vital importance for engineers because many formulas and design rules used in engineering standards are based upon the principles of this subject. The knowledge of this subject has been developed based on the theoretical, numerical and experimental methods.

§2. EXTERNAL LOADS, CONNECTIONS, SUPPORTS AND SUPPORT REACTIONS

External Loads

External loads are loads from outside of the bodies and act on the bodies. External loads (forces) can be classified into two types: surface forces and body forces.

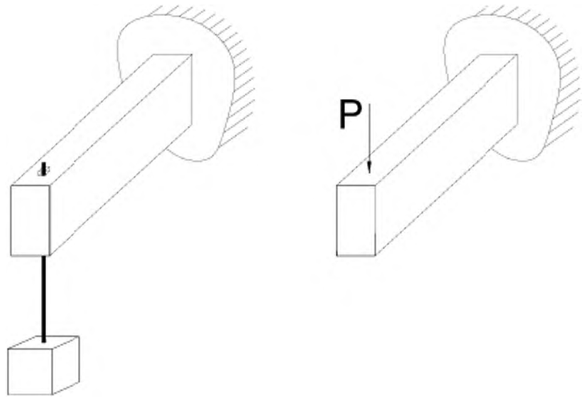
2.1. Surface loads (forces)

Surface loads are loads on the surface of bodies. Surface loads can be devided into:

a) Point (concentrated) loads

If the area on which the force is acting is small in comparison with the total surface area of the body, then the surface force can be idealized as a concentrated force or point force, which is applied to a point on the body. The unit of the point load is N, kN, etc.

For example, the weight of a car acts on the surface of a bridge desk is on the areas under the contact surfaces under the four tires. The area of the contact surface under the tire is so small compared to the area of the bridge desk. Therefore, the load from car acting on the bridge desk can be considered as a point load or concentrated load. Other examples of point loads can be support reaction, some live loads, etc. A point load can be drawn as an arrow as shown in **Figure 1.1b**.

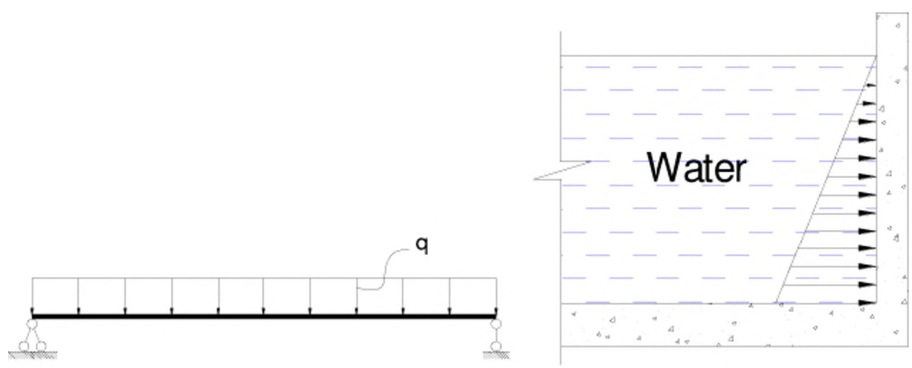


a) b)

Figure 1.1. Example of point loads

b) Distributed loads

If a surface load is applied along a narrow strip of area, that surface load can be idealized as a distributed load w(z). This distributed load is measured by the ratio of load intensity to the length of the strip. Distributed loads are graphically presented by a series of arrows along a line as shown in **Figure 1.2.** The magnitude of distributed load can vary or be constant along the distributed line.



a) b)

Figure 1.2. Example and graphical display of distributed load

The resultant force of the distributed load w(z) is equal to the area under the distributed loading curve. The resultant force acts through the centroid or geometric center of this area. The loading along the length of a beam is a typical example of distributed loads as shown in **Figure 1.2a**. Other examples of distributed load are water pressure as shown in **Figure 1.2b**. The unit of distributed load can be N/mm, kN/m, N/mm^2, kN/mm^2, etc.

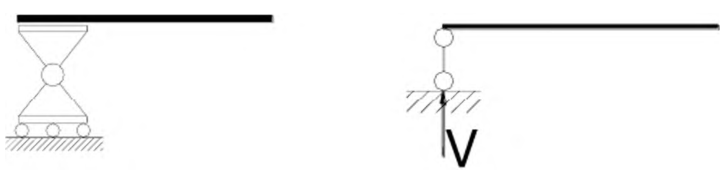
2.2. Body forces

A **body force** is developed when one body exerts a force on another body without direct physical contact between the bodies. Examples include the effects caused by the earth's gravitation or electromagnetic field. Although body forces affect each of the particles composing the body, these forces are normally represented by a single concentrated force acting on the body. In the case of gravitation, this force is called the **weight** of the body and acts through the body's center of gravity.

Supports and support reactions

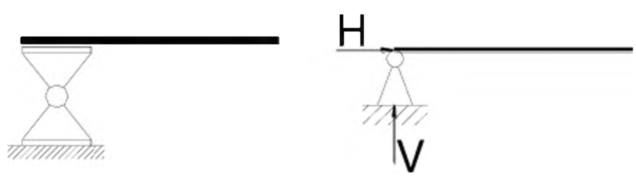
Roller support: Figure 1.3a shows compositions of a roller support. This support carries only vertical force. The roller support allows rotation and lateral displacement of the support point. Therefore, if the displacement along the x axis is called u, the displacement along y axis is called v and the rotation is called ϕ. The roller support has $\mathbf{u \neq 0; v = 0}$ and $\boldsymbol{\phi \neq 0}$. **Figure 1.3b** shows the symbolized roller support.

The surface forces that develop at the supports or points of contact between bodies are called **support reactions** or **reactions** (for short). Because the vertical displacement is prevented, the vertical reaction V is created as shown in **Figure 1.3b**.



a) b)

Figure 1.3. Supports and support reactions



a) b)

Figure 1.4. Supports and support reactions

Hinge (pin) support: Figure 1.4a shows a hinge support. This support carries vertical and lateral forces but not moment. The hinge support allows rotation around the hinge, but two displacements are equal zero: **u = 0**; **v = 0** and **ϕ ≠ 0.** Because the vertical and horizontal displacements are prevented, the vertical and horizontal reactions are created as shown in **Figure 1.4b**.

Fixed support: This support carries moment, vertical and lateral forces. This support does not allow any displacements or rotation: **u = 0**; **v = 0** and **ϕ = 0**. Because the vertical and horizontal displacements and moment are prevented, the vertical and horizontal reactions and moment reaction are created as shown in **Figure 1.5b**.

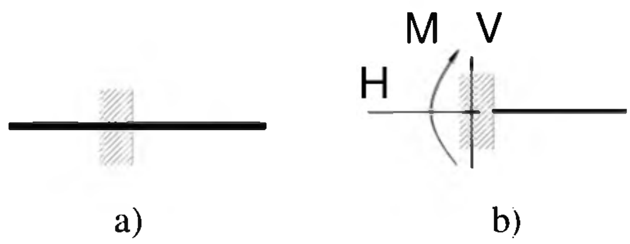


Figure 1.5. Supports and support reactions

General rule can be withdrawn as follows: **if the support prevents a translation in a given direction, then a reaction with respect to that direction must be developed.**

§3. EQUATIONS OF EQUILIBRIUM

Equilibrium of a body requires balance of forces and balance of moments as follows:

- **balance of forces** to prevent the body from translating or having accelerated motion along a straight or curved path. The balance of force is expressed in Equation 1.1, in which, $\sum F$ represents the sum of all the forces acting on the body.
- **balance of moments** to prevent the body from rotating. The balance of moments is expressed in Equation 1.2, in which, $\sum M_{/o} = 0$ is the sum of moments of all forces about any point O either on or off the body.

$$\sum F = 0 \tag{1.1}$$

$$\sum M_{/o} = 0 \tag{1.2}$$

Equations 1.1 and 1.2 are expressed in general forms. The details of these equations for the three-dimensional (3D) and two dimensional (2D) problems are as follows:

3D problem: If an Oxyz coordinate system is established with the origin O, the force and moment vectors can be decomposed into components along each coordinate axis and the above Equations (1.1) and (1.2) can be written in scalar form as six equations:

$$\sum F_x = 0; \sum F_y = 0; \sum F_z = 0; \quad (1.3)$$

$$\sum M_x = 0; \sum M_y = 0; \sum M_z = 0 \quad (1.4)$$

2D problem: 2D problems are common in engineering practice. Loads on a body can be represented as a system of forces in a plane. If the Oxy plane is considered, the conditions for equilibrium of a body can be specified with only three scalar equilibrium equations as expressed in Equations (1.5) or (1.6) or (1.7).

$$\sum X = 0; \sum Y = 0; \sum M_{I_O} = 0 \text{ (x and y are not parallel)} \quad (1.5)$$

or $$\sum M_A = 0; \sum M_B = 0; \sum M_C = 0 \text{ (A, B, C are not on a straight line)} \quad (1.6)$$

or $$\sum X = 0; \sum M_A = 0; \sum M_B = 0 \text{ (AB is not perpendicular to x)} \quad (1.7)$$

§4. MODELLING

The main idea of modelling is to create mathematical models of real structures. The mathematic models are convenient for analysis and calculation. It should be noted that designers are responsible for the structural models that they are created.

To create a model, some following rules can be used:

- Bar members can be expressed by their center lines. Plates and shells can be replaced by their middle surfaces.
- The cross sections are replaced by their sectional properties such as the cross-sectional area A, inertia modulus I, etc.
- Materials can be replaced by the properties of materials such as modulus E, stress-strain behaviour, etc.
- Connections and supports are idealized as rigid, non-friction, elastic, etc.
- Further simplifications can be used if it is necessary, such as hinges, joints, brick wall, floors, etc.

To illustrate the above rules, a model of real column structures shown in **Figure 1.6** can be examined.

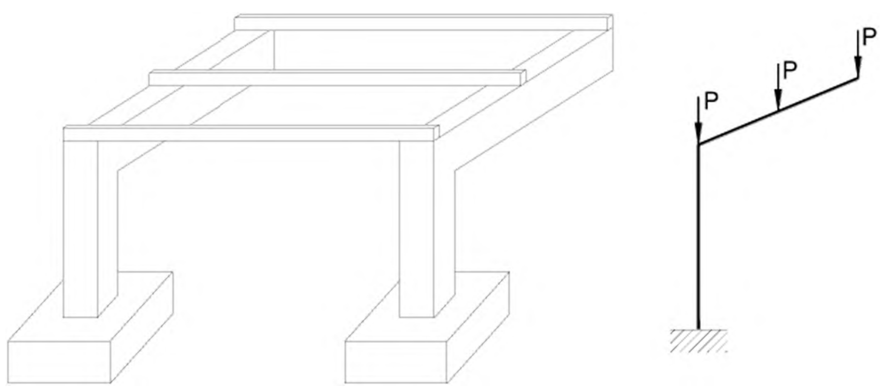


Figure 1.6. Examples of creating a column model

§5. ASSUMPTIONS

To simplify the calculation with reasonable accuracy, the three following assumptions are used in this subject.

Assumption 1: Real objects are replaced by models.

For example, a real reactangular beam is replaced by its axial axis and the section is replaced by its sectional properties.

Assumption 2: Materials are continuous, homogeneous, isotropic and linearly elastic.

Homogeneous material has the same physical and mechanical properties throughout its volume. **Isotropic material** has the same properties in all directions. Many engineering materials can be approximated as being both homogeneous and isotropic. **Linearly elastic material** has linear relationship between stress and strain. In other words, behaviour of linear elastic material follows Hooke's law (**Figure 1.7**).

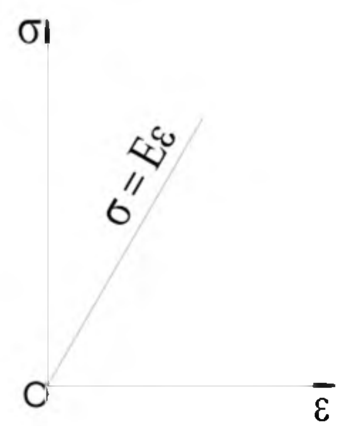


Figure 1.7. Linearly elastic relationship between the strain and stress (Hooke's law)

Assumption 3: Strain and displacement are small

The purpose of this assumption is to use infinitesimal quantities in mathematics. For example, for small angle, the approximate relationships $\sin\varphi \approx \tan\varphi \approx \varphi$, $\cos\varphi = 1$ can be used. When displacements and deformations are assumed to be small, the initial shapes (instead of the deformed shapes) of structures can be used to establish equilibrium equations for the structures.

§6. PRINCIPLE OF SUPERPOSITION

When the previous three assumptions are adopted, the principle of superposition can be used. The principle of superposition is stated as follows:

The total displacement or internal forces at a point in a structure subjected to several external loads can be determined by adding together the displacements or internal forces caused by each of the external loads acting separately.

Figure 1.8 can be used to illustrate the principle of superposition. A cantilever beam subjected to the loads P_1 and P_2 as shown in **Figure 1.8a**. The vertical displacement at the free end is Δ. The same beam subjected to the load P_1 as shown in **Figure 1.8b** has the vertical displacement Δ_1. This beam subjected to the load P_2 as shown in **Figure 1.8c** has the vertical displacement Δ_2. Based on the principle of superposition, the relationship of the displacements is wrriten as shown in Equation 1.8.

$$\Delta = \Delta_1 + \Delta_2 \tag{1.8}$$

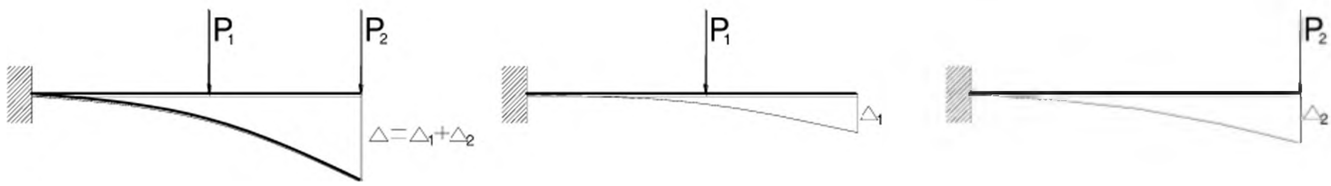


Figure 1.8. Illustration of supperposition principle

§7. EXERCISES

7.1. Exercise 1

Consider the simply supported beam AB (**Figure 1.9**) subjected to the load P. The distance from the load P to A and B are a and b, respectively. Given: P = 10 kN, a = 1 m, b = 3 m. Determine support reactions at A and B.

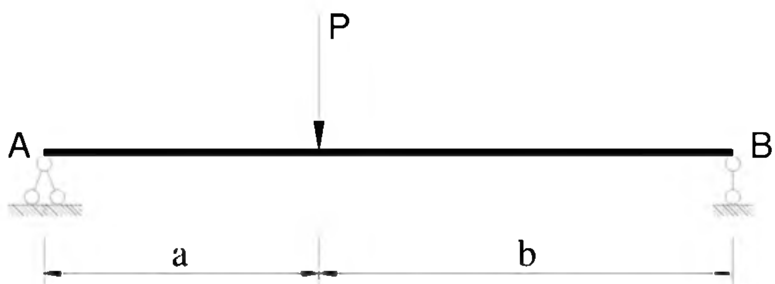


Figure 1.9

Solution:

Figure 1.10 shows the support reactions at A and B. The reactions at the hinge support A include H_A and R_A and the reaction at the roller B is R_B.

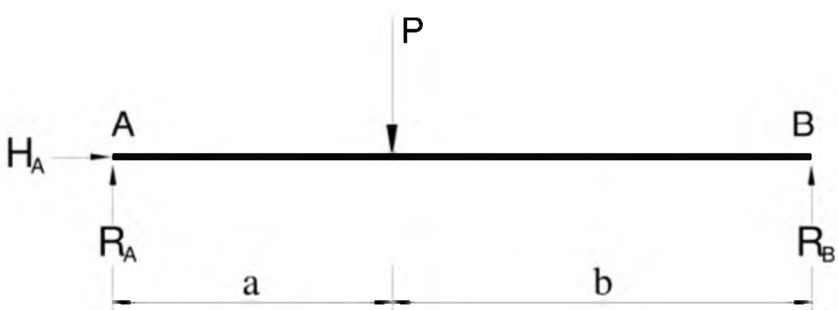


Figure 1.10

The total force with respect to the horizontal axis:

$$\sum F_{/z} = 0 \Rightarrow H_A = 0$$

The total moment with respect to B:

$$\sum M_{/B} = 0$$

$$\Rightarrow \quad R_A(a+b) - Pb = 0$$

$$\Rightarrow \quad R_A = \frac{Pb}{a+b}$$

$$R_A = \frac{10\times 3}{1+3} = 7.5kN$$

The total moment with respect to A:

$$\sum M_{/A} = 0$$

$$\Rightarrow \quad Pa - R_B(a+b) = 0$$

$$\Rightarrow \quad R_B = \frac{Pa}{a+b}$$

$$R_B = \frac{10\times 1}{1+3} = 2.5kN$$

How can we know that the results are correct? We can use the equilibrium of the forces with respect to the vertical axis:

$$\sum F_{/y} = 0$$

$$R_A + R_B - P = 0$$

$$7.5 + 2.5 - 10 = 0 \rightarrow OK$$

Remark: The lateral reaction HA is zero because there is no lateral load.

7.2. Exercise 2

Consider the simply supported beam AB subjected to the load P and the moment M_o (**Figure 1.11**). Given: P = 10 kN, M_o = 20 kNm, a = 1 m, b = 2 m. Determine support reactions at A and B.

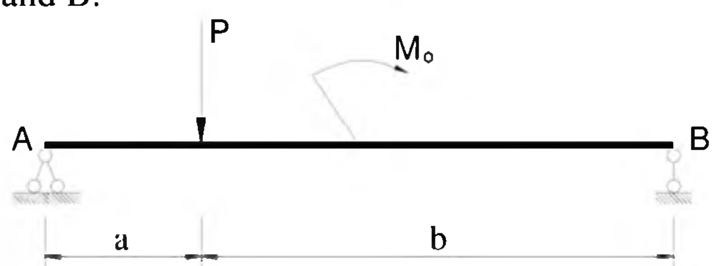


Figure 1.11

Solution:

Figure 1.12 shows the support reactions at A and B. The reactions at the hinge support A include H_A and R_A and the reaction at the roller B is R_B. Using the result of previous exercise, we can conclude that $H_A = 0$ because there is no lateral force.

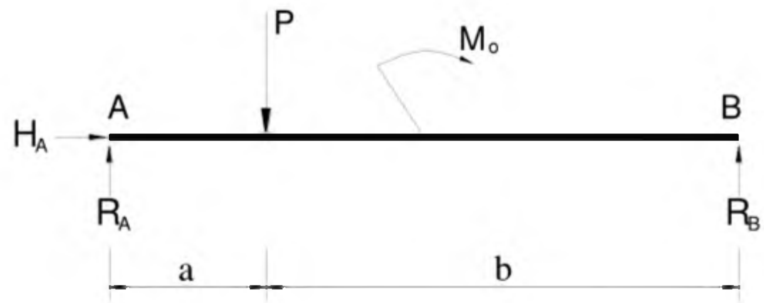


Figure 1.12

The total moment with respect to B:

$$\sum M_{/B} = 0$$

$$\Rightarrow \quad R_A(a+b) - Pb + M_o = 0$$

$$\Rightarrow \quad R_A = \frac{Pb - M_o}{a+b}$$

$$R_A = \frac{10\times 3 - 20}{1+3} = 2.5kN$$

The total moment with respect to A:

$$\sum M_{/A} = 0$$

$$\Rightarrow \quad Pa + M_o - R_B(a+b) = 0$$

$$\Rightarrow \quad R_B = \frac{Pa + M_o}{a+b}$$

$$R_B = \frac{10\times 1 + 20}{1+3} = 7.5kN$$

How can we know that the results are correct? We can use the equilibrium of the forces with respect to the vertical axis:

$$\sum F_{/y} = 0$$

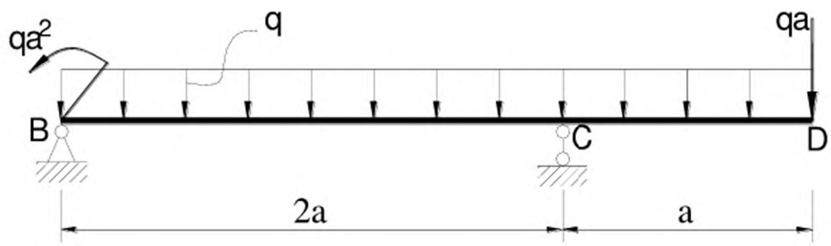
$$R_A + R_B - P = 0$$

$$2.5 + 7.5 - 10 = 0 \to OK$$

PROBLEMS

Determine the support reactions of the following structures:

1



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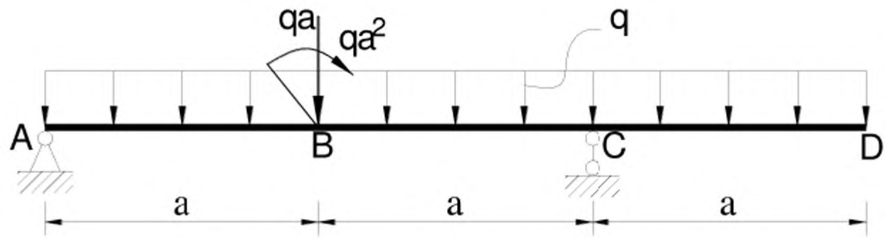

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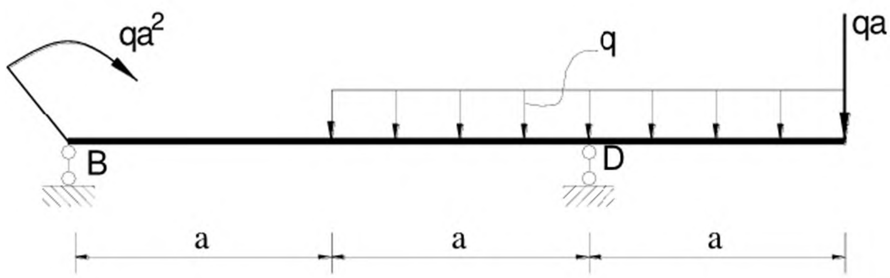

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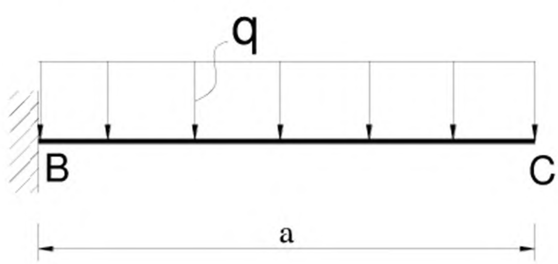

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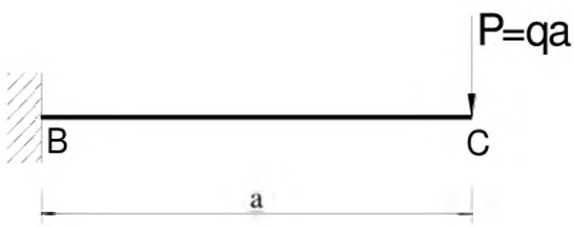

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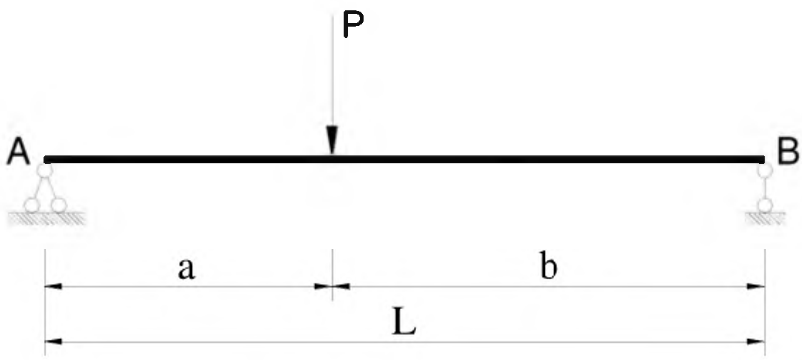

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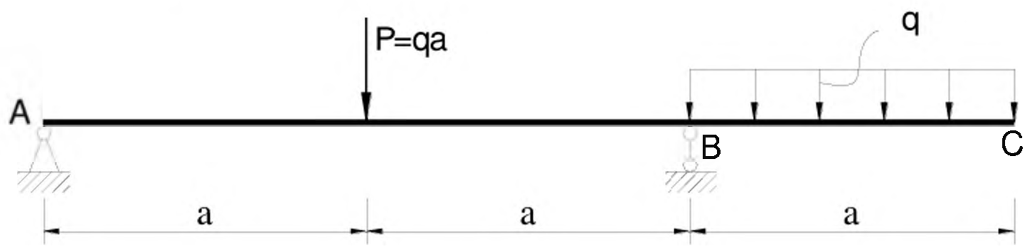

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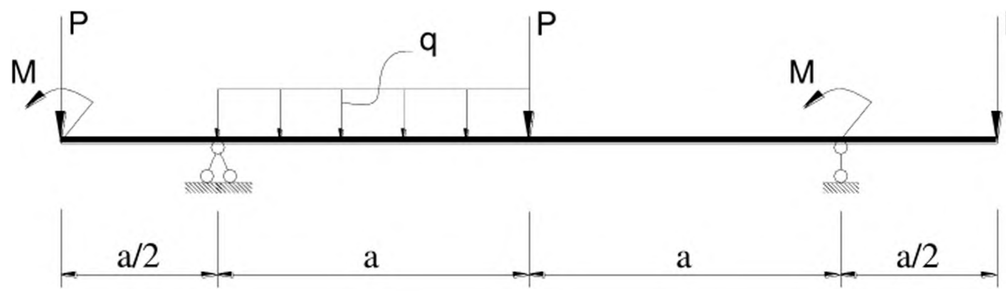
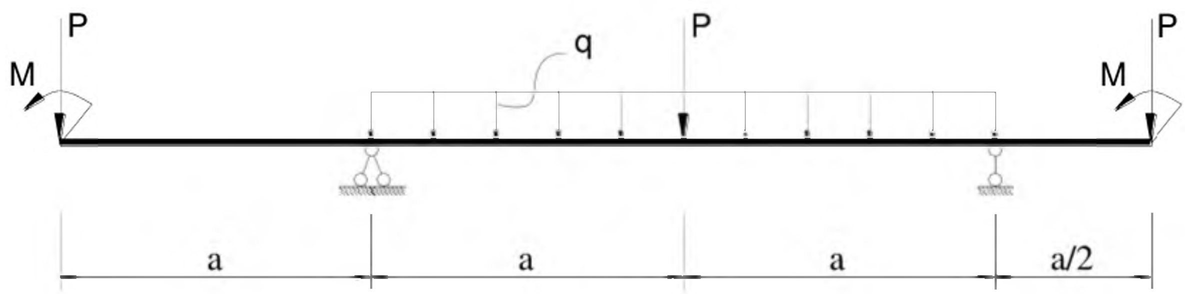


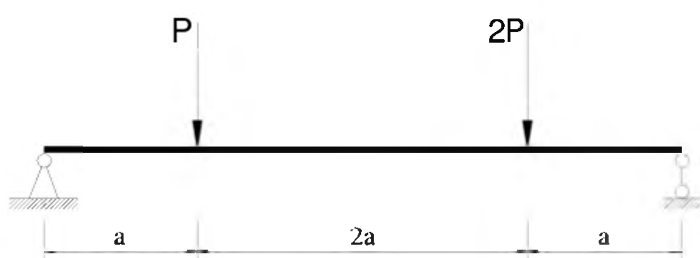
Given: P=qa, M=qa².

9



Given: P=qa, M=qa².

10



Chapter **2**

INTERNAL FORCES

§1. GENERAL CONCEPTS

External forces are forces from environment or other objects that act on the considered objects. The external force consists of loads and support reactions.

Objects have their own certain shapes and sizes because their material elements have connection forces. These forces are known as molecular bonds. When an object is subjected to an external force, these forces change to counter or balance with external forces. If this balance is not satisfied, the object will fail or be destroyed. Internal force is generally defined as follows:

Internal force is the amount of connection force changed when an object is subjected to external forces.

Mechanics of materials deals with problems related to only this change of connection force, namely internal force. It should be noted that the initial molecular bonds are not in the scope of Mechanics of Materials. If the external force is zero, then the internal force is zero.

To make it simple, it can be easy to remember that internal force is the force inside the object while external force is the force from outside the object.

Example 1

Figure 2.1 shows the spring under compression. The force acting on the spring is the external force. The force that the spring produces to balance with the external force is called the internal force.

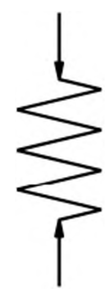


Figure 2.1. Spring under external force

Example 2: Figure 2.2 shows load (person's weight) acting on the chair.

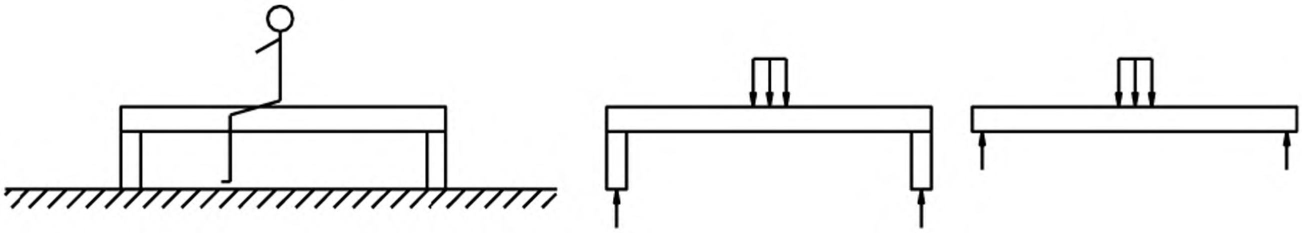


Figure 2.2. Load acting on chair

The concepts of internal and external forces are also relative. It depends on the object that is being considered. For example, when the whole chair is considered as an object, the increased force between the chair poles and the seat plate is an internal force. However, when the seat plate is considered as an object, this interaction force is the external force because it is outside the object.

§2. METHOD TO DETERMINE INTERNAL FORCES

2.1. Method to determine internal forces

Considering an object subjected to external forces $\vec{P_1}$, $\vec{P_2}$, ..., $\vec{P_n}$ and in balance, as shown in **Figure 2.3**. To determine the internal force, we use the cross-sectional method. Imagine that the object is cut into two parts by the plane π (**Figure 2.3**). The object becomes two parts: the left part and the right part (**Figure 2.4**). To make the left side work in the same condition with itself in the object before cutting, the effect of the right part on the left must be considered. The right part acts on the left part by a distributed force on the whole section. The sum of this distributed force is the internal force $\vec{P}$ (**Figure 2.4a**). Reversely, according to Newton's law 3, the left part also acts on the right part a similar force but opposite direction as shown in **Figure 2.4b**.

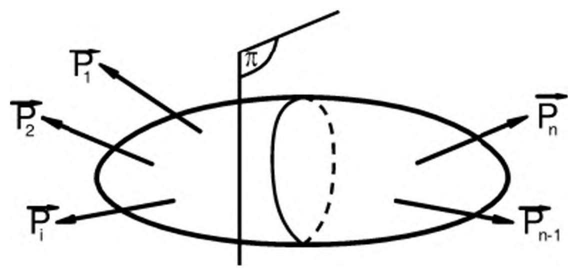


Figure 2.3. Object in balance

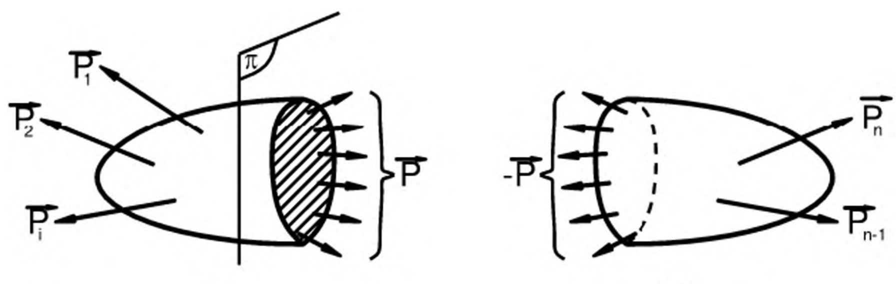


Figure 2.4. Internal force

Because the left part is under balance, the relationship between the internal force $\vec{P}$ and the external loads acting on the left part is expressed by Equation 2.1, in which, $\sum \vec{P}_{i,left}$ is the total of external forces on the left part.

$$\sum \vec{P}_{i,left} + \vec{P} = 0 \tag{2.1}$$

2.2. Components of internal force

The internal force $\vec{P}$ is equal to the force $\vec{R}$ and moment $\vec{M}$ acting at the center O. The coordinate system Oxyz is set at the cross section as shown in **Figure 2.5.** The direction of the moment can be determined by the screw rule or right-hand rule.

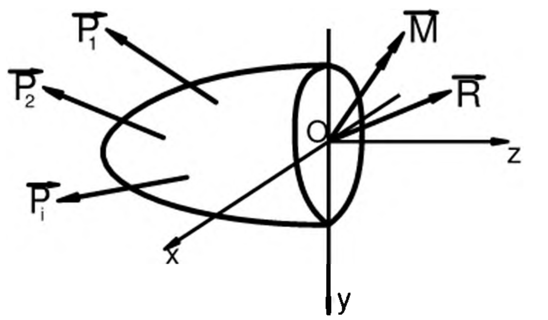


Figure 2.5. Internal forces at center

The force $\vec{R}$ is separated into three components $\overrightarrow{N_z}$, $\overrightarrow{Q_x}$, $\overrightarrow{Q_y}$ as shown in **Figure 2.6**. The relationship between these components and the force $\vec{R}$ is shown in Equation 2.2, in which, $\overrightarrow{N_z}$ is the axial force, $\overrightarrow{Q_x}$, $\overrightarrow{Q_y}$ are the shear forces corresponding to the axes x and y, respectively.

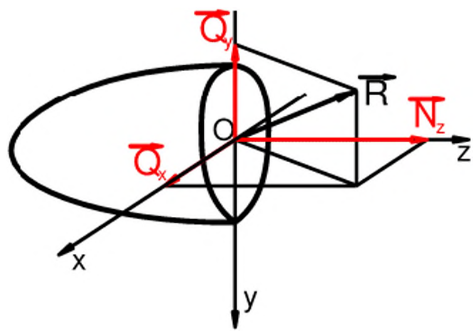


Figure 2.6

$$\vec{R} = \overrightarrow{N_z} + \overrightarrow{Q_x} + \overrightarrow{Q_y} \tag{2.2}$$

Similarly, moment $\overrightarrow{M}$ is separated into three components $\overrightarrow{M_x}$, $\overrightarrow{M_y}$, $\overrightarrow{M_z}$ as shown in **Figure 2.7**. The relationship between $\overrightarrow{M}$ and $\overrightarrow{M_x}$, $\overrightarrow{M_y}$, $\overrightarrow{M_z}$ is shown in Equation 2.3, in which, $\overrightarrow{M_x}$, $\overrightarrow{M_y}$, $\overrightarrow{M_z}$ are moments about the axis x, y, z, respectively.

Based on the effect of these moments, $\overrightarrow{M_x}$, $\overrightarrow{M_y}$ are called the bending moments while $\overrightarrow{M_z}$ is called torsional moment or torque.

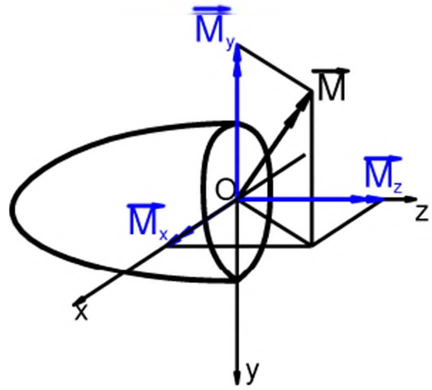
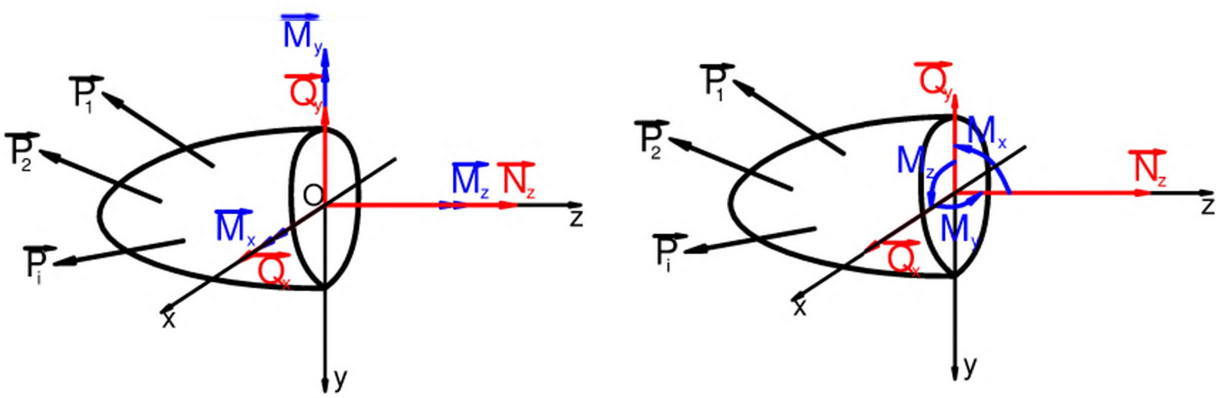


Figure 2.7

$$\overrightarrow{M} = \overrightarrow{M_x} + \overrightarrow{M_y} + \overrightarrow{M_z} \tag{2.3}$$

Thus, $\left.\begin{array}{l} \left.\begin{array}{l} \overrightarrow{N_z} \\ \overrightarrow{Q_x} \\ \overrightarrow{Q_y} \end{array}\right\} \\ \left.\begin{array}{l} \overrightarrow{M_x} \\ \overrightarrow{M_y} \\ \overrightarrow{M_z} \end{array}\right\} \end{array}\right\}$ are six components of internal forces on a section. These six components are shown in **Figure 2.8a**. Alternatively, these six components are shown in **Figure 2.8b**, which is commonly used in engineering practice.



a) b)

Figure 2.8. Six components of internal force on a section cut (two ways of display)

2.3. Equations to determine internal forces

To determine the six components of internal forces, six equilibrium equations can be established. These six equations include three equilibrium equations of forces and three equilibrium equations of moments.

Three equilibrium equations of forces

$$\begin{cases} \sum \vec{P}_{iz,left} + \vec{N_z} = 0 \\ \sum \vec{P}_{ix,left} + \vec{Q_x} = 0 \\ \sum \vec{P}_{iy,left} + \vec{Q_y} = 0 \end{cases} \tag{2.4}$$

in which, $\vec{P}_{iz,left}$, $\vec{P}_{ix,left}$, $\vec{P}_{iy,left}$ are components of $P_{i,left}$ acting on the left part corresponding to axes z, x, y, respectively.

Three equilibrium equations of moments

$$\begin{cases} \sum \overrightarrow{M}_{x\,Pi,left} + \overrightarrow{M_x} = 0 \\ \sum \overrightarrow{M}_{y\,Pi,left} + \overrightarrow{M_y} = 0 \\ \sum \overrightarrow{M}_{z\,Pi,left} + \overrightarrow{M_z} = 0 \end{cases} \tag{2.5}$$

in which, $\overrightarrow{M}_{x\,Pi,left}$, $\overrightarrow{M}_{y\,Pi,left}$, $\overrightarrow{M}_{z\,Pi,left}$ are moments due to forces P_i acting on the left part corresponding to x, y, z axes, respectively.

From the above six equilibrium equations, six components of internal forces are found.

2.4. Two-dimensional (2D) problem

In 2D problem, the external forces including forces and torque are in a plane, assuming the plane Oyz. Thus, internal forces are also in the Oyz plane. In 2D problem, there are three components of internal forces. If $\vec{P_1}$, $\vec{P_2}$, ..., $\vec{P_i}$ are in the plane Oyz, the internal forces N_z, Q_y, M_x are also in this plane. Other components of internal forces are zeros. These internal forces N_z, Q_y, M_x are shown in **Figure 2.9**.

Conventional signs of internal forces are shown **Figure 2.9** and are described as follows:

+ N_z is positive when it is outward direction of section (tension).
+ Q_y is positive when it tends to rotate the consideration part in the clockwise direction.
+ M_x is positive when it makes tension to the y^+ fibre.

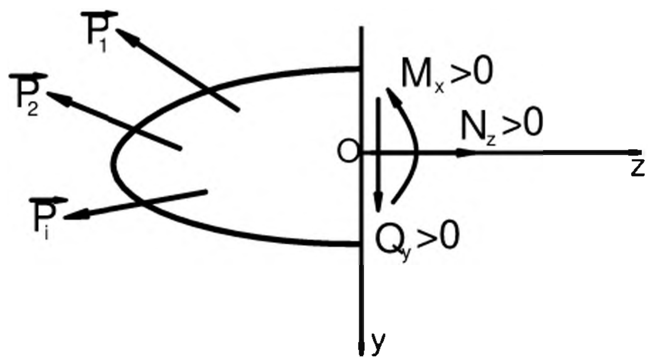


Figure 2.9. Conventional signs of internal forces

§3. STRESS

3.1. Definition of stress

Based on the assumption of continuity of materials, the internal force can be assumed to be continuously distributed on the entire section. Consider the point C with any coordinates (x, y) on the cross section, we surround this point with an infinite area ΔA as shown in **Figure 2.10**. The sum of internal forces on the area ΔA is $\Delta\vec{P}$.

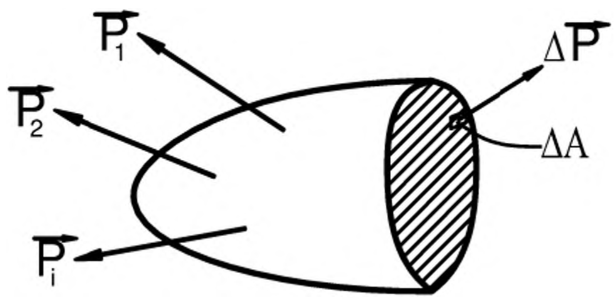


Figure 2.10. Internal force $\Delta\vec{P}$ on the infinite area ΔA

The ratio $\dfrac{\Delta\vec{P}}{\Delta A}$ is defined as the average stress at C.

Let ΔA approach 0 but still surround the point C:

$\lim\limits_{\Delta A\to 0}\dfrac{\Delta\vec{P}}{\Delta A}=\dfrac{d\vec{P}}{dA}=\vec{p}$ is called the total stress at C.

Definition: Stress is the internal force per unit area.

The unit of stress is force/area such as kN/m^2, N/mm^2, etc.

The total stress $\vec{p}$ can be decomposed into two components:

$$\vec{p}=\vec{\sigma}+\vec{\tau} \tag{2.6}$$

in which, τ is the shear stress which is the stress on the section.

σ is the normal stress which is the stress perpendicular to the section.

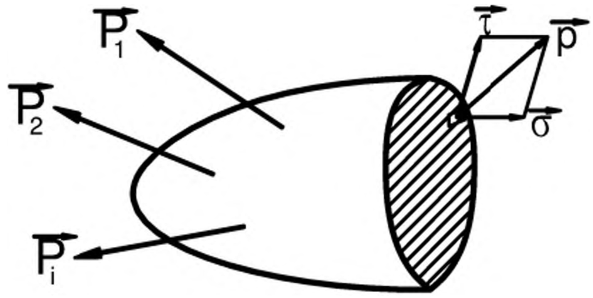


Figure 2.11. Shear stress and normal stress

3.2. Relationship between internal force and stresses

Analysing the shear stress into two components which are parallel to the x-axis and y-axis, the relationship is shown in Equation 2.7.

$$\vec{p} = \vec{\sigma} + \vec{\tau} = \overrightarrow{\sigma_z} + \overrightarrow{\tau_{zx}} + \overrightarrow{\tau_{zy}} \tag{2.7}$$

in which,

$\overrightarrow{\tau_{zx}}$ is the shear stress component in the plane perpendicular to the z axis, with the direction of x axis.

$\overrightarrow{\tau_{zy}}$ is the shear stress component in the plane perpendicular to the z axis, with the direction of y axis.

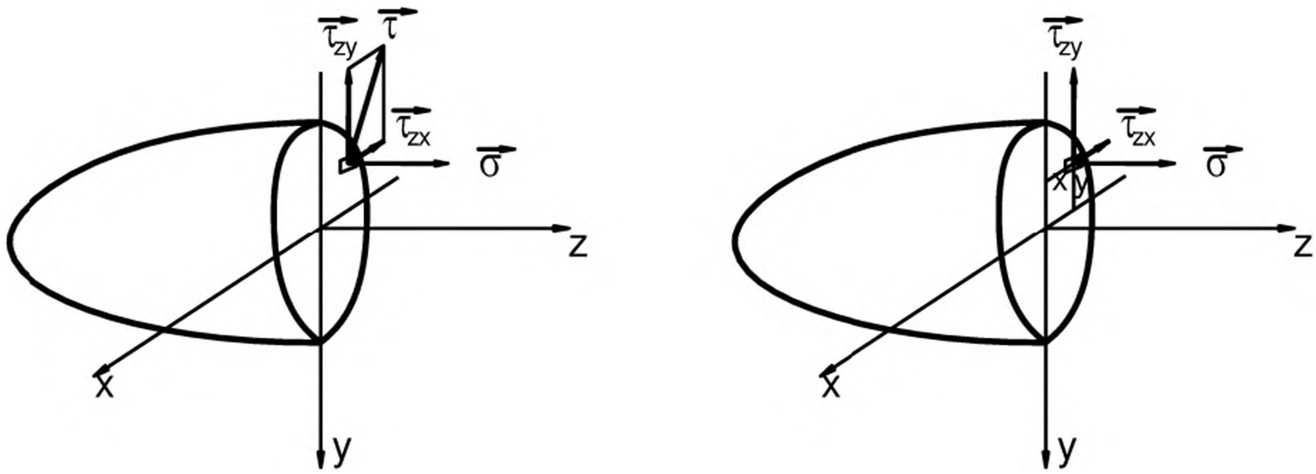


Figure 2.12. Components of shear stress

The relationship between internal forces and stresses are shown as follows.

$$N_z = \int_A dN_z = \int_A \sigma_z dA \tag{2.8}$$

$$Q_x = \int_A dQ_x = \int_A \tau_{zx} dA \tag{2.9}$$

$$Q_y = \int_A dQ_y = \int_A \tau_{zy} dA \tag{2.10}$$

$$M_x = \int_A dM_x = \int_A (\sigma_z dA) y = \int_A \sigma_z y dA \tag{2.11}$$

$$M_y = \int_A dM_y = \int_A (\sigma_z dA) x = \int_A \sigma_z x dA \tag{2.12}$$

$$M_z = \int_A dM_z = \int_A \left(\tau_{zy} dA\right) x - \left(\tau_{zx} dA\right) y = \int_A \left(\tau_{zy} x - \tau_{zx} y\right) dA \tag{2.13}$$

§4. EXERCISES

4.1. Exercise 1

Given P = 2qa, M = 2qa^2. Determine the internal forces at the section 1-1:

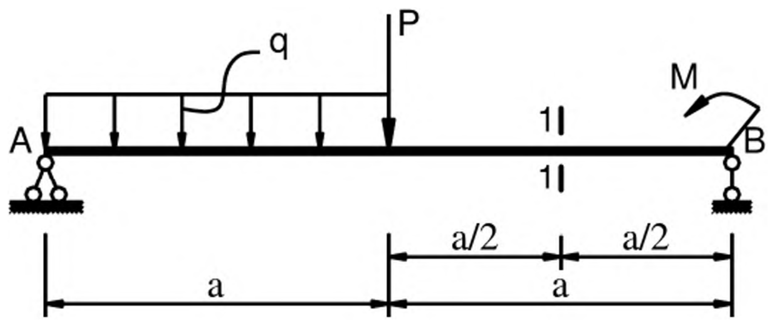
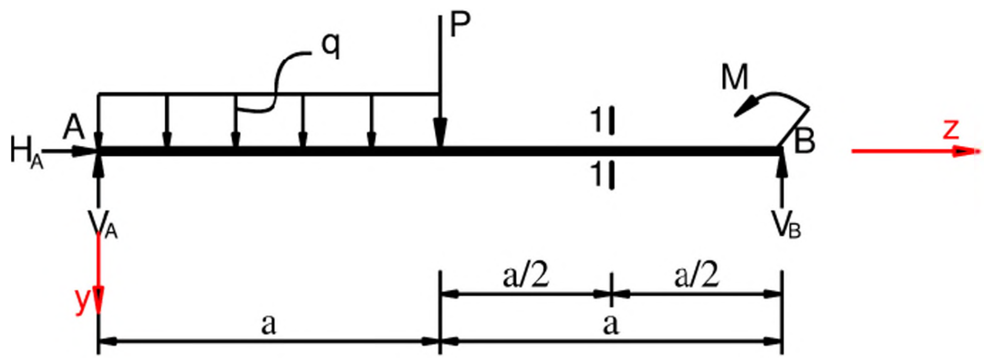


Figure 2.13

Solution

Step 1: Determine the support reactions:



$$\sum F_x = 0$$

$$\Rightarrow \quad H_A = 0$$

$$\sum {M_x}/_{B} = 0$$

$$\Rightarrow \quad V_A(2a) = qa\frac{3a}{2} + (2qa)a + 2qa^2$$

$$\Rightarrow \quad V_A = \frac{11qa}{4}$$

$$\sum M_x/_A = 0$$

$$\Rightarrow \quad \frac{qa^2}{2} + (2qa)a = 2qa^2 + V_B(2a)$$

$$\Rightarrow \quad V_B = \frac{qa}{4}$$

Check: $\sum F_y = 0 \Leftrightarrow V_A + V_A - qa - 2qa = 0$ OK.

Step 2: Use the cross-sectional method to determine the internal forces at the section 1-1:

We can examine the left part or the right part.

Consider the left part:

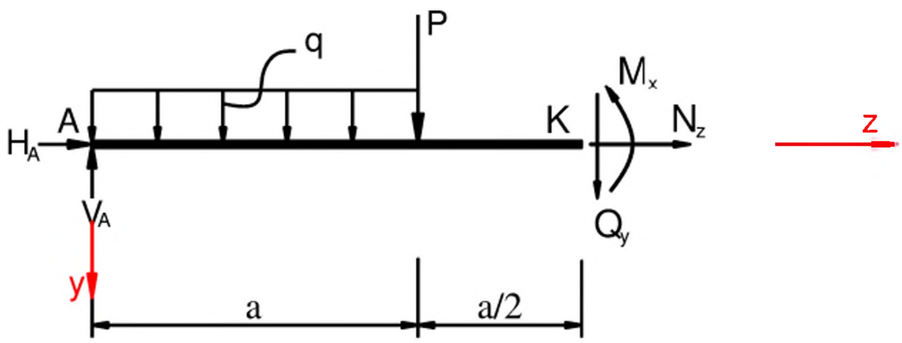


Figure 2.14

$$\sum F_x = 0$$

$$\Rightarrow \quad N_Z = H_A = 0$$

$$\sum M_x/_K = 0$$

$$\Rightarrow \quad V_A \frac{3a}{2} = qa(a) + (2qa)\frac{a}{2} + M_x$$

$$\Rightarrow \quad M_x = \frac{17qa^2}{8}$$

$$\sum M_x/_A = 0$$

$$\Rightarrow \quad \frac{qa^2}{2} + (2qa)a + Q_y\frac{3a}{2} = M_x$$

$$\frac{qa^2}{2} + (2qa)a + Q_y\frac{3a}{2} = \frac{17qa^2}{8}$$

$$\Rightarrow \quad Q_y = -\frac{qa}{4}$$

Check: $\sum F_y = 0 \Leftrightarrow V_A = qa + 2qa + Q_y$

$$\frac{11qa}{4} = qa + 2qa + (-\frac{qa}{4}) \text{ OK}$$

If we consider the right part:

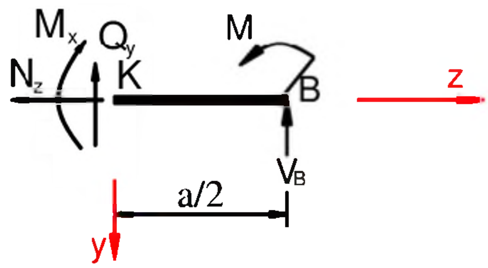


Figure 2.15

$$\sum F_z = 0$$

$$\Rightarrow \quad N_z = 0$$

$$\sum M_x/_K = 0$$

$$\Rightarrow \quad M_x = M + V_B \frac{a}{2}$$

$$\Rightarrow \quad M_x = 2qa^2 + \frac{qa}{4}\frac{a}{2} = \frac{17qa^2}{8}$$

$$\sum M_x/_B = 0$$

$$\Rightarrow \quad Q_y \frac{a}{2} + M_x = M$$

$$\Leftrightarrow \quad Q_y \frac{a}{2} + \frac{17qa^2}{8} = 2qa^2$$

$$\Rightarrow \quad Q_y = -\frac{qa}{4}$$

Check: $\sum F_y = 0 \Leftrightarrow V_B + Q_y = 0$

$$\frac{qa}{4} - \frac{qa}{4} = 0 \text{ OK}$$

The calculation is reliable.

§5. DIAGRAMS OF INTERNAL FORCES

Each time the cross-sectional method is applied, the internal forces at a given cross section are determined.

If we make an infinity of cross-sections, you get an infinity of internal forces.

The graph representing the infinite number of internal force values is called the internal force diagram or diagram of internal force.

In other words, ***the diagram of internal force is a graph representing the variation of the internal force along the axial axis.***

Steps to draw the diagram of internal forces

- Identification of support reactions.
- Use cross section method to determine internal forces:
  + Take a "cut' at the coordinate z.
  + Establish equilibrium equations to compute the internal force in accordance with z.
- Draw a line expressing the internal force on the axis z, the vertical axis representing the internal force magnitudes.
- Check the internal force diagram (this section will be discussed in the differential relations presented in §6).

5.1. Exercise 2

Draw diagrams of shear force and bending moment for the cantilever beam subjected to the point load P at the end of the beam as shown in **Figure 2.16**.

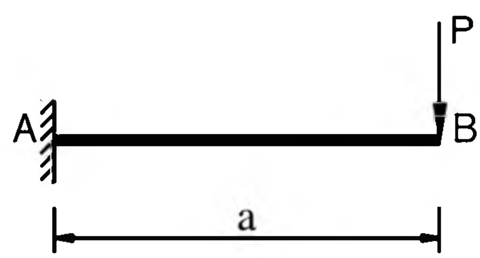


Figure 2.16

Solution

Take a cut I-I at K, the distance AK is z (**Figure 2.17a**). Consider the right part (**Figure 2.17b**):

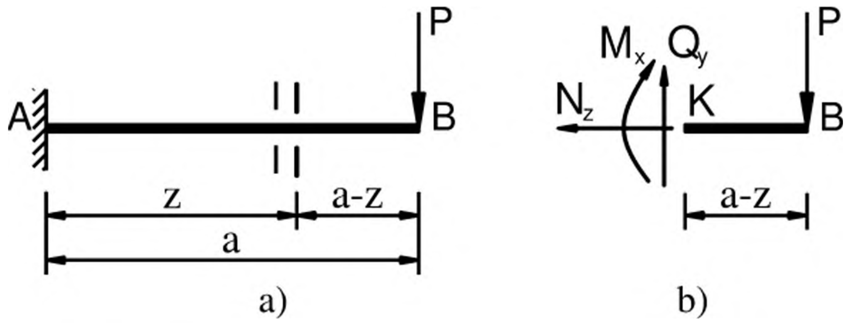


Figure 2.17

$$\sum F_x = 0 \Rightarrow N_z = 0$$

$$\sum F_y = 0 \Rightarrow Q_y = P$$

$$\sum M_x/_K = 0 \Rightarrow M_x + P(a-z) = 0 \Rightarrow M_x = -P(a-z)$$

The coordinate z varies from 0 to a, the diagrams of shear force and bending moment are drawn as shown in **Figure 2.18**.

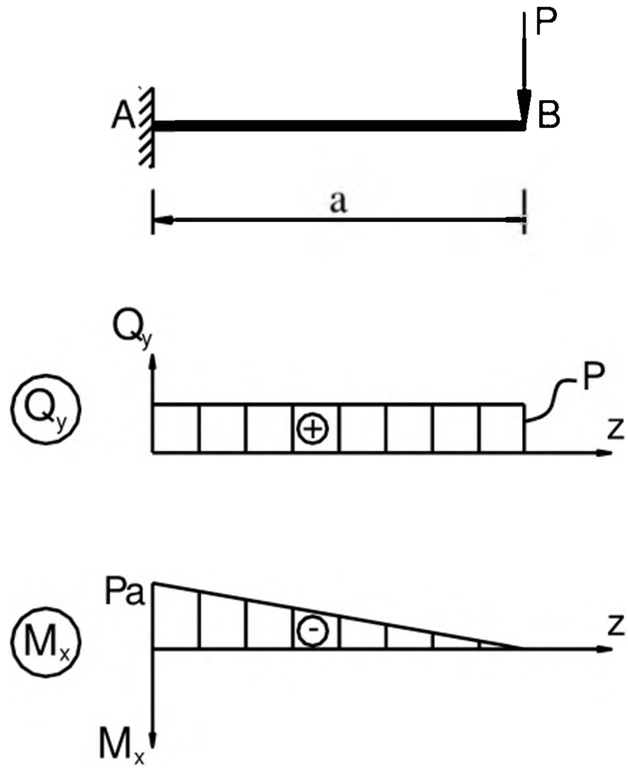


Figure 2.18

Remark:

For the shear force diagram Q_y, the axis Q_y is upward. There are signs (±) on the shear force diagram.

For bending moment diagram, the positive moment is below the horizontal axis. If we use the absolute value of M to plot, we draw the bending moment on the tension side. Signs (±) are not needed for the diagram of bending moment.

5.2. Exercise 3

Draw the diagrams of shear force and bending moment for a cantilever beam subjected to distributed load q as shown in. The length of the beam is l.

Solution

Take a cut 1-1 at the location z, consider the left part (see **Figure 2.19**).

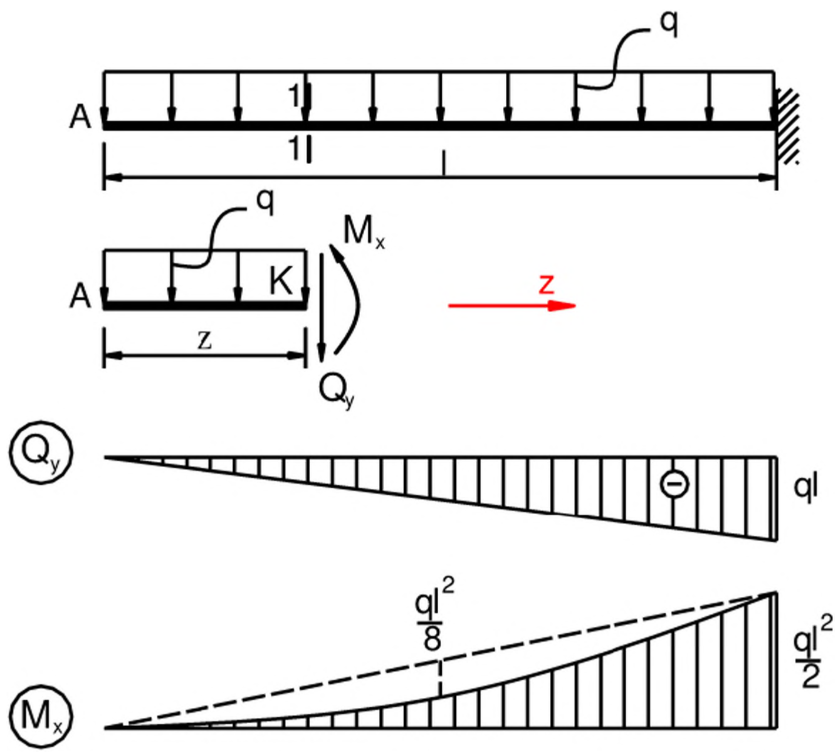


Figure 2.19

Establish the equilibrium equations, we have:

$$Q_y = -qz$$

$$M_x = -qz \times \frac{z}{2} = -\frac{qz^2}{2}$$

Draw the diagrams of shear force and bending moment (see **Figure 2.19**).

5.3. Exercise 4

Draw the diagrams of shear force and bending moment for the simply supported beam AB with the length *l* subjected to the distributed load q.

Solution

Step 1: Determine the support reactions

$$V_A = V_B = \frac{ql}{2}$$

Step 2: Take the cut I-I at K, the distance from A to K is z. Consider the left part as shown in **Figure 2.20**.

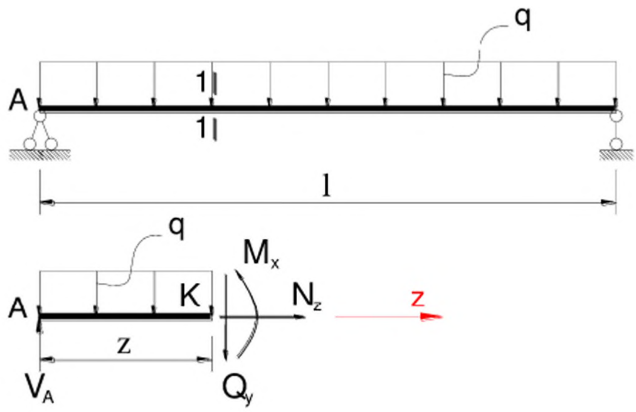


Figure 2.20

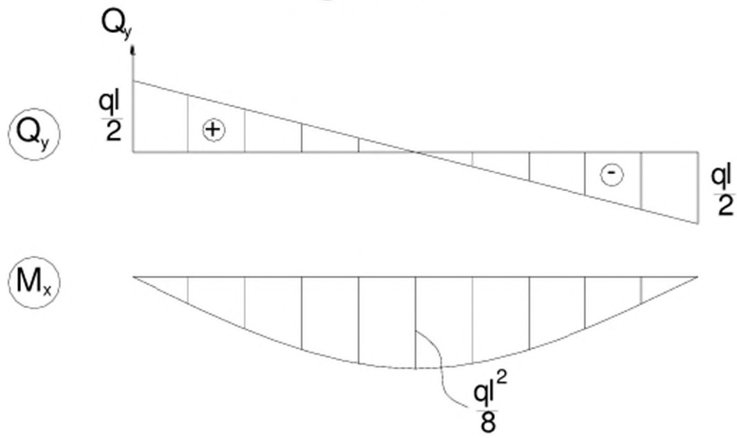


Figure 2.21

$$\sum F_x = 0 \Rightarrow N_z = 0$$

$$\sum F_y = 0 \Rightarrow Q_y + qz = \frac{ql}{2}$$

$$\Rightarrow \quad Q_y = \frac{ql}{2} - qz = q\left(\frac{l}{2} - z\right)$$

$$\sum M_x / K = 0$$

$$\Rightarrow \quad \frac{ql}{2} z + qz\frac{z}{2} = M_x$$

$$\Rightarrow \quad M_x = \frac{qz}{2}(l - z)$$

Step 3: Draw the diagrams of bending moment and shear force

Draw the diagrams of the above equations, the diagrams are shown in **Figure 2.21**.

Comments: M_{max} at the middle of the beam, Q_{max} at the support.

5.4. Exercise 5

Draw the diagrams of internal forces for a simple beam subjected to force P as shown in **Figure 2.22**. The distances from A and B to the load P are a and b, respectively. Given l = a + b.

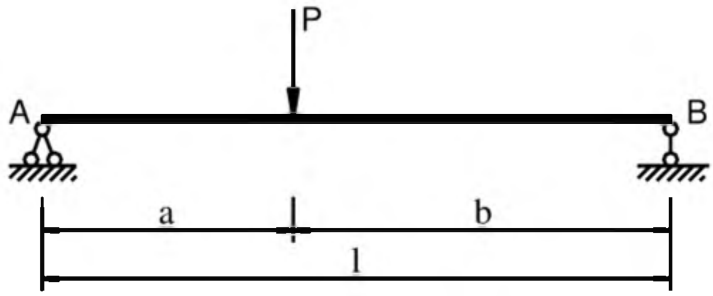


Figure 2.22

Solution

Step 1: Determine the support reactions:

$$V_A = \frac{Pb}{l};\ V_B = \frac{Pa}{l}$$

Step 2: Take the cut 1-1, consider the left part (z = 0 → a), we have:

$$Q_y = V_A = \frac{Pb}{l}$$

$$M_x = V_A z = \frac{Pb}{l} z$$

Take the cut 2-2, consider the right part 2-2 (z = a → l), we have:

$$Q_y = -V_B = -\frac{Pa}{l}$$

$$M_x = V_A (l - z) = \frac{Pa}{l}(l - z)$$

Step 3: Draw the diagrams of bending moment and shear force

The diagrams of shear force and bending moment are shown in **Figure 2.23**.

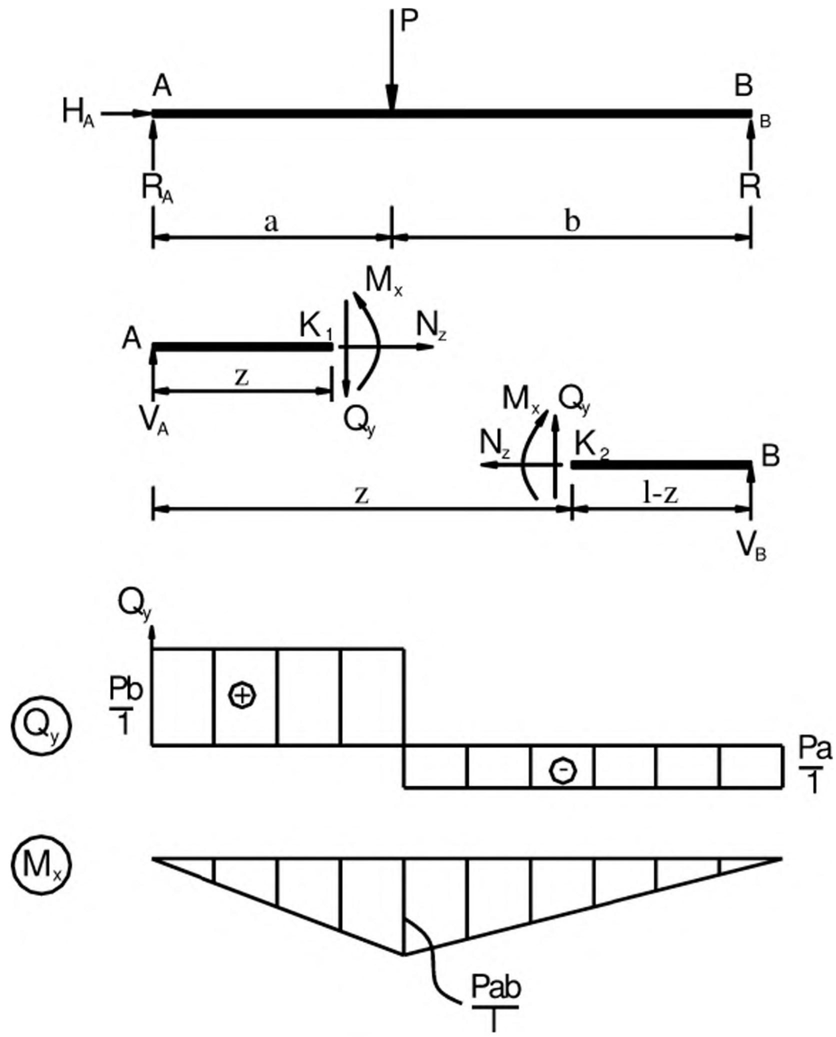


Figure 2.23

5.5. Exercise 6

Draw the diagrams of shear fore and bending moment for the beam subjected to the concentrated moment M_o as shown in **Figure 2.24**.

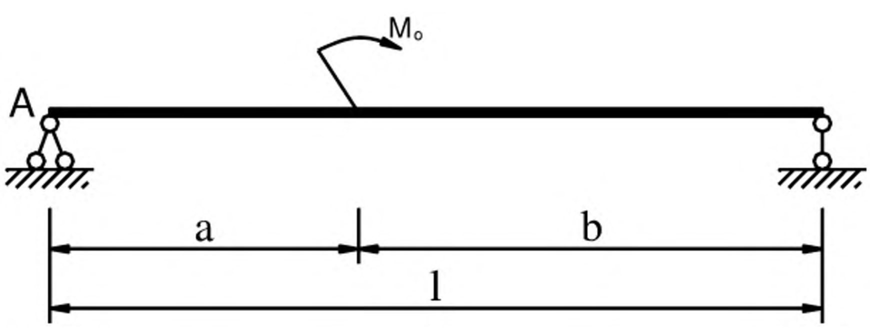


Figure 2.24

Solution

Step 1: Determine the support reactions:

$$V_A = V_B = \frac{M_o}{l}$$

Step 2: Take the cut 1-1, consider the left part (z = 0 → a), we have:

$$Q_y = -V_A = -\frac{M_o}{l}$$

$$M_x = -V_A z = -\frac{M_o}{l} z$$

Take the cut 2-2, consider the right part 2-2 (z = a → l), we have:

$$Q_y = -V_B = -\frac{M_o}{l}$$

$$M_x = V_A (l - z) = \frac{M_o}{l}(l - z)$$

Step 3: Draw the diagrams of bending moment and shear force

The diagrams of shear force and bending moment are shown in **Figure 2.25**.

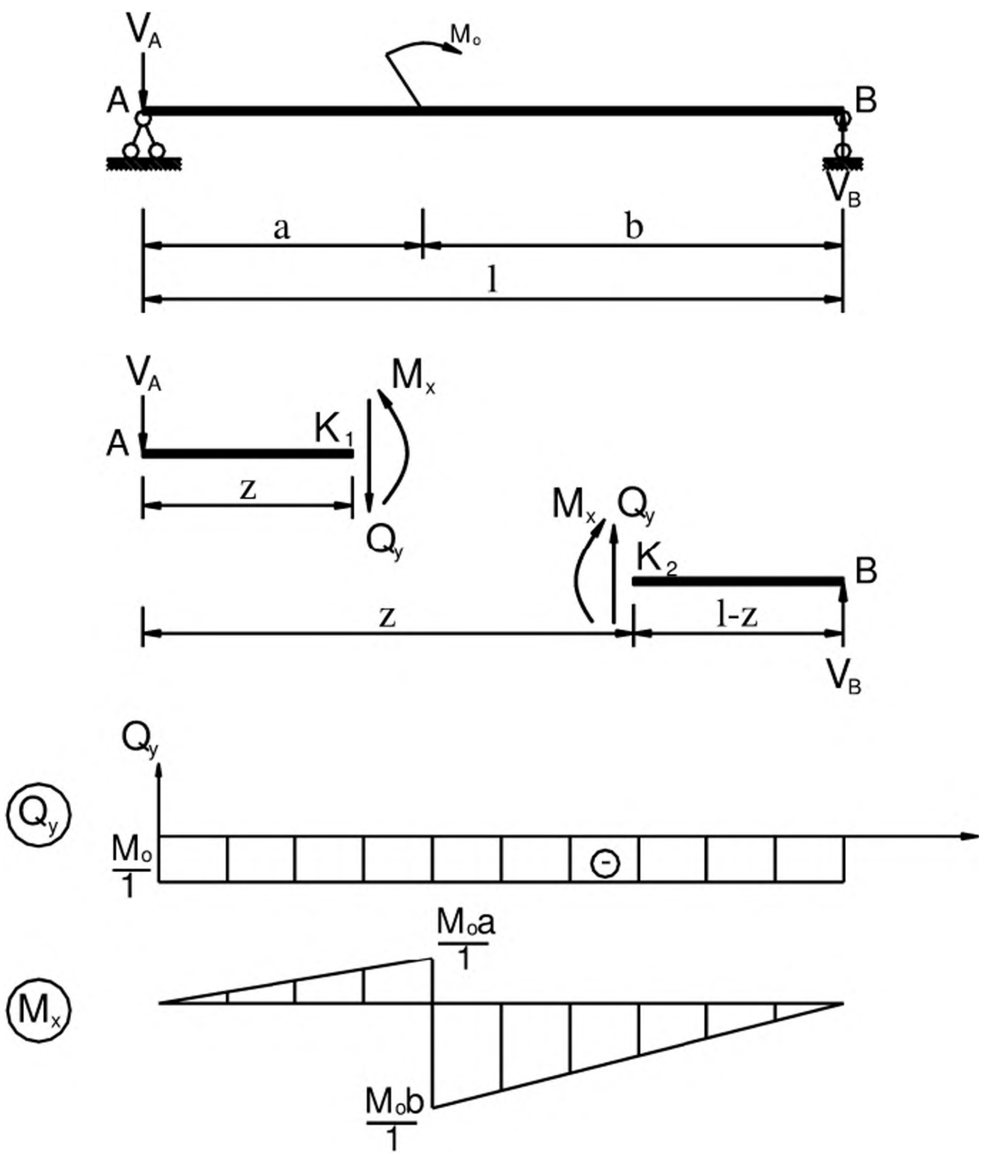


Figure 2.25

5.6. Important remarks

At the location of point load, the shear force diagram has a jump. The value of the jump is equal to the value of the point load.

At the location of concentrated moment, the bending moment diagram has a jump. The value of the jump is equal to the value of the moment.

5.7. Exercise 7

Consider the beam AB subjected to loads as shown in **Figure 2.26**.

a) Determine the support reactions.

b) Draw the diagrams of shear force and bending moment for the beam AB.

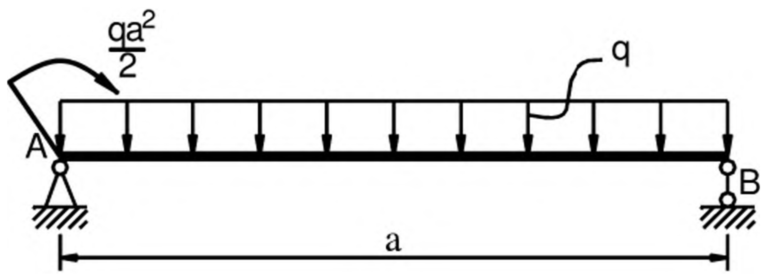


Figure 2.26

Solution

a) Determine the support reactions:

$$\sum M_{/B} = 0$$

$$\Rightarrow \quad R_A a + \frac{qa^2}{2} = \frac{qa^2}{2}$$

$$\Rightarrow \quad R_A = 0$$

$$\sum M_{/A} = 0$$

$$\Rightarrow \quad R_B\, a = \frac{qa^2}{2} + \frac{qa^2}{2}$$

$$\Rightarrow \quad R_B = qa$$

b) Draw the diagrams of shear force and bending moment for the beam AB:

The procedure for calculating the internal forces is similar to the previous exercise.

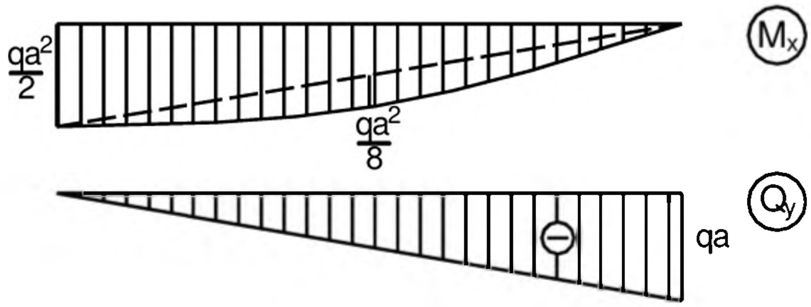


Figure 2.27

§6. DIFFERENTIAL RELATIONSHIP BETWEEN INTERNAL FORCES AND EXTERNAL LOADS

The results of the previous exercises show that the relationship between the distributed load q(z), shear force Q_y and bending moment M_x at any cross section:

- The derivative of the bending moment is the shear force.
- The derivative of the shear force is the distributed load.

The relationships are proven as follows. Consider a beam subjected to a distributed load q(z) as shown in **Figure 2.28**. Consider a infinitesimal length dz between the two sections 1-1 and 2-2 as shown in **Figure 2.28**. The internal force on the section 1-1 is Q_y and M_x. The internal force on the section 2-2 is $Q_y + dQ_y$ and $M_x + dM_x$. The distributed load has the magnitude q(z) in the positive direction as shown in **Figure 2.29**. Since dz is very small, it is possible to consider as a uniformly distributed load on the dz segment.

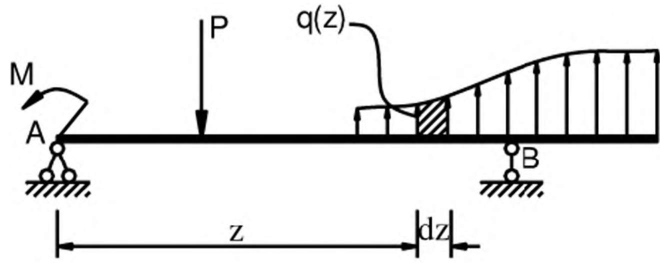


Figure 2.28

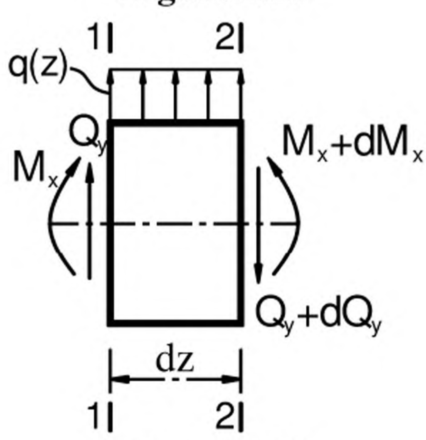


Figure 2.29

Equilibrium equation of vertival forces:

$$\sum F_y = 0$$

$$Q_y + q(z)dz - \left(Q_y + dQ_y\right) = 0$$

$$q(z) = \frac{dQ_y}{dz} \tag{2.14}$$

Equation 2.14 shows the differential relationship between the shear force and the distributed load. This equation can be stated that the derivative of the shear force is equal to the distributed load perpendicular to the axial axis.

Note: In the above relationship, the positive of distributed load is upward.

The equilibrium equation of moment corresponding to the center of cross section 2-2, we have:

$$\sum M = 0$$

$$Q_y dz + q(z)dz\frac{dz}{2} + M_x - \left(M_x + dM_x\right) = 0$$

Ignoring the extremely small differential, we have:

$$Q_y = \frac{dM_x}{dz} \tag{2.15}$$

Equation 2.15 shows the differential relationship between the bending moment and shear force. This equation can be stated that the Thus: The derivative of the bending moment is equal to the shear force.

From these two relationships (Equations 2.14 and 2.15), we have:

$$\frac{d^2M_x}{dz^2} = q(z) \tag{2.16}$$

Equation 2.16 shows the differential relationship between the bending moment and the distributed load. This equation can be stated that the quadratic (second order) derivative of the bending moment at a section is equal to the distributed load at that section.

6.1. Exercise 8

Draw diagrams of internal force for the simple beam subjected to first order distributed load as shown in **Figure 2.30**.

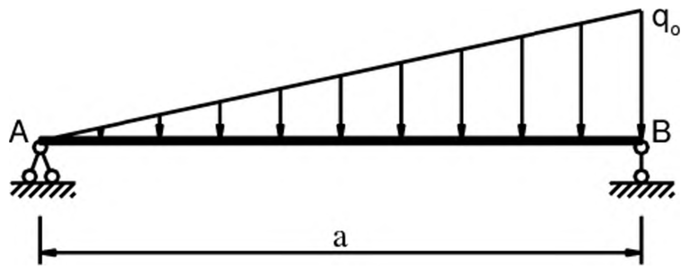


Figure 2.30

Solution

Determine the support reactions:

$$\sum M_x/_B = 0 \Rightarrow V_A l = \left(\frac{1}{2}q_o l\right)\frac{l}{3} \Rightarrow V_A = \frac{q_o l}{6}$$

$$\sum M_x/_A = 0 \Rightarrow V_B l = \left(\frac{1}{2}q_o l\right)\frac{2l}{3} \Rightarrow V_B = \frac{q_o l}{3}$$

Check: $\sum F_y = 0 \Rightarrow V_A + V_A - \frac{1}{2}q_o l = 0 \Rightarrow$ OK.

Let A be the origin. The distance from A to the cut 1-1 is z (see **Figure 2.30**). The magnitude of load at the location z is q(z):

$$q(z) = q_o \frac{z}{l}$$

The distributed load is the first order. Therefore, diagram of shear force is the second order and the diagram of bending moment is the third order.

Take the cut 1-1, consider the left part:

$$\sum F_y = 0 \Rightarrow Q_y = V_A - q(z)\frac{z}{2} = \frac{q_o l}{6} - \frac{q_o z^2}{2l} \quad \text{(the second order)}$$

Take the total moment with respect to the center of the cut 1-1:

$$\begin{aligned} M_x &= \frac{q_o l}{6}z - \left(\frac{1}{2}q(z)z\right)\frac{z}{3} \\ &= \frac{q_o l}{6}z - \left(\frac{1}{2}\left(q_o\frac{z}{l}\right)z\right)\frac{z}{3} = \frac{q_o l}{6}z - \frac{q_o z^3}{6l} \end{aligned} \quad \text{(the third order)}$$

From the two expressions above, the diagrams of shear force and bending moment for the beam are plotted in **Figure 2.31**. These diagrams have the following characteristics.

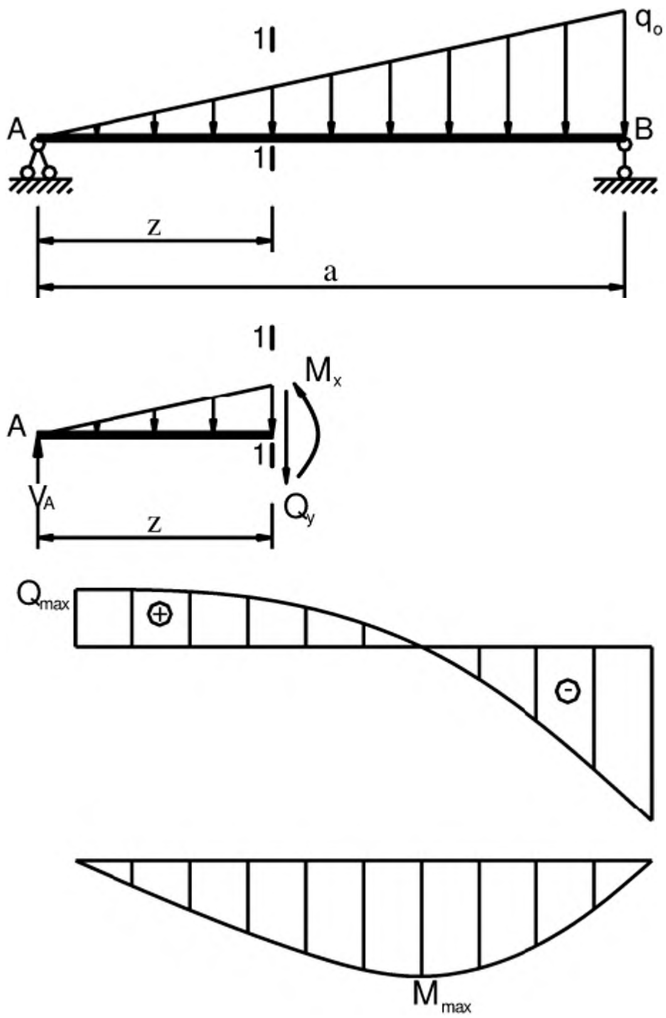


Figure 2.31

The diagram of shear force is of second order. At the location z = 0, q(z) = 0 or $q(z)=\frac{dQ_y}{dz}=0$. Therefore, Q_y reaches the extreme at this location:

$$Q_{y,z=0}=Q_{\max}=\frac{q_o l}{6}$$

The diagram of bending moment M_x is of order 3. When $Q_y = 0$, $\frac{q_o l}{6}-\frac{q_o z^2}{2l}=0$ $\Rightarrow z=\frac{l}{\sqrt{3}}$. At this location, M_x reaches the extreme:

$$M_{x,z=l/\sqrt{3}}=M_{\max}=\frac{q_o l^2}{9\sqrt{3}}$$

6.2. Exercise 9

Draw diagrams of internal forces for the beam shown in the **Figure 2.32**. Given: $L_1 = L_2 = L_3 = a$; $P = 2qa$; $M_o = qa^2$.

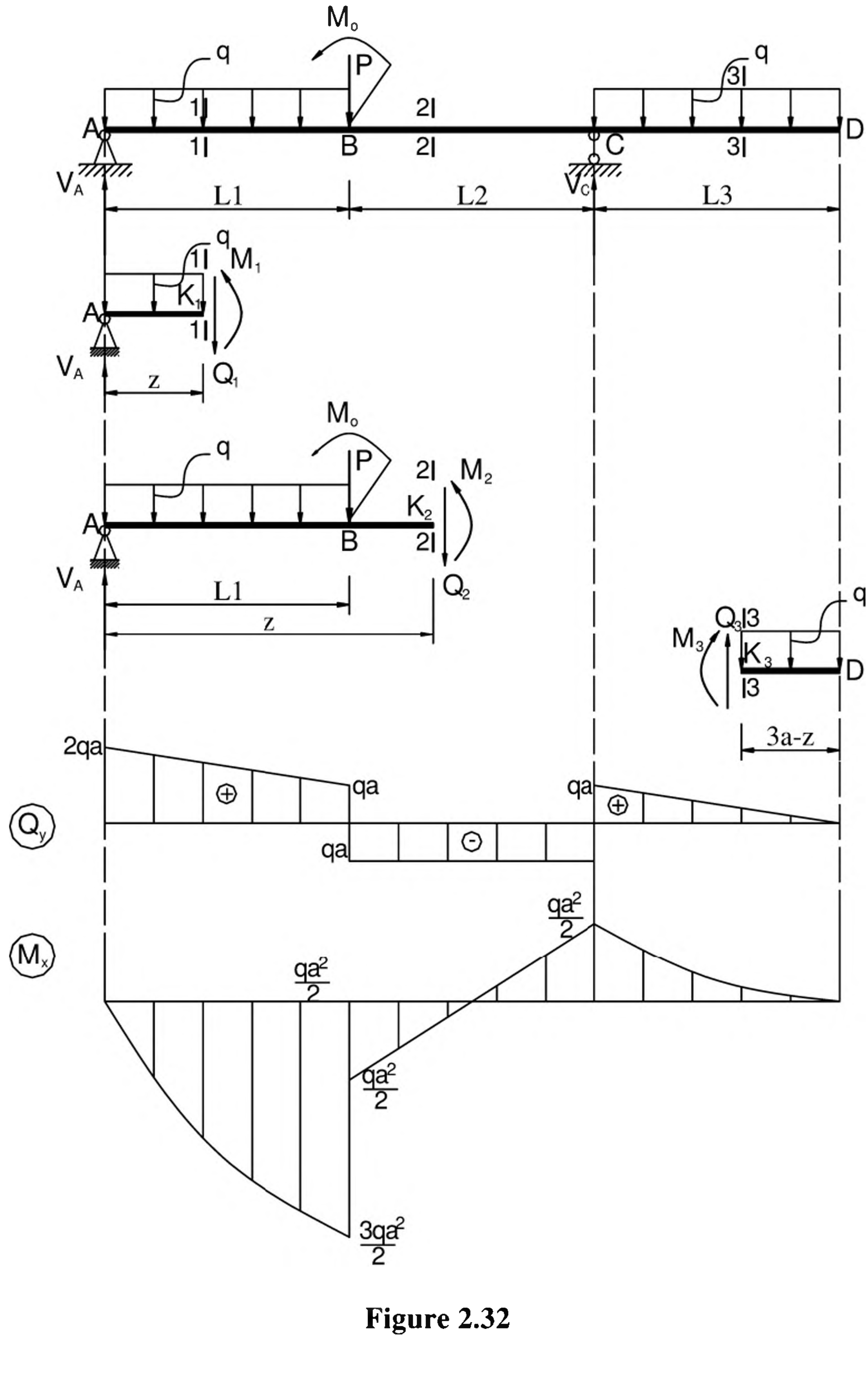

M₀
q
P
q
1I
2I
3I
A
B
C
D
1I
2I
3I
V_A
V_C
L1
L2
L3
q
1I
M_1
K_1
A
1I
V_A
Q_1
z
M_0
q
P
2I
M_2
K_2
A
B
2I
V_A
Q_2
L1
z
q
Q_3
I3
M_3
K_3
D
I3
3a-z
2qa
qa
qa
qa
Q_y
$\frac{qa^2}{2}$
M_x
$\frac{qa^2}{2}$
$\frac{qa^2}{2}$
$\frac{3qa^2}{2}$

Figure 2.32

Solution

Calculate the support reactions:

$$\sum M_{/A} = 0 \Rightarrow$$

$$\frac{qa^2}{2} + (2qa)a + qa\left(2a + \frac{a}{2}\right) = M_o + V_C 2a$$

$$\frac{qa^2}{2} + (2qa)a + qa\left(2a + \frac{a}{2}\right) = qa^2 + V_C 2a$$

$$V_C = 2qa$$

$$\sum M_{/B} = 0 \Rightarrow$$

$$qa\left(a + \frac{a}{2}\right) + (2qa)a + qa^2 = V_A 2a + \frac{1}{2}qa^2$$

$$V_A = 2qa$$

Check:

$$\sum F_y = 0 \Rightarrow 2qa + 2qa = qa + 2qa + qa \text{ OK}$$

Diagrams of internal forces:

Segment AB: take the cut 1-1, consider the left part:

$$\begin{cases} \sum F_y = 0 \Rightarrow Q_1 = 2qa - qz \\ \sum M_{K_1} = 0 \Rightarrow M_1 = 2qaz - \dfrac{qz^2}{2} \end{cases} \text{ where } 0 \le z \le a$$

Segment BC: take the cut 2-2, consider the left part:

$$\begin{cases} \sum F_y = 0 \Rightarrow Q_2 = -qa \\ \sum M_{K_2} = 0 \Rightarrow M_1 = 2qaz - qa\left(z - \dfrac{a}{2}\right) - qa^2 - 2qa(z-a) = \dfrac{3}{2}qa^2 - qaz \end{cases}$$

where $a \le z \le 2a$

Segment CD: take the cut 3-3, consider the right part:

$$\begin{cases} \sum F_y = 0 \Rightarrow Q_3 = q(3a - z) \\ \sum M_{K_3} = 0 \Rightarrow M_3 = -q\dfrac{(3a-z)^2}{2} \end{cases} \text{ where } 2a \le z \le 3a$$

6.3. Exercise 10

Draw diagrams of internal forces for the structure shown in **Figure 2.33**. Given: $L_1 = L_2 = a$; $P = qa$; $M_o = qa^2$.

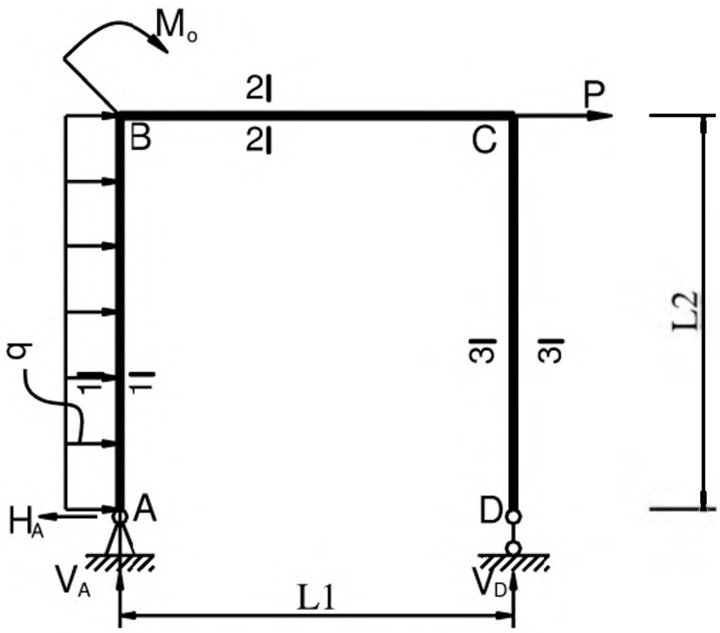


Figure 2.33

Solution

Calculate the support reactions:

$$\sum F_{/ngang} = 0 \Rightarrow H_A = 2qa$$

$$\sum M_{/A} = 0 \Rightarrow \frac{qa^2}{2} + qa^2 + (qa)a = V_D a$$

$$V_D = \frac{5}{2}qa$$

$$\sum M_{/D} = 0 \Rightarrow \frac{qa^2}{2} + qa^2 + (qa)a + V_A a = 0$$

$$V_A = -\frac{5}{2}qa$$

Check: $\sum F_y = 0 \Rightarrow V_A + V_D = -\frac{5}{2}qa + \frac{5}{2}qa = 0$ OK

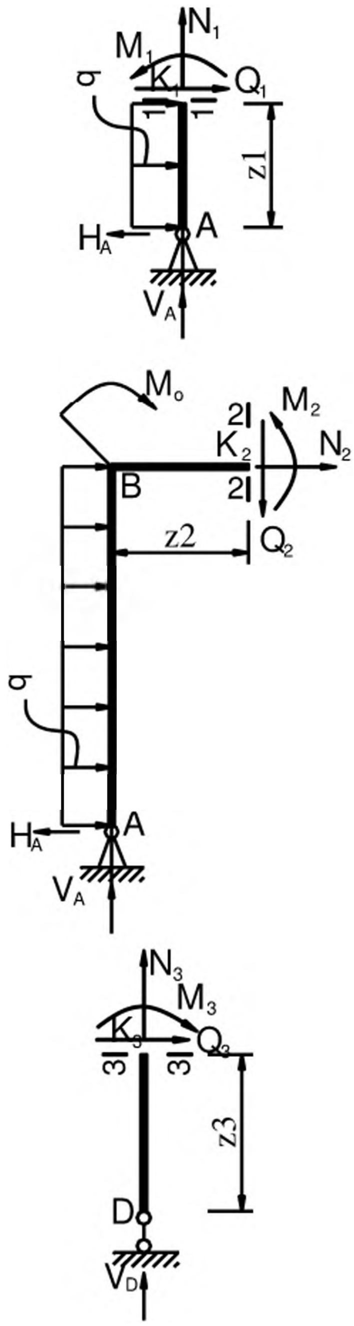


Figure 2.34

Diagrams of internal forces:

Segment AB: take the cut 1-1, consider the part below the cut:

$$\begin{cases} \sum F_z = 0 \Rightarrow N_2 = 2qa - qa = qa \\ \sum F_y = 0 \Rightarrow Q_2 = \dfrac{5}{2}qa \\ \sum M_{K_2} = 0 \Rightarrow M_2 = 2qa^2 + qa^2 - \dfrac{qa^2}{2} - \dfrac{5qaz_2}{2} = \dfrac{5}{2}qa^2 - \dfrac{5}{2}qaz_2 \end{cases}$$

where $0 \leq z_1 \leq a$

Segment BC: take the cut 2-2, consider the left part:

$$\begin{cases} \sum F_z = 0 \Rightarrow N_2 = -V_A = \dfrac{5}{2}qa \\ \sum F_y = 0 \Rightarrow Q_2 = -qa \\ \sum M_{K_2} = 0 \Rightarrow M_1 = 2qaz - qa\left(z - \dfrac{a}{2}\right) - qa^2 - 2qa(z-a) = \dfrac{3}{2}qa^2 - qaz \end{cases}$$

where $0 \leq z_2 \leq a$

Segment CD: take the cut 3-3, consider the right part:

$$\begin{cases} \sum F_z = 0 \Rightarrow N_3 = -V_D = -\dfrac{5}{2}qa \\ \sum F_y = 0 \Rightarrow Q_3 = 0 \\ \sum M_{K_3} = 0 \Rightarrow M_3 = 0 \end{cases}$$

where $0 \leq z_3 \leq a$

From the above results, the diagrams of shear force and bending moment are shown in **Figure 2.35**.

