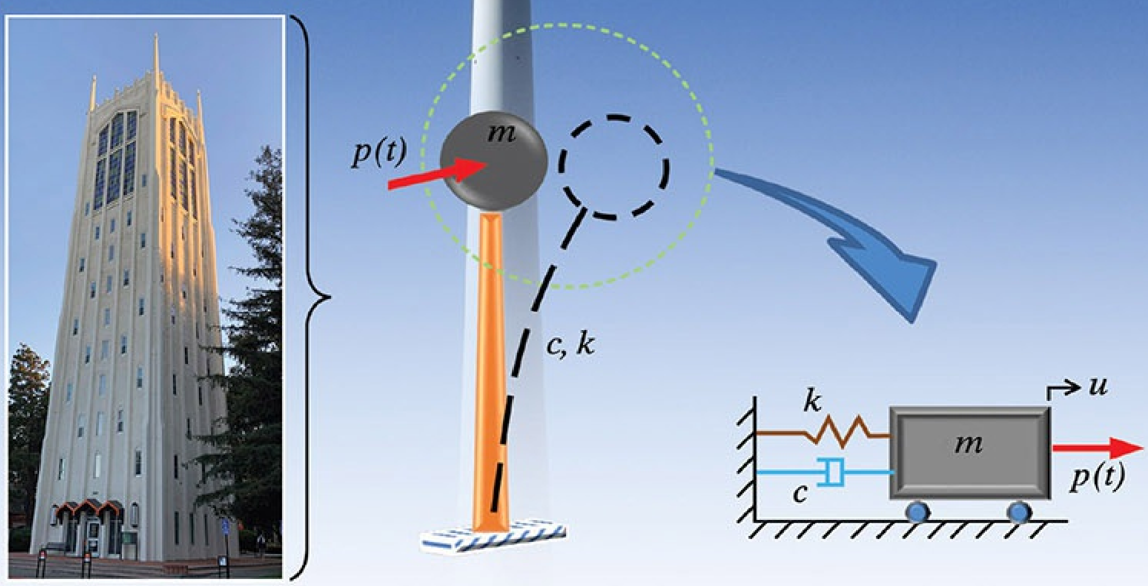
Essentials of

STRUCTURAL DYNAMICS

Contains electronic image galleries, PowerPoint presentations, MATLAB scripts, solutions manual, course outline (15weeks/10weeks), and a sample syllabus

Combines the fundamentals of structural dynamics and modern computational tools

Incorporates visual learning aids suitable for today's in-person and virtual classrooms



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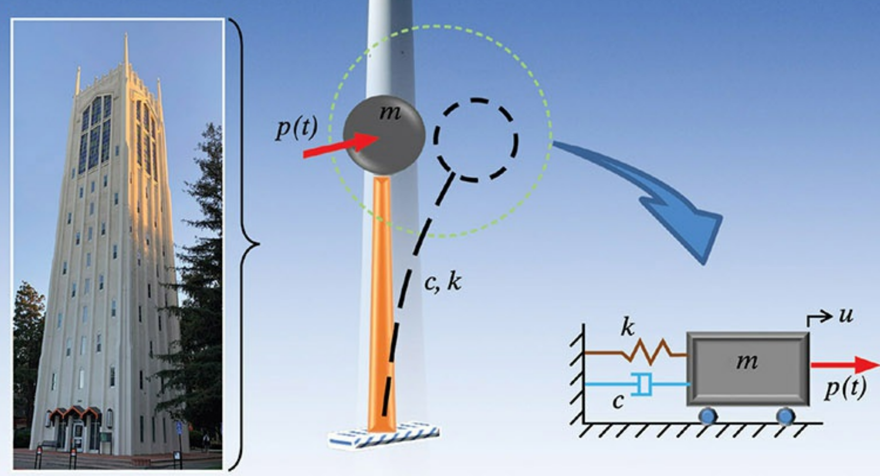

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Essentials of Structural Dynamics

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Contents

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Preface

Essentials of Structural Dynamics is intended to provide students and practitioners with a clear and concise presentation of structural dynamics. It begins with an overview of foundational concepts and methods (e.g., idealization of structures, degrees of freedom, determining mass and stiffness of structures and structural elements, types of force excitation, etc.) used throughout the book. In Chap. 2, we examine the formulation of the equation of motion for a single-degree-of-freedom (SDOF) oscillator along with the solution to the differential equation of motion for unloaded SDOF undamped and damped systems (free vibration response). Next, we present the solution for forced SDOF undamped and damped systems: harmonic forcing function in Chap. 3 and general forcing function in Chap. 4, where we also include several computational tools that are used to solve the complex mathematics involved in the solution of the equation of motion. In these first four chapters, the primary focus is on developing parameters to characterize structural system mass, stiffness, and damping, which, in turn, are used to calculate periods, frequencies, and other relevant dynamic properties such as dynamic internal forces in structures at a particular location.

We then connect the analysis of multi-degree-of-freedom (MDOF) systems to fundamental SDOF principles through the development of the generalized SDOF equations (Chap. 5). This generalized SDOF formulation can be combined with the principles associated with SDOF analyses to analyze MDOF systems, particularly multistory buildings, including finding maximum shear and bending moment in all members due to a dynamic load. The process entails computing approximate contribution factors to determine story displacements, which are then used to obtain nodal or lateral story dynamic forces that are necessary to perform a complete analysis of internal loading of the structural system.

The last part of the book, Chaps. 6 and 7, focuses on formulation and solution of the system of differential equations for MDOF systems. In Chap. 6 we focus on determining system vibration mode shapes and their associated natural periods, whereas Chap. 7 focuses primarily on a solution method known as modal analysis, which, along with the shock response spectra for general loading, can be used to establish the maximum structural response of building structures. With this maximum response, the maximum internal loads can be determined and used to design each member.

The book provides coverage of the essentials of structural dynamics, emphasizing the process of establishing and solving the equation of motion for various time-dependent loads. To that end, the reader is introduced to a wide variety of practical dynamics loading examples and end-of-chapter problems. Also, to guide the reader, each chapter begins with a list of learning outcomes. After carefully studying this textbook, a reader should be able to:

1. Demonstrate a basic understanding of the various applications of structural dynamics.
2. Identify and solve basic structural dynamics problems.
3. Perform dynamic analyses of structural systems subjected to complex dynamic loadings.

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The first objective relates to the student's ability to operate in the first two cognitive domains of Bloom's taxonomy (namely, knowledge of structural dynamics terms and comprehension of the various applications of structural dynamics). The second and third objectives primarily concentrate on the next two cognitive domains (namely, application of structural dynamics and analysis of systems subjected to various dynamic loadings). It is the hope of the authors that this along with the chapter learning objectives provides a roadmap to mastering the subject of structural dynamics.

Audience

The book is intended as a tool for a one-semester course in structural dynamics at the undergraduate or graduate levels, both in civil and architectural engineering. It provides students with an understanding of essential concepts and basic ideas relevant for solving many common and practical structural dynamics problems. Many solutions to example problems are prepared using computer programing methods (i.e., MATLAB) to enable students to solve the more complex dynamics problems. This book has also been written to support industry practitioners who are interested in gaining a working knowledge of structural dynamics and quickly gain insight into more complex analysis technics.

Acknowledgments

We would like to thank our families, especially our wives Wendy and Theresa, for their patience and understanding throughout the course of writing this book.

We would enjoy receiving any comments or suggestions from students, instructors, and practitioners regarding the contents of this book.

Hector Estrada
Luke S. Lee

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CHAPTER 1

Introduction

After reading this chapter, you will be able to:

a. Idealize structures
b. Explain the concept of degrees of freedom
c. Discretize members and systems using the lumped-mass and generalized displacement methods
d. Describe various time-dependent excitations
e. Describe the relationship between rigid-body dynamics and inertial forces and moments
f. Draw free-body diagrams of members and systems using D'Alembert's principle
g. Apply direct equilibrium to formulate the equation of motion for a single-degree-of-freedom (SDOF) system
h. Apply virtual work to formulate the equation of motion for a SDOF system
i. Determine the mass/weight of a structure
j. Determine the stiffness of a structure for various structural elements and components
k. Explain the process for determining internal forces and stresses in structural systems and members

In general, civil structural systems are not designed to move; therefore, motion of any part (or component) of the system is largely undesirable. This is entirely opposite to most mechanical engineering systems where motion is an integral part of the design. However, both disciplines cover the topic of vibration analysis following a similar approach; the subject is introduced using a simple oscillator supported by a spring and a dashpot and subjected to an initial displacement and velocity and/or a time-dependent force (or load). The vibration of the system is then characterized mathematically by writing a differential equation of motion, either following dynamic force or energy equilibrium. The solution to the differential equation is then developed analytically or numerically depending on the complexity of the load. The complexity of the solution increases exponentially when the analysis requires following several points on a system (or component), which is conducted by discretizing the system (or component) into nodes that are treated as separate oscillators. Each oscillator interacts with adjacent oscillators leading to a system of coupled differential equations, which in most practical cases is only tractable using computational methods.

The primary focus of this book is to present methods of analysis to determine deformations and stresses developed in structures when they are subjected to dynamic excitation, both from applied loading and support motion. In this context, the term dynamic implies time-varying, both excitation and response. For example, a dynamic load is any load with magnitude, direction, and/or position varying with time, which results in a response (deflections, stresses, etc.) that

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also varies with time. Therefore, structural dynamic problems require a succession of solutions corresponding to all times of interest in the response history, whereas structural analysis problems generally have a single static solution; so, structural dynamics methods are an extension of standard structural analyses methods. The main difference between the two methods is that static loading response only depends on the applied static loading (and can be determined using established principles of static equilibrium); whereas dynamic loading response depends not only on the applied dynamic loading, but also inertial forces developed to oppose the acceleration produced by the loading.

The inertial force is the most important characteristic when deciding if a problem is dynamic or static since all loads are applied over a finite period of time; slowly applied forces do not generate significant inertial forces. That is, if the inertial force is large compared to the magnitude of the applied time-dependent load, a large portion of this load must be resisted by the internal forces (axial, shear, and moment) and the dynamic character of the problem must be taken into account; however, when the inertial force is small compared to the magnitude of the applied force (motions are slow), the dynamic effect is negligible and the analysis can proceed using a static structural analysis at any desired time.

1.1 Idealization of Structures

It is impossible to perfectly model (physically or mathematically) any real engineering problem. Most dynamic analysis problems involve complex material behavior, loading, and supporting conditions, all of which make idealizations of the system necessary to be able to render the problem manageable; rigorous analysis is, however, possible using advanced computational tools such as finite element analysis or physical modeling, both of which are beyond the scope of this book. In this book we focus on mathematical models, which are a symbolic representation of the real physical system and include various simplifying assumptions. The primary objective of dynamics is to describe the motion of this mathematical model. An acceptable mathematical model must yield satisfactory approximate solutions from a safety and economic standpoint.

Consider a simply supported beam subjected to a time-dependent load that vibrates vertically; the vibration is opposed by an inertial force as the beam deflects; see Fig. 1.1. This force is directly opposite to the time-varying displacement and is proportional to the acceleration of the motion. Should we wish to determine the position of the beam at any point along the beam at any time, the analysis must then be formulated in terms of partial differential equations since both position and time are independent variables. Analytical closed-form solutions to this problem are difficult to ascertain except for relatively simple, well-defined loading cases. However, there are a number of computational methods that can be implemented to obtain solutions even for the most complex loadings. This, however, requires that we first discretize the problem into degrees of freedom.

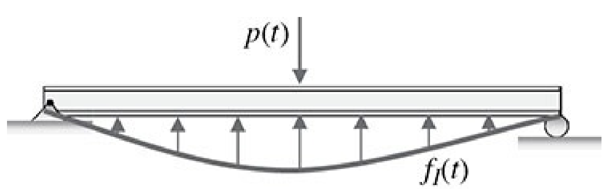

FIGURE 1.1 Inertial force in simply supported beam.

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1.2 Degrees of Freedom

To characterize the behavior of structural systems (both under static or dynamic loading), they must first be idealized as a system of connected parts, called elements or members. These members include beams, columns, and struts (in bracing systems) and can form various structural systems such as trusses and frames. To determine the displacements and internal loading necessary to design these structural systems under dynamic loading, we use structural analysis and vibration theory both of which underpin structural dynamics. The specific points on a structure used to establish displacements and internal loads in structural systems are known as nodes and are usually located at the ends of members where there are typically supports and connections (also known as joints).

A degree of freedom (DOF) in the context of structural dynamics is an independent displacement or rotation of a node to capture the effect of inertial forces. For example, consider the portal frame shown in Fig. 1.2 with rigid connections between columns and beam and fixed supports at the base of each column. The nodes at the supports and rigid connections can each be assigned three DOFs (in a two-dimensional system), for a total of 12 DOFs. However, since the nodes at the supports are "constrained" (since the base of the columns are modeled as fixed supports, their nodes are prevented from displacing or rotating), the number of DOFs can be reduced to six, three DOFs at each of the connections, since the beam and column ends at the connections must both move and rotate in sync.

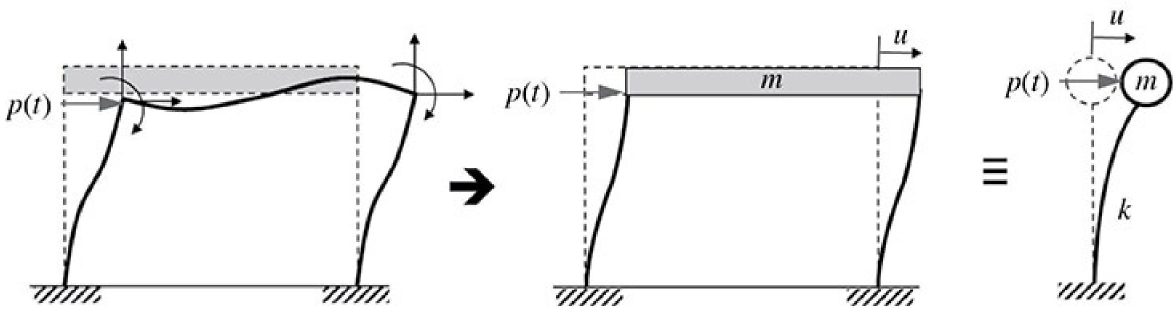

FIGURE 1.2 Idealized single degree-of-freedom (SDOF) system for a portal frame.

For our discussion in this section, the beam is assumed to be much stiffer than the columns, preventing rotations of the connections, eliminating two additional DOFs. Also, the two columns move in parallel, and if we assume that the axial deformations of the columns and beam are relatively small, we can reduce the entire system to a single-degree-of-freedom (SDOF) case as shown in Fig. 1.2; the lateral displacement of the beam, u. Also, in this model we assume that the entire mass of the frame is lumped at the top of the system. Furthermore, we assume that the stiffness of the frame is entirely contributed by the columns. Finally, note that there is no damping (i.e., energy dissipation) included in the system; without damping the system will move in perpetuity once set in motion. We will cover damping in later chapters.

As shown in Fig. 1.1, individual elements have distributed mass, which when set in motion develop a distributed inertial force. As noted earlier, the dynamic analysis of such system requires treating it as a *continuous system* with infinite number of DOFs. However, we can discretize the element into a discrete system with a finite number of DOFs using one of two procedures: the lumped-mass or the generalized displacement methods.

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1.2.1 Lumped-Mass Procedure

In the lumped-mass method the distributed element mass is concentrated at discrete points where inertial forces develop, as shown in Fig. 1.3 for the beam depicted in Fig. 1.1. Thus, independent displacements (and in some cases rotations) of these points characterize the DOFs necessary to specify the effect of the inertial force of the element. If the maximum amplitude of the displacement of the beam shown in Fig. 1.3 is relatively small, we can assume the displacements of the lumped masses to be linear and only three DOFs are required (i.e., one vertical DOF at each lumped mass). However, a large displacement of the beam would cause finite rotations of the masses (the segments of beam contributing to the lumped mass are not fully concentrated), and these rotations would have to be included as DOFs at each lumped mass for a total of six DOFs. This method is generally used in multi-degree-of-freedom (MDOF) cases where a large portion of the mass can be concentrated at discrete points to generate a more accurate effect of the inertial force, such as in shear buildings. This is the basis for the analysis of MDOF systems in Chaps. 6 and 7.

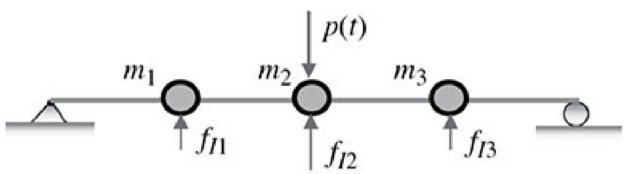

FIGURE 1.3 Simply supported beam and lumped-mass model.

1.2.2 Generalized Displacements

The generalized displacement method is most effective in modeling cases where the distributed mass is relatively uniform, and its dynamic deflected shape can be expressed as the sum of a series of specified deformed shapes of the element as shown in Fig. 1.4. Generally, we can use any shape $\psi_N(x)$ that is compatible with the geometric supports. For example, the cantilever tower shown in Fig. 1.4 has no rotation or displacement at the support so the generalized displacement shapes shown must conform to these conditions. This method is typically used in a generalized degree-of-freedom analysis of elements with distributed mass and stiffness, which is covered in Chap. 5.

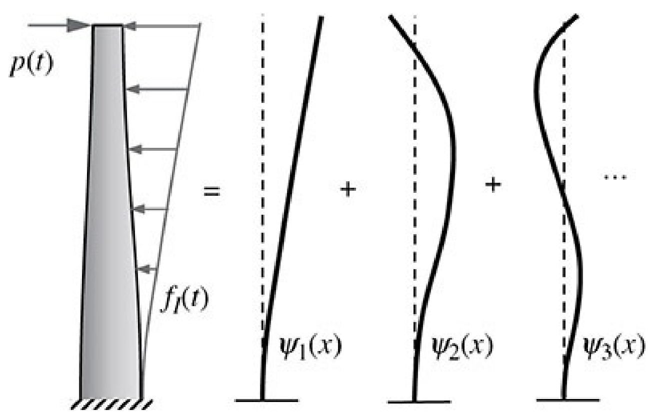

FIGURE 1.4 Cantilever tower and generalized displacements also known as structural mode shapes.

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1.3 Time-Dependent Excitations

Time-dependent excitations can be divided into two general categories: loads applied directly to a structure and motion of the support of the structure. Examples of applied loads include impact from collisions, wind-induced pressures, blast-induced pressures, wave forces on coastal and ocean structures, etc. Examples of support motion include earthquake-induced accelerations, mechanical equipment-induced vibrations, ground shock, etc. In general, we have two approaches to evaluate the response of structures to these types of loads: deterministic and nondeterministic. Deterministic analysis is used when we can assume the time-dependent excitations are known to a high degree of certainty. Whereas nondeterministic analysis is used when the time or position variations of the excitations can only be characterized in a statistical sense; that is, the excitation is random in time and/or position. In this book we focus on deterministic excitations, both applied loads and support motions. This type of excitation is further subdivided into *periodic* and *nonperiodic*.

Periodic excitations exhibit the same time variation successively for a large number of cycles; this type of loading can be represented as the sum of a series of simple harmonic (sine and cosine) components. For example, the excitation caused by an unbalanced rotating machine in a building can be represented using a simple sinusoidal variation as shown in Fig. 1.5. Nonperiodic excitation can be of short-duration (impulsive) or long-duration. For example, an air blast load can be represented as an impulsive loading as shown in Fig. 1.6.

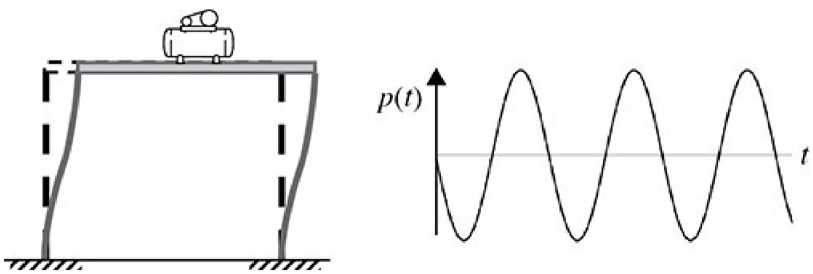

FIGURE 1.5 Unbalanced rotating machine and simple sinusoidal excitation.

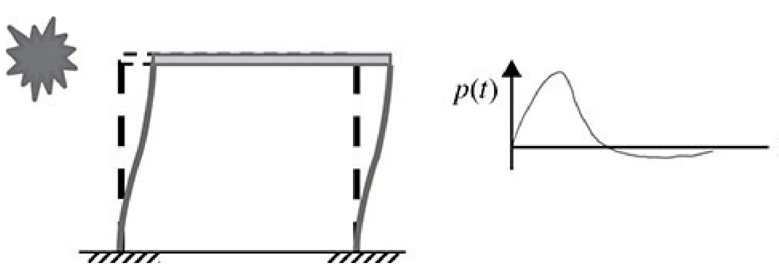

FIGURE 1.6 Air blast load and impulsive excitation.

1.4 Rigid-Body Dynamic Equilibrium

To characterize the behavior of structural systems under the action of various forces, we need fundamental analytical science principles. These principles have been developed under the umbrella of a branch of physical science known as *mechanics*. This subject is rather broad and is found in many fields of engineering. In civil engineering, mechanics focuses on structural mechanics, which describes and predicts the state of rest or motion of bodies subjected to forces. Structural mechanics is divided into two general categories: mechanics of rigid bodies and mechanics of deformable bodies. In this section we focus on the rigid-body mechanics while the rest of the book covers deformable bodies.

Rigid-body mechanics is subdivided into *statics*, which deals with equilibrium of objects at rest, and *dynamics*, which deals with objects in motion. Statics is based on Newton's first law, which states that if the resultant of a system of forces acting on a body is zero, the body will remain at rest (if originally at rest) or in uniform motion in a straight line (if originally in constant motion). Rigid-body dynamics forms the basis of vibration theory and is based on Newton's second law, which states that if the resultant of a system of forces, $\boldsymbol{F}$, acting on a particle of mass, m, is not zero, the particle will experience an acceleration, $\boldsymbol{a}$, proportional to the resultant force, $\boldsymbol{F} = m\boldsymbol{a}$. This is a vector relationship and gives the analytical relation between motion response of a *particle* and time. The motion of a rigid body can be described by two vector equations, one relating the force to the linear acceleration of the center of gravity, G, and the other relating the moments to the angular motion of the body. In plane motion (the x-y plane) the vector quantities reduce to the following three scalar relationships:

$$
\begin{aligned}
\sum F_x &= m a_{Gx} \\
\sum F_y &= m a_{Gy} \\
\sum M_G &= I_G \alpha
\end{aligned}
\tag{1.1}
$$

where

a_{Gy} and a_{Gx} are the accelerations of the center of mass G of the body.

α is the angular acceleration.

I_G is the mass moment of inertia with respect to an axis through G. This can be obtained from various statics textbooks or by the product of the square of the polar radius of gyration, $I_G = m r_p^2$ (where I_p is the polar moment of inertia, and A the cross-sectional area) and the total mass, $r_p = \sqrt{I_p / A}$.

In structural dynamics, the last of Eqs. (1.1) is seldom needed because we assume small rotations of distributed mass components. Also, the right-hand side of Eqs. (1.1) are time dependent owing to the fact that accelerations are second derivatives of displacements (or rotations) with respect to time. These relationships form the basis for dynamic equilibrium where the mass is assumed to develop an inertial force proportional to the acceleration but opposite in direction; the principle is known as D'Alembert's principle and is a convenient approach to change the effect of the acceleration of the mass into an equivalent force, the inertial force.

Although statics can be considered a special case within the area of dynamics (zero acceleration case), the subject is usually treated separately because it forms the foundation for structural analysis, which uses static equilibrium to determine the effect of applied static loading

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on structural systems and components. Static equilibrium along with D'Alembert's principle can be used to formulate dynamic equilibrium, which is used to obtain the equation of motion.

Example 1

Consider a rigid column having mass, *m*, length, *L*, and constrained to move horizontally at the top end by a spring having a stiffness *k* as shown below. Draw a free-body diagram (FBD) of the column after a time-dependent horizontal force is applied and formulate the equation of motion using D'Alembert's principle.

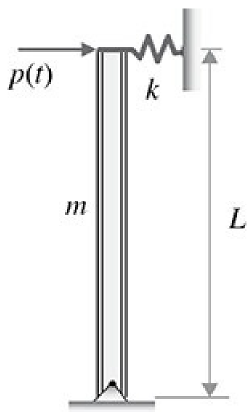

FIGURE E1.1 Rigid column model.

Solution

i. *Draw the FBD including the inertial force and moment.* The time-dependent force causes both inertial force and inertial moment at the center of gravity, *G*, which is halfway up the column, at *L*/2. For this case, we will formulate the equation of motion in terms of the rotation of the column about the pin support, θ, which is assumed to be small; thus, the displacement at the top, Δ can be approximated as the product of θ and the length, *L* (small angle approximation yields $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$). Also, the linear acceleration at *G* is proportional to the rotation, θ, and *L*/2, such that $\ddot{u}_x = \ddot{\theta}L/2$. Furthermore, the angular acceleration, is equal to the second derivative of the rotation with respect to time, $\alpha = \ddot{\theta}$. Lastly, the mass moment of inertia of the beam with respect to an axis through *G*, I_G can be obtained as follows:
The square of the radius of gyration about centroidal *x* axis for a rectangular rod with dimensions *b* and *L*:

$$r_{px}^2 = I_p / A = (bL^3 + Lb^3)/12/(bL) = L^2/12$$

Since *b* is much smaller than *L*, the polar mass moment of inertial about *G* is $I_G = mr_{px}^2 = mL^2/12$

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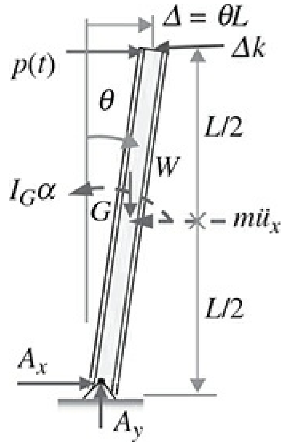

FIGURE E1.2 FBD of rigid column model.

ii. *Formulate the equation of motion by applying Eqs. 1.1 and D'Alembert's principle.* Sum forces along *x* and *y* and take moments about support *A*.

$$\sum F_x = 0;\quad A_x - m\ddot{u}_x + p(t) - \quad k = 0 \qquad A_x = \frac{m\ddot{\theta}L}{2} + \theta Lk - p(t)$$

$$\sum F_y = 0;\quad A_y - W = 0 \qquad A_y = W$$

$$\sum M_A = 0;\quad m\ddot{u}_x\ \frac{L}{2}\ \ - W\ \frac{\quad}{2}\ \ + I_G\alpha - Lp(t) + L(\quad k) = 0$$

$$\frac{m\ddot{\theta}L^2}{4} - \frac{W\theta L}{2} + \frac{mL^2\ddot{\theta}}{12} + \theta L^2 k = Lp(t)$$

This last equation leads to the equation of motion, which can be written in terms of the rotation,

$$\frac{mL}{3}\ddot{\theta} + \ Lk - \frac{W}{2}\ \ \theta = p(t)$$

Note that for this case the weight, *W* acts as a destabilizing force and must be considered in the equation of motion; when the weight equals to 2*Lk*, the column becomes unstable, and it is said to buckle. ▲

1.5 Deformable-Body Dynamic Equilibrium

As noted earlier, because of the complexity of most practical structural dynamics problems, they must be solved using approximate analyses, involving only a limited number of DOFs. The number of DOFs should be chosen such that the simplification provides sufficient accuracy from an economic and safety standpoint. Fortunately, many practical problems can be solved using a single DOF as will be shown in Chaps. 2, 3, and 4. For more complex cases, we can use a generalized SDOF analysis covered later in this chapter and in Chap. 5, or a MDOF analysis covered in Chaps. 6 and 7. But even complex cases can be formulated into a series of SDOF cases using a technique known as modal analysis. In all cases we must first start by obtaining the equations of motion using dynamic equilibrium either directly or by virtual work, both of which are introduced in this section.

As shown in Fig. 1.2 for a SDOF case, the response of the analytical model is usually determined in terms of displacements. To establish the expressions governing the response, we use all external forces (including the inertial force developed from Newton's second law), as well as internal forces that react the applied and inertial forces (which are not applicable in rigid-body equilibrium). The source of these internal forces is the system's inherent stiffness and its ability to dissipate energy (i.e., damping). The expression of equilibrium results in the *equation of motion* of the structure, the solution of which provides the response time history, usually in terms of displacement, but can also be written in terms of velocities or accelerations. The formulation of the equation of motion is the most important (and at times the most difficult) part of the entire dynamic analysis process; in general, it consists of four force terms: inertial, damping, stiffness, and excitation forces.

Consider the simplified SDOF system of the frame depicted in Fig. 1.2 and draw the FBD of the lollipop mass, as shown in Fig. 1.7. The mass can be considered a particle with negligible rotation. Intuitively the column of the lollipop provides an elastic restoring force, which can be characterized by a spring force. Recall that the restoring force in a spring is equal to the product of the elongation of the spring and the spring constant (the slope of force-displacement line, which is the stiffness, or the force required to cause a unit elongation). For this example, assume the column stiffness is k, which is a function of the column cross section and material properties (as will be shown later in this chapter). The restoring (or stiffness) force is the internal shear force in the column, V, which is the product of the displacement, u, and the stiffness, k, as shown in Fig. 1.7.