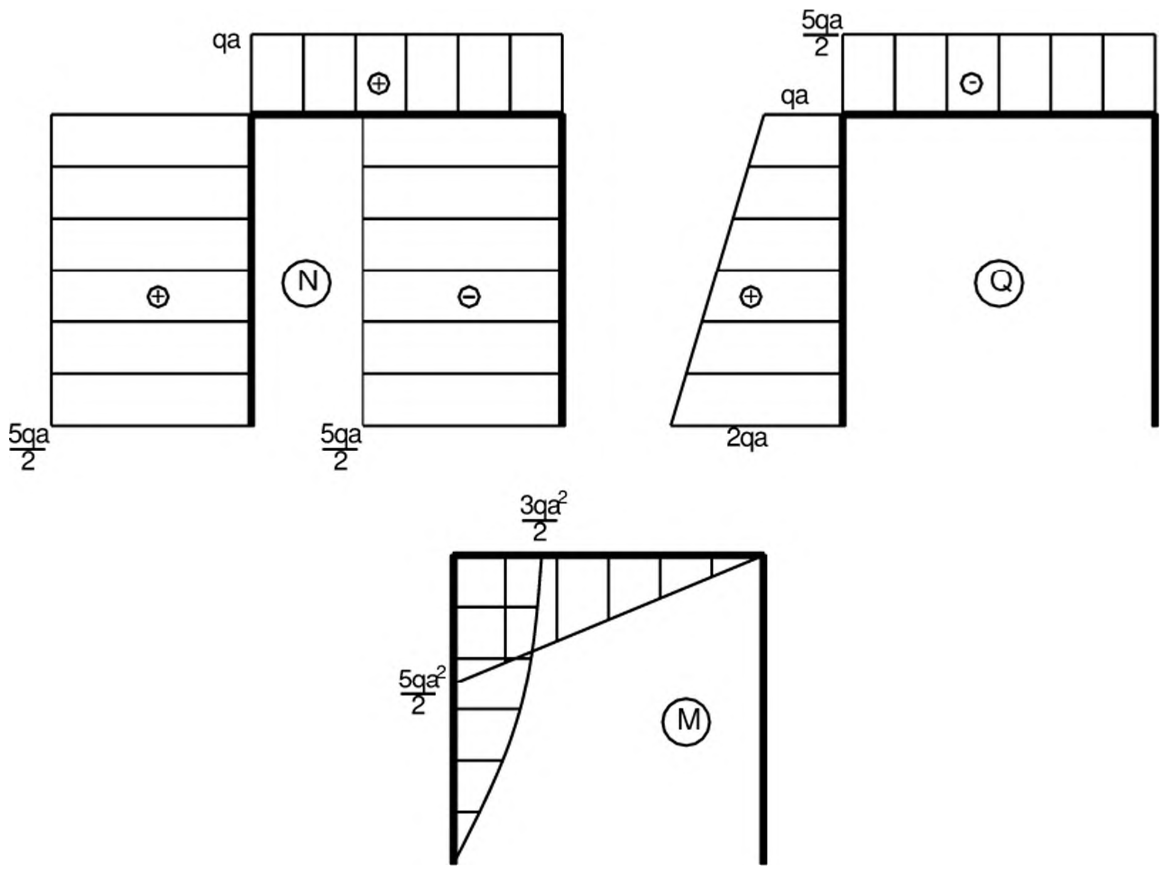


Figure 2.35

Check the balance of joint:

For the frame, the results can be checked by balancing of joint. By separating the joint from the frame and then put the concentrated external forces (if any) and the internal forces on sections, these forces must be balanced.

Figure 2.36 shows the separated joint:

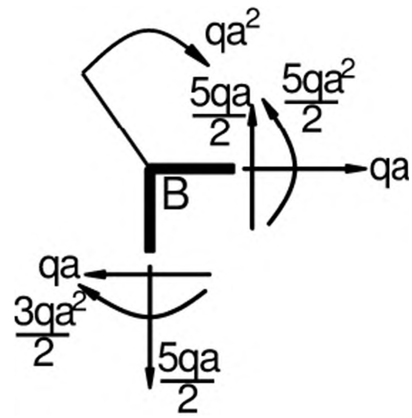


Figure 2.36

$$\sum M = qa^2 - \frac{5}{2}qa^2 + \frac{3}{2}qa^2 = 0 \Rightarrow OK$$

$$\sum F_x = qa - qa = 0 \Rightarrow OK$$

$$\sum F_y = \frac{5}{2}qa - \frac{5}{2}qa = 0 \Rightarrow OK$$

§7. ALTERNATIVE METHODS FOR DRAWING INTERNAL FORCE DIAGRAMS

7.1. Applying the superposition method

In the above sections, the internal force diagrams are drawn using methametical solutions/expressions, also known as an analytical method. If the member is subjected to a variety of loads, it is possible to draw internal force diagrams caused by individual loads and then add them up for the final internal force diagrams.

7.2. Using special points

Based on differential relations, we can determine the shapes of the internal force diagrams with respect to the given loads and thus determine the necessary number of points needed to draw the diagrams:

- If the diagram is a constant: we need 1 point.
- If the diagram is the first order: we need 2 points.
- If the diagram is the second order: we need three points: two points at two ends and one point in the middle.

If a member subjected to a distributed load, then the moment diagram tends to "welcome" the distributed load q as shown in **Figure 2.37**.

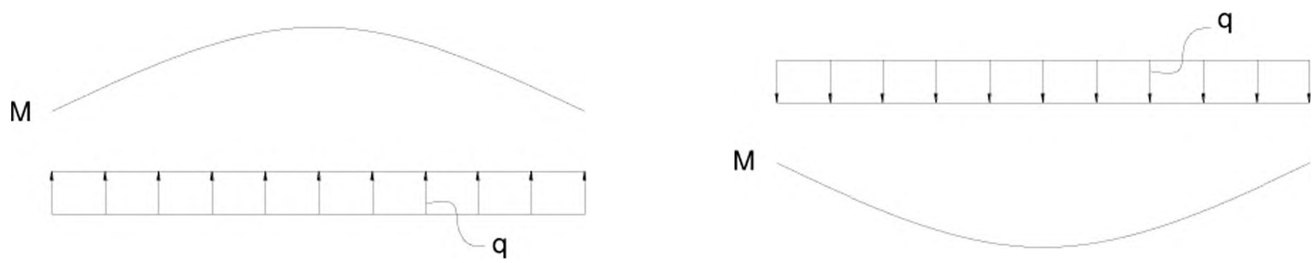


Figure 2.37

7.3. Exercise 11

Use the superposition method to draw the diagram of bending moment for the beam shown in **Figure 2.38**.

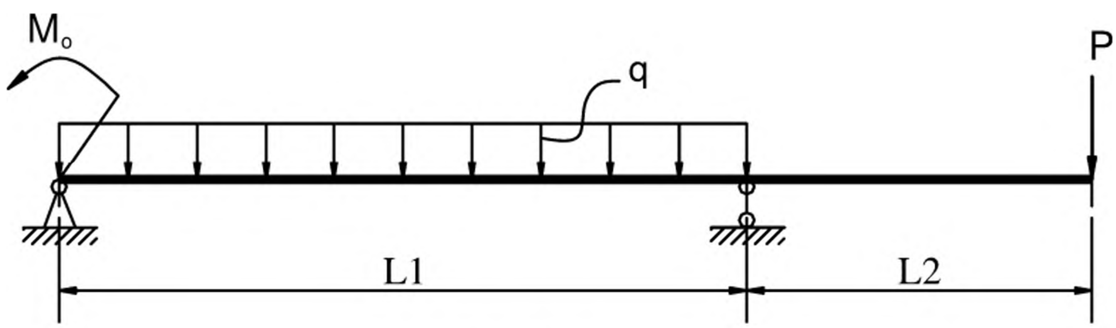


Figure 2.38

Solution

Figure 2.39a shows the diagram of bending moment caused by the moment M_o.

Figure 2.39b shows the diagram of bending moment caused by the distributed load q.

Figure 2.39c shows the diagram of bending moment caused by the concentrated load P.

Adding the three diagrams above, we get the diagram of bending moment caused by the loads on the beam.

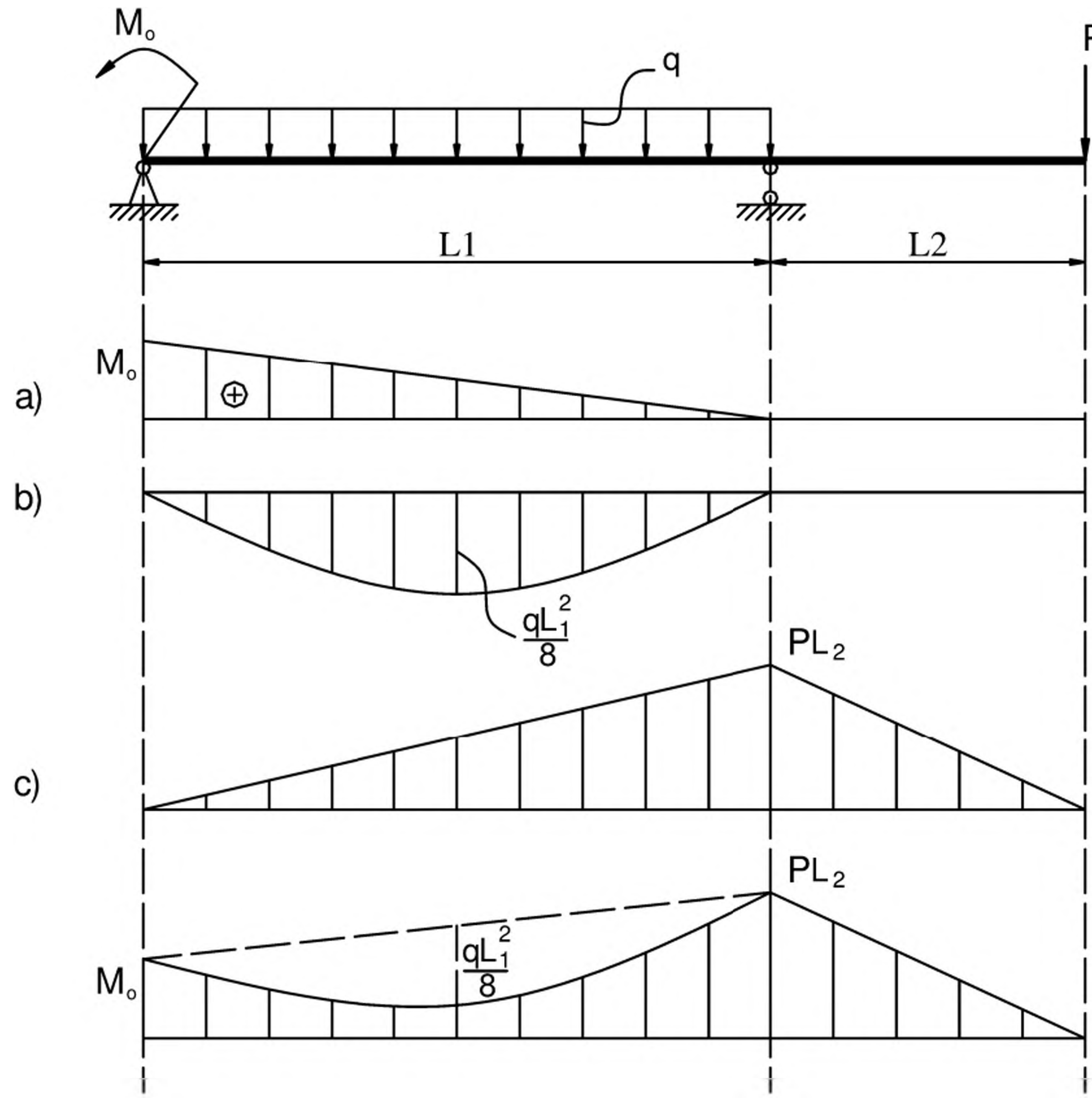


Figure 2.39

7.4. Exercise 12

Draw the diagram of bending moment for the beam shown in **Figure 2.40** by using special point method. Given: $M_o = qa^2$; $P = 2qa$; $L_1 = L_2 = L_3 = a$.

Solution

Determine the support reactions:

$$\sum M_B = 0 \Rightarrow -qa^2 + 2qa^2 + 2qa^2 - V_C 2a = 0 \Rightarrow V_C = \frac{3}{2}qa$$

$$\sum F_y = 0 \Rightarrow V_B = \frac{5}{2}qa$$

Calculate the internal forces in the beam:

Segment AB:

There is the only concentrated moment $\Rightarrow$ the shear force is 0.

The shear force = 0; therefore, the bending moment is a constant and equals to $-qa^2$.

Segment BD:

There is a constant distributed load q on this segment; therefore, the diagram of shear force is of first order and the diagram of bending moment is of second order. We need to calculate the internal forces at the ends of the segment.

At B:

$$\begin{cases} Q_B^{(BD)} = +\dfrac{5}{2}qa \\ M_B^{(BD)} = -M_o = -qa^2 \end{cases}$$

At D:

$$\begin{cases} Q_D^{(BD)} = \dfrac{5}{2}qa - qa = \dfrac{3}{2}qa \\ M_D^{(BD)} = \dfrac{5}{2}qa^2 - qa^2 - \dfrac{1}{2}qa^2 = qa^2 \end{cases}$$

Draw the shear force of this segment and we see that it does not intersect the horizontal axis. Therefore, the diagram of bending moment has no extreme. As a result, we can connect the two moments at B and D by a quadratic curve with the deflection of $qa^2/8$ so that it welcomes the distributed load q.

Segment DC:

Similarly, there is a distributed load on the segment DC; thus, the diagram of shear force is of first order and the diagram of bending is of second order.

At D:

$$\begin{cases} Q_D^{(DC)} = -\dfrac{3}{2}qa + \dfrac{1}{2}qa = -\dfrac{1}{2}qa \\ M_D^{(DC)} = \dfrac{3}{2}qa^2 - \dfrac{1}{2}qa^2 = qa^2 \end{cases}$$

At C:

$$\begin{cases} Q_C^{(DC)} = -V_C = -\dfrac{3}{2}qa \\ M_C^{(DC)} = 0 \end{cases}$$

The diagram does not intersect with the horizontal axis; thus, there is no extreme on the diagram of the bending moment.

Similar to the segment BD, we can draw the diagram of bending moment for the segment CD.

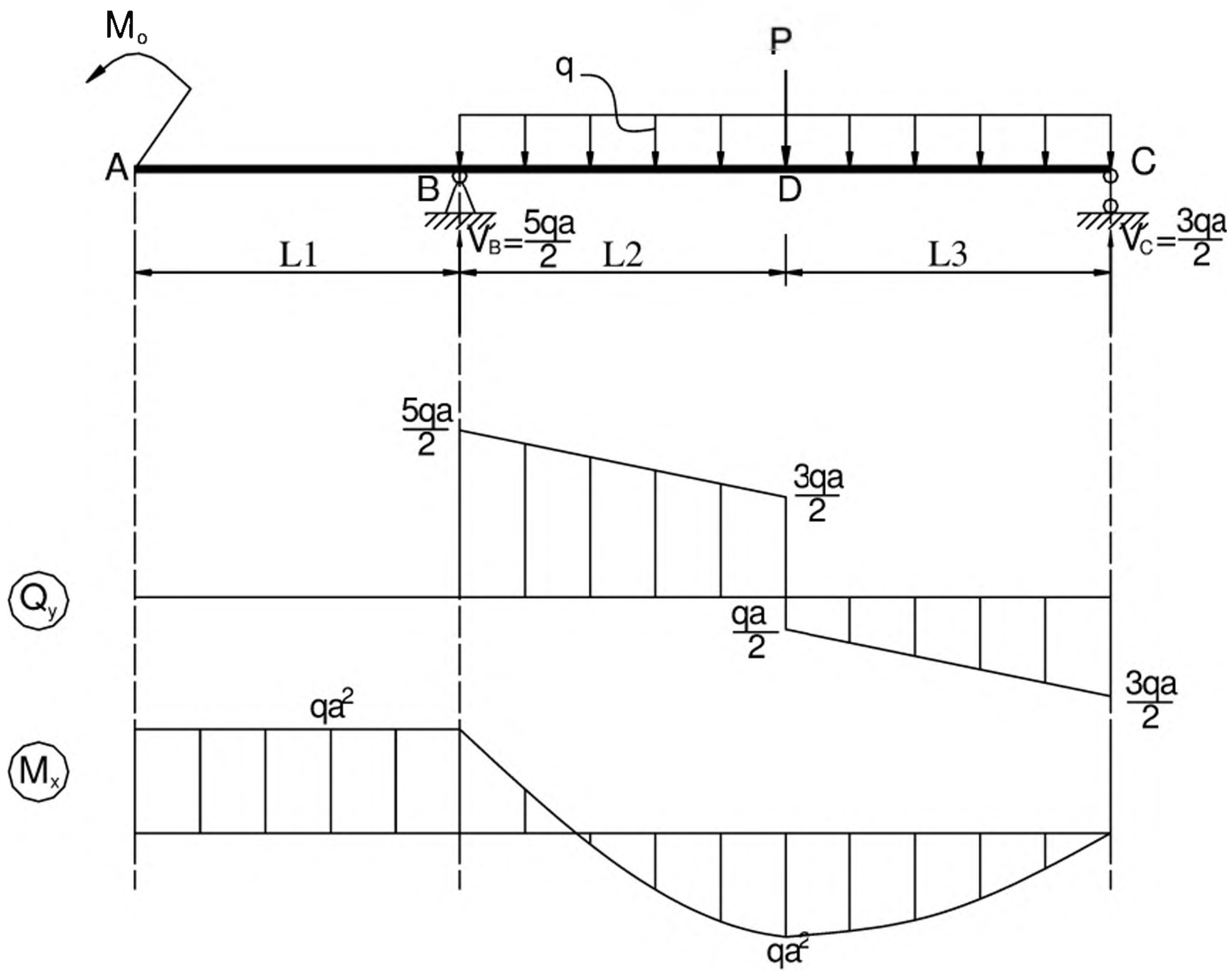


Figure 2.40

7.5. Exercise 13

Consider the beam shown in **Figure 2.41**.

a) Calculate the support reactions.

b) Draw the diagrams of shear force and bending moment.

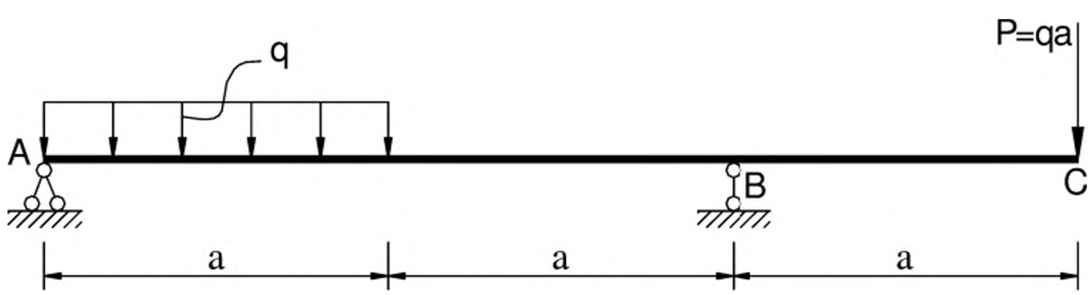


Figure 2.41

Solution

a) Calculate the support reactions:

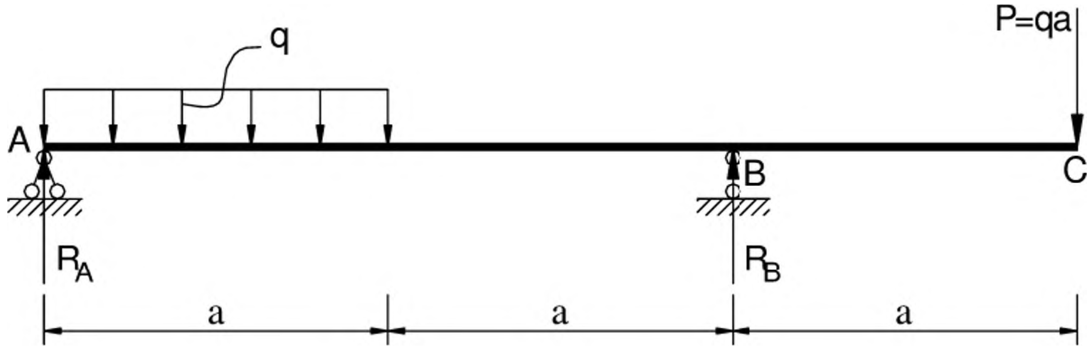


Figure 2.42

$$\sum M_{/B} = 0$$

$$\Rightarrow \quad R_A(2a) + qa(a) = qa\frac{3}{2}a$$

$$\Rightarrow \quad R_A = \frac{qa}{4}$$

$$\sum M_{/A} = 0$$

$$\Rightarrow \quad \frac{qa^2}{2} + qa(3a) + = R_B(2a)$$

$$\Rightarrow \quad R_B = \frac{7qa}{4}$$

b) Draw the diagrams of shear force and bending moment:

The calculation of the internal force is similar to previous exercises. The diagrams of bending moment and shear force are shown in **Figure 2.43**.

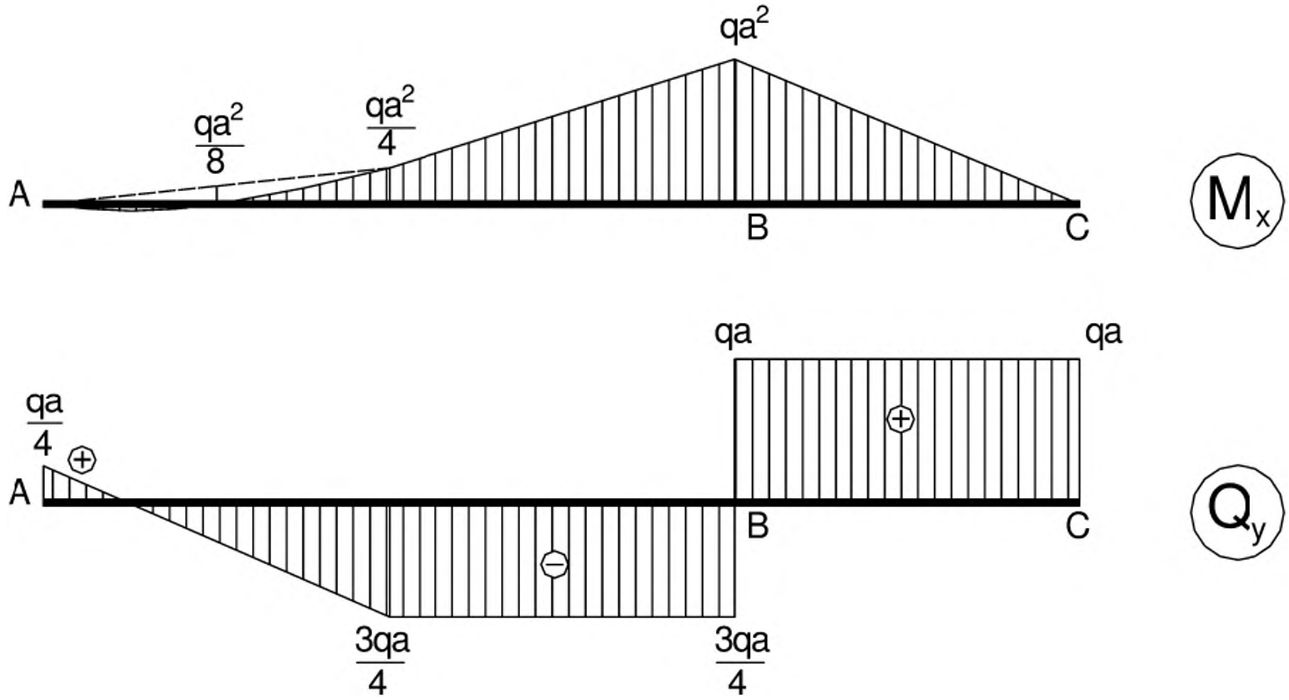


Figure 2.43

7.6. Exercise 14

Consider the structure with dimensions, coordinate system Oxyz and loads as shown in **Figure 2.44**. Draw the diagrams of N_z, M_x, M_y.

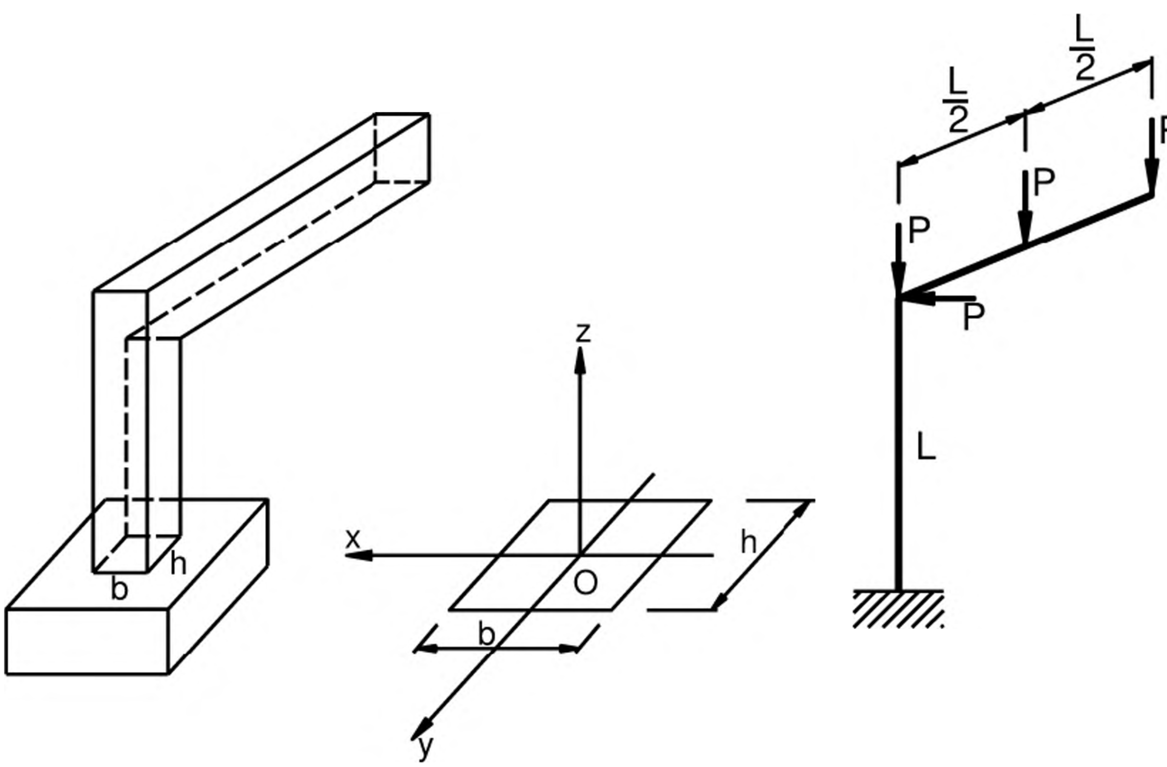


Figure 2.44

Solution

Draw the diagrams of N_z, M_x, M_y.

At the fixed end:

$$N_z = -3P$$

$$M_x = PL + \frac{PL}{2} = \frac{3PL}{2}$$

$$M_x = -PL$$

The diagrams are shown in **Figure 2.45**.

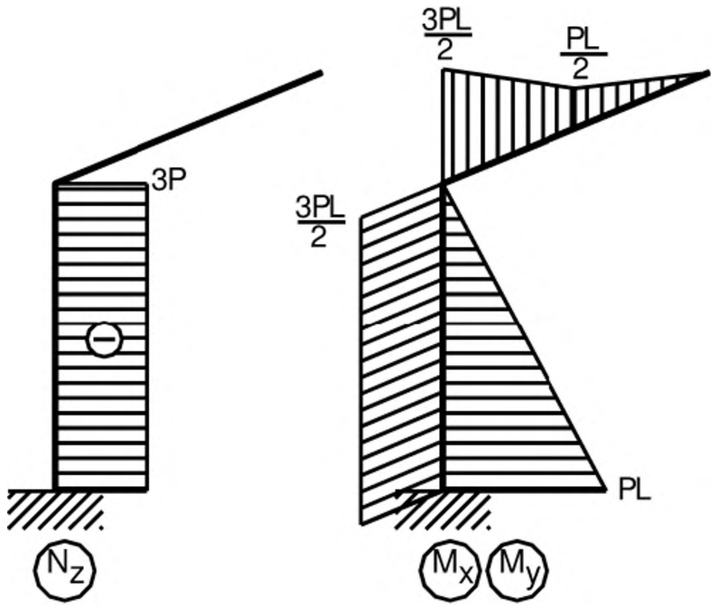


Figure 2.45

7.7. Exercise 15

Consider the structure ABC in the horizontal plane as shown in **Figure 2.46**. AB is perpendicular to BC. Draw the diagrams of bending moments, torsional moment and axial force.

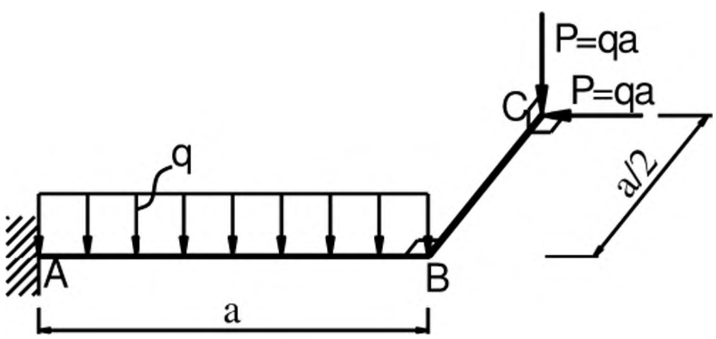
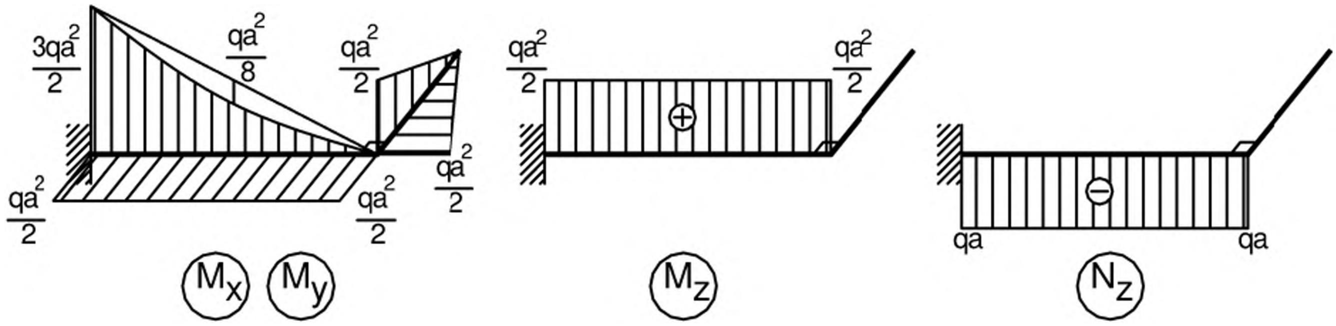


Figure 2.46

Answer



7.8. Exercise 16

Draw the diagrams of shear force and bending moment for the beam shown in **Figure 2.47**.

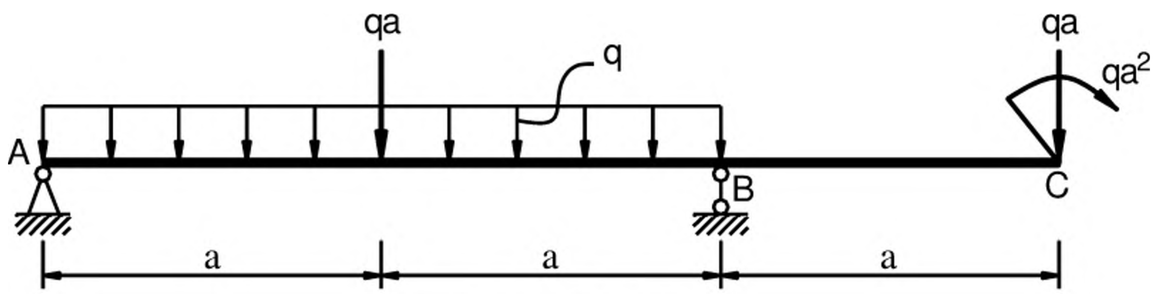


Figure 2.47

Solution

Determine the support reactions:

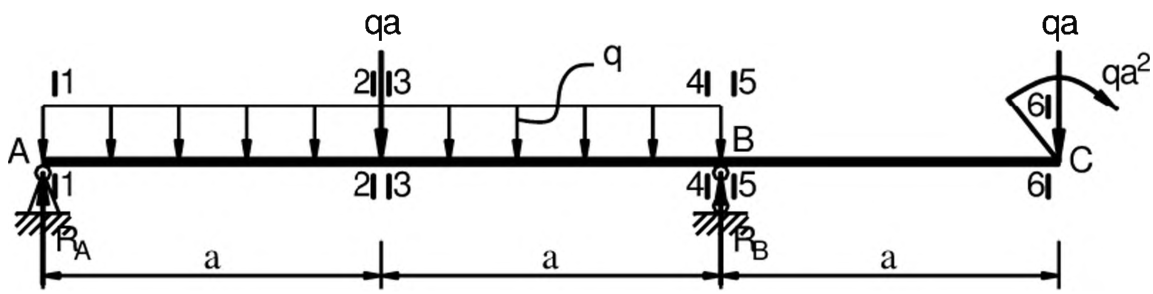


Figure 2.48

The total moment with respect to the point A:

$$\sum M_A = 0$$

$$qa.a + 2qa.a - R_B.2a + qa^2 + qa3a = 0$$

$$R_B = \frac{7}{2}qa$$

The total force with respect to the vertical axis:

$$\sum F_y = 0$$

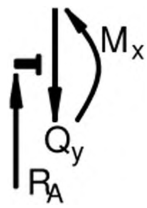
$$R_A + R_B - qa - 2qa - qa = 0$$

$$R_A = 4qa - R_B = \frac{1}{2}qa$$

Take the cut 1-1, consider the left part:

$$M_x = 0$$

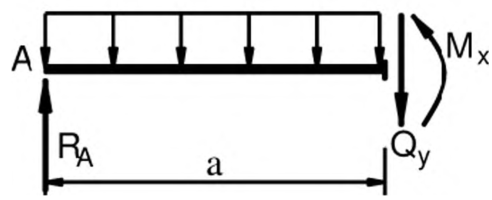
$$Q_y = R_A = \frac{qa}{2}$$



Take the cut 2-2, consider the left part:

$$M_x = R_A.a - qa\frac{a}{2} = 0$$

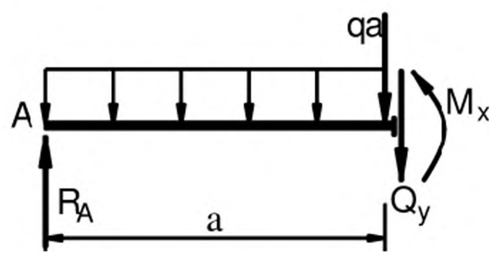
$$Q_y = R_A - qa = -\frac{qa}{2}$$



Take the cut 3-3, consider the left part:

$$M_x = R_A.a - qa\frac{a}{2} = 0$$

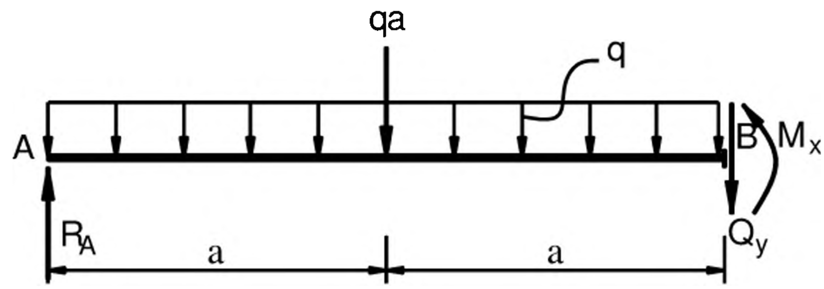
$$Q_y = R_A - qa - qa = -\frac{3qa}{2}$$



Take the cut 4-4, consider the left part:

$$M_x = R_A.2a - 2qa.a - qa.a = -2qa^2$$

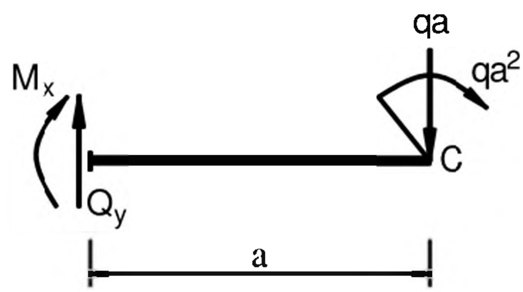
$$Q_y = R_A - 2qa - qa = -\frac{5qa}{2}$$



Take the cut 5-5, consider the right part:

$$M_x = -qa^2 - qa.a = -2qa^2$$

$$Q_y = qa$$



Take the cut 6-6, consider the right part:

$$M_x = -qa^2$$

$$Q_y = qa$$

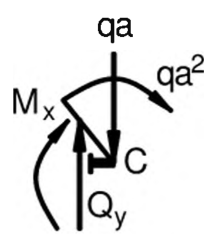


Diagram of bending moment

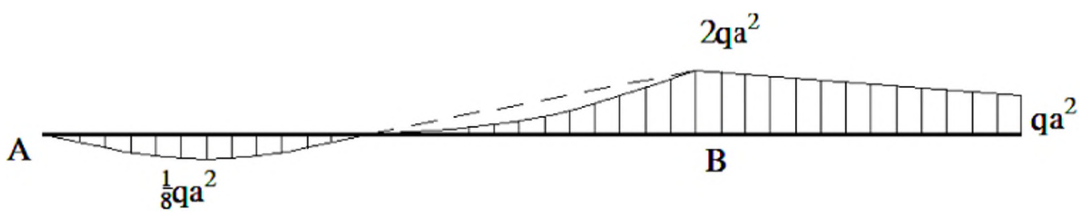
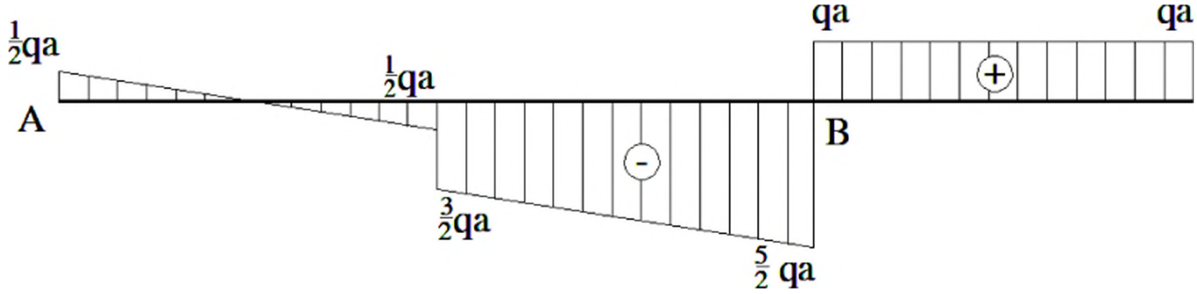


Diagram of shear force



Note: In the above solution, all the left parts of the cuts 1-1, 2-2, 3-3, 4-4 and the right parts of the cuts 5-5, 6-6 were considered. This is because there are less forces in the equilibrium equations. Readers can consider the other parts of the cuts to establish the equilibrium equations. It is worth noting that the same results were obtained.

7.9. Exercise 17

Consider the beam in **Figure 2.49**. Draw diagrams of M_x, Q_y.

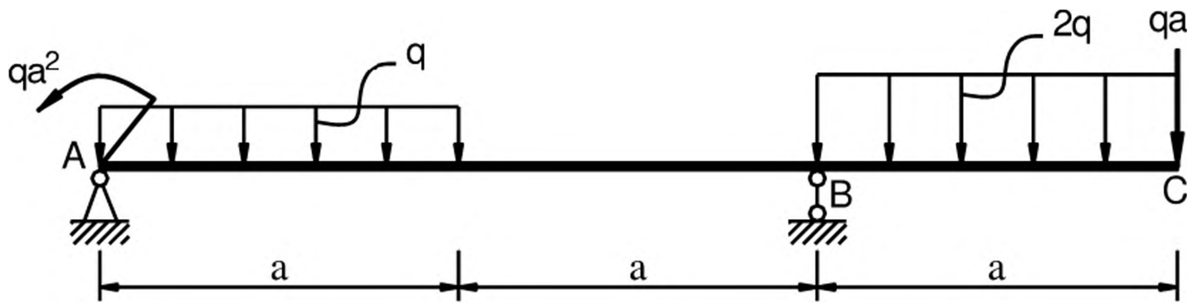


Figure 2.49

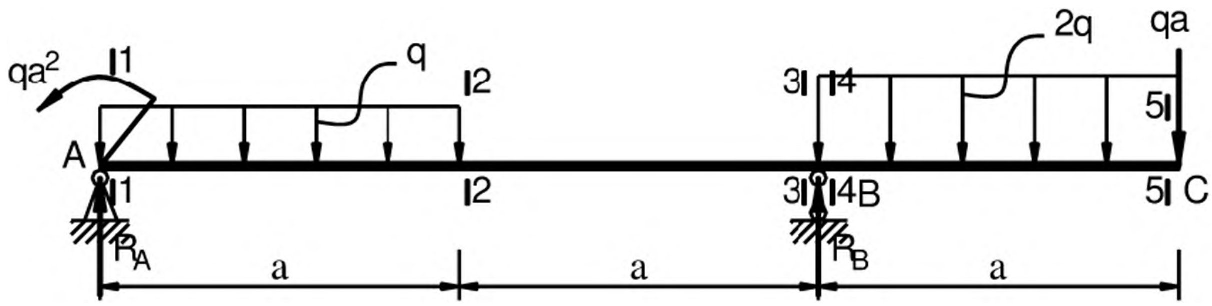


Figure 2.50

Total moment with respect to A:

$$\sum M_A = 0$$

$$qa.\frac{a}{2} - qa^2 - R_B.2a + 2qa.\frac{5a}{2} + qa.3a = 0$$

$$R_B = \frac{15}{4}qa$$

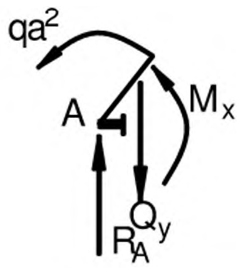
Total force with respect to the vertical axis:

$$\sum F_y = 0$$

$$R_A + R_B - qa - 2qa - qa = 0$$

$$R_A = 4qa - R_B = \frac{1}{4} qa$$

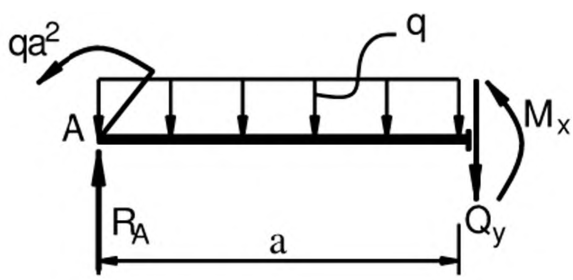
Take the cut 1-1, consider the left part:



$$M_x = -qa^2$$

$$Q_y = R_A = \frac{qa}{4}$$

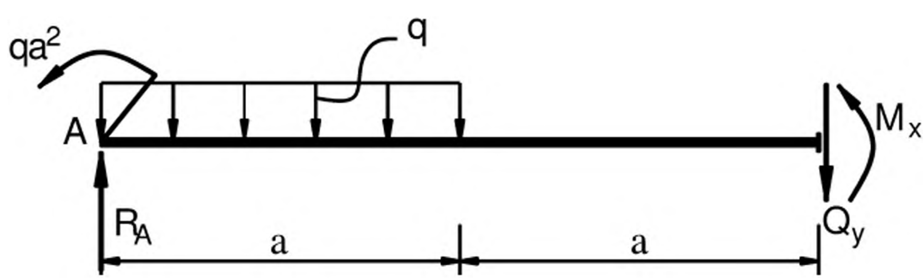
Take the cut 2-2, consider the left part:



$$M_x = R_A.a - qa.\frac{a}{2} - qa^2 = -\frac{5}{4} qa^2$$

$$Q_y = R_A - qa = -\frac{3}{4} qa$$

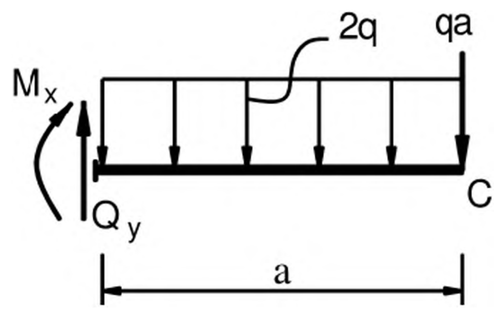
Take the cut 3-3, consider the left part:



$$M_x = R_A.2a - qa.\frac{3a}{2} - qa^2 = -2qa^2$$

$$Q_y = R_A - qa = -\frac{3}{4}qa$$

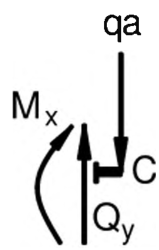
Take the cut 4-4, consider the right part:



$$M_x = -qa.a - 2qa.\frac{a}{2} = -2qa^2$$

$$Q_y = 2q.a + qa = 3qa$$

Take the cut 5-5, consider the right part:



$$M_x = 0$$

$$Q_y = qa$$

Diagram of bending moment M_x

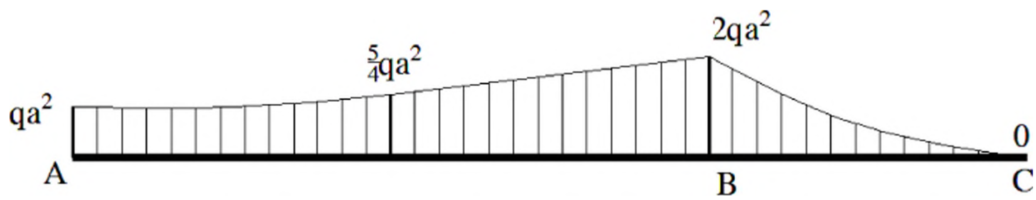


Figure 2.51

Diagram of shear force Q_y

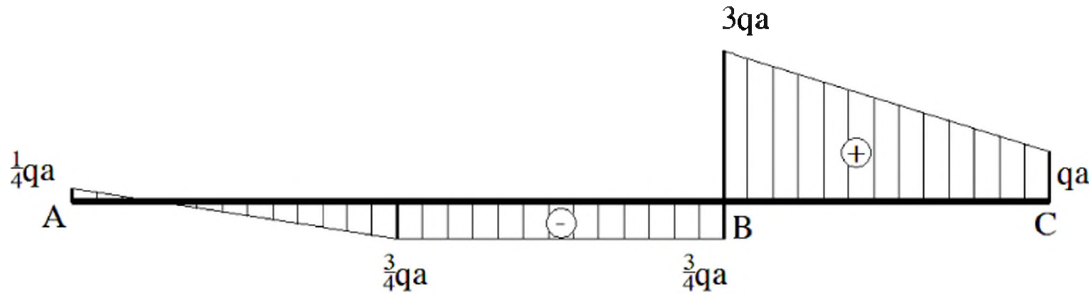
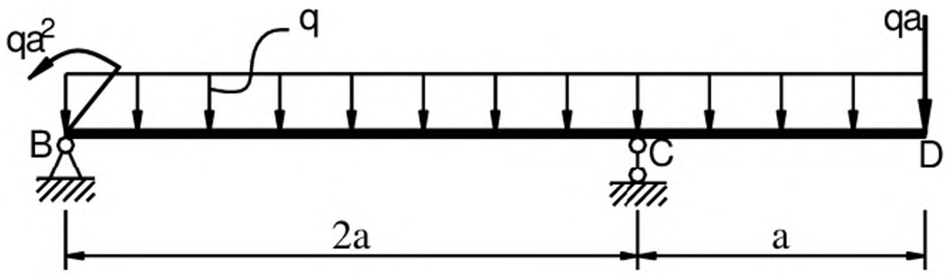


Figure 2.52

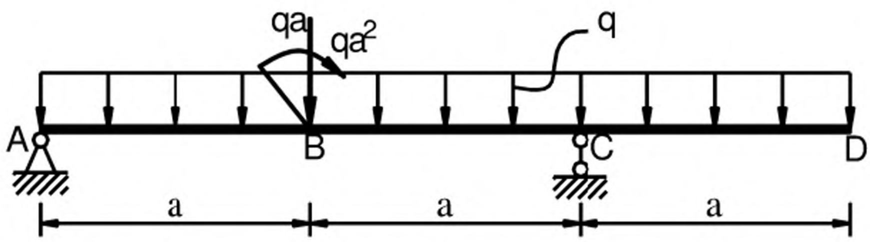
PROBLEMS

Draw the diagrams of bending moment and shear force for the following structures:

1



2



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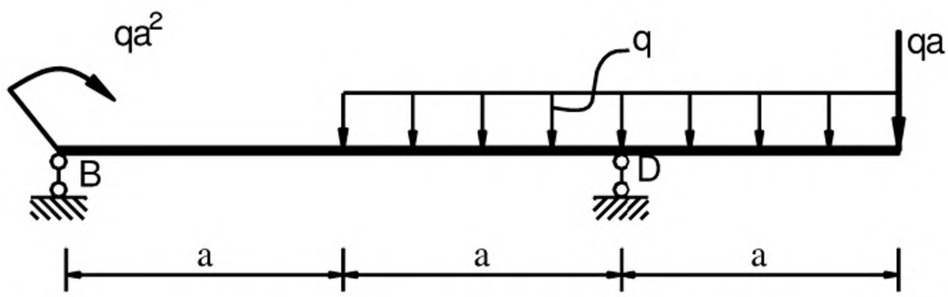

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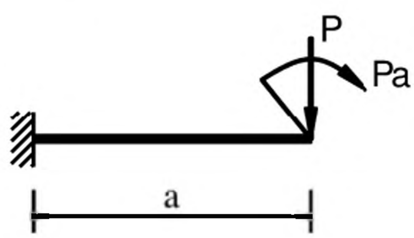
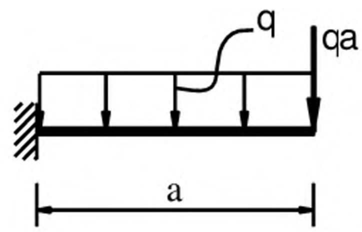
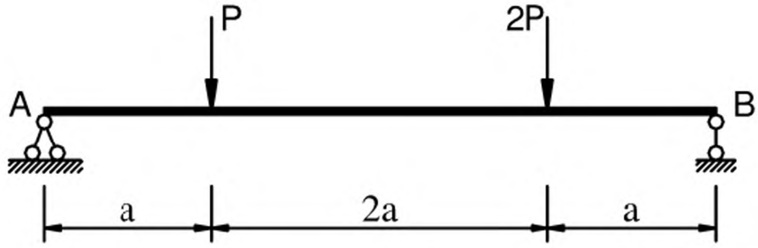


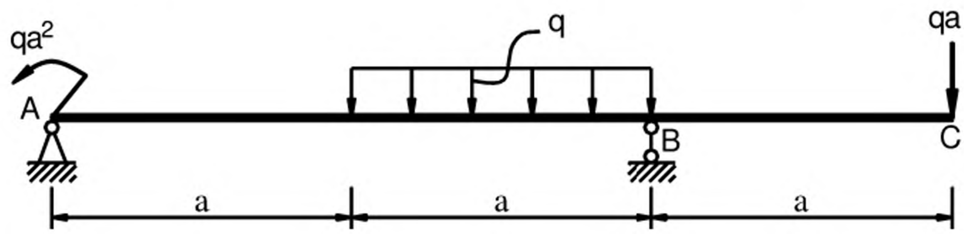
5


q
qa
a

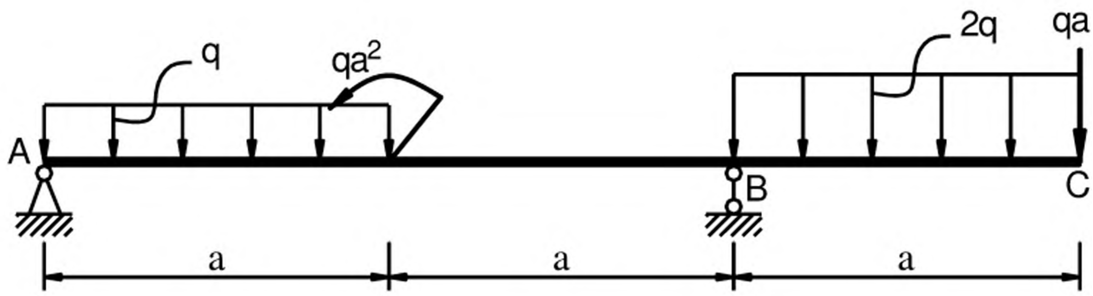
6


P
2P
A
B
a
2a
a

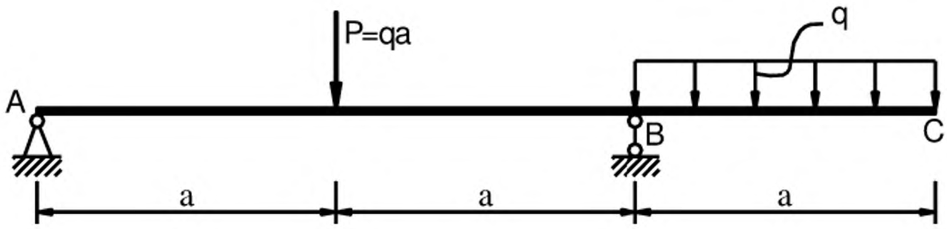
7


qa2
q
qa
A
B
C
a
a
a

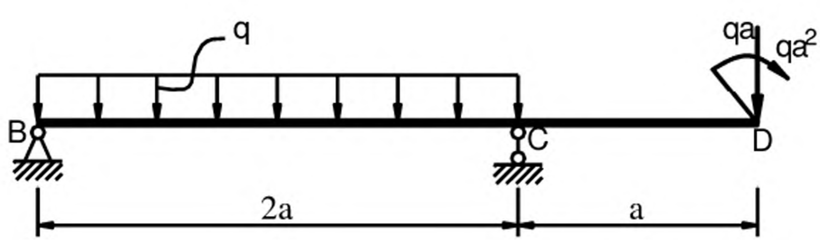
8


q
qa2
2q
qa
A
B
C
a
a
a

9


P=qa
q
A
B
C
a
a
a

10


q
qa
qa2
B
C
D
2a
a

Chapter **3**

AXIAL LOADING

§1. CONCEPTS

Figure 3.1a shows the member subjected to tension force while **Figure 3.1b** shows the member subjected to compression force. The internal force of an axially loaded member is only the axial internal force N_z.

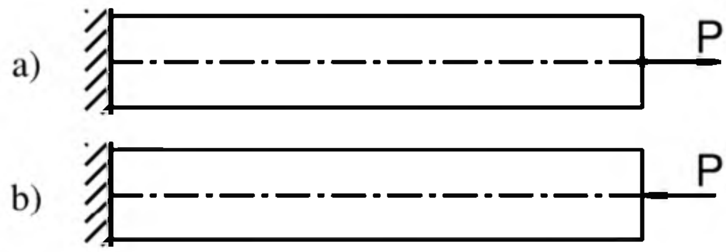


Figure 3.1. Members under axial loading

In practice, there are many structural members subjected to axial loading. Some examples are mentioned herein. **Figure 3.2** shows the My Thuan cable-stayed bridge, in which the cables are under tension. **Figure 3.3** shows the bar members under tension to support the overhang structures at Ho Chi Minh City University of Technology. In **Figure 3.4**, the columns are under compression to support the seft-weight of concrete structures. Under its selfweight, the light pole in **Figure 3.5** has axial compression force.

Figure 3.2. My Thuan cable-stayed bridge *(source: https://commons.wikimedia.org/wiki/File:C%E1%BA%A7u_M%E1%BB%B9_Thu%E1%BA%ADn_(6297673871).jpg)*

a) The front of building H2

b) Faculty of computer science and engineering

Figure 3.3. Some structures at Ho Chi Minh City University of Technology

Figure 3.4. Columns under compression

Figure 3.5. Light pole

It should be noted that, in this chapter, formulas are applied only to straight bars. Even for straight bars with partially weakened parts with grooved holes, etc., the formulas are not applied to those parts.

§2. STRESS

To establish a formula to compute the stress on the cross section, the following experiment shown in **Figure 3.6** can be considered. On the surface of the bar, we draw several lines parallel and perpendicular to the longitudinal axis. These lines create several rectangles. After the axial load is applied, the rectangles deform; however, the right angles remain unchanged.

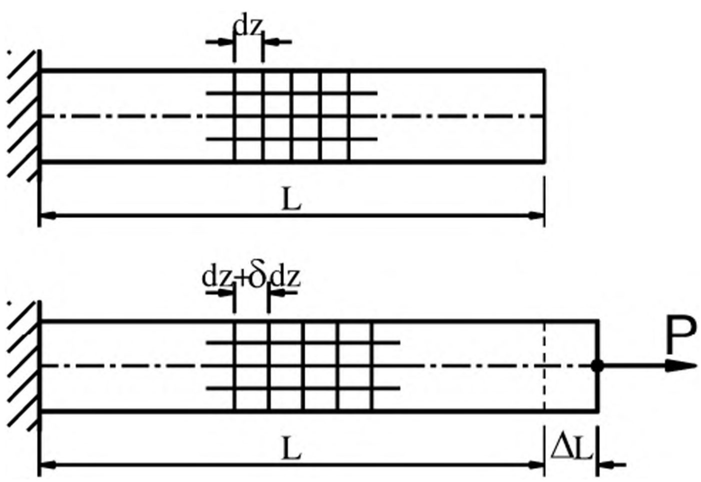


Figure 3.6. Deformation of a member under axial loading

During loading, the following aspects can be observed:

- The sections are still a plane and perpendicular to the bar axis.
- The bar only has longitudinal deformation.

Based on the obove observations, it can be concluded that there is only normal stress σ_z on the cross section. This normal stress does not change across the section.

In chapter 2, we have the following relationship:

$$N_z = \int_A dN_z = \int_A \sigma_z dA \tag{3.1}$$

Because σ_z is a constant, the above relationship becomes:

$$N_z = \sigma_z A \tag{3.2}$$

or

$$\sigma_z = \frac{N_z}{A} \tag{3.3}$$

in which, N_z is the axial force and A is the cross-sectional area.

§3. DEFORMATION OF AXIALLY LOADED BARS

3.1. Longitudinal deformation (axial deformation)

Figure 3.7 shows a bar subjected to an axial load P. Before loading, the length of the bar is L and the continuous lines represent the shape of the bar. After loading, the length of the bar is L + ΔL and the shape is represented by the dashed lines. The elongation of the bar is ΔL.

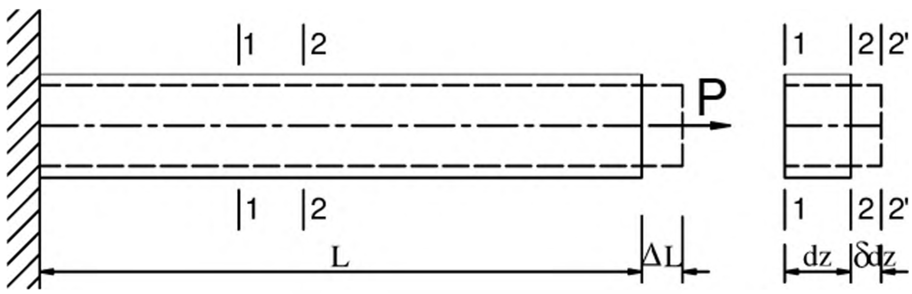


Figure 3.7. Deformation of an axially loaded bar

Consider the bar segment between sections 1-1 and 2-2. Its length of this segment is dz. After loading, the length of this segment is dz + δdz. The z-axis elongation of dz is δdz. So, relative deformation (longitudinal or axial strain) is:

$$\varepsilon_z = \frac{\delta dz}{dz} \tag{3.4}$$

According to Hooke's law, we have:

$$\varepsilon_z = \frac{\sigma_z}{E} \tag{3.5}$$

in which, E is the elastic modulus (or modulus of elasticity). It depends on the material and its unit is [force/area].

From the above two expressions, we have:

$$\frac{\delta dz}{dz} = \frac{\sigma_z}{E} = \frac{N_z}{EA} \tag{3.6}$$

$$\delta dz = \frac{N_z}{EA} dz \tag{3.7}$$

The total longitudinal deformation of element with its length L is:

$$\Delta L = \int_L \delta dz = \int_L \frac{N_z}{EA} dz \tag{3.8}$$

In case E = const, A = const, N_z = constant along the element with the length L, we have:

$$\Delta L = \frac{N_z}{EA} \int_L dz = \frac{N_z L}{EA} \tag{3.9}$$

If the bar consists of several segments of length L_i, N_{zi}, E_i, and A_i:

$$\Delta L = \sum \Delta L_i = \sum \frac{N_{zi} L_i}{E_i A_i} \tag{3.10}$$

EA is called the stiffness of the bar when pulled or compressed.

$\frac{EA}{L}$ is called relative stiffness when pulled or compressed.

3.2. Transverse deformation (lateral deformation)

Reconsider the **Figure 3.7**, the bar becomes longer while its cross section becomes smaller. Not only longitudinal deformation (axial strain ε_z) but also transverse deformations (lateral strain εx and εy) occur in bar subjected to axial loading. The relationship between these deformations are expressed in Equation 3.11.

$$\varepsilon_x = \varepsilon_y = -\nu \varepsilon_z \tag{3.11}$$

in which, ν is Poisson's ratio.

$$\nu = \left| \frac{lateral\ strain}{axial\ strain} \right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x} \tag{3.12}$$

The sign – shows the transverse and longitudinal deformation is opposite.

Table 3.1 shows the common values of elastic modulus and the Poision's ratio of some materials.

Table 3.1 Elastic modulus E of some materials

Material	E		ν
	10^{10} (N/m^2)	10^5 (MPa)	
Steel (0.15-0.2) %C	20	2	0.25-0.33
Spring steel	22	2.2	0.25-0.33
Niken Steel	19	1.9	0.25-0.33
Gray cast iron	11.5	1.15	0.23-0.27
Cu	12	1.2	0.31-0.34
Aluminum	7-8	0.7-0.8	0.32-0.36
Wood (longitudinal)	0.8-1.2	0.08-0.12	
Rubber	8	0.8	0.47

3.3. Exercise 1

Consider the bar ABC subjected to loads as shown in **Figure 3.8**. Given P = 50 kN, A_1 = 500 mm^2, A_2 = 1000 mm^2, L_1 = 2000 mm, L_2 = 1200 mm, E = 2.10^5 MPa.

a) Draw the diagram of axial force.

b) Calculate the stress in each segment.

c) Calculate the displacement of the free end A.

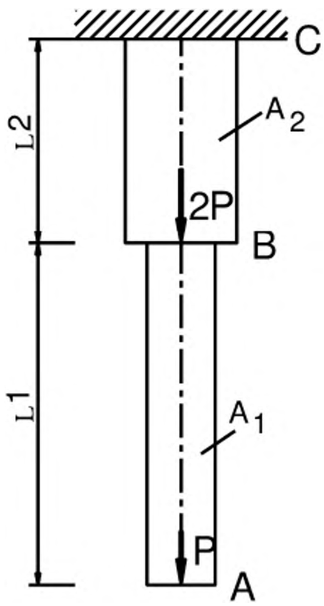


Figure 3.8

Solution

a) Draw the diagram of axial force.

Take the cuts across the segments AB and BC, consider the lower parts, we have the axial forces in these segments:

$$N_{AB} = P;\ N_{BC} = 3P$$

The diagram of axial force is as follows:

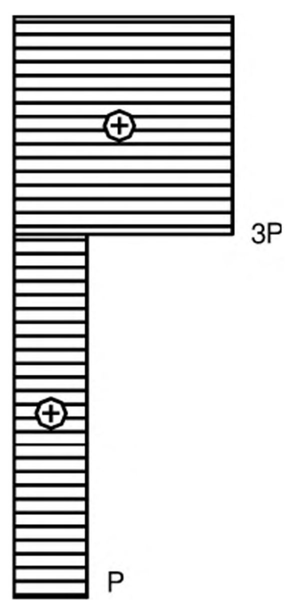


Figure 3.9

a) Calculate the stress in each segment.

$$\sigma_z^{AB} = \frac{N_{AB}}{A_{AB}} = \frac{50 \times 10^3}{500} = 100 \;\; MPa$$

$$\sigma_z^{BC} = \frac{N_{BC}}{A_{BC}} = \frac{3 \times 50 \times 10^3}{1000} = 150 \;\; MPa$$

b) Calculate the displacement of the free end A:

$$\Delta L = \frac{N_{AB} N_{AB}}{EA_{AB}} + \frac{N_{BC} N_{BC}}{EA_{BC}} \mathrm{A}$$

$$= \frac{\left(50 \times 10^3\right) \times 2000}{2 \times 10^5 \times 500} + \frac{\left(3 \times 50 \times 10^3\right) \times 1200}{2 \times 10^5 \times 1000} = 1.9 \;\; mm$$

3.4. Exercise 2

Consider the system shown in **Figure 3.10**. The bars AB and AC have the same length L, sectional area A and modulus of elasticity E. Given E = 2.10^5 MPa, L = 1000 mm; P = 300 kN; $\alpha = 30^o$; A = 1200 mm^2.

a) Determine the axial force in each element AB and AC.

b) Calculate the vertical displacement of the joint A.

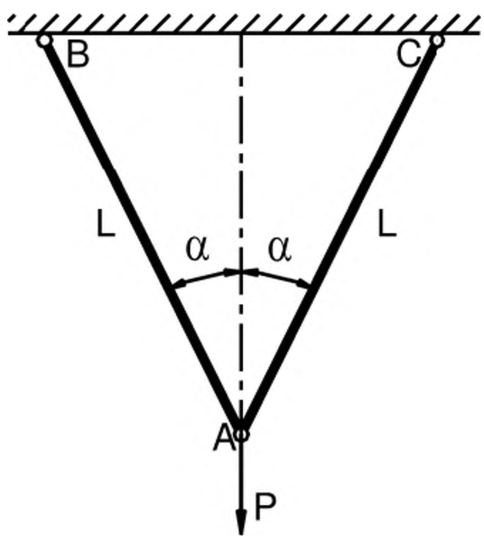


Figure 3.10

Solution

a) Determine the axial force in each element AB and AC.

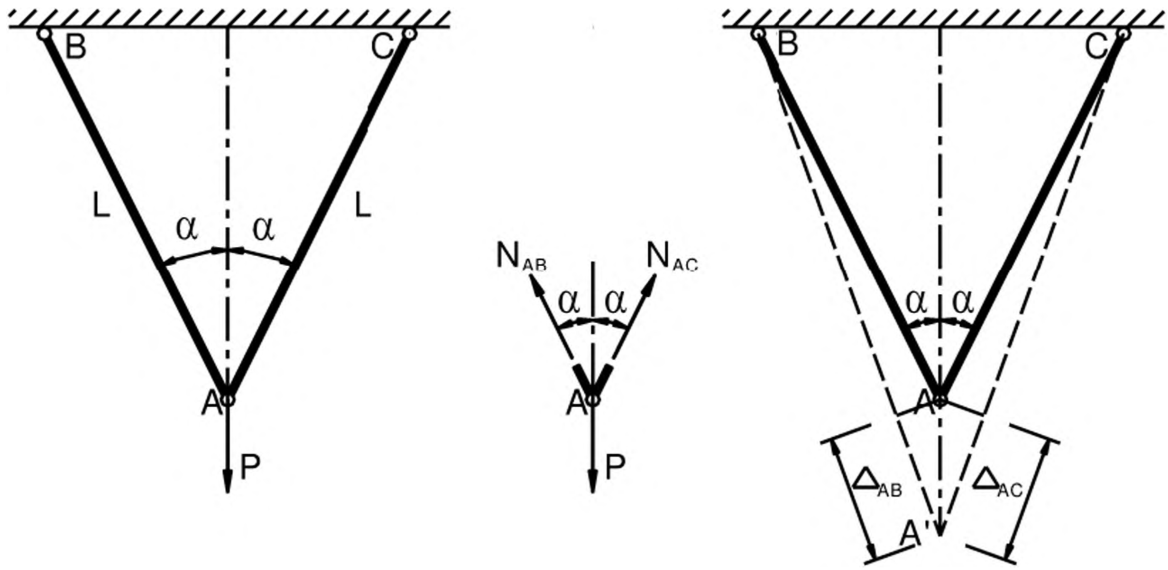


Figure 3.11

Cut the joint A, establish two equilibrium equations of force:

$$\sum F_X = 0 \Rightarrow N_{AB} = N_{AC} = N$$

$$\sum F_Y = 0 \Rightarrow 2N\cos\alpha = P$$

$$N = \frac{P}{2\cos\alpha} = \frac{300}{2\cos 30^o} = 173.2kN$$

b) Calculate the vertical displacement of the joint A.

Geometry method:

$$AA' = \frac{\Delta_{AC}}{\cos\alpha} = \frac{\frac{NL_{AC}}{EA}}{\cos\alpha} = \frac{PL}{2EA\cos^2\alpha}$$

Method using elastic strain energy:

W = U

Work of load P:

$$W = \frac{1}{2}P.AA'$$

Elastic strain energy of the system:

$$U = \frac{N^2L_{AB}}{2EA} + \frac{N^2L_{AC}}{2EA} = \frac{N^2L_{AC}}{EA}$$

$$\frac{1}{2}P.AA' = \frac{N^2 L_{AC}}{EA}$$

$$AA' = \frac{2\left(\dfrac{P}{2\cos\alpha}\right)^2 L_{AC}}{EAP} = \frac{PL_{AC}}{2EA\cos^2\alpha}$$

Substitute the values of parameters, we have the displacement.

$$AA' = \frac{300\times10^3\times1000}{2\times2\times10^5\times1200\times\cos^2 30^o} = 0.833 \text{ mm}$$

3.5. Exercise 3

Consider the bar shown in **Figure 3.12**. The segments were made from the same material.

a) Draw the diagram of axial force.

b) Calculate the stress in each segment.

c) Calculate the displacement at the free end.

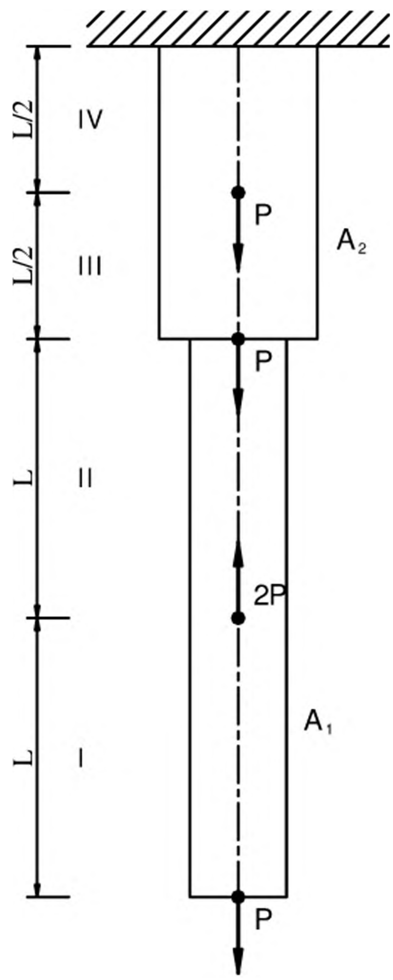
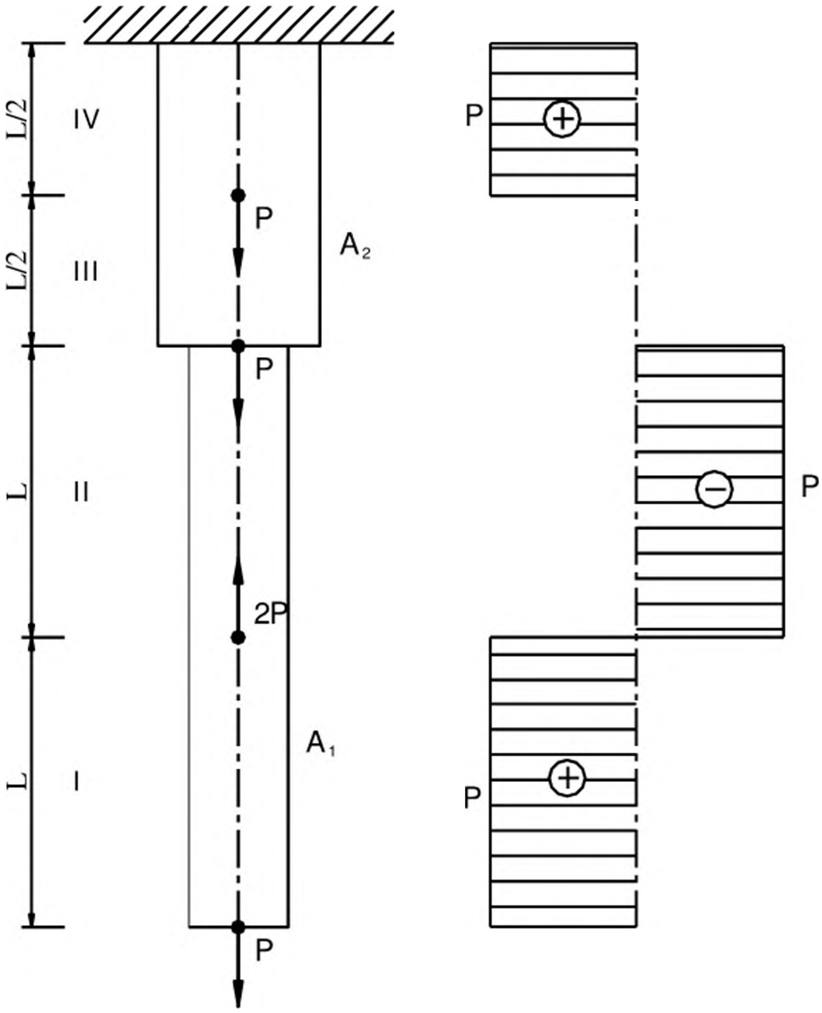


Figure 3.12

Solution

a) Draw the diagram of axial force.



b) Stress in each segment.

$$\sigma_I = \frac{N_z^I}{A_1} = \frac{P}{A_1}$$

$$\sigma_{II} = \frac{N_z^{II}}{A_1} = \frac{-P}{A_1}$$

$$\sigma_{III} = \frac{N_z^{III}}{A_2} = \frac{0}{A_2} = 0$$

$$\sigma_{IV} = \frac{N_z^{IV}}{A_2} = \frac{P}{A_2}$$

c) The displacement of the free end.

$$\Delta L = \sum \Delta L_i = \sum_{i=1}^{4} \frac{N_{zi} L_i}{E_i A_i} = \frac{PL}{EA_1} + \frac{-PL}{EA_1} + \frac{0L}{EA_2} + \frac{PL}{EA_2} = \frac{PL}{EA_2}$$

§4. MECHANICAL PROPERTIES OF MATERIALS

4.1. Concepts

Mechanical properties of materials can be determined using tension and compression experiment. Based on the mechanical properties, materials can be classified into two types: ductile and brittle materials.

- Ductile materials fail when its deformation is large, e.g. steel, aluminium, etc.
- Brittle materials fail when its deformation is small, e.g. concrete, cast iron, stone, etc.

There are four basic experiments.

4.2. Tension experiment of ductile materials (steel)

4.2.1. Test sample

Figure 3.13 shows a test sample for tension test. The length is L_o, the diameter is d_o, the cross-section area is A_o.

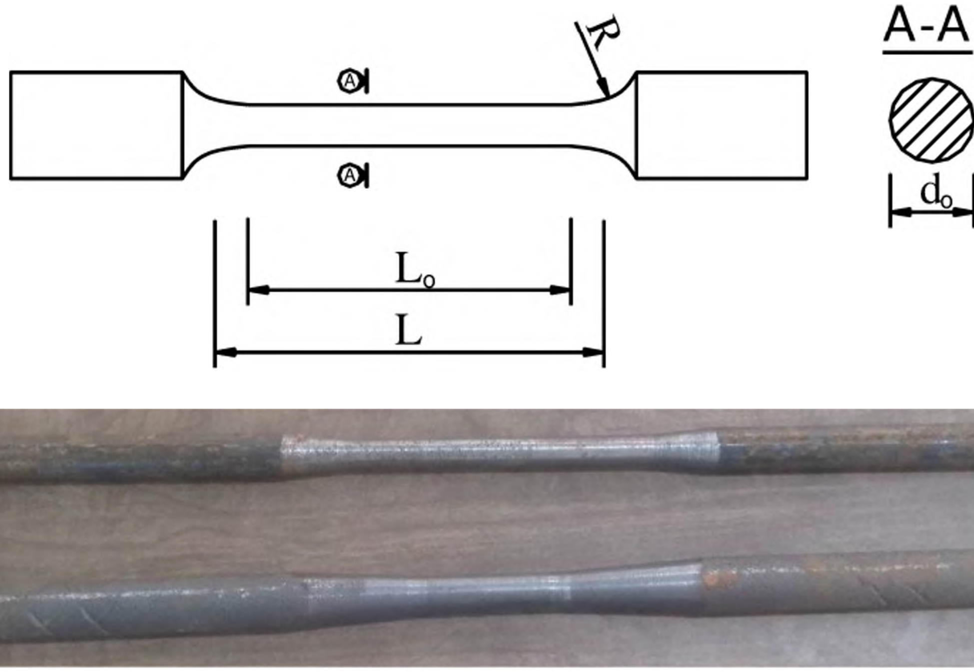


Figure 3.13. Test specimen

4.2.2. Experiment

Place the specimens on the testing machine as shown in **Figure 3.14**. Increase the tension force P from 0 until the sample fails. Relationship between longitudinal elongation and the tension force P is shown in **Figure 3.15**. The failure modes of the specimens are shown in **Figure 3.16**.

Figure 3.14. Steel specimen on testing machine

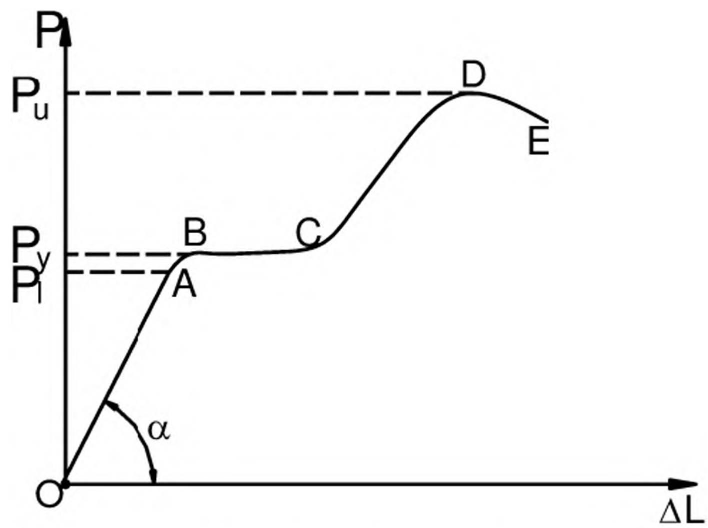


Figure 3.15. Force-deformation curve

Figure 3.16. Failure of specimen

4.2.3. Analysis

The graph shown in **Figure 3.15** can be divided into different stages as follows:

OA is elastic stage. In this stage, the following characteristics are observed:

- Relation between P and ΔL is linear.

- The largest force of this stage is the force P_l,
- The stress at the force P_l is called linear stress limit:

$$\sigma_l = \frac{P_l}{A_o} \tag{3.13}$$

AB is the transition stage.

BC is the yield stage:

- Force does not increase but the longitudinal elongation continues to increase.
- The corresponding force is called P_y.
- Yield stress:

$$\sigma_y = \frac{P_y}{A_o} \tag{3.14}$$

CD is the strain-hardening stage:

- The relationship between the force P and the longitudinal elongation ΔL is the curve.
- The maximum force is P_u.
- Ultimate stress:

$$\sigma_u = \frac{P_u}{A_o} \tag{3.15}$$

DE is the necking stage:

If the length of specimen at failure is L_1, the cross-sectional area of specimen at failure is A_1, the plasticity characteristics are defined as follows:

- Relative elongation (%)

$$\delta = \frac{L_1 - L_o}{L_o} 100\% \tag{3.16}$$

- Necking ratio (%)

$$\psi = \frac{A_o - A_1}{A_o} 100\% \tag{3.17}$$

4.2.4. Stress-strain diagram

From the P-ΔL curve shown in **Figure 3.15**, it is easy to obtain the stress-strain curve by dividing the vertical axis by A_o and dividing the horizontal axis by L_o. The vertical axis becomes the stress $\sigma_z = \frac{P}{A_o}$ while the horizontal axis becomes the strain $\varepsilon_z = \frac{\Delta L}{L_o}$. The stress-strain curve is shown in **Figure 3.17.**

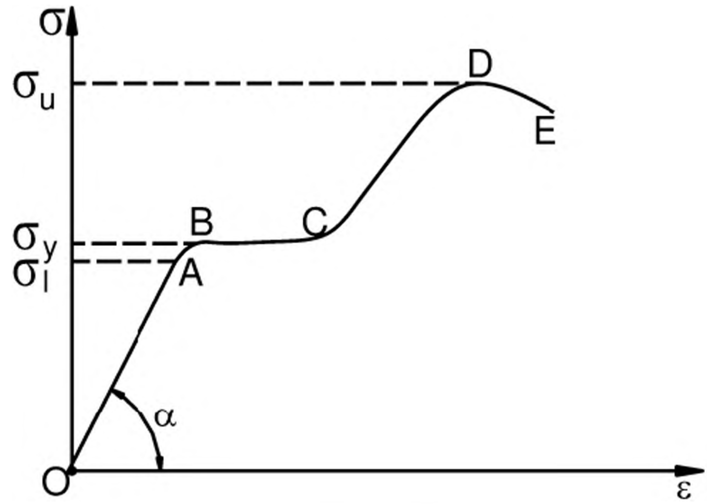


Figure 3.17. Stress-strain curve

The shape of the σ-ε curve shown in **Figure 3.17** is similar to P-ΔL curve shown **Figure 3.15**.

Modulus of elasticity (or elastic modulus) is calculated by Equation 3.18, in which, α is the angle between the horizontal axis and OA branch.

$$E = \frac{\sigma}{\varepsilon} = \tan\alpha \tag{3.18}$$

Considering the reduction of the cross-sectional area, we will have the relationship between the real stress and strain as shown by the dashed line in **Figure 3.18**.

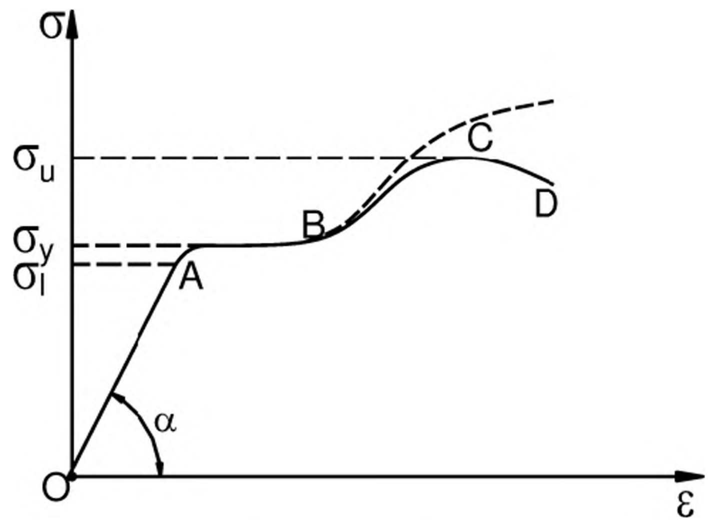


Figure 3.18. Stress-strain curves with and without considering the effect of necking

4.3. Tension experiment of brittle materials

Install the cast-iron specimen on the testing machine as shown in **Figure 3.20**. Gradually increase the load. The force-deformation curve is obtained. **Figure 3.19** shows this curve. The curve can be approximated by a straight line. There are no proportional limit and yield limit but only ultimate force P_u. The ultimate stress is calculated by Equation 3.19.

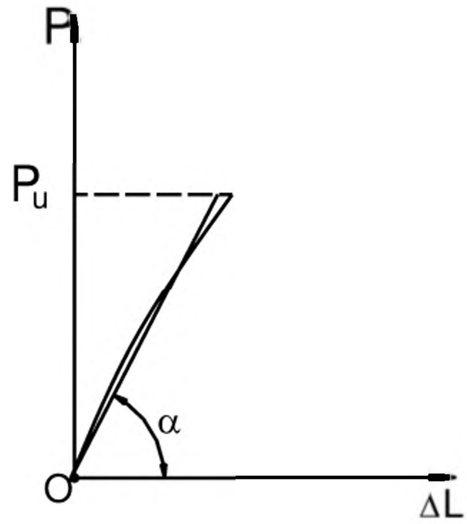


Figure 3.19. Force-deformation curve of brittle materials

Figure 3.20. Cast-iron specimen on testing machine

$$\sigma_{ult} = \frac{P_{ult}}{A_o} \tag{3.19}$$

4.4. Compression experiment of ductile materials

Samples for compression tests are usually cylindrical or cubic. The left specimen in **Figure 3.22** is a steel specimen for compression test. **Figure 3.21** shows the force-deformation curve P-ΔL which consists of the elastic regions OA, the transition region AB, the yield region BC and the strain-hardening region CD. The sample under compression will be bulged and the cross-sectional area increases. Thus, the proportional limit and yield limit can be determined; however, the ultimate limit can not be determined. The sample under compression has a drum shape. **Figure 3.22** shows the original and failed specimens.

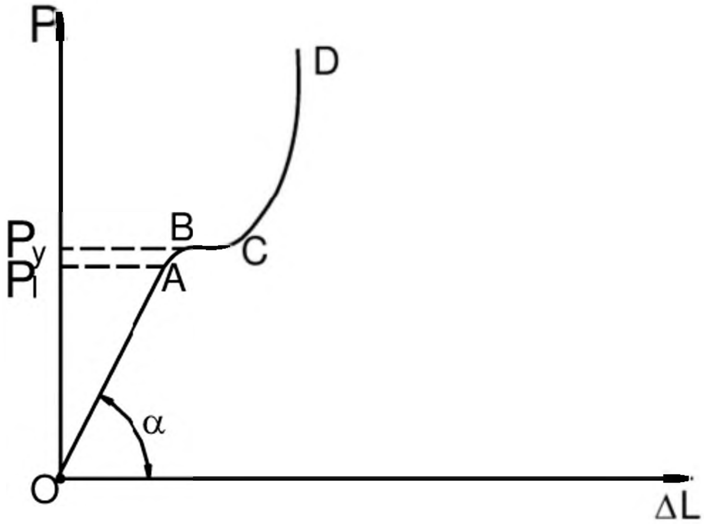


Figure 3.21. Force-deformation curve of a specimen under compression

Figure 3.22. Failure mode of steel specimens under compression

4.5. Compression experiment of brittle materials

The P - ΔL curve when compressing the brittle material is similar to it in tension test. We only determine the ultimate limit. The sample fails suddenly and the failure modes are shown in **Figure 3.23**.

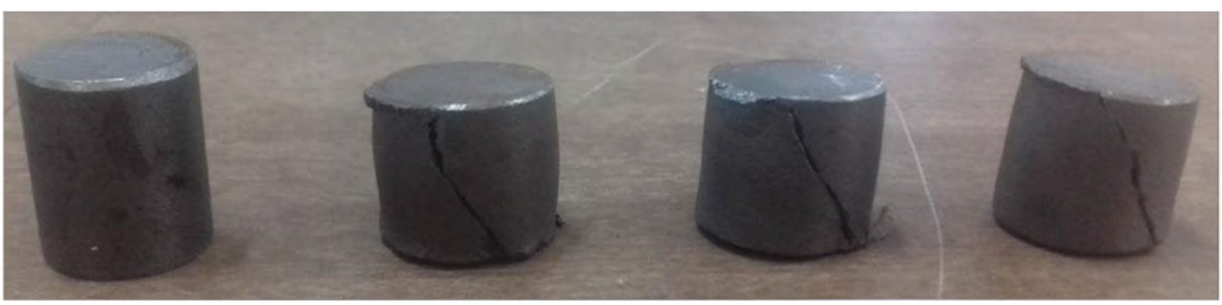


Figure 3.23. Failure mode of cast-iron specimens under compression

§5. ELASTIC STRAIN ENERGY

5.1. Concepts

Consider an axially loaded bar that works in its elastic phase as shown in **Figure 3.24a**. Increasing the force from 0 to P, axial elongation increases from 0 to ΔL. If the force is released, deformation will disappear, and the bar will recover to its original state. The recovery process occurs due to the work of the force P, which is called ***strain energy***.

5.2. Elastic strain energy

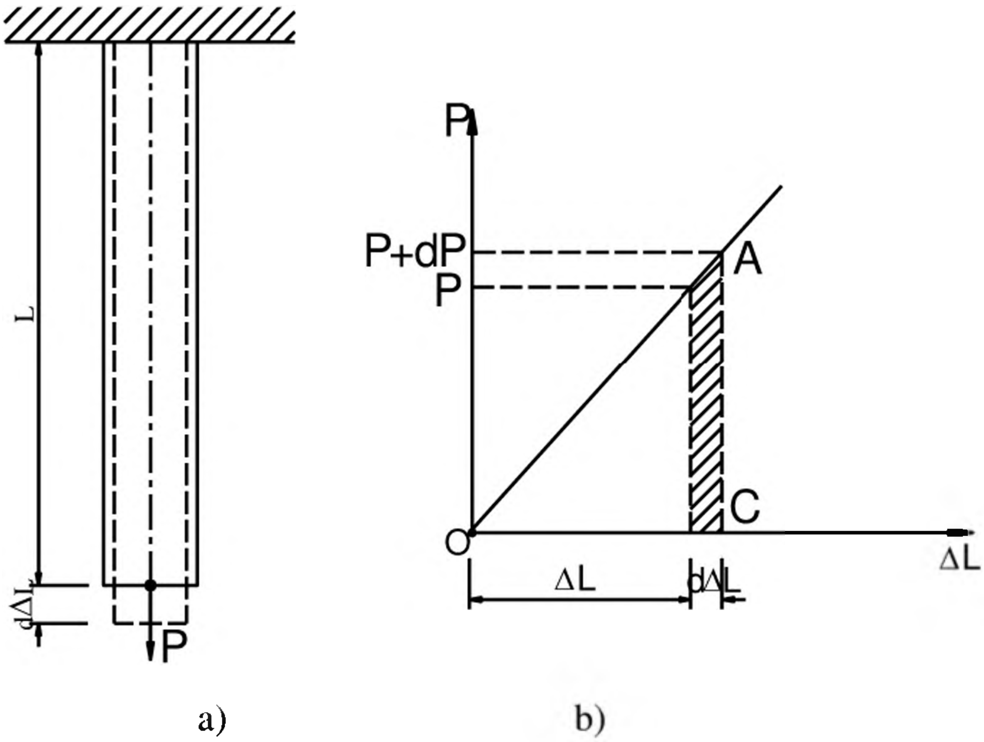


Figure 3.24. Strain energy

Relationship between pulling force P and deformation ΔL is shown in **Figure 3.24b**. We calculate the work of the force P with the strain ΔL.