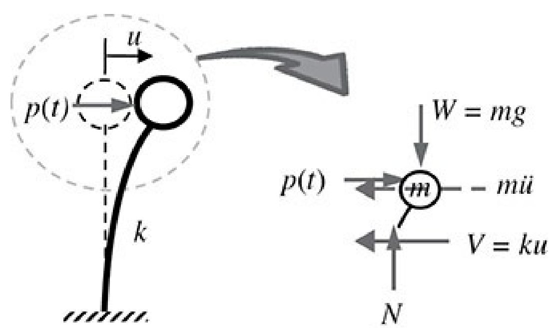

FIGURE 1.7 Free-body diagram of idealized portal frame in Fig. 1.2.

This system can be made to vibrate by applying initial displacement and/or velocity or a time-dependent excitation. The most general case can be subjected to all three forcing conditions simultaneously: initial displacement, initial velocity, and a time-dependent excitation. Application of initial conditions results in free vibration response (covered in Chap. 2) and are not included in the formulation of the equation of motion; they are implemented in the solution. Time-dependent excitation (forcing function, $p(t)$, or support motion) results in a forced response and is included in the equation of motion. The equation of motion can be formulated using several procedures; in this book we focus on direct equilibrium and the principle of virtual work, both of which use D'Alembert's principle.

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1.5.1 Direct Equilibrium

For this procedure, we need to first draw a detailed FBD, which is a sketch of the body in question isolated from all other bodies and supports showing all forces on the body. With a FBD, we can formulate the equation of motion by applying equilibrium using D'Alembert's principle by reframing dynamic equilibrium [Eqs. (1.1)] into static equilibrium. Assuming the FBD shown in Fig. 1.7 can be treated as a particle, we can write two equations of equilibrium:

$$\begin{aligned} &\overset{+}{\rightarrow} \sum F_x = 0; \quad -m\ddot{u} - ku + p(t) = 0 \Rightarrow m\ddot{u} + ku = p(t) \\ &+\!\uparrow \sum F_y = 0; \quad -W + N = 0 \Rightarrow N = mg \end{aligned} \tag{1.2}$$

where

$m\ddot{u}$ is the inertial force.
ku is the force due to the lateral deformation, u in the columns.
k is the lateral stiffness, discussed in more detail later.
$W = mg$ is the force due to gravity, g.
N is the axial force in the column.

The double dot over the u indicates double differentiation with respect to time; that is,

$$\ddot{u} = \frac{d^2u}{dt^2} \tag{1.3}$$

Horizontal equilibrium in Eq. (1.2) yields the equation of motion, which is a second-order, linear, and homogeneous differential equation with constant coefficients. In many of the problems presented in this book, this direct approach is the most convenient way to formulate the equation of motion.

Example 2
Consider a cantilever beam, which can be idealized as shown below. Draw a FBD of the mass after a time-dependent vertical force is applied and formulate the equation of motion using direct equilibrium. Also, describe the influence of gravitational force (the weight of the body) on the system. Notice that there is an initial static displacement, δ, caused by the weight.

FIGURE E2.1 Idealized cantilever beam model.

Solution

i. *Draw the FBD including the inertial force and the weight.* The time-dependent force causes additional displacement, y, which is combined with the static displacement, δ, for a total displacement, u.

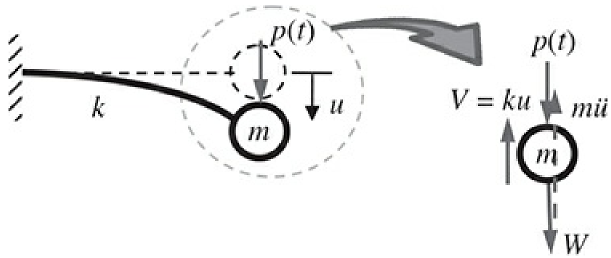

FIGURE E2.2 FBD of idealized cantilever beam model.

ii. *Formulate the equation of motion by applying vertical equilibrium.*

$$\overset{+}{\uparrow} \sum F_y = 0; \quad m\ddot{u} + ku - W - p(t) = 0 \qquad m\ddot{u} + ku = p(t) + W$$

Now express the total displacement, *u*, as the sum of the static displacement, *δ*, and the additional dynamic displacement, *y*,

$$u = \delta + y$$

And rewrite the equation of motion in terms of *δ* and *y*,

$$m\ddot{y} + k(\delta + y) = p(t) + W$$

Since the static displacement is constant in time, $m\ddot{u} = m\ddot{y}$. Also, the weight *W* causes the initial displacement, so $k\delta = W$; thus, the equation of motion can be written in terms of the dynamic displacement as,

$$m\ddot{y} + ky = p(t)$$

iii. *What influence does weight, W, have on the system?* The last equation is the same as the equation of motion for a case without gravity. Therefore, any equation of motion expressed with reference to the static equilibrium position of the system is unaffected by gravitational forces. Thus, from here on, displacements will be referenced from the static equilibrium position regardless of the notation (we will mostly use *u* to indicate dynamic displacement). Consequently, when appropriate, total displacements will be obtained by adding the corresponding static quantities to the result of the dynamic analysis. ▲

1.5.2 Principle of Virtual Work

Virtual work is an alternate method for solving equilibrium problems when direct equilibrium is difficult to apply; particularly, for complex systems involving several interconnected DOFs where the principle of virtual work greatly simplifies the formulation of the equation of motion. This principle states that a body in equilibrium under the action of external forces subjected to a virtual (imaginary) displacement, δu, that is compatible with the body's constraints, the total work done by these forces is zero. This can also be stated as the virtual work of all external forces and couples, δW_E, must be equal to the virtual work of all internal forces, δW_I, for a

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given virtual displacement of the body. That is,

$$\delta W_I = \delta W_E \tag{1.4}$$

Recall that the work of a force is equal to its magnitude times the component of displacement in the direction of the force. When both the force and displacement are given as vectors, we can obtain the work using the dot product of the two vectors. This results in a scalar quantity that is positive when both force and displacement are in the same direction and negative when they are in opposite directions. Similarly, the work of a moment is the moment magnitude times rotation in the direction of the moment. The major advantage of this method is that the virtual work contributions are scalar quantities and can be added algebraically, versus direct equilibrium where forces must be combined vectorially. See Hibbeler Engineering Mechanics: Statics, Chap. 11 for further details.

Example 3

Consider the cantilever beam given in Example 2 and draw a FBD of the mass after a time-dependent vertical force is applied and formulate the equation of motion using virtual work.

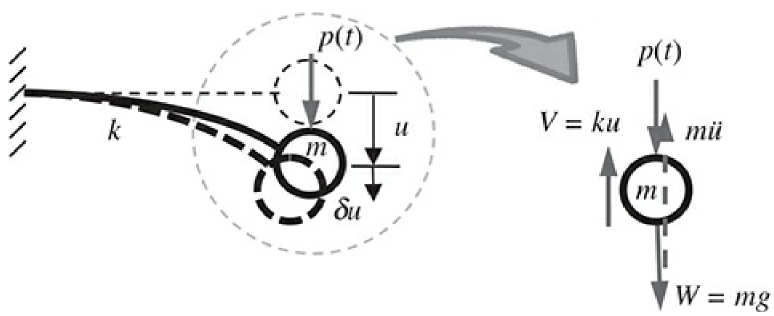

FIGURE E3.1 Idealized cantilever beam model subjected to virtual displacement and FBD.

Solution

i. *Apply a virtual displacement to the beam model and draw the FBD.* The time-dependent force causes additional displacement, *y*, which is combined with the static displacement, *δ*, for a total displacement, *u*.
ii. *Formulate the equation of motion by applying virtual work.* The virtual work of the external forces, inertial, weight, and applied forces is

$$\delta W_E = \delta u[p(t)] + \delta u(W) - \delta u(m\ddot{u})$$

The virtual work of the internal force, stiffness is

$$\delta W_I = -\delta u(ku)$$

We can now set these two equations equal [Eq. (1.4)]:

$$\delta W_E = \delta W_I$$
$$\delta u(p(t)) + \delta u(W) - \delta u(m\ddot{u}) = -\delta u(ku)$$

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Thus, since the virtual displacement is chosen arbitrarily and not equal to zero,

$$p(t) + W - m\ddot{u} = -ku$$

or

$$m\ddot{u} + ku = p(t) + W$$

Which is the same as the equation we obtained in Example 2 and can be further simplified by expressing the total displacement, u, as the sum of the static displacement, δ, and the additional dynamic displacement, y,

$$u = \delta + y$$

Thus, the equation of motions in terms of δ and y is

$$m\ddot{y} + k(\delta + y) = p(t) + W$$

Again, since the static displacement is constant in time, $m\ddot{u} = m\ddot{y}$. Also, the weight W causes the initial displacement, so $k\delta = W$; thus, the equation of motion can be written in terms of the dynamic displacement as,

$$m\ddot{y} + ky = p(t) \quad \blacktriangle$$

Example 4

Solve Example 1 using virtual work, See Fig. E4.1.

Solution

i. *Apply a virtual displacement to the beam model and draw the FBD.* Recall that the time-dependent force causes both inertial force and inertial moment at the center of gravity, G, which is halfway up the column at $L/2$. For this case, we will formulate the equation of motion in terms of the rotation of the column about the pin support, θ, and the displacement at the top, $\Delta = \theta L$. Also, the linear acceleration at G is proportional to the rotation, θ, and $L/2$, such that $\ddot{u}_x = \ddot{\theta}L/2$. Also, the angular acceleration is equal to the second derivative of the rotation with respect to time, $\alpha = \ddot{\theta}$. Lastly, mass moment of inertia of the beam with respect to an axis through G is

$$I_G = mr_{px}^2 = \frac{mL^2}{12}$$

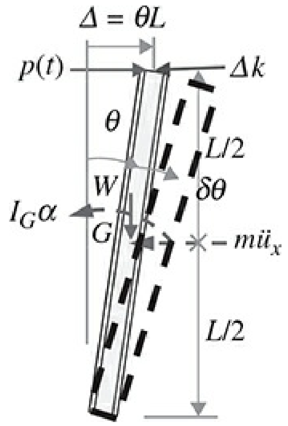

FIGURE E4.1 FBD of rigid column model subjected to virtual displacement.

Notice that the weight, W moves a distance equal to $y = L(1-\cos\theta)/2$, which has a total derivative of $\delta y = L\,\sin(\delta\theta)/2$; for small rotations, $\delta y = L\delta\theta/2$.

ii. *Formulate the equation of motion by applying virtual work.* The virtual work of the external forces, inertial, weight, and applied forces is

$$\delta W_E = \delta\ \ (p(t)) - \delta\ \ (\ \ k) - m\ddot{u}_x\ \frac{\delta}{2}\ \ - \delta\theta(\alpha I_G) + W\delta y$$

$$= L\delta\theta(p(t)) - L\delta\theta(L\theta k) - m\ddot{u}_x \frac{L\delta\theta}{2} - \delta\theta\frac{mL^2\ddot{\theta}}{12} + W\frac{L\delta\theta}{2}$$

The virtual work of the internal force is zero since we have a rigid body.

$$\delta W_I = 0$$

We can now set these two equations equal [Eq. (1.4)]:

$$\delta W_E = \delta W_I$$

$$L\delta\theta(p(t)) - L\delta\theta(L\theta k) - m\ddot{\theta}L\ \frac{L\delta\theta}{2}\ \ - \delta\theta\frac{mL^2\ddot{\theta}}{12} + W\frac{L\delta\theta}{2} = 0$$

Thus, since the virtual displacement is chosen arbitrarily and not equal to zero,

$$p(t) - \theta Lk - \frac{\ddot{\theta}mL^2}{2} + \frac{mL^2\ddot{\theta}}{12} + \frac{W}{2} = 0$$

or

$$\frac{mL}{3}\ddot{\theta} +\ \ Lk - \frac{W}{2}\ \ \theta = p(t)$$

which is the same result as Example 1. ▲

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1.6 Introduction to Generalized Single Degree of Freedom Analysis

In this section we provide the foundation for the generalized SDOF analysis that is addressed in more detail for MDOF and continuous systems in Chap. 5. First, we present details for how a system's inherent properties fundamentally characterize the vibration behavior of a structural system. As noted earlier, in this chapter we have focused on two of the three inherent properties used to define the equation of motion: mass and stiffness; the other is damping and will be covered in later chapters. In general, systems and components possess distributed properties (with infinite DOFs); however, as discussed earlier, we can simplify most practical cases into systems with a finite number of DOFs. The simplest case is the idealization of a system with localized mass and localized stiffness; this model is known as an oscillator and is depicted in Fig. 1.8. Unfortunately, this simplification is not possible for most practical systems with distributed properties; thus, we use one of three possible cases, all of which can still be reduced to a generalized SDOF case:

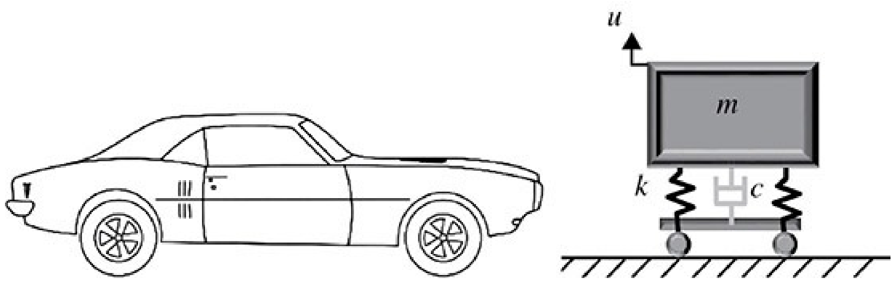

FIGURE 1.8 System with localized mass and localized stiffness known as an oscillator.

1. A system with localized mass but distributed stiffness; see Fig. 1.9.

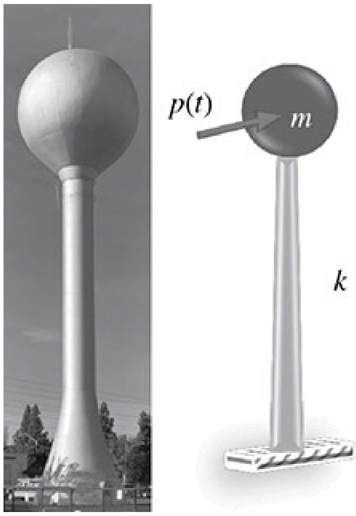

FIGURE 1.9 System with localized mass and distributed stiffness known as a lollipop.

2. A system with distributed mass but localized stiffness; see Fig. E1.1, which shows a rigid body system with a localized spring.

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3. And the most general, a system with distributed mass and distributed stiffness; see Fig. 1.10.

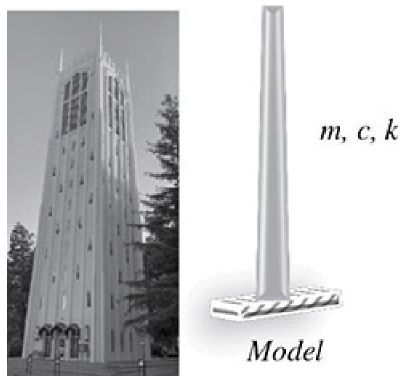

FIGURE 1.10 System with distributed mass and distributed stiffness.

As discussed in Sec. 1.2, it is necessary to reduce a problem with distributed properties to one having a finite number of DOFs. For distributed mass cases, we can lump the mass for the entire system, or various segments of the system into equivalent masses as shown in Fig. 1.3. For distributed stiffness cases, we use the definition of stiffness (the force required to cause a unit displacement) to establish a lumped parameter as shown in Fig. 1.2. These procedures work for the relatively simple cases discussed in this section; however, for more complex cases, we need a formalized procedure to lump the properties (and to analyze the resulting system), which is covered in Chap. 5. We can write the equation of motion once the distributed properties are "lumped," consisting of a rigid mass supported by a single flexible member that can be treated as an equivalent spring, with its mass lumped with the rigid mass. In the following sections we discussed simple procedures used to obtain the lumped properties.

1.6.1 Lumped Structural Mass/Weight

Equations (1.1) show that inertial force and inertial moment are intrinsically caused by the mass, which, in turn, is directly proportional to the weight; mass is equal to weight divided by the acceleration due to gravity, $g = 386.4$ ft/s^2. In order to lump the mass at the center of gravity of the body, we must determine its magnitude and mass moment of inertia, which is required in cases where the rotational inertial effects need to be included. For example, establishing the weight of a building system requires knowledge of its geometry and any possible additional loading. With the known geometry and material of the building, we can determine its weight from its various structural elements and weight of any objects permanently attached to the structure, such as walls, roofs, ceilings, and equipment: HVAC systems, plumbing fixtures, etc. During preliminary design, we must estimate the building system weight from experience or the following average weights: for timber structures use 40 to 50 lb/ft^2; for steel structures use 60 to 75 lb/ft^2; and for concrete structures use 110 to 130 lb/ft^2. These weights may have to be revised several times during the design/analysis process.

Additional guidelines for estimating weight of structural systems are provided in various design codes and standards. For example, for seismic analysis of building structures, the weight is defined as the total effective weight by ASCE-7 "Minimum Design Loads and Associated Criteria for Buildings and Other Structures" standard as,

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$$W_D = \mathrm{DL} + 0.25\,\mathrm{StL} + \mathrm{PL} + \mathrm{WPE} + 0.2\,\mathrm{SL} + \mathrm{LaL} \tag{1.5}$$

where

DL is the dead load of the structural system that is tributary to each floor.
StL is the storage load, which is a live load in areas used for storage, such as a warehouse.
PL is the larger of actual partition load or 10 psf.
WPE is the operating weight of permanent equipment.
SL is the flat roof snow load when it exceeds 30 psf, regardless of roof slope.
LaL is the landscape loads associated with roof and balcony gardens.

Once the various weights are estimated in a multistory building, we can determine the tributary weight of a floor or roof as shown in Fig. 1.11. The weight of each floor and the tributary wall loads halfway between adjacent floors is assumed to be concentrated or lumped at each floor level. The story weight, W_x, for a specific story as shown in Fig. 1.12 is given as,

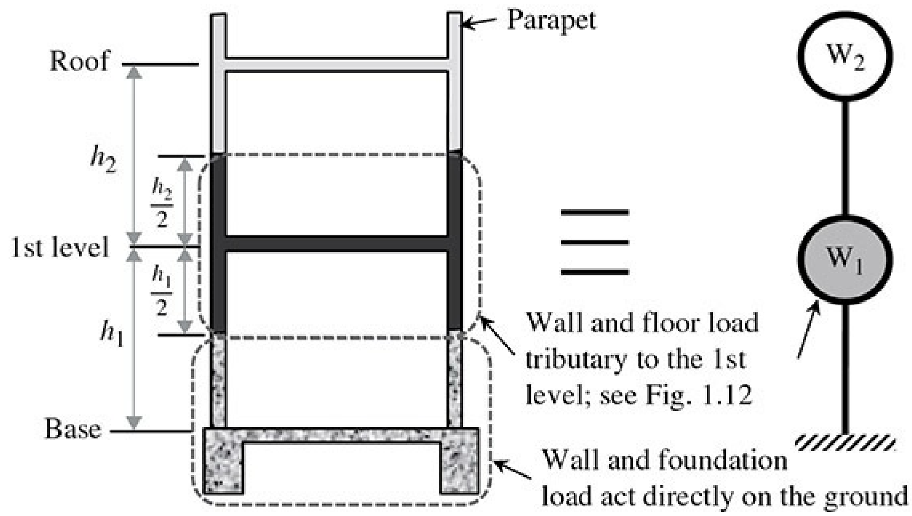

FIGURE 1.11 Building structure and lumped mass simplified model.

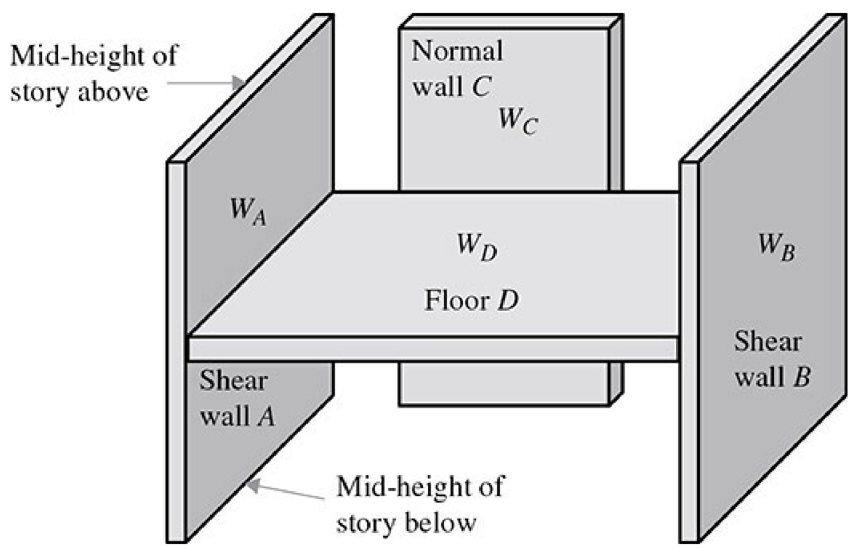

FIGURE 1.12 Story weight, W_x.

$$W_x = \text{walls}\,(A, B, C) + \text{floor}\,(D) = W_A + W_B + W_C + W_D \tag{1.6}$$

where W_D is given by Eq. (1.5). In general, as shown in Fig. 1.11, foundation weight and half of the first story wall weight are assumed to act directly on the ground and are commonly omitted in load calculations (though common practice, this is not an explicit provision in seismic design specifications).

Example 5

Determine the effective weight at the roof and first floor for the given rectangular (30 by 60 ft) two-story warehouse building (under construction) and the following loading:

Roof DL = 50 psf
Floor DL = 65 psf
Wall DL = 25 psf
Floor LL = 200 psf (storage)

Solution

i. *Identify the effective height of the first floor and roof to calculate weight of the walls at each level.* The effective height of the first level is the sum of the heights halfway to the adjacent levels as shown in Fig. E5.1—the tributary height.

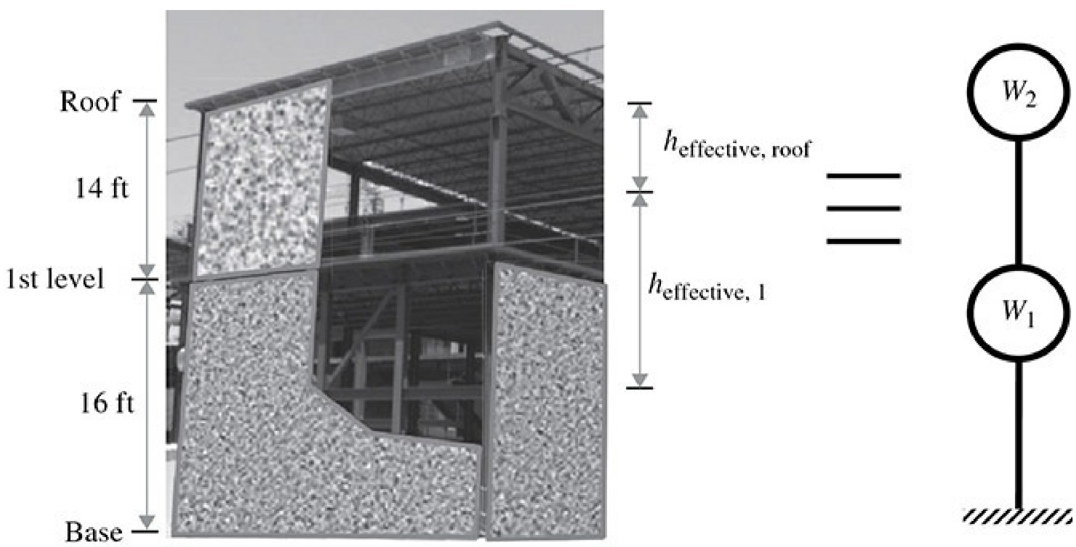

FIGURE E5.1 Two-story building and idealized structural model.

$$h_{\text{effective},1} = \frac{h_1}{2} + \frac{h_2}{2} = \frac{16\text{ ft}}{2} + \frac{14\text{ ft}}{2} = 15\text{ ft}$$

The effective height of the roof is the height halfway to the roof from first level.

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$$h_{\text{effective,roof}} = \frac{h_2}{2} = \frac{14\text{ ft}}{2} = 7\text{ ft}$$

ii. *Calculate individual weights of components associated with each level of interest.* For the first level, the weight includes dead loads from the walls and floor, as well as a portion of the storage load. For the roof, the weight includes dead loads from the walls and roof only.

First level component weights:
Total weight of walls:

$$\begin{aligned} W_{\text{walls}} &= \text{Area}_{\text{walls}} \times \text{wall DL} \\ &= (2\text{ walls} \times 30\text{ ft} + 2\text{ walls} \times 60\text{ ft})(15\text{ ft})(25\text{ psf}) = 67,500\text{ lb} \end{aligned}$$

The first level dead load:

$$W_{\text{FloorDL}} = \text{Area}_{\text{Floor}} \times \text{Floor DL} = (30\text{ ft} \times 60\text{ ft}) \times 65\text{ psf} = 117,000\text{ lb}$$

25% of the storage live load:

$$W_{\text{FloorLL}} = \text{Area}_{\text{Floor}} \times 0.25 \times \text{StL} = (30\text{ ft} \times 60\text{ ft}) \times 0.25 \times 200\text{ psf} = 90,000\text{ lb}$$

Roof level component weights:
Total weight of walls:

$$\begin{aligned} W_{\text{walls}} &= \text{Area}_{\text{walls}} \times \text{wall DL} \\ &= (2\text{ walls} \times 30\text{ ft} + 2\text{ walls} \times 60\text{ ft})(7\text{ ft})(25\text{ psf}) = 31,500\text{ lb} \end{aligned}$$

The roof dead load:

$$W_{\text{roofDL}} = \text{Area}_{\text{roof}} \times \text{Roof DL} = (30\text{ ft} \times 60\text{ ft}) \times 50\text{ psf} = 90,000\text{ lb}$$

iii. *Determine the effective seismic weight for each level.* The effective seismic weight for each level can be determined by summing the weight of the components.
Level 1 is denoted as W_1:

$$\begin{aligned} W_1 &= W_{\text{walls}} + W_{\text{FloorDL}} + W_{\text{FloorLL}} \\ W_1 &= 67{,}500\text{ lb} + 117{,}000\text{ lb} + 90{,}000\text{ lb} = 274{,}500\text{ lb} = 274.5\text{ kip} \end{aligned}$$

Roof level is denoted as W_2:

$$\begin{aligned} W_2 &= W_{\text{roof}} = W_{\text{walls}} + W_{\text{roof DL}} \\ W_2 &= 31{,}500\text{ lb} + 90{,}000\text{ lb} = 121{,}500\text{ lb} = 121.5\text{ kip} \quad \blacktriangle \end{aligned}$$

1.6.2 Lumped Structural Stiffness of Members

As noted earlier, establishing a lumped stiffness for simple cases can be accomplished using the definition of stiffness; that is, the force or moment that results in a unit displacement or rotation at a DOF, or the ratio of force (or moment) to displacement (or rotation). The lumped stiffness can be characterized once the component (or system) material and cross-sectional geometric properties are established. With the lumped mass and stiffness, we can completely characterize the equation of motion for a linear elastic SDOF system. We provide several examples of the process for determining the lumped stiffness in the remainder of this section. We also provide approximate relationships that can be established following the process used in the examples.

Example 6

Consider a simply supported beam of length, *L*, subjected to a force at midspan and determine its lateral stiffness in terms of its flexural rigidity, *EI*. Assume we can lump the mass at that point so that the systems can be idealized as a SDOF system.

Solution

i. *Determine the deflection at the point of interest (or DOF) for the given loading; in this case midspan.* The midspan deflection of the beam can be determined using one of several methods (e.g., double integration, conjugate beam, virtual work, moment-area, etc.). Also, beam deflections can be obtained from deflection tables such as those found in Part 4 of the American Institute of Steel Construction Manual. The midspan deflection for a simply supported beam with a concentrated load *F* is shown in Fig. E6.1

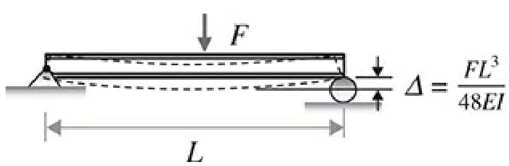


FIGURE E6.1 Midspan deflection of a simply supported beam with concentrated force.

ii. *Determine the stiffness for the given loading assuming a single DOF at midspan.* The stiffness, *k*, is force required to cause a unit displacement. The midspan deflection can be rearranged to determine the stiffness.

$$k = \frac{48EI}{L^3}$$

where the quantity *EI* is the flexural stiffness given by the product of the modulus of elasticity, *E*, and the second moment of the cross-section about the axis of bending (the moment of inertia), *I*. ▲

Following the approach of this example, we can derive equations for *k* for a number of simple structural systems that can be modeled as SDOF; see Table 1.1 for a list of cases.