Let P increase a small amount dP, the corresponding deformation is dΔL. The work dW due to the force increasing from P to P + dP is:

$$dW=(P+dP)d\Delta L=Pd\Delta L+dPd\Delta L \qquad (3.20)$$

Eliminating the small term $dPd\Delta L$, we have:

$$dW=Pd\Delta L \qquad (3.21)$$

This formula represents the hatched rectangle shown in **Figure 3.24b**.

Integrating the two sides of the Equation 3.21, we have the work done by the force increasing from 0 to P. That is denoted by the area of the triangle OAC in **Figure 3.24b**.

$$W = \frac{P\Delta L}{2} \tag{3.22}$$

This work changes to elastic strain energy U stored in the bar:

$$U = W = \frac{P\Delta L}{2} \tag{3.23}$$

Replace $\Delta L = \dfrac{PL}{EA}$

We have: $U = \dfrac{P^2 L}{2EA}$ (3.24)

Let u be the elastic strain energy per unit volume. We have:

$$u = \frac{U}{V} = \frac{P^2 L}{2EA} \tag{3.25}$$

Replace V=AL and $\sigma_z = \dfrac{P}{A}$, we obtain

$$u = \frac{\sigma_z^{\ 2}}{2E} = \frac{\sigma_z \varepsilon_z}{2} \tag{3.26}$$

Consider a bar with the length dz and internal force is N_z. We have:

$$dU = \frac{N_z^2 dz}{2EA}$$

$$U = \int_L dU = \int_L \frac{N_z^2 dz}{2EA} \tag{3.27}$$

Because $\dfrac{N_z}{2EA}$ is a constant, Equation 3.25 thus becomes:

$$U = \frac{N_z^2 L}{2EA} \tag{3.28}$$

If a system composes of many bars and L_i is the length of each bar, we have:

$$U = \sum U_i = \sum \frac{N_{zi}^2 L_i}{2E_i A_i} \tag{3.29}$$

It can be used to compute the displacement of bar systems by rearrange the Equation 3.23:

$$\Delta L = \frac{2U}{P} \tag{3.30}$$

§6. ALLOWABLE STRESS, FACTOR OF SAFETY AND THREE BASIC PROBLEMS

6.1. Allowable stress and factor of safety

Structural members or machines must be designed such that the working stresses are less than the ultimate strength of the material.

$$FS = Factor\ of\ safety$$

$$FS = \frac{\sigma_u}{\sigma_{all}} = \frac{\sigma_u}{[\sigma]} = \frac{ultimate\ stress}{allowable\ stress} \tag{3.31}$$

Factor of safety is taken into account the following considerations:

- uncertainty in material properties
- uncertainty of loadings
- uncertainty of analyses
- number of loading cycles
- types of failure
- maintenance requirements and deterioration effects
- importance of member to structures integrity
- risk to life and property
- influence on machine vibration

Condition of stress:

$$\sigma_z = \frac{N_z}{A} \le [\sigma] \tag{3.32}$$

We have three basic problems based on the above stress condition.

6.2. Three basic problems

Problem 1: Check the condition of stress

Given: Axial force, cross sectional area

Request: Check the condition of working stress

$$\sigma_z = \frac{N_z}{A} \le [\sigma](1 \pm 0.05) \tag{3.33}$$

0.05 is tolerance or allowable error 5%.

Problem 2: Find minimum cross-sectional area

Given: Axial force, allowable stress

Request: Find minimum cross-sectional area

$$A \geq \frac{N_z}{[\sigma]}(1 \pm 0.05) \tag{3.34}$$

Problem 3: Allowable axial force

Given: Allowable stress, cross-sectional area

Request: Find the allowable axial force

$$N_z \leq [\sigma] A (1 \pm 0.05) \tag{3.35}$$

or $\quad [N_z] \leq [\sigma] A$

6.3. Exercise 4

Consider the structure subjected to the load P in **Figure 3.25**. The bar AB and BC has circular cross section with the diameter D.

Given: $\alpha = 30^o; [\sigma] = 250\ MPa$; P = 25 kN, D = 20 mm.

a) Determine the axial force in the bars AB and BC.

b) Check the stress condition for the bar AB.

c) Determine the V steel to satisfy the stress condition.

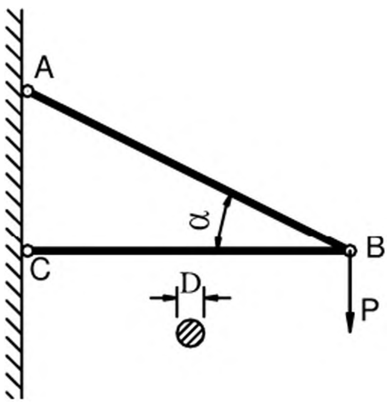
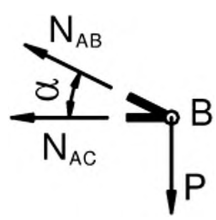


Figure 3.25

Solution

a) Determine the axial forces in the bars AB and BC.

Cut the joint B, establish the equilibrium of force:



$$\sum F_Y = 0 \Rightarrow N_{AB} \sin\alpha - P = 0$$

$$N_{AB} = \frac{P}{\sin\alpha} = \frac{25}{\sin 30^o} = 50kN \text{ (tension)}$$

$$\sum F_X = 0$$

$$\Rightarrow -N_{BC} - N_{AB}\cos\alpha = 0$$

$$\Rightarrow N_{BC} = -N_{AB}\cos\alpha = -50\cos 30^o = -43.3kN$$

(compression)

b) Check the stress condition for the bar AB and BC.

$$\sigma_{AB} = \frac{N_{AB}}{A_{AB}} = \frac{50\times10^3}{\pi 10^2} = 159MPa < [\sigma] ==> \text{OK.}$$

6.4. Exercise 5

Consider the structure in **Figure 3.26**. Determine [P] based on the stress condition of the bars 1, 2, 3.

Given: $[\sigma] = 160\ MPa$; $A_1 = 2cm^2$, $A_2 = 1cm^2$, $A_3 = 2cm^2$

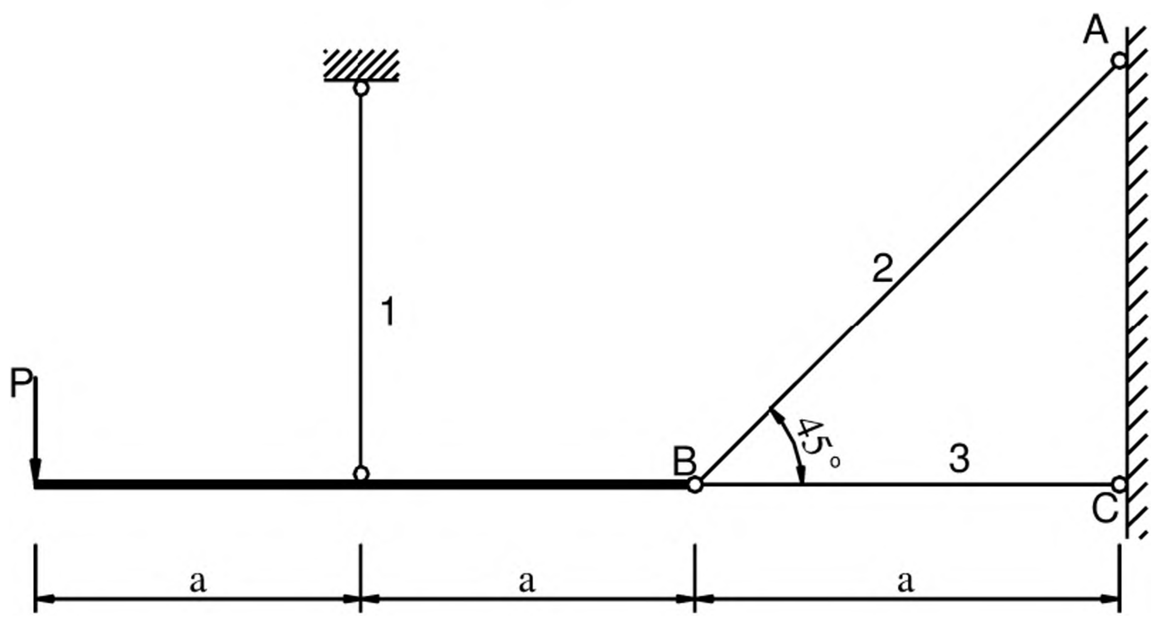


Figure 3.26

Solution

Take a cut to isolate the beam as shown in **Figure 3.27**.

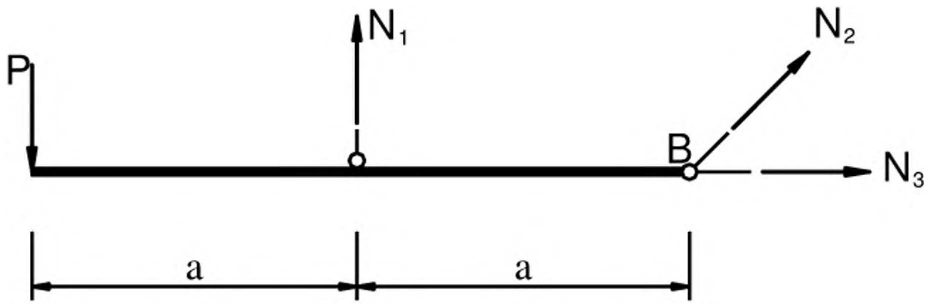


Figure 3.27

Establish the equilibrium equations:

$$\sum F_X = 0 \Rightarrow N_2 \cos 45^o + N_3 = 0$$

$$\sum F_Y = 0 \Rightarrow -P + N_1 + N_2 \sin 45^o = 0$$

$$\sum M_{/A} = 0 \Rightarrow -P2a + N_1 a = 0$$

Solve the above equations, we have:

$$N_1 = 2P; N_2 = -P\sqrt{2}; N_3 = P$$

The stress conditions:

$$\sigma_1 = \frac{N_1}{A_1} = \frac{2P}{A_1} \le [\sigma] \Rightarrow P \le \frac{[\sigma]A_1}{2} = 16 \times 2/2 = 16kN$$

$$|\sigma_2| = \frac{|N_2|}{A_2} = \frac{P\sqrt{2}}{A_2} \le [\sigma] \Rightarrow P \le \frac{[\sigma]A_2}{\sqrt{2}} = \frac{16 \times 1}{\sqrt{2}} = 11.3kN$$

$$\sigma_3 = \frac{N_3}{A_3} = \frac{P}{A_3} \le [\sigma] \Rightarrow P \le [\sigma]A_3 = 16 \times 2 = 32kN$$

Select: [P] = 11.3kN.

§7. STATIC INDETERMINACY

Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.

A structure will be statically indeterminate whenever it is of more supports than requirement to maintain its equilibrium.

Redundant reactions are replaced by unknown loads which along with the other loads must produce compatible deformations.

Deformations due to actual loads and redundant reactions are determined separately and then added or *superposed*.

7.1. Exercise 6

Consider the column subjected to the load P as shown in **Figure 3.28**. Determine the axial forces in the bars AC and BC.

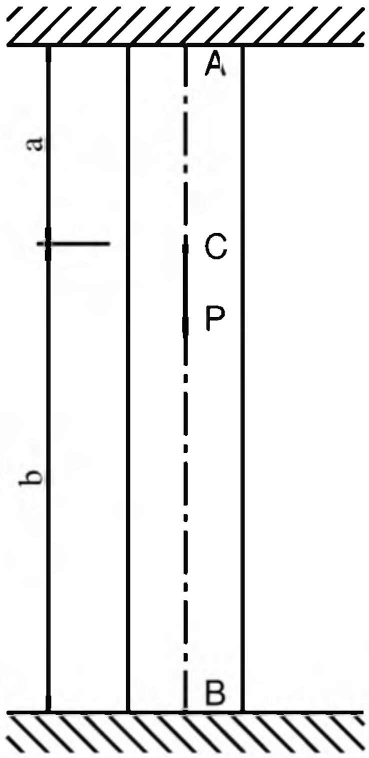
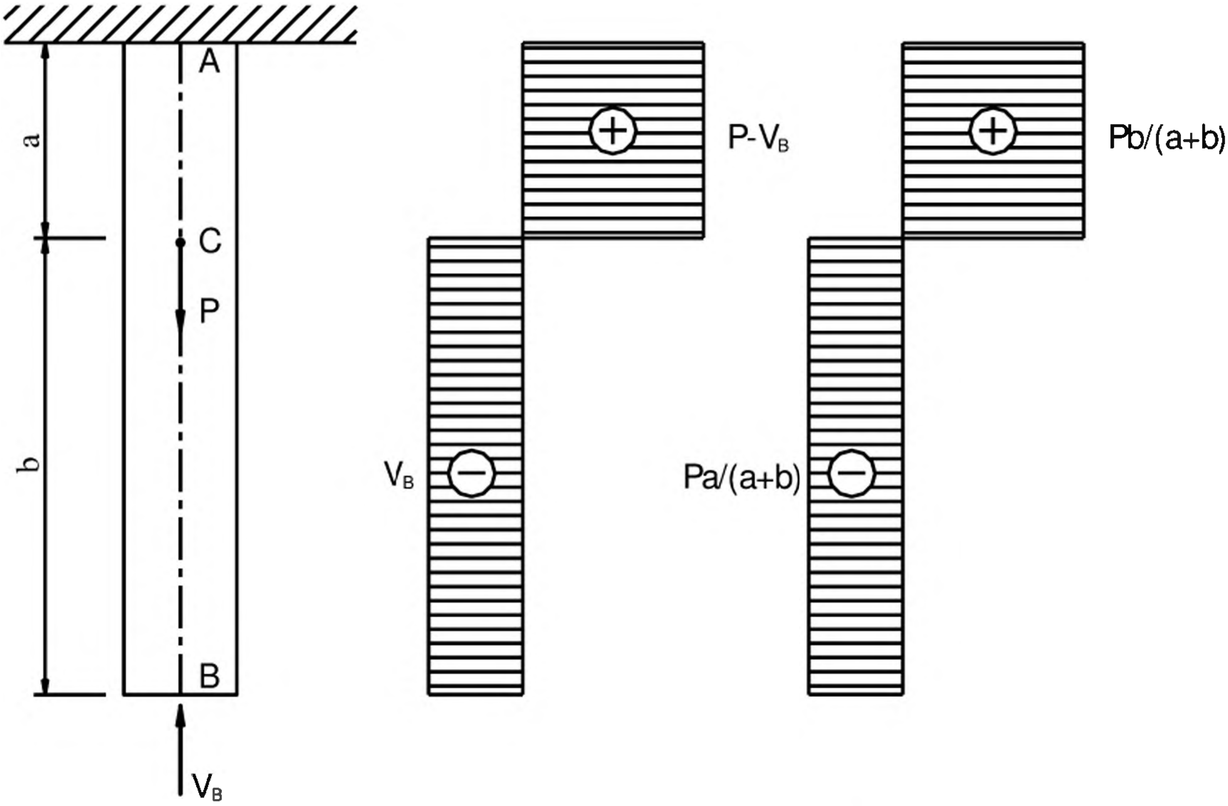


Figure 3.28

Solution

This is an indeterminate column. Remove the fixed end B and replace by the reaction V_B. Because the load is acting downward, the segment BC is in compression and the reaction V_B is shown in **Figure 3.29a**. The diagram of axial force is based on the unknown V_B as shown in **Figure 3.29b**.



a) b) c)

Figure 3.29

The boundary condition can be applied: The displacement at B must be zero because B is a fixed end.

$$\Delta L = \Delta_{BA} = \Delta_{BC} + \Delta_{CA} = 0$$

$$\Delta L = \frac{N_{BC} L_{BC}}{EA} + \frac{N_{CA} L_{CA}}{EA} = 0$$

Substitute $N_{BC} = -V_B$; $N_{CA} = P - V_B$ into the above equation, we have:

$$\Delta L = \frac{-V_B b}{EA} + \frac{(P - V_B)a}{EA} = 0$$

$$V_B = \frac{Pa}{a+b}$$

When V_B is determined, the problem becomes statically determinate.

By substituting V_B into the diagram (**Figure 3.29b**), we have the diagram in **Figure 3.29c**.

7.2. Exercise 7

Consider the structure subjected to the load P shown in **Figure 3.30**. Calculate the axial force in each bar.

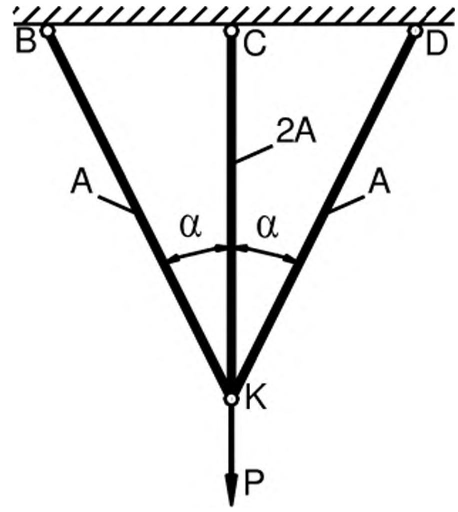


Figure 3.30

Solution

Cut the joint A as shown in **Figure 3.31a**

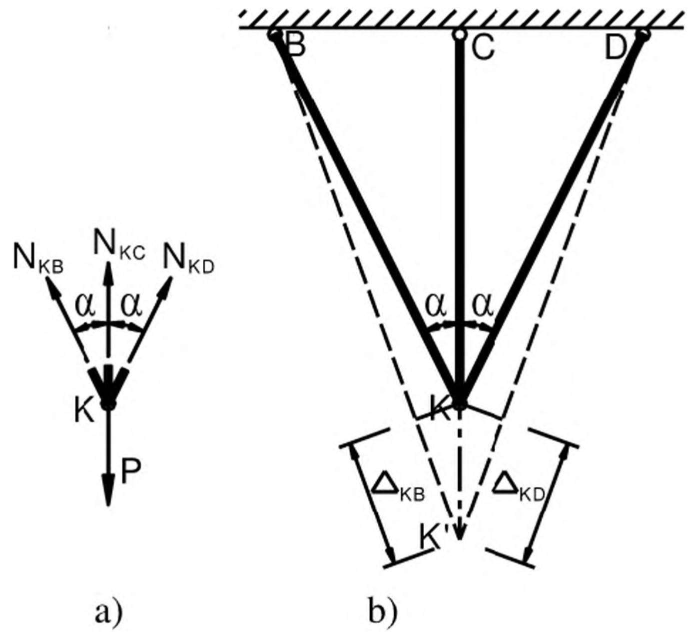


Figure 3.31

Establish two equilibrium equations:

$$\sum F_X = 0 \Rightarrow N_{AB} = N_{AD} = N$$

$$\sum F_Y = 0 \Rightarrow 2N\cos\alpha + N_{AC} = P$$

There are two equations but three unknowns. Thus, we need to establish one additional equation based on deformation. The original and deformed shape are shown in **Figure 3.31b**.

$$AA' = \frac{\Delta_{AD}}{\cos\alpha}$$

$$\frac{N_{AC}L_{AC}}{EA} = \frac{\dfrac{N_{AD}L_{AD}}{EA}}{\cos\alpha}$$

$$\frac{N_{AC}L_{AD}\cos\alpha}{EA} = \frac{\dfrac{N_{AD}L_{AD}}{EA}}{\cos\alpha}$$

$$N_{AC} = \frac{N_{AD}}{\cos^2\alpha} = \frac{N}{\cos^2\alpha}$$

Solve the system of three equations with three unknows, we have:

$$N = \frac{P\cos^2\alpha}{2\cos^2\alpha+1}$$

$$N_{AC} = \frac{P}{2\cos^2\alpha+1}$$

7.3. Exercise 8

Consider the **Figure 3.32** in which the beam BCDF is infinitely stiff. The cross-sectional areas of members CH and DG are the same. These members CH and DG also have the same elastic modulus. Determine the axial forces in members CH and DG.

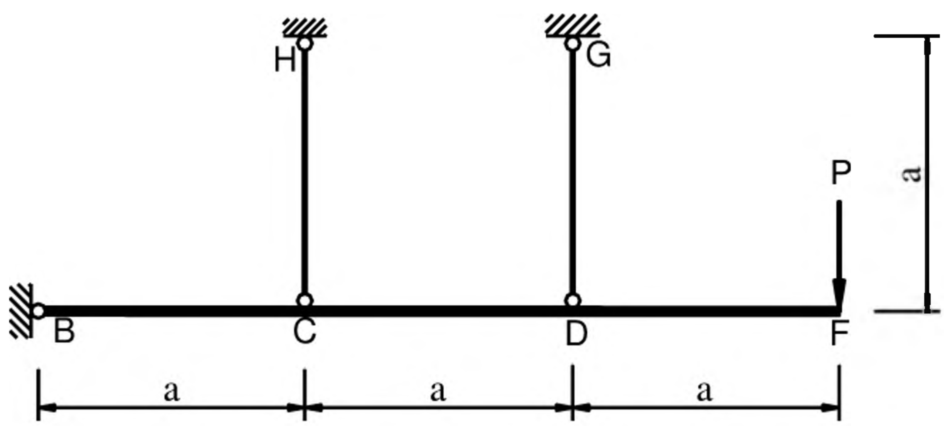


Figure 3.32

Solution

Determine the axial forces in members CH and DG.

Take the cuts across the bars CH and DG

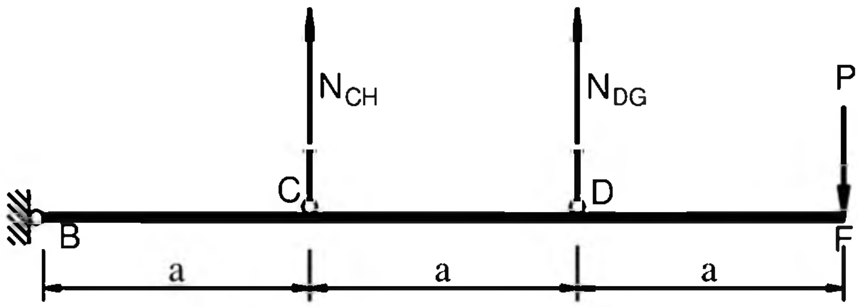


Figure 3.33

$$\sum M_{/B} = 0$$

$$\Leftrightarrow \quad N_{CH}a + N_{DG}2a = P(3a) \tag{1}$$

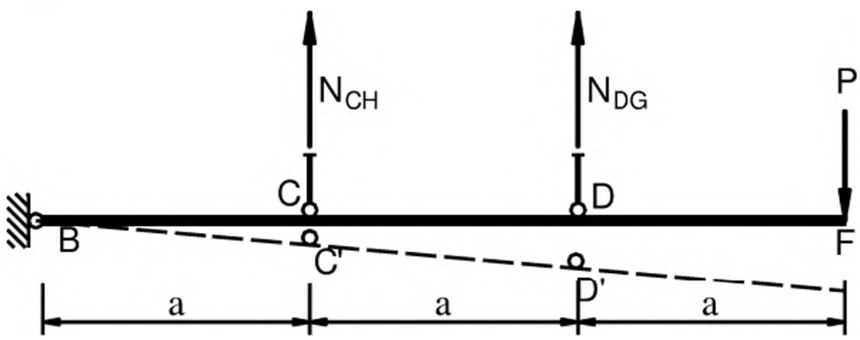


Figure 3.34

DD' = 2 CC'

$$\Delta_{DG} = 2\Delta_{CH}$$

$$\Leftrightarrow \quad \frac{N_{DG}L_{DG}}{EA_{DG}} = 2\frac{N_{CH}L_{CH}}{EA_{CH}}$$

$$\Leftrightarrow \quad N_{DG} = 2N_{CH} \tag{2}$$

From equations (1) and (2), we have:

$$N_{CH} = \frac{3P}{5}$$

$$N_{DG} = \frac{6P}{5}$$

7.4. Exercise 9

Consider the system as shown in **Figure 3.35**. The beam BCD bar is infinitely stiff. The bars BG and DH are made of the same material and have the same cross-sectional area. Determine:

a) The axial forces in the bars BG and DH.

b) The vertical displacement of B.

c) The rotation of the beam BCD.

d) Determine the allowable load [q] based on the stress condition of bars BG and DH, knowing that the bars BG and DH have allowable stress [σ].

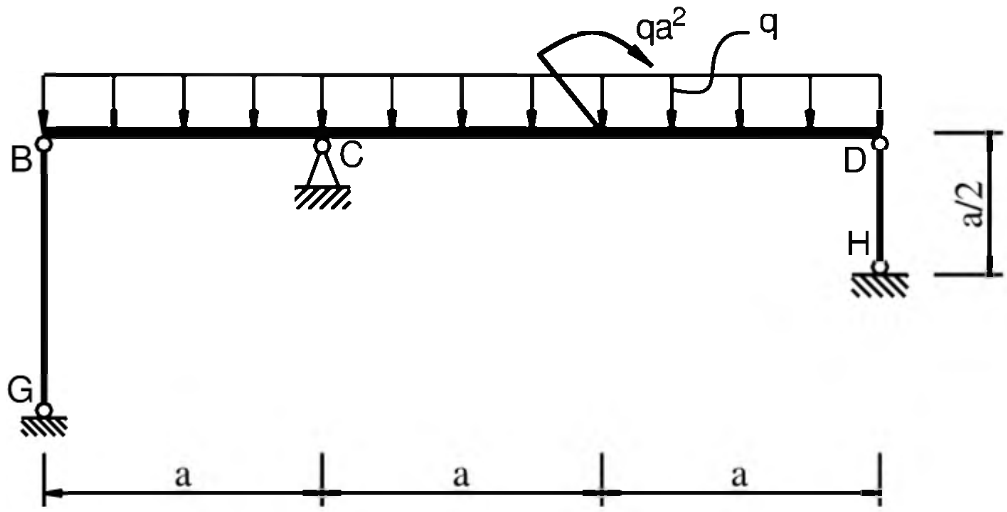


Figure 3.35

Solution

a) The axial forces in the bars BG and DH

The bar BG is under tension, the bar DH is under compression. The axial forces in these bars are shown in **Figure 3.36**.

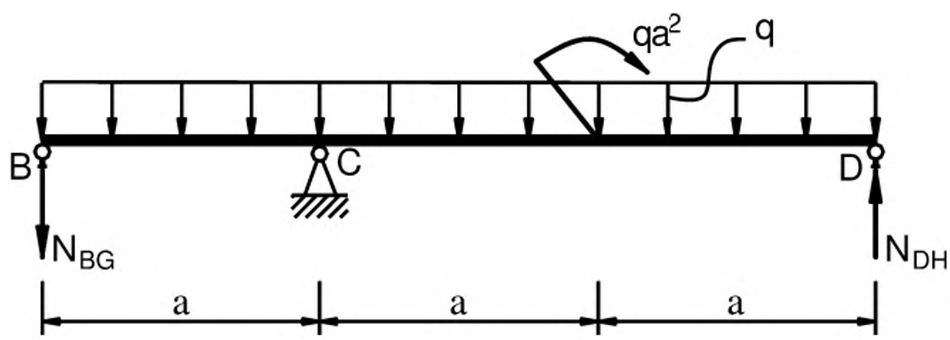


Figure 3.36

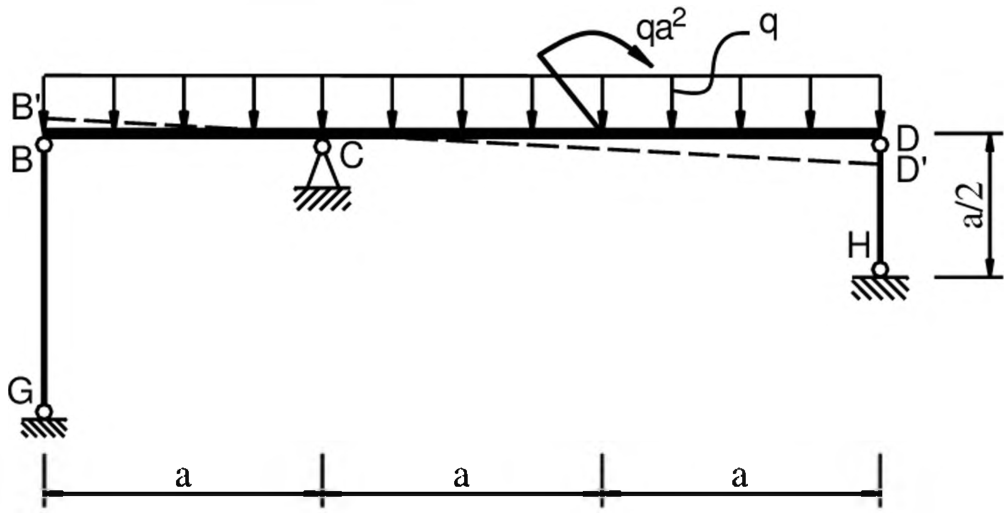
Establish the moment equation with respect to the point C:

$$\sum M_C = 0$$

$$N_{DH}.2a + N_{BG}.a + qa.\tfrac{a}{2} - 2qa.a - qa^2 = 0$$

$$2N_{DH} + N_{BG} = \tfrac{5qa}{2} \qquad (1)$$

Establish the compatibility condition:



$$\Delta_{DH} = 2\Delta_{BG}$$

$$\frac{N_{DH}.a}{2EA} = 2\frac{N_{BG}.a}{EA}$$

$$N_{DH} = 4N_{BG} \tag{2}$$

From equations (1) and (2), we have:

$$\begin{cases} N_{DH} = {}^{10qa}\!/_{9} \\ N_{BG} = {}^{5qa}\!/_{18} \end{cases}$$

b) The vertical displacement of B

$$\Delta_B = \Delta_{BG} = \frac{N_{BG}.a}{EA} = \frac{5}{18}\frac{qa^2}{EA}$$

c) The rotation of the beam BCD

$$tg\alpha = \frac{\Delta_{BG}}{a} = \frac{N_{BG}.a}{EA.a} = \frac{5}{18}\frac{qa}{EA}$$

d) Determine the allowable load [q] based on the stress conditions of bars BG and DH

The bar BG:

$$\sigma_{BG} \le [\sigma]$$

$$\frac{N_{BG}}{A} \le [\sigma] \leftrightarrow \frac{5}{18}\frac{qa}{A} \le [\sigma] \rightarrow q \le \frac{18}{5}\frac{[\sigma]A}{a} \tag{1}$$

The bar DH:

$$\sigma_{DH} \leq [\sigma]$$

$$\frac{N_{DH}}{A} \leq [\sigma] \leftrightarrow \frac{10}{9}\frac{qa}{A} \leq [\sigma] \rightarrow q \leq \frac{10}{9}\frac{[\sigma]A}{a} \quad (2)$$

From equations (1) and (2), we have:

$$[q] \leq \frac{10}{9}\frac{[\sigma]A}{a}$$

7.5. Exercise 10

Consider the system as shown in **Figure 3.37**. The beam BCDF bar is infinitely stiff. The bars DG and FH are made of the same material and have the same cross-sectional area. Determine:

a) The axial forces in the bars DG and FH

b) The vertical displacement of B

c) The rotation of the beam BCDF

d) Determine the allowable load [q] based on the stress condition of bars DG and FH, knowing that the bars DG and FH have allowable stress [σ].

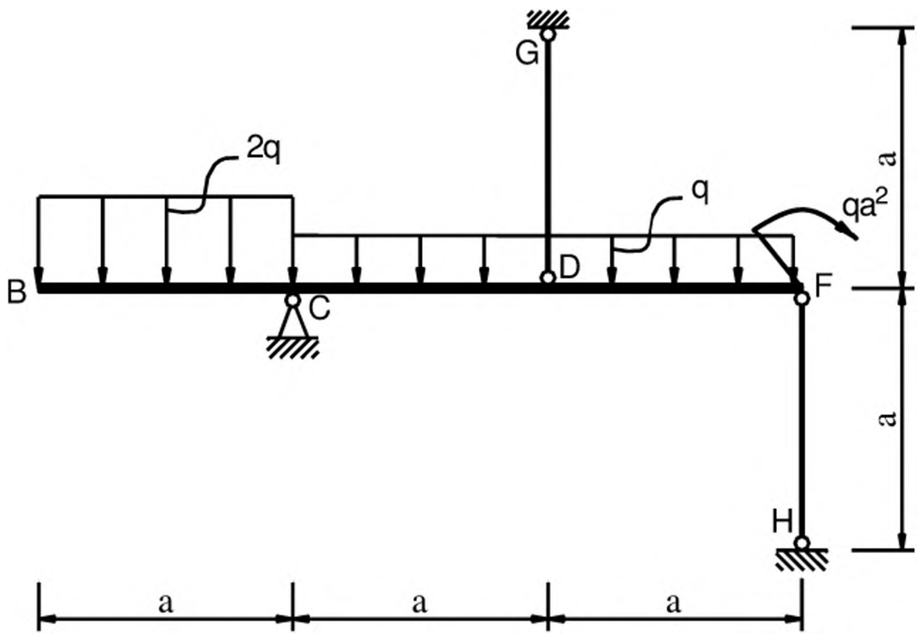


Figure 3.37

Solution

a) The axial forces in the bars DG and FH

The bar DG is in tension, and the bar FH is in compression

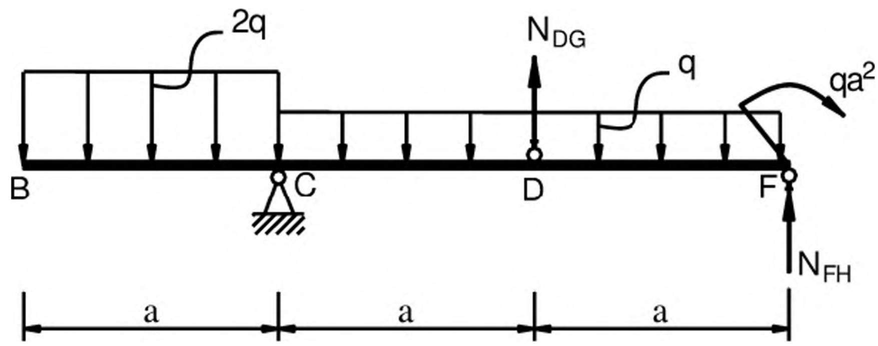


Figure 3.38

Establish the moment equation with respect to the point C:

$$\sum M_C = 0$$

$$N_{FH}.2a + N_{DG}.a + 2qa.\tfrac{a}{2} - 2qa.a - qa^2 = 0$$

$$2N_{FH} + N_{DG} = 2qa \qquad (1)$$

Establish the compatibility equation:

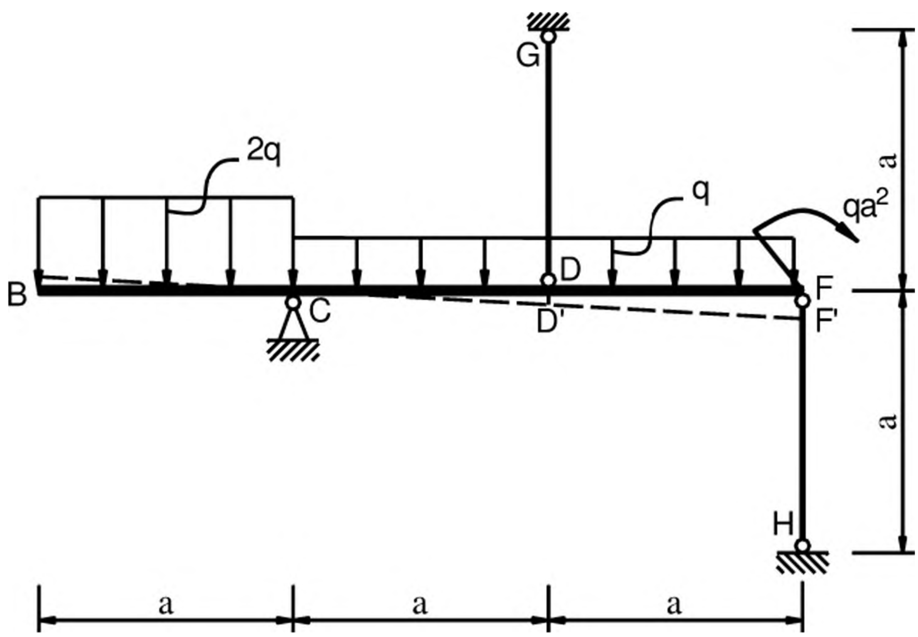


Figure 3.39

Based on **Figure 3.39**, we have:

FF' = 2DD'

$$\Delta_{FH} = 2\Delta_{DG}$$

$$\frac{N_{FH}.a}{EA} = 2\frac{N_{DG}.a}{EA}$$

$$N_{FH} = 2N_{DG} \tag{2}$$

From equations (1) and (2), we have:

$$\begin{cases} N_{FH} = {}^{4qa}/_{5} = 0.8qa \\ N_{DG} = {}^{2qa}/_{5} = 0.4qa \end{cases}$$

b) The vertical displacement of B

$$\Delta_B = \Delta_{DG} = \frac{N_{DG}.a}{EA} = \frac{2}{5}\frac{qa^2}{EA}$$

c) The rotation of the beam BCDF

$$tg\alpha = \frac{\Delta_{DG}}{a} = \frac{N_{DG}.a}{EA.a} = \frac{2}{5}\frac{qa}{EA}$$

d) Determine the allowable load [q] based on the stress condition of bars DG and FH

The bar DG:

$$\sigma_{DG} \le [\sigma]$$

$$\frac{N_{DG}}{A} \le [\sigma] \leftrightarrow \frac{2}{5}\frac{qa}{A} \le [\sigma] \rightarrow q \le \frac{5}{2}\frac{[\sigma]A}{a} \tag{1}$$

The bar FH:

$$\sigma_{FH} \le [\sigma]$$

$$\frac{N_{FH}}{A} \le [\sigma] \leftrightarrow \frac{4}{5}\frac{qa}{A} \le [\sigma] \rightarrow q \le \frac{5}{4}\frac{[\sigma]A}{a} \tag{2}$$

From equations (1) and (2), we have:

$$[q] \le \frac{5}{4}\frac{[\sigma]A}{a}$$

PROBLEMS

PROBLEM 1. Consider the column shown in **Figure 3.40**. Given: a=1 m, P=300 kN, the cross sectional area A=400 mm^2. The elastic modulus E=2×10^5 MPa.
Question 1) Draw the diagram of axial force.
Question 2) Calculate the stresses in the column.
Question 3) Calculate the displacement at the top of the column.

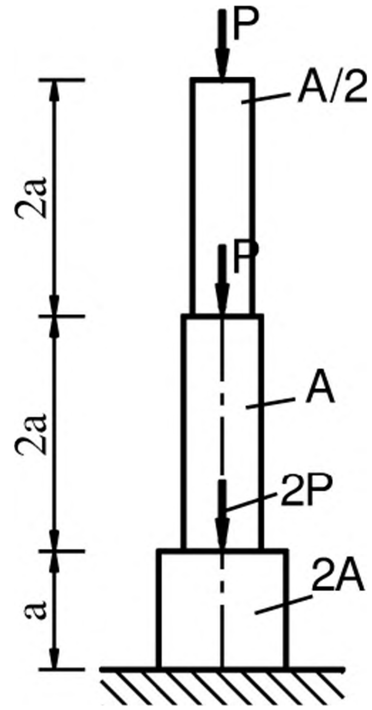


Figure 3.40

PROBLEM 2. Consider the structure shown in **Figure 3.41**. Elements CH and DG have the same cross-sectional area A, modulus of elasticity E. The stiffness of the beam BCDF is infinite. Determine axial forces in elements CH and DG.

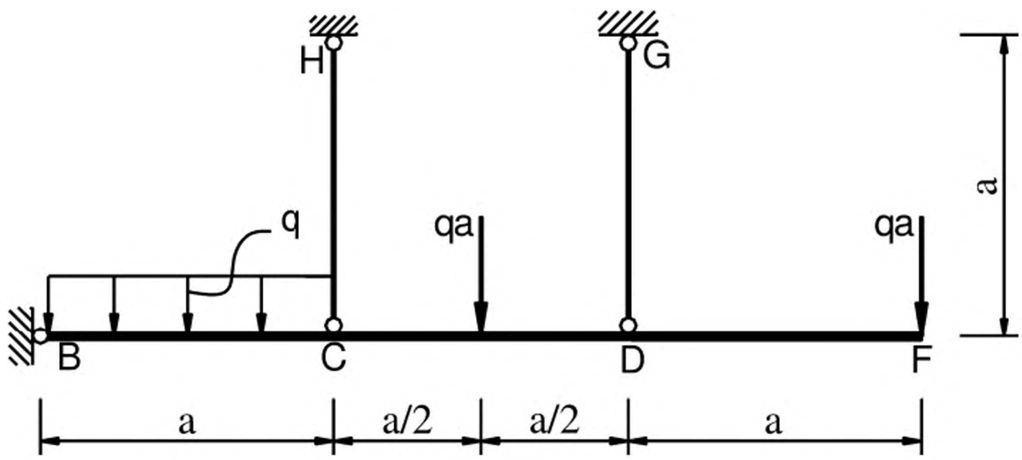


Figure 3.41

PROBLEM 3: Consider the structure in **Figure 3.42**, in which the beam BCDF is infinitely stiff. The cross-sectional areas of members CH and FG are the same, namely A. These members CH and FG also have the same elastic modulus, namely E. Given: P=20 kN, A=100 mm^2, E=2×10^5 MPa, the allowable stress is 250 MPa, a=2 m.

Question 1) Determine the axial forces in members CH and FG.

Question 2) Determine the stresses in members CH and FG.

Question 3) Check the stress conditions of the members CH and FG.

Question 4) Determine the vertical displacement at the point F.

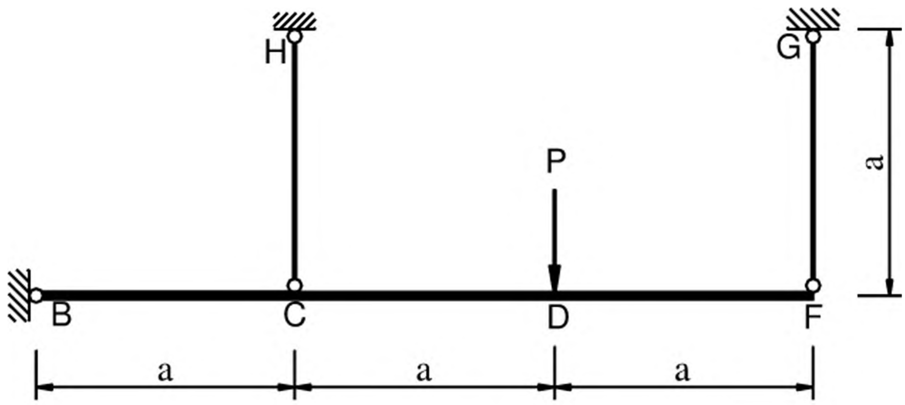


Figure 3.42

PROBLEM 4. Consider the structure in **Figure 3.43**. Elements CH and DG have the same cross-sectional area A, modulus of elasticity E. The stiffness of the beam BCD is infinite. Determine axial forces in elements CH and DG.

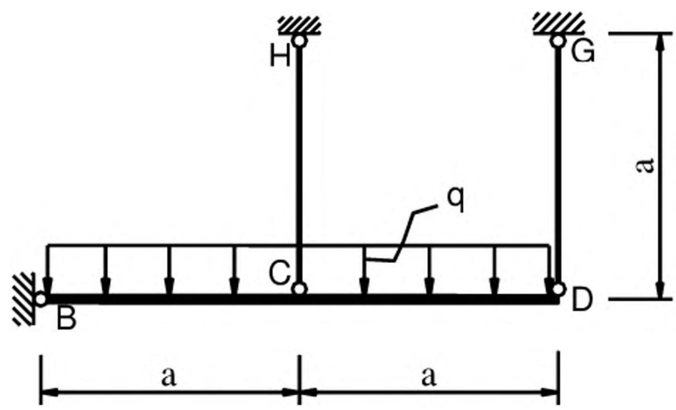


Figure 3.43

PROBLEM 5: Consider the **Figure 3.44** in which the beam BCDF is infinitely stiff. The cross-sectional areas of members CH and DG are the same. These members CH and DG also have the same elastic modulus. Determine the axial forces in members CH and DG.

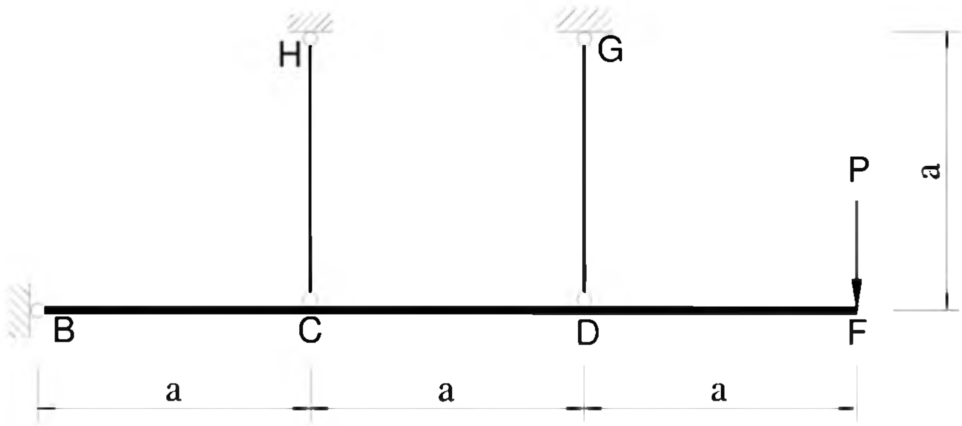


Figure 3.44

Chapter 4

STRESS STATE

§1. INTRODUCTION

This chapter presents methods to transform the stress components, which are associated with a coordinate system, into other stress components, which are associated with other coordinate system (having a different orientation).

Once the necessary transformation equations are established, we will then be able to obtain the maximum normal and maximum shear stress at a point and find the orientation of elements upon which they act.

Plane-stress transformation will be discussed since this condition is the most common in engineering practice.

1.1. Sress state

A general stress state of an element is shown in **Figure 4.1.**

"Stress state at a point is the set of all stresses acting on all surfaces passing through that point".

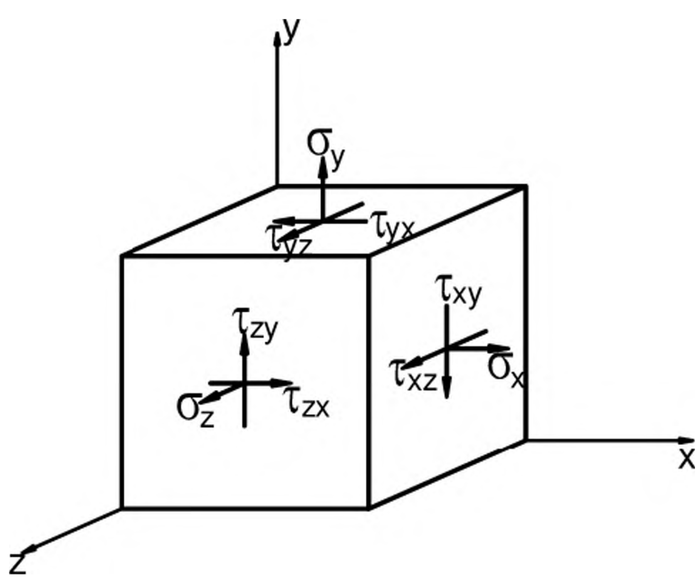


Figure 4.1. General stress state

In a general case, on each side of the element will be all three stress components. Because dx, dy, dz are extremely small, the stresses on parallel surfaces are the same.

The subscripts x, y, z of normal stresses σ_x, σ_y and σ_z indicate the directions of the stresses.

The shear stresses τ_{xy}, τ_{yz}, and τ_{zx} come with two subscripts. The first one is the axis that is perpendicular to the shear stress; the second one indicates the direction of the shear stress.

1.2. Rule of shear stress

On the two perpendicular surfaces, the magnitudes of shear stresses are equal as shown in Equation 4.1 and their directions are simultaneously inward or outward the intersection of the two surfaces as shown in **Figure 4.2**.

$$\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{yz} &= \tau_{zy} \\ \tau_{xz} &= \tau_{zx} \end{aligned} \tag{4.1}$$

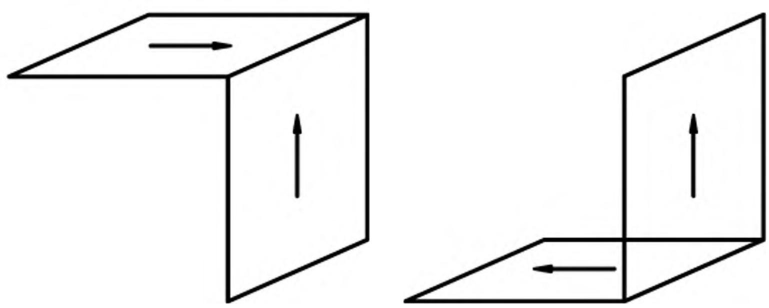


Figure 4.2. Rule of shear stress

Principle element, principle surface, principle direction and principle stresses. Classification of stress states

- **Principle element** is the element which has normal stresses while all shear stresses are zero (**Figure 4.3**).
- **Principle surfaces** are surfaces of a principle element.
- **Principle directions** are directions of normal stresses of a principle element.
- **Principle stresses** are stresses of a principle element.

The algebraic (algebraical) principle stresses σ_1, σ_2, σ_3 are conventionalized that $\sigma_1 > \sigma_2 > \sigma_3$.

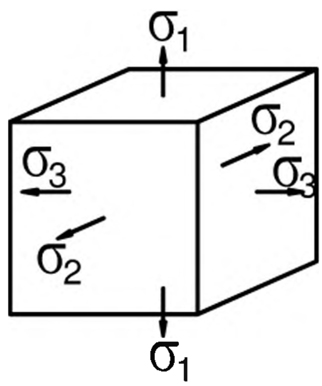


Figure 4.3. Principle element

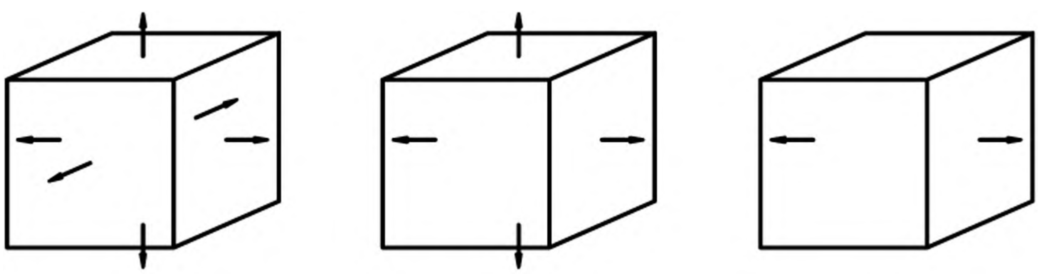
Example:

If a principle element has the principle stresses: 100 N/cm^2, 200 N/cm^2, -400 N/cm^2,

Thus, σ_1=200 N/cm^2, σ_2=100 N/cm^2, σ_3=-400 N/cm^2.

Classification of stress states

- If the three principle stresses are different from zero: 3D state of stress (**Figure 4.4a**).
- If the two principal stresses are different from zero: 2D state of stress or plane stress state (**Figure 4.4b**).
- If one principle stress is different from zero: 1D state of stress or single stress state (**Figure 4.4c**).



a) b) c)

Figure 4.4. Classification of stress states

§2. PLANE STRESS STATE

2.1. Sign convention

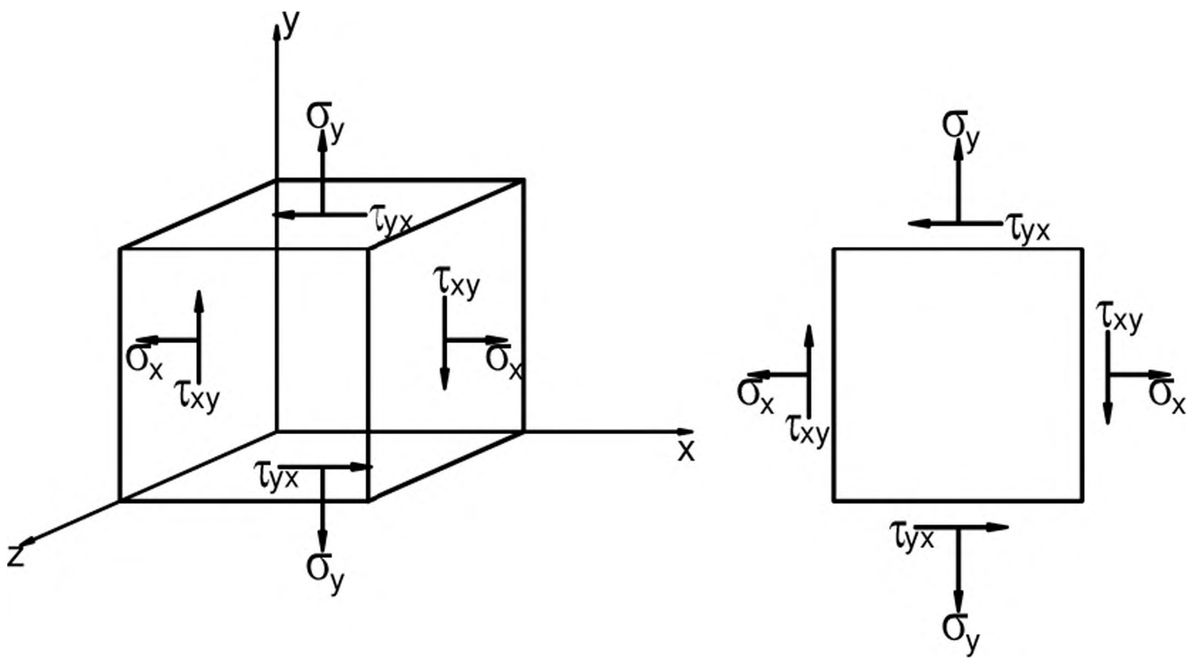


Figure 4.5. Plane stress state

The conventional signs of stresses are as follows:

$\sigma > 0$: tension

$\tau > 0$:

- Method 1: if the clockwise rotated normal vector becomes that τ.
- Method 2: if τ tends to rotate the element clockwise.

2.2. Stress transformation

In this section, formulas to compute stresses on inclined surfaces are established. A stress state of an element with a coordinate system is given. Then, the requirement is to find the stress state of any rotated element, e.g. stresses on the inclined (hatched) surface which is determined by the angle α. α is the angle between the x axis and the normal vector of the inclined surface. Conventional signs of α are as follows. The angle α is positive if it is counterclockwise when turning from the x-axis to the normal vector of the surface.

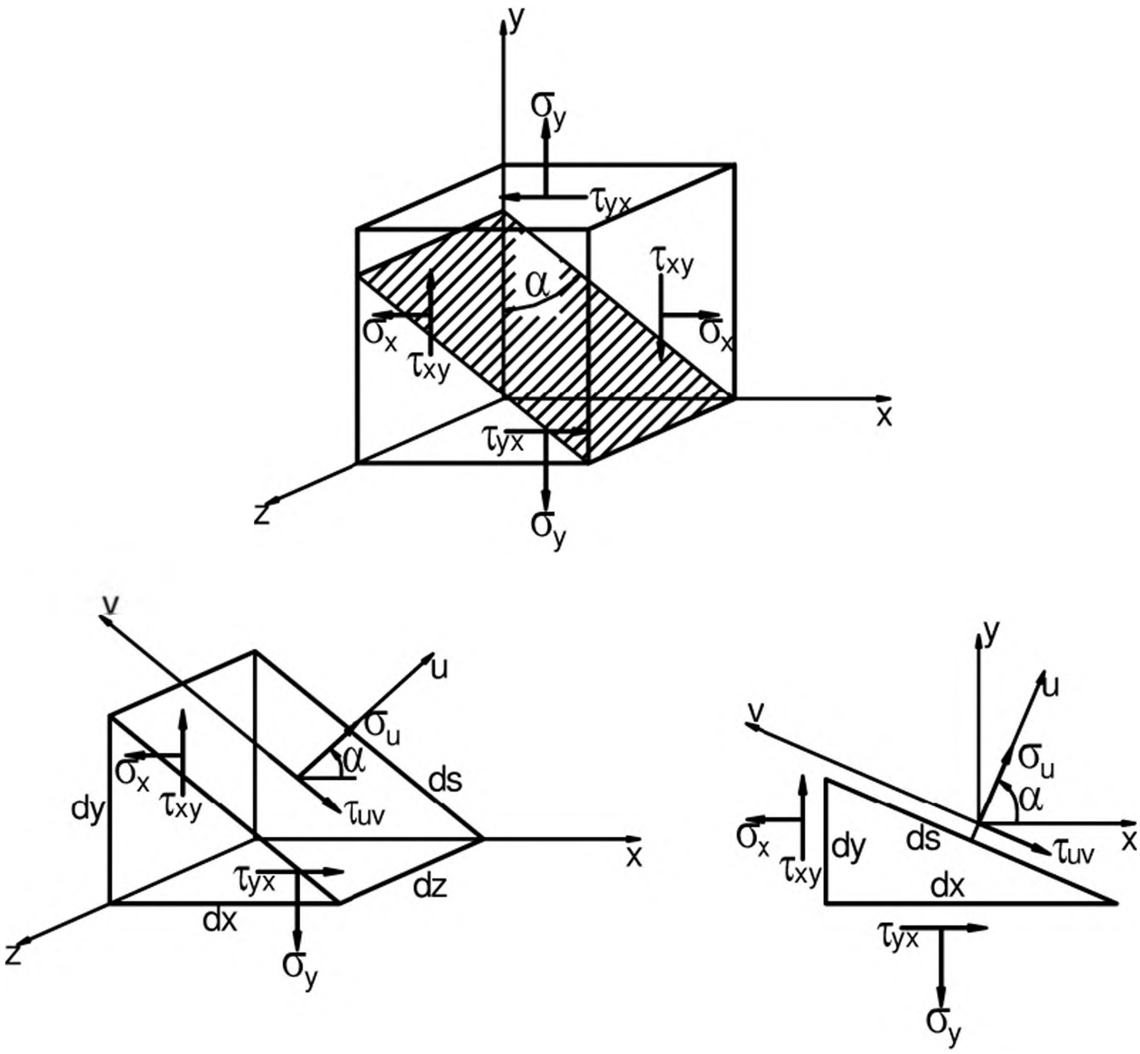


Figure 4.6. Stress transformation

Namely, σ_u and τ_{uv} are stresses on the inclined surface. They are determined by the two equilibrium equations:

$$\sum F/u = 0 \Rightarrow$$

$$\sigma_u dsdz - \sigma_x dydz\cos\alpha + \tau_{xy} dydz\sin\alpha - \sigma_y dxdz\sin\alpha + \tau_{yx} dxdz\cos\alpha = 0 \qquad (4.2)$$

$$\sum F/v = 0 \Rightarrow$$

$$\tau_{uv} dsdz - \sigma_x dydz\sin\alpha - \tau_{xy} dydz\cos\alpha + \sigma_y dxdz\cos\alpha + \tau_{yx} dxdz\sin\alpha = 0 \qquad (4.3)$$

Substitute:

$$\tau_{xy} = \tau_{yx} \quad dx = ds\sin\alpha, dy = ds\cos\alpha \qquad (4.4)$$

We obtain:

Equation 4.2:

$$\sigma_u ds - \sigma_x dy\cos\alpha + \tau_{xy} dy\sin\alpha - \sigma_y dx\sin\alpha + \tau_{yx} dx\cos\alpha = 0$$

$$\sigma_u ds - \sigma_x ds\cos^2\alpha + \tau_{xy} ds\cos\alpha\sin\alpha - \sigma_y ds\sin^2\alpha + \tau_{yx} ds\sin\alpha\cos\alpha = 0$$

$$\sigma_u = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha$$

$$\sigma_u = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha - \tau_{xy}\sin 2\alpha \tag{4.5}$$

Equation 4.3:

$$\tau_{uv} ds - \sigma_x dy\sin\alpha - \tau_{xy} dy\cos\alpha + \sigma_y dx\cos\alpha + \tau_{yx} dx\sin\alpha = 0$$

$$\tau_{uv} ds - \sigma_x ds\cos\alpha\sin\alpha - \tau_{xy} ds\cos^2\alpha + \sigma_y ds\cos\alpha\cos\alpha + \tau_{xy} ds\sin^2\alpha = 0$$

$$\tau_{uv} - \sigma_x\cos\alpha\sin\alpha - \tau_{xy}\cos^2\alpha + \sigma_y\cos\alpha\cos\alpha + \tau_{xy}\sin^2\alpha = 0$$

$$\tau_{uv} = \left(\sigma_x - \sigma_y\right)\sin\alpha\cos\alpha + \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right)$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \tag{4.6}$$

Applying the trigonometric identities:

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$$

We have:

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \tag{4.7}$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \tag{4.8}$$

These are formulas of stress transformation.

If α is substituted by $\alpha + 90^o$ as shown in the following equation, we have the stresses on the surface with normal vector v:

$$\sigma_v = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \tag{4.9}$$

$$\tau_{vu} = -\tau_{uv} \tag{4.10}$$

So, we have the relation:

$$\sigma_u + \sigma_v = \sigma_x + \sigma_y = constant \tag{4.11}$$

⇒ The total of normal stresses on the two perpendicular surfaces is constant and does not depend on the angle α.

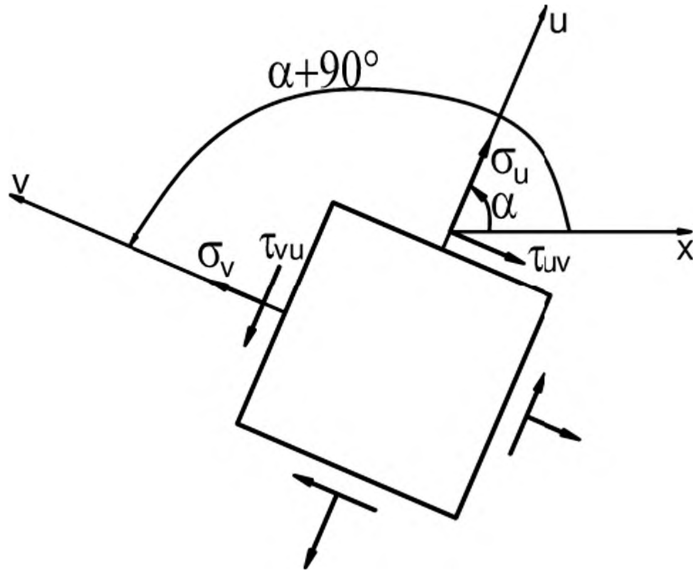


Figure 4.7. Stress state of a rotated element

2.3. Principle stress and maximum shear stress

2.3.1. Principle stress and principle direction

Let α_o be the angle between the x axis and the principle direction. The condition to find α_o is that the shear stress on that surface is zero:

$$\tau_{uv} = 0$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha_o + \tau_{xy}\cos 2\alpha_o = 0 \lim_{x \to \infty}$$

$$\tan 2\alpha_o = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{4.12}$$

This equation is also found by taking the derivative of σ_u with respect to α, and then let it be equal to 0.

The principle stresses are found by substituting the value α_o into the formula σ_u, we obtain:

$$\sigma_{\substack{\max \\ \min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{4.13}$$

Note: $\sin 2\alpha_o = \pm\frac{\tan 2\alpha_o}{\sqrt{1+\tan^2 2\alpha_o}}$; $\cos 2\alpha_o = \pm\frac{1}{\sqrt{1+\tan^2 2\alpha_o}}$

2.3.2. Maximum shear stresses

The maximum shear stresses can be determined by taking derivative of τ_{uv} with respect to α and let it be 0:

$$\frac{d\tau_{uv}}{d\alpha} = \left(\sigma_x - \sigma_y\right)\cos 2\alpha - 2\tau_{xy}\, sin2\alpha = 0$$

$$\Rightarrow \qquad \tan 2\alpha = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{4.14}$$

Thus, we have:

$$\tan 2\alpha = -\frac{1}{\tan 2\alpha_o} \tag{4.15}$$

Thus: $\quad 2\alpha = 2\alpha_o \pm k90^o$

Or: $\quad \alpha = \alpha_o \pm k45^o \qquad (4.16)$

$\Rightarrow$ the angle between the surface of maximum shear stress and the principle surface is 45°.

Substitute α into τ_{uv}, we get:

$$\tau_{\substack{\max\\ \min}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{4.17}$$

If the principle stresses are the stresses on the surfaces which are perpendicular to the axis x, y. In other words, assuming x, y are principle directions:

$$\sigma_x = \sigma_u = \sigma_1$$

$$\sigma_y = \sigma_v = \sigma_2$$

$$\tau_{xy} = \tau_{uv} = 0$$

$$\tau_{\substack{\max\\ \min}} = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + 0^2}$$

$$\tau_{\substack{\max\\ \min}} = \pm\frac{\sigma_1 - \sigma_2}{2} \tag{4.18}$$

2.3.3. Special cases

1) Special plane stress state

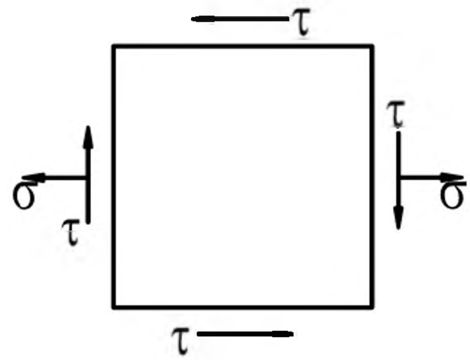


Figure 4.8. Element under plane stress state

$$\sigma_x = \sigma$$
$$\sigma_y = 0$$
$$\tau_{xy} = \tau$$

The principle stresses:

$$\sigma_{\substack{\max \\ \min}} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \tag{4.19}$$

For example: member subjected to bending.

2) Pure shear stress state

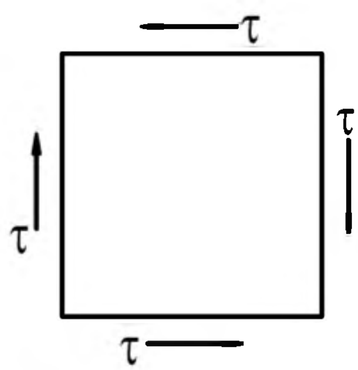


Figure 4.9. Element under pure shear stress state

$$\sigma_x = 0$$
$$\sigma_y = 0$$
$$\tau_{xy} = \tau$$

Principle stresses:

$$\sigma_{\substack{\max \\ \min}} = \pm\tau$$

$$\tan 2\alpha_o = \infty$$

Or $$\alpha_o = \frac{\pi}{4} + k\frac{\pi}{2}$$

Thus, the principle direction is 45° with x and y axis.

2.4. Exercise 1

A bar with the cross-sectional area A subjected to the axial load P as shown in **Figure 4.10**. Determine the stresses on the surfaces as shown in **Figure 4.10** with.

a) β = 30°

b) β = -30°

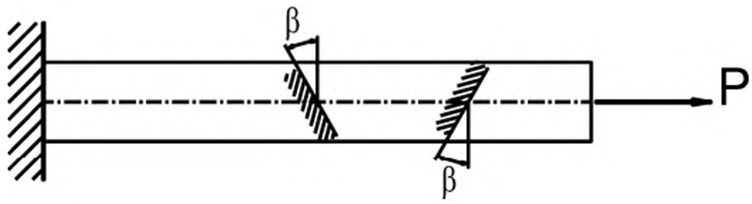


Figure 4.10

Solution

a) β = 30°

We have:

$$\sigma_x = \frac{P}{A}; \sigma_y = 0; \tau_{xy} = 0\,;\ \alpha = 30^o$$

Therefore,

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\sigma_u = \frac{P/A+0}{2} + \frac{P/A-0}{2}\cos 2.30^o = \frac{P}{2A}\left(1+\cos 2.30^o\right)$$

$$= \frac{P}{2A}\left(1+\frac{1}{2}\right) = \frac{3}{4}\frac{P}{A}$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

$$\tau_{uv} = \frac{P/A-0}{2}\sin 2.30^o = \frac{\sqrt{3}}{4}\frac{P}{A}$$

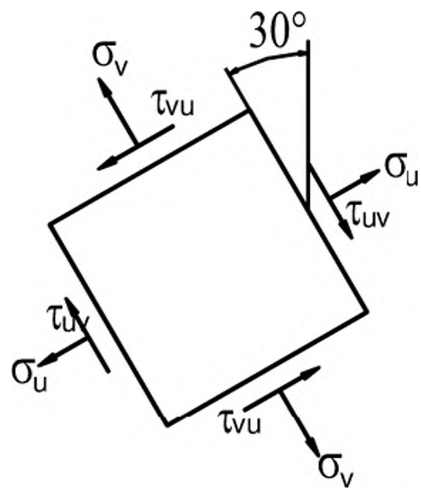


Figure 4.11

b) On the surface with $\alpha = -30^o$

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\sigma_u = \frac{P/A+0}{2} + \frac{P/A-0}{2}\cos\left(2\times(-30^o)\right)$$

$$= \frac{P}{2A}\left(1+\cos\left(2\times(-30^o)\right)\right) = \frac{P}{2A}\left(1+\frac{1}{2}\right) = \frac{3}{4}\frac{P}{A}$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

$$\tau_{uv} = \frac{P/A-0}{2}\sin 2.(-30^o) = -\frac{\sqrt{3}}{4}\frac{P}{A}$$

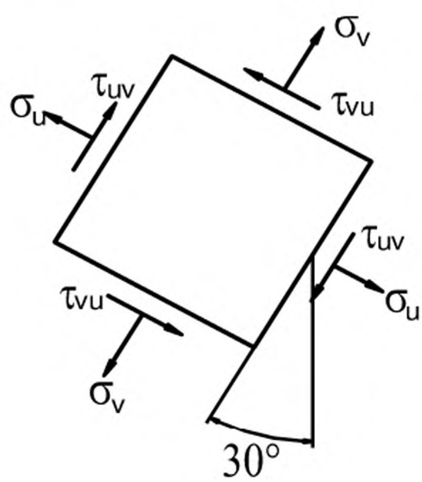
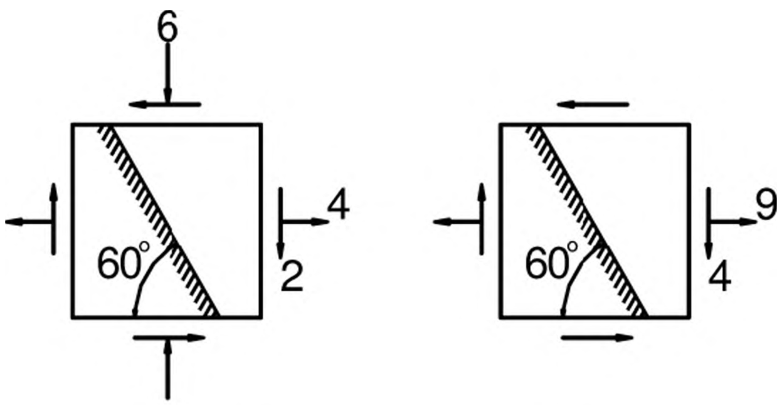


Figure 4.12

2.5. Exercise 2

Calculate the normal stress and shear stress on the surfaces shown in **Figure 4.13**. The unit of stress is MPa.



a) b)

Figure 4.13

Solution

a) $\sigma_x - 4; \sigma_y - -6; \tau_{xy} - 2$ (MPa); $\alpha = 30^o$

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha = -0.23MPa$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha = 5.3MPa$$

b) $\sigma_x = 9; \sigma_y = 0; \tau_{xy} = 4$ (MPa); $\alpha = 30^o$

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha = 3.29MPa$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha = 5.9MPa$$

2.6. Exercise 3

Calculate the normal stress and shear stress on the surfaces shown in **Figure 4.14**. The unit of stress is MPa.

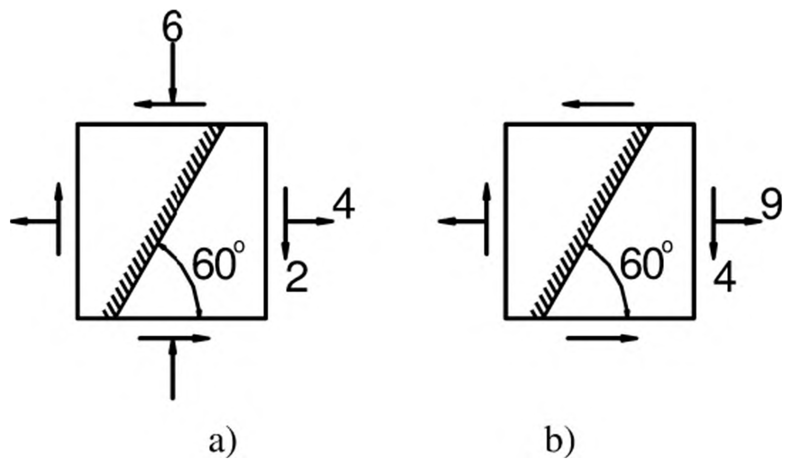


Figure 4.14

Solution

a) $\sigma_x = 4; \sigma_y = -6; \tau_{xy} = 2$ (MPa); $\alpha = -30^o$

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha = 3.2MPa$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha = 3.3MPa$$

b) $\sigma_x = 9; \sigma_y = 0; \tau_{xy} = 4$ (MPa); $\alpha = -30^o$

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha = 10.2MPa$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha = -1.9MPa$$

2.7. Exercise 4

Given: $\sigma_y = 3; \tau_{yx} = -5; \tau_{uv} = 6$. The unit of stress is MPa. Calculate σ_x.

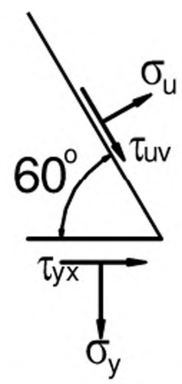


Figure 4.15

Solution

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

Substitute

$$\sigma_y = 3; \tau_{xy} = 5; \tau_{uv} = 6; \alpha = 30^o$$

We have: $\sigma_x = 11.07$ MPa.

2.8. Exercise 5

Determine the principle stresses and principle directions for the element with stress state shown in **Figure 4.16**. The unit of stress is MPa.

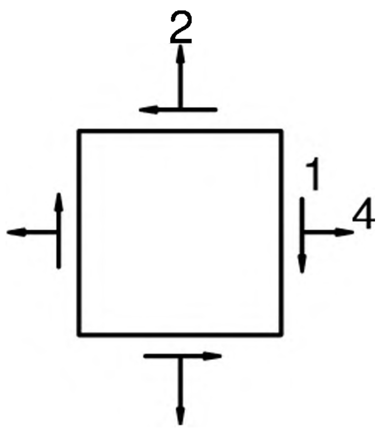


Figure 4.16

Solution

We have:

$$\sigma_x = 4\,; \sigma_y = 2\,; \tau_{xy} = 1 \text{ MPa}$$

The principle directions are determined by the equation:

$$\tan 2\alpha_o = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2(1)}{4-2} = -1$$

$$2\alpha_o = -45^o + k180^o$$

We have two solutions:

$$\alpha_o = -22^o30'; \ \alpha_o = 67^o30'$$

Substitute the found α_o into the formula σ_u, we have:

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\sigma_u = \frac{4+2}{2} + \frac{4-2}{2}\cos 2\alpha - 1\sin 2\alpha$$

$$\alpha_{o1} = -22^o30' \Rightarrow \sigma_u = \sigma_1 = 4.41 \text{ MPa}$$

$$\alpha_{o2} = 67^o30' \Rightarrow \sigma_u = \sigma_2 = 1.58 \text{ MPa}$$

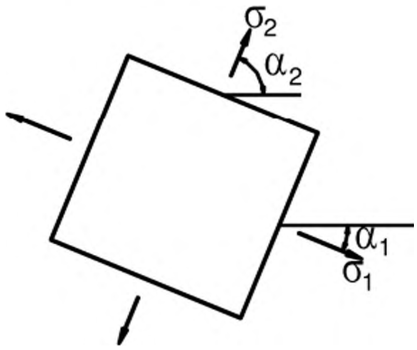


Figure 4.17

Alternatively, the maximum and minimum normal stresses are computed. Then, the angle is checked to know the corresponding stress.

The normal stresses:

$$\sigma_{\substack{\max\\\min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{\substack{\max\\\min}} = \frac{4+2}{2} \pm \sqrt{\left(\frac{4-2}{2}\right)^2 + 1^2} = 3 \pm \sqrt{2} = \begin{cases} 4.41 \\ 1.56 \end{cases} \text{MPa}$$

To know which stress goes with which angle, simply substitute the value of α_o (e.g. $\alpha_o = -22^o30'$) into the formula of σ_u.

2.9. Exercise 6

Figure 4.18 shows the stresses on surfaces. Given $\sigma_y = 3; \tau_{yx} = -5; \tau_{uv} = 6$. The unit of stress is MPa.

a) Calculate σ_x

b) Principle direction

c) Principle stresses

d) Maximum shear stress

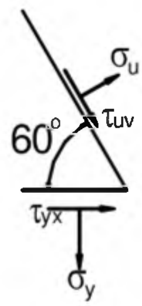


Figure 4.18

Solution

a) Calculate σ_x

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

Substitute $\sigma_y = 3; \tau_{xy} = 5; \tau_{uv} = 6; \alpha = 30^o$ into the formula, we have $\sigma_x = 11.07$ MPa.

b) The principle directions are determined by:

$$\tan 2\alpha_o = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2(5)}{11,07-3} = -1,239$$

We have:

$$\alpha_o = -25,55^o;\ \alpha_o = 64.45^o$$

Substitute the angle α_o above into the formula σ_u, the principle stresses were obtained:

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\sigma_u = \frac{11,07+3}{2} + \frac{11,07-3}{2}\cos 2\alpha - 5\sin 2\alpha$$

$\alpha_o = -25.55^o \Rightarrow \sigma$ = 13.46 MPa (max)

$\alpha_o = 64.45^o \Rightarrow \sigma$ = 0.61 MPa (min)

Determine the minimum and maximum shear stresses.

$\Rightarrow$ **The surface with maximum/minimum shear stress is 45° the principle surface.**

$\alpha = \alpha_o + 45^o = -25.55^o + 45^o \Rightarrow \tau$ = 6.425 MPa (max)

$\alpha = \alpha_o + 45^o = 64.45^o + 45^o \Rightarrow \tau$ = -6.425 MPa (min)

§3. MOHR'S CIRCLE FOR PLANE STRESS STATE

3.1.1. Mohr's circle for plane stress state

Mohr's circle shows the relationship between the normal stress σ and shear stress τ. We have:

$$\sigma_u - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\tau_{uv} = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

The parameter α can be eliminated by squaring two sides of the above equations and then add the equations together. The result is

$$\left(\sigma_u - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{uv}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \quad (4.20)$$

Set: $$c = \frac{\sigma_x + \sigma_y}{2} \quad (4.21)$$

$$R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \quad (4.22)$$

We have: $$(\sigma_u - c)^2 + \tau_{uv}^2 = R^2 \quad (4.23)$$

This is called the Mohr's circle for plane stress state:

- Horizontal axis σ and vertical axis τ.
- Center: C(c,0)
- Radius: R

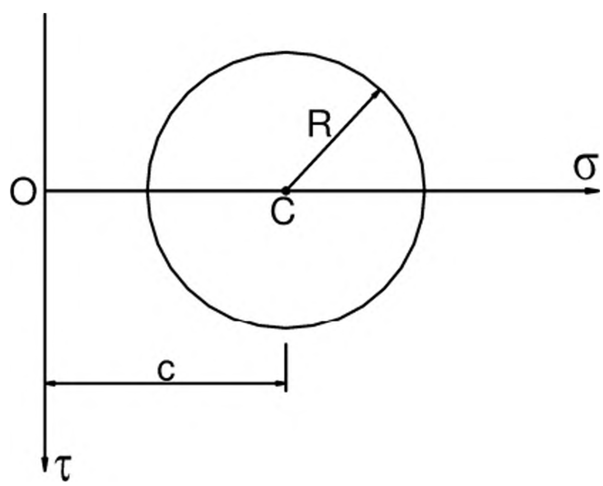


Figure 4.19. Mohr's circle

The coordinate of any point on the Mohr's circle is the normal stress and the shear stress on the surfaces paralled to the z axis.

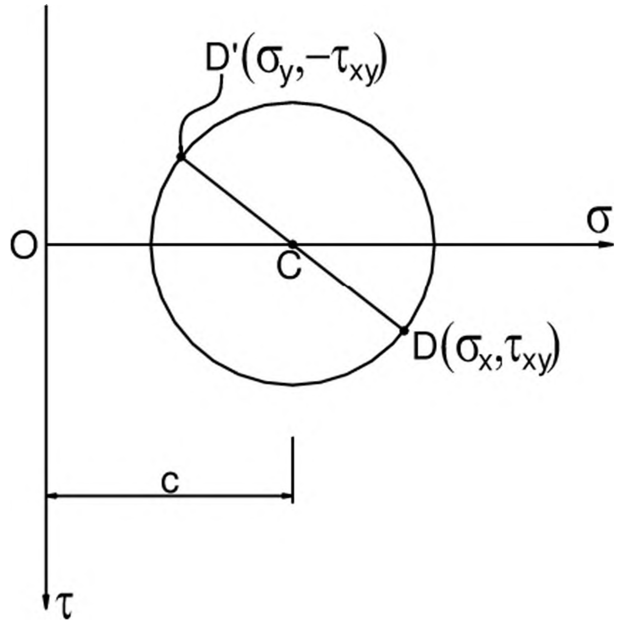


Figure 4.20. Diameter of Mohr's circle

How to draw Mohr's circle:

- Draw the coordinate system $\sigma O \tau$:
  + Horizontal axis $O\sigma$ which is parallel to the axis x
  + Vertical axis $O\tau$ which is parallel to the axis y, the positive direction is downward.
- Draw two points D and D' as shown in **Figure 4.20**:
  + $D(\sigma_x, \tau_{xy})$: The stresses on the surface perpendicular to the axis x ($\alpha = 0$).
  + $D'(\sigma_y, -\tau_{xy})$: The stresses on the surface perpendicular to the axis y ($\alpha = 90^o$).
- Determine the center C: The mid point of DD' or the intersection of DD' and the axis x.
- Draw the circle with the center C and the diameter DD'.

3.1.2. Plane stress transformation using Mohr's circle

Mohr's circle can be used to find the stresses on the surface which the angle between its normal vector u and axis x is α.

The following procedure can be used to find stresses on surfaces using Mohr's circle:

- Determine the point $P(\sigma_y, \tau_{xy})$.
- From P, draw a line paralled to the normal vector u. This line intersects the circle at a point, namely M.
- The coordinate of the point M is the found stresses σ_u and τ_{uv}.

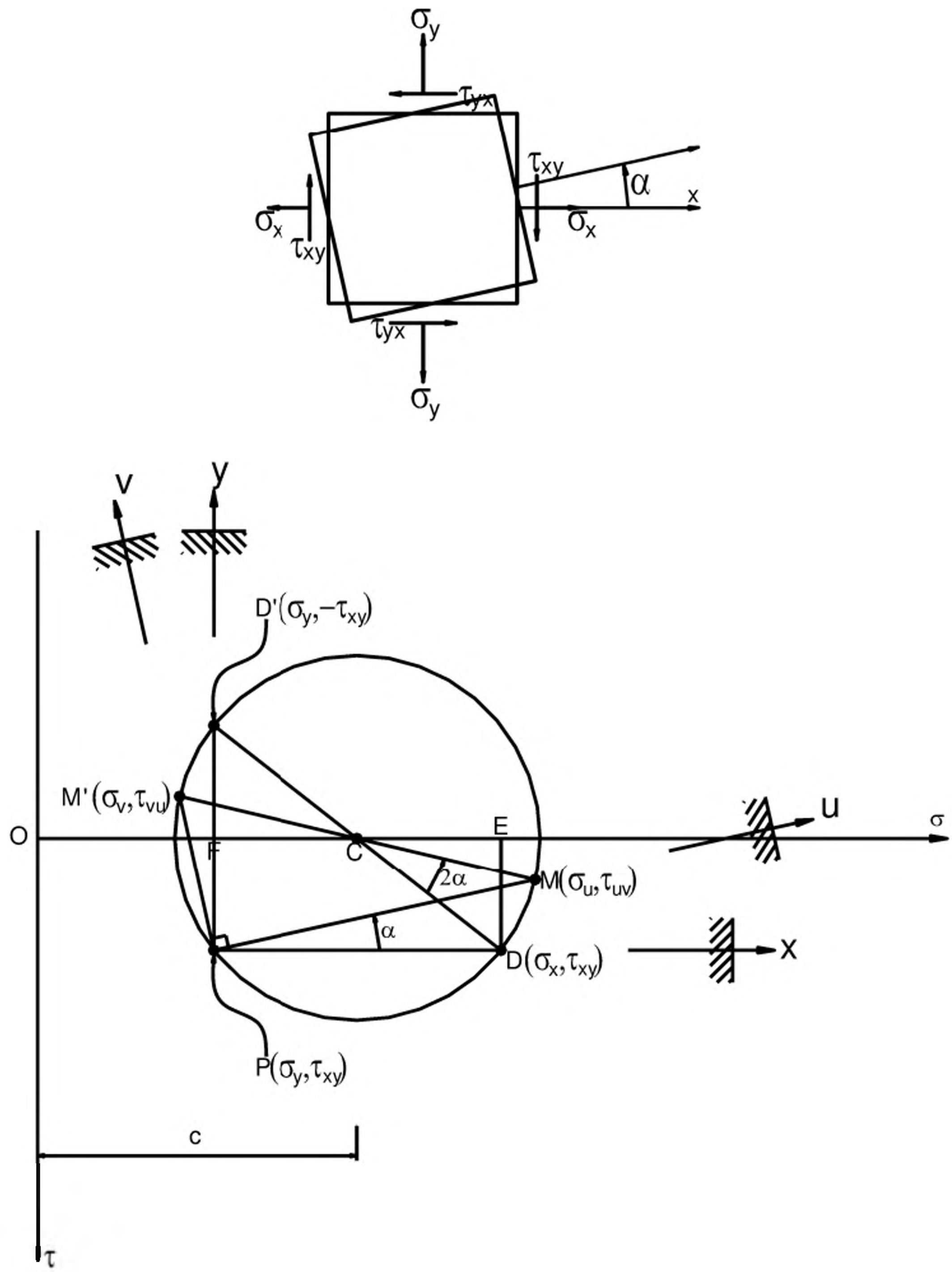


Figure 4.21. Plane stress transformation using Mohr's circle

3.1.3. Principle stress and principle direction

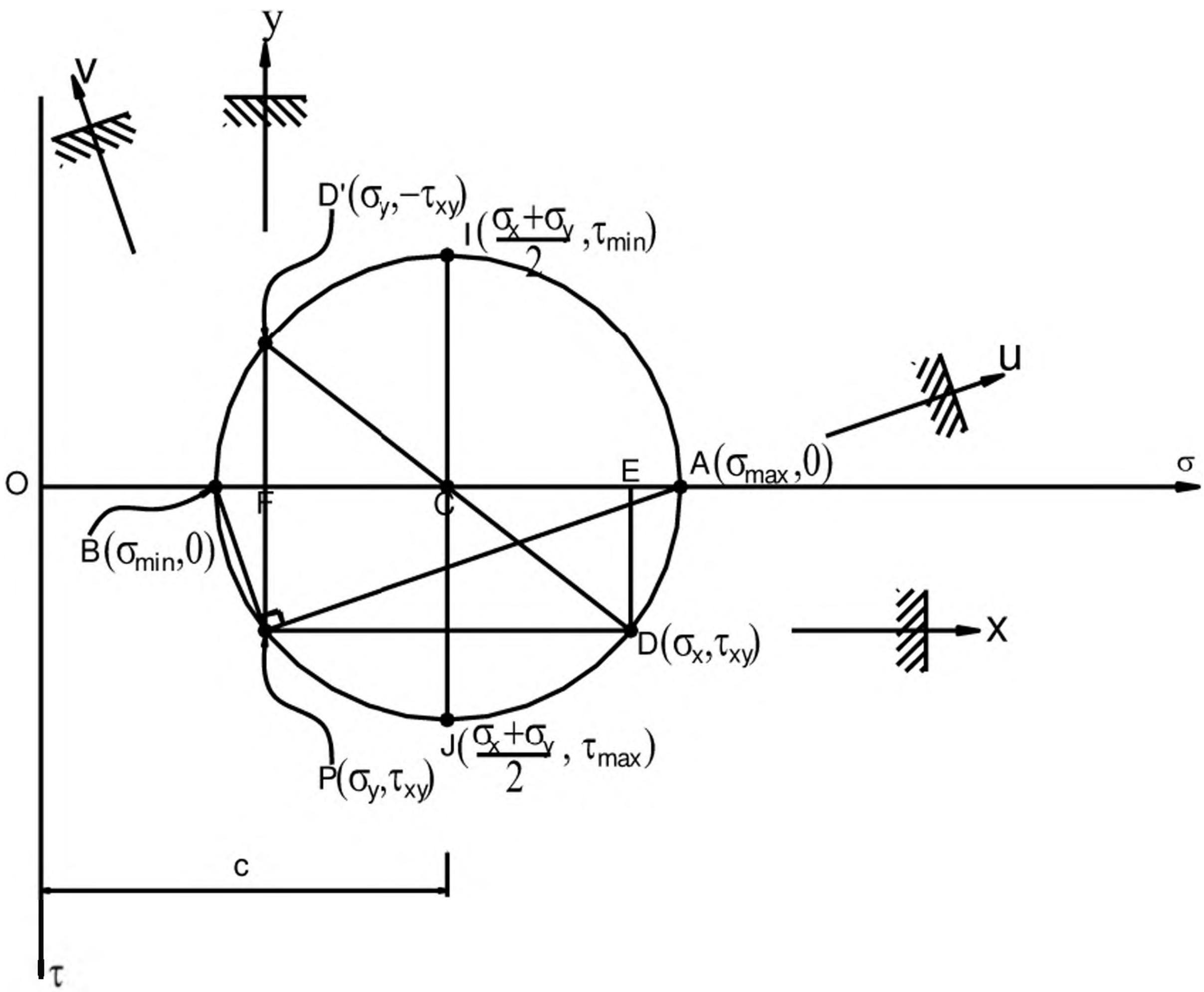


Figure 4.22. Principle stress and principle direction (the normal stresses are maximum and minimum, the shear stress is 0)

In **Figure 4.21**, let α vary $0 \rightarrow 180^{o}$

Consequently, 2α varies $0 \rightarrow 360^{o}$; and M will move one cycle.

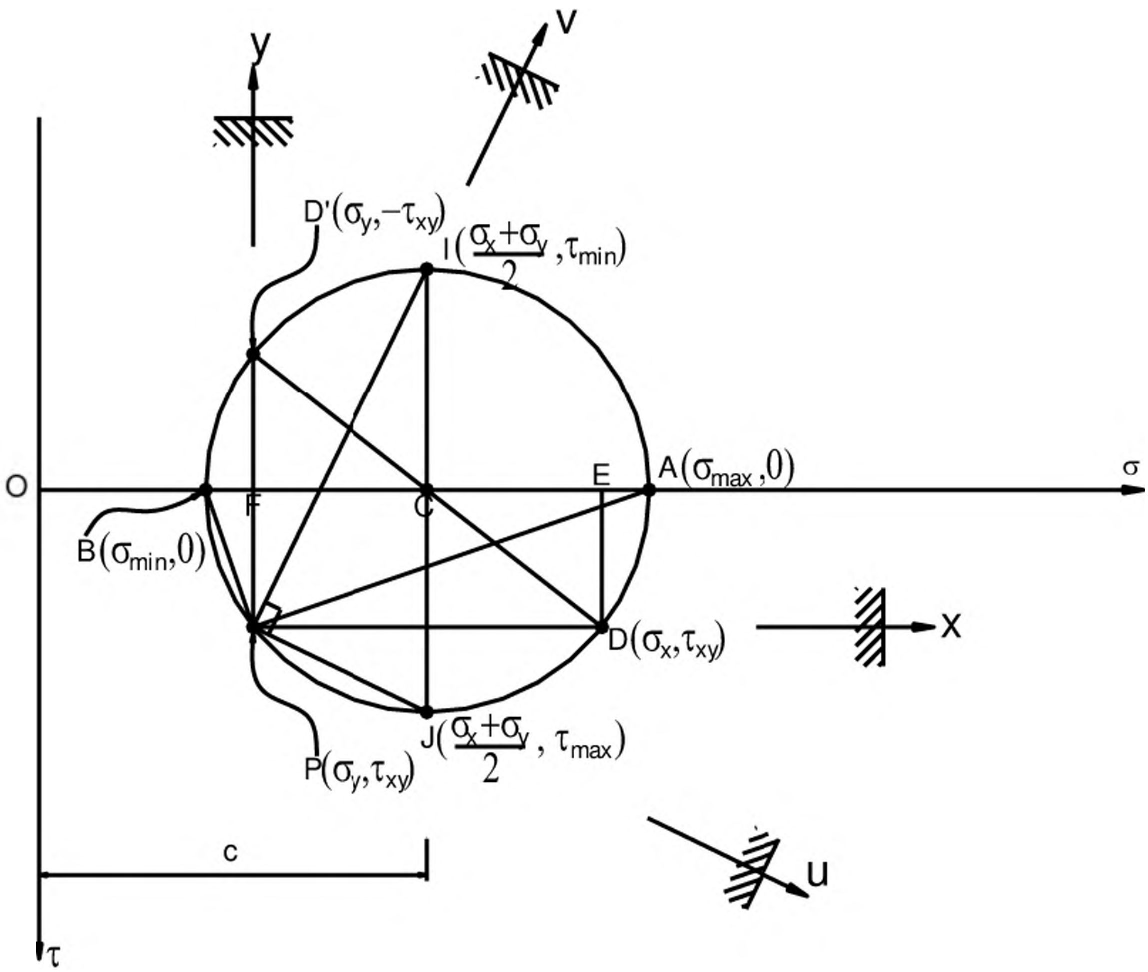


Figure 4.23. Direction and plane with zero shear stress

There are four special points:

- Point $A(\sigma_{\max},0)$: σ is maximum, $\tau = 0$.
- Point $B(\sigma_{\min},0)$: σ is minimum, $\tau = 0$.
- Point $I\left(\dfrac{\sigma_x+\sigma_y}{2},\tau_{\max}\right)$: σ is average, τ is maximum.
- Point $J\left(\dfrac{\sigma_x+\sigma_y}{2},\tau_{\min}\right)$: σ is average, τ is minimum.
- PA and PB are called principle directions.
- The surfaces perpendicular to PA and PB are called principle surfaces/planes.
- PI and PJ are normal vectors of the minimum shear stresses.
- The surfaces with normal vectors PI, PJ have minimum shear stresses.
- By observing the Mohr's circle, we have:

$$\tau_{max} = R\ ;\ \tau_{min} = -R$$

- The angle between the minimum shear stress surfaces and principle surfaces is 45°.