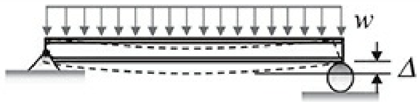
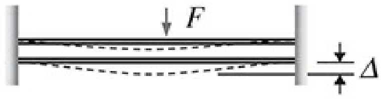
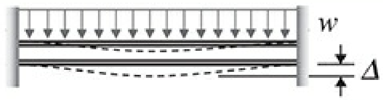
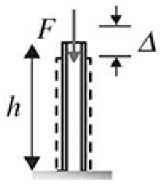
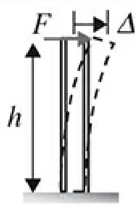
Case	Max Deflection, Δ	Stiffness, k
	$\dfrac{5wL^4}{384EI}$	$\dfrac{384EI}{5L^3}$
	$\dfrac{FL^3}{192EI}$	$\dfrac{192EI}{L^3}$
	$\dfrac{wL^4}{384EI}$	$\dfrac{384EI}{L^3}$
Axially loaded bar 	$\dfrac{Fh}{AE}$	$\dfrac{AE}{h}$
Fixed-pinned column 	$\dfrac{Fh^3}{3EL}$	$\dfrac{3EI}{h^3}$
Fixed-fixed column 	$\dfrac{Fh^3}{12EI}$	$\dfrac{12EI}{h^3}$

TABLE 1.1 Equivalent Stiffness Constants, k

Example 7

Given the water tank supported on a slender column with constant flexural rigidity, EI, as shown in Fig. E7.1, idealize the column and weight into a SDOF system and determine its stiffness for a horizontal force applied at the top of the column.

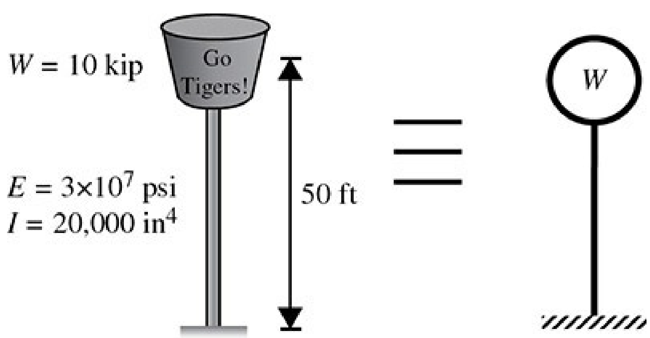


FIGURE E7.1 Water tank supported on a slender column and the idealized structural model.

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Solution

i. *Idealize the structural system.* For this case, we can assume the weight of the column to be negligible compared to the weight of the water tank. Considering only the horizontal displacement of the water tank, the lumped mass model is shown in Fig. E7.1.

ii. *Determine the stiffness prarmeters.* From Table 1.1, the lateral stiffness of the SDOF system is based on a fixed-pinned column model with horizontal load using the same process described in Example 6.

$$k = \frac{3EI}{L^3} = \frac{3(3\times10^7\text{ psi})(20{,}000\text{ in}^4)}{50\text{ ft}\times 12\,\frac{\text{in}}{\text{ft}}^{\;3}} = 8{,}333\,\frac{\text{lb}}{\text{in}} \quad \blacktriangle$$

Example 8

The space grid roof structure shown in Fig. E8.1 weighs 20 psf and is assumed to be rigid. A plan view of the structure is also shown in Fig. E8.1. The side sheathing of the building (walls) weighs 10 psf. All steel (E = 29,000 ksi) columns are W10 × 30 (I = 170 in^4). Idealize the system as a SDOF system and determine the equation of motion. Notice that the first pair of columns in Fig. E8.1 is fixed-pinned (column line A), the second is pinned-pinned (column line B), and the last pair is fixed-fixed (column line C).

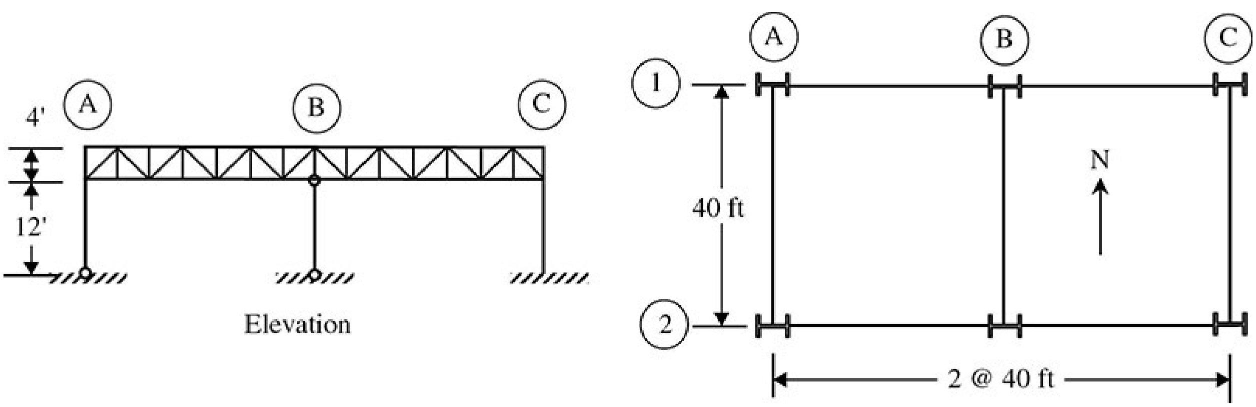

FIGURE E8.1 Elevation and plan view of space grid roof structure.

Solution We can model the system as a SDOF by assuming the roof framing to be rigid.

i. *Determine the mass (proportional to the weight) of the system.* The weight of the SDOF system includes the 40- by 80-ft roof with weight of 20 psf and the 10 psf wall sheathing around the 240-ft perimeter of the building. The weight due to sheathing includes the tributary height of the roof, which is half of the building height plus the parapet height for a total of 10 ft. The total weight is

$$W = 20\frac{\text{lb}}{\text{ft}^2}(40\text{ ft}\times 80\text{ ft}) + 10\frac{\text{lb}}{\text{ft}^2}(10\text{ ft}\times 240\text{ ft}) = 88{,}000\text{ lb} = 88\text{ kip}$$

The mass is

$$m=\frac{W}{g}=\frac{88{,}000\text{ lb}}{386.4\text{ in/s}^2}=227.7\,\frac{\text{lb}\cdot\text{s}^2}{\text{in}}$$

ii. *Determine stiffness parameters.* The lateral stiffness of the columns depends on the boundary conditions and connections to the roof beams. In this case, columns A1 and A2 are pinned at the foundation and fixed at connection; columns B1 and B2 are pin-pin connected at their ends; and columns C1 and C2 are fixed-fixed connected at their ends. A1 and A2 column stiffnesses:

$$k_{\text{A1}}=k_{\text{A2}}=\frac{3EI}{h^3}$$

B1 and B2 column stiffnesses:

$$k_{\text{B1}}=k_{\text{B2}}=0$$

C1 and C2 column stiffnesses:

$$k_{\text{C1}}=k_{\text{C2}}=\frac{12EI}{h^3}$$

The total stiffness of the SDOF system is the sum of column stiffnesses in the E-W direction.

$$K_{\text{E-W}}=2\times\frac{3EI}{h^3}+2\times\frac{12EI}{h^3}=\frac{30EI}{h^3}=\frac{30(29{,}000\text{ ksi})(170\text{ in}^4)}{(144\text{ in})^3}=49{,}500\,\frac{\text{lb}}{\text{in}}$$

iii. *Since there is no applied force, the equation of motion is given by a homogeneous differential equation.*

$$227.7\,\frac{\text{lb}\cdot\text{s}^2}{\text{in}}\ddot{u}+49{,}500\,\frac{\text{lb}}{\text{in}}u=0\ \blacktriangle$$

1.6.3 Lumped Structural Stiffness of Lateral Force Resisting Systems

Although the building structural system presented in Example 8 is a lateral force resisting system, it was analyzed as three separate members. The most common dynamic lateral force categories include seismic, wind, and blast (both accidental and intentional), and water waves; static lateral loads include earth and water pressure. In this section we present an introduction to the three different categories of lateral force resisting systems used in building design: *unbraced frames, braced frames*, and *shear walls*. Following are simplified derivations of the equivalent lateral stiffness constants for single-story systems of each of these three cases.

Unbraced frames have rigid connections between columns and beams that carry lateral load by developing moments in the members. If we consider the portal frame depicted in Fig. 1.2, we can derive the equivalent stiffness of the frame by assuming that each column contributes equally to the frame stiffness. For example, for a portal frame with a much stiffer beam than the

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supporting columns and assuming fixed supports to the foundation and rigid connections between columns and beam results in the total stiffness of,

$$K = (2)\frac{12EI}{h^3} = \frac{24EI}{h^3}$$

where h is the height of the frame, and EI is the flexural stiffness of the columns, which was defined in the sixth row of Table 1.1. Similarly, if the supports are pin connected to the foundation, the total stiffness is

$$K = (2)\frac{3EI}{h^3} = \frac{6EI}{h^3}$$

Note that the second case is four times more flexible than the fixed-fixed case. Also, these two cases assume that the beam provides infinite stiffness, which is not practical. A more general case where beam stiffness is included in the total stiffness can be derived using standard matrix structural analysis. For the frame with constant modulus of elasticity and fixed supports, the total stiffness is given by,

$$K = \frac{12EI_c}{h^3}\left(\frac{6\alpha+\kappa}{3\alpha+2\kappa}\right) \tag{1.7}$$

where $\alpha = I_b/I_c$ is the beam-to-column stiffness ratio and $\kappa = L_b/h$ is the beam-span to column-height ratio. Similarly, if the supports are pin connected to the foundation, the total stiffness is

$$K = \frac{12EI_c}{h^3}\left(\frac{\alpha}{2\alpha+\kappa}\right) \tag{1.8}$$

Example 9

Determine the total stiffness of the following steel (E = 29,000 ksi) frame with W10 × 33 columns (I_x = 171 in^4) and assuming: a) a rigid beam and b) a W18 × 65 beam (I_x = 1070 in^4).

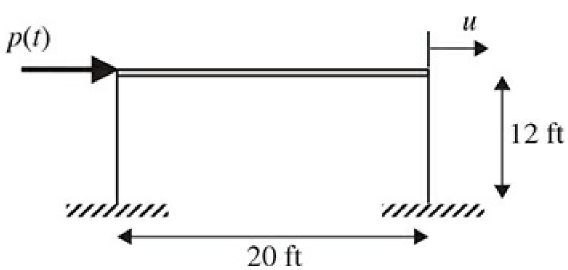

FIGURE E9.1 Portal frame system.

Solution

i. *Determine stiffness for the rigid beam case.* The lateral stiffness of the columns depends on the boundary conditions and connections to the beam. In this case, the end conditions of the columns can be considered fixed-fixed, and the stiffness parameters for each column

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can be obtained from Table 1.1.

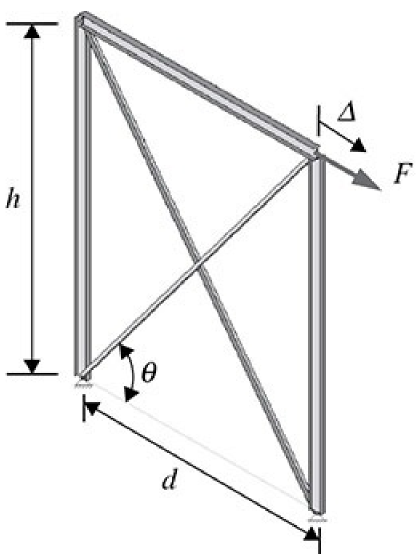
$$K = \frac{24\,EI}{h^3} = \frac{24(29{,}000\text{ ksi})(171\text{ in}^4)}{(12 \cdot 12\text{ in})^3} = 39.9\,\frac{\text{kip}}{\text{in}}$$

ii. *Determine stiffness including the stiffness of the W18 × 65 beam.* The lateral stiffness of an unbraced frame with fixed supports is given by Eq. (1.7).

$$\alpha = I_b/I_c = 1070\text{ in}^4/171\text{ in}^4 = 6.26$$
$$\kappa = L_b/h = 20\text{ ft}/12\text{ ft} = 1.67$$
$$K = \frac{12EI_c}{h^3}\,\frac{6\alpha+\kappa}{3\alpha+2\kappa} = \frac{12(29{,}000)(171)}{(12\cdot 12)^3}\,\frac{6(6.26)+1.67}{3(6.26)+2(1.67)} = 35.4\,\frac{\text{kip}}{\text{in}}$$

Note that there is an 11% decrease in the stiffness when this flexible beam is used, which is more realistic since there are no infinitely rigid members. ▲

Braced frames are assumed to behave as cantilever beams that carry the lateral load by developing an internal shear force (known as panel shear) and a bending moment (known as overturning moment). Bracing can be added to a portal frame to create the portal braced frame shown in Fig. 1.13. This type of lateral force resisting system is assumed to carry the panel shear by the diagonal members (one axial compression and the other axial tension), and the overturning moment by the axial forces in the columns.



$$\sin\theta = \frac{h}{\sqrt{h^2+d^2}}$$

$$\cos\theta = \frac{d}{\sqrt{h^2+d^2}}$$

FIGURE 1.13 Braced portal frame model.

The braced frame shown in Fig. 1.13 can be modeled as a statically indeterminate truss in which case the number of unknown reactions, *r*, and unknown member forces, *b*, exceeds the number of equations of equilibrium (two times the number of joints), 2*j*. To solve for the additional unknown in Fig. 1.13, we can use compatibility of displacements. However, because the member sizes are initially unknown, we cannot apply displacement compatibility as a first step. Therefore, we must make some simplifying assumptions to render the truss statically determinate and perform an approximate analysis. First, the diagonal members can be assumed to be stiff (in which case the diagonals can carry both tension and compression) or they can be assumed slender (in which case the member in axial compression cannot support axial force because it buckles). In the case of stiff diagonal members, we assume that the tension and compression diagonals each carry half of the panel shear; while in the case of slender diagonals, we assume that the panel shear is resisted entirely by the tension diagonal. The assumption of slender diagonal members is used to derive the equation for the stiffness of a braced portal frame following the principle of virtual work in Example 10.

But first, we review the principle of virtual work to establish the displacement of a truss joint in any direction. The process entails determining the internal forces in each member, N_i caused by the applied loads, and determining the virtual internal forces in each member, n_i caused by a unit virtual load. With these internal loads, the contribution from each member to the truss deflection is summed to determine the total displacement,

$$\Delta = \sum_{i=1}^{m} n_i \frac{N_i L_i}{E_i A_i} \tag{1.9}$$

where

N_i is the axial force in each member caused by the actual loading.

n_i is the axial force in each member caused by a unit virtual load applied at the joint and in the direction in question.

L_i is the length of each member.

A_i is the cross-sectional area of each member.

E_i is the modulus of elasticity of each member.

Example 10

Derive an equation for the lateral stiffness of a braced portal frame (see Fig. 1.13) when the diagonal members are slender.