3.1.4. Special cases

Special plane stress state: $\sigma_y = 0$

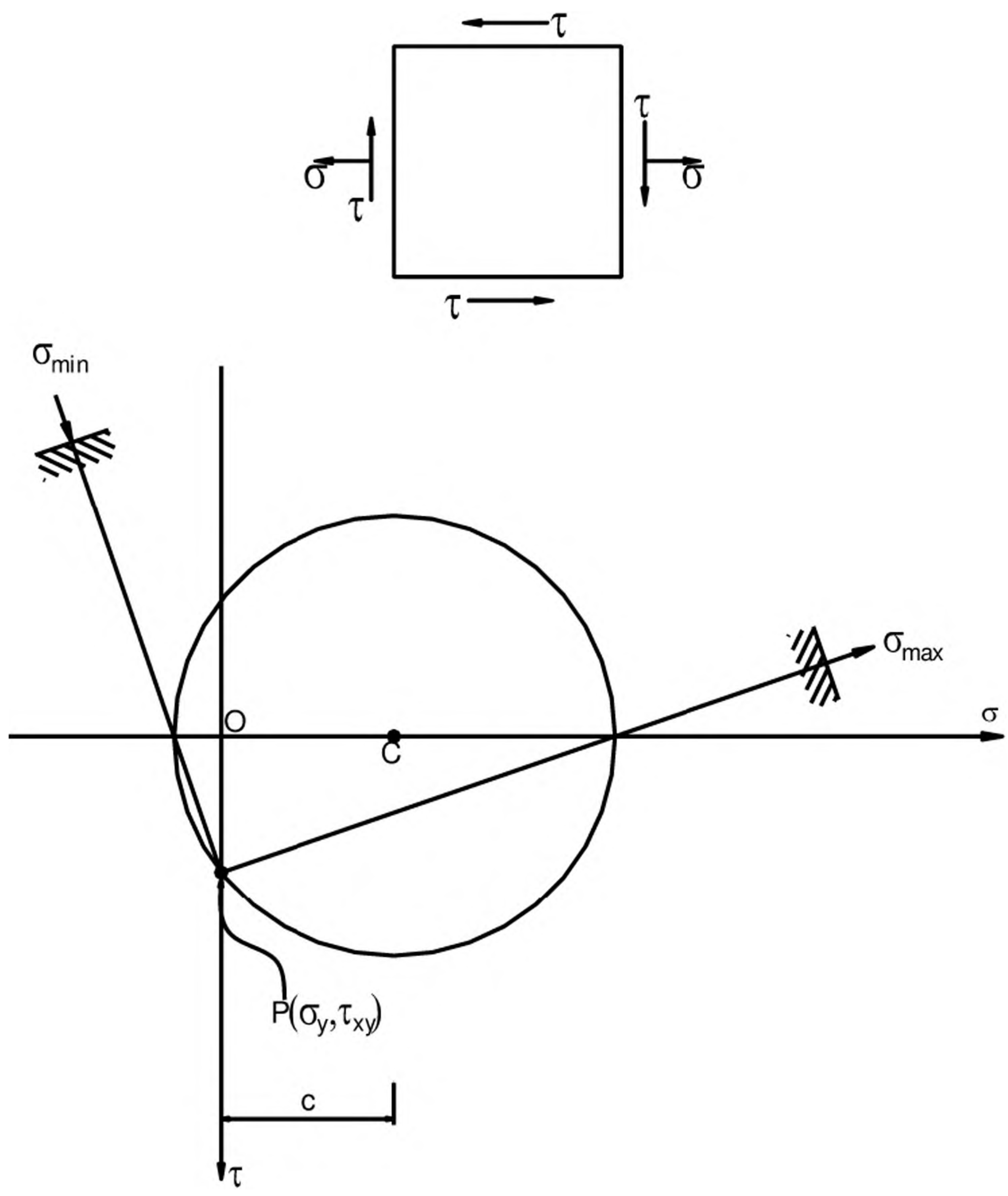


Figure 4.24. Mohr's circle of stress for the element under plane stress state

Pure shear stress state: $\sigma_x = 0,\ \sigma_y = 0$

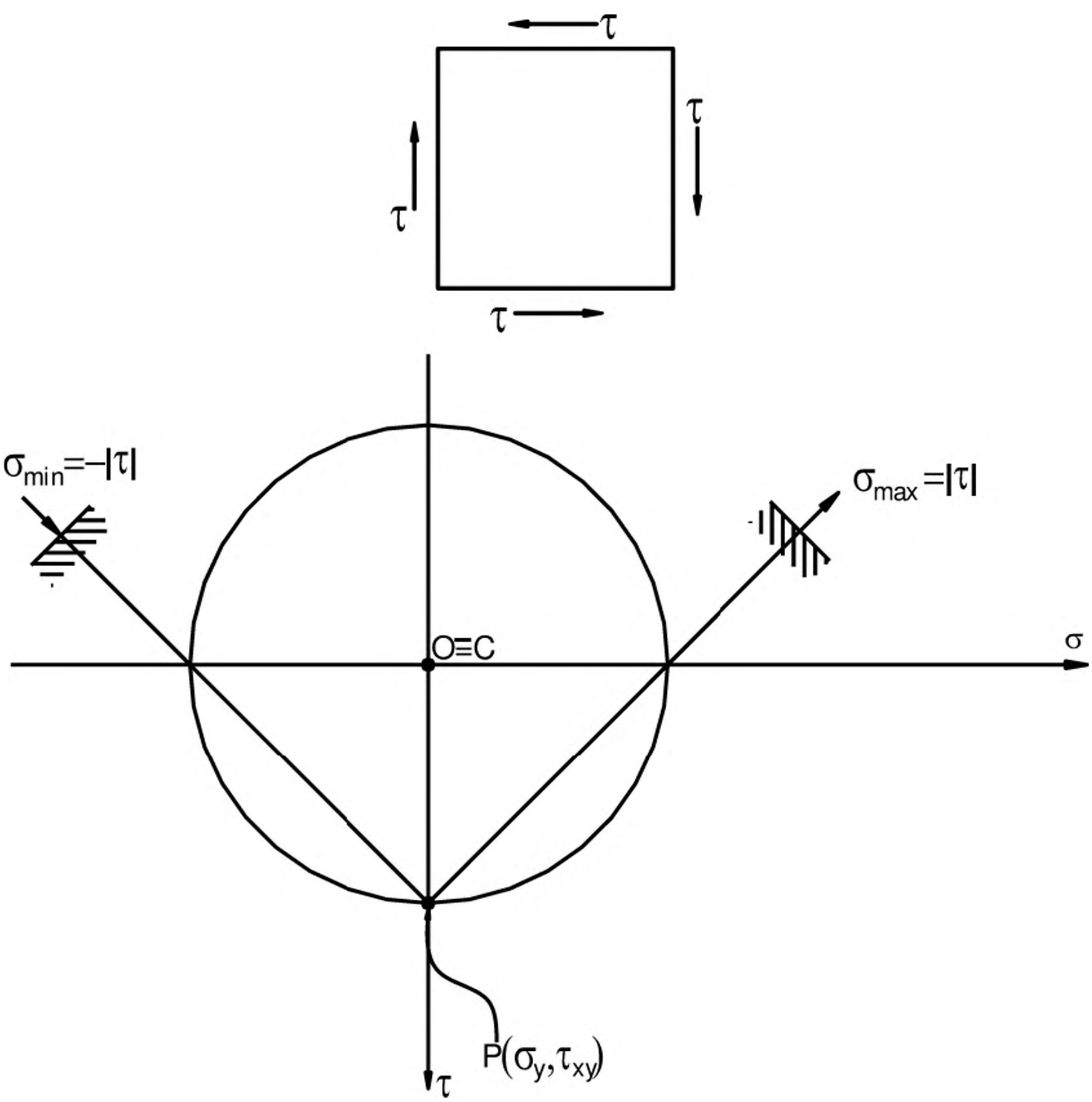


Figure 4.25. Mohr circle of stress for the element under pure shear stress state

$$\Rightarrow \quad \sigma_{max} = |\tau|; \sigma_{min} = -|\tau|$$

3.2. Exercise 7

Consider the element with plane stress state as shown in **Figure 4.26**. The unit of stress is MPa. Using Mohr's circle to determine:

a) Stresses on the surface with $\alpha = 45^o$

b) Principle stresses and principle directions

c) Maximum and minimum stresses

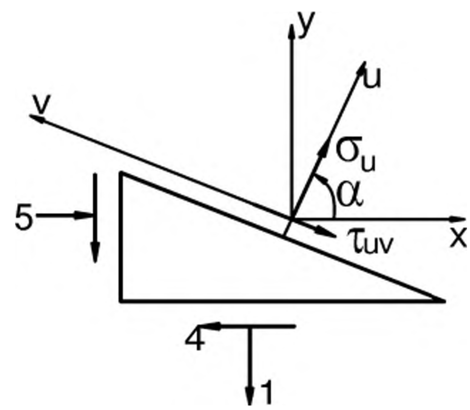


Figure 4.26

Solution

Based on the given information, we have:

$$\sigma_x = -5 \ MPa$$
$$\sigma_y = 1 \ MPa$$
$$\tau_{xy} = -4 \ MPa$$

$$c = \frac{\sigma_x + \sigma_y}{2} = \frac{-5+1}{2} = -2$$

The center of Mohr's circle: (-2,0)

The radius of Mohr's circle:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-5-1}{2}\right)^2 + (-4)^2} = 5 \ MPa$$

$D(\sigma_x, \tau_{xy})$ represents the stresses on the surface perpendicular to x axis ($\alpha = 0$).

$D'(\sigma_y, -\tau_{xy})$ represents the stresses on the surface perpendicular to y axis ($\alpha = 90^o$).

$P(\sigma_y, \tau_{xy})$ or $P(1, -4)$

a) From the point P, draw the line paralleled to u axis. This line intersects with the Mohr's circle at M. The coordinate of the point M represents the stresses on the surface with $\alpha = 45^o$.

$$\sigma_u = -6 \ MPa$$
$$\tau_{uv} = 3 \ MPa$$

b) The coordinates of the points A, B represent the principle stresses:

$$\sigma_1 = \sigma_A = 3 \ MPa$$
$$\sigma_3 = \sigma_B = -7 \ MPa$$

Two principle directions are determined by the angles:

$$\alpha_o^{(1)} = -67^o 42'$$

$$\alpha_o^{(3)} = -26^o 36'$$

c) The coordinates I and J correspond to the maximum and minimum shear stresses:

$$\tau_{\max} = 5kN / cm^2$$

$$\tau_{\min} = -5kN / cm^2$$

These stresses are on the inclined surface with the following angles:

$$\alpha_1^{(1)} = 71^o 36'$$

$$\alpha_1^{(3)} = 161^o 36'$$

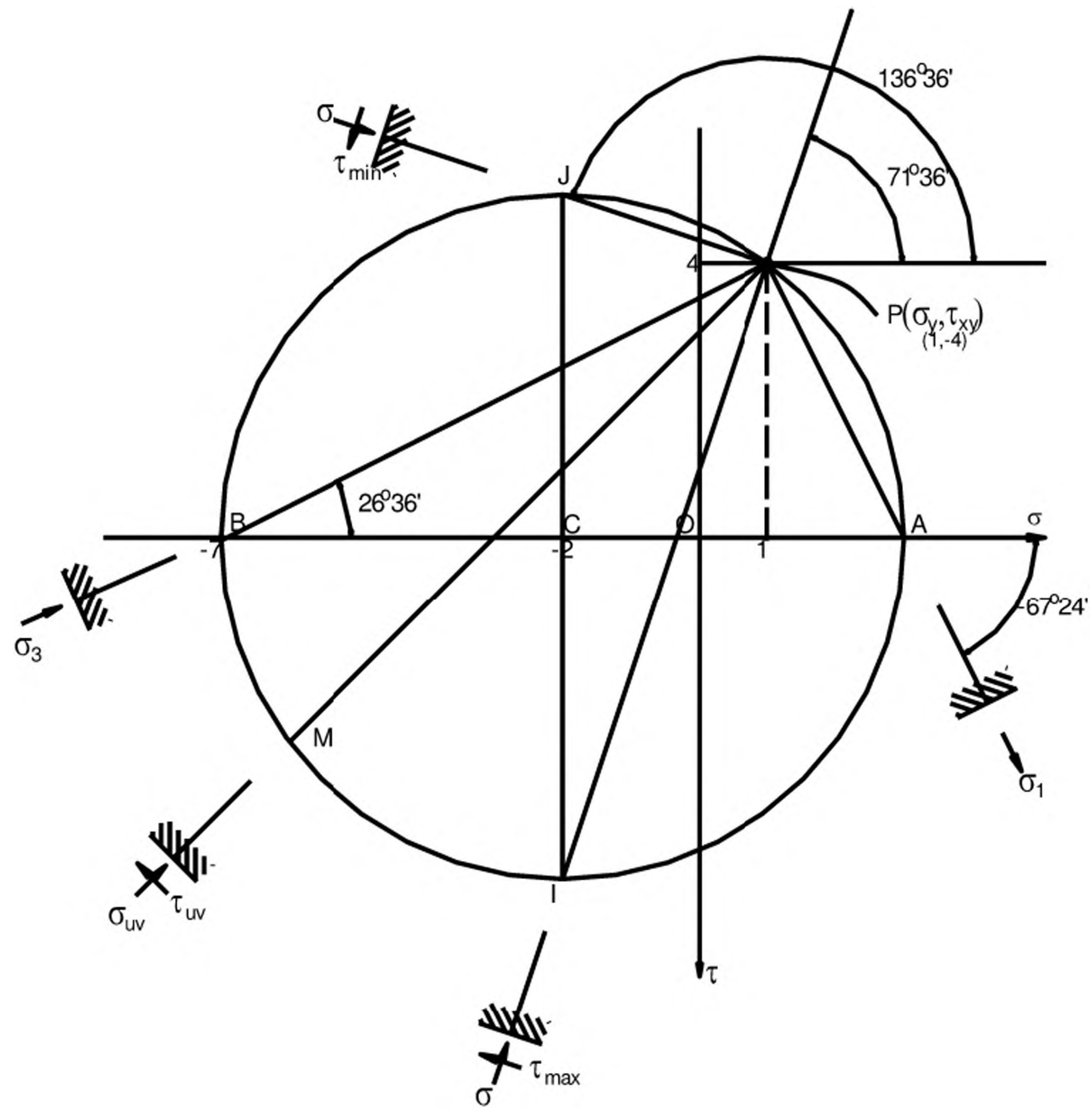


Figure 4.27. Principle surfaces

3.3. Exercise 8

Consider the element with stresses shown in **Figure 4.28**. The unit of stress is MPa.

a) Draw the Mohr's circle

b) Find the principle stresses

c) Find the principle directions

d) Find the extreme of shear stresses

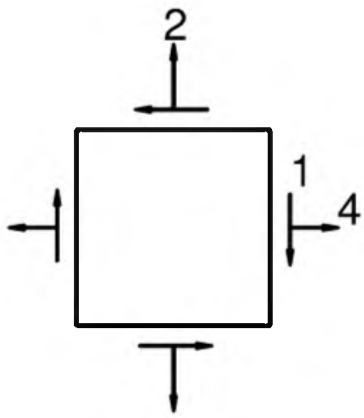


Figure 4.28

Solution

Based on the given information, we have:

$$\sigma_x = 4MPa$$

$$\sigma_y = 2MPa$$

$$\tau_{xy} = 1MPa$$

a) Draw the Mohr's circle

$$c = \frac{\sigma_x + \sigma_y}{2} = \frac{4+2}{2} = 3 \text{ MPa}$$

The center of Mohr's circle: (3,0)

The radius of Mohr's circle:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{4-2}{2}\right)^2 + (1)^2} = \sqrt{2} \text{ MPa}$$

b) Find the principle stresses

$$\sigma_{\max} = c + R = 3 + \sqrt{2}MPa$$

$$\sigma_{\min} = c - R = 3 - \sqrt{2}MPa$$

c) Find the principle directions

$$tg\alpha_1 = \frac{1}{1+\sqrt{2}} \Rightarrow \alpha_1$$

$$\alpha_2 = \alpha_1 + 90^o$$

d) Find the extreme of shear stresses

$$\tau_{max} = R = \sqrt{2} \text{ MPa}$$

$$\tau_{min} = -R = -\sqrt{2} \text{ Mpa}$$

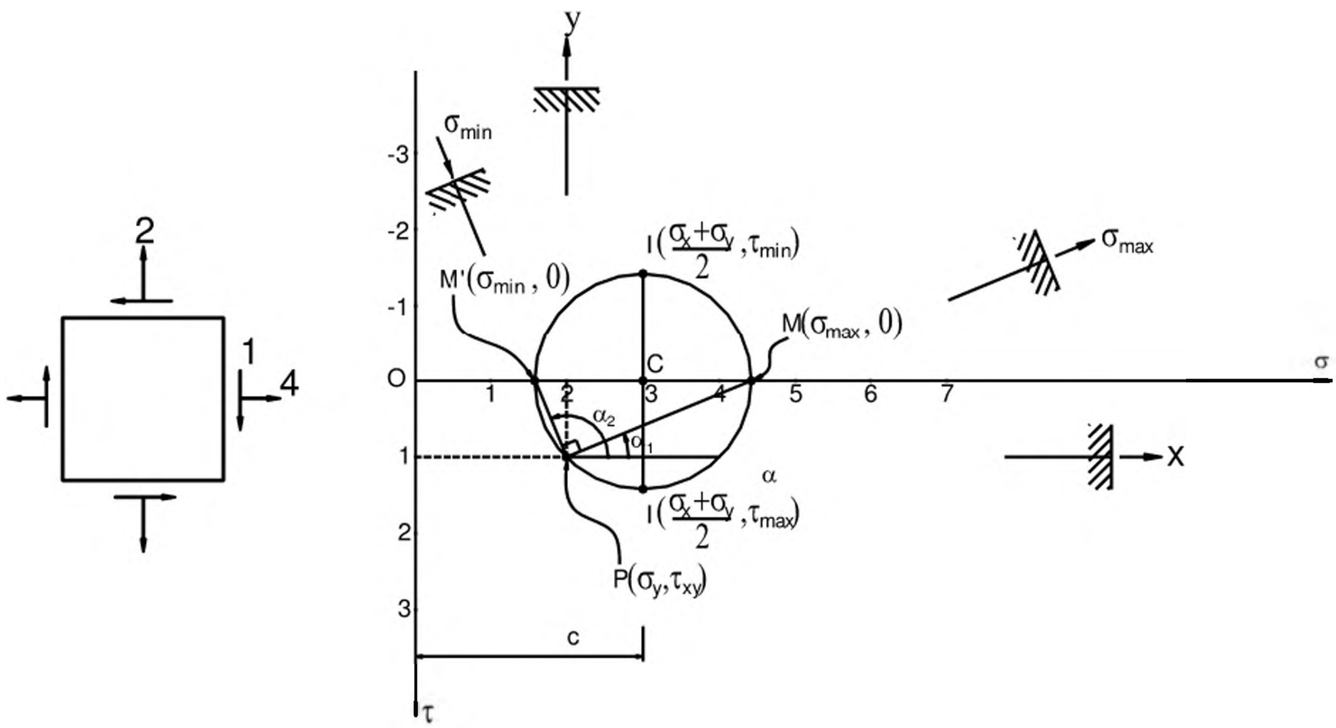


Figure 4.29

§4. RELATIONSHIP BETWEEN STRESS AND STRAIN

4.1. General Hooke's law

The general Hooke's law includes six equations: Three equations express the relationship between the strain ε and normal stress σ and three equations express the relationship between the strain γ and shear stress τ.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \tag{4.24}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \tag{4.25}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \tag{4.26}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{4.27}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{4.28}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} \tag{4.29}$$

Relation between the modulus of elasticity E and the shear modulus G:

$$G = \frac{E}{2(1+\nu)} \tag{4.30}$$

in which, ν is Poisson's ratio

4.2. Volumetric Hooke's law

Consider a principle element with the sides da_1, da_2, da_3.

The original volume:

$$V_o = da_1 da_2 da_3 \tag{4.31}$$

Under the principle stresses, the sides da_1, da_2, da_3 deform Δda_1, Δda_2, Δda_3, respectively.

The volume after deformation is:

$$V_1 = \left(da_1 + \Delta da_1\right)\left(da_2 + \Delta da_2\right)\left(da_3 + \Delta da_3\right) \tag{4.32}$$

By eliminating the high order terms, the above equation becomes:

$$V_1 = da_1 da_2 da_3\left(1 + \frac{\Delta da_1}{da_1} + \frac{\Delta da_2}{da_2} + \frac{\Delta da_3}{da_3}\right) \tag{4.33}$$

$$V_1 = V_o\left(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3\right) \tag{4.34}$$

Relative change of volume:

$$\frac{V_1 - V_o}{V_o} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{4.35}$$

The above formula is correct for all materials whether they follow Hooke's law or not. The above formula also shows that angular strain does not affect volumetric strain.

For elastic materials that comply with Hooke's law, substitute the formulas ε into the above formula, we have:

$$\theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) \tag{4.36}$$

set: $\Sigma = (\sigma_1 + \sigma_2 + \sigma_3)$

$$\Sigma = \frac{E}{1-2\nu}\theta \tag{4.37}$$

This is the formula of volumetric Hooke's law representing the linear relationship between total normal stress and relative volumetric strain.

The above formula shows that θ depends on the total normal stresses while it does not depend on the individual normal stresses. If normal stresses are replaced by mean stress, the relative volumetric strain θ remains constant.

$$\sigma_{ave} = \frac{\Sigma}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{4.38}$$

Thus, the relative volumetric strain θ can be computed based on the principle stresses:

$$\theta = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1-2\nu}{E}\Sigma \tag{4.39}$$

Or, the relative volumetric strain θ can be computed based on the average of principle stresses:

$$\theta = \frac{1-2\nu}{E}(\sigma_{tb} + \sigma_{tb} + \sigma_{tb}) = \frac{1-2\nu}{E}\Sigma \tag{4.40}$$

Meaning: For a cube element, in the two cases above (Equation 4.34 and 4.35), relative volumetric strain θ are equal. However, there are differences as follows:

In the first case: Both volume and shape change. The element changes to a boxy element.

In the second case: Only the volume changes, the shape does not change. The element still in a cube shape.

§5. ELASTIC STRAIN ENERGY

In Chapter 3, we have specific elastic strain energy of the element in the state of single stress:

$$u = \frac{\varepsilon\sigma}{2} \tag{4.41}$$

Using the principle of superposition, the elastic strain energy of an element in the state of 3D stress state is:

$$u = \frac{\varepsilon_1\sigma_1}{2} + \frac{\varepsilon_2\sigma_2}{2} + \frac{\varepsilon_3\sigma_3}{2}$$

$$u = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]\frac{\sigma_1}{2} + \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right]\frac{\sigma_2}{2} + \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]\frac{\sigma_3}{2}$$

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^3 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \tag{4.42}$$

This energy is classified into two components:

$$u = u_{volume} + u_{shape} \tag{4.43}$$

in which:

u_{volume} is the energy due to change of volume.

u_{shape} is the energy due to change of shape.

The given stress state is equal to the total two stress states:

Stress state 1: Element is subjected to 3 average principle stresses $\sigma_{tb} = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ $\Rightarrow$ change of volume.

Stress state 2: Element is subjected to 3 stresses $\sigma_1 - \sigma_{tb}$, $\sigma_2 - \sigma_{tb}$, $\sigma_3 - \sigma_{tb} \Rightarrow$ change of shape.

Thus:

$$u_{volume} = 3\frac{1-2\nu}{2E}\sigma_{tb}^2 = \frac{1-2\nu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 \tag{4.44}$$

u_{hd} can be calculated by the two following methods: $u_{hd} = u - u_{tt}$ or substitute the 3 stresses of the stress state 2 into the formula to calculate u, we get:

$$u_{shape} = \frac{1+\nu}{3E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right) \tag{4.45}$$

In the simplest case, when the element is under a simple stress state: $\sigma_1 = \sigma$, $\sigma_2 = 0$, $\sigma_3 = 0$, we have:

$$u = \frac{\sigma^2}{2E}\sqrt{a^2 + b^2} \tag{4.46}$$

$$u_{tt} = \frac{1-2\nu}{6E}\sigma^2 \tag{4.47}$$

$$u_{hd} = \frac{1+\nu}{3E}\sigma^2 \tag{4.48}$$

PROBLEMS

PROBLEM 1. Calculate the normal stress and shear stress on the surfaces shown in **Figure 4.30**. The unit of stress is MPa.

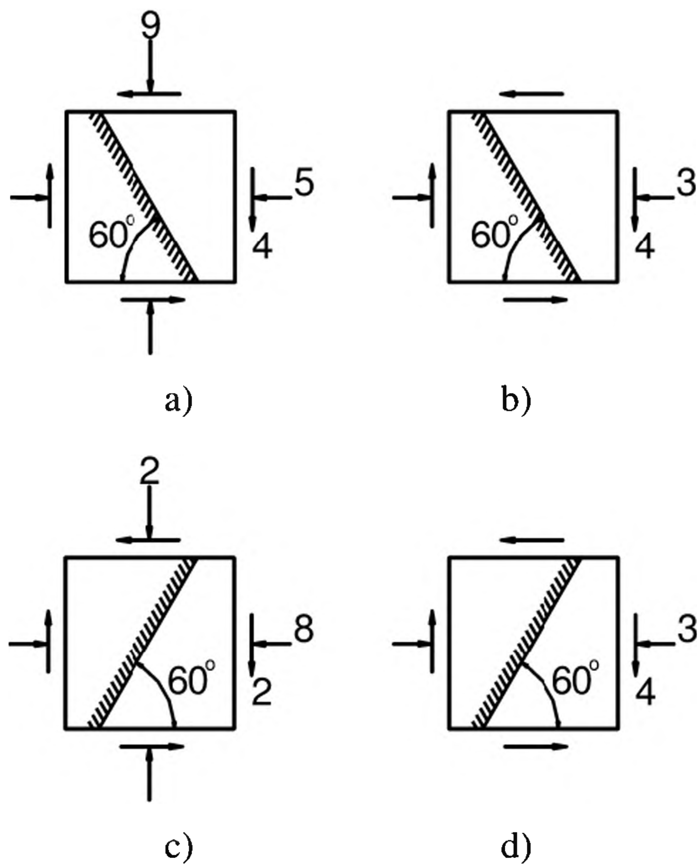


Figure 4.30

PROBLEM 2. Consider **Figure 4.31**. Given: $\sigma_y = 10; \tau_{yx} = -5; \tau_{uv} = 3$. The unit of stress is MPa. Calculate σ_x.

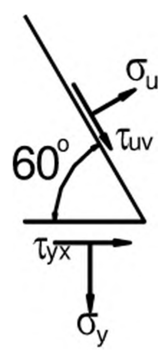


Figure 4.31

PROBLEM 3. Determine the principle stresses and principle directions for the element with stress state shown in **Figure 4.32**. The unit of stress is MPa.

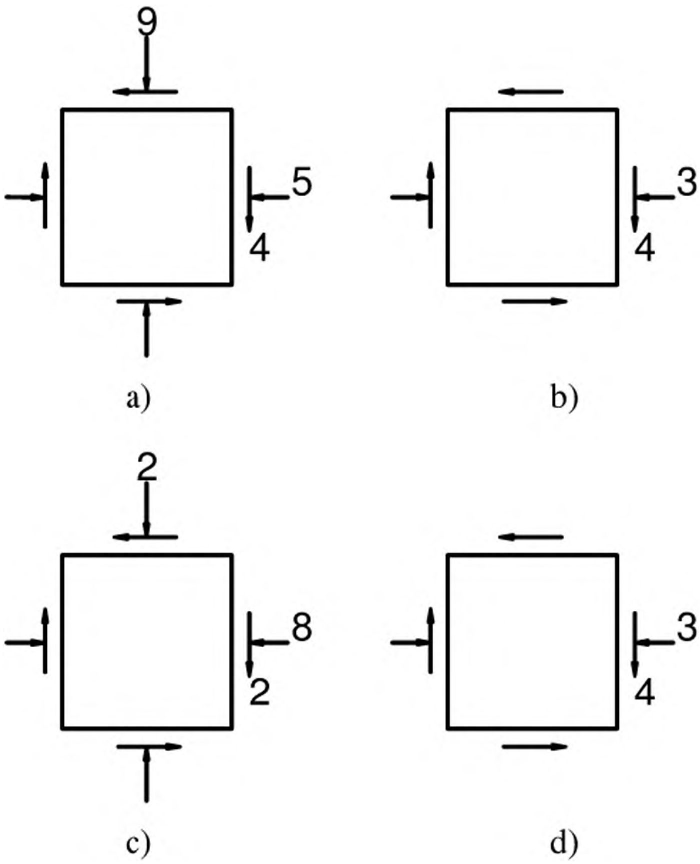


Figure 4.32

PROBLEM 4. Consider the element with plane stress state as shown in **Figure 4.33**. The unit of stress is MPa. Using Mohr's circle to determine:

a) Stresses on the surface with $\alpha = 45^o$

b) Principle stresses and principle directions

c) Maximum and minimum stresses

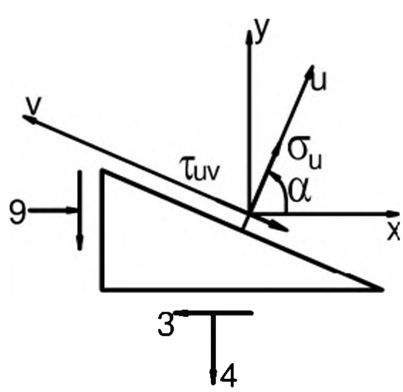


Figure 4.33

PROBLEM 5. Consider the element with stresses shown in **Figure 4.34**. The unit of stress is MPa.

a) Draw the Mohr's circle

b) Find the principle stresses

c) Find the principle directions

d) Find the extreme of shear stresses

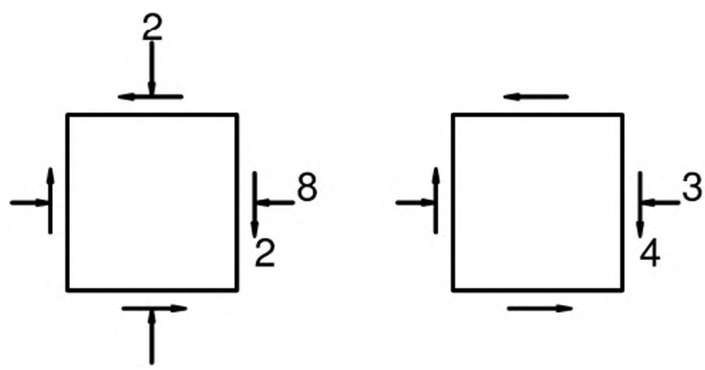


Figure 4.34

Chapter **5**

FAILURE THEORIES

§1. CONCEPTS

In case of axial loading in Chapter 3, the working condition is:

$$\sigma_{max} = \sigma_1 \le [\sigma]_{tension} \tag{5.1}$$

$$|\sigma_{min}| = |\sigma_3| \le [\sigma]_{compression} \tag{5.2}$$

$[\sigma]_{tension}$ and $[\sigma]_{compression}$ are allowable tension and compression stresses:

$$[\sigma] = \frac{\sigma_o}{n} \tag{5.3}$$

In which:

$[\sigma]$ is allowable stress.

n is a factor of safety.

Ductile material: $\sigma_o = \sigma_y$

Brittle material: $\sigma_o = \sigma_y$

To check the stress condition of a point in the complex stress state, we need to do experiments in similar destructive conditions, but it is difficult and infeasible because:

- Ultimate stress depends on not only the magnitude of the principle stresses but also the ratio of these stresses. Thus, the number of experiments is large.
- Three-dimensional compression testing requires sophisticated equipment while tension and compression testing is very popular due to its simplicity.

Therefore, it is necessary to have theories to evaluate the causes of material failures, which is known as *failure theories.*

Failure theory is the theory about causes of material failure; it does not depend on the stress state of the material, so that we can assess the durability of the materials at every state of stress when the mechanical properties of materials are known based on tension/compression tests.

That is, any state with principle stresses σ_1, σ_2 and σ_3, the stress σ_t, which is a function of σ_1, σ_2 and σ_3, is computed and used compared with the allowable stress $[\sigma]_{tension}$ or $[\sigma]_{compression}$ determined from tension/compression tests.

So, the condition is:

$$\sigma_t = f(\sigma_1, \sigma_2, \sigma_3) \leq [\sigma] \tag{5.4}$$

σ_t the equivalent stress or theoretical stress based on failure theory.

The function f depends on failure theory.

§2. FAILURE THEORIES

2.1. Failure theory based on maximum normal stress (Failure theory 1)

The failure theory based on maximum normal stress (Failure theory 1) is also known as maximum principle stress. This failure theory was developed by W.J.M. Rankine (1820-1872). Therefore, this failure theory is also called Rankine's theory. This is the simplest and oldest theory of failure. According to this failure theory, materials fail when the the maximum principle stress reaches the tension strength of materials under axial tension or the minimum principle stress reaches the compression strength of materials under axial tension. These conditions are shown in Equations (5.5) and (5.6).

$$\sigma_{t1} = \sigma_1 \leq \frac{\sigma_o}{n} = [\sigma]_{tension} \tag{5.5}$$

$$\sigma_{t1} = |\sigma_3| \leq \frac{\sigma_o}{n} = [\sigma]_{compression} \tag{5.6}$$

in which, σ_{t1} is the equivalent stress or stress based on the failure theory 1.

2.2. Failure theory based on maximum normal strain (Failure theory 2)

Failure theory based on maximum normal strain (Failure theory 2) is sometimes called maximum principle strain theory. This theory of failure was proposed by Saint Venant; therefore, the theory is also called Saint Venant's theory. According to this theory, materials fail when the maximum principle strain reaches the yield strain in the axial tension tests. This condition is expressed by Equation (5.7).

$$\sigma_{t2} = \sigma_1 - \nu(\sigma_2 + \sigma_3) \leq \frac{\sigma_o}{n} = [\sigma]_{tension} \tag{5.7}$$

2.3. Failure theory based on maximum shear stress (Failure theory 3)

Failure theory based on maximum shear stress (Failure theory 3) was proposed by Henri Edouard Tresca (1814-1885). According to this theory, materials fail when the maximum shear stress reaches the yield stress in axial tension tests. The condition is expressed by Equation (5.8).

$$\sigma_{t3} = \sigma_1 - \sigma_3 \le [\sigma]_{tension} \tag{5.8}$$

2.4. Failure theory based on maximum shear strain energy (Failure theory 4)

Failure theory based on maximum shear strain energy (Failure theory 4) was proposed by Richard von Mises (1883-1953); therefore, it is also called von Mises's theory. Materials fail when the maximum shear strain energy per unit volume reaches the maximum shear strain energy at yield in the axial tests.

$$\sigma_{t4} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \le [\sigma]_{tension} \tag{5.9}$$

Failure theory based on ultimate stress state (Failure theory 5 or Mohr theory)

This theory was developed based on experiments.

Condition:

$$\sigma_{t5} = \sigma_1 - \alpha\sigma_3 \le [\sigma]_{tension} \tag{5.10}$$

In which: $\alpha = \dfrac{[\sigma]_{tension}}{[\sigma]_{compression}}$ (5.11)

§3. APPLICATIONS OF THE FAILURE THEORIES

- 1-D stress state: Use failure theory 1.
- 2-D or 3-D stress state: Use failure theory 2 for brittle materials; use failure theory 3, 4 for ductile materials.

3.1. Exercise 1

Write the stress condition based on the failure theories of shear stress and elastic strain energy for the plane stress state as shown in **Figure 5.1**.

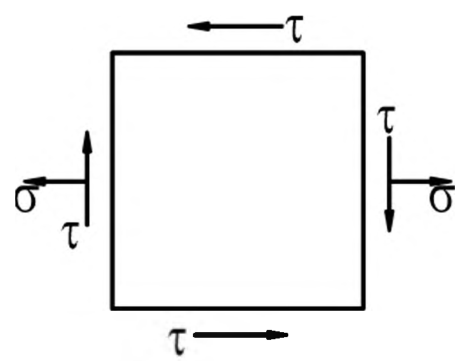


Figure 5.1

Solution

We have:

$$\sigma_{1,3} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \tag{5.12}$$

$$\sigma_2 = 0 \tag{5.13}$$

Apply the failure theory 3:

$$\sigma_{t3} = \sigma_1 - \sigma_3 = \sqrt{\sigma^2 + 4\tau^2} \le [\sigma] \tag{5.14}$$

Apply the failure theory 4:

$$\sigma_{t4} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \le [\sigma] \tag{5.15}$$

$$\sigma_{t4} = \sqrt{\sigma^2 + 3\tau^2} \le [\sigma] \tag{5.16}$$

3.2. Exercise 2

Write the stress condition based on the failure theories of shear stress and elastic strain energy for the pure shear stress state as shown in **Figure 5.2**.

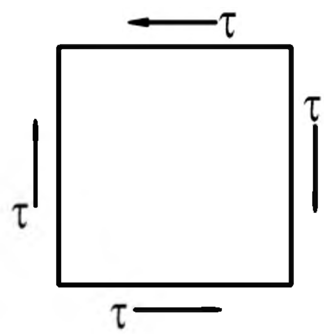


Figure 5.2

Solution

We have:

$$\sigma_1 = -\sigma_3 = |\tau|; \sigma_2 = 0 \tag{5.17}$$

Apply the failure theory 3:

$$\sigma_{t3} = \sigma_1 - \sigma_3 = 2|\tau| \le [\sigma] \tag{5.18}$$

Apply the failure theory 4:

$$\sigma_{t4} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \le [\sigma] \tag{5.19}$$

$$\sigma_{t4} = \sqrt{3\tau^2} \le [\sigma] \tag{5.20}$$

Chapter ***6***

GEOMETRIC PROPERTIES OF CROSS SECTIONS

§1. CONCEPTS

When a bar is subjected to axial force, the stress depends only on the cross-sectional area A. When a member is subjected to bending, twisting, etc., the stress depends not only on the cross-section A, but also the shape and layout of the cross-section. In other words, the stress depends on factors called *geometric properties of cross sections*.

Consider a cantilever beam subjected to a vertical point load P at the end of the beam as shown in **Figure 6.1**. It is easy to know that deflection in **Figure 6.1a** is larger than that in **Figure 6.1b**. In other words, the member in **Figure 6.1b** is stiffer than that in **Figure 6.1a**. Therefore, in addition to the cross-sectional area A, there are other geometric properties of the cross section, which are presented in the next lessons.

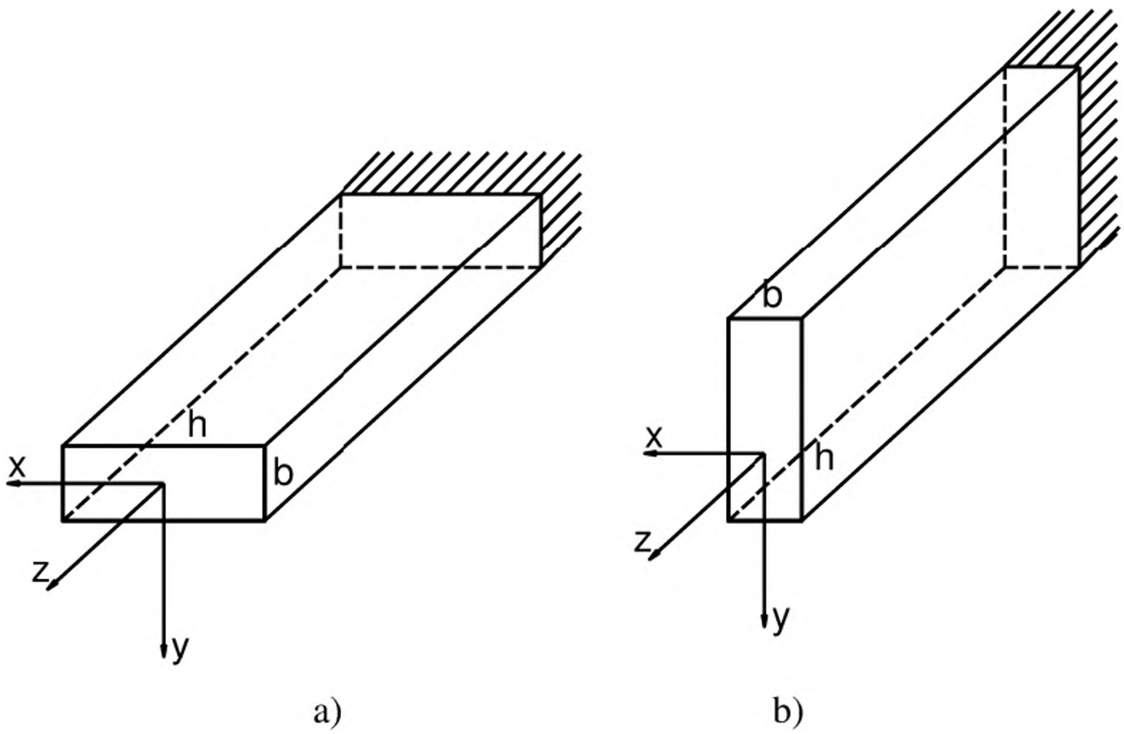


Figure 6.1. Illustration of geometric properties of cross sections

§2. FIRST MOMENT OF CROSS SECTIONS ABOUT AN AXIS

Consider an arbitrary section in the coordinate system Oxy as shown in **Figure 6.2.**

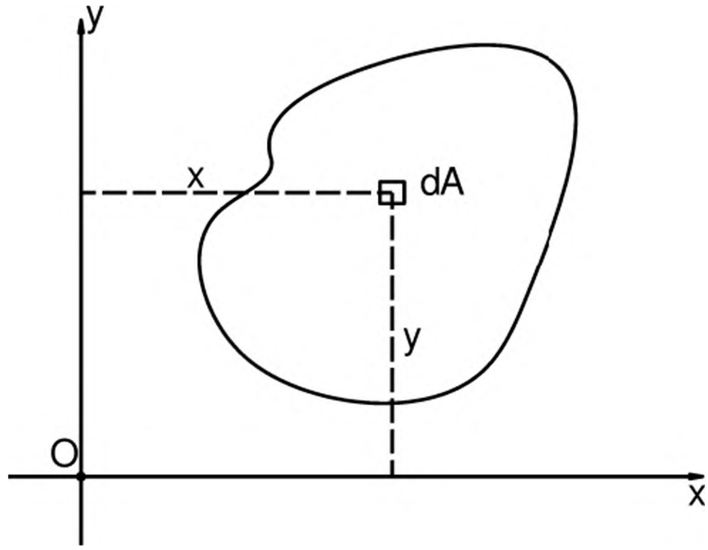


Figure 6.2. First moment of a cross section about an axis

Set Sx, Sy be the ***first moments*** of a cross section A about an axis Ox and Oy, respectively. These first moments are defined by Equation 6.1:

$$S_x = \int_A ydA; \quad S_y = \int_A xdA \tag{6.1}$$

x, y can be negative or positive; thus, S_x, S_y can also be negative or positive.

Unit of first moment is *[length]*^{3}.

If $S_u = 0$, the axis u is called ***central axis*** u.

The ***symmetric axis*** is a central axis because the first moment about the symmetry axis will be zero.

Centroid is the intersection of at least two central axes.

First moment of a cross section about any central axis is 0. From this, we can find the center as follows:

- Draw Cx'y' axis system going through the center C and paralleling to the x, y axes of the original axis system Oxy.

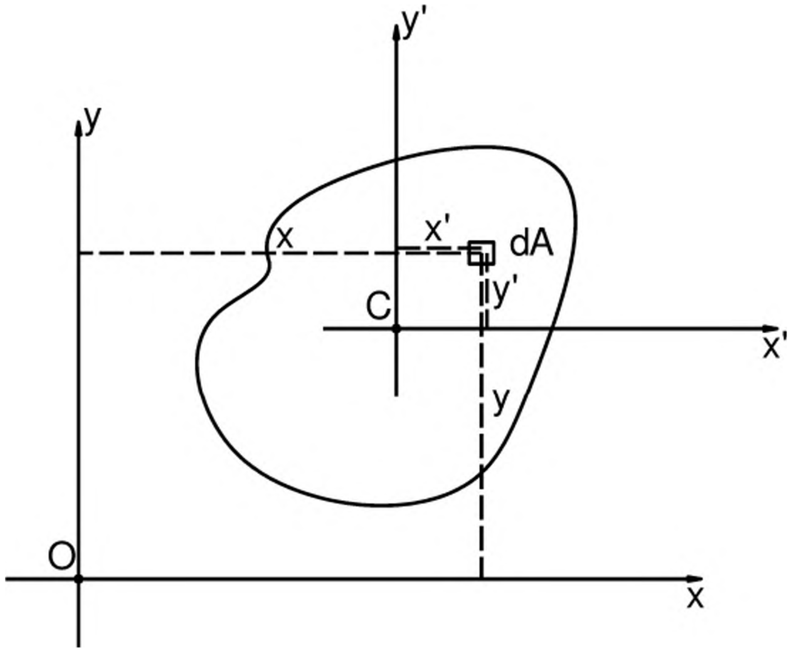


Figure 6.3. First moment of a cross section about different axes

We have:

$$x' = x - x_C\,;\;\; y' = y - y_C \tag{6.2}$$

or

$$x = x_C + x'\,;\;\; y = y_C + y' \tag{6.3}$$

Thus:

$$S_x = \int_A y dA = \int_A (y_C + y')dA = \int_A y_C dA + \int_A y' dA = y_C A + S_{x'} \tag{6.4}$$

x' is a central axis, so $S_{x'} = 0$

Thus: $$S_x = y_C A \tag{6.5}$$

Similarly: $$S_y = x_C A \tag{6.6}$$

The coordinate of the centroid C:

$$x_C = \frac{S_y}{A} \tag{6.7}$$

$$y_C = \frac{S_x}{A} \tag{6.8}$$

Thus, the first moment of a complex cross section is equal to the algebraic sum of first moments of simple cross sections.

When calculating the first moment of a complex cross section, we divide the cross section into many simple areas, and then algebraically sum first moments of simple areas.

$$S_x = A_1 y_1 + A_2 y_2 + ... + A_n y_n = \sum_{i=1}^{n} A_i y_i \tag{6.9}$$

$$S_y = A_1 x_1 + A_2 x_2 + ... + A_n x_n = \sum_{i=1}^{n} A_i x_i \tag{6.10}$$

in which,

A_i are area of simple area i

x_i, y_i are the coordinates of the centroid of simple area i

n is number of simple areas.

The centroid coordinate of a complex cross section:

$$x_C = \frac{S_y}{A} = \frac{\sum_{i=1}^{n} A_i x_i}{\sum_{i=1}^{n} A_i} \tag{6.11}$$

$$y_C = \frac{S_x}{A} = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i} \tag{6.12}$$

§3. INERTIA MOMENT AND RADIUS OF GYRATION

3.1. Inertia moment of a cross section about an axis

Inertia moment of a cross section about an axis x:

$$I_x = \int_A y^2 dA \tag{6.13}$$

Inertia moment of a cross section about an axis y:

$$I_y = \int_A x^2 dA \tag{6.14}$$

Th unit of the inertia moment is [length]^4.

Ix, Iy are always positive.

Thus, the inertia moment of a complex cross section is equal to the sum of inertia moments of simple cross sections.

3.2. Polar inertia moment

Polar inertia moment of a cross section about a point O:

$$I_\rho = \int_A \rho^2 dA \tag{6.15}$$

in which: ρ is the distance from the center of dA to the point O.

The unit of polar inertia moment is [length]^4.

Polar inertia moment is always positive: $I_\rho > 0$

We have: $\rho^2 = x^2 + y^2$

Thus: $$I_\rho = I_x + I_y \tag{6.16}$$

If two any perpendicular axes u and v intersects at the origin O, we have: $I_\rho = I_u + I_v$

So: $$I_x + I_y = I_u + I_v = I_\rho = const \tag{6.17}$$

Therefore, *the total inertia moments for two perpendicular axes is a constant.*

3.3. Product of inertia moment for a cross section

The product of inertia for a cross section with respect to Oxy system is defined as follows.

$$I_{xy} = \int_A xydA \tag{6.18}$$

I_{xy} can be positive, negative or zero.

3.4. Principle inertia moment

If $I_{xy} = 0$, the axis system Oxy is a principle axes of inertia.

If the principle axes going through the center ($S_x = 0$, $S_y = 0$), the axes are called central principle axes of inertia.

Any symmetrical axis is a central principle axis of inertia.

3.5. Radius of gyration

Radius of gyration is defined by Equation 6.19 and 6.20:

$$r_x = \sqrt{\frac{I_x}{A}} \tag{6.19}$$

$$r_y = \sqrt{\frac{I_y}{A}} \tag{6.20}$$

The unit of radius of gyration is *[length]*.

§4. PRINCIPLE INERTIA MOMENTS OF SIMPLE CROSS SECTIONS

4.1. Rectangle

Consider the rectangle with the coordinate system shown in **Figure 6.4**. The width is b, the height is h. Take a finite strip dy and the coordinate of the strip dy is y.

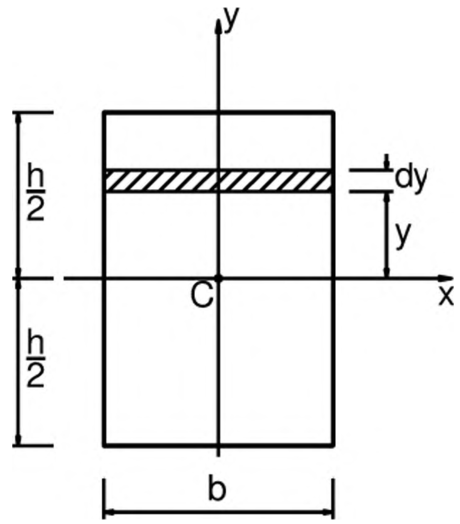


Figure 6.4. Principle inertia moment of a rectangular section

$$I_x = \int_A y^2 dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b dy = b\frac{y^3}{3}\Bigg]_{-\frac{h}{2}}^{\frac{h}{2}} = b\frac{\left(\frac{h}{2}\right)^3}{3} - b\frac{\left(-\frac{h}{2}\right)^3}{3} = \frac{2hb^3}{24} = \frac{hb^3}{12} \tag{6.21}$$

Principle inertia moment about x axis:

$$I_x = \frac{bh^3}{12} \tag{6.22}$$

Similarly for I_y. Principle inertia moment about y axis:

$$I_y = \frac{hb^3}{12} \tag{6.23}$$

4.2. Triangle

Figure 6.5 shows a triangle section with the dimensions are b and h.

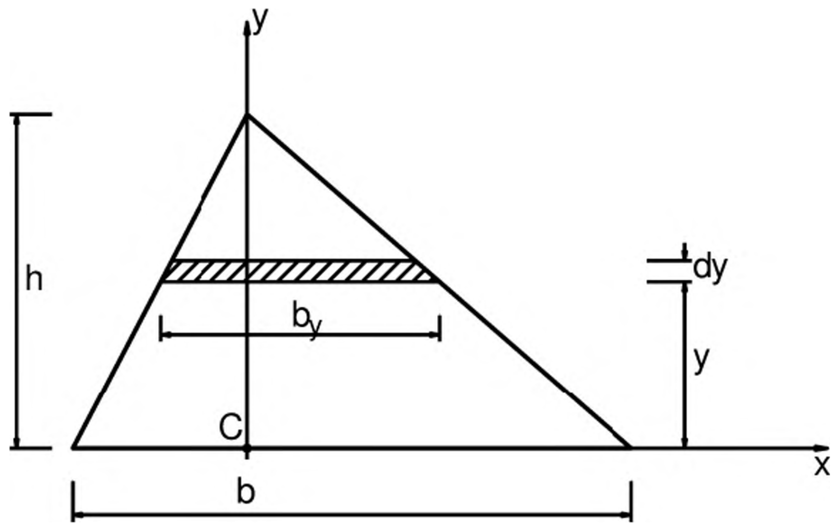


Figure 6.5. Principle inertia moment of a triangular section

We have:

$$b_y = \frac{b(h-y)}{h} \tag{6.24}$$

$$I_x = \int_A y^2 dA = \int_0^h y^2 \frac{b(h-y)}{h} dy = \frac{b}{h}\int_0^h y^2 (h-y) dy = \frac{b}{h}\left(\frac{hy^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^h = \frac{bh^3}{12} \tag{6.25}$$

$$I_x = \frac{bh^3}{12} \tag{6.26}$$

4.3. Circle

Consider the circular section with the coordinate system shown in **Figure 6.6**. The radius of the circle is R. The diameter D = 2R.

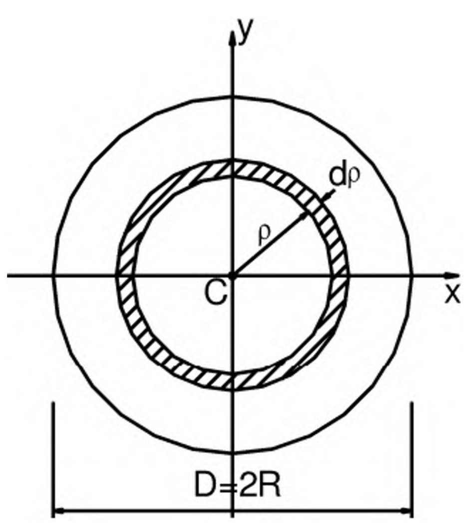


Figure 6.6. Principle inertia moment of a circular section

We have:

$$dA = 2\pi\rho d\rho$$

$$I_\rho = \int_A \rho^2 dA = \int_0^R 2\pi\rho^3 d\rho = \frac{\pi R^4}{2} \tag{6.27}$$

$$I_\rho = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} \approx 0,1D^4 \tag{6.28}$$

Due to symmetry:

$$I_x = I_y = \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \approx 0,05D^4 \tag{6.29}$$

Note: $I_\rho = I_x + I_y = 2I_x = 2I_y$

$$I_x = I_y = \frac{I_\rho}{2} \tag{6.30}$$

4.4. Donut

A donut section with the inner radius r and the outter radius R as shown in **Figure 6.7**.

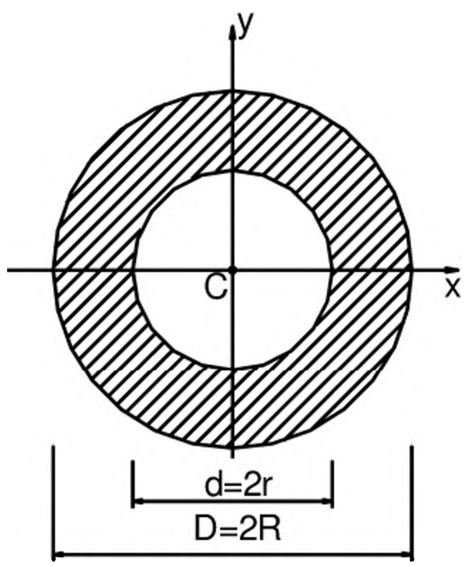


Figure 6.7. Principle inertia moment of a donut section

We have:

$$I_{x,\ donut} = I_{x,\ large\ circle} - I_{x,\ small\ circle}$$

$$I_{x,\ large\ circle} = \frac{\pi R^4}{2}$$

$$I_{x,\ small\ circle} = \frac{\pi r^4}{2}$$

$$I_x = I_y = \frac{\pi}{4}\left(R^4 - r^4\right) \tag{6.31}$$

Or $$I_x = I_y = 0,05D^4 - 0,05d^4 = 0,05D^4\left(1 - \eta^4\right) \tag{6.32}$$

in which, $\eta = \frac{d}{D}$

§5. PARALLEL AXIS THEOREM

Consider an arbitrary section shown in **Figure 6.8**. There are two paralled coordinate systems Oxy and OXY. The properties of the cross section in the Oxy system are known. Find the properties of the cross section in the parallel axes O'XY.

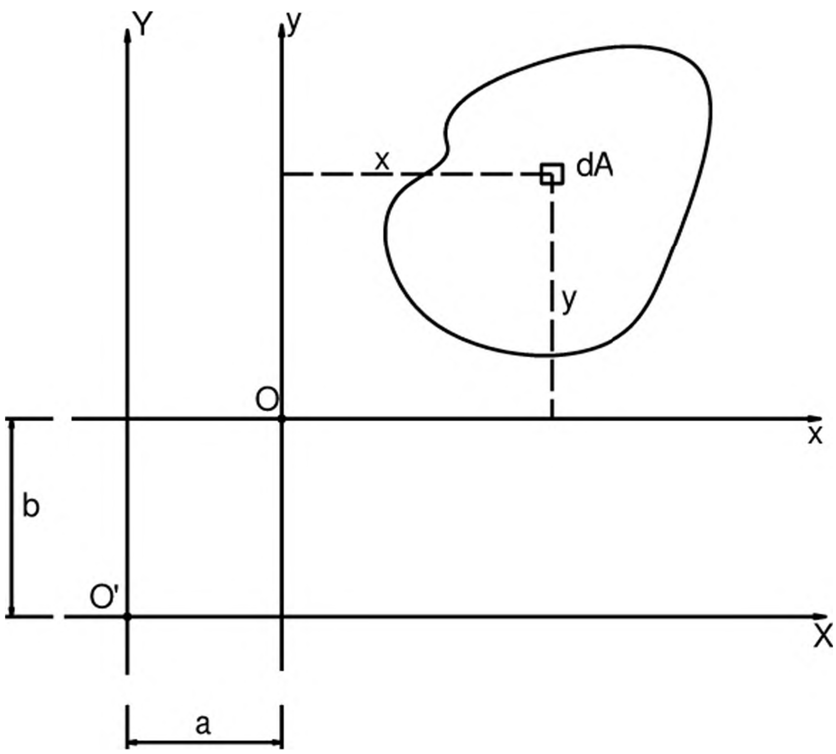


Figure 6.8. Properties of a cross section in the parallel axis system O'XY

We have: $X = x + a$

and $Y = y + b$

$$I_X = \int_A Y^2 dA = \int_A (y+b)^2 dA$$

$$= \int_A y^2 dA + 2b\int_A y dA + \int_A b^2 dA = I_x + 2bS_x + b^2 A$$

$$I_X = I_x + 2bS_x + b^2 A \tag{6.33}$$

Similarly:

$$I_Y = I_y + 2aS_y + a^2 A \tag{6.34}$$

We also have:

$$I_{XY} = \int_A XYdA = \int_A (x+a)(y+b)dA$$

$$= \int_A xydA + b\int_A xdA + a\int_A ydA + ab\int_A dA$$

$$I_{XY} = I_{xy} + bS_y + aS_x + abA \tag{6.35}$$

$$I_{XY} = I_{xy} + aS_x + bS_y + abA \tag{6.36}$$

If Oxy is central principle axes:

$$I_X = I_x + b^2 A \tag{6.37}$$

$$I_Y = I_y + a^2 A \tag{6.38}$$

$$I_{XY} = I_{xy} + abA \tag{6.39}$$

Comment: the inertia moment about central axis is minimum.

§6. INCLINED AXIS THEOREM (Transformation of inertia moment)

Consider an arbitrary section with the coordinate system Oxy and the inclined coordinate system Ouv. The inclined angle between the two coordinate systems is α. The rotation angle α is positive if counterclockwise as shown in **Figure 6.9**. The properties of the cross section in system Oxy are known. Find the properties of the cross section in the inclined axes Ouv.

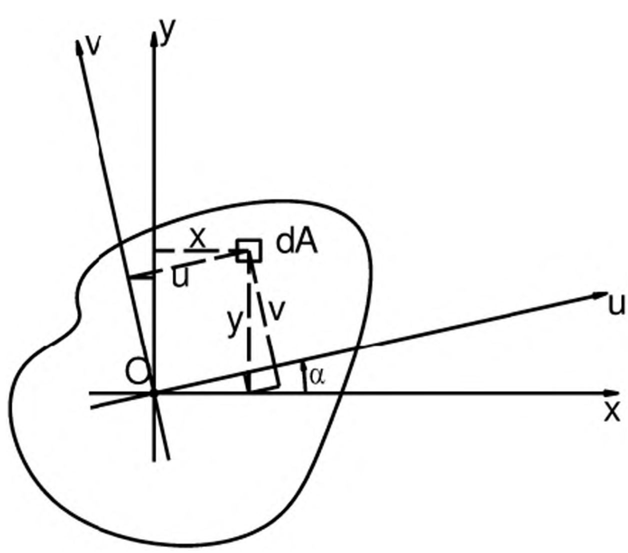


Figure 6.9. Properties of a cross section in inclined axes Ouv

We have: $u = y\sin\alpha + x\cos\alpha$ and $v = y\cos\alpha - x\sin\alpha$

$$I_u = \int_A v^2 dA = \int_A (y\cos\alpha - x\sin\alpha)^2 dA$$

$$= \cos^2\alpha \int_A y^2 dA + \sin^2\alpha \int_A x^2 dA - 2\sin\alpha\cos\alpha \int_A xydA$$

$$I_u = I_x\cos^2\alpha + I_y\sin^2\alpha - 2I_{xy}\sin\alpha\cos\alpha$$

Use: $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$; $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$

and $2\sin\alpha\cos\alpha = \sin 2\alpha$

We have:

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\alpha - I_{xy}\sin 2\alpha \quad (6.40)$$

Similarly:

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\alpha + I_{xy}\sin 2\alpha \quad (6.41)$$

We have: $I_u + I_v = I_x + I_y = const$ (6.42)

Analyse I_{uv}, we obtain:

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\alpha + I_{xy}\cos 2\alpha \quad (6.43)$$

§7. MOHR'S CIRCLE FOR INERTIA MOMENT

7.1. Method to determine the principle coordinate system

Principle axis system has $I_{uv} = 0$

To find the angle α so that the moment of inertia reaches its extreme, we take the derivative of

I_u with respect to α. Then, set it equal to zero.

$$\frac{dI_u}{d\alpha} = -2\frac{I_x - I_y}{2}\sin 2\alpha - 2I_{xy}\cos 2\alpha = 0$$

$$\Rightarrow \quad \tan 2\alpha = -\frac{2I_{xy}}{I_x - I_y} \quad (6.44)$$

where, α is the principle angle. There are two principle α with 90^o difference.

Thus, for a principle axis system, moment of inertia reaches its extreme, namely, principle moment of inertia.

Substitute the angle α obtained from the above Equation 6.44 into the equation of I_u, then, use the trigonometric relationship:

$$\sin 2\alpha_o = \pm \frac{\tan 2\alpha_o}{\sqrt{1+\tan^2 2\alpha_o}}$$

$$\cos 2\alpha_o = \pm \frac{1}{\sqrt{1+\tan^2 2\alpha_o}}$$

we have:

$$I_{\substack{max \\ min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{6.45}$$

7.2. Mohr's circle for inertia moment

Mathematically, there are similarities between the transformation equation for moment of inertia and that for stress as shown in **Table 6.1**.

Table 6.1. Comparison between stress and moment of inertia in Mohr's circle

Transformation equation for moment of inertia	Transformation equation for stress
$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\alpha - I_{xy}\sin 2\alpha$	$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$
$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\alpha + I_{xy}\sin 2\alpha$	$\sigma_v = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$
$I_{uv} = \frac{I_x - I_y}{2}\sin 2\alpha + I_{xy}\cos 2\alpha$	$\tau_{uv} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$

I_u corresponding to σ_u

I_v corresponding to σ_v

I_x corresponding to σ_x

I_y corresponding to σ_y

I_{xy} corresponding to τ_{xy}

I_{uv} corresponding to τ_{uv}

Therefore, if we use a coordinate system in which the horizontal axis represents I_u and the vertical axis represent I_{uv}, then the relationship between I_u and I_{uv} is a circle, namely, Mohr's circle of inertia moment. Similar to the Mohr's circle of stress; however, I_{uv} axis is upward and the circle is always on right side of the vertical axis because I_u is always positive.

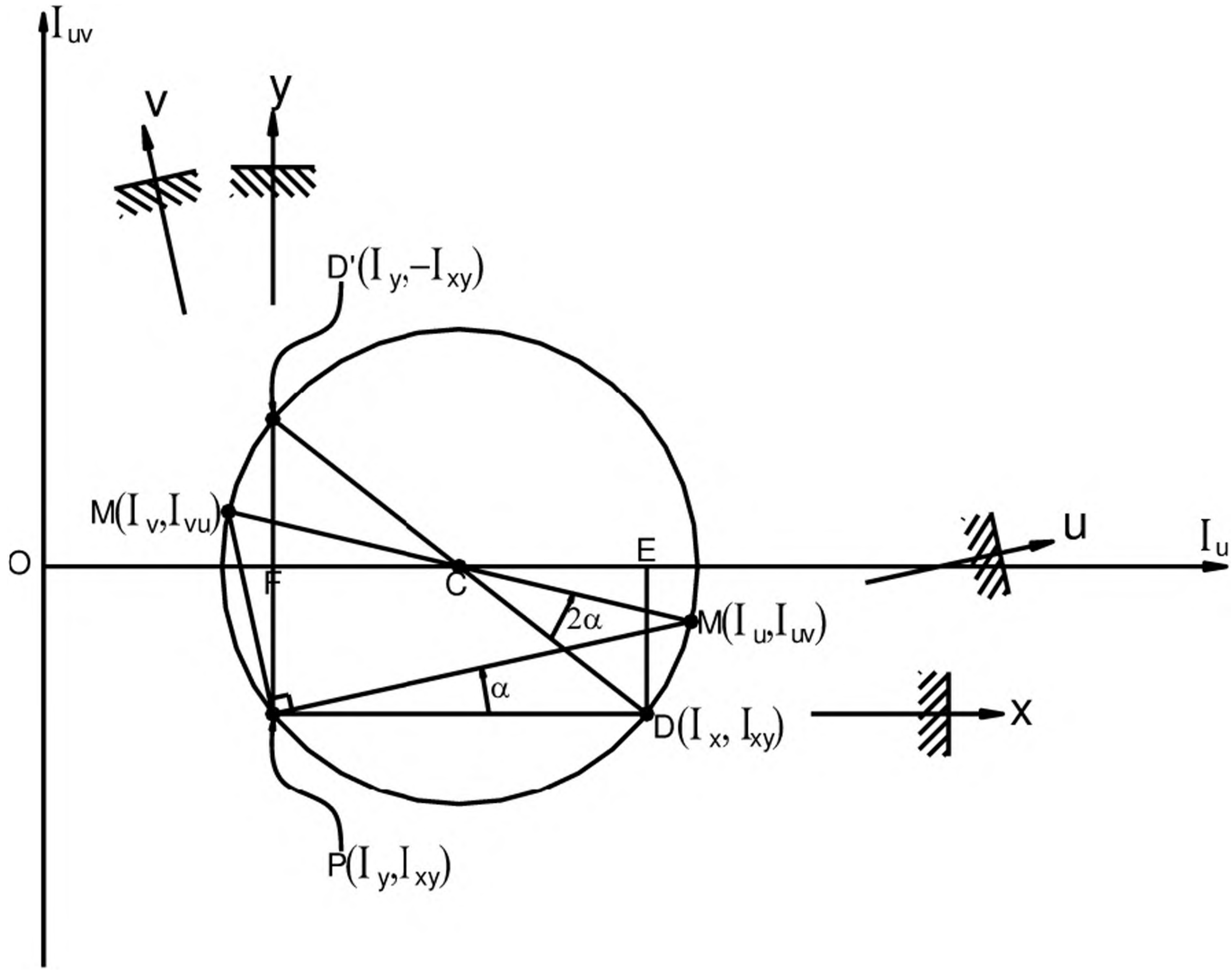


Figure 6.10. Mohr's circle of inertia moment

7.3. Method to determine the principle axes of sections

Generally, the principle axes of sections are determined as follows:

- Choose any Oxy axis system. Determine the coordinate of the center of the section in this axis system.
- Move the axis system to the center of the section. Calculate the inertia moments for the section in the principle axis system.
- Rotate axis system to find the principle direction passing through the center of the section.

§8. EXERCISES

8.1. Exercise 1

Consider the cross section shown in **Figure 6.11**. Given: a = 100 mm, t = 20 mm.

Calculate: I_x, I_y, r_x, r_y.

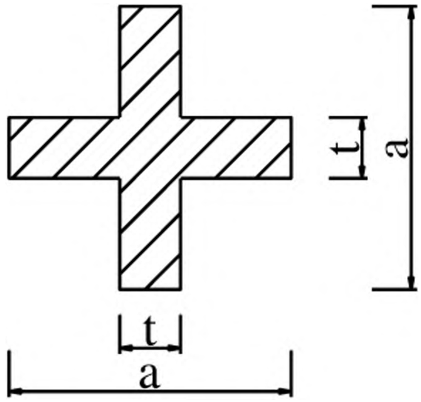


Figure 6.11

Solution

Divide the section into parts I and II.

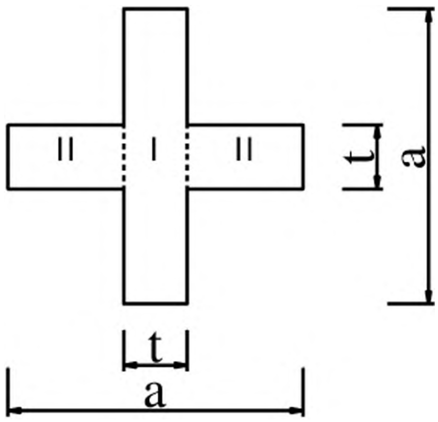


Figure 6.12

$$I_x = I_y = \frac{ta^3}{12} + \frac{(a-t)t^3}{12} = \frac{20\times100^3}{12} + \frac{(100-20)20^3}{12} = 1.72\times10^6 mm^4$$

$$A = 100\times20 + (100-20)\times20 = 3600mm^2$$

$$r_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{1.72\times10^6}{3600}} = 21.9mm$$

8.2. Exercise 2

Consider the section shown in **Figure 6.13**. Given: b = 50 mm, t = 50 mm, h = 300 mm, h_1 = 250 mm.

a) Determine the center of the section.

b) Determine the principle moment of inertia.

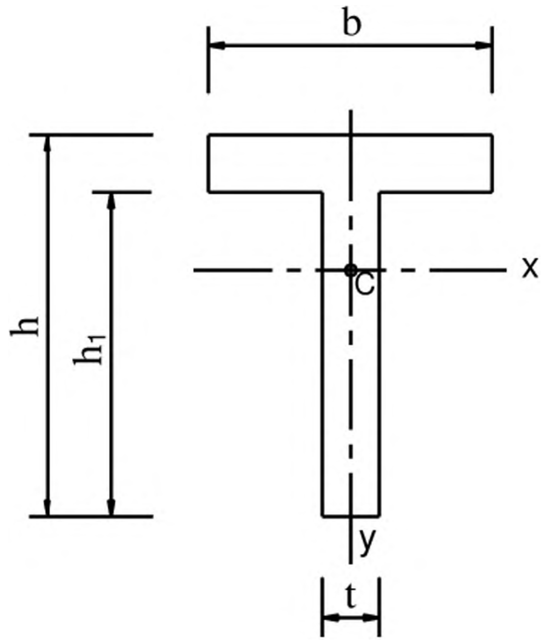


Figure 6.13

Solution

a) Determine the center of the section.

The center C has to be on the symmetric axis, therefore, $x_c = 0$. The center C is on the y axis and has a distance y_c from the bottom side. Divide the section into two parts: I and II.

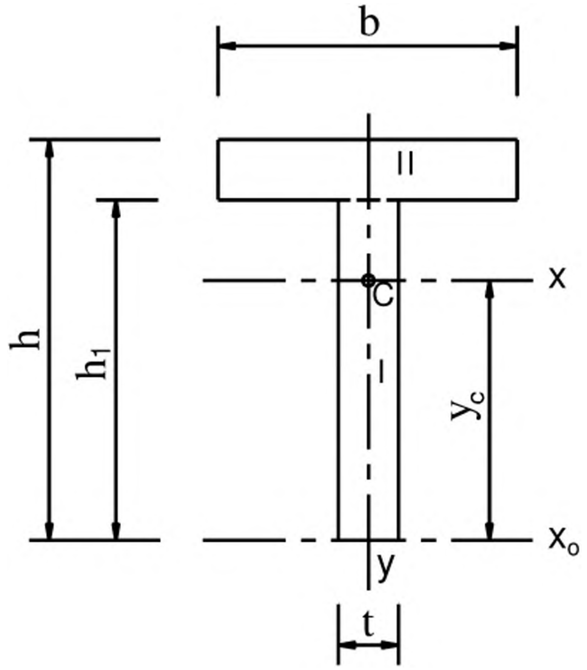


Figure 6.14

$$y_c = \frac{S_{x_o}}{A}$$

$$= \frac{250 \times 50 \times \frac{250}{2} + 50 \times 150\left(250 + \frac{50}{2}\right)}{250 \times 50 + 150 \times 50} = 181.25mm$$

b) Determine the principle moment of inertia

$$I_x = \left[\frac{50 \times 250^3}{12} + 50 \times 250\left(\frac{250}{2} - 181.25\right)^2\right]$$

$$+ \left[\frac{150 \times 50^3}{12} + 150 \times 50\left(300 - 181.25 - \frac{50}{2}\right)^2\right]$$

$$= 172.135 \times 10^6 mm^4$$

$$I_y = \frac{50 \times 150^3}{12} + \frac{250 \times 50^3}{12} = 16.67 \times 10^6 mm^4$$

8.3. Exercise 3

Consider the section shown in **Figure 6.15**.

a) Determine the center of the section.

b) Determine the principle moment of inertia

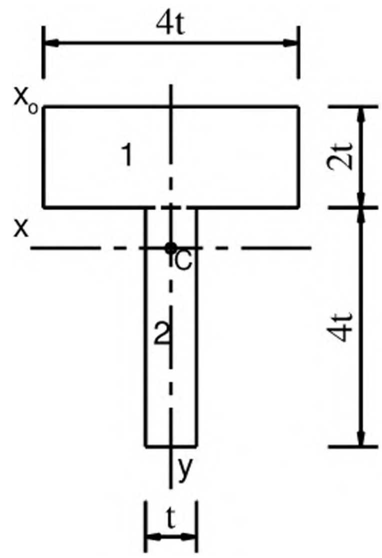


Figure 6.15

Solution

The center C has to be on the symmetric axis, therefore, $x_c = 0$

Divide the section into two parts as shown in **Figure 6.15**.

Option 1: Choose the axis x_o across the top side.

$$y_c = \frac{S_{x_o}}{A} = \frac{S_{x_o}^{I} + S_{x_o}^{II}}{A} = \frac{(4t \times 2t)t + (4t \times t)4t}{(4t \times 2t) + (4t \times t)} = 2t$$

$$I_x = \left[\frac{4t \times (2t)^3}{12} + 4t \times 2t \times t^2\right] + \left[\frac{t \times (4t)^3}{12} + (t \times 4t)(2t)^2\right]$$

$$= \frac{32t^4}{3} + \frac{64t^4}{3} = 32t^4$$

$$I_y = \frac{2t \times (4t)^3}{12} + \frac{4t \times t^3}{12} = 11t^4$$

Option 2: Choose the axis x_o across the center of part 1:

$$y_c = \frac{S_{x_o}}{A} = \frac{S_{x_o}^{I} + S_{x_o}^{II}}{A} = \frac{0 + (4t \times t)3t}{(4t \times 2t) + (4t \times t)} = t$$

Then, I_x is calculated. The result is the same.

8.4. Exercise 4

Consider the section shown in **Figure 6.16**. The dimensions are in mm.

a) Determine the center of the section.

b) Determine the principle moment of inertia

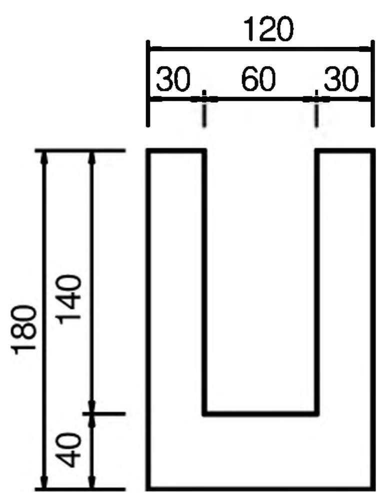


Figure 6.16

Solution

The center C has to be on the vertical symmetric axis, therefore, $x_c = 0$

Divide the section into three parts: I, II and III as shown in **Figure 6.17**

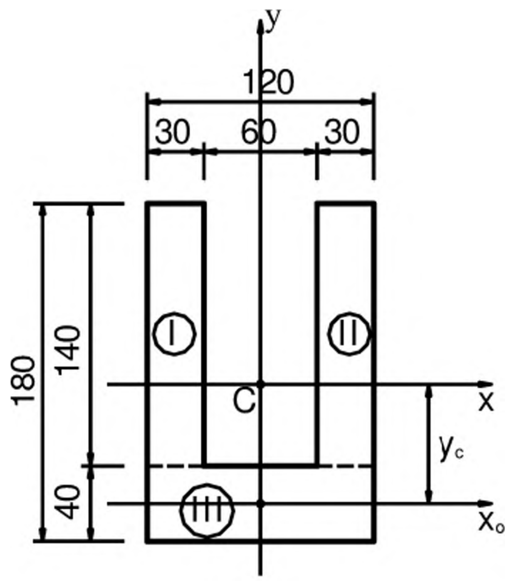


Figure 6.17

Option 1: Choose the axis x_o across the center of part III.

$$y_c = \frac{S_{x_o}}{A} = \frac{S_{x_o}^{I} + S_{x_o}^{II} + S_{x_o}^{III}}{A}$$

$$S_{x_o}^{I} = S_{x_o}^{II} = 140 \times 30 \times \left(\frac{140}{2} + \frac{40}{2}\right) = 378 mm^3$$

$S_{x_o}^{III} = 0$ because the center of part III on the axis x_o.

$$A = 2 \times (140 \times 30) + (40 \times 120) = 132 mm^2$$

$$y_c = \frac{S_{x_o}}{A} = \frac{S_{x_o}^{I} + S_{x_o}^{II} + S_{x_o}^{III}}{A} = 5.72 \text{ mm}$$

$$I_x = I_x^{I} + I_x^{II} + I_x^{III} = 2I_x^{I} + I_x^{III}$$

$$= 2 \times \left[\frac{30 \times (140)^3}{12} + 120 \times 30 \times \left(\frac{140}{2} - 57.2\right)^2\right]$$

$$+ \left[\frac{120 \times (40)^3}{12} + 120 \times 40 \times (57.2)^2\right]$$

$$= (2 \times 1138 + 1635) \times 10^4 = 3911 \times 10^4 \text{ mm}^4.$$

$$I_y = I_y^I + I_y^{II} + I_y^{III} = 2I_y^I + I_y^{III}$$

$$= 2\times\left[\frac{140\times(30)^3}{12} + 140\times 30\times(45)^2\right] + \left[\frac{40\times(120)^3}{12}\right]$$

$$= (2\times 882 + 576)\times 10^4 = 2340\times 10^4 \text{ mm}^4.$$

Alternatively, we can have another method for this problem: The given section is equal to the rectangular section abcd subtracted the section efgh as shown in **Figure 6.18**.

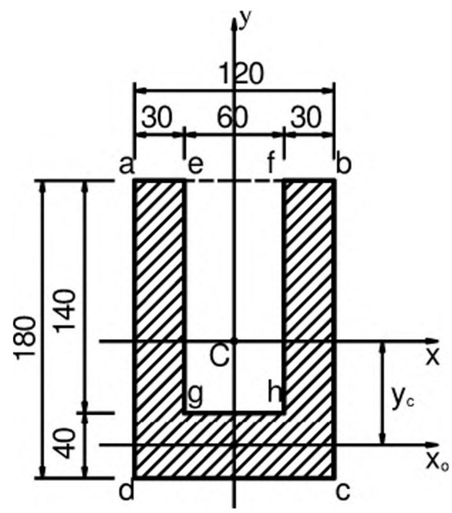


Figure 6.18

$$I_y = I_y^{abcd} - I_y^{efgh} = \left[\frac{180\times(120)^3}{12}\right] - \left[\frac{60\times(140)^3}{12}\right]$$

$$= (2592 - 252)\times 10^4 = 2340\times 10^4 \text{ mm}^4.$$

8.5. Exercise 5

Consider the steel section shown in **Figure 6.19**. Determine the principle moment of inertia with respect to b, t and h.

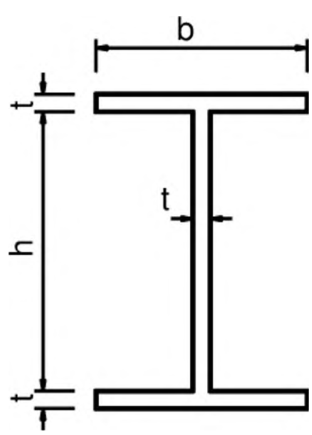


Figure 6.19

Solution

This is a symmetric section; thus, the principle is shown in **Figure 6.20**. Divide the section into three parts: I, II and III.

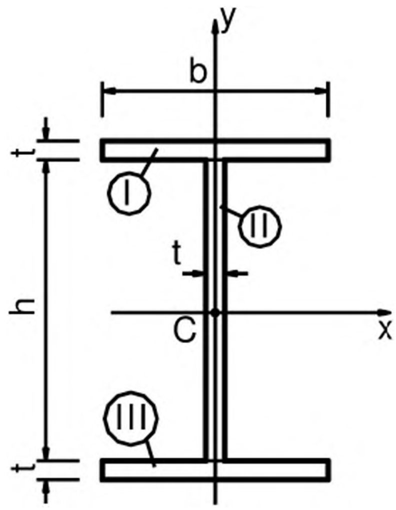


Figure 6.20

$$I_x = I_x^I + I_x^{II} + I_x^{III} = 2I_x^I + I_x^{III}$$

$$= 2\times\left[\frac{b\times t^3}{12} + b\times t\times\left(\frac{h}{2}-\frac{t}{2}\right)^2\right] + \left[\frac{t\times h^3}{12}\right]$$

$$I_y = I_y^I + I_y^{II} + I_y^{III} = 2I_y^I + I_y^{III}\sqrt{a^2+b^2}$$

$$= 2\times\left[\frac{t\times b^3}{12}\right] + \left[\frac{h\times t^3}{12}\right]$$

PROBLEMS

PROBLEM 1. Consider the section shown in **Figure 6.21**. Given: b = 100 mm, t = 10 mm, h = 300 mm, h_1 = 280 mm.

a) Determine the center of the section.

b) Determine the principle moment of inertia.

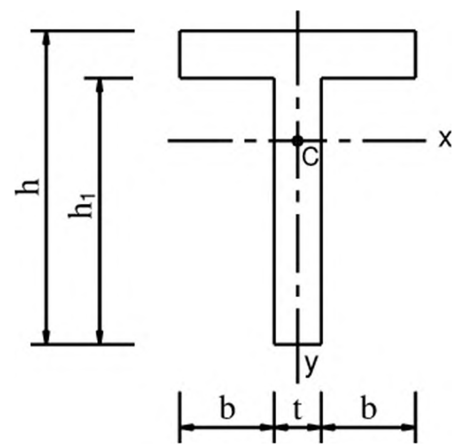


Figure 6.21

PROBLEM 2. Consider the sections shown in **Figure 6.22 – Figure 6.26**.

a) Determine the center of the section.

b) Determine the principle moment of inertia

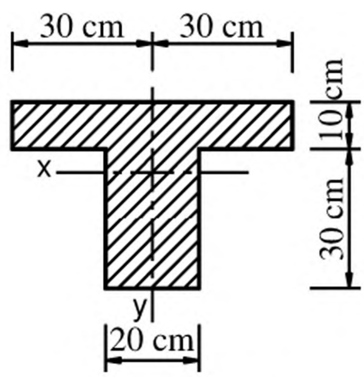


Figure 6.22

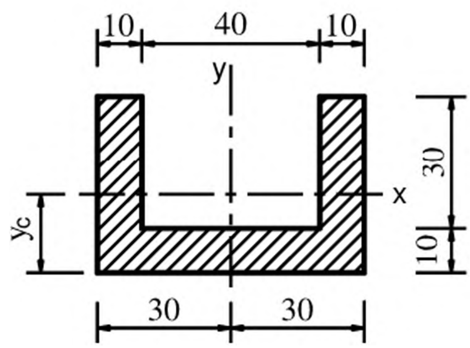


Figure 6.23 (Dimensions are in mm)

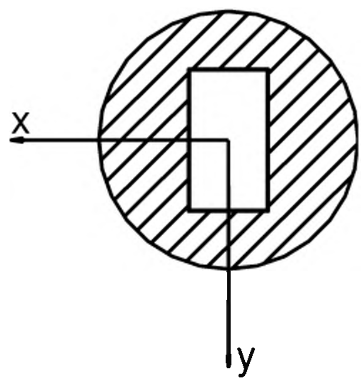


Figure 6.24

Given: The radius of the circle is R = 100 mm, the dimensions of the rectangle is b×h = 20×60 mm.

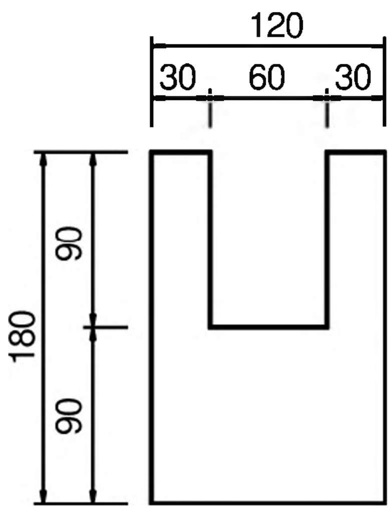


Figure 6.25

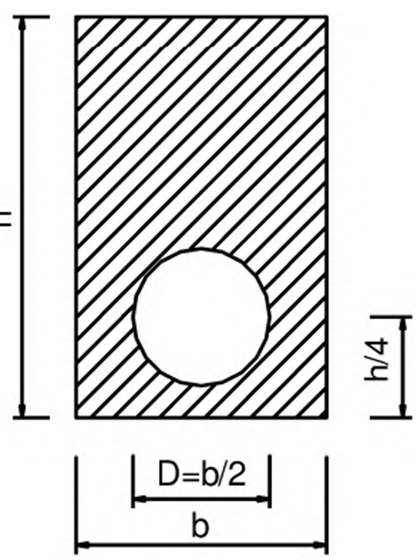


Figure 6.26

Chapter 7

BENDING AND SHEARING

§1. CONCEPTS

Consider a member subjected to loads as shown in **Figure 7.1**. The loads are perpendicular to the axial axis and in a plane. The member is bent and called 'beam'.

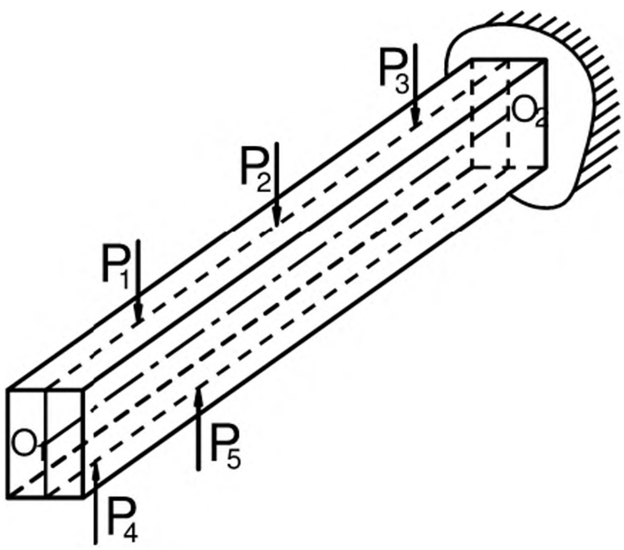


Figure 7.1. Member subjected to loads

Some definitions:

Load plane is plane containing all loads and the axial axis.

Load line is the intersection of the load plane and the cross section.

In this chapter, we only investigate the following cases:

- The cross section has at least one symmetry axis.
- This symmetry axis is also a central principle inertia axis.
- The loads lie in the load plane containing the axial axis and the central principle inertia axis.

After deformation, the axial axis is still in the load plane and the beam is in *plane bending*.

Figure 7.2 shows some examples of beams under bending.

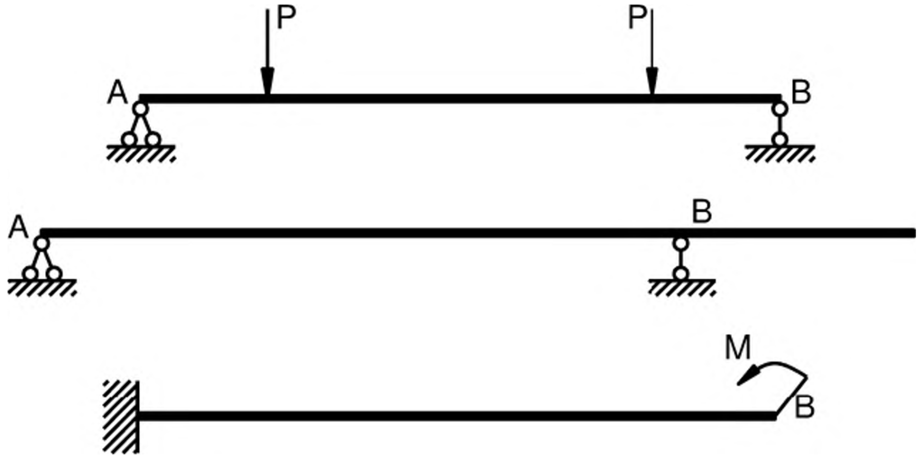


Figure 7.2. Bending in beams

Plane bending can be classified into:

- *Pure bending*: Bending moment is the only one component of internal force on the cross section.
- *Shear and bending*: Shear force and bending moment are two components of internal force on the cross section.

Consider the beam shown in **Figure 7.3**. Its shear force and bending moment diagrams are also shown in this figure. It can be seen that the segment b has only bending moment while the shear force is zero.

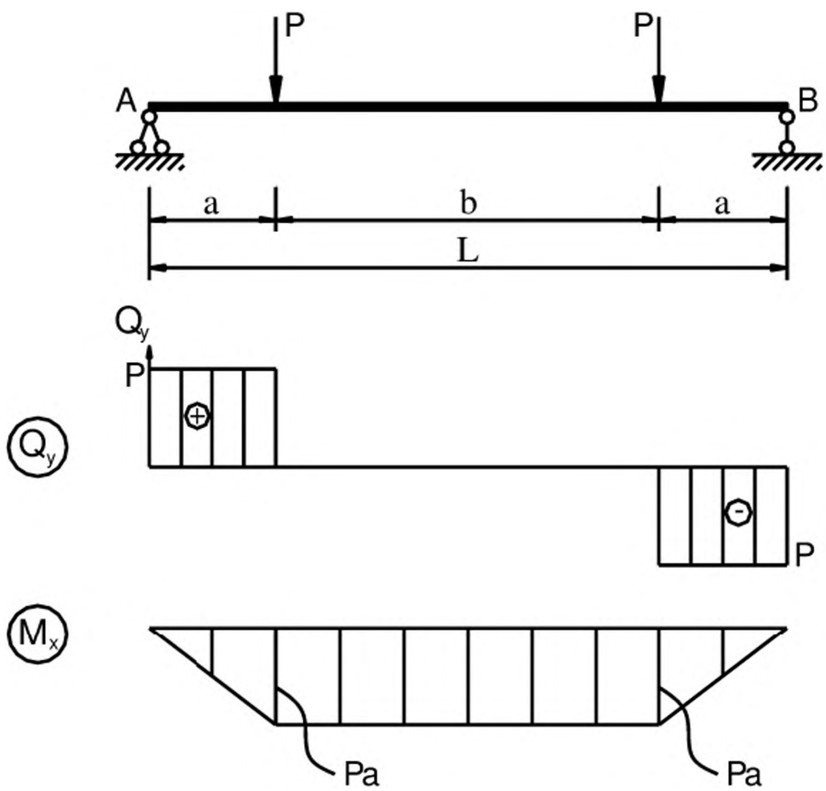


Figure 7.3. Bending and shear

Figure 7.4 shows other examples of pure bending, in which the member has only one component of internal force of bending moment.