Solution

i. *Apply equilibrium to determine internal forces for actual and virtual loading.* Slender diagonals cannot carry compression load; thus, from statics we can deduce that the left-hand column and beam are also zero-force members, leaving us with the following cases for both the actual and virtual loading,

   Use the method of joints to determine the internal forces in the remaining members by first drawing a FBD of the upper right-hand joints and applying equilibrium to the resulting concurrent force systems; that is,

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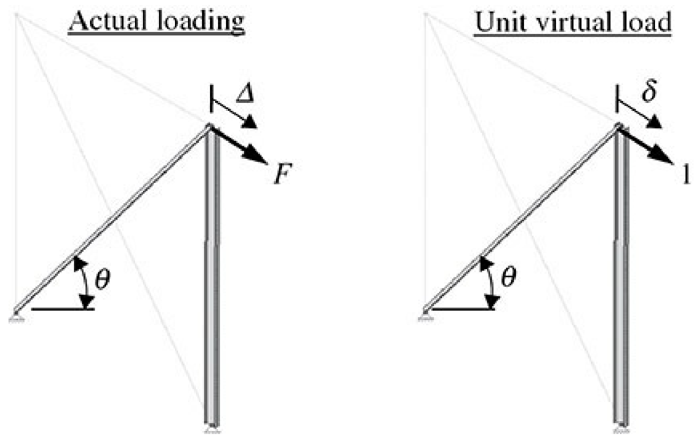

FIGURE E10.1 Actual and virtual loading with slender diagonal elements assumed.

Equilibrium:

$$\Sigma F_x = 0;\ F - N_1 \cos\theta = 0$$
$$N_1 = F/\cos\theta$$
$$\Sigma F_y = 0;\ -N_2 - N_1 \sin\theta = 0$$
$$N_2 = -F\tan\theta$$

Equilibrium:

$$\Sigma F_x = 0;\ 1 - n_1 \cos\theta = 0$$
$$n_1 = 1/\cos\theta$$
$$\Sigma F_y = 0;\ -n_2 - n_1 \sin\theta = 0$$
$$n_2 = -\tan\theta$$

FIGURE E10.2 FBDs of joints due to the actual loading (left) and virtual loading (right).

ii. *Apply virtual work [Eq. (1.9)] to determine the displacement at the top of the frame,* Δ. Since the axial displacement of the column contributes little to the lateral displacement of the frame, it's customary to ignore it.

$$= \sum_{i=1}^{1} n_i \frac{N_i L_i}{E_i A_i} = \frac{(1/\cos\theta)(F/\cos\theta)\sqrt{h^2 + d^2}}{E_1 A_1} = F\ \frac{(h^2 + d^2)^{3/2}}{d^2 E_1 A_1}$$

iii. *Determine the stiffness (ratio of force to displacement) of the frame:* The frame lateral deflection equation can be rearranged to determine the stiffness as,

$$k = \frac{d^2 EA}{(h^2 + d^2)^{3/2}}$$

where the quantity EA is the axial stiffness given by the product of the modulus of elasticity, E and the cross-sectional area, A. ▲

Example 11

In this example, we introduce the concept of structural period (the time it takes for the system to oscillate through a complete cycle), which will be derived in Chap. 2. For this example, we

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consider the platform shown in Fig. E11.1. The system is used at a stadium to film sporting events and has reported experiencing large dynamic excitations. Preliminary investigations indicate that the natural period is 0.9 second. Camera personnel is recommending that the period be limited to 0.3 second. Determine the required diameter of a system of diagonal ties (steel wires with E = 29,000 ksi) to retrofit the system to the new specifications.

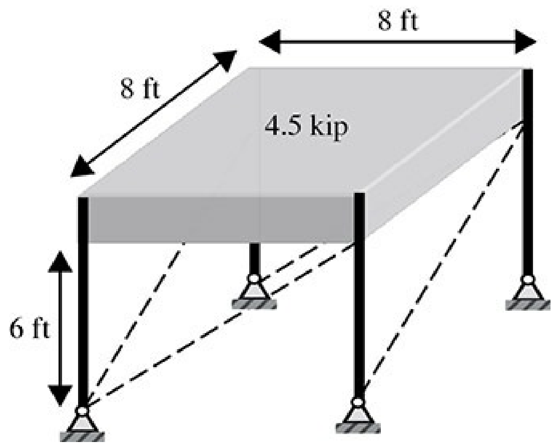

FIGURE E11.1 Schematic of structural platform.

Solution The platform can be idealized as a SDOF system. Also, since the platform is square, only one direction of motion needs to be considered, which is retrofitted with four diagonal ties (two along each side, but only the tension one is shown with a dash line in Fig. E11.1).

i. *Determine the existing stiffness of the system using the observed natural period.* With the system mass,

$$m = W/g = 4.5\ \text{kip}/(386.4\ \text{in/s}^2) = 0.01164\ \text{kip·s}^2/\text{in}$$

and the existing natural period of the SDOF system (T_n = 0.9 second), we can determine the available stiffness using the following period equation that will be derived in Chap. 2,

$$T_n = 2\pi\sqrt{\frac{m}{k}} \qquad k = m\,{\frac{2\pi}{T_n}}^{2}$$

Which yields

$$k = 0.01164\,\frac{\text{kip}\cdot\text{s}^2}{\text{in}}\,{\frac{2\pi}{0.9\ \text{s}}}^{2} = 0.5676\,\frac{\text{kip}}{\text{in}}$$

ii. *Determine the stiffness necessary to reduce the period to 0.3 s.*

$$k_{\text{new}} = m\,{\frac{2\pi}{T_n}}^{2} = 0.01164\,\frac{\text{kip·s}^2}{\text{in}}\,{\frac{2\pi}{0.3\ \text{s}}}^{2} = 5.108\,\frac{\text{kip}}{\text{in}}$$

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iii. *Determine the required diameter of the diagonal ties needed to increase the stiffness of the system.* Determine the difference between the available and the new required stiffnesses. This must be provided by the remedial diagonal ties. The change in stiffness given by the equation derived in Example 10 is used to determine the required radius of the diagonal ties.

$$k_{\text{new}} - k = 2\frac{d^2EA}{(h^2+d^2)^{3/2}}$$

where $d = 8$ ft, $h = 6$ ft, and $A = \pi r^2$, so

$$5.108 - 0.5676 = 2\frac{(8\cdot 12)^2(29{,}000)\pi r^2}{[(6\cdot 12)^2+(8\cdot 12)^2]^{3/2}}$$

Solving for the radius of the wire,

$$r = \sqrt{\frac{(5.108-0.5676)[(6\cdot 12)^2+(8\cdot 12)^2]^{3/2}}{2(8\cdot 12)^2(29{,}000)\pi}} = 0.0685 \text{ in}$$

Thus, the required diameter is $2r = 0.137$ in, which can be provided by a 3/16 in wire. ▲

Shear walls can be assumed to behave as deep cantilever beams that carry the lateral load by developing internal shear force and bending moment. That is, a shear wall has deflections that are affected by both moment and shear (unlike regular beams, where most of the deflection is caused by flexure). We can once again use the principle of virtual work to establish the lateral displacement of the top of the shear wall. The process entails determining the internal shear and moment functions (diagrams) caused by horizontal lateral loads applied at the top of the wall, and determining the virtual internal shear force and moment functions (diagrams) caused by a horizontal lateral unit virtual load at the top of the wall. With the internal shear force and moment functions, the deflection of the top of the wall can be determined using the following virtual work equation,

$$\Delta = \int_0^L m_f \frac{M_f}{EI}dx + \int_0^L \mu\left(\frac{vV}{GA}\right)dx \qquad (1.10)$$

where

m_f is the internal virtual moment caused by an external virtual unit load.
M_f is the internal moment caused by the lateral load.
v is the internal shear force caused by an external virtual unit load.
V is the internal shear force caused by the lateral load.
L is the length of the member.
I is the moment of inertia of the member.
E is the modulus of elasticity of the member.
A is the area of the member.

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G is the shear modulus of elasticity of the member.

μ is a constant that takes the following values depending on the cross-section of the shear wall:

μ = 1.2 for rectangular sections.

μ = 10/9 for circular sections.

$\mu = A/A_{web}$ (ratio of total area to area of web) for I or box sections.

Example 12

Derive equations for the stiffness of the cantilever shear walls shown in Fig. E12.1:

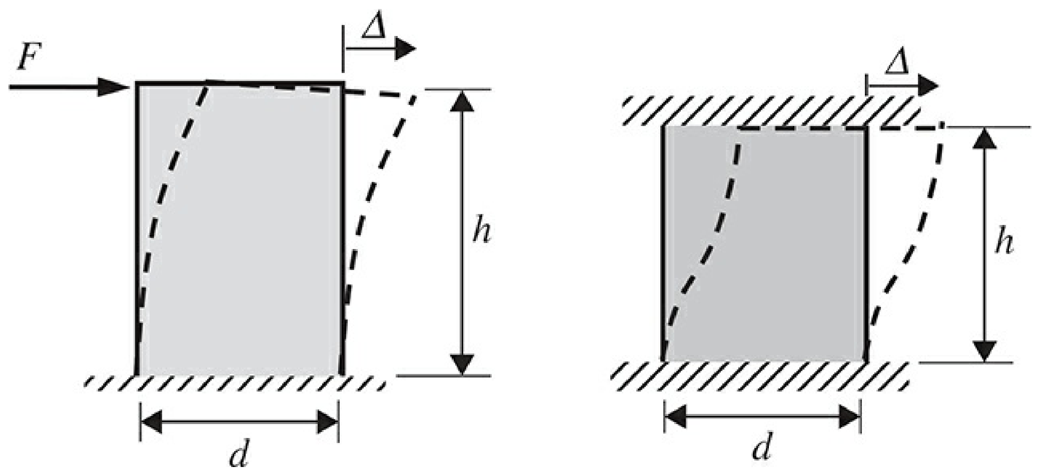

FIGURE E12.1 Cantilever shear walls.

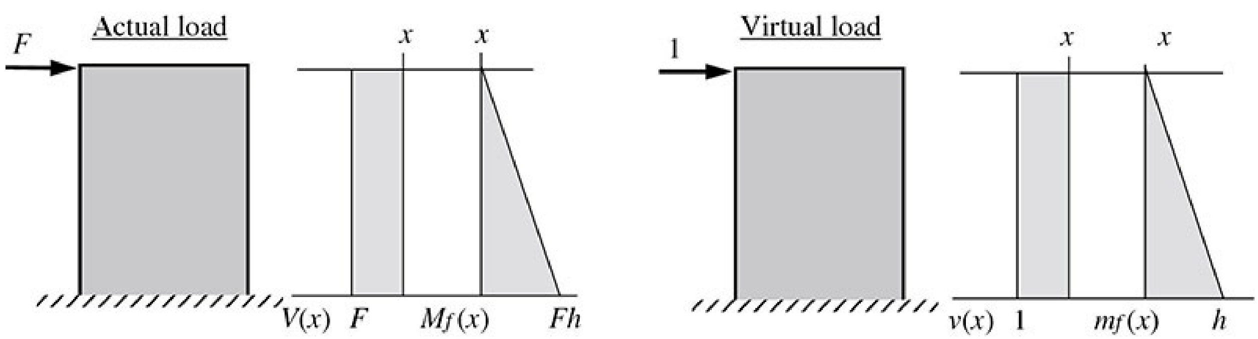

FIGURE E12.2 Internal shear and bending moment diagrams for actual loading and virtual loading.

Solution

i. *Apply equilibrium to determine shear force and bending moment functions for actual and virtual loading.* Use structural analysis to determine the shear force and bending moment diagrams for a cantilever beam, for both real and virtual loading; that is,

ii. *Determine the displacement, Δ at the top of the wall.* Apply virtual work [Eq. (1.10)] to determine the displacement at the top of the wall, Δ. Since the moments vary lineally and the shears are constant, we get the following result,

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$$= \frac{1}{EI}\int_0^L m_f(x)M_f(x)dx + \frac{}{GA}\int_0^L v(x)V(x)dx = \frac{1}{EI3}(h)(Fh)h + \frac{}{GA}(1)(F)h$$

$$= \frac{Fh^3}{3\,EI} + \frac{Fh}{GA}$$

iii. *Determine the stiffness (ratio of force to displacement) of the wall:*
The wall lateral deflection equation can be rearranged to determine the stiffness, which for a rectangular (μ = 1.2) shear wall is

$$K = \frac{3\,AGEI}{AGh^3 + 3.6\,EIh} \tag{1.11}$$

iv. *Following a similar process, we can also derive the equation for a shear wall that is fixed at the top and bottom.*

$$K = \frac{12\,AGEI}{AGh^3 + 14.4\,EIh} \tag{1.12}$$

where the quantity EI is the flexural stiffness given by the product of the modulus of elasticity, E, and the moment of inertia of the area, I; and AG is the shear stiffness given by the product of the shear modulus of elasticity, G, and the area, A. ▲

Example 13
Consider the building structure introduced in Example 8 (repeated here for convenience) and determine the total stiffness in the north-south direction, which has a 20-ft shear wall lateral force resisting system (one on each side) consisting of ½-in plywood (E = 1,000 ksi and G = 500 ksi).

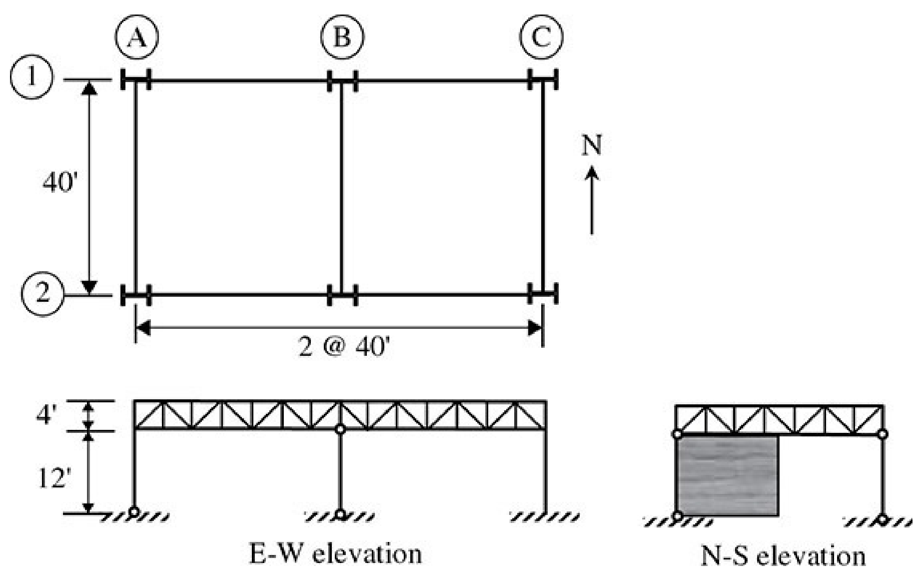

FIGURE E13.1 Plan and elevation views of structural system.