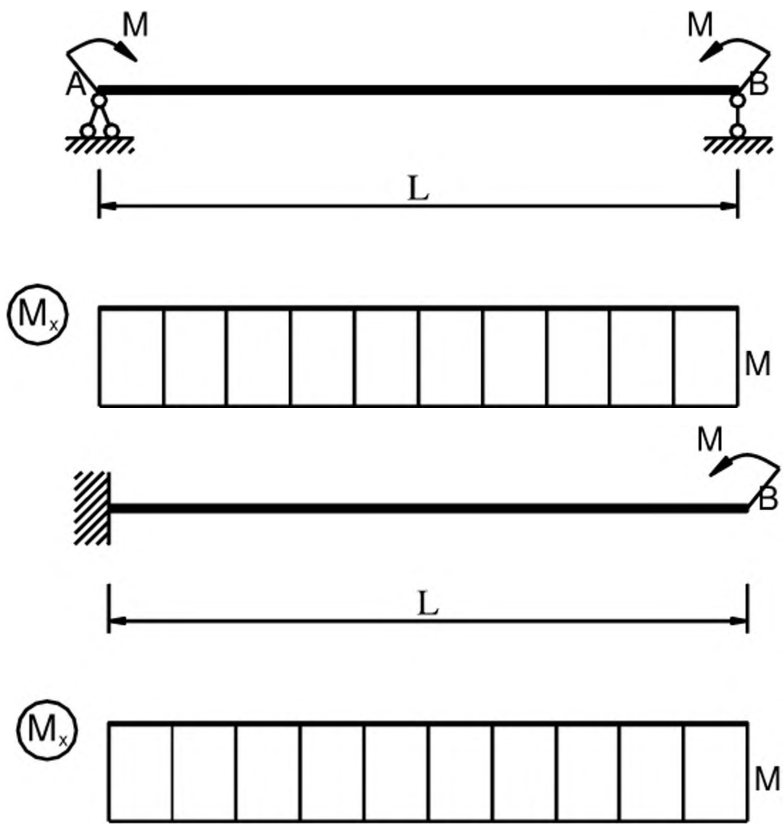
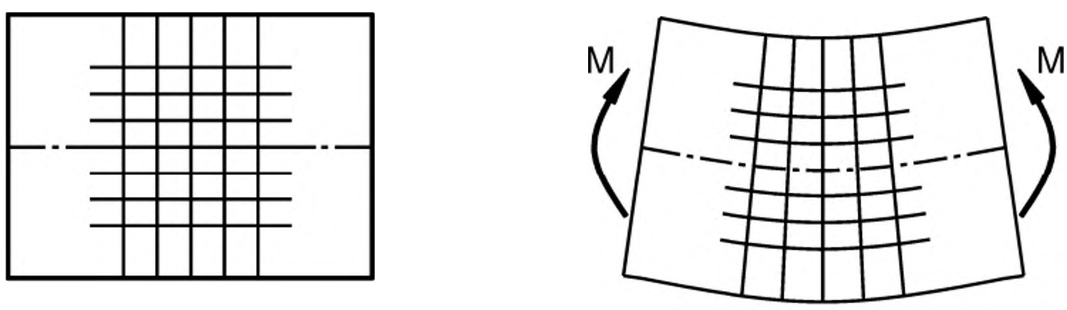


Figure 7.4. Pure bending

§2. PURE BENDING

The objective of this lesson is to determine stress at any point on the cross section caused by bending moment. To achieve this objective, an experiment can be conducted as follows. On the side of a beam, we draw lines which are parallel to the axial axis. These parallel lines represent longitudinal fibres. We also draw lines perpendicular to the axial axis. These perpendicular lines represent sections. These parallel and perpendicular lines are shown in **Figure 7.5a**. The beam is then subjected to bending moment and it is deformed as shown in **Figure 7.5b**.



a) Before bending b) After bending

Figure 7.5. A beam segment before and after bending

Observations after bending can be described as follows:

- Lines perpendicular to the axial axis are still lines perpendicular to the axial axis.
- Lines parallel to the axial axis become curves parallel to the curve axial axis.
- Top fibres are shortened, bottom fibres are elongated.
- The fibre whose length does not change is call *neutral fibre*.
- All *neutral fibres* make *neutral layer*.
- Intersection of neutral layer and a cross section is called *neutral line* or *neutral axis*.

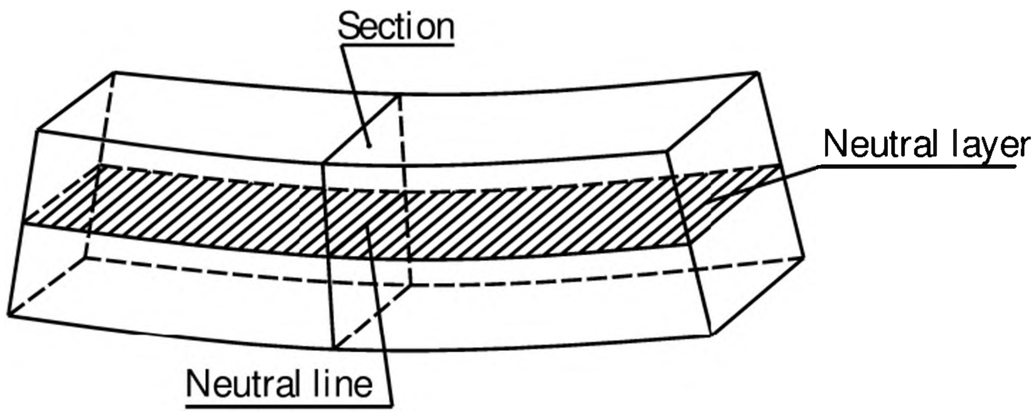
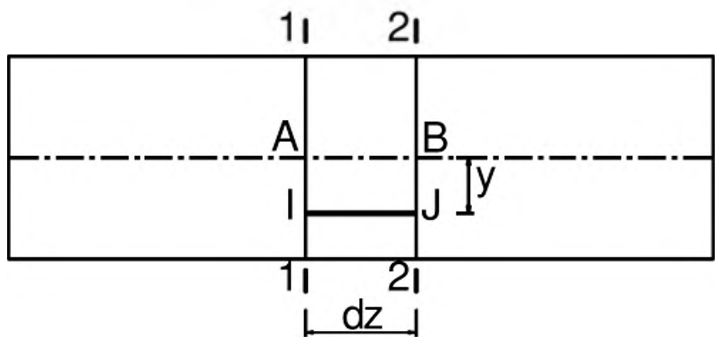
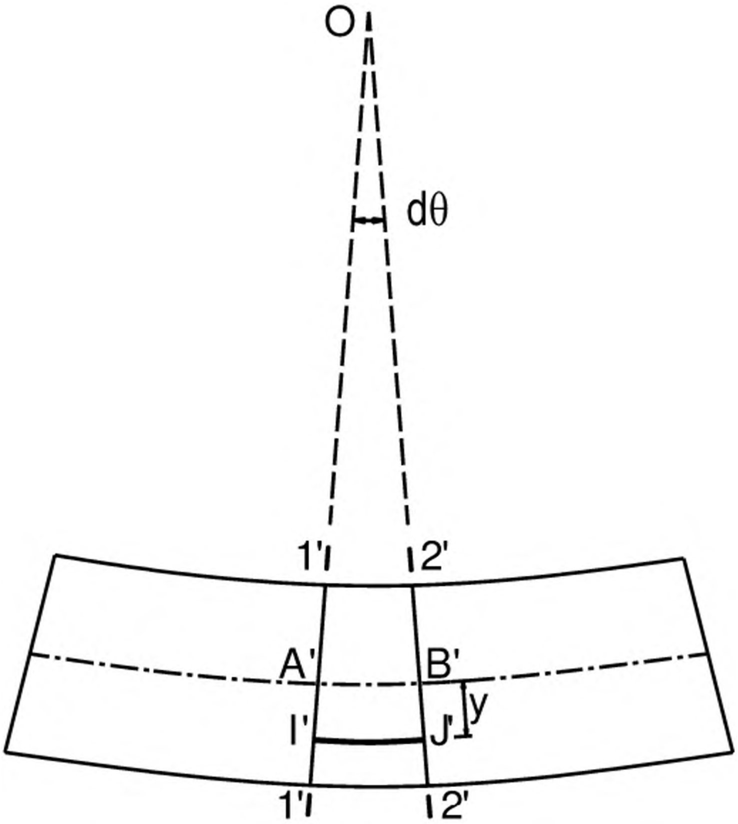


Figure 7.6. Beams under bending

Considering a segment between sections 1-1 and 2-2 as shown in **Figure 7.7a**. The length of this segment is dz. AB is the fibre at the neutral axis. IJ is the fibre at the coordinate y from the neutral axis. After bending, these sections become 1'-1' and 2'-2' and are still perpendicular to the axis. The angle between these sections is $d\theta$. The line AB becomes the curve A'B' and the line IJ becomes the curve I'J'. Set the radius OA' = OB' be equal to ρ.



a)



b)

Figure 7.7. Deformation of beam under bending

For the fibre AB:

$$AB = dz = \rho d\theta \tag{7.1}$$

For the fibre IK having the distance y from the neutral axis, its strain is:

$$\varepsilon_z = \frac{I'K' - IK}{IK} = \frac{(\rho + y)d\theta - \rho d\theta}{\rho d\theta} = \frac{y}{\rho} = \kappa y \tag{7.2}$$

in which: $\kappa = \dfrac{1}{\rho}$ is curvature.

The above expression shows that the train is proportional to the curvature and the distance y from neutral axis.

Each fibre is in tension or compression (1D stress state). Based on Hooke's law:

$$\sigma_z = E\varepsilon_z = E\kappa y \tag{7.3}$$

Equilibrium equation:

$$\sum M_x = 0$$

$$\Leftrightarrow \quad M_x = \int_A dM_x = \int_A \sigma_z y dA = \int_A E\kappa y^2 dA = E\kappa \int_A y^2 dA \tag{7.4}$$

$$M_x = E\kappa I_x$$

$$\kappa = \frac{1}{\rho} = \frac{M_x}{EI_x} \tag{7.5}$$

Therefore, the curvature is proportional to the bending moment and inversely proportional to EI_x which is called the bending stiffness of beams.

From the above, we have:

$$M_x = (E\kappa)I_x = \frac{\sigma_z}{y} I_x \tag{7.6}$$

$$M_x = \frac{\sigma_z}{y} I_x$$

Or $$\sigma_z = \frac{M_x}{I_x} y \tag{7.7}$$

in which,

M_x is the bending moment. M_x is positive when it causes tension of y+ fibre.

I_x is the inertia moment about x axis.

y is the distance from neutral axis to the point of computed stress.

This expression shows that the normal stress is proportional to the bending moment M_x and y, and inversely proportional to I_x.

We can use the absolute values to calculate the stress as shown in the formula (7.8):

$$\sigma_z = \pm \frac{|M_x|}{I_x} |y| \tag{7.8}$$

Take + sign if M_x causes tension to the point of computed stress.

Take – sign if M_x causes compression to the point of computed stress.

The stress is maximum or minimum when y is maximum or minimum, respectively:

$$\sigma_{max} = \frac{|M_x|}{I_x} \left| y_{max}^{tension} \right| = \frac{|M_x|}{W_x^{tension}} \tag{7.9}$$

$$\sigma_{min} = \frac{|M_x|}{I_x} \left| y_{max}^{compression} \right| = \frac{|M_x|}{W_x^{compression}} \tag{7.10}$$

in which,

$$W_x^{tension} = \frac{I_x}{\left|y_{\max}^{tension}\right|};\ W_x^{compression} = \frac{I_x}{\left|y_{\max}^{compression}\right|}$$

These are called inertia modulus of section under bending.

If $\left|y_{\max}^k\right| = \left|y_{\max}^n\right| = \dfrac{h}{2}$

We have $W_x^k = W_x^n = W_x = \dfrac{I_x}{\left(\dfrac{h}{2}\right)}$ (7.11)

After determining the maximum and minimum normal stresses, we can check the stress conditions.

Stress conditions

$$\sigma_{\max} = \frac{|M_x|}{W_x^k} \le [\sigma]_k \tag{7.12}$$

$$|\sigma_{\min}| = \frac{|M_x|}{W_x^n} \le [\sigma]_n \tag{7.13}$$

Inertia modulus of section under bending of simple sections:

Rectangular sections

Principle inertia moment about x axis:

$$I_x = \frac{hb^3}{12} \tag{7.14}$$

Principle inertia moment about y axis:

$$I_y = \frac{hb^3}{12} \tag{7.15}$$

$$\left|y_{x,\max}^k\right| = \left|y_{x,\max}^n\right| = \frac{h}{2}$$

$$\left|y_{y,\max}^k\right| = \left|y_{y,\max}^n\right| = \frac{b}{2}$$

Therefore,

$$W_x^k = W_x^n = W_x = \frac{\frac{bh^3}{12}}{\left(\frac{h}{2}\right)} = \frac{bh^2}{6} \tag{7.16}$$

$$W_y^k = W_y^n = W_y = \frac{\frac{hb^3}{12}}{\left(\frac{b}{2}\right)} = \frac{hb^2}{6} \tag{7.17}$$

Circular sections

$$I_x = I_y = \frac{\pi R^4}{4} \tag{7.18}$$

$$\left|y_{\mathrm{x,max}}^k\right| = \left|y_{\mathrm{x,max}}^n\right| = R$$

$$\left|y_{\mathrm{y,max}}^k\right| = \left|y_{\mathrm{y,max}}^n\right| = R$$

$$W_x^k = W_x^n = W_x = \frac{\frac{\pi R^4}{4}}{R} = \frac{\pi R^3}{4} \tag{7.19}$$

$$W_y^k = W_y^n = W_y = \frac{\frac{\pi R^4}{4}}{R} = \frac{\pi R^3}{4} \tag{7.20}$$

Three basic problems

Problem 1: Check the condition of stress

Given:

- Cross section
- Load
- Allowable stresses

Request: Check the condition of stress.

Solution: From the given data, we determine M_x at the cross section to be checked, calculate W_x, stresses and then compare with the allowable stress.

2.1. Exercise 1

The cross section of an inverted T-beam subjected to a bending moment M_x = 7200 Nm. Given I_x = 5313.5×10^4 mm^4, made of materials with: $[\sigma]_{tension} = 20MPa$; $[\sigma]_{compression} = 30MPa$. Check the stress condition.

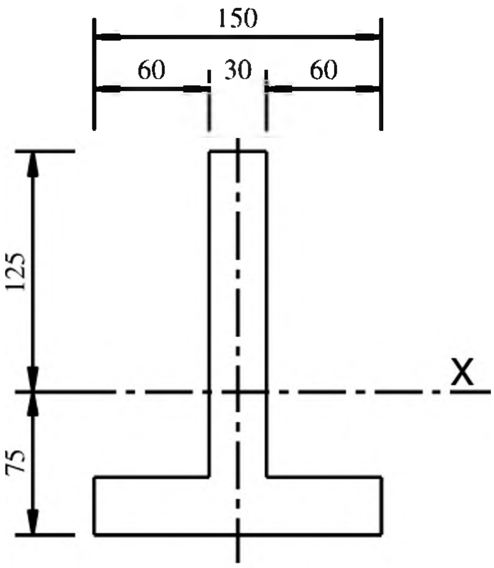


Figure 7.8

Solution

We have:

$$\left|y_{\max}^{tension}\right| = 75 \ \ mm$$

$$\left|y_{\max}^{compression}\right| = 125 \ \ mm$$

$$W_x^{tension} = \frac{I_x}{y_{\max}^{tension}} = \frac{53125 \times 10^4}{75} = 70.83 \times 10^4 mm^3$$

$$W_x^{compression} = \frac{I_x}{y_{\max}^{compression}} = \frac{5312.5 \times 10^4}{125} = 42.5 \times 10^4 mm^3$$

$$\sigma_{\max} = \frac{|M_x|}{W_x^{tension}} = \frac{7200 \times 10^3}{70.83 \times 10^4} = 10.2MPa < [\sigma]_k$$

$$|\sigma_{\min}| = \frac{|M_x|}{W_x^{compression}} = \frac{7200 \times 10^3}{42.5 \times 10^4} = 17MPa \le [\sigma]_{compression}$$

Conclusion: The stress condition is ok.

Problem 2: Select cross section

Given:

- Cross section
- Load
- Allowable stresses

Request: Determine the minimum cross-sectional property.

Solution:

From the given data, we determine M_x at the cross section to be checked, calculate W_x and then determine the dimensions of the cross section.

2.2. Exercise 2

For a pure bending beam with M_x = 60 kNm. The beam is made of steel with $[\sigma]_{tension} = [\sigma]_{compression} = 160MPa$. Choose the steel beam size to meet the stress condition. Given:

a) Beams made of 2 steel I

b) Beams made of 1 steel I

Solution

I beam is symmetric, $[\sigma]_{tension} = [\sigma]_{compression} = 160MPa$, thus,

$$W_x \geq \frac{|M_x|}{[\sigma]} = \frac{60 \times 10^6}{160} = 3.75 \times 10^5 mm^3$$

a) Choose 2I20 (Appendix I-1 page 302): $W_x = 2 \times 184 = 368cm^3$. Check the stress condition:

$$\sigma_{max} = \frac{|M_x|}{W_x^{tension}} = \frac{60 \times 10^6}{368 \times 10^3} = 163MPa$$

Tolerance:

$$\frac{163 - 160}{160} 100\% = 1.9\%$$

Conclusion: Select 2I20

b) 1I 27 has $W_x = 371$ cm^3 which is larger than the W_x found in the question a); therefore, the stress condition is satisfied.

Problem 3: Determine the allowable load

Given:

- Cross section
- Types and forms of load
- Allowable stresses

Request: Determine the allowable load value

Solution: From the given data, we calculate M_x. From M_x we can determine the value of allowable load.

2.3. Exercise 3

A cast iron beam with size, shape and bending moments is shown in **Figure 7.9**. Given: $I_x = 254.7\times10^6$ mm^4 and the allowable tensile load is 15 MPa. Request:

a) Determine the allowable bending moment.

b) Determine the maximum compressive stress in the beam.

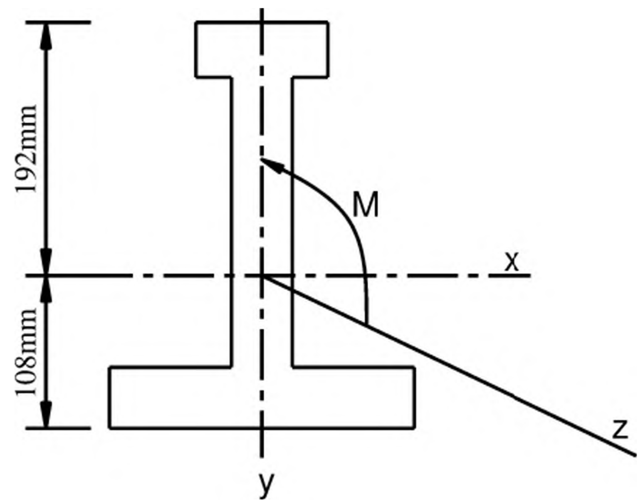


Figure 7.9

Solution

a) We have:

$$|M_x| = [\sigma]_{tension} W_x^{tension} = [\sigma]_{tension} \frac{I_x}{y_{\max}^{tension}}$$

$$= 15\times10^6 \frac{25470\times10^{-8}}{108\times10^{-3}} = 3,54\times10^4\, Nm$$

b) Determine the maximum compressive stress in the beam.

$$\sigma_{\min} = -\frac{|M_x|}{I_x}\left|y_{max}^{compression}\right| = \frac{3.54\times10^4}{25470\times10^{-8}}192\times10^{-3} = -26MPa$$

This compressive stress is larger than the allowable stress; therefore, the stress condition is not satisfied.

2.4. Exercise 4

Consider a simple beam with L = 4 m, b = 200 mm, h = 250 mm, q = 10 N/mm as shown in **Figure 7.10**, $[\sigma]$ = 12 MPa. Check the stress condition caused by bending moment.

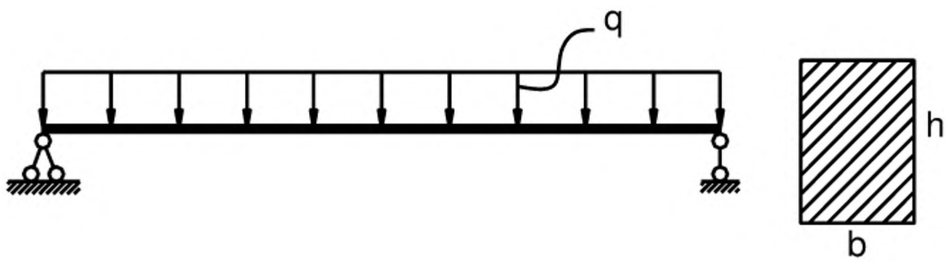


Figure 7.10

Solution

The maximum bending moment is at the middle of the beam and its value is

$$M = \frac{qL^2}{8}$$

We have: $W = \frac{bh^2}{6}$

The stress is computed as

$$\sigma_{\substack{max \\ min}} = \frac{\dfrac{qL^2}{8}}{\dfrac{bh^2}{6}} = \frac{3qL^2}{4bh^2}$$

Substitute the values of parameters into the above equation, we have

$$\sigma_{\substack{max \\ min}} = \pm\frac{3\times 10\times 4000^2}{4\times 200\times 250^2} = 9.6 \text{ MPa} \Rightarrow \sigma_{max} \le [\sigma] \text{ OK}$$

§3. SHEARING AND BENDING

If a section has both bending moment and shear force, the assumption that the deformed section is still in a plane is not true as can be seen in **Figure 7.11**. However, experiments show that the normal stress does not change significantly compared to the case of pure bending. Hence, the formula for normal stress is also used for the case of combined shear and bending.

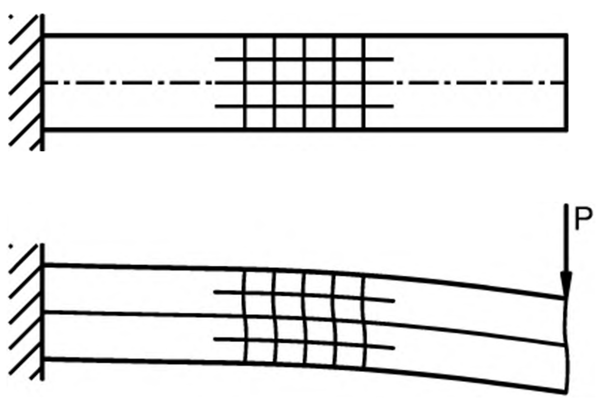


Figure 7.11. A cantilever beam under bending and shear

- M_x causes normal stress.
- Shear force causes shear stress.

Consider a rectangular cross section with the width b and the height h (> b).

D. I. Zhuravskii's assumptions:

- Direction of shear stress is the direction of the shear force.
- The shear stress is the same for every point which has the same distance from neutral axis.

Consider the segment dz as shown in **Figure 7.12**.

To investigate the shear stress at K located at y_o, take a cut cross K and paralleled to the neutral plane.

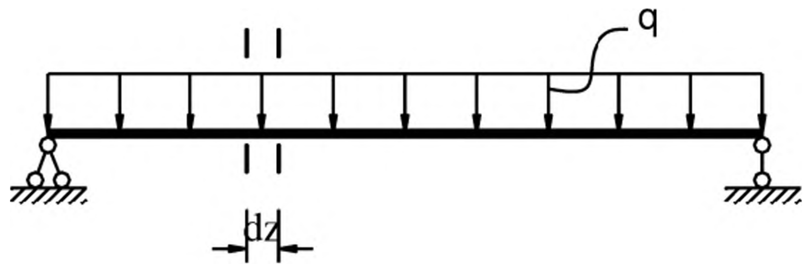


Figure 7.12. A beam subjected to distributed load

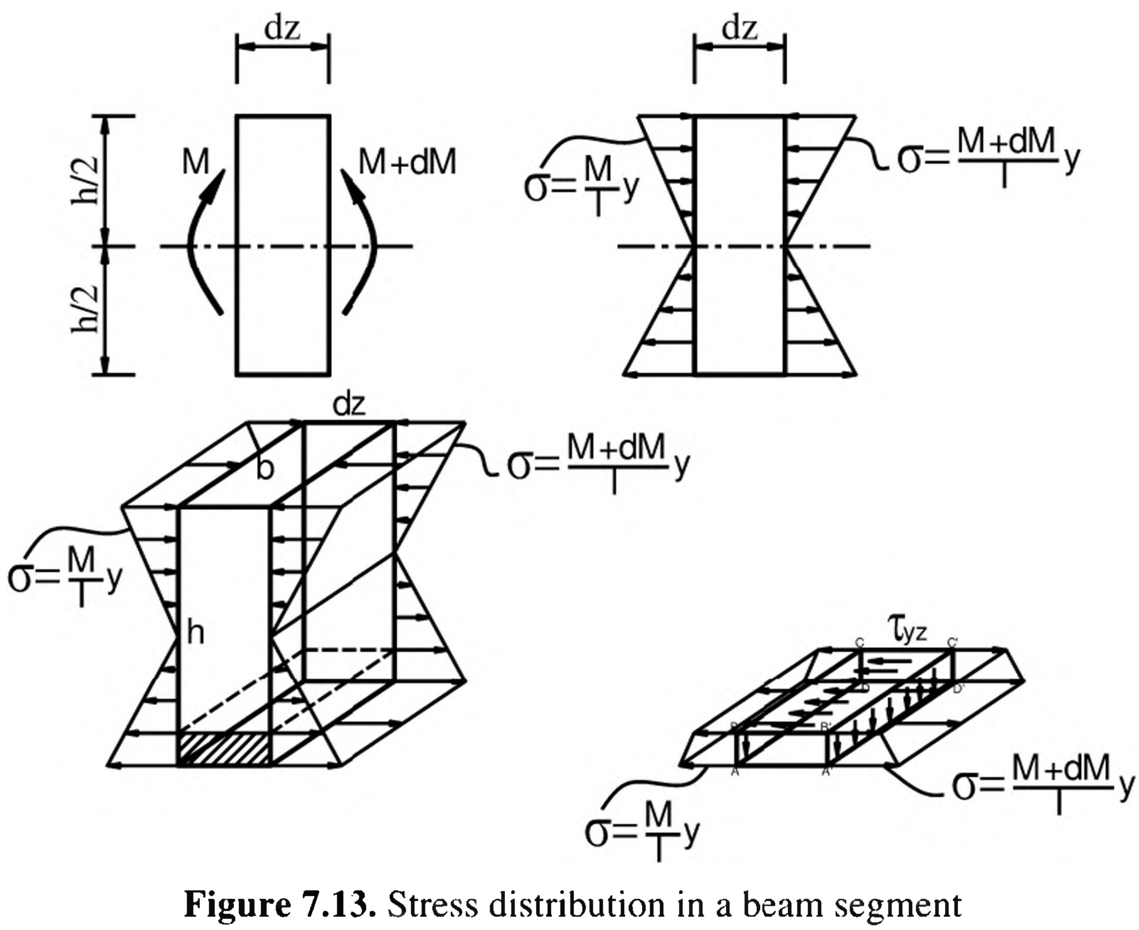


Figure 7.13. Stress distribution in a beam segment

Consider the element ABCDA'B'C'D'

Equilibrium equation:

$$N_1 - N_2 + T = 0 \tag{7.21}$$

in which,

N_1 is the force on the surface ABCD:

$$N_1 = \int_{ABCD} \sigma_z dA = \int_{ABCD} \frac{M_x}{I_x} y dA \tag{7.22}$$

N_2 is the force on the surface A'B'C'D':

$$N_2 = \int_{A'B'C'D'} \sigma_z' dA = \int_{A'B'C'D'} \frac{M_x + dM_x}{I_x} y dA \tag{7.23}$$

T is the force on the surface BB'C'B:

$$T = \tau_{yz} b dz \tag{7.24}$$

The equilibrium Equation 7.21 becomes:

$$\int_{ABCD} \frac{M_x}{I_x} y dA - \int_{A'B'C'D'} \frac{M_x + dM_x}{I_x} y dA + \tau_{yz} b dz = 0$$

$$\tau_{yz} b dz = \int_{A'B'C'D'} \frac{dM_x}{I_x} y dA$$

$$\tau_{yz} = \frac{dM_x}{dz} \frac{1}{I_x b} \int_{A'B'C'D'} y dA$$

We have the differential relationship:

$$Q_y = \frac{dM_x}{dz} \tag{7.25}$$

$\int_{A'B'C'D'} y dA$ is the first moment of ABCD or A'B'C'D'.

$$\int_{A'B'C'D'} y dA = S_x^c$$

$$\tau_{yz} = \tau_{zy} = \frac{Q_y S_x^c}{I_x b^c} \tag{7.26}$$

For a rectangular cross section:

$$S_x^c = b\left(\frac{h}{2} - y\right)\left(y + \frac{\frac{h}{2} - y}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$\tau_{zy} = \frac{Q_y}{2I_x}\left(\frac{h^2}{4} - y^2\right) \tag{7.27}$$

⇒ The shear stress is a second order function of y. The shear stress is zero at the top or the bottom (y = ± h/2) and maximum at the neutral axis (y = 0). The maximum shear stress is:

$$\tau_{\max} = \frac{Q_y h^2}{8I_x} = \frac{3}{2}\frac{Q_y}{A} \tag{7.28}$$

Note that the above stress formula applies to beams with the width b less than the height h. When b = h, the real shear stress is 13% greater than the calculated value using the above formula.

3.1. Exercise 5

A simply supported beam with a rectangular section b = 180 mm, h = 270 mm, span length L = 4 m, load q = 12 N/mm. Materials available: $[\sigma] = 11\ MPa;\ [\tau] = 22\ MPa$.

Requirements: Check the conditions of maximum normal stress and maximum shear stress.

Solution:

Maximum moment at the middle of the beam is:

$$M_{\max} = \frac{ql^2}{8} = \frac{12 \times 4000^2}{8} = 24 \times 10^6\,Nmm$$

Shear force at support:

$$Q_{\max} = \frac{ql}{2} = \frac{12 \times 4000}{2} = 24 \times 10^3\,N$$

Maximum stresses:

$$\sigma_{\max} = \frac{M_{\max}}{W} = \frac{24 \times 10^6}{\frac{180 \times 270^2}{6}} = 10.97MPa < 11MPa$$

$$\tau_{\max} = \frac{3Q_{\max}}{2bh} = \frac{3 \times 24 \times 10^3}{2 \times 180 \times 270} = 0.74MPa < 22MPa$$

For I and T sections

Since the I or T cross sections can be considered to be composed of rectangles, with a certain degree of accuracy, the deduction formulas for rectangular beams are also applicable to these types of sections. The shear stresses at any point with a distance from the neutral axis as shown in **Figure 7.14** are calculated by the formula 7.27.

$$\tau = \frac{Q_y.S_x^c}{I_x.b^c} \tag{7.29}$$

In the case of a shear stress at a point in the web, the width b^c is the width of the web t. S_x^c is the static moment of the slash area below the neutral axis x. The maximum stress is at the mid height of the section. It is zero at the top and bottom fibers. At the connection between the web and the flanges, there is a jump because b^c suddenly changes from b^c = t in the web to b^c = t in the flange.

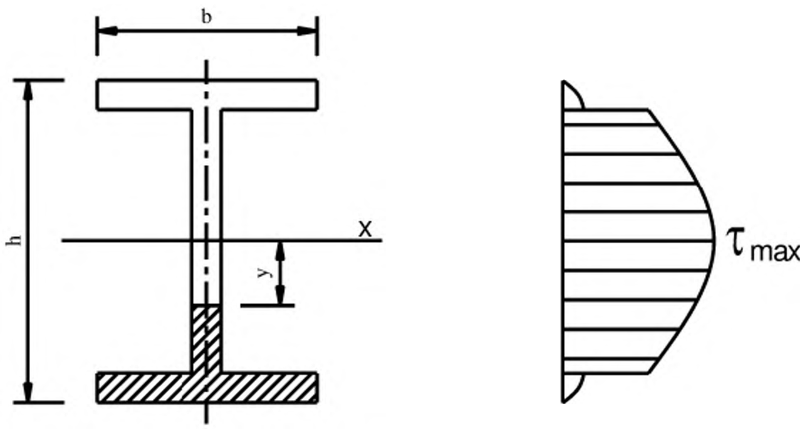


Figure 7.14. Distribution of shear stress in I beam

3.2. Exercise 6

Determine the maximum shear stress in the given T beam with b = 80 mm, t = 20 mm, h = 160 mm, h_l = 140 mm and Q_y = 20 kN.

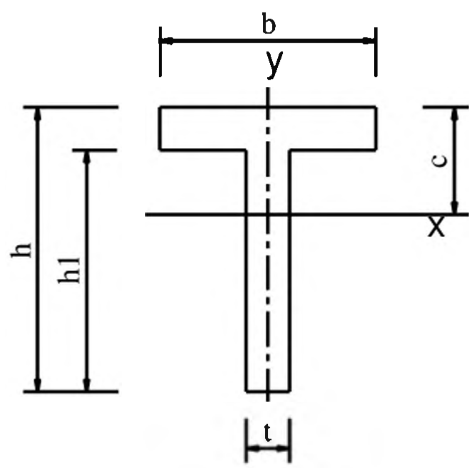


Figure 7.15

Solution

Determine the center:

$$c = \frac{80\times20\times10+140\times20\times90}{80\times20+140\times20} = 60.9mm$$

Inertia moment:

$$I_x = \frac{80\times20^3}{12} + (80\times20)(60.9-10)^2 + \frac{20\times140^3}{12} + (140\times20)(99.1-70)^2$$
$$= 11.143\times10^6 mm^4$$

Maximum shear stress at neutral axis.

We have first moment of the cut-off part:

$$S_x^c = 2(160-60.9)\frac{(160-60.9)}{2} = 982.08\times10^5 mm^3$$

Thus: $$\tau_{\max} = \frac{Q_y S_x^c}{I_x b^c} = \frac{20\times10^3\times982.08\times10^5}{11.143\times10^6\times20} = 8.81MPa$$

For circular and donut sections

When the beam has a circular cross-section, the shear stress on the cross-section is no longer parallel to the shear force. If no force is applied to the outer surface of the beam, the shear stresses on the two infinitesimal areas at points 1 and 2 on the area adjacent to the circumference of the cross-section must be oriented tangentially to this circumference as shown in **Figure 7.16**.

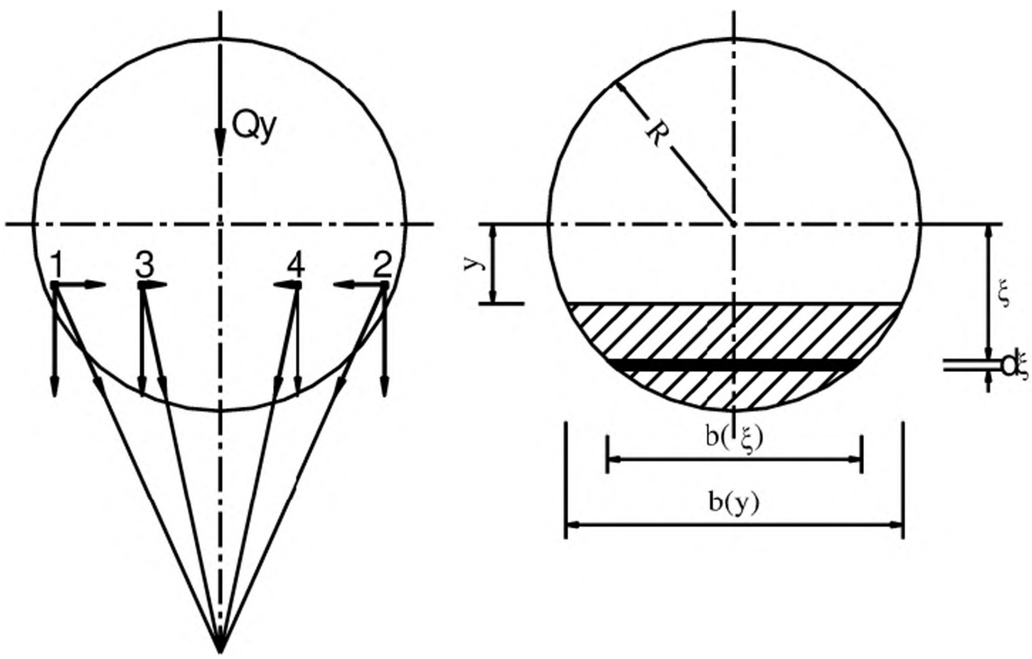


Figure 7.16

These tangents will converge at point C on the direction of the shear force. Because the shear force Q_y is the total of shear stresses, the shear stresses at the infinitesimal areas at 3 and 4 with the same distance to the neutral axis will have a traverse direction across C.

Each of these shear stresses has two components: the vertical component and the horizontal component. The horizontal components of the left and right sections will balance themselves due to the symmetry. The total of vertical components is the shear force Q_y.

Thus, in circular cross-section beams, vertical components play the role of shear stresses which is similar to the shear stresses in rectangular cross-section beams.

First moment:

$$S_x^c = \int_{A_c} dA\xi = \int_{A_c} \xi b(\xi)d\xi$$

We have:

$$b^c = b(\xi) = 2\sqrt{R^2 - y^2}$$

So:

$$S_x^c = \int_y^R 2\sqrt{R^2 - y^2}\xi d\xi = \frac{2}{3}\left(R^2 - y^2\right)^{3/2} \infty$$

$$\tau_{zy} = \frac{Q_y S_x^c}{I_x b^c} = \frac{Q_y \frac{2}{3}\left(R^2 - y^2\right)^{3/2}}{\frac{\pi R^4}{4} 2\sqrt{R^2 - y^2}} = \frac{4}{3}\frac{Q_y}{A}\left(1 - \frac{y^2}{R^2}\right)$$

The maximum shear stress on the neutral axis with y = 0

$$\tau_{zy} = \frac{4}{3}\frac{Q_y}{A} \tag{7.30}$$

For donut section:

$$S_x^c = \frac{2}{3}\left(R^2 - r^2\right)$$

$$I_x = \frac{\pi}{4}\left(R^4 - r^4\right)$$

$$I_x = 2(R - r)$$

The maximum shear stress:

$$\tau_{zy} = \frac{4}{3}\frac{Q_y}{A}\frac{R^2 + Rr + r^2}{R^2 + r^2} \tag{7.31}$$

3.3. Exercise 7

Consider the rectangular beam shown in **Figure 7.17**. Given q = 5 kN/m, a = 1 m, b = 30 cm, h = 40 cm.

a) Draw the diagrams of bending moment and shear force.

b) Calculate σ_{max}, σ_{min}.

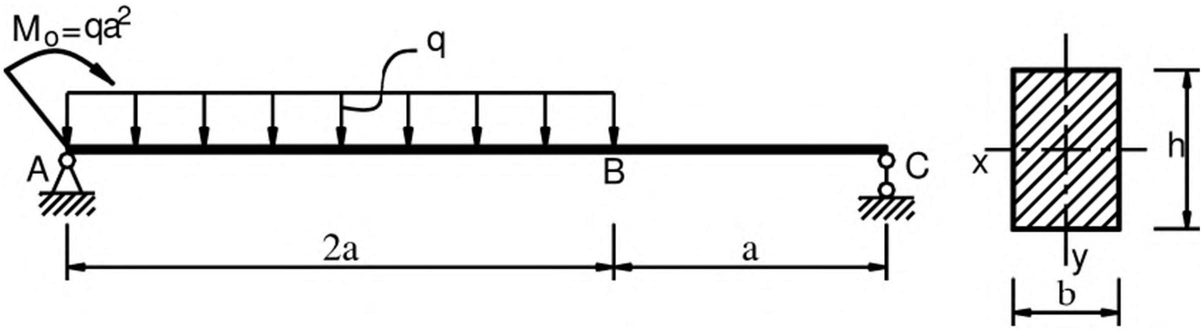


Figure 7.17

Solution

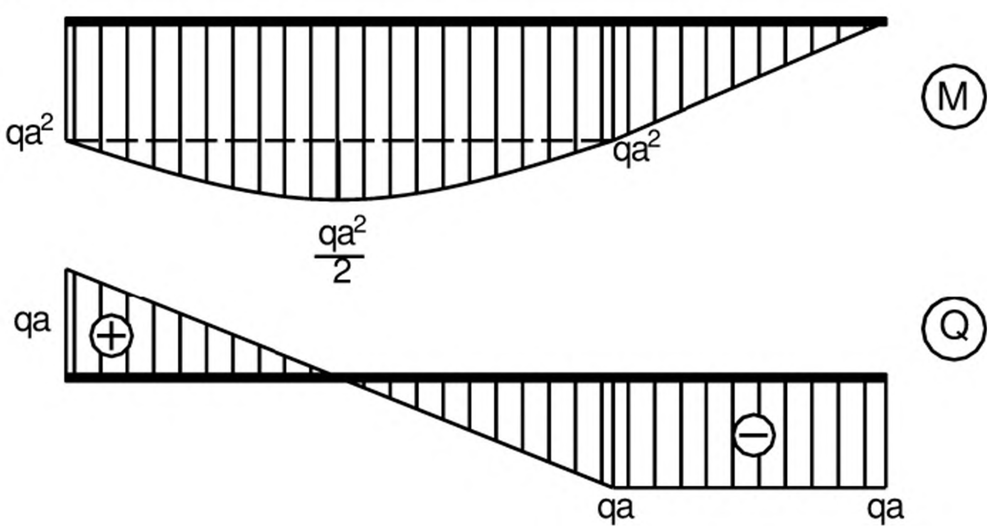
a) Draw the diagrams of bending moment and shear force.

Support reactions:

$$\sum M_{/A} = 0 \Leftrightarrow q(2a)a + qa^2 = V_C 3a \Rightarrow V_C = qa$$

$$\sum M_{/C} = 0 \Leftrightarrow V_A(3a) + qa^2 = q(2a)(2a) \Rightarrow V_A = qa$$

Results:



$$M_{\max} = \frac{3qa^2}{2} = \frac{3\times5\times1^2}{2} = 7,5kNm = 750kNcm$$

$$Q_{\max} = qa = 5\times1 = 5kN$$

Calculate σ_{max}, σ_{min}.

$$I_x = \frac{30\times40^3}{12} = 1,6.10^5 cm^4$$

$$W_x = \frac{30\times40^2}{6} = 8.10^3 cm^3$$

$$\sigma_{\substack{\max\\ \min}} = \pm\frac{|M_x|}{W_x} = \pm\frac{750}{8.10^3} = 0,09375kN/cm^2$$

$$\tau_{\max} = \frac{3}{2}\frac{Q_y}{A} = \frac{3}{2}\frac{5}{30\times40} = 0,00625kN/cm^2$$

Or:

$$S_x^c = 20\times30\times\frac{20}{2} = 6000cm^3$$

$$\tau_{\max} = \frac{Q_y S_x^c}{I_x b^c} = \frac{5\times6000}{1,6.10^5\times30} = 0,00625kN/cm^2$$

§4. STRESS CONDITIONS

There are three stress states in beams subjected to shear and bending as shown in **Figure 7.18**.

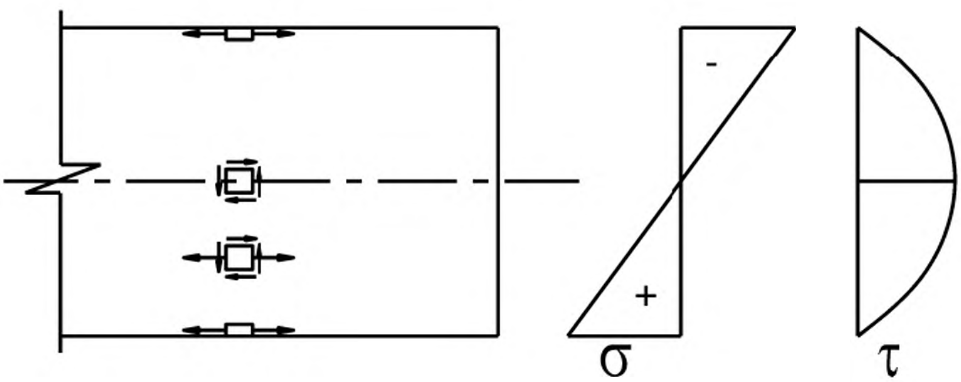


Figure 7.18. Elements for checking stress condition

- At the top and bottom fibres: There is only normal stress.

For ductile materials $[\sigma]_{tension} = [\sigma]_{compression} = [\sigma]$, the stress condition is

$$Max|\sigma| \leq [\sigma] \tag{7.32}$$

For brittle materials $[\sigma]_{tension} \neq [\sigma]_{compression}$, the stress condition is

$$\sigma_{\max} \leq [\sigma]_{tension} \tag{7.33}$$

$$\sigma_{\min} \leq [\sigma]_{compression} \tag{7.34}$$

• At the neutral fibre: There is only shear stress and the element is under pure shear

For ductile materials:

Failure theory 3:

$$\tau_{\max} \leq [\tau] = \frac{[\sigma]}{2} \tag{7.35}$$

Failure theory 4:

$$\tau_{\max} \leq [\tau] = \frac{[\sigma]}{\sqrt{3}} \tag{7.36}$$

For brittle materials: Use Mohr's theory.

• Other fibres: There are both normal stress and shear stress (plane stress state). For sections I, C, ... the stress at the connection between flange and web is normally checked.

The stresses σ_z and τ_{zy} are computed from M_x and Q_y, then we compute σ_1 and σ_3 using the following fomula:

$$\sigma_{\substack{\max \\ \min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = \frac{\sigma_z}{2} + \sqrt{\left(\frac{\sigma_z}{2}\right)^2 + \tau_{zy}^2}$$

$$\sigma_3 = \frac{\sigma_z}{2} - \sqrt{\left(\frac{\sigma_z}{2}\right)^2 + \tau_{zy}^2}$$

$$\sigma_2 = 0$$

For ductile materials:

Use failure theory 3

$$\sigma_{t3} = |\sigma_1 - \sigma_3| \leq [\sigma]_k \tag{7.37}$$

in which: $$\sigma_1 - \sigma_3 = 2\sqrt{\left(\frac{\sigma_z}{2}\right)^2 + \tau_{zy}^2}$$

Use failure theory 4:

$$\sigma_{t4} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \le [\sigma]_k$$

$$\sigma_{t4} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_3\sigma_1} \le [\sigma] \qquad (7.38)$$

For brittle materials: Use Mohr's theory.

4.1. Exercise 8

Determine the No. of the cross-section steel as shown in **Figure 7.19a**. Given: $[\sigma]$=160 MPa, P = 60 kN, L_1 = 1 m, L_2 = 6 m.

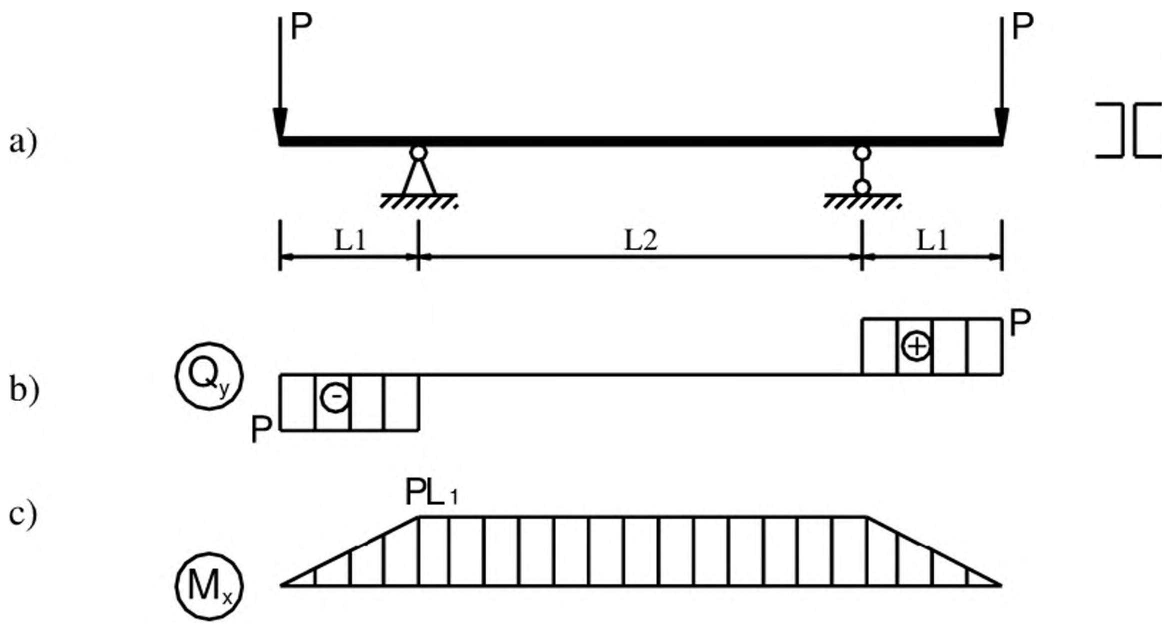


Figure 7.19

Solution

M_{max} and Q_{max} at the support:

$$M_{max} = PL_1 = 60 \times 10^3.1000 = 60 \times 10^6\, Nmm$$

$$Q_{max} = P = 60 \times 10^3\, N$$

Minimum sectional modulus required:

$$W = \frac{M_{max}}{[\sigma]} = \frac{60 \times 10^6}{160} = 3.75 \times 10^5\, mm^3 = 375 cm^3$$

Based on Table of sectional properties of steel C, TCVN1654-75, we can choose 2[22. For 1 [22: h = 22cm, b = 8.2cm, d = 0.54cm, t = 0.95cm, A = 26.7cm^2, I_x = 2110cm^4, W_x = 192cm^3, S_x = 110cm^3.

Check the normal stress:

W_x of 2[22 is 2×192 = 384 cm^3 > W_x needed ==> OK.

Check the shear stress:

$$\tau_{max} = \frac{Q_y S_x^c}{I_x b^c} = \frac{60\times10^6\times(2\times110\times10^3)}{(2\times2110\times10^4)\times(2\times5.4)} = 29MPa$$

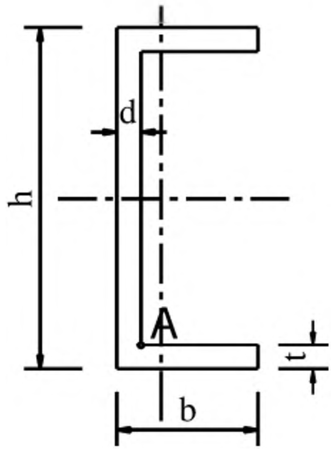


Figure 7.20

Based on the failure theory of maximum shear stress:

$$[\tau] = \frac{[\sigma]}{2} = \frac{160}{2} = 80MPa > \tau_{max} = 29MPa$$

The element at the location between web and flange (point A):

$$\sigma_A = \frac{60\times10^6}{(2\times2110\times10^4)}(110-9.5) = 143MPa$$

$$S_x^c = 2\times(82\times9.5)\left(110-\frac{9.5}{2}\right) = 163.98\times10^3 mm^3$$

$$\tau_A = \frac{Q_y S_x^c}{I_x b^c} = \frac{60\times10^3\times163.98\times10^3}{(2\times2110\times10^4)\times(2\times5.4)} = 21.6MPa$$

Based on the failure theory of maximum shear stress:

$$\sigma_{t3} = \sqrt{\sigma^2+4\tau^2} = \sqrt{143^2+4\times21.6^2} = 149.4MPa < [\sigma]$$

= > OK.

4.2. Exercise 9

Determine the allowable load [P] for the beam I10 as shown in **Figure 7.21a**. Given a = 80 cm, $[\sigma] = 160$ kN/cm^2.

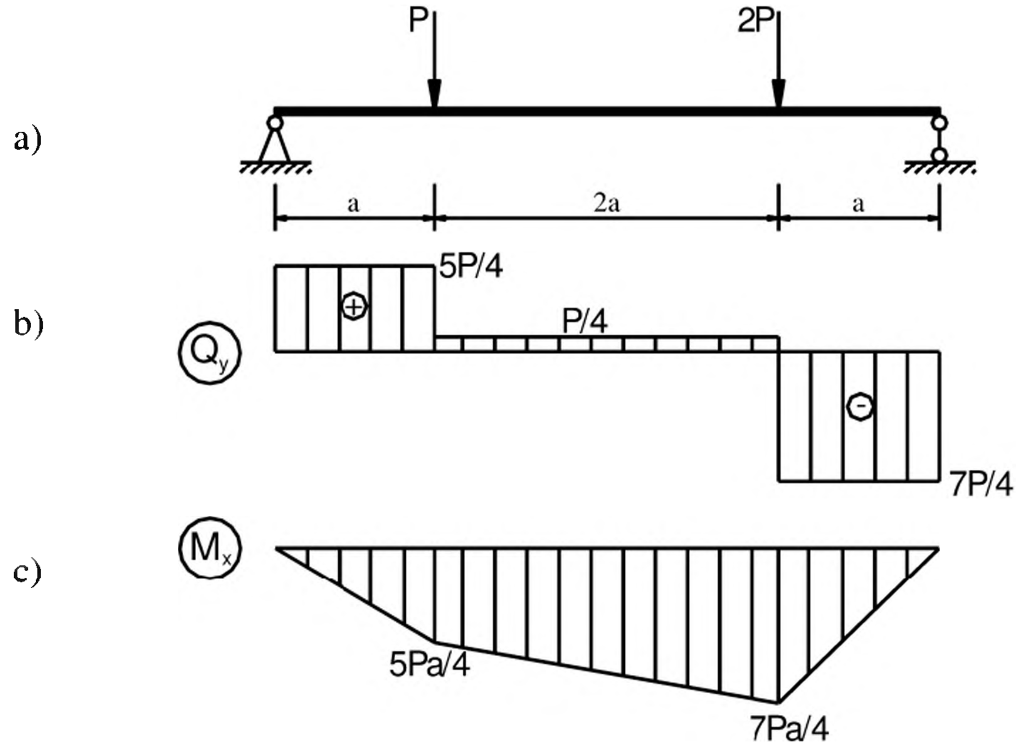


Figure 7.21

The diagrams of shear force and bending moment is shown in **Figure 7.21b** and c, respectively. The critical section has:

$$M_x = \frac{7}{4}Pa, \quad Q_y = \frac{7}{4}P$$

I10 has: h = 10cm, b = 5.5cm, d = 0.45cm, t = 0.72cm, A = 12cm^2, $I_x = 198$cm^4, $W_x = 39.7$cm^3, $S_x = 23$cm^3.

The stress condition:

$$\sigma_{max} = \frac{M_{max}}{W_x} \leq [\sigma]$$

$$\frac{7Pa}{4W_x} \leq [\sigma]$$

$$\Rightarrow \quad P \leq \frac{4W_x[\sigma]}{7a} = \frac{4 \times 39.7 \times 16}{7 \times 0.8} = 4.54kN$$

Select: [P] = 4.54 kN.

For the allowable load [P], we check the stress state for other element at pure shear and special plane stress state.

- Element at pure shear (at the neutral axis and $Q_y = 7P/4$).

$$S_x^c = S_x = 23cm^3$$

$$b^c = d = 0,45cm$$

$$\tau = \frac{Q_y S_x^c}{I_x b^c} = \frac{(7\times 4.54/4)\times 23}{198\times 0.45} = 2.05kN/cm^2 < [\tau] = \frac{[\sigma]}{2} = 8kN/cm^2$$

=> OK.

- Element at special plane stress state (at the location between web and flange):

$$M_x = \frac{7}{4}Pa = \frac{7}{4}4.54\times 0.8 = 6.35kNm$$

$$Q_y = \frac{7}{4}4.54 = 7.94kN$$

$$S_x^c = 5.5\times 0.72\times \frac{10-0.72}{2} = 18.37cm^3$$

$$b^c = d = 0.45cm$$

$$\tau = \frac{Q_y S_x^c}{I_x b^c} = \frac{(7\times 4.54/4)\times 18.37}{198\times 0.45} = 1.64kN/cm^2$$

$$\sigma = \frac{6.35\times 10^2}{198}(10/2-0.72) = 13.73kN/cm^2$$

Based on the failure theory of maximum shear stress:

$$\sigma_{t3} = \sqrt{\sigma^2+\tau^2} = \sqrt{13.73^2+4\times 1.64^2} = 14.12kN/cm^2 < [\sigma]$$

=> OK.

Conclude: [P] = 4.54 kN.

PROBLEMS

PROBLEM 1. Consider a simply supported beam subjected to the concentrated load P = 60 kN at the middle of the beam. The length of the beam is L = 6 m. The dimensions of the cross section are b = 200 mm, h = 400 mm, [σ] = 12 MPa. Check the stress condition caused by bending moment.

PROBLEM 2. Consider the beam BCD subjected to loads as shown in **Figure 7.22**. The cross section of the beam BCD has the width b = 300 mm and the height h = 400 mm. Given: q = 10 kN/m, a = 2 m.

a) Determine the support reactions.

b) Calculate and draw the diagrams of shear force and bending moment for the beam BCD.

c) Calculate the maximum and minimum normal stresses in the beam BCD.

d) Calculate the maximum absolute value of shear stress in the beam BCD.

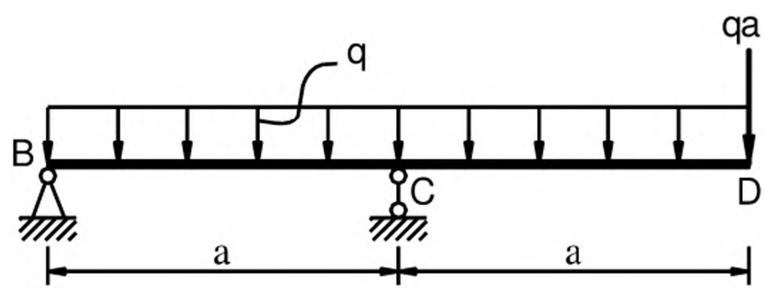


Figure 7.22

PROBLEM 3. Consider the rectangular beam shown in **Figure 7.23**. Given: q = 10 kN/m, a = 5 m, b = 30 cm, h = 40 cm.

a) Determine the support reactions.

b) Calculate and draw the diagrams of shear force and bending moment for the beam.

c) Calculate the maximum and minimum normal stresses in the beam.

d) Calculate the maximum absolute value of shear stress in the beam.

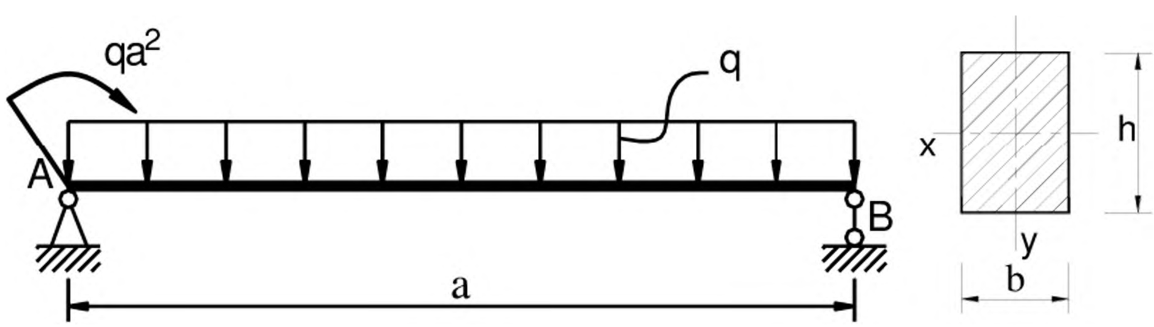


Figure 7.23

PROBLEM 4. Consider the beam subjected to the two loads P as shown in **Figure 7.24.** Determine the allowable load [P] for the beam I20. Given: a = 100 cm, $[\sigma] = 160\ kN/cm^2$.

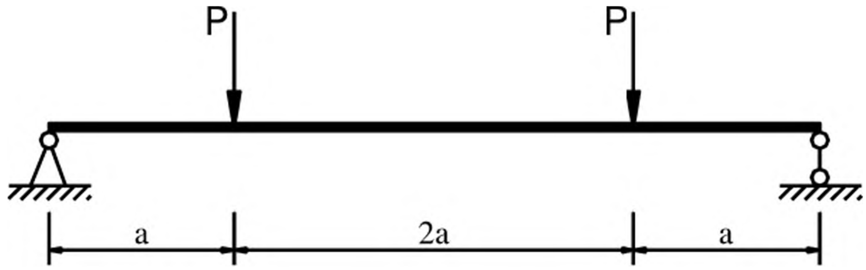


Figure 7.24

Chapter 8

DEFLECTION OF BEAMS

§1. CONCEPTS

A beam subjected to loads must satisfy not only the stress condition (durability) but also the deflection condition (deformation condition). The stress condition was studied in previous chapters. The deflection condition is presented in this chapter.

Under the effect of the load, a beam is bent. The longitudinal axis beam is also bent and is called the *elastic curve*.

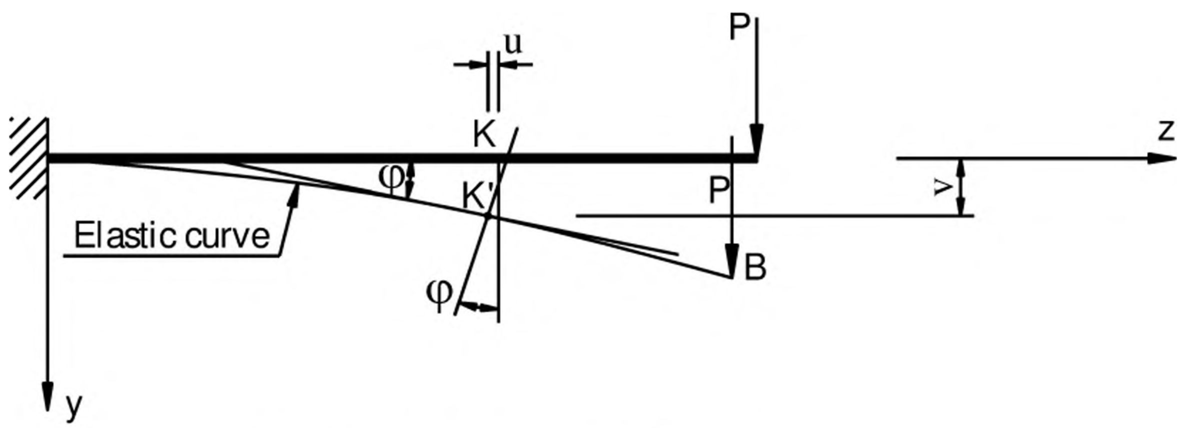


Figure 8.1. Deflection of a cantilever beam

After deformation, the point K moves to its new position K'. The displacement KK' can be separated into two components:

- The component v is perpendicular to the axial axis and is called vertical displacement.
- Component u is parallel to the axial axis and is called horizontal displacement.
- The cross section is rotated an angle φ which is called rotation.

Thus,

$$\begin{cases} u \\ v \\ \varphi \end{cases}$$ are the three displacement components of the section K.

When the deformation of the beam is small, consequently, u is small and can be neglected. Thus, KK' ≈ v. That is, KK' is perpendicular to the axial axis.

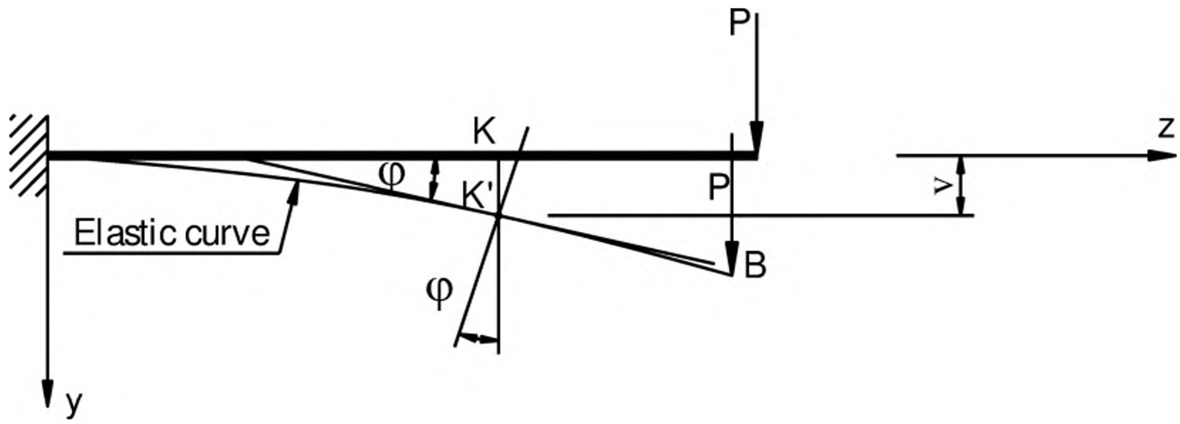


Figure 8.2

We have:

$$\varphi \approx \tan \varphi = \frac{dv}{dz}$$

In the coordinate system Oyz:

Equation of elastic curve:

$$y(z) = v(z)$$

Equation of rotation:

$$\varphi(z) = \frac{dv}{dz} = \frac{dy}{dz} = y'(z) \tag{8.1}$$

Deflection conditions

$$\left[\frac{f}{L}\right] = \frac{1}{300} \div \frac{1}{1000}$$ depends on structures.

in which,

f is the deflection of beam.

L is the length of beam.

§2. EQUATIONS OF ELASTIC CURVES

2.1. The differential equation of elastic curves

Consider a cantilever beam BC subjected to the vertical load P as shown in **Figure 8.3a**. The beam was deformed, and the elastic curve is BC'. Take the infinite segment dz, after deformation, it becomes ds. The angle between the horizontal axis and the tangent of the elastic curve at z is φ. Consequently, the angle between the two tangents of the sections at z and z + dz is dφ. The vertical distance from the section at z to the section at z + dz is dy as shown in **Figure 8.3b**.

In Chapter 7, we have: $\frac{1}{\rho} = \frac{M_x}{EI_x}$

Consider the infinite segment ds:

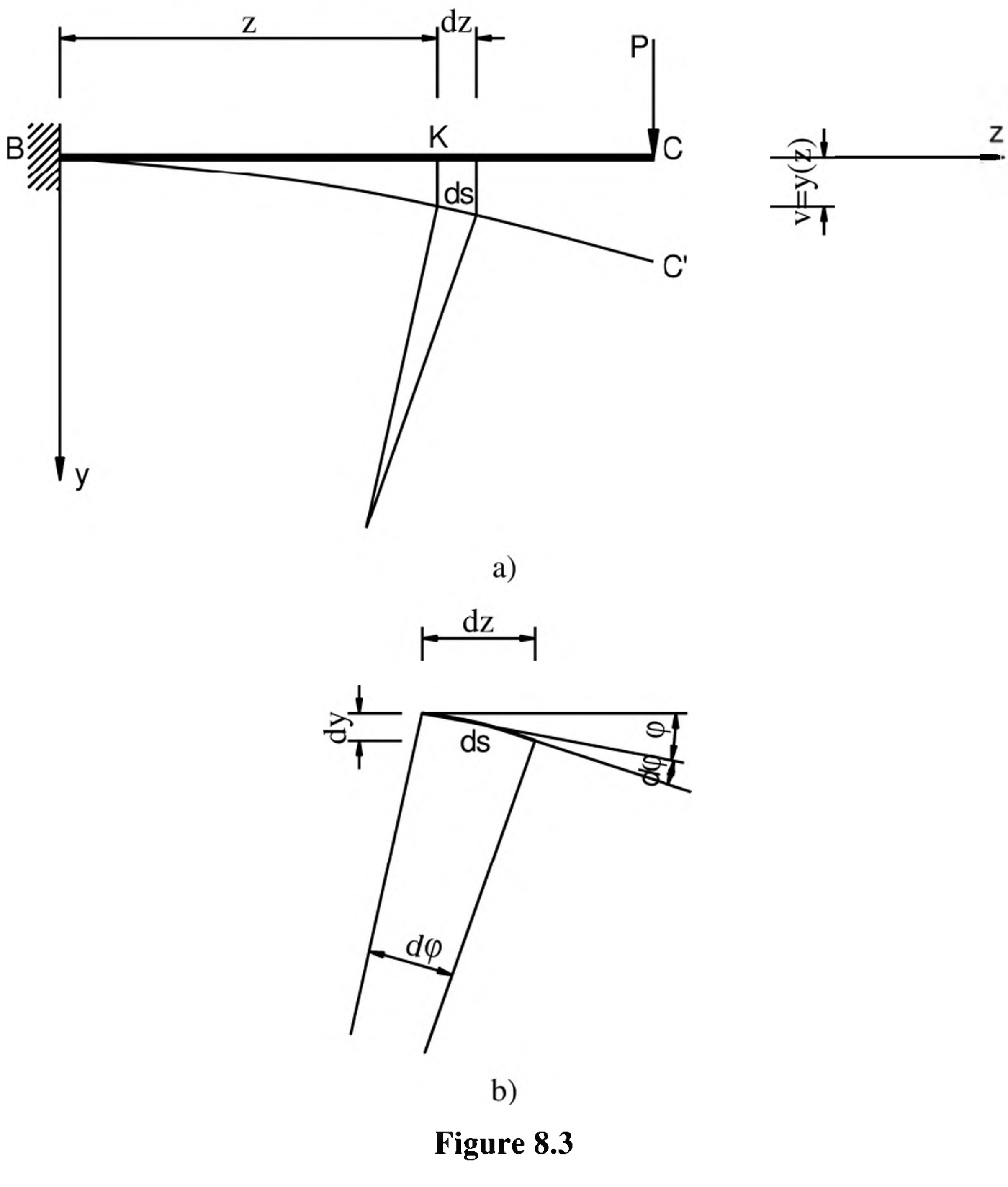


Figure 8.3

We have the following geometric relationship:

$$ds = \rho d\varphi \;\Rightarrow\; \frac{1}{\rho} = \frac{d\varphi}{ds} = \frac{d\varphi}{dz}\frac{dz}{ds} \tag{8.2}$$

$$\tan\varphi = \frac{dy}{dz}$$

$$\Rightarrow \quad \varphi = \arctan\frac{dy}{dz} = \arctan y'$$

$$\Rightarrow \quad \frac{d\varphi}{dz} = \frac{d}{dz}(\arctan y') = \frac{y''}{1+(y')^2} \tag{8.3}$$

$$ds = \sqrt{dz^2 + dy^2} = \sqrt{1+\left(\frac{dy}{dz}\right)^2}dz = \sqrt{1+(y')^2}dz$$

$$\Rightarrow \quad \frac{dz}{ds} = \frac{1}{\sqrt{1+(y')^2}} \tag{8.4}$$

From the three above equations, we have:

$$\frac{1}{\rho} = \frac{y''}{\left[1+(y')^2\right]^{3/2}} \tag{8.5}$$

or:

$$\frac{M_x}{EI_x} = \frac{y''}{\left[1+(y')^2\right]^{3/2}} \tag{8.6}$$

Because the sign of y'' is opposite to M_x. Thus:

$$\frac{y''}{\left[1+(y')^2\right]^{3/2}} = -\frac{M_x}{EI_x} \tag{8.7}$$

Figure 8.4 shows the sign convention of the curvature.

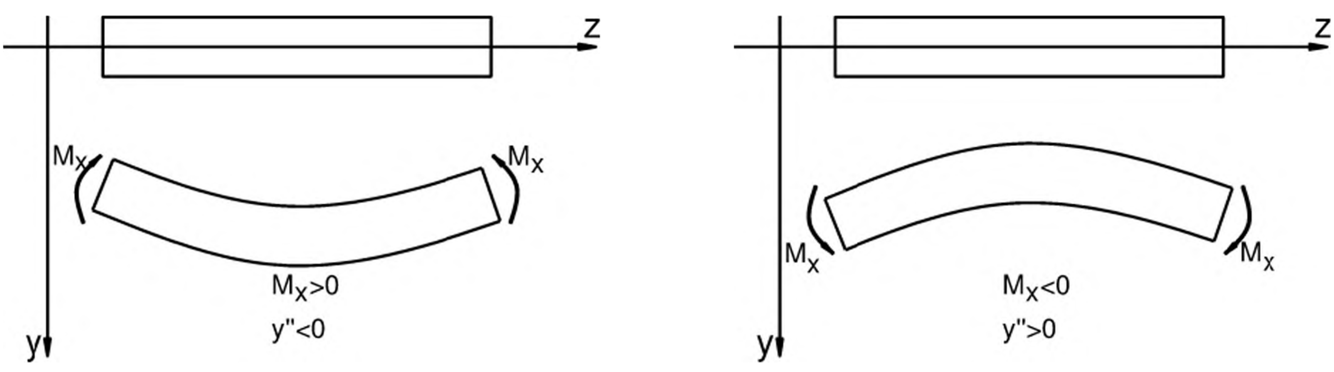


Figure 8.4. Signs of curvature

With the assumption of small deformation, y is thus small. Consequently, the rotation φ or y' is also small. Therefore, y'^2 can be neglected, and we have:

$$y'' = -\frac{M_x}{EI_x} \tag{8.8}$$

in which EI_x is the stiffness of the beam.

2.2. Equation of elastic curves using direct integration method

Integrating the two sides of the differential equation of elastic curves (Equation 8.8):

$$\varphi = y' = \int -\frac{M_x}{EI_x} dz + C \tag{8.9}$$

$$y = \int \left(\int -\frac{M_x}{EI_x} dz + C \right) dz + D \tag{8.10}$$

C and D are two constants, which are determined based on the boundary conditions.

2.3. Exercise 1

Write the equations of elastic curve and rotation for the console beam subjected to the point load P at the beam end. The length of the beam is L, the stiffness EI = const. From the established equations, determine the maximum deflection and rotation.

Solution

Moment at z:

$$M_x = -Pz$$

The second order equation of the elastic curve:

$$y'' = -\frac{M_x}{EI_x} = \frac{Pz}{EI_x}$$

$$\varphi = y' = \int \frac{Pz}{EI_x} dz = \frac{Pz^2}{2EI_x} + C \tag{8.11}$$

$$y = \int \left(\frac{Pz^2}{2EI_x} + C \right) dz = \frac{Pz^3}{6EI_x} + Cz + D \tag{8.12}$$

Boundary condition: y = 0 and y' = 0 when z = L

=> $$C = -\frac{PL^2}{2EI_x}; D = \frac{PL^3}{3EI_x}$$

Thus: $$\varphi = y' = \frac{Pz^2}{2EI_x} - \frac{PL^2}{2EI_x}$$

$$y = \frac{Pz^3}{6EI_x} - \frac{PL^2}{2EI_x}z + \frac{PL^3}{3EI_x} \tag{8.13}$$

The maximum deflection and rotation at the free end:

$$y_{max} = \frac{PL^3}{3EI_x}$$

$$\varphi_{max} = y' = -\frac{PL^2}{2EI_x}$$

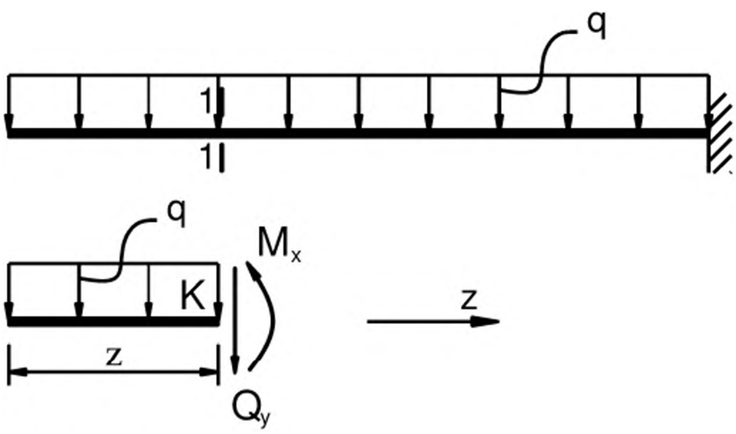
$y_{max} > 0$ => the deflection is downward.

$\varphi < 0$ => the direction of rotation is anticlockwise.

2.4. Exercise 2

Write the equations of elastic curve and rotation for the console beam subjected to the distributed load q. The length of the beam is L, the stiffness EI = const. From the established equations, determine the maximum deflection and rotation.

Solution



Moment at the cut z:

$$M_x = -qz^2/2$$

We have:

$$y'' = -\frac{M_x}{EI_x} = \frac{qz^2}{2EI_x}$$

$$\varphi = y' = \int \frac{qz^2}{2EI_x}dz = \frac{qz^3}{6EI_x} + C$$

$$y = \int \left(\frac{qz^3}{6EI_x} + C \right) dz = \frac{qz^4}{24EI_x} + Cz + D$$

Boundary conditions:

At z = L, we have: y = 0, y' = 0

We get:

$$C = -\frac{qL^3}{6EI_x}$$

$$D = \frac{qL^4}{8EI_x}$$

Thus: $$\varphi = y' = \frac{qz^3}{6EI_x} - \frac{qL^3}{6EI_x}$$

$$y = \frac{qz^4}{24EI_x} - \frac{qL^3}{6EI_x}z + \frac{qL^4}{8EI_x}$$

Maximum deflection and rotation at the free end:

$$y_{\max} = \frac{qL^4}{8EI_x} \tag{8.14}$$

$$\varphi = y' = -\frac{qL^3}{6EI_x} \tag{8.15}$$

$y_{max} > 0$ => the deflection is downward.

$\varphi < 0$ => the rotation is anticlockwise.

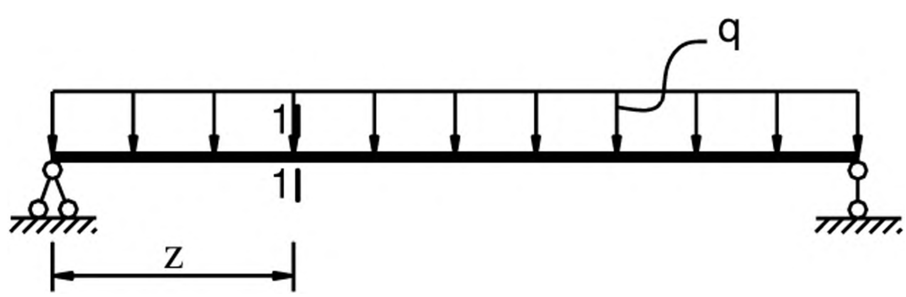
2.5. Exercise 3

Determine the equations of deflection and rotation for a simply supported beam subjected to a distributed load q.

Then, determine the maximum deflection and rotation.

Given the beam length is L; EI = const.

Solution



Moment at the cut z:

$$M_x = \frac{qL}{2}z - \frac{qz^2}{2} = \frac{q}{2}\left(Lz - z^2\right)$$

We have:

$$y'' = -\frac{M_x}{EI_x} = -\frac{q}{2EI_x}\left(Lz - z^2\right)$$

$$\varphi = y' = \int -\frac{q}{2EI_x}\left(Lz - z^2\right)dz = -\frac{q}{2EI_x}\left(\frac{Lz^2}{2} - \frac{z^3}{3}\right) + C$$

$$y = \int\left(-\frac{q}{2EI_x}\left(\frac{Lz^2}{2} - \frac{z^3}{3}\right) + C\right)dz = -\frac{q}{2EI_x}\left(\frac{Lz^3}{6} - \frac{z^4}{12}\right) + Cz + D$$

Boundary conditions:

At z = 0, we have y = 0,

At z = L, we have y = 0,

We get:

$$D = 0$$

$$C = \frac{qL^3}{24EI_x}$$

Thus:

$$\varphi = y' = -\frac{q}{2EI_x}\left(\frac{Lz^2}{2} - \frac{z^3}{3}\right) + \frac{qL^3}{24EI_x}$$

$$y = -\frac{q}{2EI_x}\left(\frac{Lz^3}{6} - \frac{z^4}{12}\right) + \frac{qL^3}{24EI_x}z$$

Maximum deflection and rotation at the middle of the beam (z = L/2):

$$y_{\max} = \frac{5}{384}\frac{qL^4}{EI_x} \tag{8.16}$$

$$\varphi = y' = 0$$

$y_{max} > 0$ => the deflection is downward.

2.6. Exercise 4

Write the equations of elastic curve and rotation for the simply supported beam subjected to the point load P as shown in **Figure 8.5**. The length of the beam is L, the stiffness EI = const. From the established equations, determine the maximum deflection and rotation.

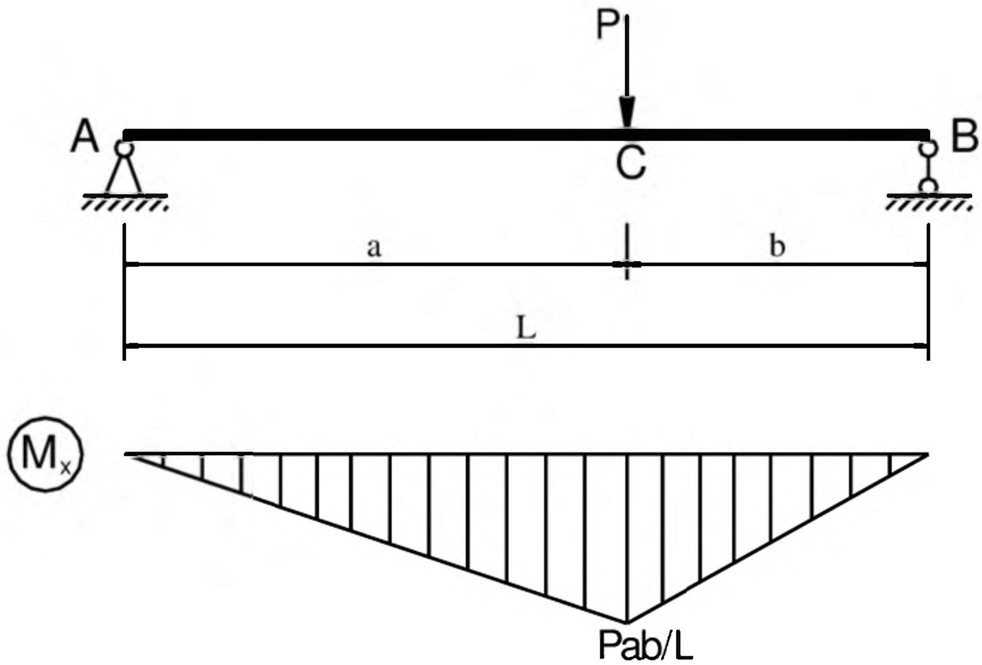


Figure 8.5

Solution

Moment at the section of segments:

Segment AC: $0 \le z_1 \le a$:

$$M_{x(AC)} = \frac{Pb}{L} z_1$$

Segment CB: $a \le z_2 \le L$:

$$M_{x(CB)} = \frac{Pb}{L} z_2 - P(z_2 - a)$$

The second order differential equation:

Segment AC:

$$y_1'' = -\frac{M_x}{EI_x} = -\frac{Pb}{LEI_x} z_1$$

Segment CB:

$$y_2'' = -\frac{M_x}{EI_x}$$

$$= -\frac{\frac{Pb}{L} z_2 - P(z_2 - a)}{EI_x} = -\frac{Pb}{LEI_x} z_2 + \frac{P(z_2 - a)}{EI_x}$$

Integrate the above equations, we have:

Segment AC:

$$\varphi = y' = \int -\frac{Pb}{LEI_x} z_1 dz = -\frac{Pb}{2LEI_x} z_1^2 + C_1$$

$$y = \int\left(-\frac{Pb}{2LEI_x} z_1^2 + C \right)dz = -\frac{Pb}{6LEI_x} z_1^3 + C_1 z + D_1$$

Segment CB:

$$\varphi = y' = \int\left(-\frac{Pb}{LEI_x} z_2 + \frac{P(z_2 - a)}{EI_x} \right)dz$$

$$= -\frac{Pb}{2LEI_x} z_2^2 + \frac{P(z_2 - a)}{EI_x} z_2 + C_2$$

$$y = \int\left(-\frac{Pb}{2LEI_x} z_2^2 + \frac{P(z_2 - a)}{EI_x} z_2 + C_2 \right)dz$$

$$= -\frac{Pb}{6LEI_x} z_2^3 + \frac{P(z_2 - a)}{EI_x} z_2^2 + C_2 z_2 + D_2$$

Boundary conditions:

$y_1 = 0$ when $z_1 = 0$,

$y_2 = 0$ when $z_2 = L$,

$y_1 = y_2$, $y'_1 = y'_2$ when $z_1 = z_2 = a$

We have:

$$D_1 = 0$$

$$-\frac{Pb}{6LEI_x} L^3 + \frac{P(L-a)}{EI_x} L^2 + C_2 L + D_1$$

$$-\frac{Pb}{6LEI_x} a^3 + C_1 a + D_1 = -\frac{Pb}{6LEI_x} a^3 + C_2 a + D_2$$

$$-\frac{Pb}{2LEI_x} a^2 + C_1 = -\frac{Pb}{6LEI_x} a^3 + C_2 a + D_2$$

Solved:

$$D_1 = D_2 = 0$$

$$C_1 = C_2 = \frac{Pb}{6LEI_x}\left(L^2 - b^2\right)$$

Thus:

Segment AC:

$$\varphi = y' = -\frac{Pb}{2LEI_x} z_1^2 + \frac{Pb}{6LEI_x}\left(L^2 - b^2\right)$$

$$y = -\frac{Pb}{6LEI_x} z_1^3 + \frac{Pb}{6LEI_x}\left(L^2 - b^2\right) z_1$$

Segment CB:

$$\varphi = y' = -\frac{Pb}{2LEI_x} z_2^2 + \frac{P(z_2 - a)}{EI_x} z_2 + \frac{Pb}{6LEI_x}\left(L^2 - b^2\right)$$

$$y = -\frac{Pb}{6LEI_x} z_2^3 + \frac{P(z_2 - a)}{EI_x} z_2^2 + \frac{Pb}{6LEI_x}\left(L^2 - b^2\right) z_2$$

The deflection and rotation at the middle of the beam (z = L/2):

At the maximum defection, the rotation is 0.

Assume that a > b, the rotations at the two ending sections of the segment AC:

$$z_1 = 0$$

$$\Rightarrow \quad \varphi_{1A} = \frac{Pb}{6LEI_x}\left(L^2 - b^2\right) > 0$$

$$z_1 = a$$

$$\Rightarrow \quad \begin{aligned} \varphi_{1C} &= -\frac{Pb}{2LEI_x} a^2 + \frac{Pb}{6LEI_x}\left(L^2 - b^2\right) \\ &= \frac{Pb}{6LEI_x}\left(-3a^2 + (a+b)^2 - b^2\right) = \frac{Pab(b-a)}{3LEI_x} < 0 \end{aligned}$$

Thus, the rotation changes its sign when moving from A to C. This means that there is a section in AC segment has the rotation of zero, which is corresponding to the maximum deflection.

Set $$\varphi = -\frac{Pb}{2LEI_x} z_1^2 + \frac{Pb}{6LEI_x}\left(L^2 - b^2\right) = 0$$

$$z_1 = \sqrt{\frac{L^2 - b^2}{3}}$$

Substitute into the deflection equation, we have:

$$y_{max} = -\frac{Pb}{6LEI_x}\left(\frac{L^2 - b^2}{3}\right)^{3/2} + \frac{Pb}{6LEI_x}\left(L^2 - b^2\right)\sqrt{\frac{L^2 - b^2}{3}}$$

$$y_{\max} = \frac{Pb\left(L^2-b^2\right)^{3/2}}{6LEI_x}\left[-\left(\frac{1}{3}\right)^{3/2}+\sqrt{\frac{1}{3}}\right]$$

$$= \frac{Pb\left(L^2-b^2\right)^{3/2}}{6LEI_x}\frac{2}{(3)^{3/2}} = \frac{Pb\left(L^2-b^2\right)^{3/2}}{3LEI_x}\frac{1}{(3)^{3/2}}$$

$$y_{\max} = \frac{\sqrt{3}Pb\left(L^2-b^2\right)^{3/2}}{27LEI_x}$$

Remarks

If the load P located at the middle of the beam (a = b = L/2), the deflection is maximum:

$$y_{\max} = \frac{PL^3}{48EI_x} \tag{8.17}$$

When the load P moving from the middle of the beam to the support B, we will get the coordinates of the section of zero rotation varies from 0.5L to 0.577L. From this small difference, it is common in practice that when a load located at any position on a simply supported beam, the maximum deflection can approximately be computed by the deflection at the middle of the beam.

Thus, substitute z = L/2 into the equation of y, we have:

$$y_{\max} = \frac{Pb}{48EI_x}\left(3L^2-4b^2\right) \tag{8.18}$$

The difference between the results obtained from Equation 8.17 and 8.18 are negligible.

2.7. Exercise 5

Consider the beam with its cross section in **Figure 8.6**. Given: P = 65 kN, b = 15 cm, t = 5 cm, h = 30 cm, h_1 = 25 cm, E = 2.10^5 MPa.

a) Determine the center C and inertia moment I_x of the section.

b) Calculate the deflection of the beam.

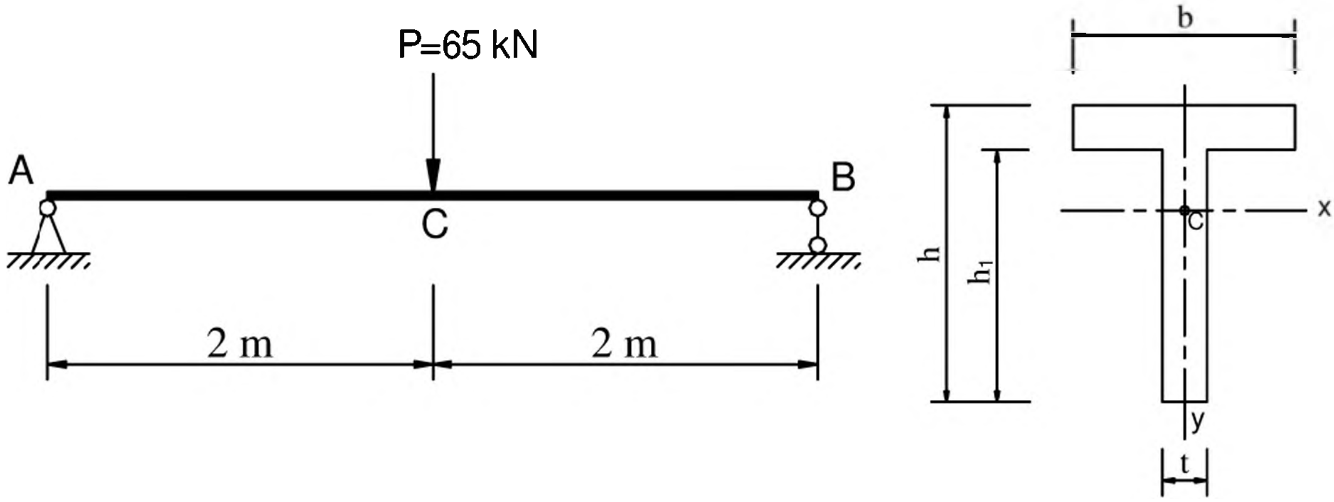


Figure 8.6

a) Determine the center C, inertia moment I_x of the section.

$$y_c = \frac{S_{x_o}}{A} = \frac{25\times5\times\frac{25}{2}+5\times15\left(25+\frac{5}{2}\right)}{25\times5+15\times5} = 18,125\,cm$$

$$I_x = \left[\frac{5\times25^3}{12}+5\times25\left(\frac{25}{2}-18,125\right)^2\right]+\left[\frac{15\times5^3}{12}+15\times5\left(30-18,125-\frac{5}{2}\right)^2\right]$$

$$= 17213,5\,cm^4$$

b) Calculate the deflection of the beam (vertical displacement of C).

$$f = \frac{PL^3}{48EI_x} = \frac{65\times400^3}{48\times\left(2\times10^4\right)\times17213,5} = 0,25\,cm$$

PROBLEMS

PROBLEM 1. Write the equations of elastic curve and rotation for the console beam subjected to the point load M at the beam end. The length of the beam is L, the stiffness EI = const. From the established equations, determine the maximum deflection and rotation.

PROBLEM 2. Write the equations of elastic curve and rotation for the console beam subjected to the point load P and moment M at the beam end. The length of the beam is L, the stiffness EI = const. From the established equations, determine the maximum deflection and rotation.

PROBLEM 3. Write the equations of elastic curve and rotation for the simply supported beam subjected to the point load P at the middle of the beam. The length of the beam is L, the stiffness EI = const. From the established equations, determine the maximum deflection and rotation.

PROBLEM 4. Write the equations of elastic curve and rotation for the console beam subjected to the point load P at the middle of the beam and the distributed load q. The length of the beam is L, the stiffness EI = const. From the established equations, determine the maximum deflection and rotation.

Chapter 9

PURE TORSION (TORQUE)

§1. CONCEPTS

1.1. Definition

A member is under pure torsion if M_z is the only component of internal forces.

Sign convention for M_z: positive when looking at the cross section, M_z rotates clockwise, as shown in **Figure 9.1**.

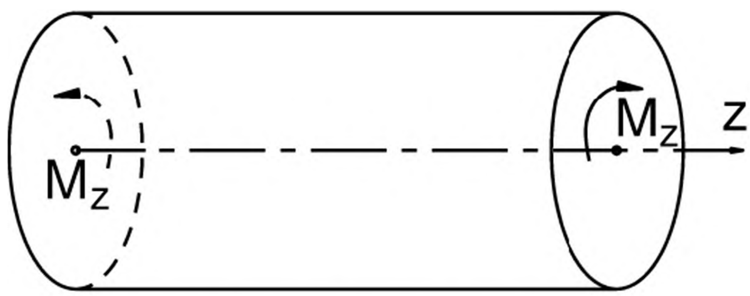


Figure 9.1. Positive M_z

1.2. Diagram of torsional moment

To draw diagram of torsional moment, we use the sectional method and the equilibrium condition $\sum M_{/z} = 0$.

1.3. Engine power capacity and torsional moment

M_o (Nm) causes the shaft to rotate and angle α (radian) in the time duration t (s):

- The work is:

$$A = M_o\alpha \tag{9.1}$$

- The power capacity (Nm/s) is:

$$W = \frac{A}{t} = \frac{M_o\alpha}{t} = M_o\frac{\alpha}{t}$$

$$W = M_o\omega \tag{9.2}$$

$$M_o = \frac{W}{\omega} \tag{9.3}$$

in which, $\omega = \frac{\alpha}{t}$ is the rotational velocity (rad/s).