Solution

i. *Determine stiffness parameters.* The lateral stiffness of the shear walls depends on the moment of inertia and area of the footprint of the wall.
Moment of inertia of shear wall,

$$I = bh^3/12 = 0.5 \text{ in } (20 \times 12 \text{ in})^3/12 = 576{,}000 \text{ in}^4$$

Area of shear wall,

$$A = bh = 0.5 \text{ in } (20 \times 12 \text{ in}) = 120 \text{ in}^2$$

The total stiffness in the north-south direction is given by Eq. (1.12).

$$\begin{aligned} K &= 2\left[\frac{12AGEI}{AGh^3 + 14.4EIh}\right] \\ &= 2\left[\frac{12(120 \text{ in}^2)(500 \text{ ksi})(1{,}000 \text{ ksi})(576{,}000 \text{ in}^4)}{(120 \text{ in}^2)(500 \text{ ksi})(144 \text{ in})^3 + 14.4(1{,}000 \text{ ksi})(576{,}000 \text{ in}^4)(144 \text{ in})}\right] \\ &= 604 \text{ kip/in} \quad \blacktriangle \end{aligned}$$

1.7 Flexural and Shear Stresses in Lateral Force Resisting Portal Systems

In the previous subsection, we covered lumped stiffness of lateral force resisting (or portal) systems (*unbraced frames*, *braced frames*, and *shear walls*), which was obtained from the force required to cause a unit displacement. Consequently, if we know the actual displacement, we can determine an equivalent lateral force by multiplying the stiffness times the actual displacement. Thus, after determining the maximum dynamic displacement, either from force or base motion excitation (both of which are covered in Chap. 3), we can determine an equivalent static lateral force and conduct a static structural analysis to determine element internal loading (bending moment, shear force, and axial force) and stresses (using standard strength of materials procedures) needed for design; no additional dynamic analysis is necessary. For all three portal systems shown in Fig. 1.14, we can follow one of two different approaches to conduct this structural analysis: equivalent static force or element level analysis for simple cases. The first is applicable to all three cases, while the second must consider each case separately by analyzing the panel quantities at the base, the shear, V_b, and the moment, M_b, shown in Fig. 1.14.

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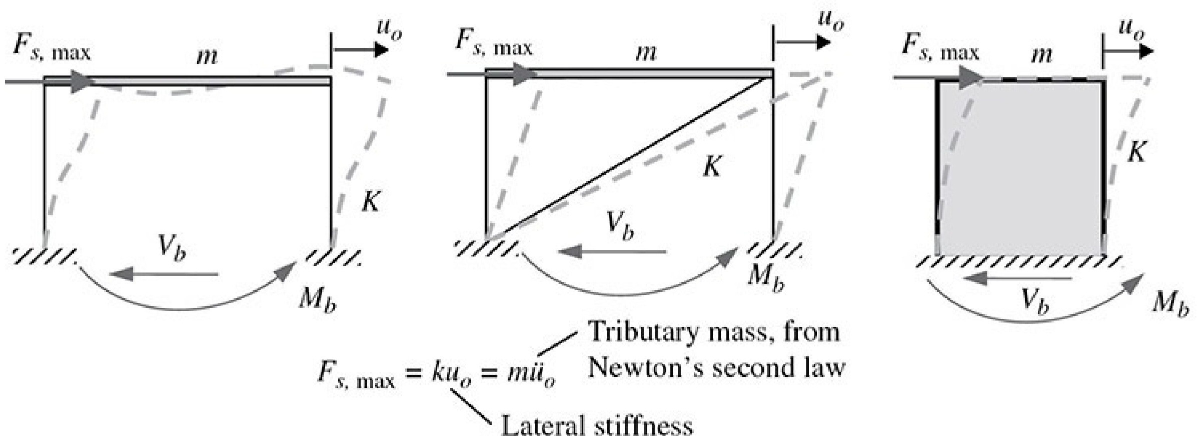

FIGURE 1.14 Equivalent static force in portal systems.

1.7.1 Equivalent Static Force Analysis

Equivalent static force entails replacing the dynamic effect with a slowly applied force that produces deformation $u_{max} = u_o$ (or acceleration $\ddot{u}_{max} = \ddot{u}_o$). As will be shown in Chap. 3, this displacement can equally be produced by a dynamically applied force or a support excitation. After determining the equivalent static force, $F_{s,max}$ as shown in Fig. 1.14, we can determine all internal forces and stresses.

1.7.2 Element Level Analysis

Element level analysis can be used to obtain the forces (normal force, shear force, and moment) in each element by using the shear and moment developed in each of the portal systems as shown in Fig. 1.14. For the first two cases, the overturning moment, M_b, is carried by the columns by developing axial forces (tension in one side and compression in the other); while for the shear wall, the chord elements (boundary elements) carry this moment by developing axial forces (again, tension in one side and compression in the other). The axial forces in the boundary elements (columns or chords) can be obtained by assuming the forces in the elements form a couple as shown in Fig. 1.15; the magnitude of each of these forces is the moment, M_b, divided by the span, L, of the portal system.

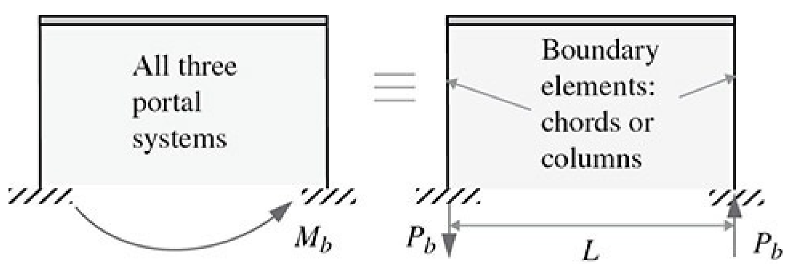

FIGURE 1.15 Equivalent couple for the overturning moment of all portal systems.

With the maximum axial forces, we can compute the *maximum normal stress*, which must be combined with the normal flexural stress to obtain the total normal stress,

$$\sigma_{max} = \frac{P_{max}}{A} \tag{1.13}$$

where

P_{max} is maximum axial force in the member.
A is the cross-sectional area of the member.

The panel shear (also known as base shear, V_b) is caried by the columns in unbraced frames by developing internal shear force and bending moment, the diagonal element in the braced frames by developing an axial force in the diagonal element, and the interior of the shear wall (which is called sheathing in timber shear walls) by developing shear flow or unit shears (shear force per unit length).

Unbraced frames can have fixed-pinned or fixed-fixed columns; Fig. 1.16 depicts both cases. The first case shows the maximum internal shear and moment at the fixed side; but in the case of a pinned supported frame with rigid beam to column connections the internal shear and moments shown would be located at the connection. The second case has the same maximum shear and moment at the top and bottom of the column.

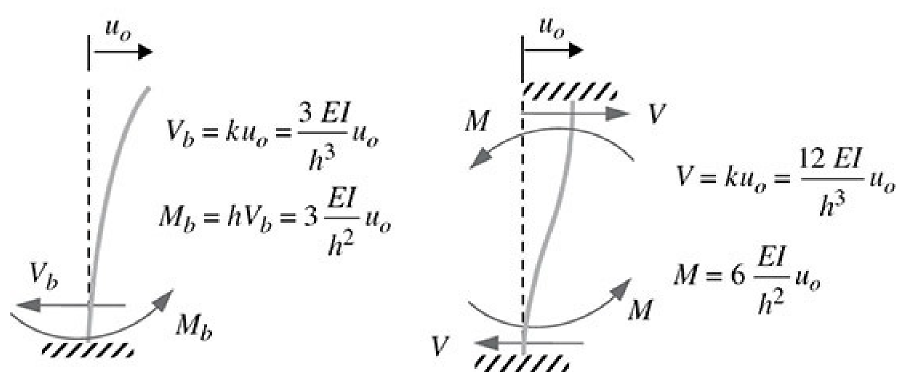

FIGURE 1.16 Internal shear force and moment for pinned-fixed and fixed-fixed columns.

In both cases, maximum stresses (shear and flexural) can be determined after the internal shear force and moment have been computed. The *shear stress* for a general section is

$$\tau = \frac{VQ}{Ib} \tag{1.14}$$

where

Q is the first moment of the area above the plane in question.
I is the moment of inertia of the cross section.
b is the width of the section at the plane in question.

This equation yields the following maximum shear stresses for:

Rectangular section,

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$$\tau_{\max} = 1.5\frac{V_{\max}}{A} \tag{1.15}$$

I-shaped section,

$$\tau_{\max} \approx \frac{V_{\max}}{A_{\text{web}}} \tag{1.16}$$

where

A is the area of the entire rectangular cross section.
A_{web} is the area of the web of I-shaped section; in practice, this is equal to the product of the depth of section and thickness of the web.

The maximum flexural stress for columns can be determined using the flexure formula

$$\sigma_{\max} = \frac{M_{\max}}{S} \tag{1.17}$$

where S is the section elastic modulus.

Braced frames can have two diagonal elements, but for this case we assume a single tensile element. Also, we assume a very small displacement such that the angle the diagonal element makes with respect to the ground does not change when the frame is deformed. This leads to the deformed shape shown in Fig. 1.17 where the extension of the diagonal element, u_d, and the lateral deformation, u_o, can be assumed to form the adjacent and hypotenuse, respectively. Using the cosine function we can obtain the relationship between these two quantities,

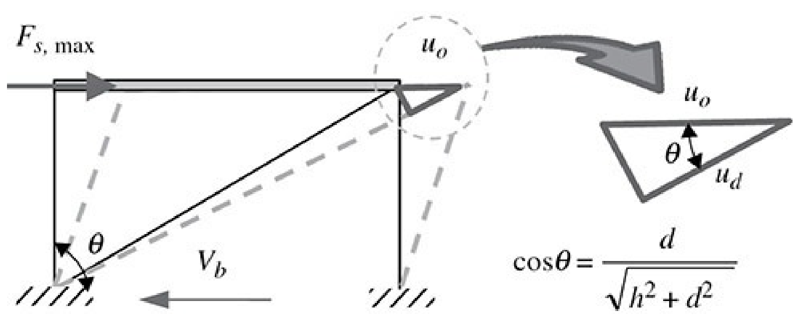


FIGURE 1.17 Internal axial force in diagonal element of braced frame.

$$\cos\theta = \frac{u_d}{u_o} = \frac{d}{\sqrt{h^2+d^2}} \Rightarrow u_d = \frac{d(u_o)}{\sqrt{h^2+d^2}} \tag{1.18}$$

The axial force is given as the product of the stiffness (equation derived in Example 10) and u_d,

$$P_{\max} = ku_d = \frac{d^3EAu_o}{(h^2+d^2)^2} \tag{1.19}$$

And the normal stress is given by Eq. 1.13, the axial force divided by the cross-sectional area of the diagonal element.

Shear walls can be fixed-pinned or fixed-fixed when walls are part of a multistory frame. In both cases, the unit shear is equal to the shear force, (V or V_b) divided by the span, L, of the wall as shown in Fig. 1.18. This is known as the unit shear and is equivalent to the average shear flow.

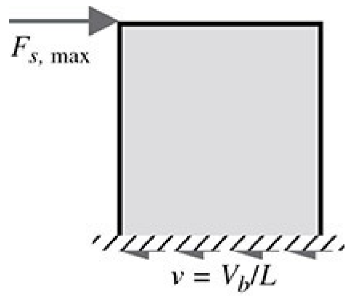
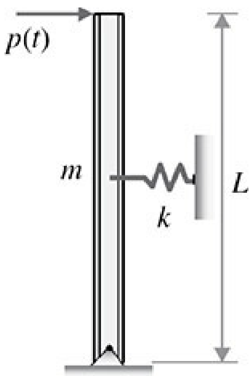


FIGURE 1.18 Unit shear in shear wall.

1.8 Problems

1.1 Consider a rigid column having mass m, length L, and constrained to move horizontally at mid-height by a spring having a stiffness k as shown below. Draw a FBD of the column after a time-dependent horizontal force is applied and formulate the equation of motion using D'Alembert's principle.



1.2 Consider a rigid column having mass m, length L, and constrained to move horizontally at the base by a rotational spring having a stiffness $k\theta$ as shown below. Draw a FBD of the column after a time-dependent horizontal force is applied and formulate the equation of motion using D'Alembert's principle.

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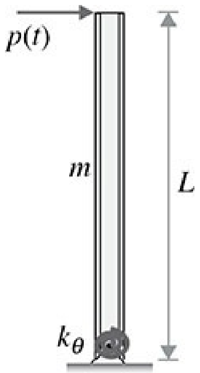
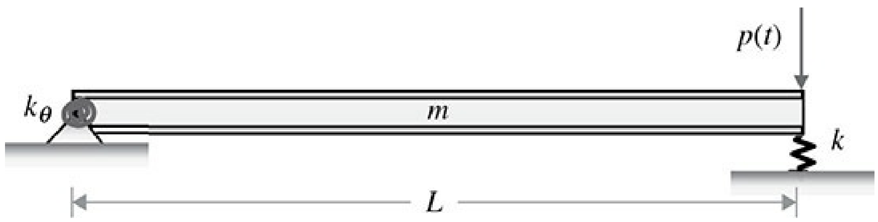


1.3 Consider a rigid beam having mass m, span L, and constrained to move vertically at the right support by a linear spring having a stiffness k and constrained at the left support by a rotational spring having a stiffness $k\theta$ as shown below. Draw a FBD of the beam after a time-dependent vertical force is applied and formulate the equation of motion in terms of the rotation of the beam using D’Alembert’s principle.



1.4 Determine the effective weight at the roof and floor levels for a 60 ft by 150 ft, three-story office building with equal story heights of 10 ft and the following loading:

Roof DL = 20 psf

Roof LL = 20 psf

Floor DL = 20 psf

Floor LL = 60 psf

Wall DL = 10 psf

1.5 The bridge beam depicted below supports a rigid deck that weighs 200 kip. Assuming the weight is lumped at midspan, determine its mass and stiffness (E = 29,000 kip/in^2 and I_x = 13,000 in^4).

1.6 Given a water tank weighing 50 kips and supported on a slender (I = 10,000 in^4), steel (E = 29 × 10^9 psi) column as shown, determine its mass and stiffness.