Set n is the number of circular frequency (cycles per minute), we have:

$$\omega = \frac{n2\pi}{60} = \frac{n\pi}{30} \tag{9.4}$$

Thus,
$$M_o = \frac{W}{\omega} = \frac{30W}{n\pi} \tag{9.5}$$

$$M_o = \frac{30W}{n\pi} \tag{9.6}$$

Note: In the above formulas, W is the power in the unit of Nm/s, n is the number of rotational cycles per minute (circular frequency). The unit of n is cycles per minute or revolutions per minute (rpm).

- If the unit of W is in hourse power (CV, HP):

$$1\text{ HP} = 750\text{ Nm/s} = 0.736\text{ kW}$$

$$M_o = \frac{30W}{n\pi} = \frac{30 \times 750 \times W}{n\pi} = 7162\frac{W}{n}$$

$$M_o = 7162\frac{W}{n} \tag{9.7}$$

- If the unit of W is in kW:

$$1\text{kW} \approx 1020\text{ Nm/s}$$

$$M_o = \frac{30W}{n\pi} = \frac{30 \times 1020 \times W}{n\pi} = 9740\frac{W}{n}$$

$$M_o = 9740\frac{W}{n} \tag{9.8}$$

1.4. Exercises

Exercise 1

Draw the diagram of torsional moment for the shaft shown in **Figure 9.2**.

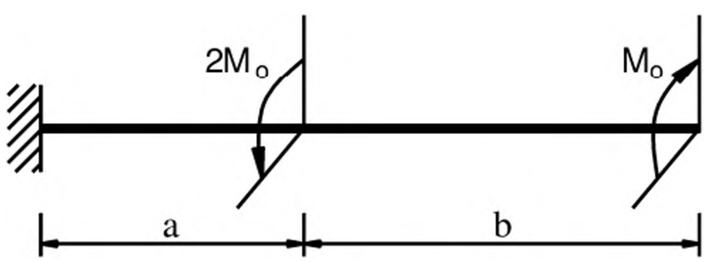


Figure 9.2

Solution

Take the cut 1-1, consider the right part:

$$M_z = M_o$$

Take the cut 2-2, consider the right part:

$$M_z = -M_o$$

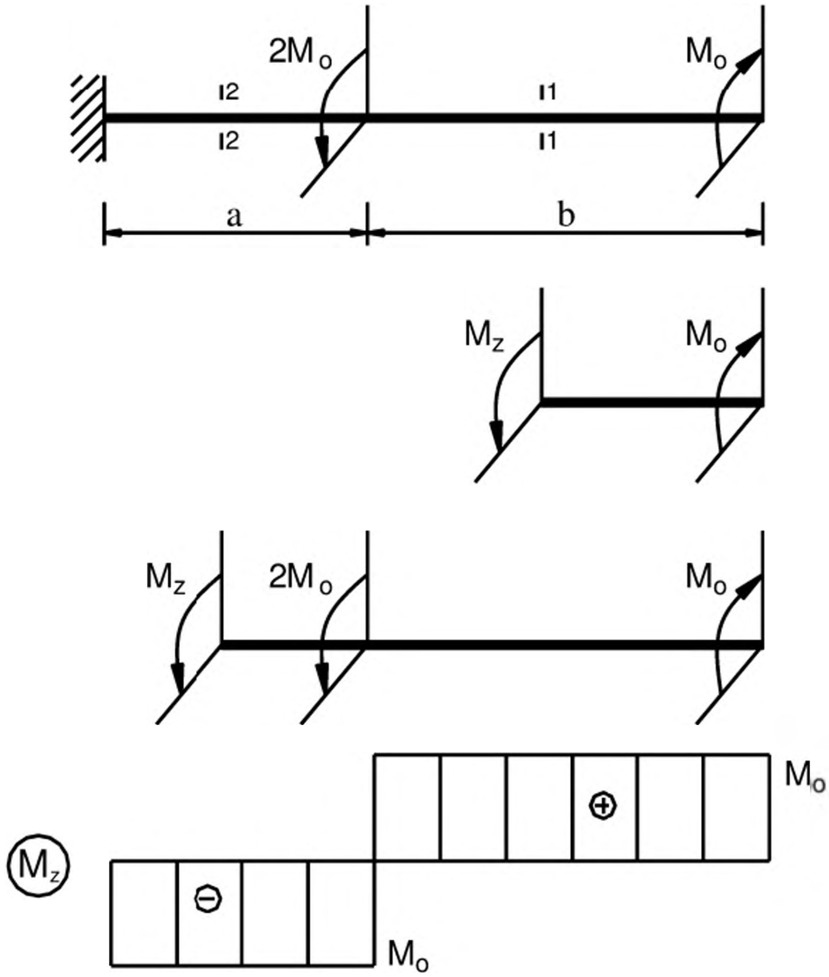


Figure 9.3

Exercise 2

Consider the shaft shown in **Figure 9.4**

a) Determine the reaction at D.

b) Draw the diagram of torsion moment M_z.

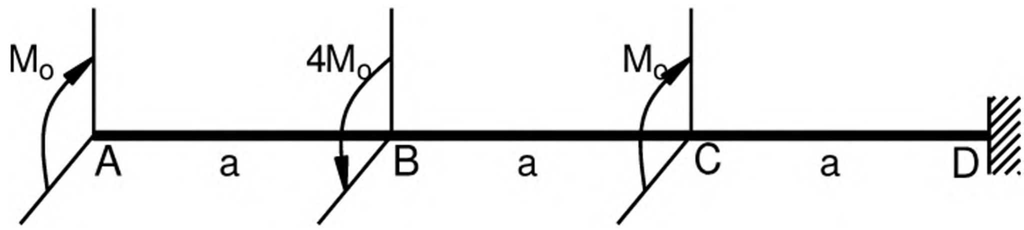
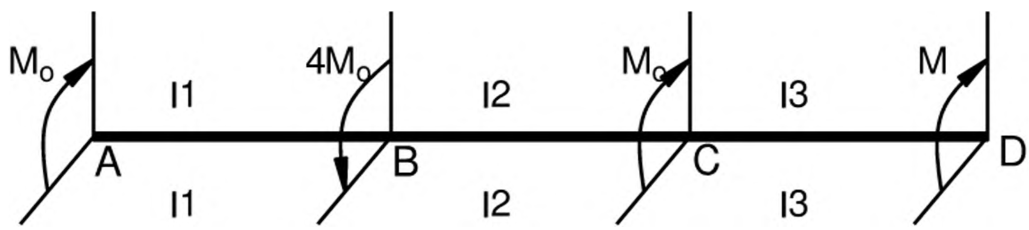


Figure 9.4. A shaft subjected to torsion

Solution

a) Take the cut at D, consider the left part:

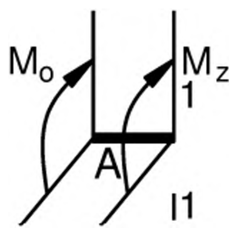


$$M_z + M_o - 4M_o + M_o = 0$$

$=>$ $M_z = 2M_o$.

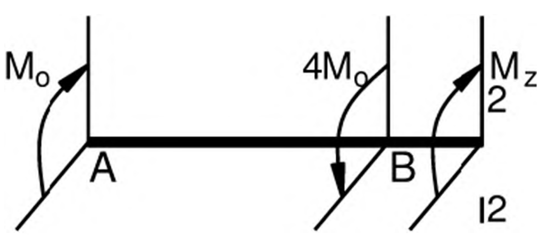
b) Draw the diagram M_z:

- Take the cut 1-1 between A and B, consider the left part:



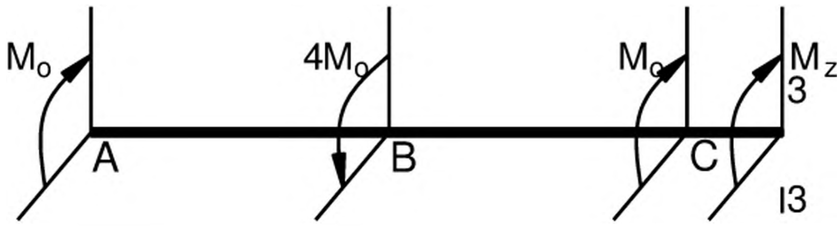
$$M_z = -M_o$$

- Take the cut 2-2 between B and C, consider the left part:



$$M_z = 3M_o$$

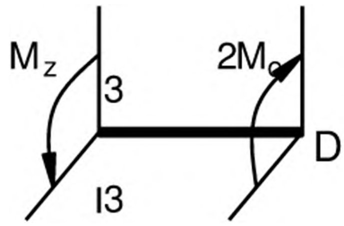
- Take the cut 3-3 between C and D, consider the left part:



$$M_z + M_o - 4M_o + M_o = 0$$

$$=> \quad M_z = 2M_o$$

Or consider the right part, we also have the same result $M_z = 2M_o$.



c) Diagram M_z is shown in **Figure 9.5**

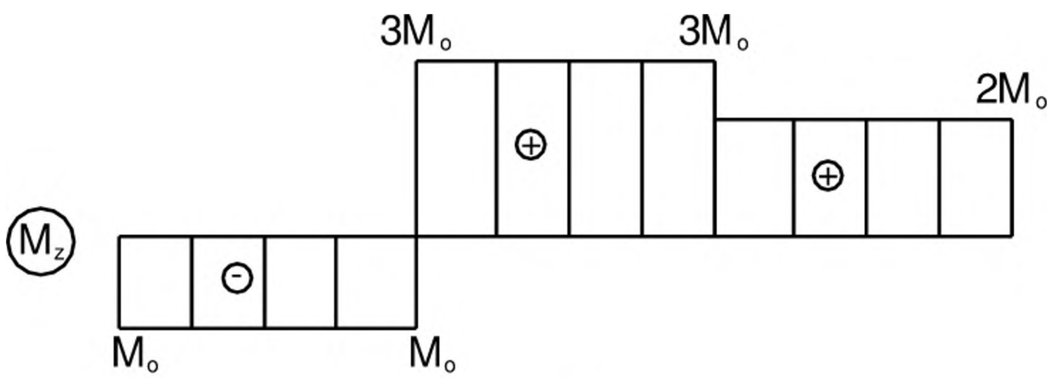


Figure 9.5

Exercise 3

Draw the diagram of torsional moment M_z.

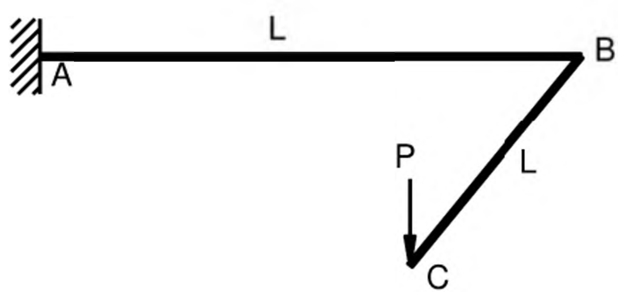
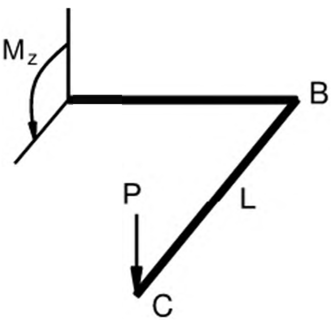


Figure 9.6

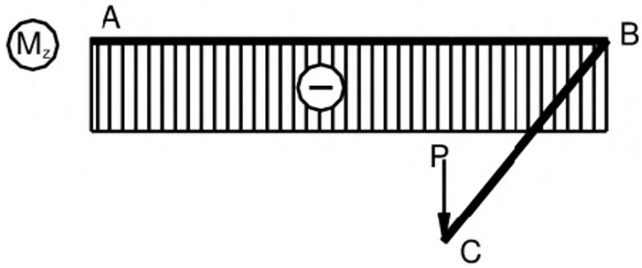
Solution

Only AB segment has torsional moment.

Take the cut across the segment AB, consider the left part:



M = -PL



Exercise 4

A shaft subjected to a torsional moment M_o and distributed torsional moment m as shown in **Figure 9.7**. Given m = 5 kNm/m; M_o = 10 kNm; L = 1 m.

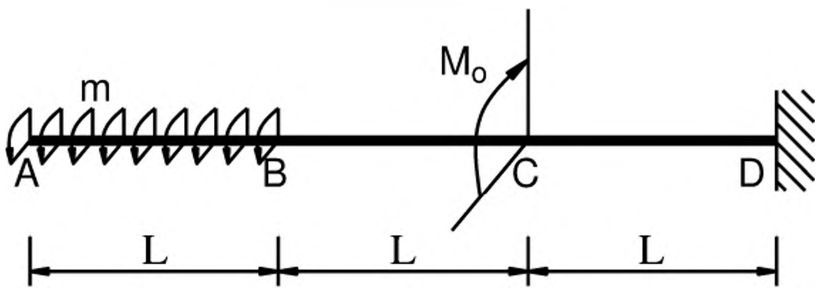
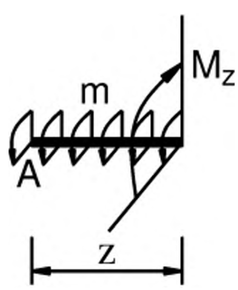


Figure 9.7. A shaft subjected to a torsional moment M_o and distributed torsional moment m

Solution

• Take the cut 1-1 between A and B, consider the left part:

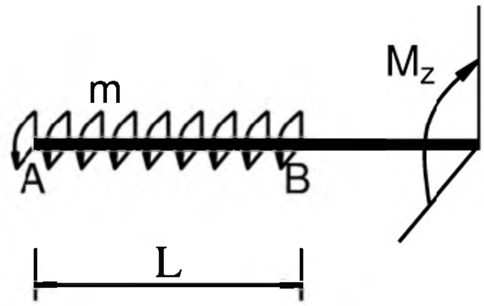


$$M_z = mz$$

At A: z = 0 => $M_z = 0$

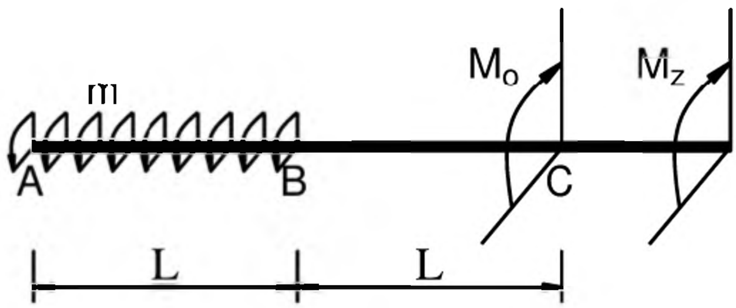
At B: z = L => $M_z = mL$

• Take the cut 2-2 between B and C, consider the left part:



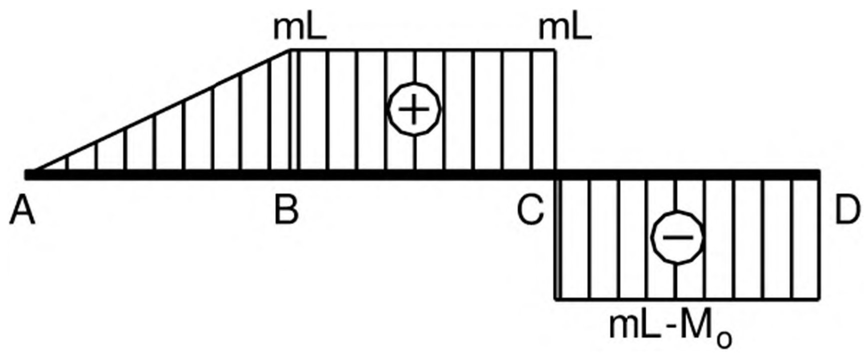
$$M_z = mL$$

• Take the cut 3-3 between C and D, consider the left part:



$$M_z = mL - M_o$$

Diagram M_z:



Note: The principle of superposition can be applied to draw the diagram of internal moment M_z.

Exercise 5

Use the principle of supperposition to draw the diagram of internal moment M_z for the shaft in Exercise 1.

Solution

We can draw the diagrams of M_z due to each moment as shown below. **Figure 9.8** shows the diagram M_z due to the torsional moment M_o. **Figure 9.9** shows the diagram M_z due to the torsional moment $2M_o$. Then, these torsional moment diagrams were summed up to obtain the diagram of M_z.

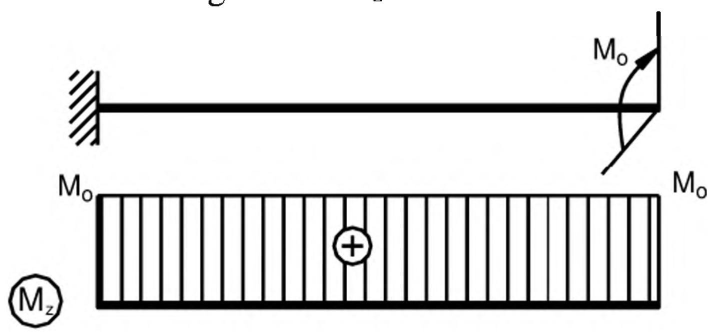


Figure 9.8. Diagram of torsional moment caused by the load M_o

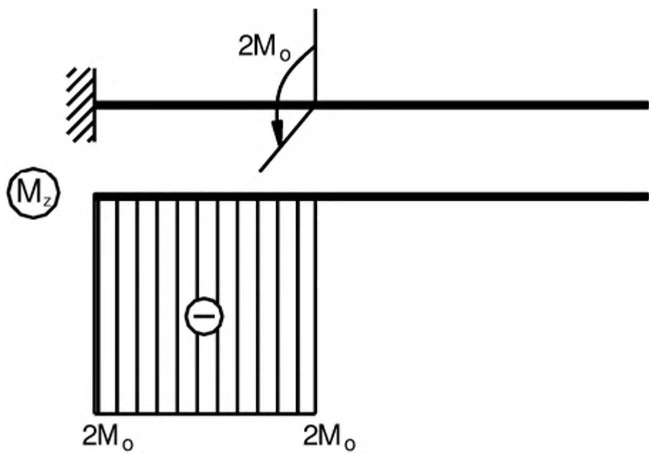


Figure 9.9. Diagram of torsional moment caused by the load M_o

Add the **Figure 9.8** and **Figure 9.9** together, we have the diagram of torsional moment for the shaft under the given loads as shown in **Figure 9.10**.

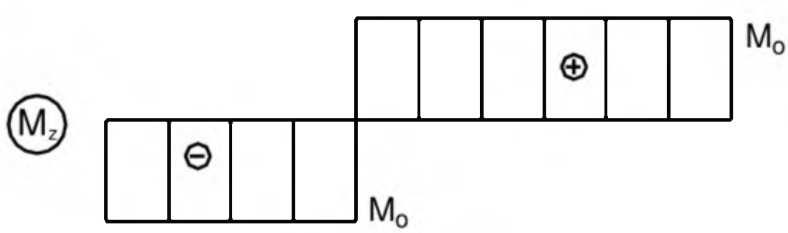


Figure 9.10

§2. TORSION OF STRAIGHT CIRCULAR MEMBERS

2.1. Experiment

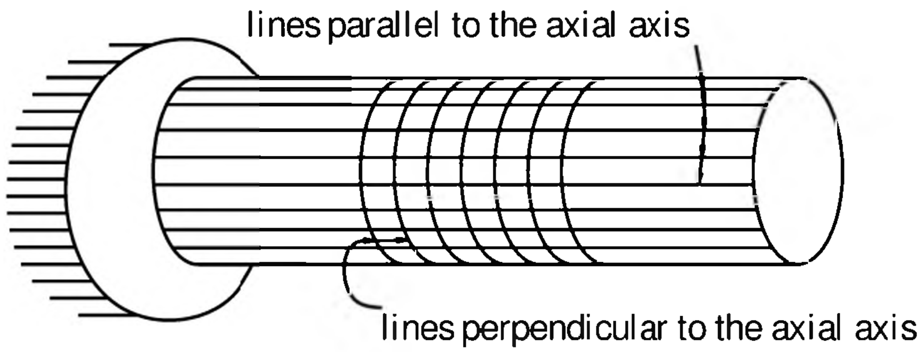
Consider a solid shaft with a circular cross section. The radius of the cross section is R. An experiment is conducted as follows. Before subjecting to a torsional moment, we draw lines on the surface:

- Lines parallel to the axial axis
- Lines perpendicular to the axial axis

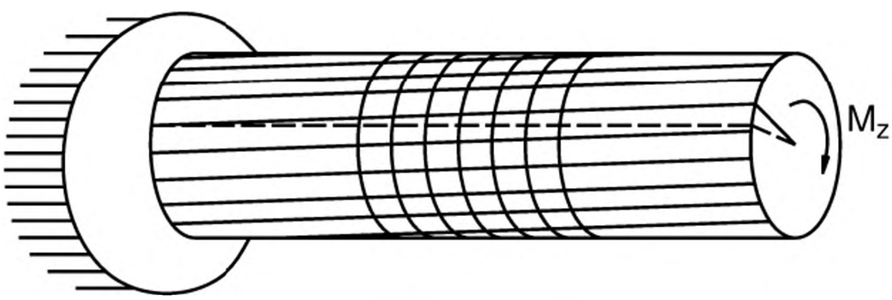
The shaft before loading is shown in **Figure 9.11a**. Then, the shaft is subjected to a torsional moment. Its deformation is shown in **Figure 9.11b** with the following characteristics:

- The axial axis is still straight.
- The length of the member is unchanged.
- Lines ⊥ the axial axis are still lines ⊥ the axial axis and are in planes perpendicular to the axis.
- Lines // the axial axis become spirals.
- Rectangles become parallelograms.

Based on the above observations, some assumptions can be drawn as presented in Section 2 (below).



a)Before deformation



b) After deformation

Figure 9.11. Circular member under torsion

2.2. Assumptions

During the deformation:

a) Cross sections are still in planes and are ⊥ the axial axis.

b) Cross sections do not have longitudinal displacement. Radii are still straight lines and their lengths remain unchange.

2.3. Shear stress

Consider the element taken by three pairs of planes as shown in **Figure 9.12**:

- Two planes (1-1) and (2-2). The distance between the two planes is dz.
- Two planes whose intersection is the axis z and the angle between these two planes is dα.
- Two cylinders whose axis is the axis z and the radii are ρ and ρ + dρ.

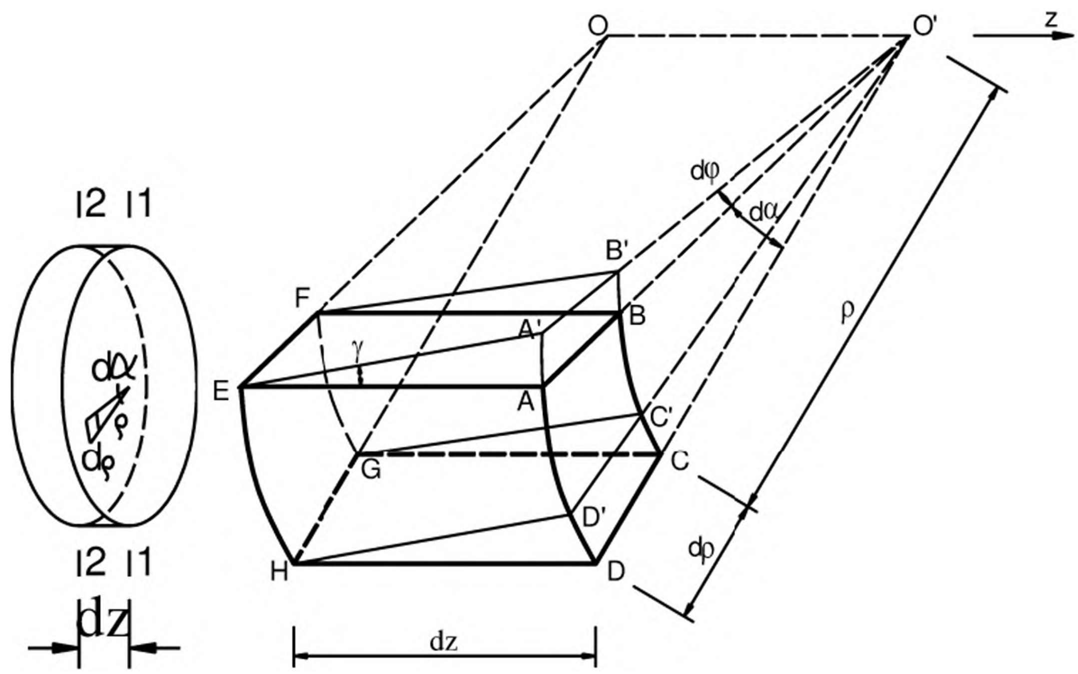


Figure 9.12. An element under torsion

We have:

- Points A, B, C, D → A', B', C', D'.
- The plane A'B'C'D' ∈ 2-2.
- OA'B' are on a straight line; OC'D' are on a straight line.
- Set dφ = the angle AOA'
  = the twisted angle between planes 1-1 and 2-2;
  = the twisted angle between cross sections 1-1 and 2-2.
- Set γ = the angle AEA' = shear strain (of the element).

$$\tan\gamma \approx \gamma = \frac{AA'}{AE} = \frac{\rho d\varphi}{dz} \tag{9.9}$$

Comments

- *Based on the assumption a) → on ABCD và EFGH, there is no shear stress on the direction to the axial axis z due to the right angles are unchanged.*

- *Based on the assumption b) → no normal stresses on surfaces of elements.*
- *Based on the assumption b) → on ABFE and CDHG surfaces, there is no shear stress on the direction to the axial axis z because radii remain straight.*

Thus, on the cross section, only shear stress, which is perpendicular to radii, namely τ_ρ, and element is in shear stress state as shown in **Figure 9.13**.

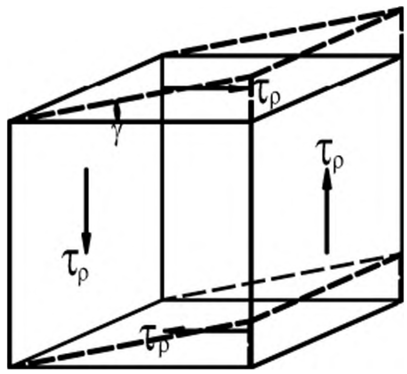


Figure 9.13. Element under pure shear

Apply the Hooke's law:

$$\tau_\rho = G\gamma \tag{9.10}$$

We have

$$\tan\gamma \approx \gamma = \frac{AA'}{AE} = \frac{\rho d\varphi}{dz} \tag{9.11}$$

So:

$$\tau_\rho = G\left(\rho\frac{d\varphi}{dz}\right) \tag{9.12}$$

On the surface 1-1, we have:

+ dA is the area around the considered point.
+ $\tau_\rho dA$ is infinite shear force on the infinite area dA.
+ Moment of the above infinite force about the axis z is $\tau_\rho dA\rho$.
+ Total moment:

$$M_z = \int_A \tau_\rho \rho dA = \int_A \left[G\rho\frac{d\varphi}{dz}\right]\rho dA = \int_A G\rho^2\frac{d\varphi}{dz}dA \tag{9.13}$$

We have, $G\dfrac{d\varphi}{dz} = const$ for any point on the cross section A, thus:

$$M_z = G\frac{d\varphi}{dz}\int_A \rho^2 dA = G\frac{d\varphi}{dz}I_\rho \tag{9.14}$$

$$\frac{d\varphi}{dz} = \frac{M_z}{GI_\rho} \tag{9.15}$$

or $\theta = \dfrac{M_z}{GI_\rho}$ is the relative rotation (rad/m). (9.16)

Substitute Equation 9.15 into Equation 9.8, we have:

$$\tau_\rho = G\rho\frac{M_z}{GI_\rho}$$

$$\tau_\rho = \frac{M_z}{I_\rho}\rho \tag{9.17}$$

τ min when ρ min; $\rho_{min} = 0$ (circle) or $= r$ (donut)

τ max when ρ max; $\rho_{max} = R$

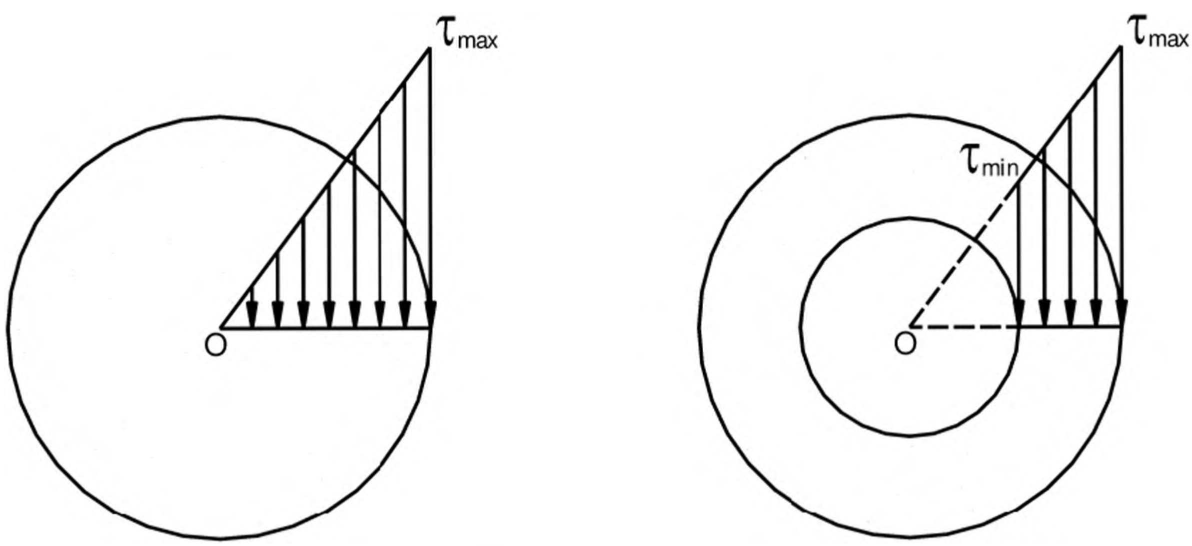


Figure 9.14. Distribution of shear stress

$$\tau_{max} = \frac{M_z}{I_\rho}R = \frac{M_z}{\left(\frac{I_\rho}{R}\right)}$$

$$\tau_{max} = \frac{M_z}{W_\rho} \tag{9.18}$$

in which, $W_\rho = \dfrac{I_\rho}{R}$ is called sectional resistance modulus.

- For circular cross section with radius R:

$$I_\rho = \frac{1}{2}\pi R^4 \tag{9.19}$$

$$W_\rho = \frac{I_\rho}{R} = \frac{\frac{\pi R^4}{2}}{R} = \frac{\pi R^3}{2} = \frac{\pi D^3}{16} \approx 0.2D^3 \tag{9.20}$$

- For donut section with outer radius R and inner radius r:

$$I_\rho = \frac{1}{2}\pi R^4 - \frac{1}{2}\pi r^4 \tag{9.21}$$

$$W_\rho = \frac{I_\rho}{R} = \frac{1}{R}\left[\frac{1}{2}\pi R^4 - \frac{1}{2}\pi r^4\right] \tag{9.22}$$

Exercise 6

The solid shaft is fixed at A and subjected to the torsional moments M_B = 300 Nm and M_C = 800 Nm as shown in **Figure 9.15**. The radius of the shaft is 35 mm. Determine the stress at the following points:

a) Point D on the surface of the segment AB

b) Point H in the segment BC. The distance from the poni H to the axial axis is 20 mm.

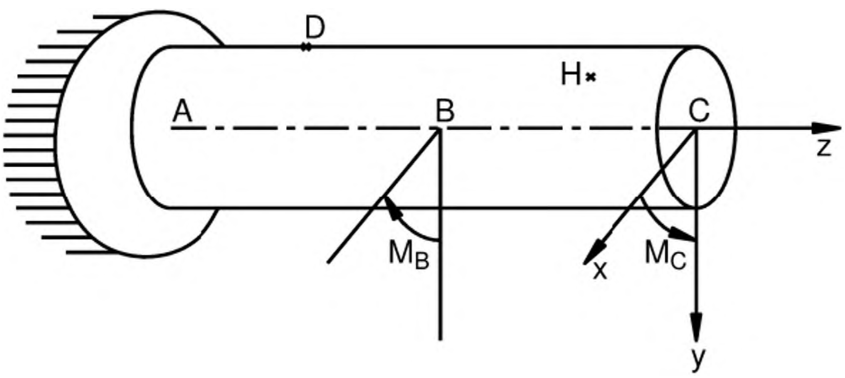
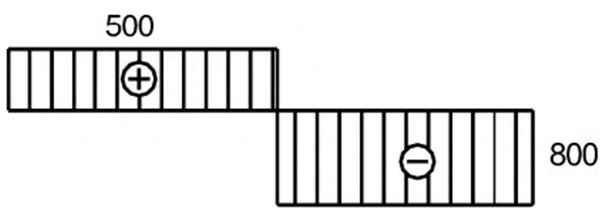


Figure 9.15

Solution

The diagram of torsional moment:



$$I_\rho = \frac{1}{2}\pi R^4 = \frac{1}{2}\pi 35^4 = 2.357\times 10^6 mm^4$$

Point D on the surface of the segment AB:

$$\tau_\rho = \frac{M_z}{I_\rho}\rho = \frac{500\times 10^3}{2.357\times 10^6}\times 35 = 7.42MPa$$

Point H on the surface of the segment BC:

$$\tau_\rho = \frac{M_z}{I_\rho}\rho = \frac{800\times 10^3}{2.357\times 10^6}\times 20 = 6.8MPa$$

2.4. Rotation angle

We have:

$$\frac{d\varphi}{dz} = \frac{M_z}{GI_\rho}$$

$$\leftrightarrow \quad d\varphi = \frac{M_z}{GI_\rho}dz$$

The rotation angle (the angle of twist) of segment with the length L:

$$\varphi = \int_0^L d\varphi = \int_0^L \frac{M_z}{GI_\rho}dz \tag{9.23}$$

If many segments included, each segment has $\frac{M_z}{GI_\rho}$ = const:

$$\varphi = \sum_i \left(\frac{M_z L}{GI_\rho}\right)_i \tag{9.24}$$

Sign of φ ***≡ sign of*** M_z

2.5. Conditions of stress and deformation

Stress condition

$$\tau_{max} \le [\tau] \tag{9.25}$$

Based on failure theory 3 (failure theory based on maximum shear stress 3):

$$\sigma_1 - \sigma_3 \le [\sigma]$$

Substitute $\sigma_1 = |\tau|; \sigma_3 = -|\tau|$, we get: $2|\tau| \le [\sigma]$

Or $$|\tau| \le \frac{[\sigma]}{2} \tag{9.26}$$

Based on failure theory 4 (failure theory based on maximum deformation energy)

$$\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \le [\sigma]$$

Substitute

$$\sigma_1 = |\tau|; \sigma_2 = 0; \sigma_3 = -|\tau|$$

We get: $$|\tau| \le \frac{[\sigma]}{\sqrt{3}} \tag{9.27}$$

Deformation condition

$$\theta_{max} \le [\theta] \tag{9.28}$$

Unit of θ: (rad/m).

Exercise 7

Consider the shaft subjected to the torsional moment shown in **Figure 9.16**. The shaft has a circular section with diameter D. Given M_A = 50 kNm, M_B = 80 kNm, $[\sigma] = 160 MPa$, $[\theta] = 0.25^o / m$, $G = 8 \times 10^4 MPa$, L = 500 mm.

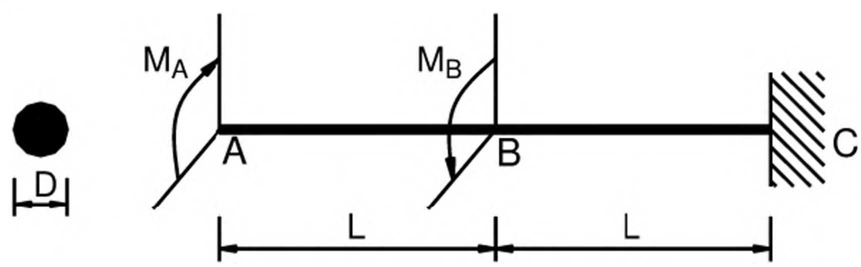


Figure 9.16

a) Draw M_z.

b) Determine D_{min},

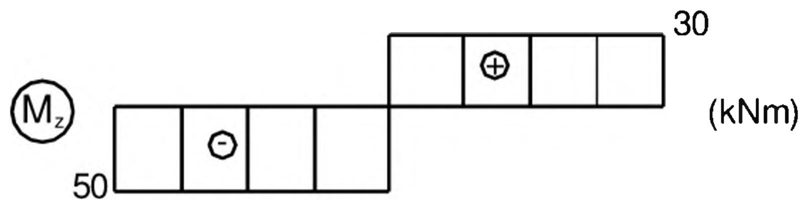
c) Calculate φ_{AB}, φ_{BC} corresponding to the selected D_{min}.

Solution

a) Draw M_z

Take a cut in the segment AB and then establish the equilibrium, we can easily have M_z = -50 kNm.

Similarly, take a cut in the segment BC and then establish the equilibrium, we can easily have M_z = 30 kNm.



b) Determine D_{min}

Stress condition:

$$\tau_{max} \le [\tau]$$

$$\frac{M_z}{W_\rho} = \frac{M_z}{\frac{1}{2}\pi R^3} \le [\tau]$$

$$R \ge \sqrt[3]{\frac{2M_z}{\pi[\tau]}}$$

in which, $[\tau] = \frac{[\sigma]}{2} = 80MPa$; $M_z = 50kNm$

Therefore,

$$R \ge \sqrt[3]{\frac{2M_z}{\pi[\tau]}} = \sqrt[3]{\frac{2\times 50\times 10^6}{\pi\times 80}} = 73.6 \text{ mm}$$

The deformation condition:

$$\theta_{max} \le [\theta]$$

$$\frac{M_z}{GI_\rho} = \frac{M_z}{G\frac{1}{2}\pi R^4} \le [\theta]$$

$$R \ge \sqrt[4]{\frac{2M_z}{\pi G[\theta]}}$$

$$[\theta] = 0.25^o/m = 4.36\times 10^{-3} rad/m = 4.36\times 10^{-6} rad/mm$$

We have:

$$D \ge \sqrt[4]{\frac{2\times 50\times 10^6}{\pi\times 8.10^4 \times 4.36\times 10^{-6}}} = 97.7mm$$

From the two conditions of stress and stiffness, it can be selected with R = 100 mm.

Calculate φ_{AB}, φ_{BC} corresponding to the selected D_{min}.

$$I_\rho = \frac{1}{2}\pi R^4 = \frac{1}{2}\pi \times 100^4 = 1.395 \times 10^8 mm^4$$

$$\varphi_{AB} = \frac{M_z L}{GI_\rho} = \frac{50 \times 10^6 \times 500}{8 \times 10^4 \times 1.395 \times 10^8} = 0.0022 rad$$

Exercise 8

An engine transfers a torsional moment at A as shown in **Figure 9.17**. The torsional moments received at B and C are 20 Nm and 50 Nm, respectively. The rotational frequency is n = 1400 cycles/minute.

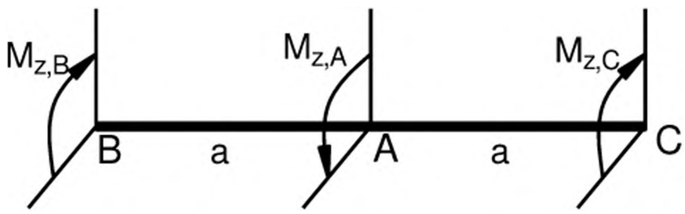


Figure 9.17

Determine:

a) The torsional power received at B and C

b) The power of the engine

Solution

a) The power received at B and C

$$M_o = 9740\frac{W}{n} \rightarrow W = \frac{nM_o}{9740}$$

$$W_B = \frac{1400 \times 20}{9740} = 2.87 kW$$

$$W_C = \frac{1400 \times 50}{9740} = 7.19 kW$$

b) The power of the engine

$$W_A = \frac{1400 \times 70}{9740} = 10.06 kW$$

Exercise 9

An engine with a power of 10 kW transmits a torque to a circular shaft with diameter D at the section A. The rotational velocity of the shaft is 1400 rpm. Assume that the engine efficiency is 100%, then, sections B and C receive the transmission power of 3 kW and 7 kW, respectively, as shown in **Figure 9.18**. Determine the mimimum diameter D and the angle of twist of the segment AC. Given: $[\sigma]=160MPa$; $[\theta]=0.25^o/m$; $G=8\times10^4 MPa; a=0.5m.\pi$

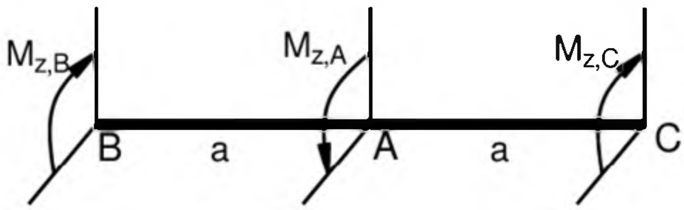


Figure 9.18

Solution

We have:

$$M_o = 9740\frac{W}{n}$$

$$M_{z,A} = 9740\frac{10}{1400} = 69.57Nm$$

$$M_{z,B} = 9740\frac{3}{1400} = 20.87Nm$$

$$M_{z,C} = 9740\frac{7}{1400} = 48.7Nm$$

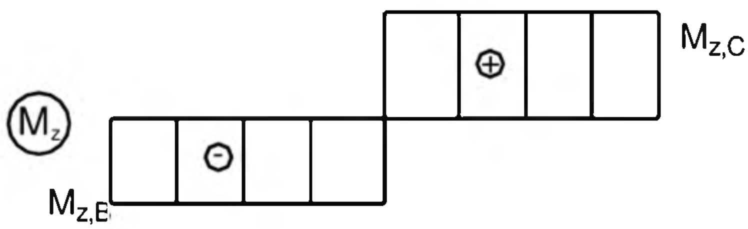


Figure 9.19

The condition of shear stress:

$$\tau_{max} \leq [\tau]$$

$$\frac{M_z}{W_\rho} = \frac{M_z}{0,2D^3} \leq [\tau]$$

$$D \geq \sqrt[3]{\frac{M_z}{0.2[\tau]}}$$

in which, $[\tau]=\dfrac{[\sigma]}{2}=80MPa$; $M_z = M_{z,C} = 48.7Nm = 48700Nmm$

Thus, $D \ge 144.9mm$

The deformation condition:

$$\theta_{max} \le [\theta]$$

$$\frac{M_z}{GI_\rho} = \frac{M_z}{G \times 0.1D^4} \le [\theta]$$

$$D \ge \sqrt[4]{\frac{M_z}{G \times 0.1[\theta]}}$$

$$[\theta] = 0.25^o / m = 0.00436 rad / m; M_z = 48.7Nm = 48700Nmm$$

Thus,

$$D \ge 111.7mm$$

Based on the two above conditions, it can be selected D = 150 mm

$$\varphi_{AC} = \frac{M_z L}{GI_\rho} = \frac{48700}{8 \times 10^4 \times 0.1 \times 150^4} = 0.006 rad$$

2.6. Elastic strain energy

In Chapter 4, the elastic strain energy of an element is:

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^3 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

Because:

$$\sigma_1 = |\tau|; \sigma_2 = 0; \sigma_3 = -|\tau|;\ E = \frac{2G}{1+\nu}$$

Thus: $$u = \frac{1+\nu}{E}\tau_\rho^2 = \frac{1}{2}\frac{\tau_\rho^2}{G} \tag{9.29}$$

The elastic strain energy of a segment dz is:

$$dU = \int_V u dV = \int_V u dA dz$$

$$dU = \int_V \frac{1}{2}\frac{\tau_\rho^2}{G} dA dz = \int_V \frac{1}{2}\frac{\left(\dfrac{M_z}{I_\rho}\rho\right)^2}{G} dA dz$$

$$dU = \frac{1}{2G}\left(\frac{M_z}{I_\rho}\right)^2 dz\int_V \rho^2 dA$$

$$dU = \frac{1}{2G}\left(\frac{M_z}{I_\rho}\right)^2 dzI_\rho = \frac{1}{2}\frac{M_z^2}{GI_\rho}dz$$

The elastic strain energy of length L is:

$$U = \frac{1}{2}\int_0^L \frac{M_z^2}{GI_\rho}dz \tag{9.30}$$

The elastic strain energy of a segment L with $\frac{M_z}{GI_\rho} = const$:

$$U = \frac{1}{2}\frac{M_z^2}{GI_\rho}L \tag{9.31}$$

The elastic strain energy of several segments L with $\frac{M_z}{GI_\rho} = const$:

$$U = \frac{1}{2}\sum_i\left(\frac{M_z^2 L}{GI_\rho}\right)_i \tag{9.32}$$

2.7. Failure modes of materials

Many studies on the stress state of shafts under torsion shows that an element on the outer surface (**Figure 9.20a**) is under pure shear as shown in **Figure 9.20b**. The shear stress is maximum (τ_{max}). For this stress state, the principle tensile and compressive stresses $\sigma_1 = -\sigma_3 = |\tau|$ are on two directions 45° compared to the shaft axis.

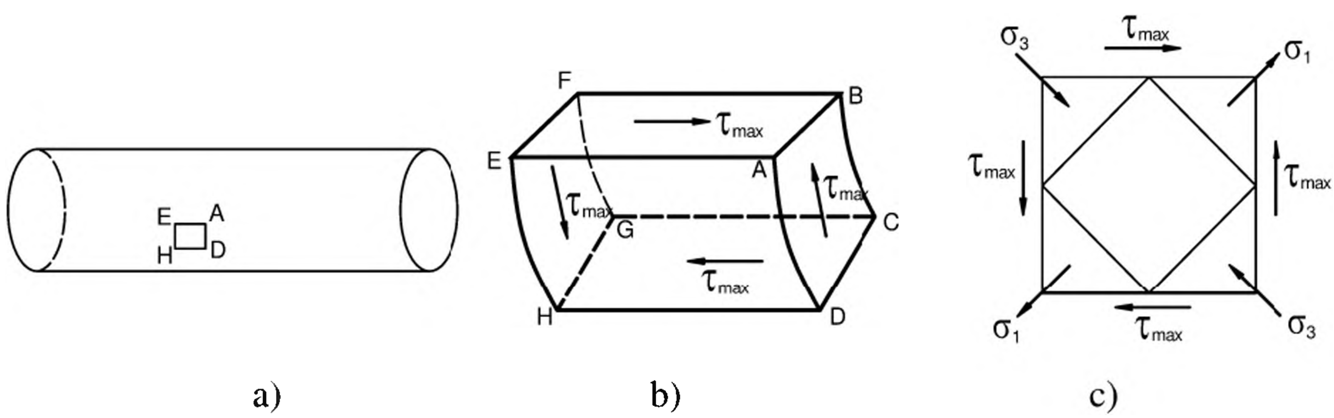


Figure 9.20

Figure 9.21 – **Figure 9.23** show examples of faiure modes of materials under torsion.

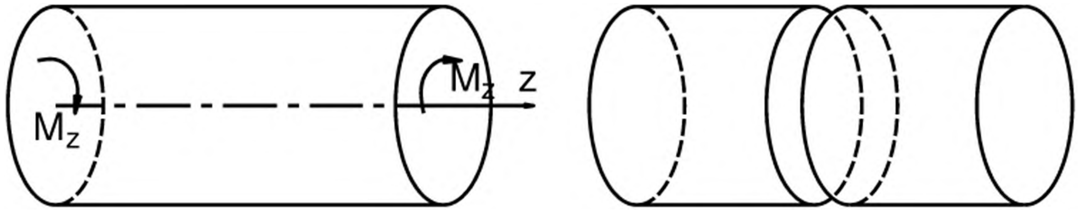


Figure 9.21

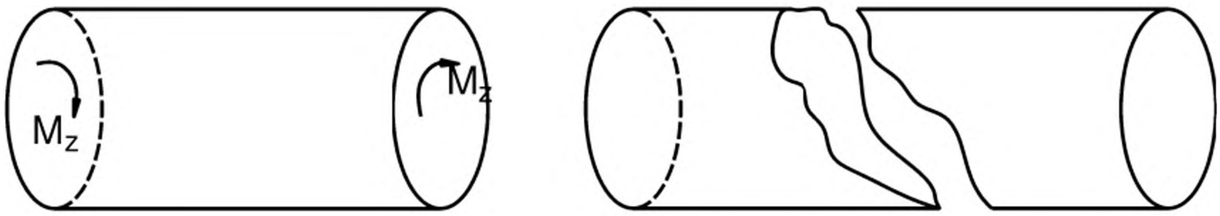


Figure 9.22

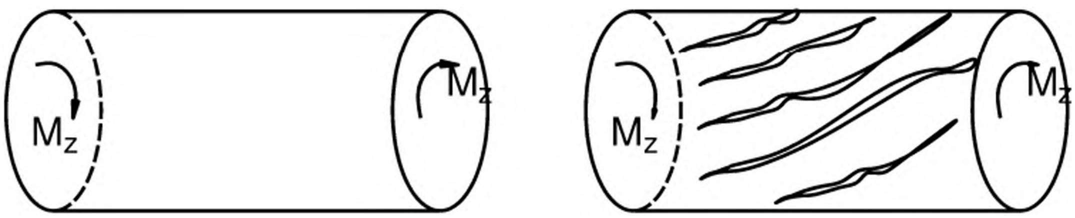


Figure 9.23

2.8. Indeterminate circular members subjected to torsion

Approach

Using both compatability and equilibrium equations to solve the problems.

Exercise 10

The shaft is fixed at its ends A and B and subjected to a torque M_o = 300 Nm as shown in **Figure 9.24**. The shaft radius is 20 mm. Given L = 0.8 m. Determine the maximum shear stresses in the regions AC and CB of the shaft.

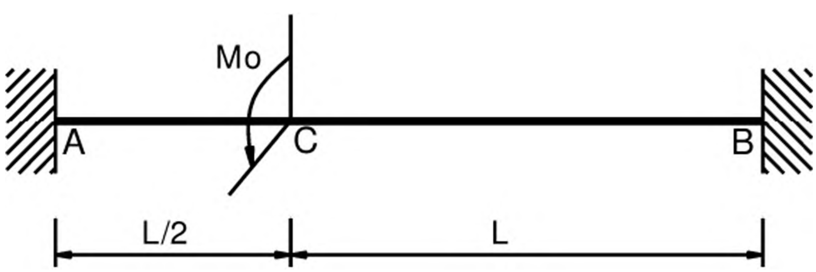


Figure 9.24

Solution

Approach 1: Release the two fixed ends and replace by the torsional moments M_A and M_B, and then draw the diagram of torsional moment as shown in **Figure 9.25**.

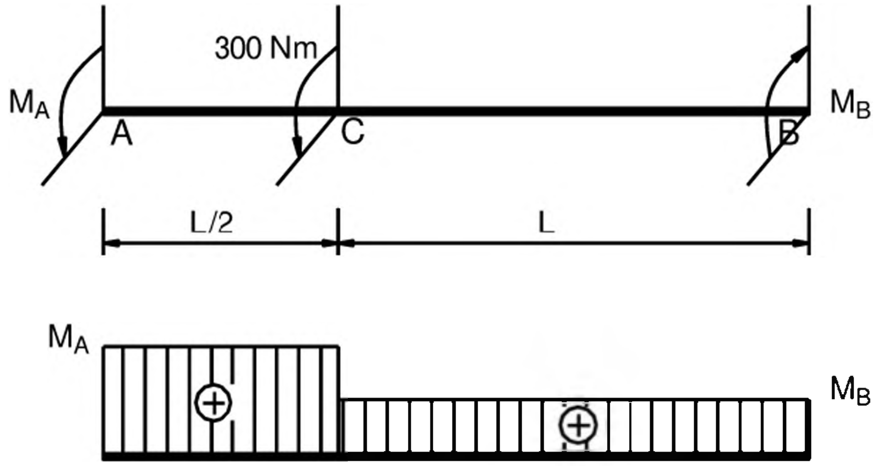


Figure 9.25

Equilibrium equation:

$$M_A + 300 - M_B = 0 \qquad (1)$$

Compatibility condition:

$$\varphi_{CA} = -\varphi_{CB}$$

(The angle of twist at C of the two segments are equal in magnitude but opposite direction).

$$\frac{M_A L_{AC}}{GI_\rho} = -\frac{M_B L_{BC}}{GI_\rho}$$

or $\varphi_{A/B} = 0$ (The angle of twist at A with respected to that at B is 0).

$$\varphi_{A/C} + \varphi_{C/B} = 0$$

$$\frac{M_A L_{AC}}{GI_\rho} + \frac{M_B L_{BC}}{GI_\rho} = 0$$

$$\frac{M_A \times 0.4}{GI_\rho} = -\frac{M_B \times 0.8}{GI_\rho}$$

$$M_A = -2M_B \qquad (2)$$

(1) and (2) => $M_A = -200Nm$; $M_B = 100Nm$

After having the torsional moments M_A and M_B, the diagram of torsional moment in **Figure 9.25** becomes the diagram of torsional moment shown in **Figure 9.26**.

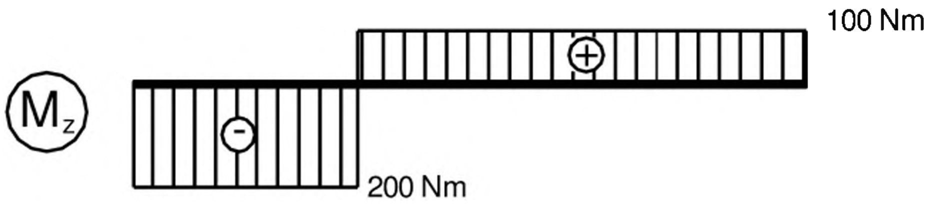


Figure 9.26

$$I_\rho = \frac{1}{2}\pi R^4 = \frac{1}{2}\pi 20^4 = 251200 \ \ (\mathrm{mm}^4)$$

$$\tau_{\max,AC} = \frac{M_A}{I_\rho} R = \frac{200\times10^3}{257200} 20 = 15.6 MPa$$

$$\tau_{\max,CB} = \frac{M_B}{I_\rho} R = \frac{100\times10^3}{257200} 20 = 7.8 MPa$$

Approach 2:

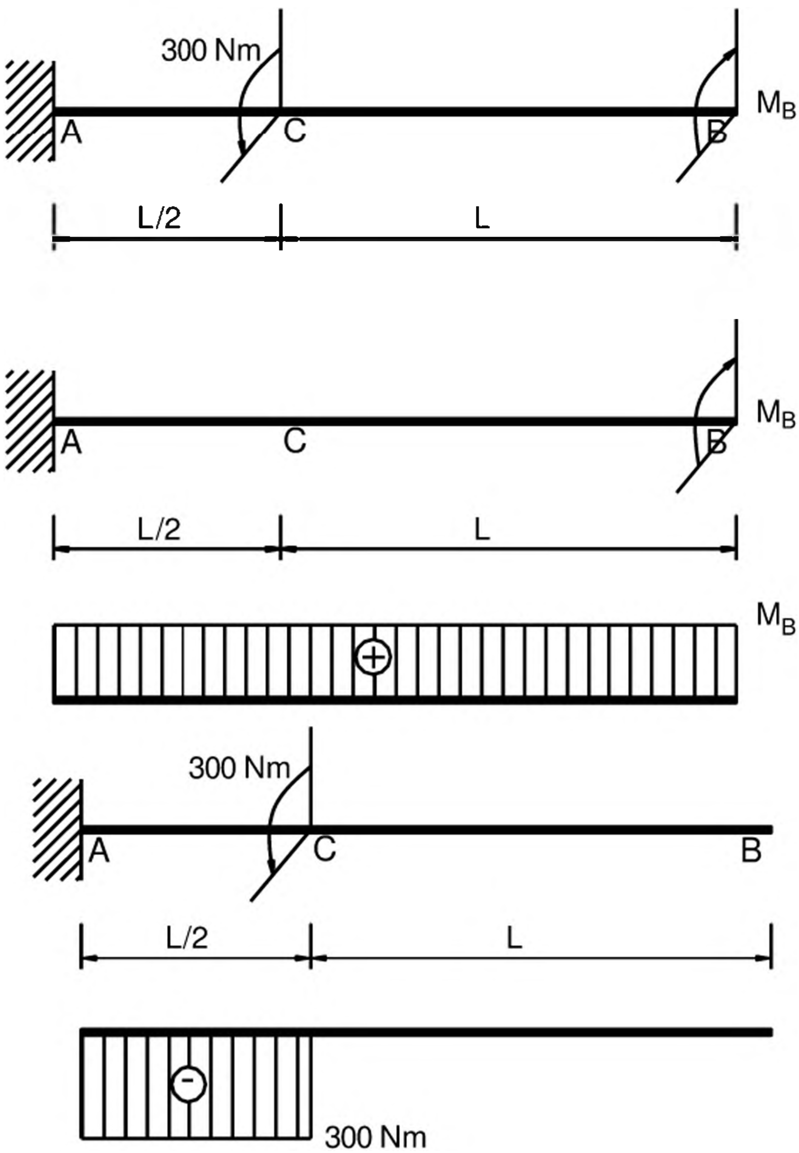


Figure 9.27

Release the fixed end B and replaced by the torsional moment M_B as shown in **Figure 9.27a**.

The condition is that the rotation at B is zero

$$\varphi_{B/A} = 0$$

$$\frac{M_B L_{BA}}{GI_\rho} + \frac{(-300)L_{AC}}{GI_\rho} = 0 \text{ (remember to take the signs of moment)}$$

$$M_B(1.2) + (-300)(0.4) = 0$$

$$M_B = 100Nm$$

Using the principle of supperposition, we have the diagram which is similar to the approach 1.

Approach 3:

- Release the fixed end B
- Draw the diagram of torsion moment

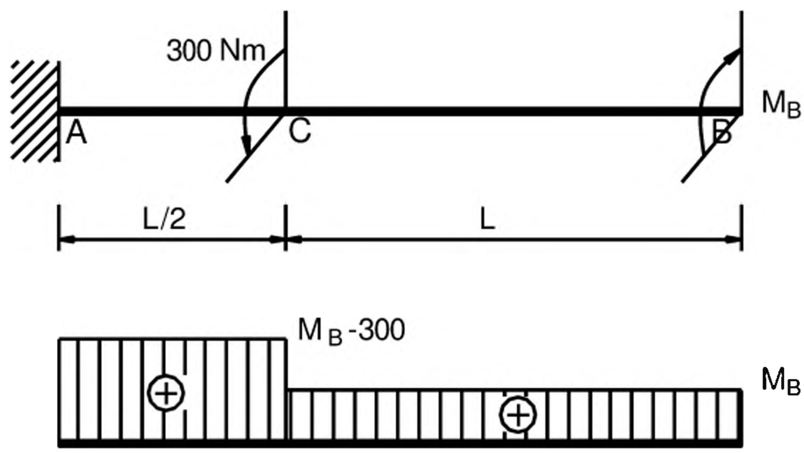


Figure 9.28

The boundary condition is that the rotation at B is zero

$$\varphi_{B/A} = 0$$

$$\frac{M_B L_{BC}}{GI_\rho} + \frac{(M_B - 300)L_{CA}}{GI_\rho} = 0 \text{ (remember to take the signs of moment)}$$

$$M_B(0.8) + (M_B - 300)(0.4) = 0$$

$$2M_B + (M_B - 300) = 0$$

$$M_B = 100Nm$$

Redraw the diagram of torsion moment

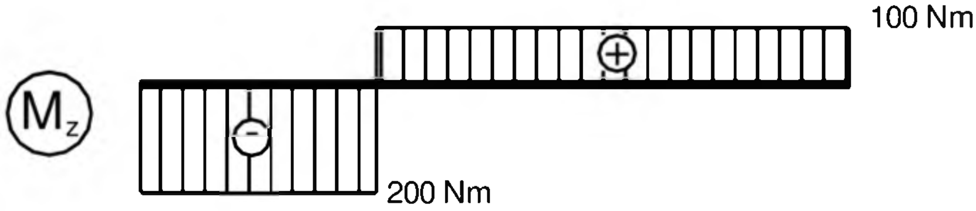


Figure 9.29

$$I_\rho = \frac{1}{2}\pi R^4 = \frac{1}{2}\pi 20^4 = 251200\ (\text{mm}^4)$$

$$\tau_{\max,AC} = \frac{M_A}{I_\rho}R = \frac{200\times 10^3}{257200}20 = 15.6MPa$$

$$\tau_{\max,CB} = \frac{M_B}{I_\rho}R = \frac{100\times 10^3}{257200}20 = 7.8MPa$$

Exercise 11

The shaft is fixed at its ends A and E and subjected to torsional moment as shown in **Figure 9.30**. The cross section of the shaft is circular. The allowable shear stress of material is [τ].

a) Draw the diagram of torque.

b) Determine the radius R with respect to M_o and [τ].

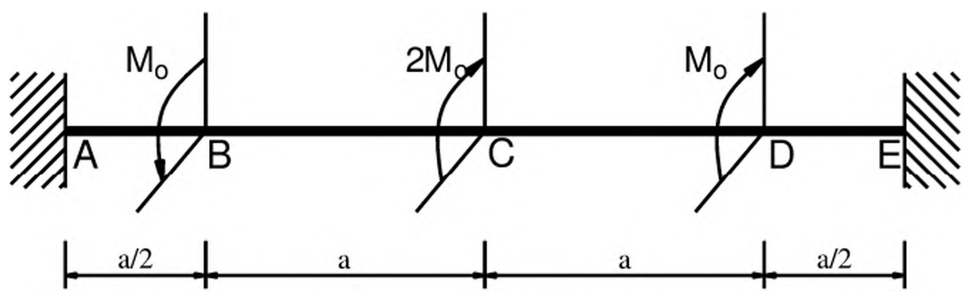


Figure 9.30

Solution

a) Draw the diagram of M_z.

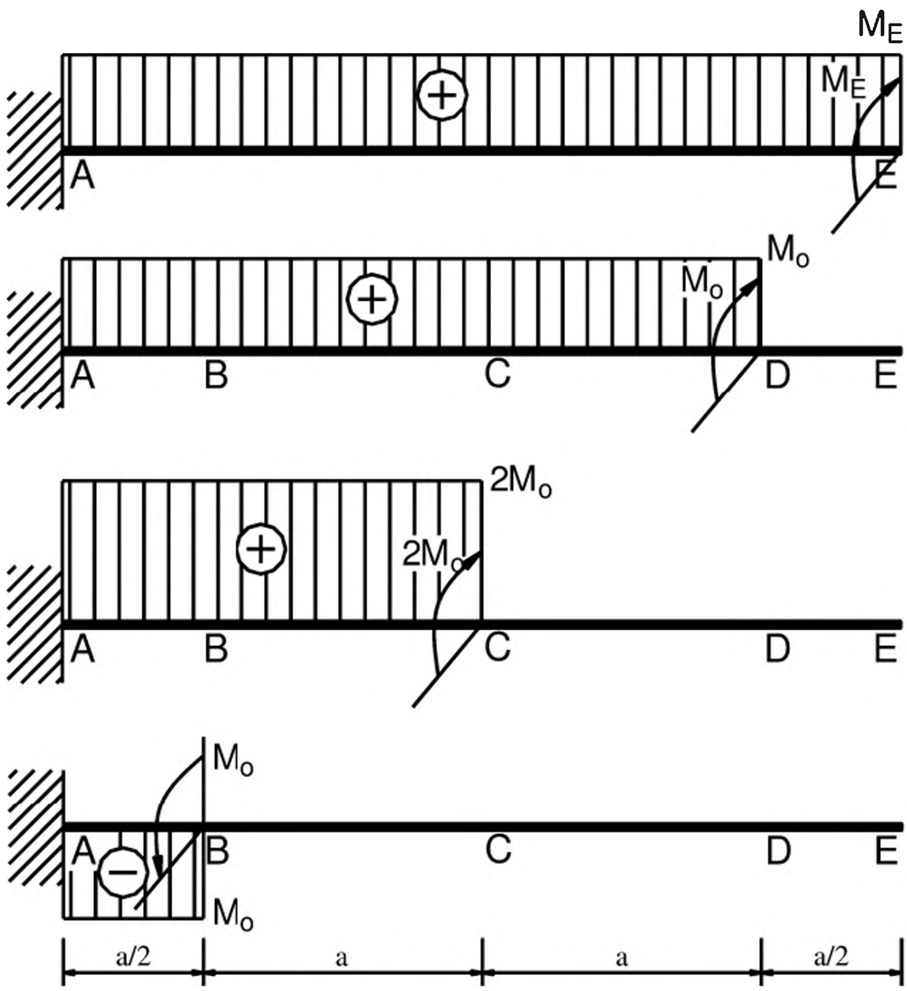


Figure 9.31

$$\varphi_E = \varphi_{EA} = \sum \left(\frac{M_z L}{GI_\rho} \right)_i$$

$$= \frac{M_E 3a}{GI_\rho} + \frac{M_o 2,5a}{GI_\rho} + \frac{2M_o 1,5a}{GI_\rho} - \frac{M_o 0,5a}{GI_\rho} = 0$$

(remember to take signs of the torsional moment)

$$M_E = -\frac{5}{3} M_o$$

($M_E = \frac{5}{3} M_o$ and the direction is opposite).

Other approach:

- Release the fixed end B
- Draw the diagram of torsion moment

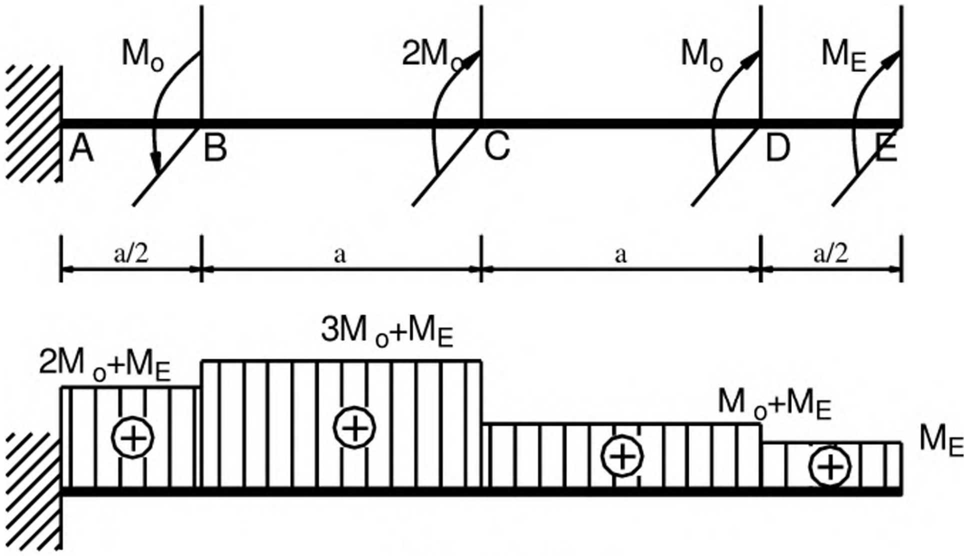


Figure 9.32

The boundary condition is that the rotation at B is zero

$$\varphi_E = \varphi_{EA} = \sum\left(\frac{M_z L}{GI_\rho}\right)_i$$

$$= \frac{(2M_o + M_E)a/2}{GI_\rho} + \frac{(3M_o + M_E)a}{GI_\rho} + \frac{(M_o + M_E)a}{GI_\rho} + \frac{M_E a/2}{GI_\rho}$$

$$= 0$$

$$(2M_o + M_E)\frac{1}{2} + (3M_o + M_E) + (M_o + M_E) + M_E\frac{1}{2} = 0$$

$$5M_o + 3M_E = 0$$

$$M_E = -\frac{5}{3}M_o$$

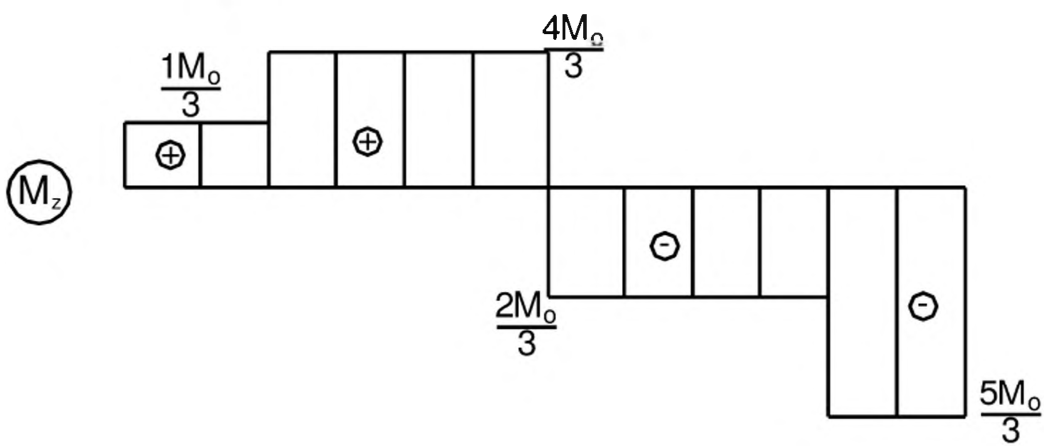


Figure 9.33

The diagram is the same as the approach 2.

b) Determine the radius R with respect to M_o and $[\tau]$:

$$M_{\max} = \frac{5}{3} M_o$$

Stress condition:

$$\tau_{\max} \le [\tau]$$

$$\frac{\frac{5M_o}{3}}{\frac{1}{2}\pi R^4} R \le [\tau]$$

$$\frac{10 M_o}{3\pi R^3} \le [\tau]$$

$$R \ge \sqrt{\frac{10}{3\pi}\frac{M_o}{[\tau]}}$$

Exercise 12

Consider the shaft shown in **Figure 9.34**. The cross section is circular and the allowable shear stess is $[\tau]$. Draw the diagram of M_z, determine the radius R with respect to M_o and $[\tau]$ for the following cases:

a) E is a free end.

b) E is a fixed end.

c) The fixed end E has a compulsary rotation $\varphi = \dfrac{2M_o a}{GI_\rho}$.

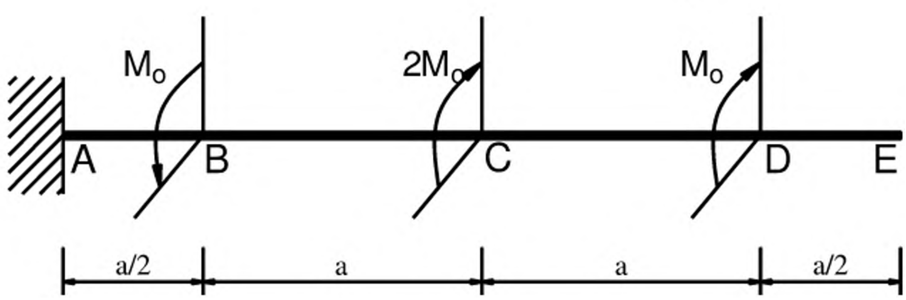


Figure 9.34

Solution

The solutions for questions a) and b) are similar to those of the previous exercises. Therefore, only solution for the question c) is presented herein.

The fixed end E has a compulsary rotation $\varphi = \dfrac{2M_o a}{GI_\rho}$.

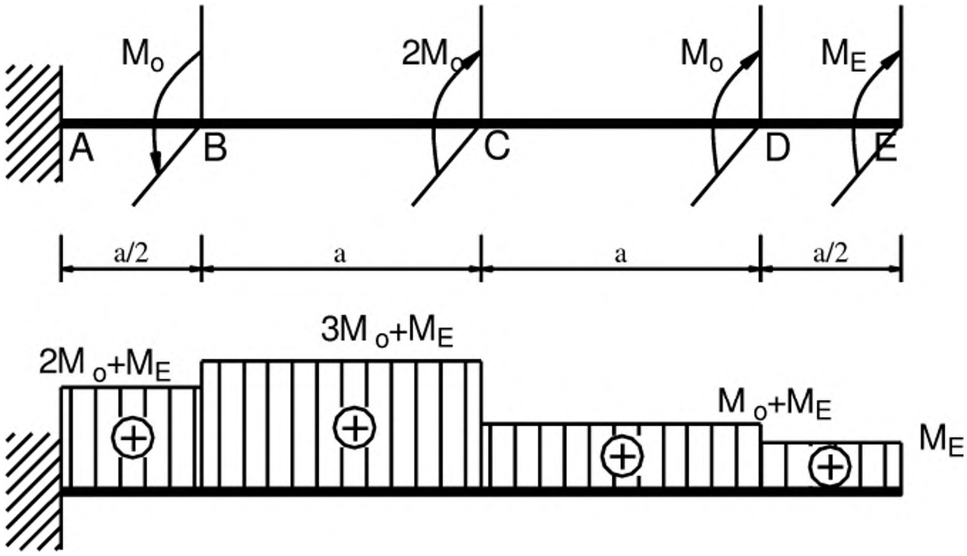


Figure 9.35

$$\varphi_E = \varphi_{EA}$$
$$= \sum \left(\frac{M_z L}{G I_\rho} \right)_i$$
$$= \frac{(2M_o + M_E)a/2}{GI_\rho} + \frac{(3M_o + M_E)a}{GI_\rho} + \frac{(M_o + M_E)a}{GI_\rho} + \frac{M_E a/2}{GI_\rho}$$
$$= \frac{2M_o a}{GI_\rho}$$
$$M_o + M_E = 0$$
$$M_E = -M_o$$

Then, the radius can be determined as $R \geq \sqrt{\frac{10}{3\pi} \frac{M_o}{[\tau]}}$

§3. TORSION OF RECTANGULAR CROSS SECTION MEMBERS

If a rectangular section member subjected to torsion, the assumption that the section is still in a plane is no longer true. The study results of rectangular cross section member showed that:

- There is only shear stress on the cross section.
- The shear stresses at center and corner are zero.

Figure 9.36 shows the distribution of shear stress on the section. The diagrams of shear stress are curves: increasing from 0 at the center to maximum at the mid points of the rectangular sides.

At the mid point of the height, the shear stress is maximum (τ_{max}). At the mid point of the width side, the shear stress is $\tau_1 < \tau_{max}$.

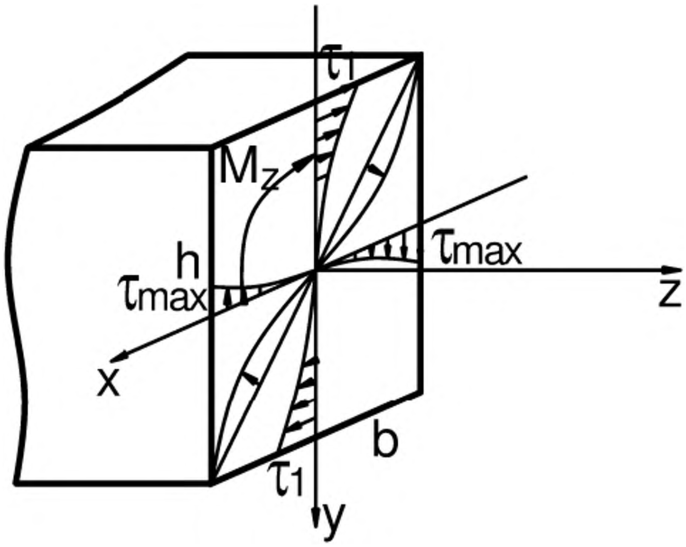


Figure 9.36. Distribution of shear stress on section

- Shear stress:

At the mid point of the height:

$$\tau_{max} = \frac{M_z}{\alpha h b^2} \tag{9.33}$$

At the mid point of the width:

$$\tau_1 = \gamma \tau_{max} \tag{9.34}$$

- Torsional angle (angle of twist):

$$\theta = \frac{M_z}{\beta G h b^3} \tag{9.35}$$

α, β, γ are coefficients depending on ratio of height to width, which are shown in **Table 9.1.**

Table 9.1. Coefficients α, β, γ

h/b		1	1.5	1.75	2.0	2.5	3.0	4.0	6	8	10	>10
Coefficient	α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
	β	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333
	γ	1.000	0.859	0.820	0.795	0.766	0.753	0.745	0.743	0.742	0.742	0.742

Exercise 13

Consider the solid shaft ABCD with circular section. The radius of the cross section is R and the allowable shear stresss is [τ]. Draw the diagram of torsional moment and determine R based on M_o and [τ] for the following cases:

a) D is a free end.

b) D is a fixed end.

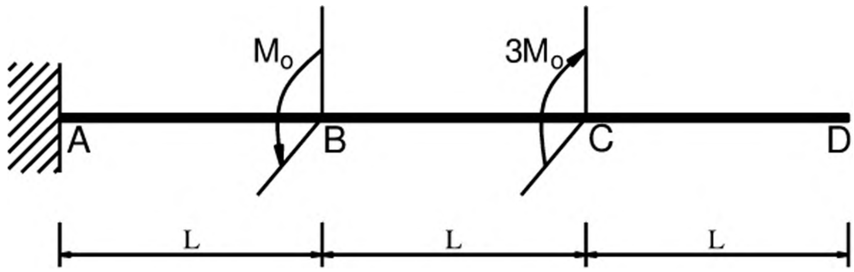


Figure 9.37

Solution

D is a free end. The diagram of torsional moment is shown in **Figure 9.38**

Figure 9.38

The condition of stress:

$$\tau_{max} \leq [\tau]$$

$$\frac{M_{max}}{\frac{1}{2}\pi R^4} R \leq [\tau]$$

$$R \geq \sqrt{\frac{2M_{max}}{\pi[\tau]}}$$

$$R \geq \sqrt{\frac{6M_o}{\pi[\tau]}}$$

D is a fixed end.

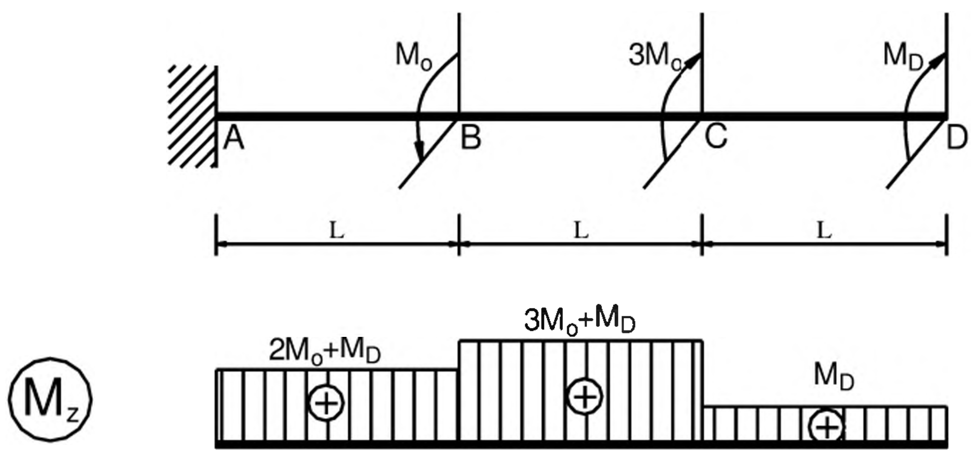


Figure 9.39

$$\varphi_D = \varphi_{DA} = \sum \left(\frac{M_z L}{GI_\rho} \right)_i = \frac{(2M_o + M_D)L}{GI_\rho} + \frac{(3M_o + M_D)L}{GI_\rho} + \frac{M_D L}{GI_\rho} = 0$$

$$5M_o + 3M_D = 0$$

$$M_D = -\frac{5}{3} M_o$$

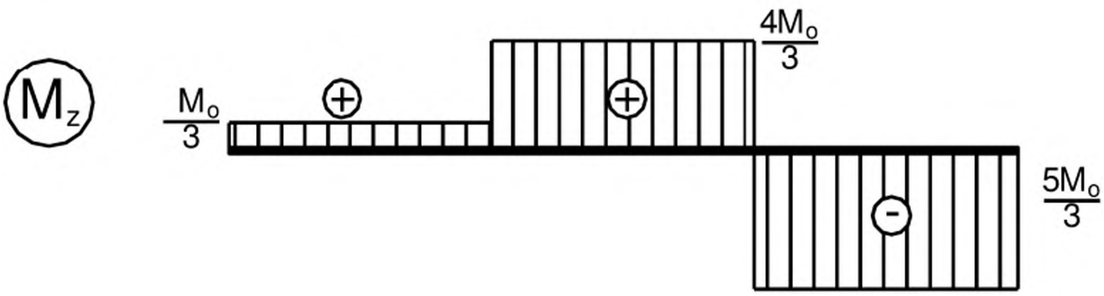


Figure 9.40

$$M_{\max} = \frac{5}{3} M_o$$

$$R \geq \sqrt{\frac{2M_{\max}}{\pi[\tau]}}$$

$$R \geq \sqrt{\frac{10M_o}{3\pi[\tau]}}$$

PROBLEMS

PROBLEM 1. The circular cross-sectional shaft BCD is subjected to the torsional moments at C and D as shown in **Figure 9.41**. The radius of the cross section of the shaft BCD is R.

a) Draw the diagram of torsion moment for the shaft BCD.

b) Determine the shear stresses in segments BC and CD.

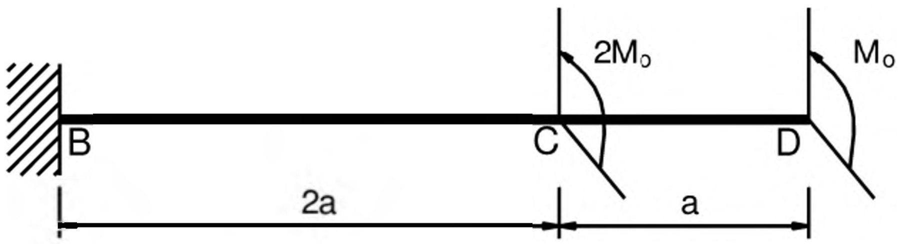


Figure 9.41

PROBLEM 2. The circular cross-sectional shaft BCD is subjected to the torsional moments at C and D as shown in **Figure 9.42**. The radius of the cross section of the shaft BCD is R. Given: $M_o = 300$ Nm and $R = 10$ mm.

a) Draw the diagram of torsion moment of the shaft BCD.

b) Determine the shear stresses in segments BC and CD.

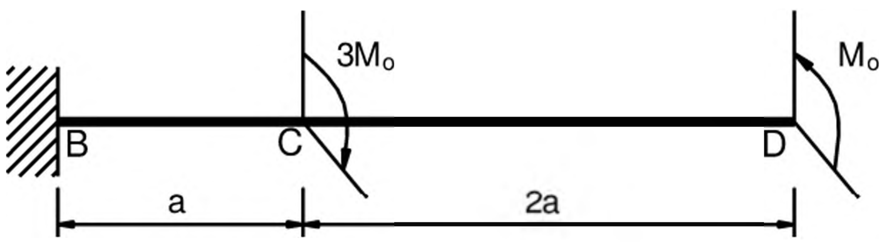


Figure 9.42

PROBLEM 3. The circular cross-sectional shaft BCD is subjected to the torque as shown in **Figure 9.43**. The radius of the cross section is R = 4 cm. Draw the diagrams of torsion moment and determine the maximum shear stresses for the following cases:

a) B is a free end.

b) B is a fixed end.

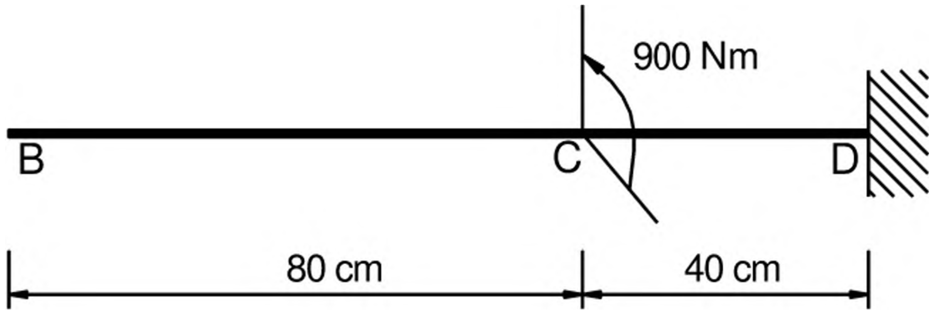


Figure 9.43

PROBLEM 4. The circular cross-sectional shaft ABC is subjected to the torque as shown in **Figure 9.44**. The radius of the cross section is R = 5cm. Draw the diagrams of torsion and determine the maximum shear stresses for the following cases:

a) A is a free end.

b) A is a fixed end.

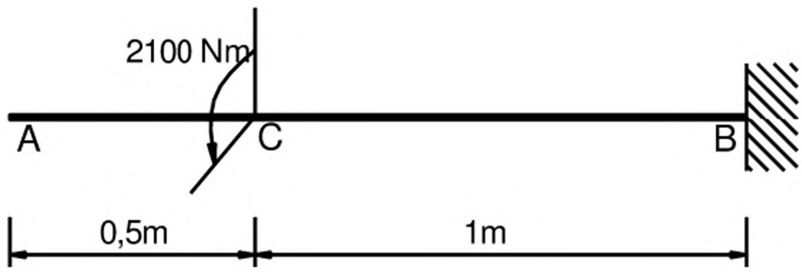


Figure 9.44

Chapter 10

COMBINED LOADINGS

§1. CONCEPTS

In this chapter, members subjected to combined loads are studied. The internal forces of these members include N_z, M_x, M_y, M_z.

+ N_z is the axial force
+ M_x is the bending moment about the x axis
+ M_y is the bending moment about the y axis
+ M_z is the torsion moment
+ The effects of shear forces are neglected.

Figure 10.1 shows an example of rectangular sections subjected to combined loads.

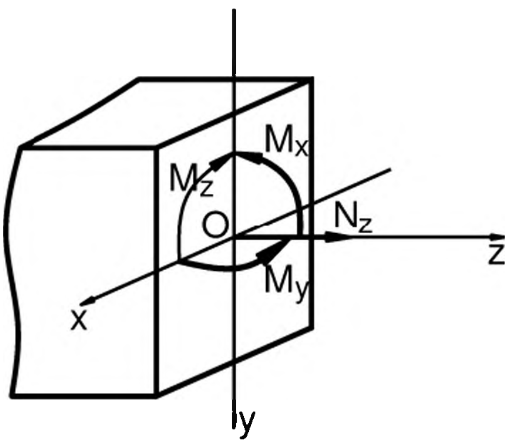


Figure 10.1. Combined loading on a cross section

§2. ASYMMETRIC BENDING

2.1. Definition

If a member is under asymmetric bending, there are two components of internal forces M_x and M_y on cross sections.

Figure 10.2 shows an example of a section subjected to bending moments M_x and M_y.

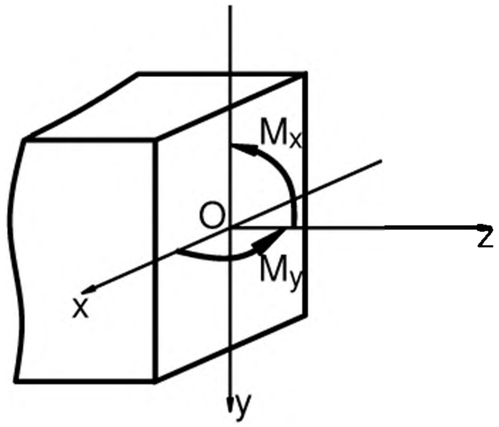


Figure 10.2. Asymmetric bending

Alternatively, M_x and M_y can be represented by vectors as shown in **Figure 10.3**. The sum of M_x and M_y is M_u.

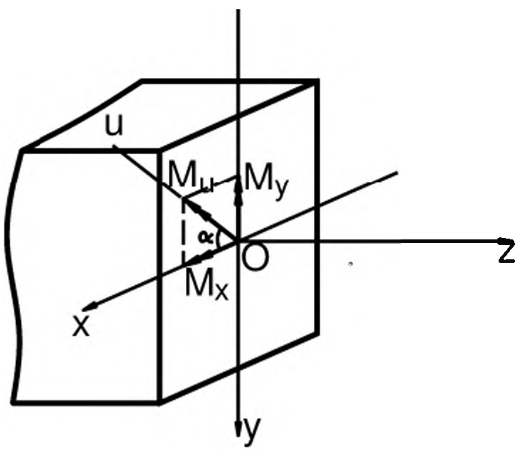


Figure 10.3. Asymmetric bending (using vector of moment)

in which,

$$\begin{cases} M_x = M_u \cos\alpha \\ M_y = M_u \sin\alpha \end{cases} \tag{10.1}$$

u is an asymmetric axis, thus it is called asymmetric bending.

Note:

- If $M_x = 0$ or $M_y = 0$, u will become y or x axis, respectively, and it is *symmetric bending*.
- If the cross section is circle, u is always a *symmetric* axis, thus it is always *symmetric bending* and

$$\sigma_{\substack{\max \\ \min}} = \pm\frac{M_u}{I_u}R = \pm\frac{M_u}{W_u} = \pm\frac{\sqrt{M_x^2 + M_y^2}}{W_u} \tag{10.2}$$

2.2. Normal stress

2.2.1. Formula

At point A(x,y):

M_x causes the normal stress:

$$\sigma_z^{M_x} = \frac{M_x}{I_x} y \tag{10.3}$$

M_y causes the normal stress:

$$\sigma_z^{M_y} = \frac{M_y}{I_y} x \tag{10.4}$$

Stress due to both M_x and M_y is:

$$\sigma_z^{(M_x,M_y)} = \sigma_z^{M_x} + \sigma_z^{M_y} = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x \tag{10.5}$$

2.2.2. Signs of stress

There are two methods to determine the signs of stress:

Method 1: The sign of stress depends on the sign of M_x, M_y, x, y.

- M_y is positive if it causes tension to the x+ fibers.
- M_x is positive if it causes tension to the y+ fibers.

Method 2: Use absolute values to calculate the stress value by Equation (10.6). Then, the signs are selected.

$$\sigma_z = \pm \frac{|M_x|}{I_x}|y| \pm \frac{|M_y|}{I_y}|x| \tag{10.6}$$

Take + if the moment causes tension to the point of computed stress.

Take – if the moment causes compression to the point of computed stress.

Figure 10.4 shows the sign of the terms in the Formula 10.6 corresponding to the bending moments.

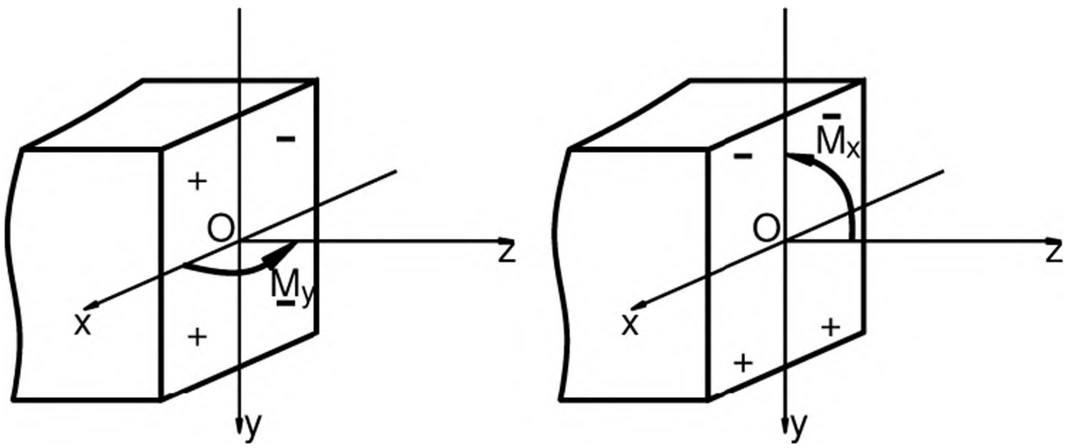


Figure 10.4. Sign of stress

2.2.3. Exercise 1

Consider the section in **Figure 10.5**. Given: b = 12cm, h = 30 cm, M_x = 100 kNm, M_y = 20 kNm. Calculate the normal stresses at A, B, C and D as shown in **Figure 10.5**. The coordinates of C and D are (3,-7.5) and D(-3,-7.5), respectively.

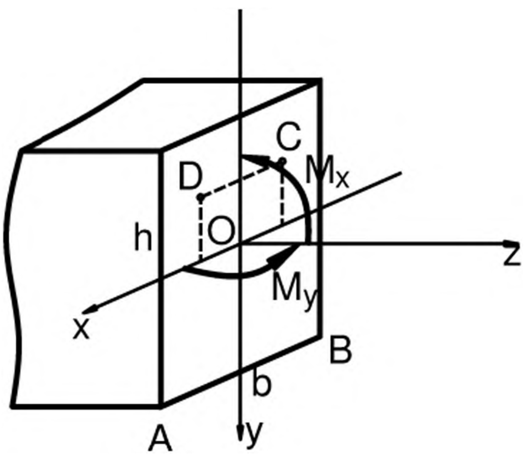


Figure 10.5

Solution

We have:

$$I_x = \frac{12(30)^3}{12} = 27000cm^4$$

$$I_y = \frac{30(12)^3}{12} = 4320cm^4$$

Table 10.1. Calculation results of Exercise 1

Point	Coordinate	Method 1: use $\sigma_z = \frac{M_x}{I_x}y + \frac{M_y}{I_y}x$	Method 2: use $\sigma_z = \pm\frac{\lvert M_x\rvert}{I_x}\lvert y\rvert \pm \frac{\lvert M_y\rvert}{I_y}\lvert x\rvert$
A	(6,15)	$= \frac{100\times10^2}{27000}(15) + \frac{20\times10^2}{4320}(6)$ $= 5,6 + 2,8 = 8,4kN/cm^2$	$= \frac{100\times10^2}{27000}(15) + \frac{20\times10^2}{4320}(6)$ $= 5,6 + 2,8 = 8,4kN/cm^2$
B	(-6,15)	$= \frac{100\times10^2}{27000}(15) + \frac{20\times10^2}{4320}(-6)$ $= 5,6 - 2,8 = 2,8kN/cm^2$	$= \frac{100\times10^2}{27000}\lvert 15\rvert - \frac{20\times10^2}{4320}\lvert -6\rvert$ $= 5,6 - 2,8 = 2,8kN/cm^2$
C	(-3,-7,5)	$= \frac{100\times10^2}{27000}(-7,5) + \frac{20\times10^2}{4320}(-3)$ $= -2,8 - 1,4 = -4,2kN/cm^2$	$= -\frac{100\times10^2}{27000}\lvert 7,5\rvert - \frac{20\times10^2}{4320}\lvert 3\rvert$ $= -2,8 - 1,4 = -4,2kN/cm^2$
D	(3,-7,5)	$= \frac{100\times10^2}{27000}(-7,5) + \frac{20\times10^2}{4320}(3)$ $= -2,8 + 1,4 = -1,4kN/cm^2$	$= -\frac{100\times10^2}{27000}\lvert 7,5\rvert + \frac{20\times10^2}{4320}\lvert 3\rvert$ $= -2,8 + 1,4 = -1,4kN/cm^2$

2.2.4. Exercise 2

Consider the beam shown in the **Figure 10.6**. Given: L = 2 m, P = 2400 N and perpendicular to the axial axis, $\alpha = 30^o$, b = 12 cm, h = 20 cm, $[\sigma] = 10$ MN/m^2.

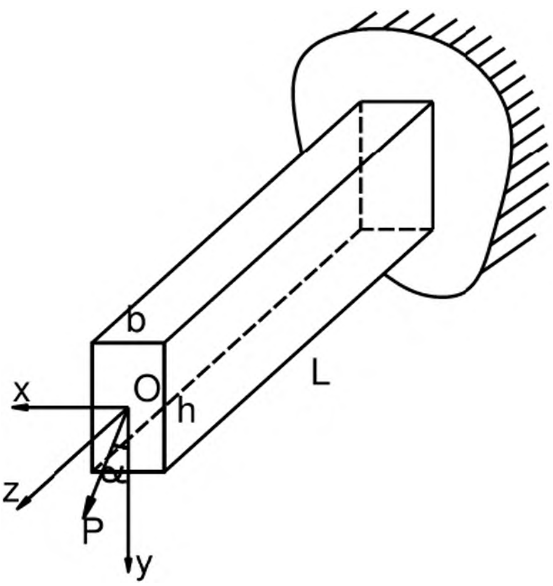


Figure 10.6

a) Draw diagrams of M_x, M_y.

b) Calculate the maximum and minimum stresses.

c) Check the stress condition.

Solution

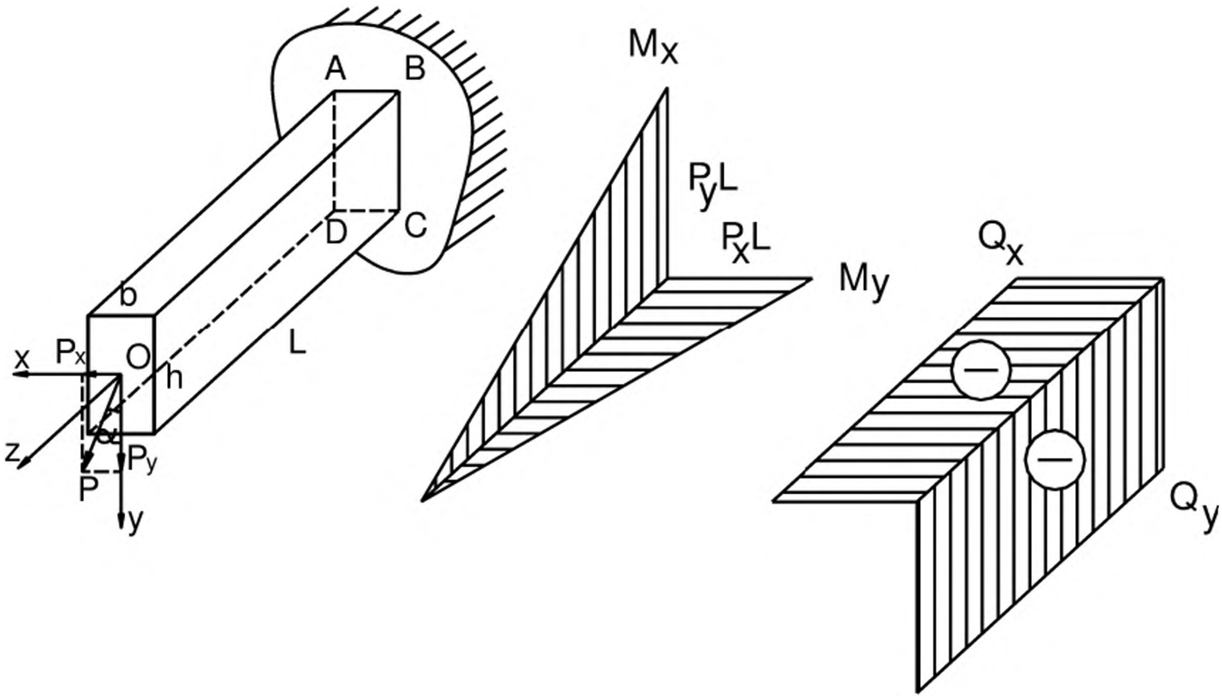


Figure 10.7

Decompose the force P into two components:

$$P_x = P \times \sin 30^o = 2400 \times 0.5 = 1200 \text{ N}$$

$$P_y = P \times \cos 30^o = 2400 \times 0.866 = 2078 \text{ N}$$

Bending moments at the fixed end:

$$M_x = P_y L = 2078 \times 2 = 4156 \text{ Nm}$$

$$M_y = P_x L = 1200 \times 2 = 2400 \text{ Nm}$$

Inertia moment:

$$I_x = \frac{12 \times 20^3}{12} = 8000 cm^3$$

$$I_y = \frac{20 \times 12^3}{12} = 2880 cm^3$$

Consider the corner points A, B, C, D as shown in **Figure 10.8**. The coordinates of these points and the calculation of stresses are presented in **Table 10.2**.

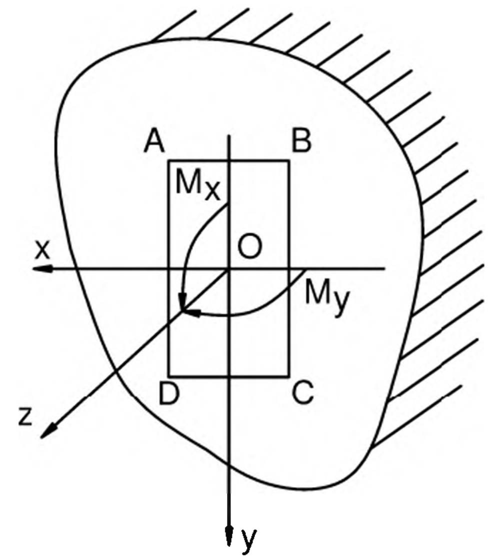


Figure 10.8

Table 10.2. Calculation results of Exercise 2

Point	Coordinate	Method 1: use $\sigma_z = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x$	Method 2: use $\sigma_z = \pm\frac{\lvert M_x\rvert}{I_x}\lvert y\rvert \pm \frac{\lvert M_y\rvert}{I_y}\lvert x\rvert$
A	(6,-10)	$= \frac{-4156\times10^2}{8000}(-10) + \frac{-2400\times10^2}{2880}6$ $= 519{,}5 - 500 = 19{,}5 N/cm^2$	$= +\frac{4156\times10^2}{8000}(10) - \frac{2400\times10^2}{2880}6$ $= 519{,}5 - 500 = 19{,}5 N/cm^2$
B	(-6,-10)	$= \frac{-4156\times10^2}{8000}(-10) + \frac{-2400\times10^2}{2880}(-6)$ $= 519{,}5 + 500 = 1019{,}5 N/cm^2$	$= +\frac{4156\times10^2}{8000}(10) + \frac{2400\times10^2}{2880}6$ $= 519{,}5 + 500 = 1019{,}5 N/cm^2$
C	(-6,10)	$= \frac{-4156\times10^2}{8000}(10) + \frac{-2400\times10^2}{2880}(-6)$ $= -519{,}5 + 500 = -19{,}5 N/cm^2$	$= -\frac{4156\times10^2}{8000}(10) + \frac{2400\times10^2}{2880}6$ $= -519{,}5 + 500 = -19{,}5 N/cm^2$
D	(6,10)	$= \frac{-4156\times10^2}{8000}(10) + \frac{-2400\times10^2}{2880}(6)$ $= -519{,}5 - 500 = -1019{,}5 N/cm^2$	$= -\frac{4156\times10^2}{8000}(10) - \frac{2400\times10^2}{2880}6$ $= -519{,}5 - 500 = -1019{,}5 N/cm^2$

2.3. Neutral line and diagram of stress

2.3.1. Neutral line (neutral axis)

The equation $\sigma_z = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x$ is a plane in the coordinate system Oxyz, and is called *stress plane*. In **Figure 10.9**, the plane ABCD is the stress plane.

The intersection of the stress plane and the cross section is called *neutral line* or *neutral axis*. In **Figure 10.9**, the line EF is the neutral line or neutral axis.

The normal stress of any point on the neutral line is zero.

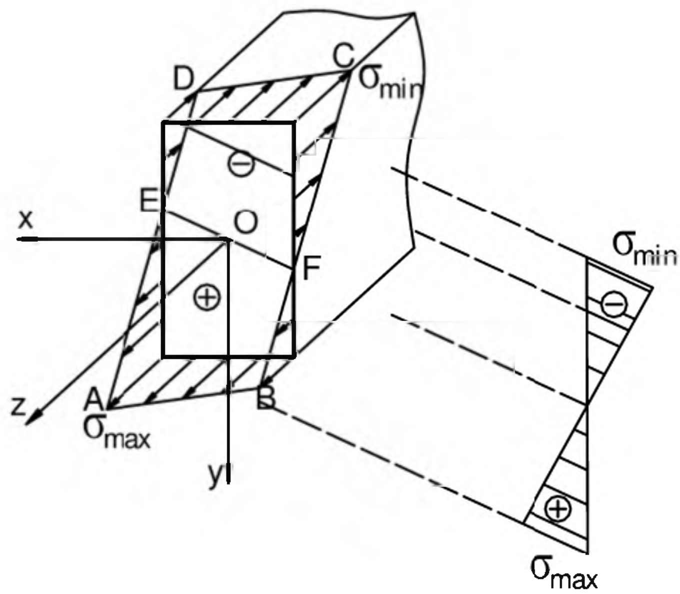


Figure 10.9. Distribution of normal stress on the cross section

Equation of neutral axis:

$$\sigma_z = 0$$

$$\frac{M_x}{I_x} y + \frac{M_y}{I_y} x = 0$$

$$y = -\frac{M_y}{M_x} \frac{I_x}{I_y} x \qquad (10.7)$$

This equation is in form of y = ax. Therefore, it goes through the origin O and has an angle β with the x axis, where

$$tg\beta = -\frac{M_y}{M_x} \frac{I_x}{I_y} \qquad (10.8)$$

Comments:

- Neutral axis divides the area into two regions: compression and tension (see **Figure 10.9**).
- Points located on a straight line parallel to neutral axis have the same stress value.
- The value of stress at a point increases with the increase of the distance from that point to the neutral axis, the greater the stress value is. The farthest point on tension region has maximum normal stress $\sigma_{z,max}$; the farthest point on compression region has minimum normal stress $\sigma_{z,min}$.

2.3.2. Diagram of stress

How to draw the diagram of stress

Step 1: Extend the neutral line.

Step 2: Draw a line ⊥ neutral axis.

Step 3: From the farthest points, draw lines // the neutral axis.

Step 4: Draw maximum stress, minimum stress.

Step 5: Connect the maximum stress and minimum stress.

Meaning

A point lies on the line // the neutral axis has the stress that is y coordinate of the stress diagram corresponding to that point.

2.3.3. Maximum and minimum stresses and stress condition

In **Figure 10.9**

A is the farthest point on the tension region. The normal stress at A is:

$$\sigma_{z,\max}^{A} = \sigma_{z,\max} = +\frac{|M_x|}{I_x}|y_A| + \frac{|M_y|}{I_y}|x_A| \tag{10.9}$$

$$\sigma_{z,\max}^{A} = \sigma_{z,\max} = \frac{|M_x|}{W_x^{tension}} + \frac{|M_y|}{W_y^{tension}} \tag{10.10}$$

where $W_x^{tension} = \frac{I_x}{y_A}; W_y^{tension} = \frac{I_y}{x_A}$

C is the farthest point on the compression region. The normal stress at C is:

$$\sigma_{z,\min}^{C} = \sigma_{z,\min} = -\frac{|M_x|}{I_x}|y_C| - \frac{|M_y|}{I_y}|x_C| \tag{10.11}$$

$$\sigma_{z,\min}^{C} = \sigma_{z,\min} = -\frac{|M_x|}{W_x^{compression}} - \frac{|M_y|}{W_y^{compression}} \tag{10.12}$$

where, $W_x^{compression} = \frac{I_x}{y_C}; W_y^{compression} = \frac{I_y}{x_C}$

For a symmetric cross section:

$$W_x^{tension} = W_x^{compression}$$

$$W_y^{tension} = W_y^{compression}$$

The stress condition:

$$\sigma_{z,\max} \le [\sigma]_{tension}$$

$$|\sigma_{z,\min}| \le [\sigma]_{compression}$$

2.3.4. Exercise 3

Consider the hollow circular beam shown in **Figure 10.10**. Given: D = 16 cm, b = 7 cm, h = 11 cm, α = 30°, q = 15 kN/m, L = 1.2m, $[\sigma]_k = 3{,}5kN/cm^2$, $[\sigma]_n = 13kN/cm^2$.

a) Determine the equation of neutral axis.

b) Determine the maximum and minimum normal stresses.

c) Draw the diagram of normal stress.

d) Check the stress condition.

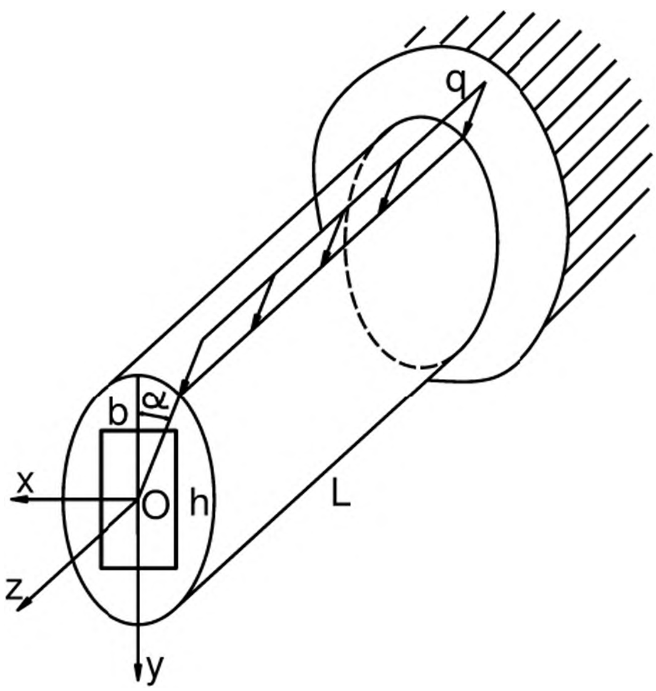


Figure 10.10. A cantilever beam

Solution

a) Determine the equation of neutral axis.

$$q_x = q\sin\alpha$$

$$q_y = q\cos\alpha$$

Bending moment at the fixed end:

$$M_x = -\frac{q_y L^2}{2} = -\frac{q\cos\alpha . L^2}{2} = -\frac{15\times\cos 30^o \times 1,2^2}{2} = -9.35kNm$$

$$M_y = -\frac{q_x L^2}{2} = -\frac{q\sin\alpha . L^2}{2} = -\frac{15\times\sin 30^o \times 1,2^2}{2} = -5.4kNm$$

Inertia moment:

$$I_x = \frac{\pi R^4}{4} - \frac{bh^3}{12} = \frac{3,14\times 8^4}{4} - \frac{7\times 11^3}{12} = 2439cm^4$$

$$I_y = \frac{\pi R^4}{4} - \frac{hb^3}{12} = \frac{3,14\times 8^4}{4} - \frac{11\times 7^3}{12} = 2901cm^4$$

The equation of neutral axis:

$$y = -\frac{M_y}{M_x}\frac{I_x}{I_y}x = -\frac{-5.4}{-9.35}\frac{2439}{2901}x = 0.49x$$

$$\beta = \arctan(0.49) = 26.1^o$$

The result is shown in **Figure 10.11**.

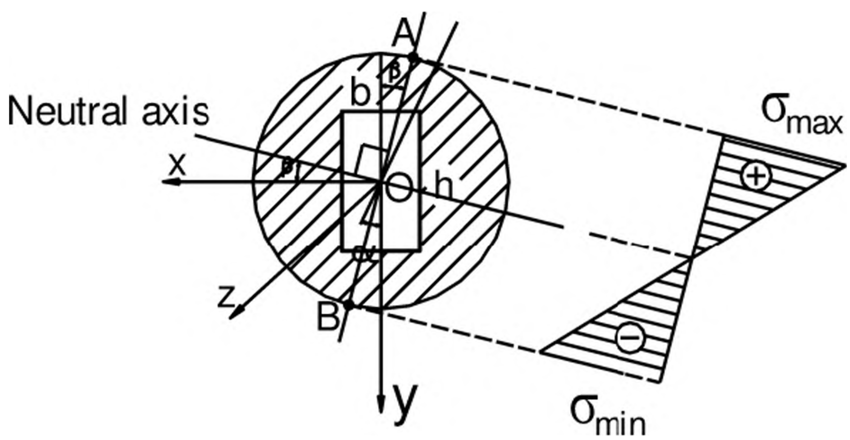


Figure 10.11

b) Determine the maximum and minimum normal stresses.

The points A, B are farthest from the neutral axis:

$$|x_A| = |x_B| = R\sin\beta = 8\times\sin 26.1^o = 8\times 0.44 = 3.52cm$$

$$|y_A| = |y_B| = R\sin\beta = 8\times\cos 26.1^o = 8\times 0.9 = 7.2cm$$

$$\sigma_{\max}^{A}=+\frac{|M_x|}{I_x}|y_A|+\frac{|M_y|}{I_y}|x_A|=\frac{9.35\times10^2}{2439}7.2+\frac{5.4\times10^2}{2901}3.52=3.41kN/cm^2$$

$$\sigma_{\max}^{B}=-\frac{|M_x|}{I_x}|y_A|-\frac{|M_y|}{I_y}|x_A|=-\frac{9,35\times10^2}{2439}7.2-\frac{5.4\times10^2}{2901}3.52=-3.41kN/cm^2$$

c) Diagram of stress: see **Figure 10.11**.

d) Stress condition:

$$\sigma_{\max}^{A}=3.41kN/cm^2<[\sigma]_k \Rightarrow \text{OK}$$

$$\left|\sigma_{\max}^{B}\right|=3.41kN/cm^2<[\sigma]_n \Rightarrow \text{OK}$$

2.3.5. Exercise 4

Consider the T beam shown in **Figure 10.12**. Given: q = 4 kN/m, L = 2 m, P = qL in the plane Oxy and inclined angle $\alpha = 30^o$, b = 5 cm, t = 5 cm, h = 30 cm, h_1 = 25 cm.

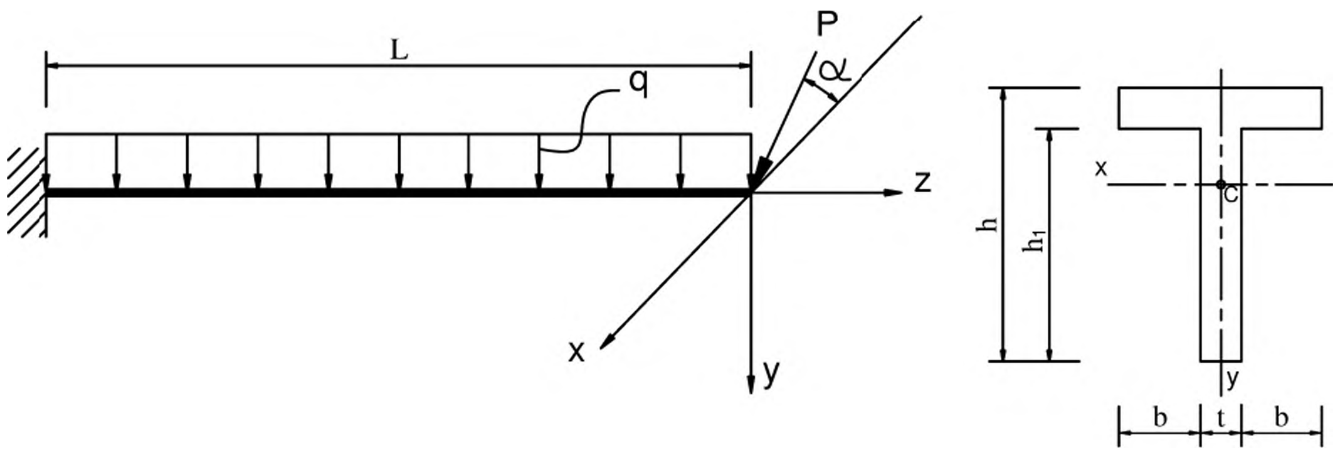


Figure 10.12

a) Determine the center C, principle moment of inertia of the section.

b) Draw M_x, M_y.

c) Determine the neutral axis of the section at the fixed end.

d) Draw the diagram of stress.

e) Calculate the maximum and minimum normal stresses.

Solution

a) Determine the center C, principle moment of inertia of the section I_x, I_y.

$$y_c=\frac{S_{x_o}}{A}=\frac{25\times5\times25/2+5\times15(25+5/2)}{25\times5+15\times5}=18.125cm$$

$$I_x = \left[\frac{5\times 25^3}{12} + 5\times 25\left(\frac{25}{2} - 18.125\right)^2\right] + \left[\frac{15\times 5^3}{12} + 15\times 5\left(30 - 18.125 - \frac{5}{2}\right)^2\right]$$

$$= 17213.5 cm^4$$

$$I_y = \frac{5\times 15^3}{12} + \frac{25\times 5^3}{12} = 1667 cm^4$$

Draw diagrams of M_x, M_y

P = qL

$$P_x = P\cos\alpha = qL\times\cos 30^o = qL\frac{\sqrt{3}}{2}$$

$$P_y = P\sin\alpha = qL\times\sin 30^o = \frac{qL}{2}$$

$$M_x = -P_y L - \frac{qL^2}{2} = -\frac{qL^2}{2} - \frac{qL^2}{2} = -qL^2 = 16 kNm$$

$$M_y = --P_x L = -\frac{qL\sqrt{3}}{2}L = -\frac{\sqrt{3}qL^2}{2} = -13.8 \text{ kNm}$$

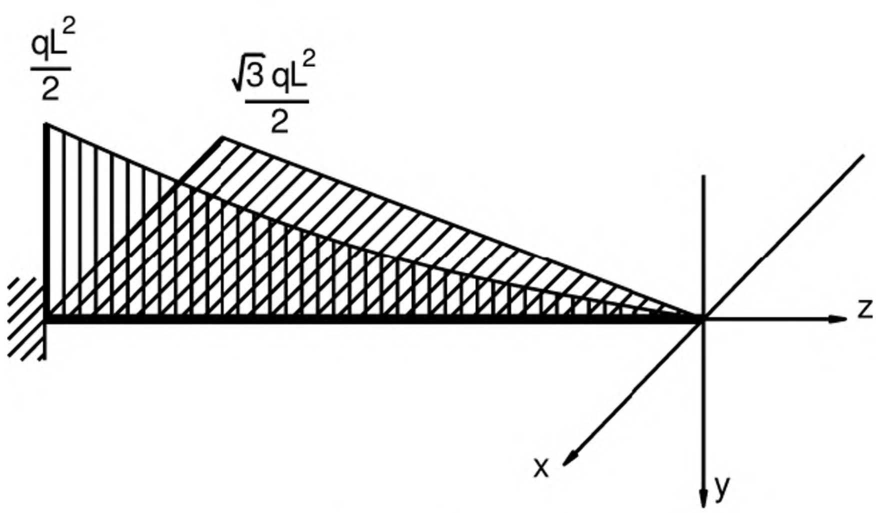


Figure 10.13

c) Equation of the neutral axis:

$$y = -\frac{M_y}{M_x}\frac{I_x}{I_y}x = -\frac{\sqrt{3}}{2}\frac{17213.5}{1667}x = -8.94x$$

$$\beta = \arctan(-8.94) = -83.6^o$$

The neutral axis is shown in **Figure 10.14**

d) Draw the diagram of stress.

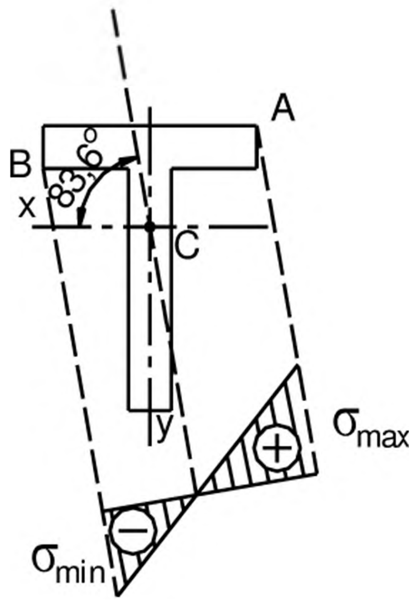


Figure 10.14

e) Calculate the maximum and minimum normal stresses.

The coordinate of the point A which has maximum stress:

$x_A = -b-t/2 = -5-5/2 = -7.5$ cm

$y_A = h-y_C = 30-18.125 = 11.875$ cm

The coordinate of the point B which has minimum stress:

$x_B = b + t/2 = 5 + 5/2 = 7.5$ cm

$y_B = -(h-y_C) = -(25-18.125) = -6.875$ cm

$$\sigma_{max}^{A} = +\frac{|M_x|}{I_x}|y_A| + \frac{|M_y|}{I_y}|x_A| = \frac{16\times10^2}{17213.5}11.875 + \frac{13.8\times10^2}{1667}7.5$$

$$= 7.34 kN/cm^2$$

$$\sigma_{min}^{B} = -\frac{|M_x|}{I_x}|y_B| - \frac{|M_y|}{I_y}|x_B| = -\frac{16\times10^2}{17213.5}6.875 - \frac{13.8\times10^2}{1667}7.5$$

$$= -5.6 kN/cm^2$$

2.3.6. Exercise 5

Consider the I20 beam shown in **Figure 10.15**. Given: q = 4 kN/m, P = qL, L = 2 m, $\alpha = 60^o$. The load P is in the plane Oxy.

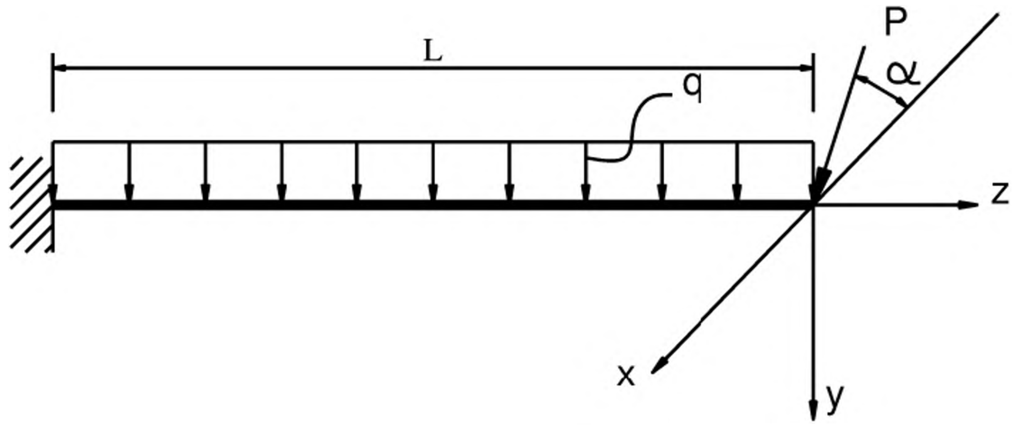


Figure 10.15

a) Draw diagrams of M_x, M_y.

b) Determine the neutral axis of the section at the fixed end.

c) Draw the diagram of stress.

d) Calculate the maximum and minimum normal stresses.

Solution

a) Draw diagrams of M_x, M_y.

$$P = qL = 4\times2 = 8\ kN$$

$$P_x = P\cos\alpha = 8\times\cos 60^o = 4kN$$

$$P_y = P\sin\alpha = 8\times\sin 60^o = 4\sqrt{3}kN$$

Moment at the fixed end:

$$M_x = -\left(\frac{qL^2}{2} + P_yL\right) = -\left(\frac{4\times2^2}{2} + 4\sqrt{3}\times2\right) = -21.856kNm$$

$$M_y = -P_xL = -4\times2 = -8kNm$$

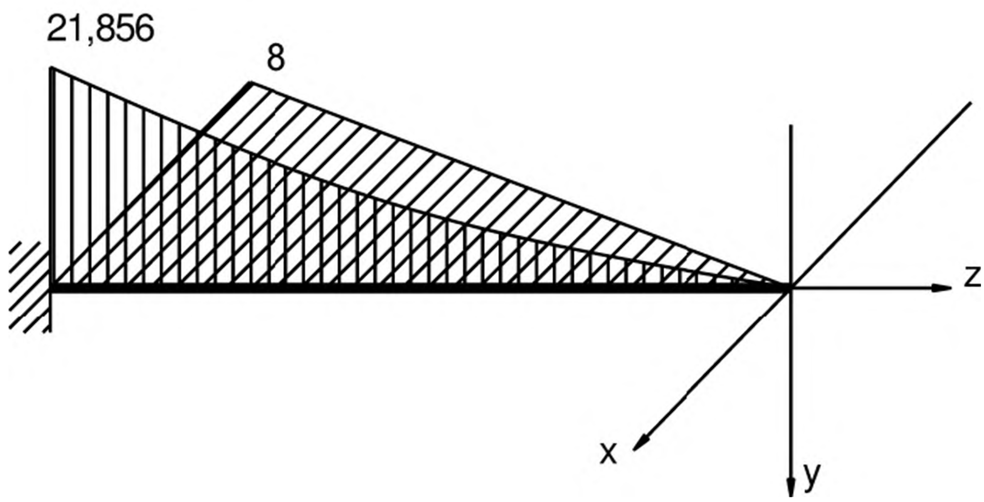


Figure 10.16

b) Determine the neutral axis of the section at fixed end.

I20 → A=26.8 cm^2, I_x =1840 cm^4, W_x = 184 cm^3, I_y = 115 cm^4, W_y = 23.1 cm^3.

$$y=-\frac{M_y}{M_x}\frac{I_x}{I_y}x=-\frac{-21.856}{-8}\frac{1840}{115}x=-5.85x$$

$$\beta=\arctan(-5.85)=-80.3^o$$

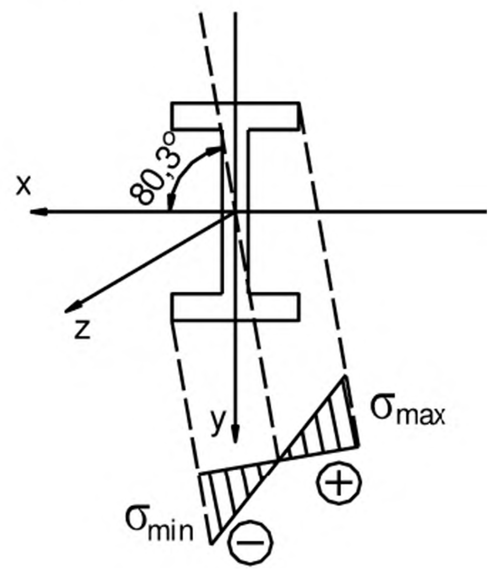


Figure 10.17

c) Draw the diagram of stress.

d) Calculate the maximum and minimum normal stresses.

Due to the symmetry, we have:

$$W_x^k=W_x^n=W_x$$

$$W_y^k=W_y^n=W_y$$

$$\sigma_{z,\max}=\frac{|M_x|}{W_x^k}+\frac{|M_y|}{W_y^k}=\frac{2185.6}{1840}+\frac{800}{184}=46.5kN/cm^2$$

$$\sigma_{z,\min}=-\frac{|M_x|}{W_x^n}-\frac{|M_y|}{W_y^n}=-\frac{2185.6}{1840}-\frac{800}{184}=-46.5kN/cm^2$$

2.3.7. Exercise 6

The beams in the two Exercises above are replaced by a hollow round steel girder with the outer radius R = 7 cm and inner radius r = 4 cm. Calculate the maximum and minimum stress on section of the fixed end.

Solution

$$I_u = I_x = I_y = \frac{1}{4}\pi\left(R^4 - r^4\right) = \frac{1}{4}\pi\left(7^4 - 4^4\right) = 1394cm^4$$

$$M_u = \sqrt{M_x^2 + M_y^2}$$

$$\sigma_{\substack{\max\\ \min}} = \pm\frac{M_u}{I_u}R$$

M_x =16 kNm, M_y = 13.8 kNm

$$=> \quad \sigma_{\substack{\max\\ \min}} = \pm\frac{M_u}{I_u}R = 10.61kN/cm^2$$

M_x = 21.856 kNm, M_y = 8 kNm

$$=> \quad \sigma_{\substack{\max\\ \min}} = \pm\frac{M_u}{I_u}R = 11.68kN/cm^2$$

2.3.8. Exercise 7

Assume that you are working for a design company. Your boss asks you to design a steel [beam supported by the two concrete walls as shown in **Figure 10.18**. Assume that the beam is simply supported. Given L = 5 m, q = 6 kN/m, $[\sigma]_n = [\sigma]_k$ =16 kN/cm^2, $\alpha = 30^o$.

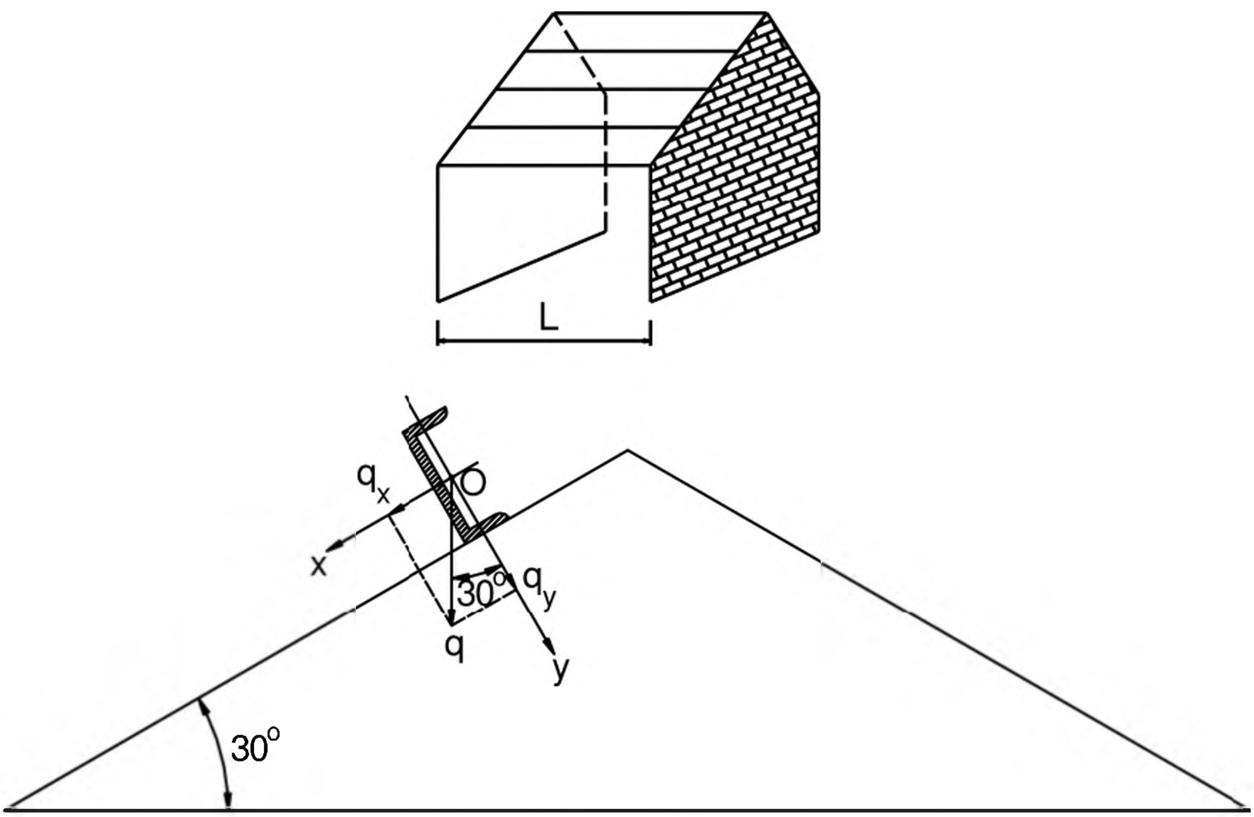


Figure 10.18

Solution

We have the simply supported beam shown in **Figure 10.19**.

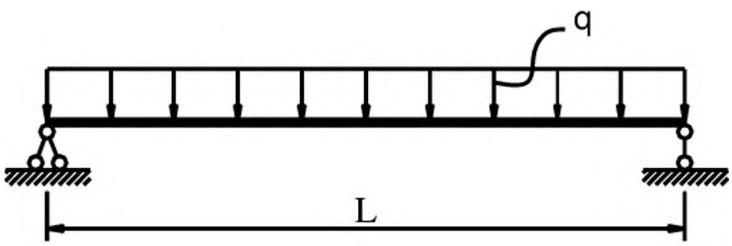


Figure 10.19

$$q_x = q\sin\alpha$$

$$q_y = q\cos\alpha$$

Moment at the middle of the beam:

$$M_x = \frac{q_y L^2}{8} = \frac{q\cos\alpha . L^2}{8} = \frac{6\times\cos 30^o \times 5^2}{8} = 16.24 kNm$$

$$M_y = \frac{q_x L^2}{8} = \frac{q\sin\alpha . L^2}{8} = \frac{6\times\sin 30^o \times 5^2}{8} = 9.375 kNm$$

The moments M_x and M_y are shown in **Figure 10.20**.

- Point A has the smallest stress while point B has the largest stress.
- $\left|\sigma_z^A\right| \geq \left|\sigma_z^B\right|$ because the distance from the axis x to A is larger than that to B.
- $[\sigma]_n = [\sigma]_k$

=> Therefore, the stress at A needs to be checked.

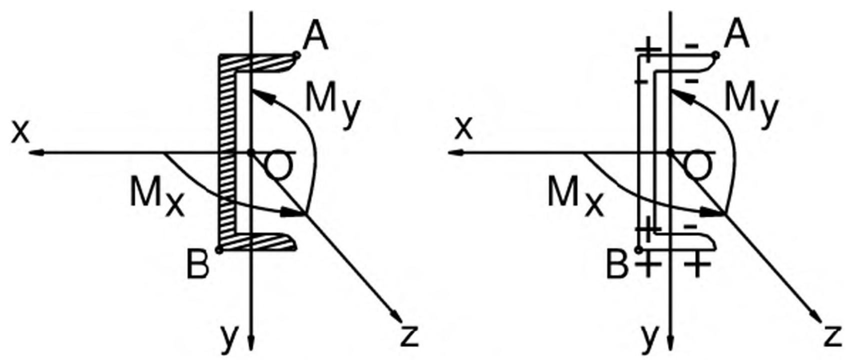


Figure 10.20

$$\left|\sigma_{min}\right| = \frac{\left|M_x\right|}{W_x} + \frac{\left|M_y\right|}{W_y^{canh}} \leq [\sigma]$$

$$\frac{16.24\times10^2}{W_x}+\frac{9.375\times10^2}{W_y^{flange}}\le 16$$

(the unit is in kN, cm)

$$\frac{16.24}{W_x}+\frac{937.5}{W_y^{flange}}\le 16$$

Use trial and error method:

Select steel [36, which has

$$W_x = 601cm^3;\ W_y^{flange} = 61.7cm^3$$

$$\frac{1624}{601}+\frac{937.5}{61.7}=17.9kN/cm^2 > 16kN/cm^2 \text{ => Not OK}$$

Increase the section size by selecting [40, which has

$$W_x = 760cm^3;\ W_y^{flange} = 73.4cm^3$$

$$\frac{16.24}{760}+\frac{937.5}{73.4}=14.9kN/cm^2 \le 16kN/cm^2 \text{ => OK}$$

Thus, select [40.

§3. BENDING AND TENSION OR COMPRESSION (ECCENTRIC AXIAL LOADINGS)

3.1. Definition

A member is in bending and tension or compression if the internal forces include M_u (or M_x and M_y) and N_z.

3.2. Normal stress

3.2.1. Formula

Apply the superposition method:

$$\begin{aligned}\sigma_z^{(N_z,M_x,M_y)} &= \sigma_z^{N_z}+\sigma_z^{M_x}+\sigma_z^{M_y}\\ &= \frac{N_z}{A}+\frac{M_x}{I_x}y+\frac{M_y}{I_y}x\end{aligned} \tag{10.13}$$

or:

$$\sigma_z = \pm\frac{|N_z|}{A}\pm\frac{|M_x|}{I_x}|y|\pm\frac{|M_y|}{I_y}|x| \tag{10.14}$$

3.2.2. Exercise 8

Consider the rectangular section shown in **Figure 10.21**. Given: b = 120 mm, h = 300 mm, M_x = 100 kNm, M_y = 20 kNm, N_z = 1000 kN. Calculate the stresses at points A, B at corners and C(-30, -75), D(30, 75).

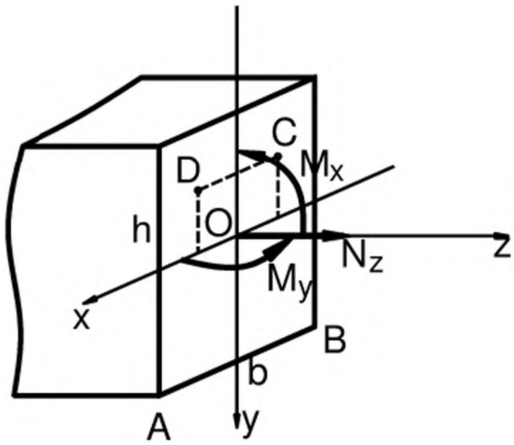


Figure 10.21

Solution

We have:

A = 12.30 = 360 cm^2;

$$I_x = \frac{12(30)^3}{12} = 27000cm^4$$

$$I_x = \frac{30(12)^3}{12} = 4320cm^4$$

The calculation results are shown in **Table 10.3**.

Table 10.3. Calculation results of Exercise 2

Point	Coordinate	Method 1: use $\sigma_z = \frac{N_z}{A} + \frac{M_x}{I_x}y + \frac{M_y}{I_y}x$	Method 2: use $\sigma_z = \pm\frac{N_z}{A} \pm \frac{\lvert M_x\rvert}{I_x}\lvert y\rvert \pm \frac{\lvert M_y\rvert}{I_y}\lvert x\rvert$
A	(6,-15)	$= \frac{1000}{360} + \frac{100\times10^2}{27000}(15) + \frac{20\times10^2}{4320}(6)$ $= 2.8 + 5.6 + 2.8 = 11.1kN/cm^2$	$= \frac{1000}{360} + \frac{100\times10^2}{27000}(15) + \frac{20\times10^2}{4320}(6)$ $= 2.8 + 5.6 + 2.8 = 11.1kN/cm^2$
B	(-6,15)	$= \frac{1000}{360} + \frac{100\times10^2}{27000}(15) + \frac{20\times10^2}{4320}(-6)$ $= 2.8 + 5.6 - 2.8 = 5.6kN/cm^2$	$= \frac{1000}{360} + \frac{100\times10^2}{27000}(15) + \frac{20\times10^2}{4320}(-6)$ $= 2.8 + 5.6 - 2.8 = 5.6kN/cm^2$

C	(-3,-7,5)	$=\frac{1000}{360}+\frac{100\times10^2}{27000}(-7,5)$ $+\frac{20\times10^2}{4320}(-3)$ $=2.8-2.8-1.4=-1.4kN/cm^2$	$=\frac{1000}{360}\pm\frac{100\times10^2}{27000}(7,5)$ $-\frac{20\times10^2}{4320}(3)$ $=2.8-2.8-1.4=-1.4kN/cm^2$
D	(3,-7,5)	$=\frac{1000}{360}+\frac{100\times10^2}{27000}(-7,5)$ $+\frac{20\times10^2}{4320}(3)$ $=2.8-2.8+1.4=+1.4kN/cm^2$	$=\frac{1000}{360}\pm\frac{100\times10^2}{27000}(7,5)$ $+\frac{20\times10^2}{4320}(3)$ $=2.8-2.8+1.4=1.4kN/cm^2$

3.3. Neutral axis, stress diagram and stress condition

Similar to symmetric bending:

$$\sigma_z^{(N_z,M_x,M_y)}=\frac{N_z}{A}+\frac{M_x}{I_x}y+\frac{M_y}{I_y}x=0$$

$$y=-\frac{M_y}{M_x}\frac{I_x}{I_y}x-\frac{N_z}{A}\frac{I_x}{M_x} \tag{10.15}$$

The above equation is in the form of y = ax + b

This is a line intersecting the vertical axis at $b=-\frac{N_z}{A}\frac{I_x}{M_x}$

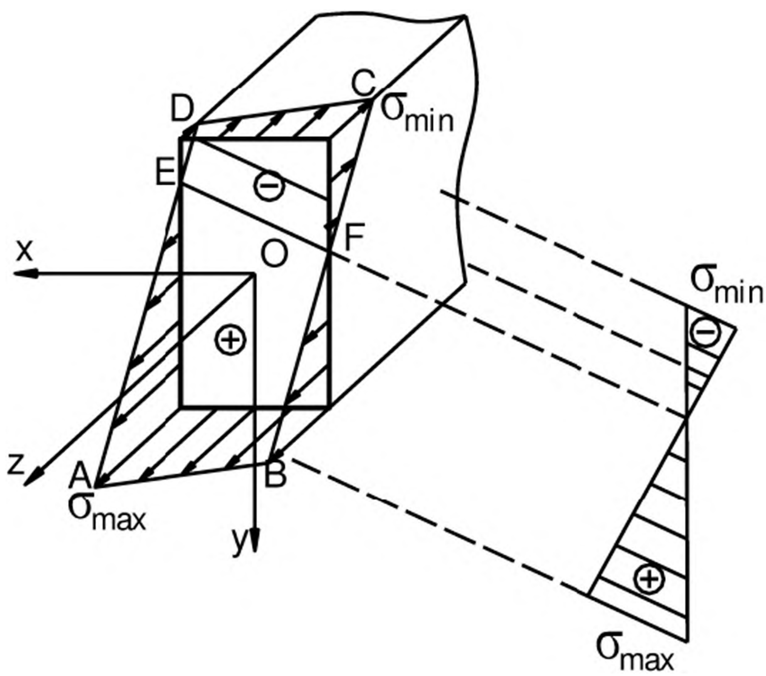


Figure 10.22

Stress condition:

$$\sigma_{\max} \leq [\sigma]_{tension} \tag{10.16}$$

$$|\sigma_{\min}| \leq [\sigma]_{compression} \tag{10.17}$$

3.3.1. Exercise 9

Given: **Figure 10.23**, $q_x = 10$ kN/m, $P_y = 20$ kN, $P_z = 100$ kN, L = 5m, b = 30 cm, h = 40 cm.

a) Draw diagrams N_z, M_x, M_y.

b) Determine the neutral axis at the fixed end.

c) Draw the diagram of stress.

d) Determine maximum and minimum stresses.

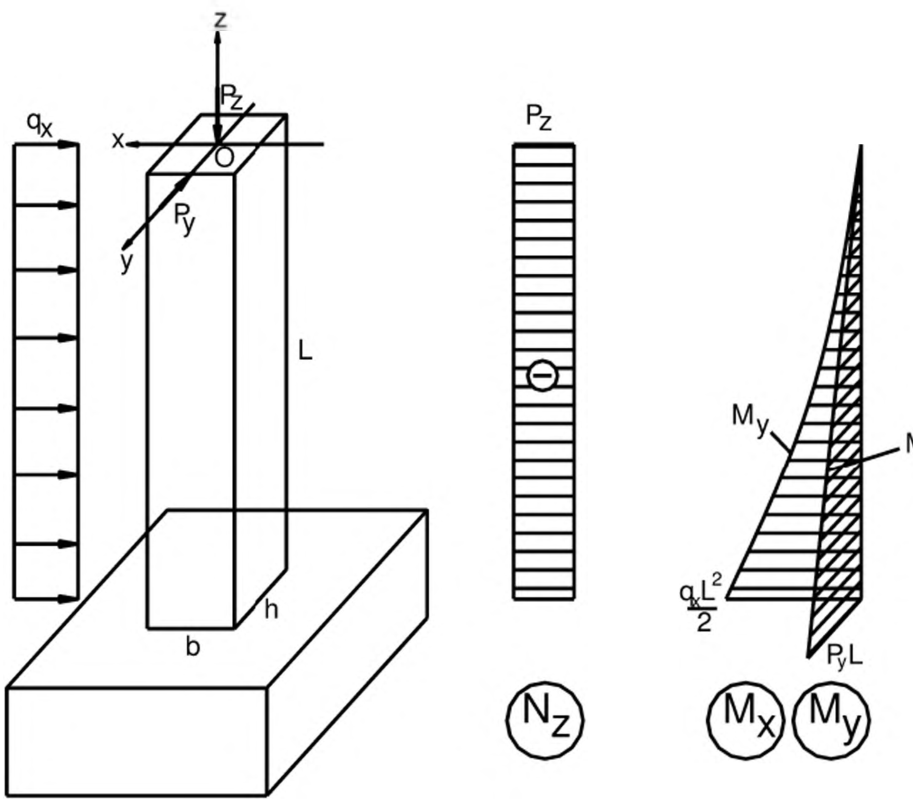


Figure 10.23

Solution

a) Draw diagrams N_z, M_x, M_y.

At the fixed end:

$$N_z = -P_z = -100 \text{ kN}$$

$M_x = P_y.L = 20.5 = 100$ kNm

$M_y = q_x.L^2/2 = 10.5^2/2 = 125$ kNm

b) Determine the neutral axis at the fixed end.

$A = b \times h = 30 \times 40 = 1200 \text{ cm}^2$

$$I_x = \frac{30(40)^3}{12} = 16.10^4 cm^4$$

$$I_y = \frac{40(30)^3}{12} = 9.10^4 cm^4$$

$$y = -\frac{M_y}{M_x}\frac{I_x}{I_y}x - \frac{N_z}{A}\frac{I_x}{M_x}$$

$$= -\frac{125}{100}\frac{16.10^4}{9.10^4}x - \frac{-100}{1200}\frac{16.10^4}{100.10^2} = -2.22x + 1.33$$

$$\beta = \arctan(-2.22) = -65.8^o$$

c) Draw the diagram of stress.

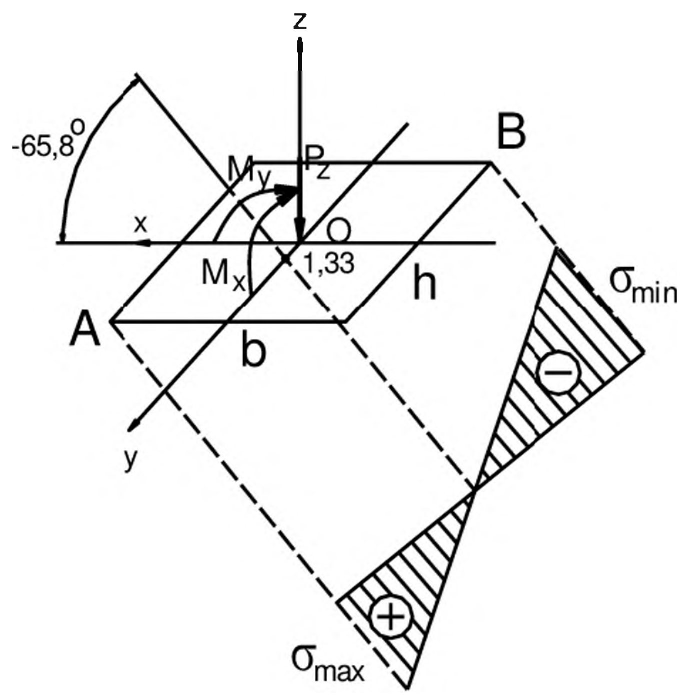


Figure 10.24

d) Determine maximum and minimum stresses.

At A(15,20)

$$\sigma_{max} = \frac{N_z}{A} + \frac{M_x}{I_x}y + \frac{M_y}{I_y}x = -\frac{100}{1200} + \frac{100.10^2}{16.10^4}20 + \frac{125.10^2}{9.10^4}15$$

$$= -0.83 + 1.25 + 2.08 = 3.25 kN/cm^2$$

At B(-15,-20)

$$\sigma_{min} = \frac{N_z}{A} + \frac{M_x}{I_x}y + \frac{M_y}{I_y}x = -\frac{100}{1200} + \frac{100.10^2}{16.10^4}(-20) + \frac{125.10^2}{9.10^4}(-15)$$

$$= -0.83 - 1.25 - 2.08 = 3.42 kN/cm^2$$

3.4. Eccentric axial loading

The eccentric load P located at K(x_K,y_K) is equal to the three following compoments at the center:

$$N_z = P \qquad (10.18)$$

$$M_x = N_z y_K \qquad (10.19)$$

$$M_y = N_z x_K \qquad (10.20)$$

And the problem becomes the problem of members subjected to combination of bending and tension or compression.

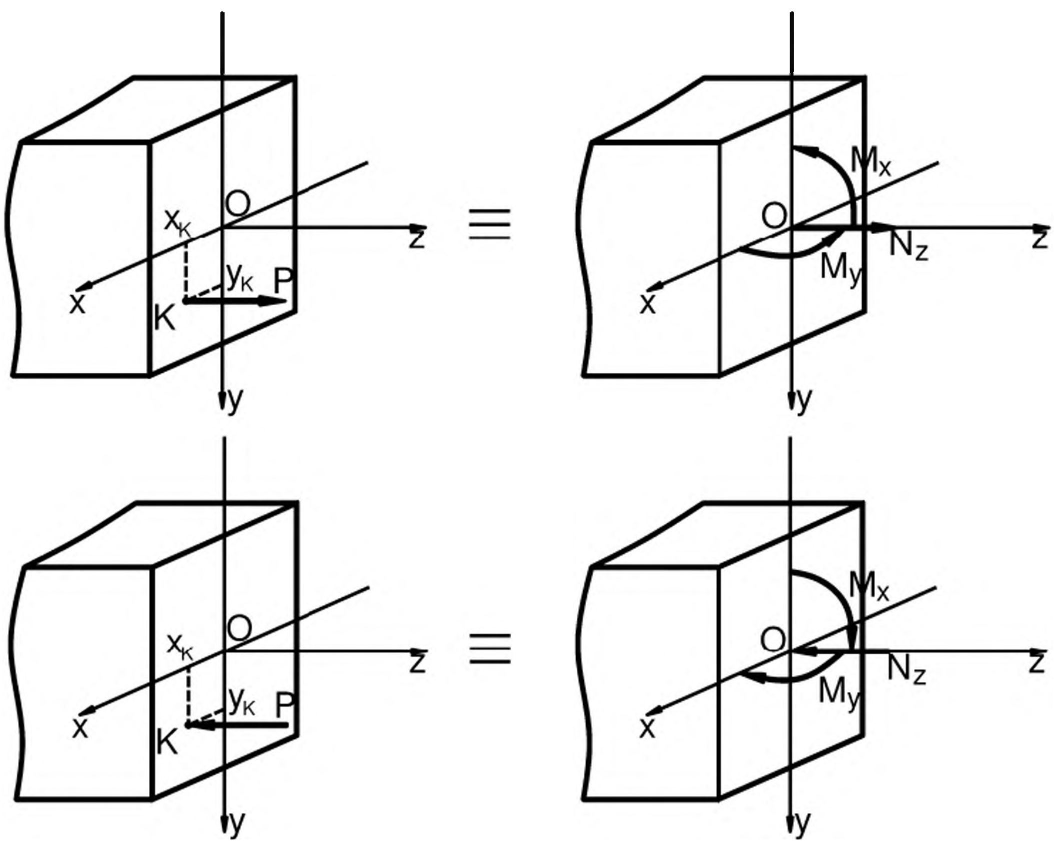


Figure 10.25

3.4.1. Exercise 10

Consider the column shown in **Figure 10.26**. Given: the load P_z = 1000 kN located at K(-7,5,-10), b = 30 cm, b = 40 cm.

a) Draw diagrams N_z, M_x, M_y.

b) Determine the neutral axis at the fixed end.

c) Draw the diagram of stress.

d) Determine maximum and minimum stresses.

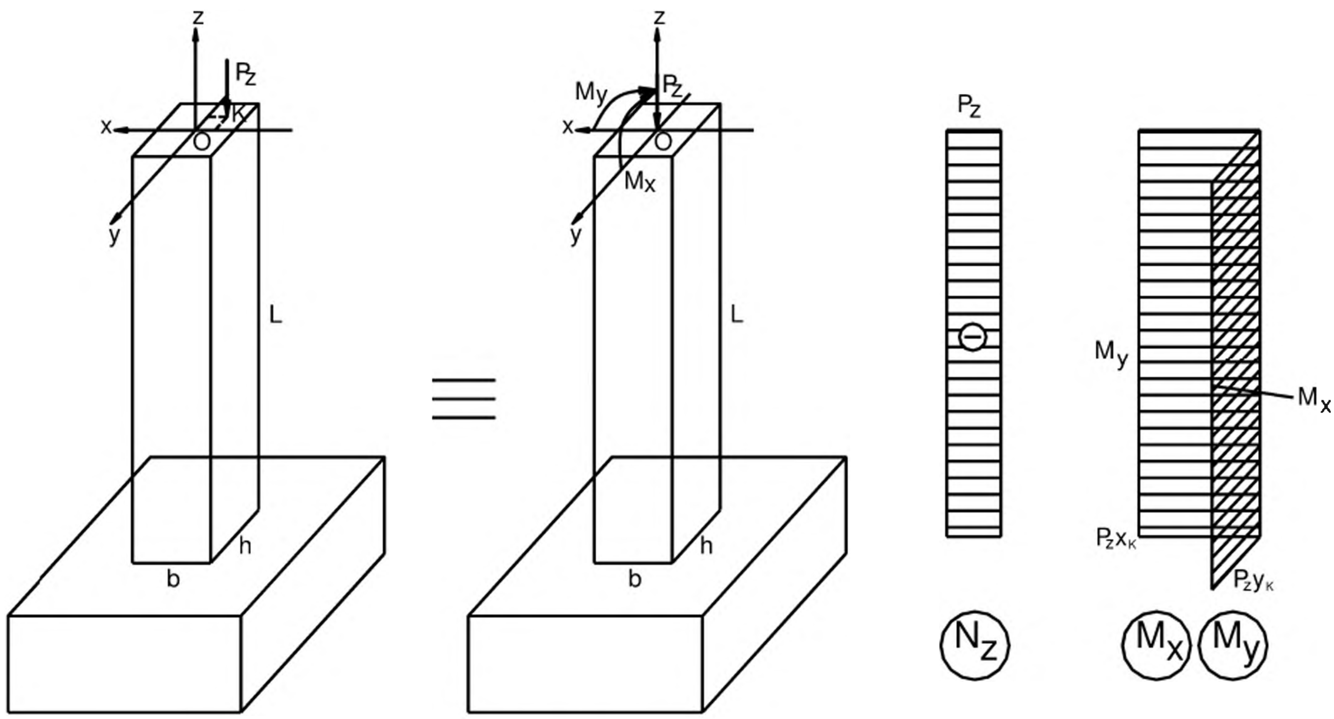


Figure 10.26

Solution

a) Draw diagrams N_z, M_x, M_y.

At the fixed end:

$$N_z = -P_z = -1000 \text{ kN}$$

$$M_x = N_z y_K = -1000 \times (-10) = 10000 kNcm$$

$$M_y = N_z x_K = -1000 \times (-7{,}5) = 7500 kNcm$$

b) Determine the neutral axis at the fixed end.

$$A = bh = 30.40 = 1200 \text{ cm}^2$$

$$I_x = \frac{30(40)^3}{12} = 16.10^4 cm^4$$

$$I_y = \frac{40(30)^3}{12} = 9.10^4 cm^4$$

$$y = -\frac{M_y}{M_x}\frac{I_x}{I_y}x - \frac{N_z}{A}\frac{I_x}{M_x}$$

$$= -\frac{7500}{10000}\frac{16.10^4}{9.10^4}x - \frac{-1000}{1200}\frac{16.10^4}{10000} = -1.33x + 13.33$$

$$\beta = \arctan(-1.33) = -53.1^o$$

c) Draw the diagram of stress.

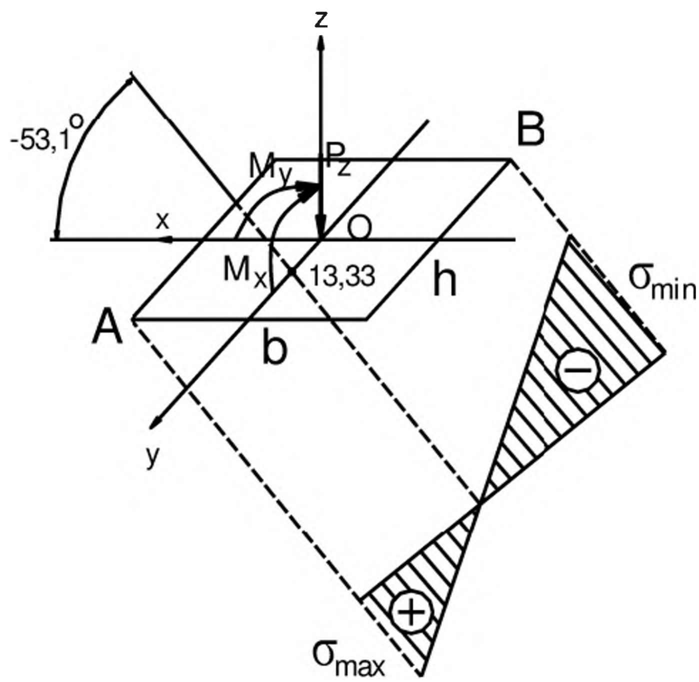


Figure 10.27

d) Determine maximum and minimum stresses.

At A(15,20)

$$\sigma_{max} = \frac{N_z}{A} + \frac{M_x}{I_x}y + \frac{M_y}{I_y}x$$

$$= -\frac{1000}{1200} + \frac{10000}{16.10^4}20 + \frac{7500}{9.10^4}15 = -0.83 + 1.25 + 1.25 = 1.67 kN / cm^2$$

At B(-15,-20)

$$\sigma_{min} = \frac{N_z}{A} + \frac{M_x}{I_x} y + \frac{M_y}{I_y} x$$

$$= -\frac{1000}{1200} + \frac{10000}{16.10^4}(-20) + \frac{7500}{9.10^4}(-15)$$

$$= -0.83 - 1.25 - 1.25 = -3.33 kN / cm^2$$

3.5. Core of section

3.5.1. Definition

Core of section is is a set of points on which the load P causes only compressive stress on the whole section.

For materials with large compressive stress $[\sigma]_{compression}$ but small tensile stress $[\sigma]_{tension}$ such as concrete, brick, stone, etc, it is necessary to locate the load P in the core of section.

3.5.2. Method to determine the core of section

In order to have only compressive stress on the section, the neutral line must not cross the section.

Set K(x_K, y_K) is the coordinate of the point load.

Equation of neutral axis:

$$\sigma_z = \frac{N_z}{A} + \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = 0 \tag{10.21}$$

$$\frac{N_z}{A} + \frac{N_z y_K}{I_x} y + \frac{N_z x_K}{I_y} x = 0$$

$$\frac{N_z}{A}\left[1 + \frac{A y_K}{I_x} y + \frac{A x_K}{I_y} x\right] = 0$$

Set:

$$i_x = \sqrt{\frac{I_x}{A}}$$

$$i_y = \sqrt{\frac{I_y}{A}}$$

$$1 + \frac{y_K}{i_x^2} y + \frac{x_K}{i_y^2} x = 0 \tag{10.22}$$

Set: $$a = -\frac{i_y^2}{x_K} \quad (10.23)$$

$$a = -\frac{i_x^2}{y_K} \quad (10.24)$$

Equation of neutral axis:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Characteristics of the neutral axis:

- Intersect the axis x at a, and axis y at b
- Does not go through the quadrant containing the point K (because a and b are always opposite to x_K and y_K, respectively).
- The closer the point of applied force to the center O of the section is, the farther the neutral line (to the center O) is. This is because the decreases x_K and y_K lead to the increases of a and b.
- When the neutral line is outside the cross-section, the stress on the cross-section has only one sign.

Method to determine the core of sections:

- Let the neutral line become the edges of the section $\rightarrow$ a, b $\rightarrow$ x_K, y_K.

$$x_K = -\frac{i_y^2}{a}$$

$$y_K = -\frac{i_x^2}{b}$$

- Similar equations can be established for other edges.
- Connect the above points to have the core of section.

3.5.3. Exercise 11

Consider the rectangular section ABCD shown in **Figure 10.28**. The width is b and the height is h. Determine the core of section.

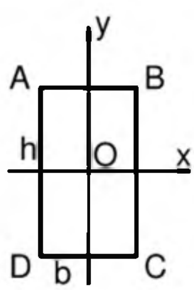


Figure 10.28

Solution

When the neutral axis becomes the side AB:

- Intersect the horizontal axis at ∞:

$$a = -\frac{i_y^2}{x_K} = \infty \rightarrow x_K = 0$$

- Intersect the vertical axis at h/2:

$$b = -\frac{i_x^2}{y_K} = \frac{h}{2} \rightarrow y_K = -\frac{2i_x^2}{h} = -\frac{2i_x^2}{h} = -\frac{2\frac{bh^3/12}{bh}}{h} = -\frac{h}{6}$$

When the neutral axis becomes the side BC:

- Intersect the horizontal axis at b/2:

$$a = -\frac{i_y^2}{x_K} = \frac{b}{2} \rightarrow x_K = -\frac{2i_y^2}{b} = -\frac{2i_y^2}{b} = -\frac{2\frac{hb^3/12}{bh}}{b} = -\frac{b}{6}$$

- Intersect the vertical axis at ∞:

$$b = -\frac{i_x^2}{y_K} = \infty \rightarrow y_K = 0$$

Due to symmetry, the core of section is shown in **Figure 10.29**.

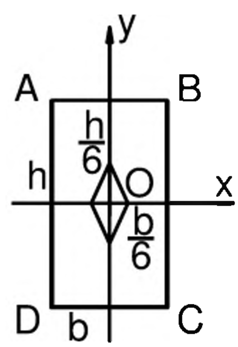


Figure 10.29

3.5.4. Exercise 12

Determine the core of circular section with diameter D.

Solution: The section core is a circular section with diameter of D/8 as shown in **Figure 10.30**.

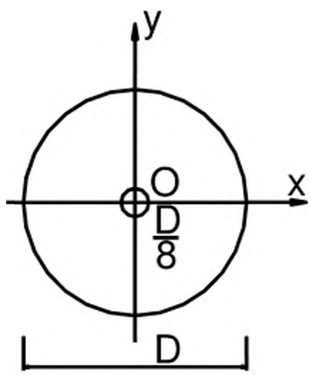


Figure 10.30

§4. BENDING AND TORSION

4.1. Definition

A member is under bending and torsion if the components of internal forces include M_u (M_x and M_y) and M_z.

4.2. Circular cross section members

An example for circular cross section members subjected to bending is moment transmission shaft: torque, belt tension, self-weight, pulley, etc.

Components of internal forces of a member under combinations of bending and torsion include M_x, M_y and M_z. Alternatively, the two bending moments M_x and M_y can be represented by the bending moment M_u and M_z, in which M_u is the resultant of M_x and M_y.

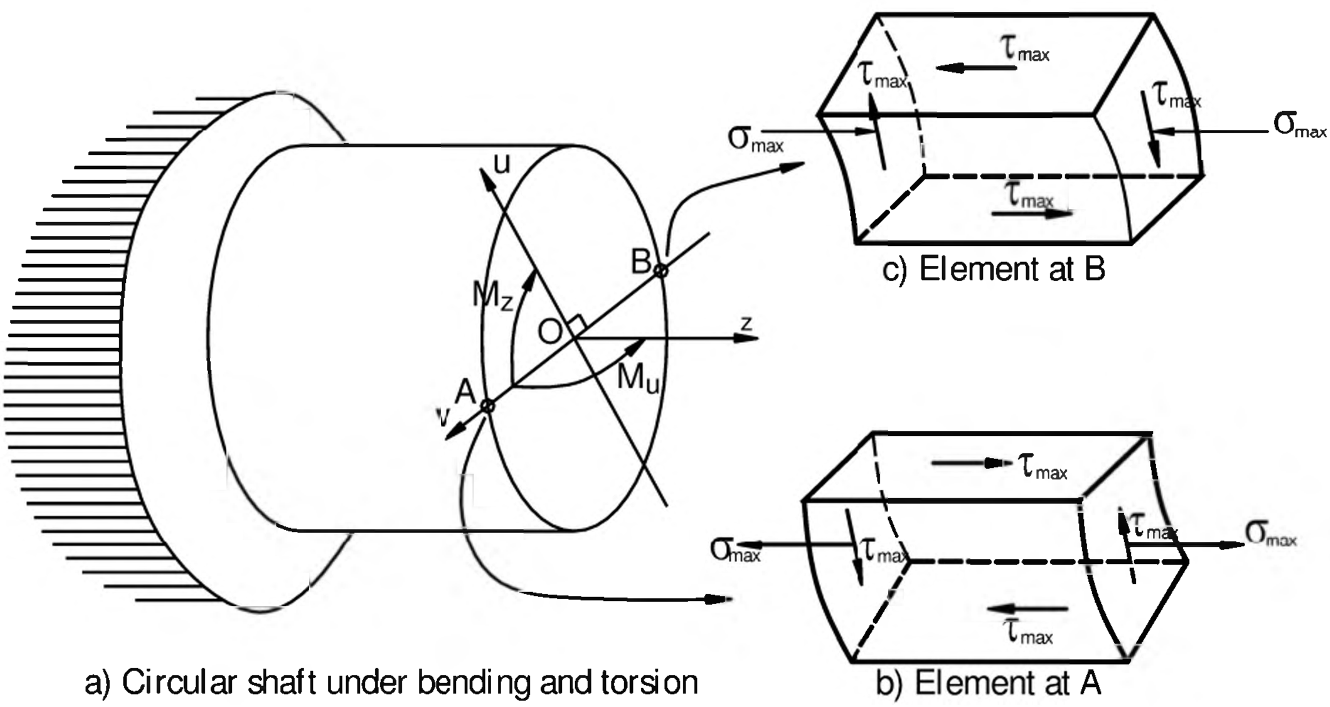


Figure 10.31

The farthest points A and B to the axis u:

Maximum normal stress due to M_u:

$$\sigma_{\substack{max\\min}} = \pm\frac{|M_u|}{W_u} = \frac{\sqrt{M_x^2 + M_y^2}}{W_u} \tag{10.25}$$

Shear stress due to M_z:

$$\tau_{max} = \frac{M_z}{W_\rho} \tag{10.26}$$

in which

$$W_u = \frac{1}{4}\pi R^3,\ W_\rho = \frac{1}{2}\pi R^3 \text{ for circular cross section.}$$

$$W_u = \frac{\frac{1}{4}\pi\left(R^4 - r^4\right)}{R},\ W_\rho = \frac{\frac{1}{2}\pi\left(R^4 - r^4\right)}{R} \text{ for donut.}$$

Stress condition:

Failure theory 3:

$$\sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

Failure theory 4:

$$\sqrt{\sigma^2 + 3\tau^2} \le [\sigma]$$

4.2.1. Exercise 13

Consider the beam in **Figure 10.32**. Given: $P_x = 20$ kN, $P_y = 10$ kN, $M_o = 1200$ kNcm, L = 0.5 m, the beam with circular section, the radius is R = 6 cm, $[\sigma] = 12$ kN/cm^2.

a) Draw the diagrams of moments.

b) Check the stress condition.

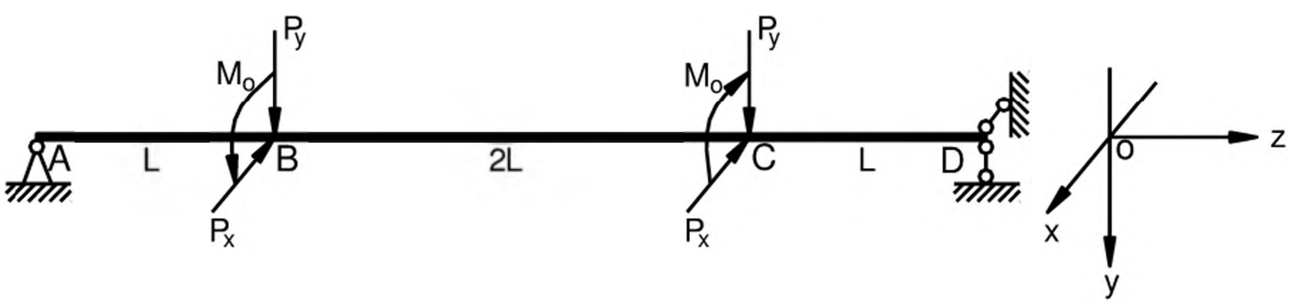


Figure 10.32

Solution

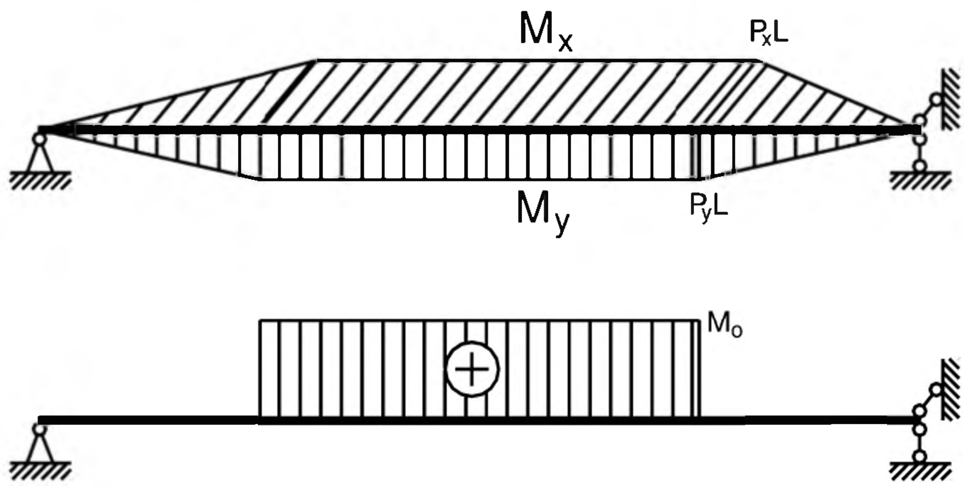


Figure 10.33

$$M_x = P_y L = 10 \times 50 = 500 kNcm$$

$$M_y = P_x L = 20 \times 50 = 1000 kNcm$$

$$M_z = M_o = 1200 kNcm$$

$$W_u = \frac{1}{4}\pi R^3 = \frac{1}{4}\pi \times 6^3 = 169.5 cm^3$$

$$W_\rho = \frac{1}{2}\pi R^3 = \frac{1}{2}\pi \times 6^3 = 339 cm^3$$

$$\sigma_{max} = \frac{M_u}{W_u} = \frac{\sqrt{M_x^2 + M_y^2}}{W_u} = \frac{\sqrt{500^2 + 1000^2}}{169.5} = 6.6 kN / cm^2$$

$$\tau_{max} = \frac{M_z}{W_\rho} = \frac{1200}{339} = 3.54 kN / cm^2$$

$$\sqrt{\sigma^2 + 4\tau^2} = \sqrt{6.6^2 + 4 \times 3.54^2} = 9.67 kN / cm^2 \le [\sigma]$$

4.3. Rectangular cross section members

Figure 10.34a shows the rectangular section subjected to bending and torsion. Using the superposition method, it can be separated into **Figure 10.34b** and **Figure 10.34c**. **Figure 10.34b** is the rectangular section subjected to asymmetric bending while **Figure 10.34c** shows the section under pure torsion. The stresses caused by the bending moments in **Figure 10.34b** was computed by formulas presented in the previous lesson. The stresses caused by the torsional moments in **Figure 10.34c** was computed by formulas presented in Chapter 9. The stress distribution in the rectangular section is shown in **Figure 10.35**.

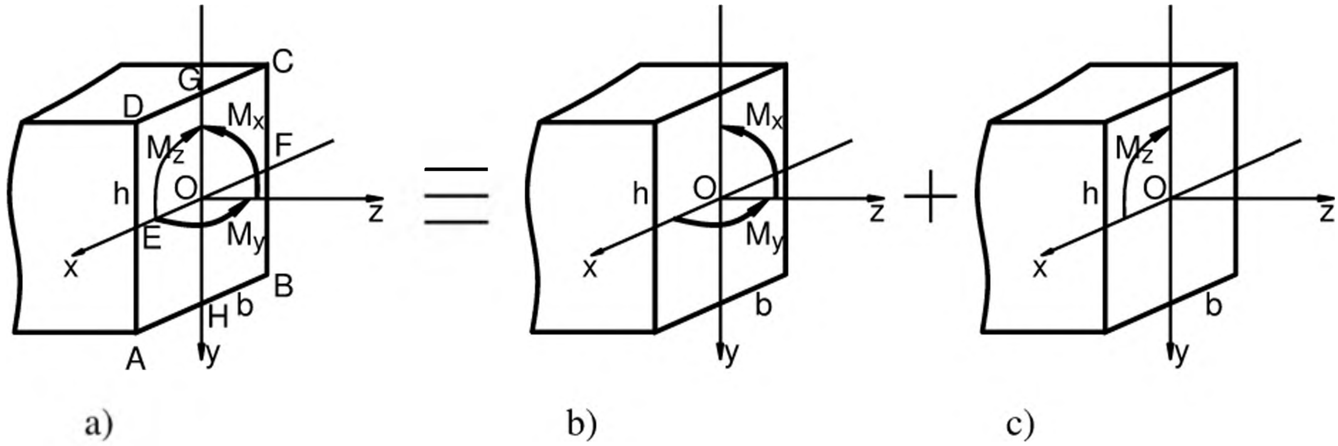


Figure 10.34. Rectangular section under bending and torsion

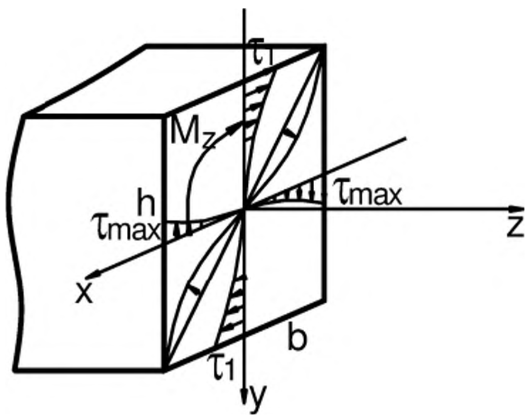


Figure 10.35. Stress distribution in section under torsion

Normal stress at points A - G or any point with coordinate (x,y):

$$\sigma_z = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x \tag{10.27}$$

Shear stress:

+ At A, B, C, D:

$$\tau = 0$$

+ At E, F (mid points of the height):

$$\tau_{\max} = \frac{M_z}{\alpha h b^2}$$

+ At G, H (mid points of the width):

$$\tau_1 = \gamma \tau_{\max}$$

where $$W_\rho = \alpha h b^2$$

Angle of twist:

$$\theta = \frac{M_z}{\beta G h b^3} \tag{10.28}$$

α, β, γ are coefficients depending on ratio of height to width. These coefficients are shown in **Table 10.4**.

Table 10.4. Table of coefficients α, β, γ

h/b		1	1.5	1.75	2.0	2.5	3.0	4.0	6	8	10	>10
Coefficient	α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
	β	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333
	γ	1.000	0.859	0.820	0.795	0.766	0.753	0.745	0.743	0.742	0.742	0.742

Stress condition

Stress condition for points (A, B, C, D) which has only normal stress:

$$\sigma_{z,\max} \le [\sigma]_{tension} \tag{10.29}$$

$$\left|\sigma_{z,\min}\right| \le [\sigma]_{compression} \tag{10.30}$$

Stress condition for points (E, F, G, H) which has both normal stress and shear stress:

+ Failure theory 3:

$$\sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

+ Failure theory 4:

$$\sqrt{\sigma^2 + 3\tau^2} \le [\sigma]$$

4.3.1. Exercise 14

Consider the section shown in **Figure 10.36**. Given: M_x = 1500 kNm, M_z = 1200 kNm, b = 40 cm, h = 60 cm, [σ] = 13 kN/cm^2.

a) Determine the stresses at points A to H.

b) Check the stress condition.

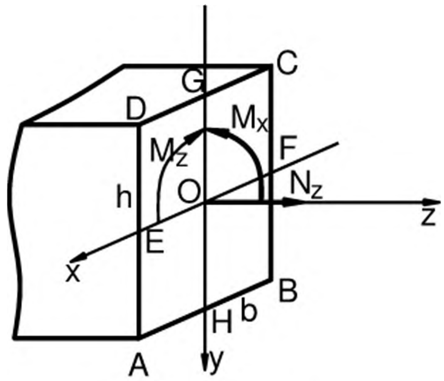


Figure 10.36

Solution

Sectional properties:

$$A = b \times h = 40 \times 60 = 2400 \text{ cm}^2$$

$$I_x = \frac{40(60)^3}{12} = 72.10^4 cm^4$$

$$I_y = \frac{60(40)^3}{12} = 32.10^4 cm^4$$

Because h/b = 60/40 =1.5. From **Table 10.4** => $\alpha = 0.231$; $\gamma = 0.859$

We have:

$$W_\rho = \alpha h b^2 = 0.231 \times 60 \times 40^2 = 22176 cm^3$$

$$\sigma_{\substack{max \\ min}} = \pm \frac{|M_x|}{i_x}|h/2| = \pm \frac{1500 \times 10^2}{72 \times 10^4} 30 = \pm 6.25 kN/cm^2$$

$$\tau_{max} = \frac{M_z}{W_\rho} = \frac{1200 \times 10^2}{22176} = 5.41 kN/cm^2$$

$$\tau_1 = \gamma \tau_{max} = 0.859 \times 5.41 = 4.65 kN/cm^2$$

At A, B: Tension

At C, D: Compression

E, F: $\tau_{max} = 5.41 kN/cm^2$

G, H: sigma + $\tau_1 = 4.65 kN/cm^2$

+ Failure theory 3:

$$\sqrt{\sigma^2 + 4\tau^2} = \sqrt{6.25^2 + 4 \times 4.65^2} \le [\sigma] ==> \text{OK}$$

4.3.2. Exercise 15

Consider the beam shown in **Figure 10.37**. Given: q = 4 kN/m, P = 2qL, L = 1.2m, b = 20 cm, h = 30 cm, [σ] = 1 kN/cm^2. ABC is on the horizontal plane. AB is perpendicular to BC.

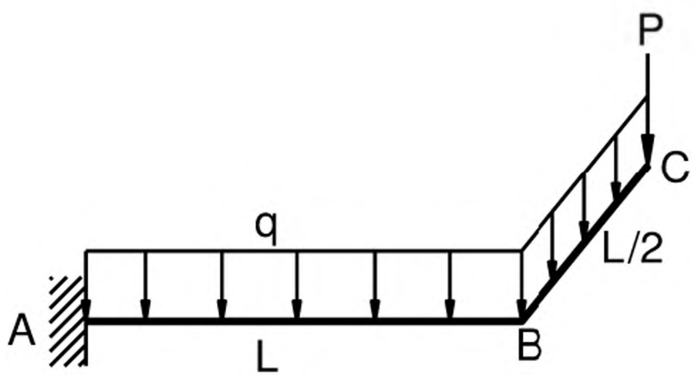


Figure 10.37

a) Draw the diagrams of moments.

b) Check the stress condition for the section at fixed end.

Solution

$M_x = (2qL)L + (qL/2)L + qL^2/2 = 3qL^2 = 3\times4\times1.2^2\times100$

$M_z = P.a/2 + (qa/2)a/4 = 9qa^2/8$

$$W_x = \frac{bh^2}{6} = \frac{20\times30^2}{6} = 3000cm^3$$

h/b = 30/20 = 1.5 → α = 0.231; γ = 0.859

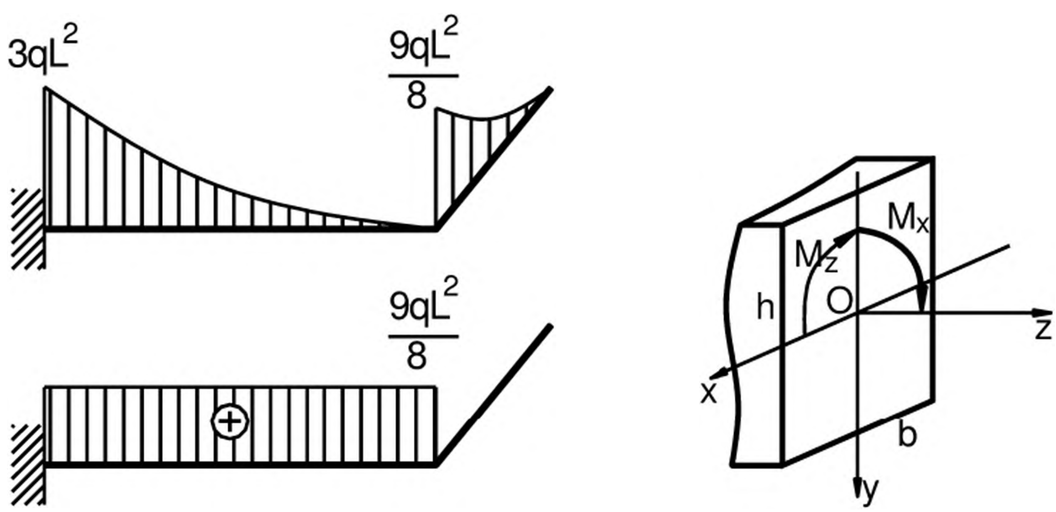


Figure 10.38

$$\sigma_{\substack{max\\min}} = \frac{M_x}{W_x} = \frac{1728}{3000} = 0.576kN/cm^2$$

$$\tau_{max} = \frac{M_z}{\alpha hb^2} = \frac{648}{0.231\times 30\times 20^2} = 0.233kN/cm^2$$

$$\tau_1 = \gamma\tau_{max} = 0.859\times 0.233 = 0.2kN/cm^2$$

At the long side:

$$\sigma = \sigma_{max} \text{ và } \tau = \tau_1$$

The stress condition:

$$\sqrt{\sigma^2 + 4\tau^2} = \sqrt{0.576^2 + 4\times 0.2^2} = 0.7kN/cm^2 < [\sigma]$$

At the middle of the short side:

$$\sigma = 0 \text{ và } \tau_{max}$$

Stress condition:

$$\sqrt{\sigma^2 + 4\tau^2} = \sqrt{0^2 + 4\times 0.233^2} = 0.466kN/cm^2 < [\sigma]$$

4.3.3. Exercise 16

Consider the beam shown in **Figure 10.39**. Given: q = 50 kN/m, P = 2qL, L = 4 m, b = 40 cm, h = 60 cm, [σ] = 13 kN/cm^2.

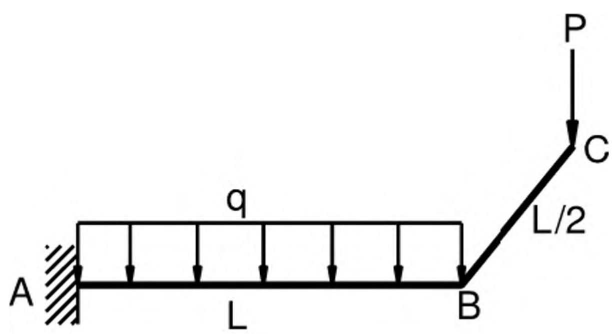


Figure 10.39

a) Draw diagrams of moments.

b) Check the stress condition for the section at the fixed end.

Solution

We have:

$M_x = (2qL)\times L + qL^2/2 = 5qL^2/2 = 5\times 50\times 4^2/2\times 100 = 2\times 10^5$ kNcm

$M_z = 2qL\times L/2 = qL^2 = 50\times 4^2\times 100 = 8\times 10^4$ kNcm

$$W_x = \frac{bh^2}{6} = \frac{40 \times 60^2}{6} = 24000 cm^3$$

h/b = 60/40 = 1.5 → α = 0.231; γ = 0.859

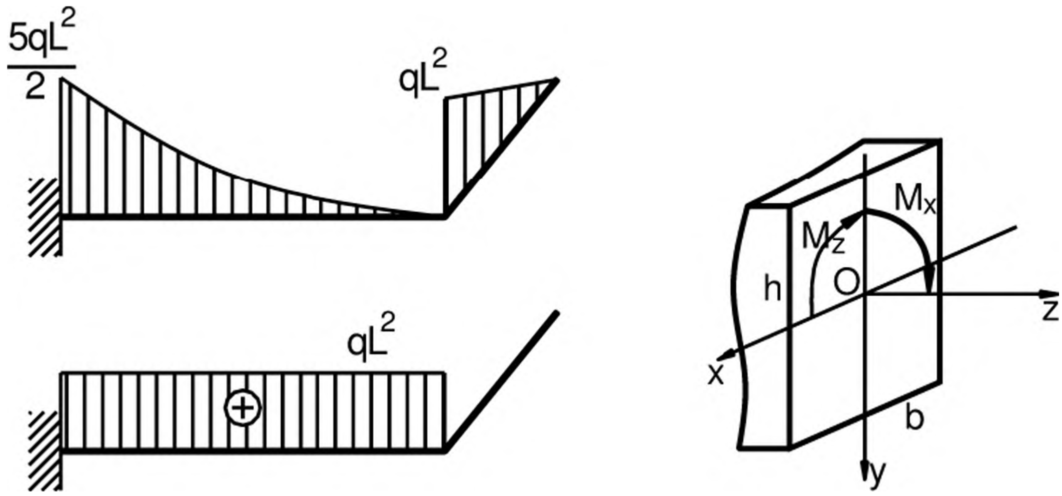


Figure 10.40

$$\sigma_{\substack{max \\ min}} = \frac{M_z}{W_x} = \frac{2.10^5}{24000} = 8.33 kN/cm^2$$

$$\tau_{max} = \frac{M_z}{\alpha h b^2} = \frac{8.10^4}{0.231 \times 60 \times 40^2} = 3.6 kN/cm^2$$

$$\tau_1 = \gamma \tau_{max} = 0.859 \times 3.6 = 3.1 kN/cm^2$$

At the middle of the long side:

$$\sigma = \sigma_{max} \text{ and } \tau = \tau_1$$

Stress condition:

$$\sqrt{\sigma^2 + 4\tau^2} = \sqrt{8.33^2 + 4 \times 3.1^2} = 10.4 kN/cm^2 < [\sigma]$$

At the middle of the short side:

$$\sigma = 0 \text{ and } \tau_{max}$$

Stress condition:

$$\sqrt{\sigma^2 + 4\tau^2} = \sqrt{0^2 + 4 \times 3.6^2} = 7.2 kN/cm^2 < [\sigma]$$

§5. GENERAL COMBINED LOADINGS

5.1. Definition

A member subjected to general combined loadings if the components of internal forces include M_u (M_x and M_y), M_z and N_z.

Circular cross section members

We have:

$$M_u = \sqrt{M_x^2 + M_y^2} \tag{10.31}$$

Maximum and minimum stresses:

$$\sigma_{\substack{max \\ min}} = \pm\frac{|N_z|}{A} \pm \frac{|M_u|}{W_u} \tag{10.32}$$

Shear stress:

$$\tau_{max} = \frac{M_z}{W_\rho} \tag{10.33}$$

Stress condition:

Failure theory 3:

$$\sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

Failure theory 4:

$$\sqrt{\sigma^2 + 3\tau^2} \le [\sigma]$$

Rectangular cross section members

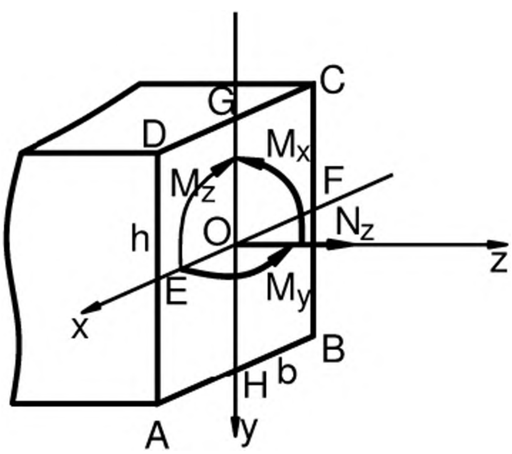


Figure 10.41

At points A, B, C, D:

+ Normal stresses:

$$\sigma_{\substack{\max\\ \min}} = \pm\frac{|N_z|}{A} \pm \frac{|M_x|}{W_x} \pm \frac{|M_y|}{W_y} \tag{10.34}$$

+ Stress condition:

$$\sigma_{z,\max} \le [\sigma]_{tension}\,;\ \left|\sigma_{z,\min}\right| \le [\sigma]_{compression} \tag{10.35}$$

At mid points of the height:

+ Normal stresses:

$$\sigma_{\substack{\max\\ \min}} = \pm\frac{|N_z|}{A} \pm \frac{|M_y|}{W_y} \tag{10.36}$$

+ Shear stress:

$$\tau_{\max} = \frac{M_z}{\alpha h b^2} \tag{10.37}$$

+ Stress condition:

Failure theory 3:

$$\sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

Failure theory 4:

$$\sqrt{\sigma^2 + 3\tau^2} \le [\sigma]$$

At mid points of the width:

+ Normal stresses:

$$\sigma_{\substack{\max\\ \min}} = \pm\frac{|N_z|}{A} \pm \frac{|M_x|}{W_x} \tag{10.38}$$

+ Shear stress:

$$\tau_1 = \gamma \tau_{\max} \tag{10.39}$$

+ Stress condition:

Failure theory 3:

$$\sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

Failure theory 4:

$$\sqrt{\sigma^2 + 3\tau^2} \le [\sigma]$$

5.4. Exercise 17

Consider the structure subjected to loads as shown in **Figure 10.42**. Given: the cross section is circular with the radius R = 85 mm, q = 4 kN/m, P = 16 kN, L = 2 m, $[\sigma]$ = 16 kN/cm^2.

a) Draw the diagrams of M_x, M_y, M_z, N_z.

b) Calculate the maximum normal and shear stresses on the section at the fixed end.

c) Check the stress condition based on the failure theory 3.

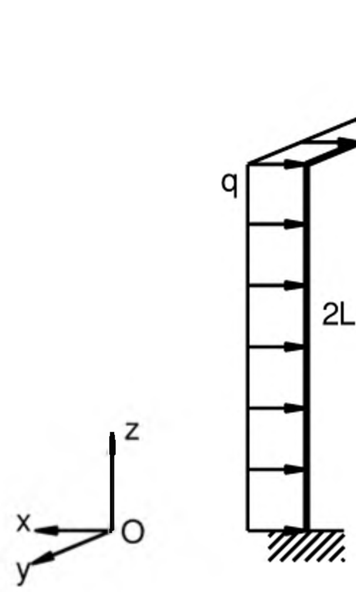


Figure 10.42

Solution

a) Draw the diagrams of M_x, M_y, M_z, N_z.

At the fixed end:

$$N_z = P = 16 \text{ kN}$$

$$M_x = PL = 16\times2 = 32 \text{ kNm}$$

$$M_y = (qL)2L+q(2L)^2/2 = 2qL^2+2qL^2 = 4qL^2 = 4\times4\times2^2 = 64 \text{ kNm}$$

$$M_z = qL^2/2 = 4\times2^2/2 = 8 \text{ kNm.}$$

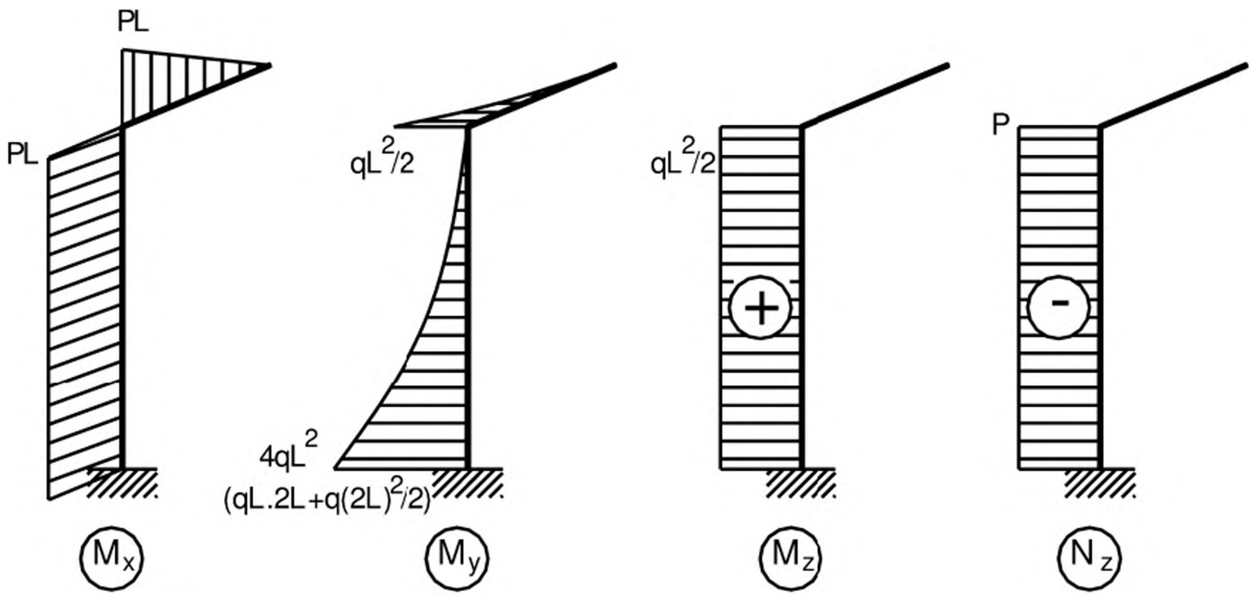


Figure 10.43

Calculate the maximum normal and shear stresses on the section at the fixed end.

$$M_u = \sqrt{M_x^2 + M_y^2} = \sqrt{32^2 + 64^2} = 71.55kNm$$

$$A = \pi R^2 = \pi 8.5^2 = 226.9cm^2$$

$$W_u = \frac{1}{4}\pi R^3 = \frac{1}{4}\pi \times 8.5^3 = 482.3cm^3$$

$$W_\rho = \frac{1}{2}\pi R^3 = \frac{1}{2}\pi \times 8.5^3 = 964.6cm^3$$

$$\sigma_{max} = \frac{|N_z|}{A} + \frac{|M_u|}{W_u}$$

$$= \frac{16}{226.9} + \frac{71.55\times 10^2}{482.3} = 0.07 + 14.8 = 14.87kN/cm^2$$

$$\tau_{max} = \frac{M_z}{W_\rho} = \frac{8\times 10^2}{964.6} = 0.83kN/cm^2$$

Check the stress condition based on the failure theory 3.

$$\sqrt{\sigma^2 + 4\tau^2} = \sqrt{14.87^2 + 4\times 0.83^2} = 14.96kN/cm^2 \le [\sigma] \text{ OK.}$$

5.5. Exercise 18

Consider the structure (in the plane Oyz) subjected to loads as shown in **Figure 10.44**. The load P_x is parallel to the x axis. Given: q = 22 kN/m, P_x = qa, L = 4a, a = 1 cm, [σ] = 16 kN/cm^2.

a) Draw the diagrams of bending moments and torsional moment.

b) Calculate the stresses at the corner and middle of sides for the section at the fixed end.

c) Check the stress condition based on the failure theory 3.

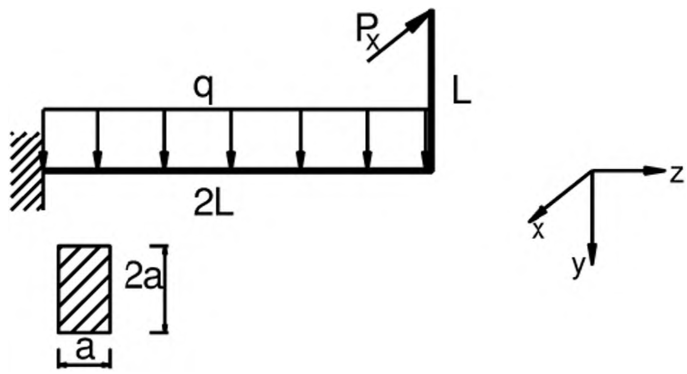


Figure 10.44

Solution

a) Draw the diagrams of bending and torsional moments.

At the fixed end:

$$M_x = 32qa^2$$

$$M_y = 8qa^2$$

$$M_z = 4qa^2$$

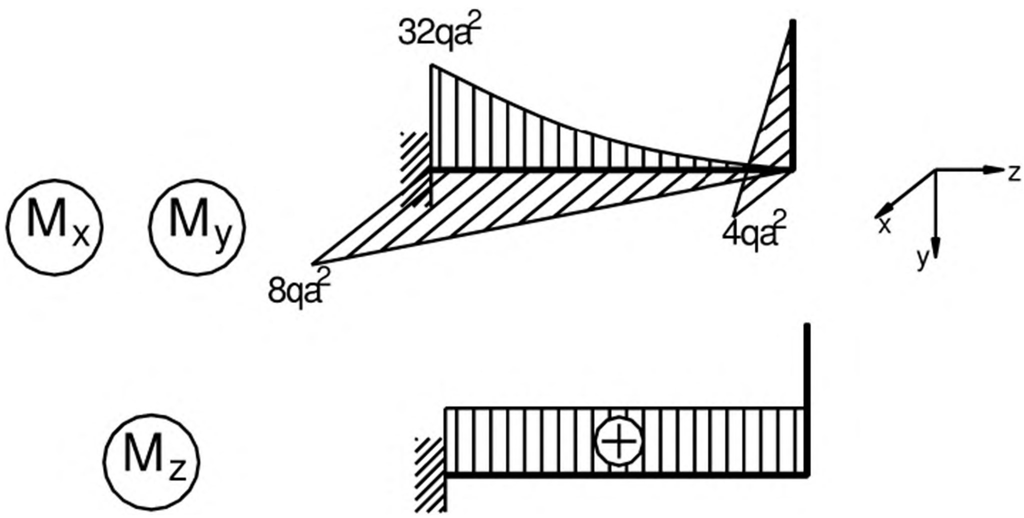


Figure 10.45

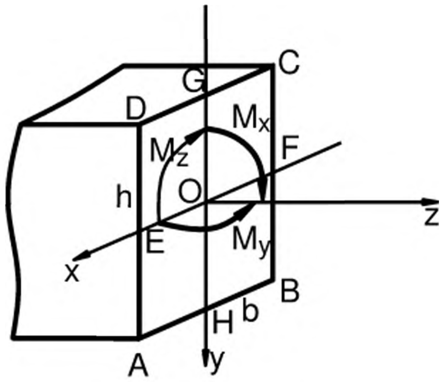


Figure 10.46

b) Calculate the stresses at the corner and middle of sides for the section at the fixed end.

$$A = a\times 2a = 2a^2$$

$$I_x = \frac{a(2a)^3}{12} = \frac{2a^4}{3}$$

$$I_y = \frac{2a(a)^3}{12} = \frac{a^4}{6}$$

$h/b = 2a/a = 2 \Rightarrow \alpha = 0.246;\ \gamma = 0.795$

$$W_p = \alpha hb^2 = 0.246\times 2a\times a^2 = 0.492a^3$$

At A (a/2, a):

$$\tau = 0$$

$$\sigma = \frac{M_x}{I_x}y + \frac{M_y}{I_y}x = \frac{-32qa^2}{\frac{2a^4}{3}}a + \frac{8qa^2}{\frac{a^4}{6}}\frac{a}{2} = \frac{-48q+24q}{a} = -\frac{24q}{a}$$

At B (-a/2, a):

$$\tau = 0$$

$$\sigma = \frac{M_x}{I_x}y + \frac{M_y}{I_y}x = \frac{-32qa^2}{\frac{2a^4}{3}}a + \frac{8qa^2}{\frac{a^4}{6}}\left(-\frac{a}{2}\right) = \frac{-48q-24q}{a} = -\frac{72q}{a}$$

At C (-a/2, -a):

$$\tau = 0$$

$$\sigma = \frac{M_x}{I_x}y + \frac{M_y}{I_y}x = \frac{-32qa^2}{\frac{2a^4}{3}}(-a) + \frac{8qa^2}{\frac{a^4}{6}}\left(-\frac{a}{2}\right) = \frac{48q-24q}{a} = \frac{24q}{a}$$

At D(a/2, -a):

$$\tau = 0$$

$$\sigma = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = \frac{-32qa^2}{\frac{2a^4}{3}}(-a) + \frac{8qa^2}{\frac{a^4}{6}} \frac{a}{2} = \frac{48q + 24q}{a} = \frac{72q}{a}$$

At E(a/2, 0):

$$\tau_{\max} = \frac{M_z}{W_\rho} = \frac{4qa^2}{0,492a^3} = \frac{8,13q}{a}$$

$$\sigma = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = \frac{-32qa^2}{\frac{2a^4}{3}}(0) + \frac{8qa^2}{\frac{a^4}{6}} \frac{a}{2} = \frac{24q}{a}$$

At F(-a/2, 0):

$$\tau_{\max} = \frac{M_z}{W_\rho} = \frac{4qa^2}{0.492a^3} = \frac{8.13q}{a}$$

$$\sigma = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = \frac{-32qa^2}{\frac{2a^4}{3}}(0) + \frac{8qa^2}{\frac{a^4}{6}} \left(-\frac{a}{2}\right) = -\frac{24q}{a}$$

At G(0, -a):

$$\tau_1 = \gamma\tau_{\max} = 0.795\frac{8.13q}{a} = \frac{6.49q}{a}$$

$$\sigma = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = \frac{-32qa^2}{\frac{2a^4}{3}}(-a) + \frac{8qa^2}{\frac{a^4}{6}} 0 = \frac{48q}{a}$$

At H(0, a):

$$\tau_1 = \gamma\tau_{\max} = 0.795\frac{8.13q}{a} = \frac{6.49q}{a}$$

$$\sigma = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = \frac{-32qa^2}{\frac{2a^4}{3}}(a) + \frac{8qa^2}{\frac{a^4}{6}} 0 = -\frac{48q}{a}$$

Check the stress condition based on the failure theory 3:

$$\sqrt{\sigma^2 + 4\tau^2} \leq [\sigma]$$

5.6. Exercise 19

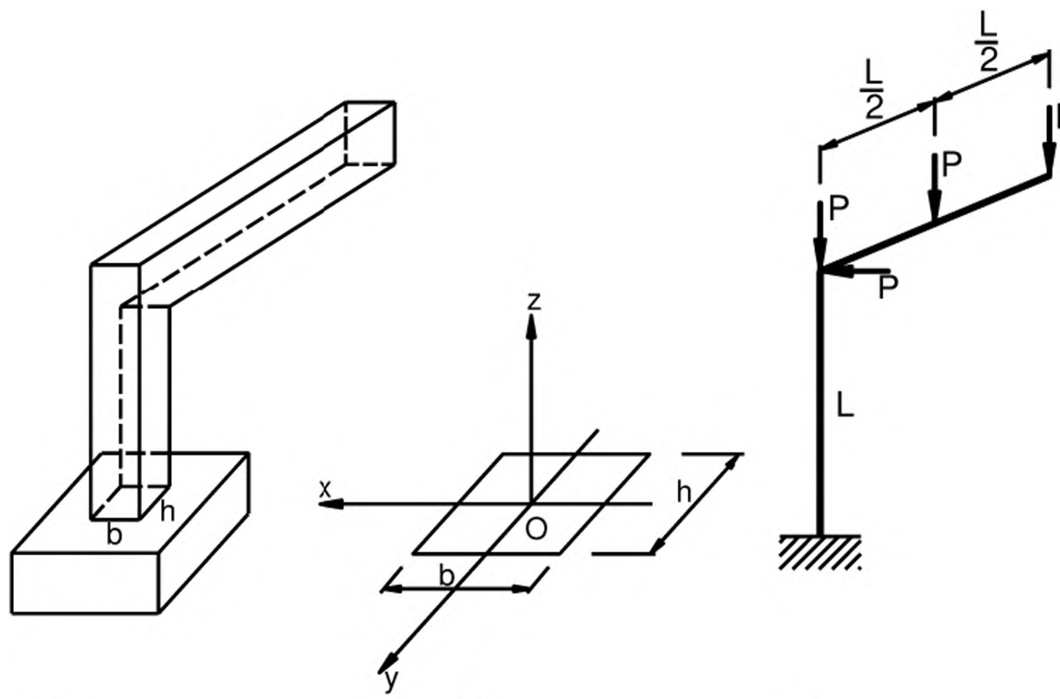
Consider the structure shown in **Figure 10.47a**. Dimensions, coordinate system Oxyz and loads are shown in **Figure 10.47b**. Given: P = 200 kN, L = 4 m, b = 40 cm, h = 60 cm.

a) Draw the diagrams of N_z, M_x, M_y.

b) Determine the equation of neutral axis and draw the neutral axis.

c) Draw the stress diagram for the section at the fixed end.

d) Calculate the maximum and minimum normal stresses on the section at fixed end.



a) b) c)

Figure 10.47

Solution

a) Draw the diagrams of N_z, M_x, M_y.

At the fixed end:

$$N_z = -3P = -3\times200 = -600 \text{ kN}$$

$$M_x = PL + \frac{PL}{2} = \frac{3PL}{2} = 3\times200\frac{400}{2} = 12.10^4\,kNcm$$

$$M_y = -PL = -200\times400 = -8.10^4\,kNcm$$

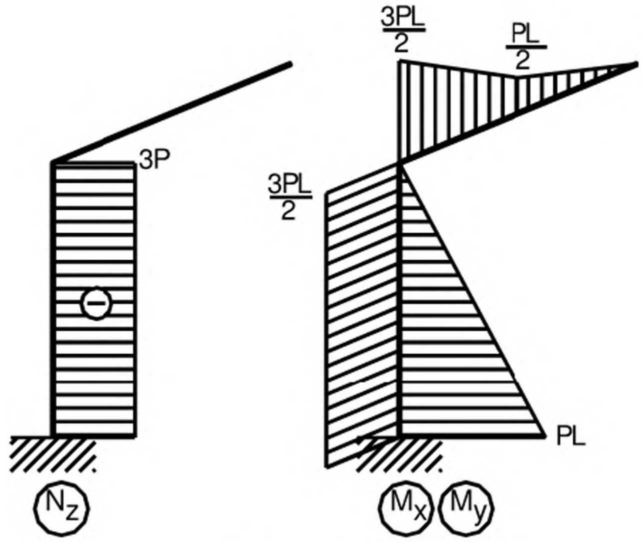


Figure 10.48

b) Determine the equation of neutral axis. Draw the neutral axis

$$A = bh = 40\times 60 = 2400 \text{ cm}^2$$

$$I_x = \frac{40(60)^3}{12} = 72\times 10^4 cm^4$$

$$I_y = \frac{60(40)^3}{12} = 32\times 10^4 cm^4$$

$$y = -\frac{M_y}{M_x}\frac{I_x}{I_y}x - \frac{N_z}{A}\frac{I_x}{M_x}$$

$$= -\frac{-8\times 10^4}{12\times 10^2}\frac{72.10^4}{32.10^4}x - \frac{-600}{2400}\frac{72\times 10^4}{12\times 10^4} = 1,5x + 1,5$$

$$\beta = \arctan(1,5) = 56,31^o$$

c) Draw the stress diagram for the section at the fixed end

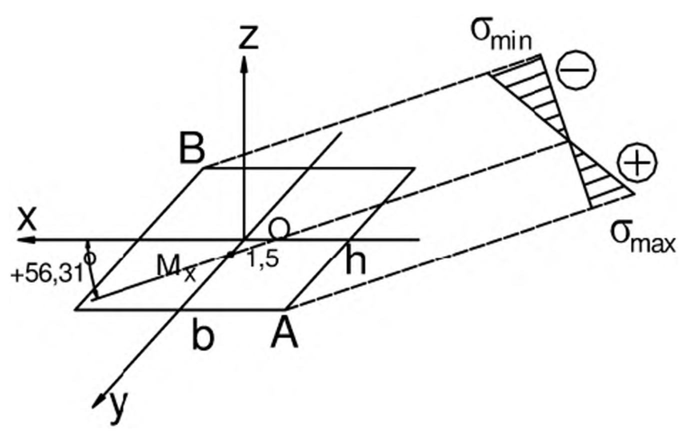


Figure 10.49

Calculate the maximum and minimum normal stresses on the section at fixed end.

Maximum stress at A(-20,30)

$$\sigma_{max} = \frac{N_z}{A} + \frac{M_x}{I_x} y + \frac{M_y}{I_y} x$$

$$= \frac{-600}{2400} + \frac{12\times10^4}{72\times10^4} 30 + \frac{-8\times10^4}{32\times10^4}(-20)$$

$$= -0,25 + 5 + 5 = 9,75 kN/cm^2$$

Maximum stress at B(20,-30)

$$\sigma_{max} = \frac{N_z}{A} + \frac{M_x}{I_x} y + \frac{M_y}{I_y} x$$

$$= \frac{-600}{2400} + \frac{12\times10^4}{72\times10^4}(-30) + \frac{-8\times10^4}{32\times10^4}(20)$$

$$= -0,25 - 5 - 5 = -10,25 kN/cm^2$$

5.7. Exercise 20

Consider the cantilever beam shown in **Figure 10.50**. The dimensions b, h and coordinate system Oxyz at the fixed end are also shown in **Figure 10.50**. Given: $P_x = P_y = 500$ N, L = 1000 mm, b = 40 mm, h = 60 mm.

a) Draw diagrams of internal forces M_x and M_y.

b) Determine maximum and minimum normal stresses.

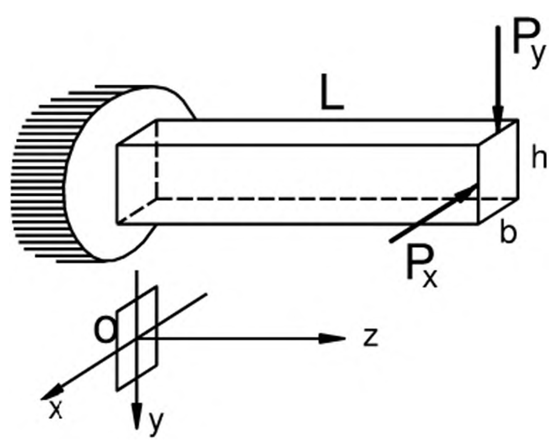


Figure 10.50

Solution

a) Draw diagrams of internal forces M_x and M_y.

At the fixed end:

$$M_x = P_y L = PL = 500\times1000 = 5\times10^5 \text{ Nmm}$$

$$M_y = P_x L = PL = 500\times1000 = 5\times10^5 \text{ Nmm}$$

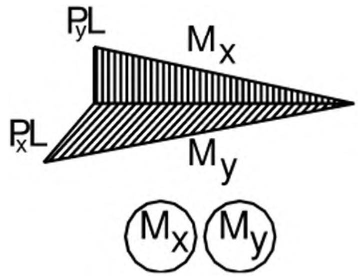


Figure 10.51

b) Determine maximum and minimum normal stresses.

$$A = b\times h = 40\times60 = 2400 \text{ mm}^2$$

$$I_x = \frac{40.60^3}{12} = 72\times10^4 mm^4$$

$$I_y = \frac{60.40^3}{12} = 32\times10^4 cm^4$$

Maximum stress at point (20, 30)

$$\sigma_{max} = \frac{M_x}{I_x}y + \frac{M_y}{I_y}x = \frac{5.10^5}{72.10^4}30 + \frac{5.10^5}{32.10^4}20$$

$$= 20.83 + 31.25 = 52.08 N/\text{mm}^2$$

Minimum stress at point (-20, -30)

$$\sigma_{max} = \frac{M_x}{I_x}y + \frac{M_y}{I_y}x = \frac{5.10^5}{72.10^4}(-30) + \frac{5.10^5}{32.10^4}(-20)$$

$$= -20.83 - 31.25 = -52.08 N/\text{mm}^2$$

c) Draw the stress diagram.

$$y = -\frac{M_y}{M_x}\frac{I_x}{I_y}x$$

$$y=-\frac{5\times10^5}{5\times10^5}\frac{72\times10^4}{32\times10^4}x=-2.25x$$

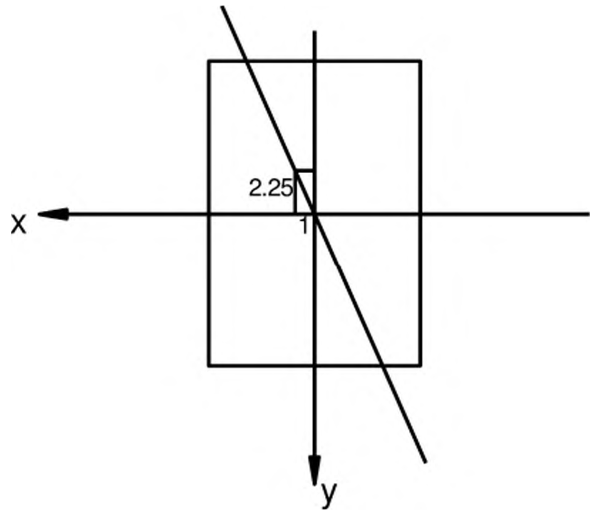


Figure 10.52

PROBLEMS

PROBLEM 1. Consider the section in **Figure 10.53**. Given: b = 20 cm, h = 40 cm, M_x = 200 kNm, M_y = 100 kNm. Calculate the normal stresses at A, B, C and D as shown in **Figure 10.53**. The coordinates of C and D are (3,-7.5) and D(-3,-7.5), respectively.

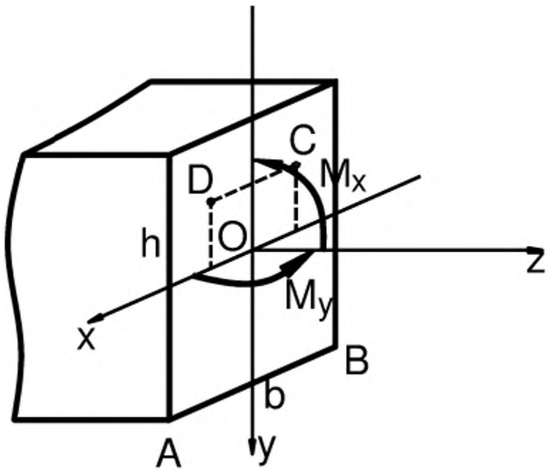


Figure 10.53

PROBLEM 2. Consider the cantilever column shown in **Figure 10.54**. The dimensions b, h and coordinate system Oxyz at the fixed end are also shown in **Figure 10.54**. Given: P_x = 500 N, P_z = 2400 N, L = 1200 mm, b = 40 mm, h = 60 mm.

a) Draw diagrams of internal forces N_z and M_y.

b) Determine maximum and minimum normal stresses.

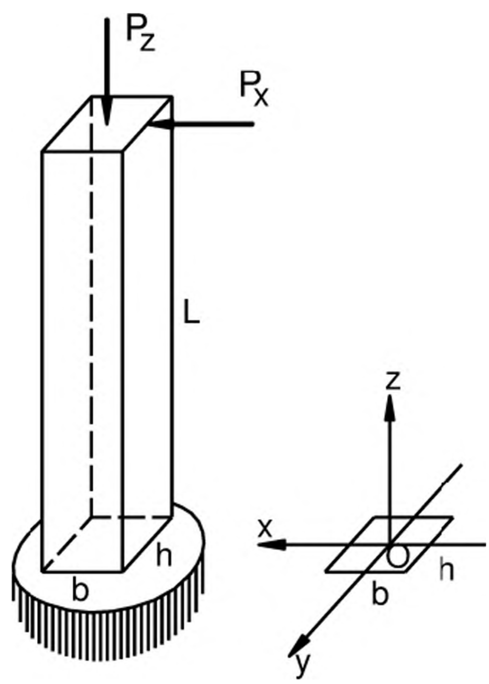


Figure 10.54

PROBLEM 3. Consider the cantilever beam shown in **Figure 10.55**. The dimensions b, h and coordinate system Oxyz at the fixed end are also shown in **Figure 10.55**. Given: $P_x = P_y = 500$ N, L = 1000 mm, b = 40 mm, h = 60 mm.

a) Draw diagrams of internal forces M_x and M_y.

b) Determine maximum and minimum normal stresses.

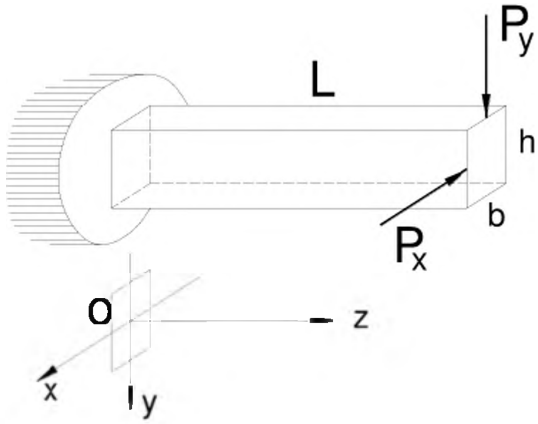


Figure 10.55

PROBLEM 4. The column with its dimensions and coordinate system Oxyz at the column bottom end is shown in **Figure 10.56**. Given: $P_x = 5$ kN, $P_y = 10$ kN, $P_z = 15$ kN, L = 4 m, b = 20 cm, h = 30 cm.

a) Draw diagrams of internal forces N_z, M_x and M_y.

b) Determine the equation of neutral axis.

c) Determine maximum and minimum normal stresses.

d) Draw the stress diagram.

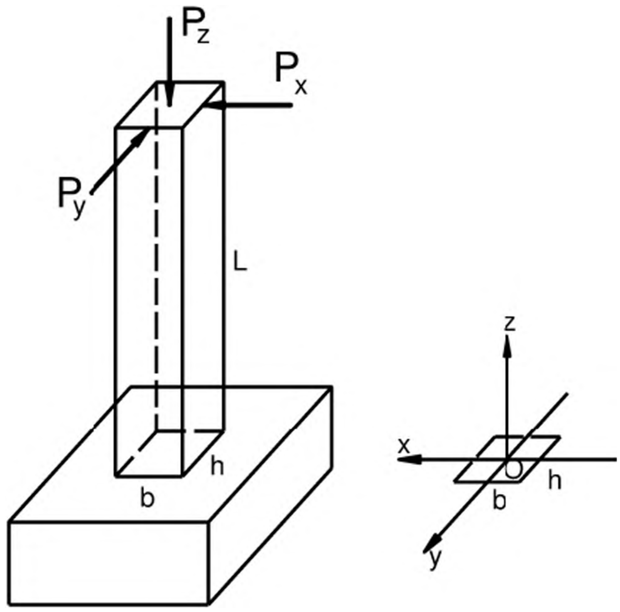


Figure 10.56

Chapter 11

STABILITY OF COLUMNS

§1. CONCEPTS

A column must satisfy the following conditions:

+ Stress condition (durability)
+ Stiffness (deflection or deformation) condition
+ Stability condition

The stress and stiffness conditions were studied in previous chapters. This chapter focuses on the stability condition.

Concept of stability can be illustrated in **Figure 11.1**.

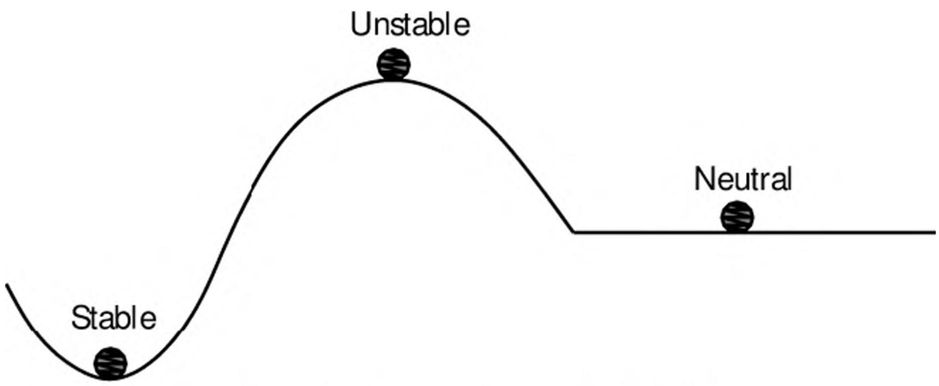


Figure 11.1. Concept of stability

Figure 11.2. Stability of column under axial loading

Considering the column subjected to the load P as shown in **Figure 11.2**, a horizontal disturbing load R makes a small lateral displacement δ. There are three cases when the load R is withdrawn:

- If $P < P_{cr}$ (critical load): The column will restore to its original state → stable.
- If $P > P_{cr}$: The horizontal displacement will increase → instable.
- If $P = P_{cr}$: The horizontal displacement remains δ → neutral equilibrium.

The nature of instability:

- Sudden and dangerous
- Instability of a bar/member may lead to structural collapse.

The scope of this chapter is the stability of bars subjected to axial loading.

§2. CRITICAL LOAD OF AXIAL COMPRESSED BARS

2.1. Simple column (column with pin supports)

Consider the simple column subjected to the critical compressive force P_{cr}. When subjected to disturbance, the column will be bent:

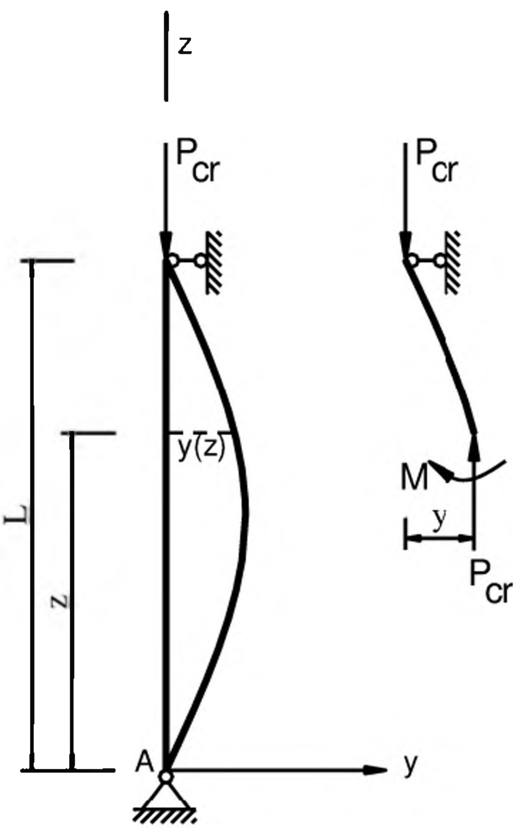


Figure 11.3

In Chapter 8, we have:

$$y'' = -\frac{M_x}{EI} \tag{11.1}$$

M is determined from the equilibrium condition:

$$M_x = P_{cr}y \tag{11.2}$$

Thus:

$$y'' = -\frac{P_{cr}y}{EI} \text{ or } y''+\frac{P_{cr}}{EI}y = 0 \tag{11.3}$$

Set:

$$\alpha^2 = \frac{P_{cr}}{EI} \tag{11.4}$$

We have:

$$y''+\alpha^2 y = 0$$

Solution:

$$y = C_1 \cos\alpha z + C_2 \sin\alpha z$$

Constants C_1 and C_2 are determined from boundary conditions:

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y(l) = 0 \Rightarrow C_2 \sin\alpha l = 0$$

Because $y(z) \neq 0$ thus $C_1 \neq 0$

$\rightarrow \quad \sin\alpha l = 0$

$\rightarrow \quad \alpha l = n\pi$ in which n = 1,2,3, ...

$\rightarrow \quad \alpha = \dfrac{n\pi}{l}$

Substitute α into Equation (11.4), we have:

$$P_{cr} = \alpha^2 EI = \left(\frac{n\pi}{l}\right)^2 EI \tag{11.5}$$

Only n=1 have practical meaning. Thus:

$$P_{cr} = \left(\frac{\pi}{l}\right)^2 EI$$

$$P_{cr} = \frac{\pi^2 EI}{l^2} \tag{11.6}$$

Equation of elastic curve:

$$y = C_2 \sin\alpha z = C_2 \sin\frac{\pi}{l}z \tag{11.7}$$

C_2 is horizontal displacement at the middle of the column.

2.2. Other columns

When applying the above method for different supports at the ends, a general formula for calculating the critical load is shown in Equation 11.8. This is called Euler's formula, where, n is a sinusoidal half-number of the elastic line when it becomes unstable, $\mu = \frac{1}{n}$ is equivalent coefficient, $l_e = \mu l$ is equivalent length of column.

$$P_{cr} = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 EI}{(\mu l)^2} \tag{11.8}$$

Values of μ are shown in **Figure 11.4**.

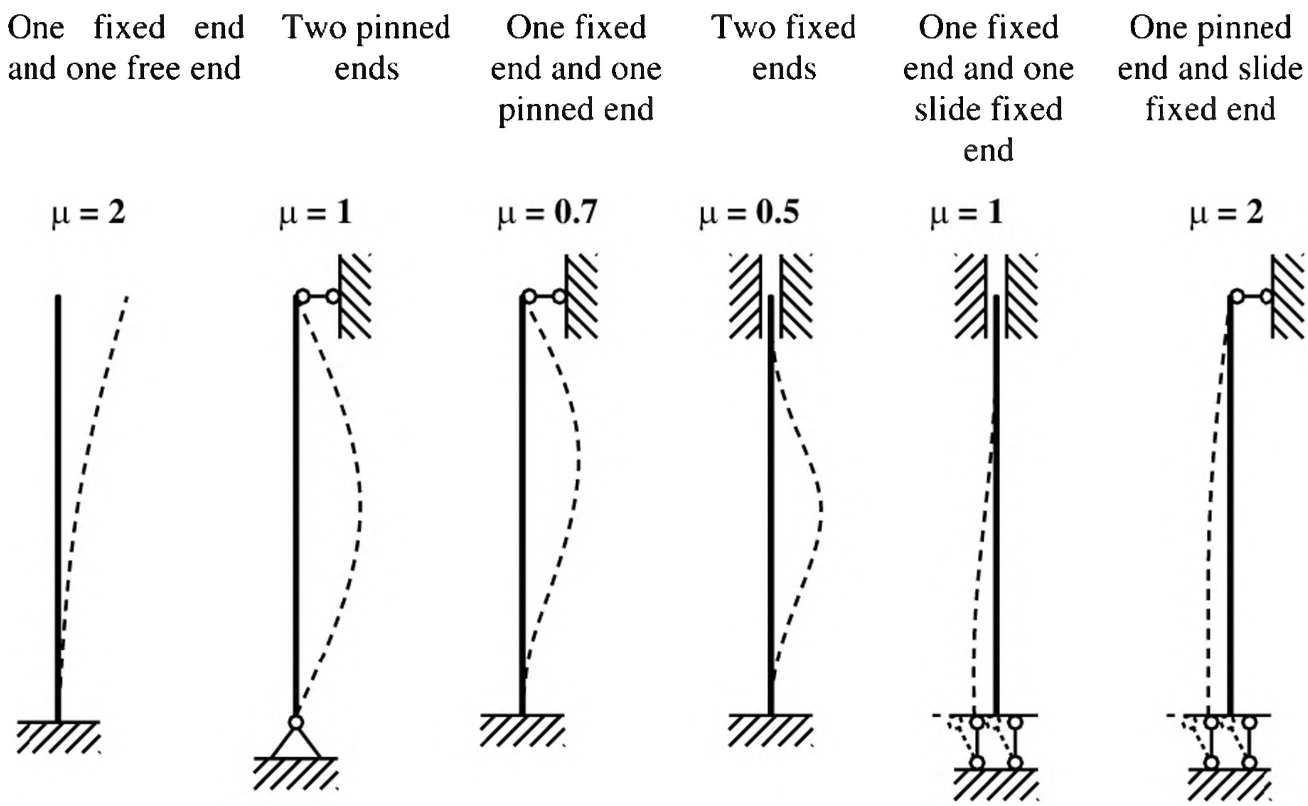


Figure 11.4. Instability and the coefficient μ.

2.3. Critical stress

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{(\mu l)^2 A} = \frac{\pi^2 E r^2}{(\mu l)^2} = \frac{\pi^2 E}{\left(\frac{\mu l}{r}\right)^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \tag{11.9}$$

in which $r = \sqrt{\dfrac{I}{A}}$

$\lambda = \dfrac{l_e}{r} = \dfrac{\mu l}{r}$ is the slenderness.

2.3. Slenderness λ has no unit

The stability of column is inversely propotional to the slenderness λ.

Limit domain for Euler

The Euler's formula is established based on the differential equation of elastic curve; thus, it is valid when:

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \leq \sigma_l$$

or:

$$\lambda \geq \sqrt{\frac{\pi^2 E}{\sigma_l}}$$

$$\lambda \geq \lambda_o$$

where $\lambda_o = \sqrt{\dfrac{\pi^2 E}{\sigma_l}}$ is the slender limit.

If member with $\lambda \geq \lambda_o$, the member is highly slender. The Euler's formula is only applied for member with high slenderness.

For each material, $\lambda_o = const$.

Example:

- Constructional steel: $\lambda_o = 100$
- Wood: $\lambda_o = 75$
- Cast iron: $\lambda_o = 80$.

§3. STABILITY DESIGN OF COLUMNS SUBJECTED TO COMPRESSIVE AXIAL LOADS

A column subjected to a compressive load must satisfy:

- Stress condition:

$$\sigma = \frac{P}{A_g} \leq [\sigma]_{compression} \tag{11.10}$$

- Stability condition:

$$\sigma = \frac{P}{A} \le [\sigma]_{buckling} \tag{11.11}$$

In which: A_g is the real cross-sectional area or reduction area (after subtracting area due to holes).

A is the cross-sectional area (without subtracting area due to holes).

$$[\sigma]_{compression} = \frac{\sigma_o}{n}$$

$$[\sigma]_{bucking} = \frac{\sigma_{cr}}{k} = \varphi[\sigma]_{compression}$$

n is factor of safety (stress).

k is factor of safety considering buckling.

φ is stress reduction factor.

Example:

Steel: k = 1.8 – 3.5;

Cast iron: k = 5 – 5.5;

Wood: k = 2.8 – 3.2.

$$\varphi = \frac{[\sigma]_{buckling}}{[\sigma]_{compression}} = \frac{\sigma_{cr}}{\sigma_o}\frac{n}{k}$$

We have:

$$\frac{\sigma_{cr}}{\sigma_o} < 1; \frac{n}{k} < 1$$

Thus: $\varphi < 1$

The factor φ depends on E, λ, k:

Table 11.1. The factor φ

Slenderness λ	The factor φ				
	Steel No. 2, 3, 4	Steel No. 5	Steel СПК	Cast iron	Wood
0	1.00	1.00	1.00	1.00	1.00
10	0.99	0.98	0.97	0.97	0.99
20	0.96	0.95	0.95	0.91	0.97
30	0.94	0.92	0.91	0.81	0.93
40	0.92	0.89	0.87	0.69	0.87
50	0.89	0.86	0.83	0.54	0.80
60	0.86	0.82	0.79	0.44	0.71
70	0.81	0.76	0.72	0.34	0.60
80	0.75	0.70	0.65	0.26	0.48
90	0.69	0.62	0.55	0.20	0.38
100	0.60	0.51	0.43	0.16	0.31
110	0.52	0.43	0.35		0.25
120	0.45	0.36	0.30		0.22
130	0.40	0.33	0.26		0.18
140	0.36	0.29	0.23		0.16
150	0.32	0.26	0.21		0.14
160	0.29	0.24	0.19		0.12
170	0.26	0.21	0.17		0.11
180	0.23	0.19	0.15		0.10
190	0.21	0.17	0.14		0.09
200	0.19	0.16	0.13		0.08

Due to $\varphi < 1$, if the buckling condition satisfies, the stress condition will also satisfy. Thus, only the buckling condition is necessary.

However, for members with reduction of area, both buckling and stress conditions are required to be checked (due to $A_g \neq A$).

Three problems

Problem 1: Checking the buckling condition:

$$\sigma = \frac{P}{A} \leq \varphi[\sigma]_{compression} \tag{11.12}$$

Problem 2: Determining the allowable load:

$$[P] \leq \varphi[\sigma]_{compression} A \tag{11.13}$$

Problem 3: Selecting the area:

$$A \geq \frac{P}{\varphi[\sigma]_{compression}} \tag{11.14}$$

Because the inequality contains two variables A and φ(A), thus, A is found by trial and error method.

Procedure:

- Select $\varphi_o = 0,5$, compute $A_o \geq \dfrac{P}{\varphi_o [\sigma]_{compression}}$ => λ_o
- From λ_o, the value $\varphi_o^{'}$ is determined from the **Table 11.1.** If $\varphi_o^{'} \neq \varphi_o$, reselect $\varphi_o = \dfrac{\varphi_o + \varphi_o^{'}}{2}$ => $A_1 = \dfrac{P}{\varphi_1 [\sigma]_{compression}}$ => λ_1, the $\varphi_1^{'}$ is determined from the **Table 11.1.**
- Repeat the above calculations about 2-3 times, if the error between 2 calculations is small enough (≤5%), then the calculation can be stopped.

§4. EXERCISES

4.1. Exercise 1

A column with two pin ends made of steel. Its cross section is I N°22a, E = $2.1.10^5$ MPa, σ_l = 210 MPa, L = 3 m. Calculate critical stress and critical load.

Solution

I N°22a → $r_{min} = r_y = 2.5$ cm, A = 32.4 cm^2.

Two pin ends → μ=1.

$$\lambda_o = \sqrt{\frac{\pi^2 E}{\sigma_l}} = \sqrt{\frac{\pi^2 \times 2.1 \times 10^5}{210}} = 99.3$$

$$\lambda = \frac{\mu l}{r_{\min}} = \frac{1 \times 3000}{25} = 120 > \lambda_o = 99.3 \text{ (Use Euler's formula)}$$

$$\sigma_{th} = \frac{\pi^2 E}{\lambda^2} = \frac{\pi^2 \times 2.1 \times 10^5}{120^2} = 143 MPa$$

$$P_{cr} = \sigma_{cr} A = 143 \times 3240 = 463.3 \times 10^3 N = 463.3 kN$$

4.2. Exercise 2

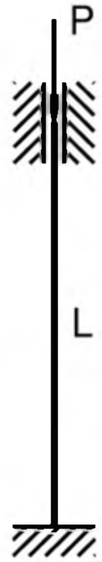
A steel column I24a with two fixed ends, its length is 7.5 m, E = 2.1×10^4 kN/cm^2, σ_l = 21 kN/cm^2, safety factor of stability is 3, P = 150 kN.

a) Calculate the critical stress.

b) Calculate the critical force.

c) Calculate the allowable force.

d) Check the stability condition.



Solution

I N°24a → $r_{min} = r_y = 2.63$ cm, A = 37.5 cm^2.

Two fixed ends → $\mu = 0.5$.

$$\lambda_o = \sqrt{\frac{\pi^2 E}{\sigma_l}} = \sqrt{\frac{\pi^2 \times 2.1\times10^4}{21}} = 99.3$$

$$\lambda = \frac{\mu l}{r_{\min}} = \frac{0.5\times750}{2.63} = 142 > \lambda_o = 99.3 \text{ (use Euler formula)}$$

Calculate the critical stress.

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} = \frac{\pi^2 \times 2.1\times10^4}{142^2} = 10.3kN/cm^2$$

Calculate the critical force.

$$P_{cr} = \sigma_{cr} A = 10.3\times37.5 = 386.25kN$$

Calculate the allowable force.

$$[P] = \frac{P_{cr}}{k} = \frac{386.25}{3} = 128.75kN$$

Check the stability condition.

$$[P] = 128.75kN < P = 150kN \text{ unstable}$$

4.3. Exercise 3

A 2 m column has donut cross section with outside diameter R = 3 cm, inside diameter r = 2 cm. One end is fixed, and one end is pinned, material E = 7.1×10^3 kN/cm^2, σ_l = 18 kN/cm^2, safety factor of stability is 3, P = 80 kN.

a) Calculate the critical stress.

b) Calculate the critical load.

c) Calculate the allowable load.

d) Check the stability condition.

Solution

We have:

$$I_x = I_y = \frac{\pi}{4}\left(R^4 - r^4\right) = \frac{\pi}{4}\left(3^4 - 2^4\right) = 51.05cm^4$$

$$A = \pi\left(R^2 - r^2\right) = \pi\left(3^2 - 2^2\right) = 15.71cm^2$$

$$r_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{51.05}{15.71}} = 1.8cm$$

One end is fixed, and one end is pinned → μ = 0.7.

$$\lambda_o = \pi\sqrt{\frac{E}{\sigma_l}} = \pi\sqrt{\frac{7.1\times10^3}{18}} = 62.4$$

$$\lambda = \frac{\mu l}{r_{\min}} = \frac{0.7\times200}{1.8} = 77.66 > \lambda_o = 62.4 \text{ (dùng công thức Euler)}$$

a) Critical stress

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} = \frac{\pi^2\times7.1\times10^4}{77.66^2} = 11.6kN/cm^2$$

b) Critical load

$$P_{cr} = \sigma_{cr}A = 11.6\times15.7 = 182.3kN$$

c) Allowable load

$$[P] = \frac{P_{cr}}{k} = \frac{182.3}{3} = 60.8kN$$

d) The stability condition

$$[P] = 60.8kN < P = 80kN \text{ => not stable}$$

4.4. Exercise 4

Consider a column with the cross section shown in **Figure 11.5**. Given: a = 10 cm, t = 2 cm, two fixed ends, L = 3 m, elastic modulus E = $7.1.10^3$ kN/cm^2, σ_l = 18 kN/cm^2, safety factor is 3, P = 160 kN.

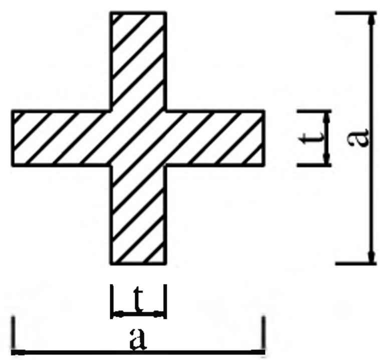


Figure 11.5

a) Calculate the critical stress.

b) Calculate the critical load.

c) Calculate the allowable load.

d) Check the stability condition.

Solution

We have:

$$I_x = I_y = \frac{ta^3}{12} + \frac{(a-t)t^3}{12} = \frac{2\times10^3}{12} + \frac{(10-2)2^3}{12} = 172cm^4$$

$$A = 10\times2 + (10-2)2 = 36cm^2$$

$$r_{min} = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{172}{36}} = 2.19cm$$

Two fixed ends → μ = 0.5

$$\lambda_o = \pi\sqrt{\frac{E}{\sigma_{tl}}} = \pi\sqrt{\frac{7.1\times10^3}{18}} = 62.4$$

$$\lambda = \frac{\mu l}{r_{\min}} = \frac{0.5\times300}{2.19} = 68.6 > \lambda_o = 62.4$$

= > use the Euler's formula

a) Calculate the critical stress.

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} = \frac{\pi^2\times7.1\times10^4}{68.6^2} = 14.86kN/cm^2$$

b) Calculate the critical load.

$$P_{cr} = \sigma_{cr}A = 14.86\times36 = 535.1kN$$

c) Calculate the allowable load.

$$[P] = \frac{P_{cr}}{k} = \frac{535.1}{3} = 178.4kN$$

d) Check the stability condition.

$$[P] = 178.4kN > P = 160kN \text{ => Stable}$$

4.5. Exercise 5

Consider the column with two fixed ends. The length L = 6m and the rectangular section with the width b = 8 cm, and the height h = 12cm. The elastic modulus is $E = 2.1\times10^4$ kN/cm^2, $\sigma_l = 21$ kN/cm^2, stability safety factor = 2, thermal coefficient $\alpha = 1.25\times10^{-5}$ (1/°C).

a) Calculate the critical stress.

b) Calculate the critical axial force.

c) Calculate the allowable axial load.

d) Determine the increment of temperature at which the column is instable.

Solution

We have:

$$r_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{\frac{hb^3}{12}}{bh}} = \frac{b}{\sqrt{12}} = \frac{8}{\sqrt{12}} = 2.31cm$$

A = b×h = 8×12 = 96 cm^2.

Two fixed ends → $\mu = 0.5$.

$$\lambda_o = \sqrt{\frac{\pi^2 E}{\sigma_l}} = \sqrt{\frac{\pi^2 \times 2.1\times10^4}{21}} = 99.3$$

$$\lambda = \frac{\mu l}{r_{\min}} = \frac{0.5\times600}{2.31} = 129.9 > \lambda_o = 99.3 \text{ (use the Euler's fomula)}$$

Calculate the critical stress.

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} = \frac{\pi^2 \times 2.1\times10^4}{129.9^2} = 12.3kN/cm^2$$

Calculate the critical axial force.

$$P_{cr} = \sigma_{cr} A = 13.3\times96 = 1180.8kN$$

Calculate the allowable axial load.

$$[P]_{buckling} = \frac{P_{cr}}{k} = \frac{1180.8}{2} = 590.4kN$$

Determine the increment of temperature at which the column is instable.

Set Δt is the increase of temperature at which the column is instable

If one end is free, the elongation of the column is $\alpha.\Delta t.l$

Because both ends are fixed; therefore, there is a reaction R in the column = > internal force N = R which shortens the column an amount: $\frac{Nl}{EA}$

Thus,

$$\alpha.\Delta t.l = \frac{Nl}{EA}$$

$$=> \quad N = \alpha.\Delta t.EA$$

When the column is instable:

$$N = [P]_{buckling}$$

$$\alpha.\Delta t.EA = [P]_{buckling}$$

$$\Delta t = \frac{[P]_{buckling}}{\alpha EA}$$

4.6. Exercise 6

A column with two pinned ends makes from steel I N°27. The elastic modulus is $E = 2.1\times10^4$ kN/cm², $\sigma_l = 21$ kN/cm², L = 4 m.

Solution

We have:

I N°27 → $r_{min} = r_y = 2.54$ cm, A = 40.2 cm².

Two pinned ends → $\mu = 1$

$$\lambda = \frac{\mu l}{r_{min}} = \frac{1\times400}{2.54} = 157.5 > \lambda_o = 100$$

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} = \frac{\pi^2\times2.1\times10^4}{157.5^2} = 8.35 kN/cm^2$$

$$P_{cr} = \sigma_{cr}A = 8.35\times40.2 = 335.6 kN$$

4.7. Exercise 7

A column with two pinned ends subjected to the axial load P = 230 kN. The length is 2 m and the steel section I. Given: $[\sigma]_{compression}$=140 MN/m².

Solution

Iteration 1: Assume $\varphi_o = 0.5$

We calculate

$$A_o \geq \frac{P}{\varphi_o[\sigma]_{compression}} = \frac{230.10^3}{0.5\times140\times10^6} = 32.8.10^{-4} m^2$$

From the catalogue of steel section I

=> select I N°22a → $r_{min} = r_y = 2.5$ cm, A = 32.4 cm².

Two pinned ends → $\mu = 1$

$$\lambda = \frac{\mu l}{r_{min}} = \frac{1\times200}{2.5} = 80 < \lambda_o = 100$$

Table 11.1 → $\varphi = 0.75$, which is a large difference with $\varphi_o = 0.5$)

Iteration 2:

$$\varphi = \frac{0.5+0.75}{2} = 0.625$$

We calculate

$$A_o \geq \frac{P}{\varphi_o [\sigma]_{compression}} = \frac{230.10^3}{0.625 \times 140 \times 10^6} = 26.2 \times 10^{-4} m^2$$

From thc cataloguc of stccl scction I

=> select I N°20 → $r_{min} = r_y = 2.06$ cm, A = 26.4 cm^2.

$$\lambda = \frac{\mu l}{r_{min}} = \frac{1 \times 200}{2.06} = 97$$

Table 11.1 → φ = 0.625, which is equal to the above assumed value

$$\sigma = \frac{P}{A} = \frac{230 \times 10^3}{0.627 \times 26.4 \times 10^{-4}}$$
$$= 139.10^6 Pa = 139 MM / m^2 \leq [\sigma]_n = 140 MM / m^2$$

Result: select I20

4.8. Exercise 8

Consider the structure in the **Figure 11.6**. The column AB is made of steel CT3 with $\lambda_o = 100$, $[\sigma]_n = 16 kN / cm^2$, the cross section is I30. L = 2 m, q = 20 kN/m, P_1 = 80 kN, P_2 = 40 kN.

Check the stability condition of the column AB.

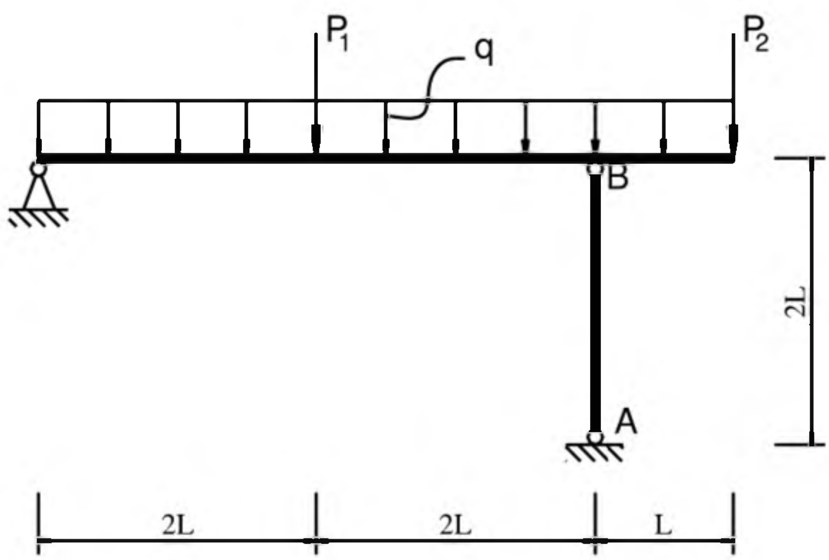


Figure 11.6

Solution

I30 → $r_{min} = r_y = 2.69$ cm, A = 46.5 cm^2.

Two pin ends → $\mu = 1$.

$$\lambda = \frac{\mu l}{r_{\min}} = \frac{1\times 400}{2.69} = 148.5$$

λ	φ
140	0.36
150	0.32

Interpolate:

$$\varphi = 0.36 - \frac{0.36-0.32}{150-140}(148.5-140) = 0.326$$

Allowable load:

$$[P] = [\sigma]_{buckling} A = \varphi[\sigma]_{buckling} A = 0.326\times 16\times 46.5 = 242kN$$

Internal force of AB:

$$\sum M_{/C} = 0$$

$$N = \frac{80\times 4 + 40\times 10 + (20\times 10)\times 5}{8} = 215kN$$

$N < [P]$ → stable.

4.9. Exercise 9

Consider the structure in **Figure 11.7**. The bar AB is made of steel CT3, with $[\sigma]_n = 14kN/cm^2$. The cross section 2L125×125×12 is assembled by rivet with diameter d_o = 30 mm. Given: L = 2.5 m, P = 300 kN.

Check the stress and stability conditions of the bar AB.

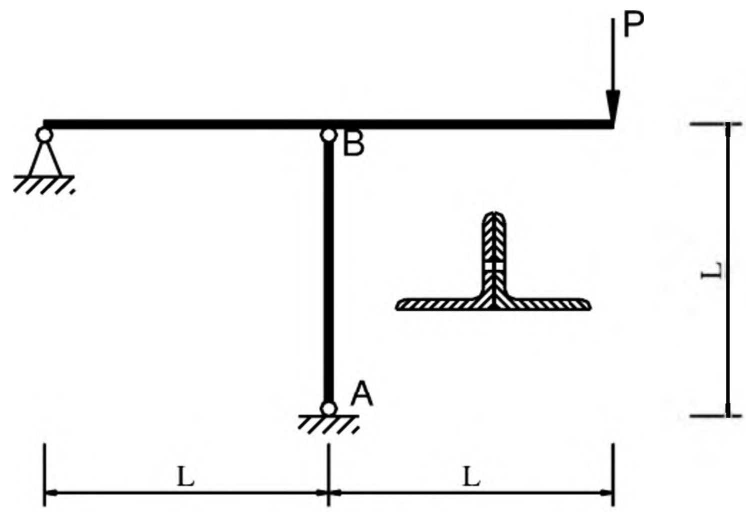


Figure 11.7

Solution

Determine the axial force in the bar AB:

$$\sum M_{/\,goitrai} = 0$$

$$N = \frac{P \times 2L}{L} = 2P = 600kN$$

The sectional properties:

L125x125x12 → A = 28.9 cm^2. $I_x = I_y$ = 422 cm^4, z_o=3.53cm

2L125x125x12 →

A = 2×28.9 = 57.8 cm^2.

$$A_g = A - 2\times1.2\times3 = 57.8\text{-}2\times1.2\times3 = 50.6cm^2$$

I_x = 2.422 = 844 cm^4

$$I_y = 2\left(I_{y.1L} + Az_o^2\right) = 2\left(422 + 28.9\times3.53^2\right) = 2\times782 = 1564cm^4$$

$$r_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{844}{57.8}} = 3.82cm$$

Two pinned ends → μ = 1

$$\lambda = \frac{\mu l}{r_{\min}} = \frac{1\times250}{3.82} = 65.4$$

λ	φ
60	0.86
70	0.81

Interpolate, we have;

$$\varphi = 0.86 - \frac{0.86 - 0.81}{70 - 60}(65.4 - 60) = 0.833$$

Stress condition:

$$\sigma = \frac{P}{A_g} \leq [\sigma]_n$$

$$\sigma = \frac{N}{A_g} = \frac{600}{50.6} = 11.86 kN/cm^2 \leq [\sigma]_{compression} = 14 kN/cm^2 \text{ (OK)}$$

Stability condition:

$$\sigma = \frac{P}{A} \leq [\sigma]_{buckling}$$

$$\sigma = \frac{N}{A} = \frac{600}{57,8} = 10,38 kN/cm^2 \leq [\sigma]_{buckling} \text{ (OK)}$$

$$= \varphi[\sigma]_n = 0,833 \times 14 = 11,69 kN/cm^2$$

Conclusion: The bar AB satisfies the conditions of stress and stability.

4.10. Exercise 10

Consider the column shown in **Figure 11.8**. Determine the most appropriate cross-section.

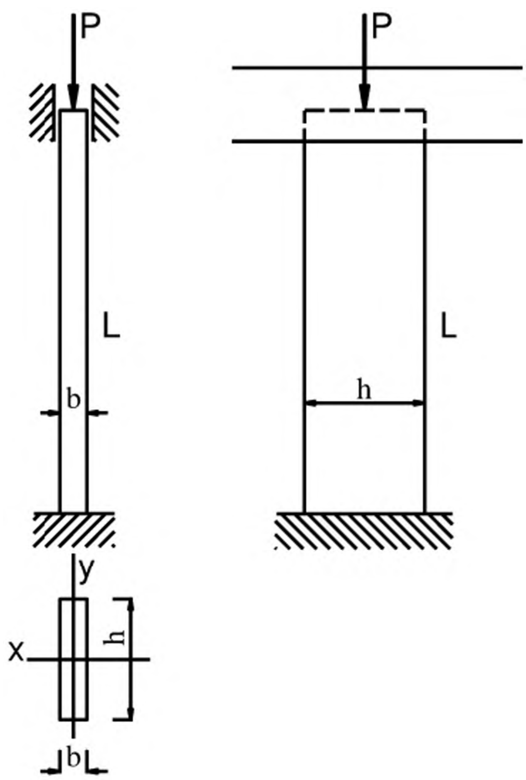


Figure 11.8

Solution

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{bh^3}{12}}{bh}} = \frac{h}{\sqrt{12}};\ r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{hb^3}{12}}{bh}} = \frac{b}{\sqrt{12}}$$

A = bh.

In the plane with smaller stiffness: Two fixed ends → $\mu_y = 0.5$;

$$\lambda_y = \frac{\mu_y l}{r_y} = \frac{0.5 \times l}{b}\sqrt{12}$$

In the plane with larger stiffness: One fixed end and one free end → $\mu_x = 2$.

$$\lambda_x = \frac{\mu_x l}{r_x} = \frac{2 \times l}{h}\sqrt{12}$$

The appropriate section: $\lambda_x = \lambda_y$

$$=> \quad \frac{0.5 \times l}{b}\sqrt{12} = \frac{2 \times l}{h}\sqrt{12} \ => \ h = 4b$$

4.10. Exercise 11

Reconsider the column in **Figure 11.8**. The cross section is b×h = 8 × 28 cm, the length is L = 3 m. Given $[\sigma] = 1kN/cm^2$. Determine the allowable load [P] so that the column satisfies the stability condition.

Solution

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{bh^3}{12}}{bh}} = \frac{h}{\sqrt{12}} = \frac{28}{\sqrt{12}} = 8.08cm$$

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{hb^3}{12}}{bh}} = \frac{b}{\sqrt{12}} = \frac{8}{\sqrt{12}} = 2.31cm$$

A = b × h = 8 × 28 = 224 cm^2

In the plane with smaller stiffness: Two fixed ends → $\mu_y = 0.5$.

$$\lambda_y = \frac{\mu_y l}{r_y} = \frac{0.5 \times 300}{2.31} = 65$$

In the plane with larger stiffness: One fixed end and one free end → $\mu_x = 2$.

$$\lambda_x = \frac{\mu_x l}{r_x} = \frac{2\times 300}{8.08} = 74.3$$

$\lambda_x > \lambda_y$ => the bar will be bent in the plane Oyz.

$$\lambda_x = 74.3 \rightarrow \varphi = 0.548$$

$$[P] = \varphi[\sigma]A = 0.548\times 1\times 224 = 122.75kN$$

PROBLEMS

PROBLEM 1. A column with two pin ends made of steel. Its cross section is I N°20, E = $2.1.10^5$ MPa, σ_l = 210 MPa, L = 2.5 m. Calculate critical stress and critical load.

PROBLEM 2. A steel column I20 with two fixed ends, its length is 5 m, E = 2.1×10^4 kN/cm^2, σ_l = 21 kN/cm^2, safety factor of stability is 3, P = 150 kN.

a) Calculate the critical stress.

b) Calculate the critical force.

c) Calculate the allowable force.

d) Check the stability condition.

PROBLEM 3. A 3 m column has donut cross section with outside diameter R = 5 cm, inside diameter r = 3 cm. One end is fixed, and one end is pinned, material E = 7.1×10^3 kN/cm^2, σ_l = 18 kN/cm^2, safety factor of stability is 3, P = 80 kN.

a) Calculate the critical stress.

b) Calculate the critical load.

c) Calculate the allowable load.

d) Check the stability condition.

PROBLEM 4. Consider the structure given in the **Figure 11.9**. The stiffness of member BCD is infinite. Members CF and DE are made of the same material and cross section of donut with outside radius R = 3 cm, inside radius r = 2 cm, $[\sigma]$ = 14 kN/cm^2, E = 2.10^4 kN/cm^2, q = 10 kN/m, a = 2 m.

a) Determine the axial forces in members CF and DE.

b) Calculate the vertical displacement at D.

c) Check the stability condition for the member DE.

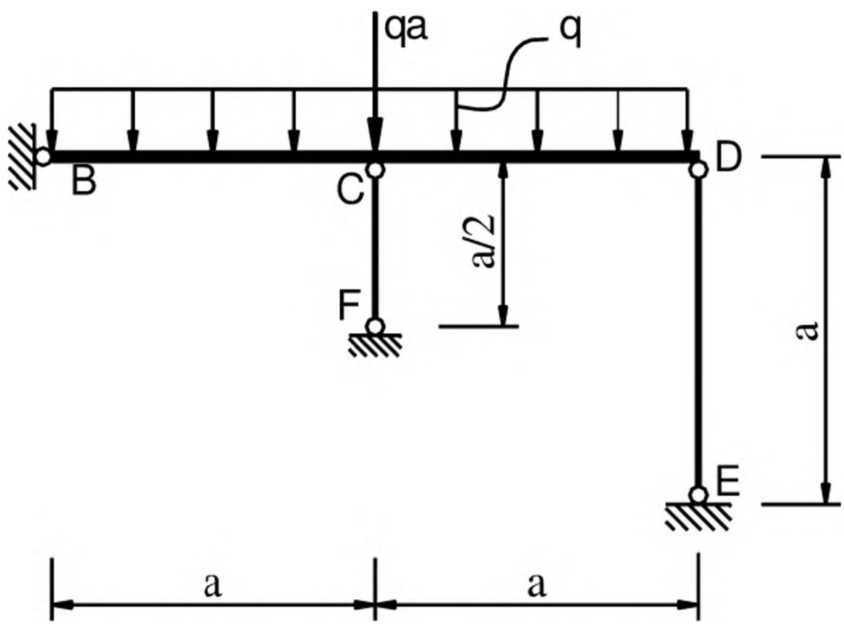


Figure 11.9

PROBLEM 5. Consider the structure in **Figure 11.10**. Given: q = 10 kN/m, a = 4 m. Steel members CK and CH have the donut section with outer radius of 8 cm and inner radius of 6 cm, [σ] = 14 kN/cm^2, E = 2.10^4 kN/cm^2.

a) Determine the axial forces in members CK and CH.

b) Determine the vertical displacement at C.

c) Check the stress condition of members CK và CH.

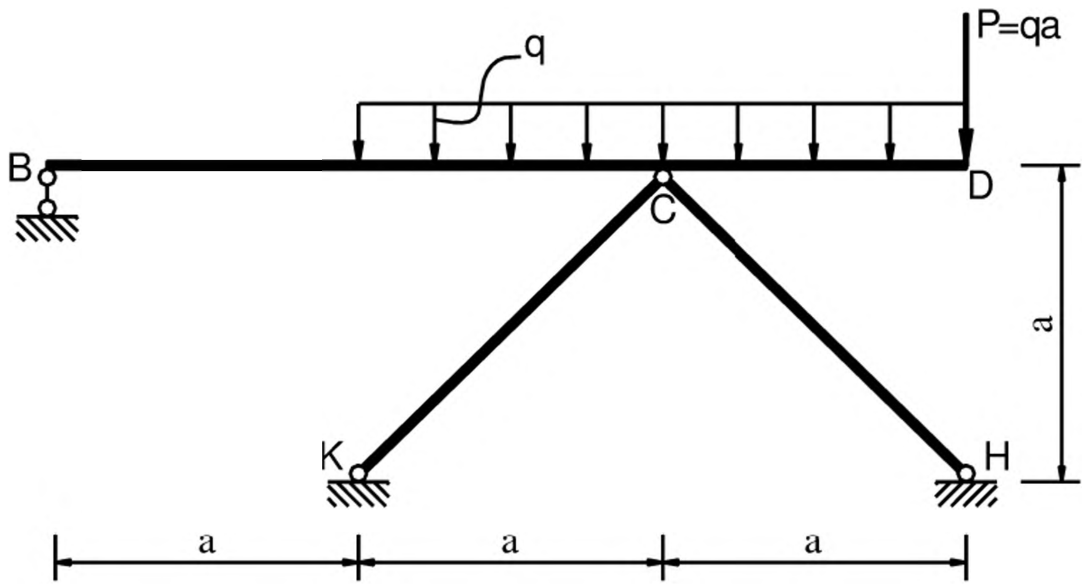


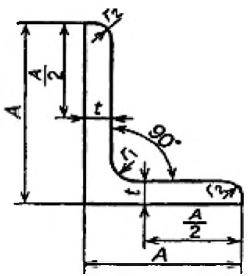
Figure 11.10

APPENDIXES

Appendix 1

Hot-rolled steel sections - Part 1: Equal-leg angles

TCVN 7571-1:2019



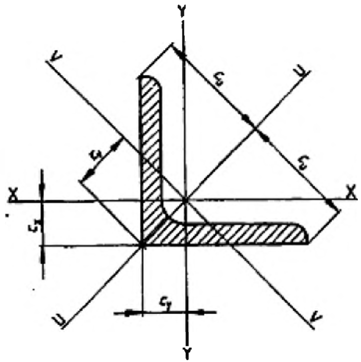
Mô men chống uốn $I = ai^2$

Bán kính quán tính $i = \sqrt{I/a}$

Mô men chống xoắn $Z = I/e$

(a - diện tích mặt cắt theo Bảng 4)

(i - bán kính quán tính)



Appendix 1. Sectional properties of hot-rolled steel sections with equal-leg angles (TCVN 7571-1:2019)

Size	b mm	t mm	r_1 mm	kg/m	A cm^2	C_x=C_y cm	C_u cm	C_v cm	I_x=I_y cm^4	r_x=r_y cm	Z_x=Z_y cm^3	I_u cm^4	r_u cm	I_v cm^4	r_v cm	W_v cm^3
20x20	20	3	3.5	0.88	1.12	0.598	1.4	0.846	0.39	0.59	0.279	0.618	0.742	0.17	0.38	0.195
25x25	25	3	3.5	1.12	1.42	0.723	1.8	1.02	0.8	0.75	0.452	1.27	0.945	0.33	0.48	0.326
	25	4	3.5	1.45	1.85	0.762	177	1.08	1.02	741	0.586	1.61	0.931	0.43	0.48	0.399
30x30	30	3	5	1.36	1.74	0.835	2.1	118	1.4	0.9	0.649	2.22	1.13	0.59	0.58	0.496
	30	4	5	1.78	2.27	0.878	2.1	1.24	1.8	0.89	0.85	2.85	1.12	0.75	0.58	607

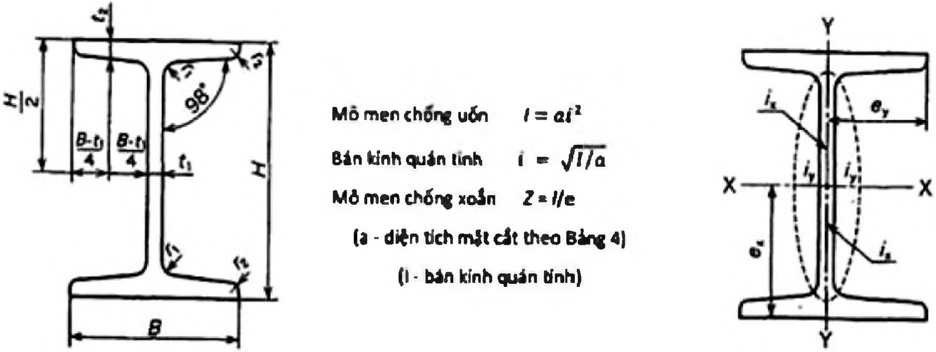
35 x 35	35	4	5	2.09	2.67	1	2.5	1.42	2.95	1.05	1.18	4.68	1.32	1.23	0.68	0.865
	35	5	5	2.57	3.28	1.04	2.5	1.48	3.56	1.04	1.45	5.64	1.31	1.49	0.68	1.01
40x40	40	3	6	1.84	2.35	1.07	2.8	1.52	3.45	1.21	1.18	5.45	1.52	1.44	0.78	0.949
	40	4	6	2.42	3.08	1.12	2.8	1.58	4.47	1.21	1.56	7.09	1.52	1.86	0.78	1.17
	40	5	6	2.97	3.79	1.16	2.8	1.64	5.43	1.2	1.91	8.6	1.51	2.26	0.77	1.38
45 x 45	45	4	7	2.74	3.49	1.23	3.2	1.75	6.43	1.36	1.97	10.2	1.71	2.68	0.88	1.53
	45	5	7	3.38	4.3	1.28	3.2	1.81	7.84	1.35	2.43	12.4	1.7	3.26	0.87	1.8
50 x 50	50	4	7	3.06	3.89	1.36	3.5	1.92	8.97	1.52	2.46	14.2	1.91	3.73	0.98	1.94
	50	5	7	3.77	4.8	1.4	3.5	1.99	11	1.51	3.05	17.4	1.9	4.55	0.97	2.29
	50	6	7	4.47	5.69	1.45	3.5	2.04	12.8	1.5	3.61	20.3	1.89	5.34	0.97	2.61
60 x 60	60	4	6.5	3.68	4.69	1.61	4.2	2.28	16	1.85	3.66	25.4	2.33	6.62	1.19	2.9
	60	5	8	4.57	5.82	1.64	4.2	2.32	19.4	1.82	4.45	30.7	2.3	8.03	1.17	3.46
	60	6	8	5.42	6.91	1.69	4.2	2.39	22.8	1.82	5.29	36.1	2.29	9.44	1.17	3.96
	60	8	8	7.09	9.03	1.77	4.2	2.5	29.2	1.8	6.89	46.1	2.26	12.2	1.16	4.86
65 x 65	65	6	9	5.91	7.53	1.8	4.6	2.55	29.2	1.97	6.21	46.3	2.48	12.1	1.27	4.74
	65	8	9	7.73	9.85	1.89	4.6	2.67	37.5	1.95	8.13	59.4	2.46	15.6	1.26	5.84
70 x 70	70	6	9	6.38	8.13	1.93	5	2.73	36.9	2.13	7.27	58.5	2.68	15.3	1.37	5.6
	70	7	9	7.38	9.4	1.97	5	2.79	42.3	2.12	8.41	671	2.67	17.5	1.36	6.28
75 x 75	75	6	9	6.85	8.73	2.05	5.3	2.9	45.8	2.29	8.41	72.7	2.89	18.9	1.47	6.53
	75	8	9	8.99	11.4	2.14	5.3	3.02	59.1	2.27	11	93.8	2.86	24.5	1.46	8.09
80 x 80	80	6	10	7.34	9.35	2.17	5.7	3.07	55.8	2.44	9.57	88.5	3.08	23.1	1.57	7.56
	80	8	10	9.63	12.3	2.26	5.7	3.19	72.2	2.43	12.6	115	3.06	29.9	1.56	9.37
80 x 80	80	10	10	11.9	15.1	2.34	5.7	3.3	87.5	2.41	15.4	139	3.03	36.4	1.55	11
90 x 90	90	7	11	9.61	12.2	2.45	6.4	3.47	92.5	2.75	14.1	147	3.46	38.3	1.77	11
	90	8	11	10.9	13.9	2.5	6.4	3.53	104	2.74	16.1	166	3.45	43.1	1.76	12.2
	90	9	11	12.2	15.5	2.54	6.4	3.59	116	2.73	17.9	184	3.44	47.9	1.76	13.3
	90	10	11	13.4	17.1	2.58	6.4	3.65	127	2.72	19.8	201	3.42	52.6	1.75	14.4

100 x 100	100	8	12	12.2	15.5	2.74	7.1	3.87	145	3.06	19.9	230	3.85	59.9	1.96	15.5
	100	10	12	15	19.2	2.82	7.1	3.99	177	3.04	24.6	280	3.83	73	1.95	18.3
	100	12	12	17.8	22.7	2.9	7.1	4.11	207	3.02	29.1	328	3.8	85.7	1.94	20.9
120 x 120	120	8	13	14.7	18.7	3.23	8.5	4.56	255	3 69	29.1	405	4.65	105	2.37	23.1
	120	10	13	18.2	23.2	3.31	8.5	4.69	313	3.67	36	497	4.63	129	2.36	27.5
	120	12	13	21.6	27.5	3.4	8.5	4.8	368	3.65	42.7	584	4.6	152	2.35	31.6
125 x 125	125	8	13	15.3	19.5	3.35	8.8	4.74	290	3.85	31.7	461	4.85	120	2.47	25.3
	125	10	13	19	24.2	3.44	8.8	4.86	356	3.84	39.3	565	4.83	146	2.46	30.1
	125	12	13	22.6	28.7	3.52	8.8	4.98	418	3.81	46.6	664	4.81	172	2.45	34.6
130 x 130	130	9	12	17.9	22.7	3.53	9.2	4.99	366	4.01	38.7	583	5.06	150	2.57	30.1
	130	12	12	23.4	29.8	3.64	9.2	5.15	467	3.96	49.9	743	5	192	2.54	37.3
	130	15	12	28.8	36.8	3.76	9.2	5.34	568	3.93	61.5	902	4.95	234	2.53	43.8
150 x 150	150	10	16	23	29.3	4.03	11	5.71	624	4.62	56.9	990	5.82	258	2.97	45.1
	150	12	16	27.3	34.8	4.12	11	5.83	737	4.6	67.7	1 170	5.8	303	2.96	52
	150	15	16	33.8	43	4.25	11	6.01	898	4.57	83.5	1 430	5.76	370	2.93	61.6
175 x 175	175	12	15	31.8	40.5	4.73	12	6.69	1170	5.38	91.8	1860	6.78	480	3.44	71.7
	175	15	15	39.4	50.2	4.85	12	6.86	1440	5.35	114	2290	6.75	589	3.42	85.9
180 x 180	180	15	18	40.9	52.1	4.98	13	7.05	1 590	5.52	122	2 520	6.96	653	3.54	92.7
	180	18	18	48.6	61.9	5.1	13	7.22	1 870	5.49	145	2 960	6.92	768	3.52	106
200 x 200	200	16	18	48.5	61.8	5.52	14	7.81	2 340	6.16	162	3 720	7.76	960	3.94	123
	200	20	18	59.9	76.3	5.68	14	8.04	2 850	6.11	199	4 530	7.7	1 170	3.92	146
	200	24	18	71.1	90.6	5.84	14	8.26	3 330	6.06	235	5 280	7.64	1 380	3.9	167
250 x 250	250	28	18	104	133	7.24	18	10.2	7 700	7.62	433	12 200	9.61	3170	4.89	309
	250	35	18	128	163	7.5	18	10.6	9 260	7.54	529	14 700	9.48	3 860	4.87	364

Appendix 2

TCVN 7571-15:2019

Hot-rolled steel sections - Part 15: I sections



Appendix 2. Sectional properties of hot-rolled steel I sections (TCVN 7571-15:2019)

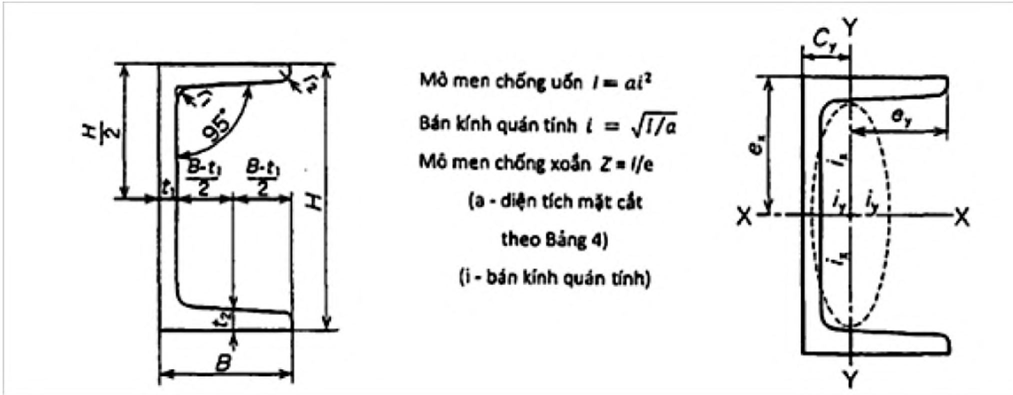
H x B	t_1	t_2	r_1	r_2	A	Mass per unit length	I_x	I_y	r_x	r_y	W_x	W_y
	mm				cm²	kg/m	cm⁴		cm		cm³	
100 x 50	4.5	6.8	7	3.5	10.9	8.57	175	12.3	4.01	1.06	35	4.93
100 x 55	4.5	7.2	7	2.5	12	9.46	198	17.9	4.06	1.22	39.7	6.49
100 x 75	5	8	7	3.5	16.43	12.9	281	47.3	4.14	1.7	56.2	12.6
120 x 60	5	7.6	8	4	14.9	11.5	342	23.5	4.83	1.27	57	7.84

120 x 64	4.5	7.2	7.5	3	14.7	11.5	350	27.9	4.88	1.38	58.4	8.72
125 x 75	5.5	9.5	9	4.5	20.45	16.1	538	57.5	5.13	1.68	86	15.3
150 x 75	5.5	9.5	9	4.5	21.83	17.1	819	57.5	6.12	1.62	109	15.3
150 x 125	8.5	14	13	6.5	46.15	36.2	1760	385	6.18	2.89	235	61.6
180 x 100	6	10	10	5	30.06	23.6	1670	138	7.45	2.14	186	27.5
200 x 100	7	10	10	5	33.06	26	2170	138	8.11	2.05	217	27.7
200 x 150	9	16	15	7.5	64.16	50.4	4460	753	8.34	3.43	446	100
250 x 125	7.5	12.5	12	6	48.79	38.3	5180	337	10.3	2.63	414	53.9
250 x 125	10	19	21	10.5	70.73	55.5	7310	538	10.2	2.76	585	86
300 x 150	8	13	12	6	61.58	48.3	9480	588	12.4	3.09	632	78.4
300 x 150	10	18.5	19	9.5	83.47	65.5	12700	886	12.3	3.26	849	118
300 x 150	11.5	22	23	11.5	97.88	76.8	14700	1080	12.2	3.32	978	143
350 x 150	9	15	13	6.5	74.58	58.5	15200	702	14.3	3.07	870	93.5
350 x 150	12	24	25	12.5	111.1	87.2	22400	1180	14.2	3.26	1280	158
400 x 150	10	18	17	8.5	91.73	72	24100	864	16.2	3.07	1200	115
400 x 150	12.5	25	27	13.5	122.1	95.8	31700	1240	16.1	3.18	1580	165
450 x 175	11	20	19	9.5	116.8	91.7	39200	1510	18.3	3.6	1740	173
450 x 175	13	26	27	13.5	146.1	115	48800	2020	18.3	3.72	2170	231
600 x 190	13	25	25	12.5	169.4	133	98400	2460	24.1	3.81	3280	259
600 x 190	16	35	38	19	224.5	176	130000	3540	24.1	3.97	4330	373

Appendix 3

TCVN 7571-11:2019

Hot-rolled steel section - Part 11: U section



Appendix 3. Sectional properties of hot-rolled steel U sections (TCVN 7571-11:2019)

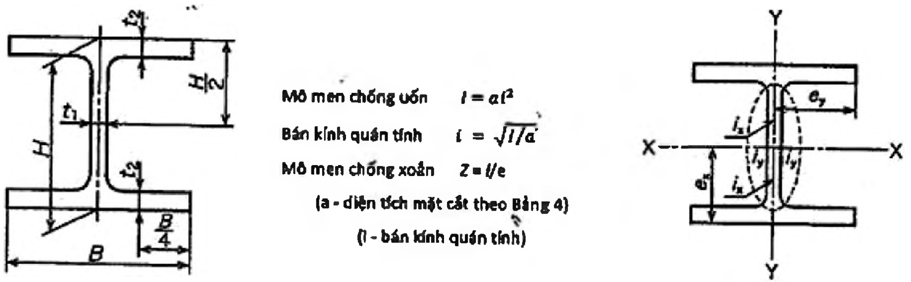
H x B	t_1	t_2	r_1	r_2	A cm^2	kg/m	C_y	I_x	I_y	i_x	i_y	W_x	W_y
75 x 40	5	7	8	4	8,818	6.92	1.28	75.3	12.2	2.92	1.17	20.1	4.47
80 x 45	5.5	7.5	8	4	10.5	8.23	1.43	102	18	3.12	1.3	25.6	5.85
100 x 50	5	7.5	8	4	11.92	9.36	1.54	188	26	3.97	1.48	37.6	7.52
120 x 55	6.3	8.5	8	4.5	16	12.5	1.6	350	39.5	4.68	1.57	58.4	10.1

125 x 65	6	8	8	4	17.11	13.4	1.9	424	61.8	4.98	1.9	67.8	13.4
140 x 60	6.7	9	9	4.5	19.2	15	1.68	570	55.3	5.45	1.67	81.4	12.8
150 x 75	6.5	10	10	5	23.71	18.6	2.28	861	117	6.03	2.22	115	22.4
	9	12.5	15	7.5	30.59	24	2.31	1050	147	5.86	2.19	140	28.3
160 x 65	7.2	10	9	5.5	23.2	18.2	1.81	900	79	6.22	1.81	113	16.8
180 x 75	7	10.5	11	5.5	27.2	21.4	2.13	1380	131	7.12	2.19	153	24.3
200 x 80	7.5	11	12	6	31.33	24.6	2.21	1950	168	7.88	2.32	195	29.1
200 x 90	8	13.5	14	7	38.65	30.3	2.74	2490	277	8.02	2.68	249	44.2
250 x 90	9	13	14	7	44.07	34.6	2.4	4180	294	9.74	2.58	334	44.5
	11	14.5	17	8.5	71.17	40.2	2.4	4680	329	9.56	2.54	374	49.9
300 x 90	9	13	14	7	48.57	38.1	2.22	6440	309	11.5	2.52	429	45.7
	10	15.5	19	9.5	55.74	43.8	2.34	7410	360	11.5	2.54	494	54.1
	12	16	19	9.5	61.9	48.6	2.28	7870	379	11.3	2.48	525	56.4
380 x 100	10.5	16	18	9	69.39	54.5	2.41	14500	535	14.5	2.78	763	70.5
	13	16.5	18	9	78.96	62	2.33	15600	565	14.1	2.67	823	73.6
	13	20	24	12	85.71	63.7	2.54	17600	655	14.3	2.76	926	87.8

Appendix 4

TCVN 7571-16:2017

Hot-rolled steel sections – Part 16: H sections



Appendix 4. Sectional properties of hot-rolled steel H sections (TCVN 7571-16:2017)

(H×B)	H	B	t_1	t_2	r	A	Mass per unit length	I_x	I_y	r_x	r_y	W_x	W_y
	mm					cm^2	kg/m	cm^4		cm		cm^3	
100×50	100	50	5	7	8	11.85	9.3	187	14.8	3.98	1.12	37.5	5.91
100×100	100	100	6	8	8	21.59	16.9	378	134	4.18	2.49	75.6	26.7
125×60	125	60	6	8	8	16.69	13.1	409	29.1	4.95	1.32	65.5	9.71
125×125	125	125	6.5	9	8	30	23.6	839	293	5.29	3.13	134	46.9
150×75	150	75	5	7	8	17.85	14	666	49.5	6.11	1.66	88.8	13.2
150×100	148	100	6	9	8	26.35	20.7	1000	150	6.17	2.39	135	30.1

150×150	150	150	7	10	8	39.65	31.1	1620	563	6.4	3.77	216	75.1
175×90	175	90	5	8	8	22.9	18	1210	97.5	7.26	2.06	138	21.7
175×175	175	175	7.5	11	13	51.43	40.4	2900	984	7.5	4.37	331	112
200×100	198	99	4.5	7	8	22.69	17.8	1540	113	8.25	2.24	156	22.9
	200	100	5.5	8	8	26.67	20.9	1810	134	8.23	2.24	181	26.7
200×150	194	150	6	9	8	38.11	29.9	2630	507	8.3	3.65	271	67.6
200×200	200	200	8	12	13	63.53	49.9	4720	1600	8.62	5.02	472	160
250×125	248	124	5	8	8	31.99	25.1	3450	255	10.4	2.82	278	41.1
	250	125	6	9	8	36.97	29	3960	294	10.4	2.82	317	47
250×175	244	175	7	11	13	55.49	43.6	6040	984	10.4	4.21	495	112
250×250	250	250	9	14	13	91.43	71.8	10700	3650	10.8	6.32	860	292
300×150	298	149	5.5	8	13	40.8	32	6320	442	12.4	3.29	424	59.3
	300	150	6.5	9	13	46.78	36.7	7210	508	12.4	3.29	481	67.7
300×200	294	200	8	12	13	71.05	55.8	11100	1600	12.5	4.75	756	160
300×300	300	300	10	15	13	118.5	93	20200	6750	13.1	7.55	1350	450
350×175	346	174	6	9	13	52.45	41.2	11000	791	14.5	3.88	638	91
	350	175	7	11	13	62.91	49.4	13500	984	14.6	3.96	771	112
350×250	340	250	9	14	13	99.53	78.1	21200	3650	14.6	6.05	1250	292
350×350	350	350	12	19	13	171.9	135	39800	13600	15.2	6.89	2280	776
400×200	396	199	7	11	13	71.41	56.1	19800	1450	16.6	4.5	999	145
	400	200	8	13	13	83.37	65.4	23500	1740	16.8	4.56	1170	174
400×300	390	300	10	16	13	133.3	105	37900	7200	16.9	7.35	1940	480

400×400	400	400	13	21	22	218.7	172	66600	22400	17.5	10.1	3330	1120
	414	405	18	28	22	295.4	232	92800	31000	17.7	10.2	4480	1530
	428	407	20	35	22	360.7	283	119000	39400	18.2	10.4	5570	1930
	458	417	30	50	22	528.6	415	187000	60500	18.8	10.7	8170	2900
	498	432	45	70	22	770.1	605	298000	94400	19.7	11.1	12000	4370
450×200	446	199	8	12	13	82.97	65.1	28100	1580	18.4	4.36	1260	159
	450	200	9	14	13	95.43	74.9	32900	1870	18.6	4.43	1460	187
450×300	440	300	11	18	13	153.9	121	54700	8110	18.9	7.26	2490	540
500×200	496	199	9	14	13	99.29	77.9	40800	1840	20.3	4.31	1650	185
	500	200	10	16	13	112.3	88.2	46800	2140	20.4	4.36	1870	214
500×300	482	300	11	15	13	141.2	111	58300	6760	20.3	6.92	2420	450
	488	300	11	18	13	159.2	125	68900	8110	20.8	7.14	2820	540
600×200	596	199	10	15	13	117.8	92.5	66600	1980	23.8	4.1	2240	199
	600	200	11	17	13	131.7	103	75600	2270	24	4.16	2520	227
600×300	582	300	12	17	13	169.2	133	98900	7660	24.2	6.73	3400	511
	588	300	12	20	13	187.2	147	114000	9010	24.7	6.94	3890	601
	594	302	14	23	13	217.1	170	134000	10600	24.8	6.98	4500	700
700×300	692	300	13	20	18	207.5	163	168000	9020	28.5	6.59	4870	601
	700	300	13	24	18	231.5	182	197000	10800	29.2	6.83	5640	721
800×300	792	300	14	22	18	239.5	188	248000	9920	32.2	6.44	6270	661
	800	300	14	26	18	263.5	207	286000	11700	33	6.67	7160	781
900×300	890	299	15	23	18	266.9	210	339000	10300	35.6	6.2	7160	687
	900	300	16	28	18	305.8	240	404000	12600	36.4	6.43	8990	842
	912	302	18	34	18	360.1	283	491000	15700	36.9	6.59	10800	1040
	918	303	19	37	18	387.4	304	535000	17200	37.2	6.67	11700	1140

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