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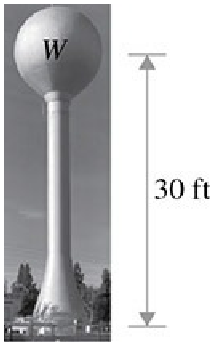
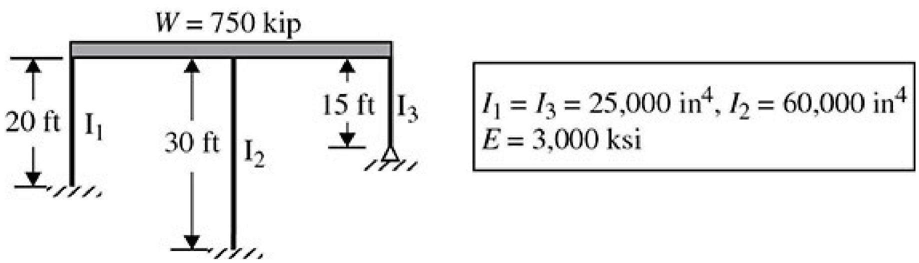
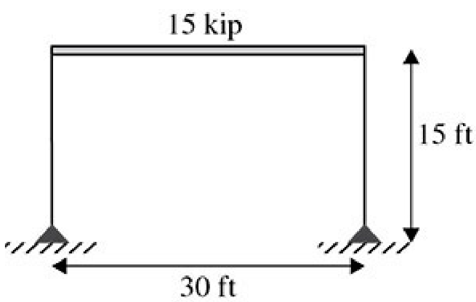


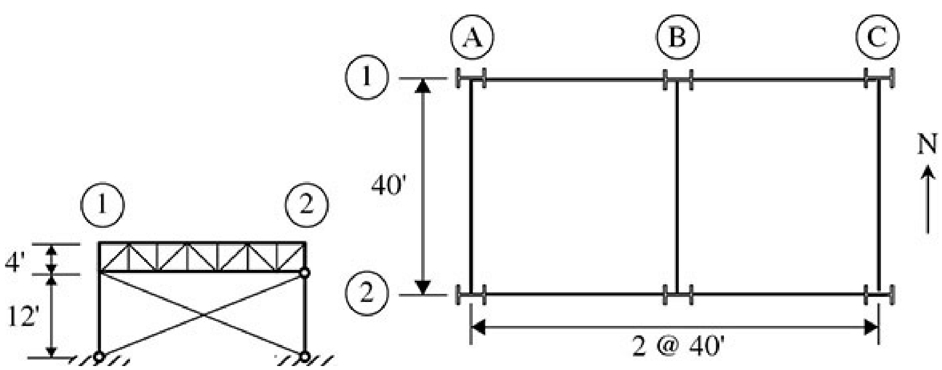
1.7 Given the following structural system and properties, determine the mass and the total stiffness.



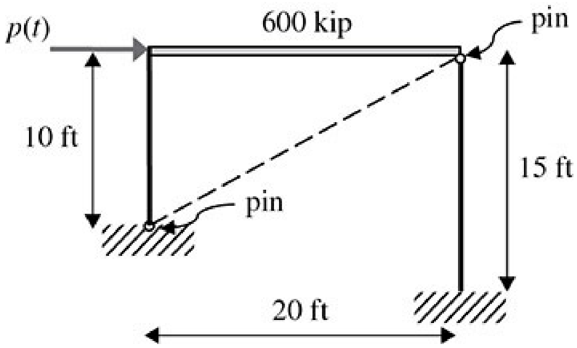
1.8 Given the following steel (E = 29,000 kip/in^2) building frame, write the equation of motion. The columns have I_x = 75 in^4 and the beam has I_x = 150 in^4.



1.9 Determine the equation of motion for the structural system depicted below, which has a roof weighing 22.5 psf and side sheathing weighing 10 psf. The total weight is assumed to be concentrated at the bottom of the roof trusses. The lateral force resisting system in the north-south direction (*one on each side*) consists of bracing with ½ in steel (E = 29,000 ksi) rods.



1.10 The following frame has a rigid beam, a diagonal ½-in-diameter steel rod (E = 29,000 ksi) brace, and two steel columns (E = 29,000 ksi and $_{Ix}$ = 82.7 in^4). Determine the total stiffness of the system.



1.11 Consider the building structure introduced in Prob. 1.9 and determine the total stiffness in the north-south direction, which has a lateral force resisting system (one on each side) consisting of a 10-ft-long, 6-in-thick concrete shear wall (E = 3,500 ksi and G = 1,500 ksi).

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CHAPTER 2

Free Vibration of Single-Degree-of-Freedom Systems

After reading this chapter, you will be able to:

a. Solve the equation of motion for an undamped SDOF system
b. Solve the equation of motion for a damped SDOF system
c. Describe the effect of viscous damping for underdamped, critically damped, and overdamped systems
d. Use logarithmic decrement to determine properties of structural systems

The free vibration response of a single-degree-of-freedom (SDOF) system is determined by its initial position and speed. That is, the response of an oscillator is caused by these two initial conditions (initial displacement and velocity), not an applied force; force-induced vibration is examined in the next chapter. To characterize the free vibration motion of a SDOF system, we first need to formulate a mathematical model that describes displacement, u, as a function of time. This can be accomplished by establishing an equation of motion for the system, as discussed in Chap. 1. This equation can be determined by applying equilibrium to a free-body diagram of the mass using D'Alembert's principle, where the inertial force developed by the mass (from Newton's second law, $F = ma$) is included in the free-body diagram as shown in Fig. 2.1. Also, as discussed in Chap. 1, the stiffness, k, is an empirical relationship obtained from Hooke's law, and it deals with the deformation of structures up to the elastic limit. For structures that experience deformation beyond the elastic limit, a different formulation is necessary.

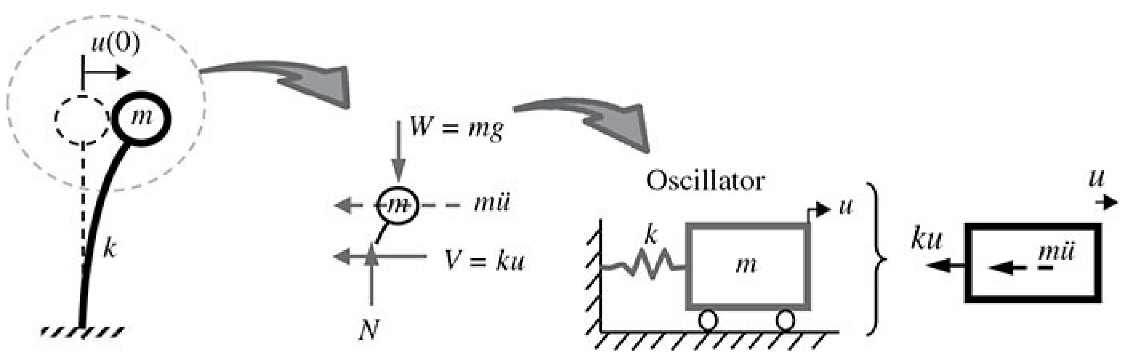

FIGURE 2.1 Free-body diagram of idealized portal frame and oscillator model.

When the inverted pendulum shown in Fig. 2.1 is released from the position shown, the initial displacement of $u(0)$, it will move back and forth along the same path, repeating itself in

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equal intervals; such periodic motion is known as oscillatory or vibratory. This type of motion will continue ad infinitum unless a force disturbs the SDOF system. That is, the motion is dissipated by an externally applied nonconservative force, internal or external friction, or other "damping"-related force; and depending on the relative magnitude of the damping force, the system may not vibrate at all as shown later in this chapter.

2.1 Free Vibration Response of Undamped SDOF Systems

Periodic motion can be expressed mathematically using sines and cosines; this is also called *harmonic motion*. As discussed in Chap. 1, motion is characterized using the position of the SDOF system (displacement) from its equilibrium position (where no net force acts) at any instant. Following a similar approach as in Sec. 1.5.1, but with zero applied excitation, we can derive the equation of motion for free vibration, which entails applying horizontal equilibrium to the free-body diagram shown in Fig. 2.1,

$$\overset{+}{\rightarrow} \sum F_x = 0; \quad -m\ddot{u} - ku = 0 \qquad m\ddot{u} + ku = 0$$

where

- m is the mass of the system.
- k is the lateral stiffness, discussed in more detail later.
- The double dot over the u indicates double differentiation with respect to time, Eq. (1.3).

The equation of motion for free vibration response, a second-order, linear, and homogeneous differential equation with constant coefficients, can be written in the following form:

$$\ddot{u} + \omega_n^2 u = 0 \tag{2.1}$$

where, for convenience, a new parameter is introduced, the natural circular frequency (from now on called natural frequency) with units of radians per second (rad/s),

$$\omega_n = \sqrt{\frac{k}{m}} \tag{2.2}$$

2.1.1 Solution to the Undamped SDOF System Equation of Motion

The solution to Eq. (2.1) is of the form:

$$u(t) = e^{\lambda t}$$

where λ is an arbitrary constant. This equation represents the horizontal displacement of the mass and can be differentiated with respect to time to obtain the velocity, $\dot{u}(t) = \lambda e^{\lambda t}$, which can be differentiated to obtain the acceleration, $\ddot{u}(t) = \lambda^2 e^{\lambda t}$. We can then solve for λ by substituting the equations for displacement and acceleration into Eq. (2.1); that is, $\lambda^2 e^{\lambda t} + \omega_n^2 e^{\lambda t} = (\lambda^2 + \omega_n^2) e^{\lambda t} = 0$. Since the exponential function is not zero, the quantity in parenthesis must be zero and thus,

$$\lambda = \pm\omega_n i$$

where i is the imaginary unit of a complex number given as $i = \sqrt{-1}$. Since Eq. (2.1) is a second order differential equation, two independent solutions are needed,

$$u(t) = A_1 e^{\omega_n it} + A_2 e^{-\omega_n it}$$

where A_1 and A_2 are constants of integration.

This equation can be expressed in polar form using Euler's identities ($e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$),

$$u(t) = A_1(\cos\omega_n t + i\sin\omega_n t) + A_2(\cos\omega_n t - i\sin\omega_n t)$$

which can be rewritten as

$$u(t) = (A_1 + A_2)\cos\omega_n t + i(A_1 - A_2)\sin\omega_n t$$

Since the two trigonometric functions are real-valued solutions to the equation of motion, we can express the general solution in real form as

$$u(t) = A\cos\omega_n t + B\sin\omega_n t \tag{2.3}$$

where A and B are arbitrary constants that can be solved by how the motion starts, that is, using initial conditions of displacement and velocity at time $t = 0$. Substituting initial displacement, $u(0)$, into Eq. (2.3), $u(0) = A\cos(0) + B\sin(0)$ yields $A = u(0)$. Also, taking a time derivative of Eq. (2.3) and substituting the initial velocity, $\dot{u}(0)$, $\dot{u}(0) = -A\omega_n \sin(0) + B\omega_n \cos(0)$, yields $B = \frac{\dot{u}(0)}{\omega_n}$. The complete solution is the free vibration of an undamped SDOF system tracking the position of the mass as a function of time as shown in Fig. 2.2,

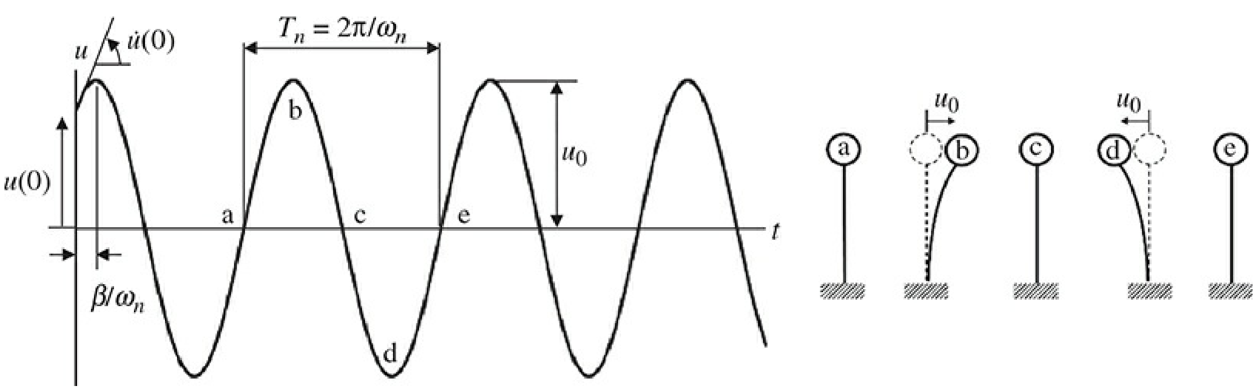

FIGURE 2.2 Free vibration of an undamped SDOF system.

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$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t \tag{2.4}$$

2.1.2 Natural Period and Frequency of Vibration

This harmonic motion repeats itself with a frequency of ω_n. As shown in Fig. 2.2, the time required to complete a full cycle of 2π is $2\pi/\omega_n$ and is known as the natural period of vibration, T_n, which is given in units of seconds (s),

$$T_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} \tag{2.5}$$

Also, the number of complete cycles per unit of time is the natural cyclic frequency, f_n (in cycles/s or hertz), which is the reciprocal of the natural period of vibration, T_n (s); thus, it is also proportional to ω_n,

$$f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi} \tag{2.6}$$

Note that these three quantities are proportional and essentially represent the same physical quantity expressed in different units.

We can also write the equation for the free vibration response [Eq. (2.4)] in terms of the period as follows,

$$u(t) = u(0)\cos\ 2\pi\frac{t}{T_n}\ + \frac{\dot{u}(0)}{\omega_n}\sin\ 2\pi\frac{t}{T_n}$$

2.1.3 Phase Angle and Maximum Amplitude of Vibration Motion

An alternative formulation for the free vibration response described by Eq. (2.4) can be used to determine the maximum displacement, known as the amplitude of vibration, u_0 (see Fig. 2.2). Also, a relationship for the lapse time from initial condition to the first amplitude of vibration peak, known as the phase angle, β is obtained from this analysis.

First, assume that constants A and B in Eq. (2.3) are related to u_0 and β as follows:

$$A = u_0\cos\beta \qquad \text{and} \qquad B = u_0\sin\beta \tag{2.7}$$

Squaring these two relationships and adding the results yields

$$A^2 + B^2 = u_0^2(\cos^2\beta + \sin^2\beta)$$

The quantity in parentheses is equal to 1.0 (trigonometric identity); thus, the amplitude of motion is

$$u_0 = \sqrt{A^2 + B^2} = \sqrt{u(0)^2 + (\dot{u}(0)/\omega_n)^2}$$

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$$\frac{B}{A} = \frac{u_0 \sin\beta}{u_0 \cos\beta} = \tan\beta$$

Also, to obtain the phase angle, β, we take the ratio of B to A, ; or,

$$\beta = \tan^{-1} \frac{B}{A} = \tan^{-1} \frac{\dot{u}(0)}{u(0)\omega_n}$$

Substituting Eq. (2.7) into Eq. (2.3) yields the equation of motion in terms of u_0 and β; that is,

$$u(t) = A\cos\omega_n t + B\sin\omega_n t = u_0 \cos\beta\cos\omega_n t + u_0 \sin\beta\sin\omega_n t$$

which can be simplified using a trigonometric transformation with the following identities: $\cos\beta\cos\omega t = \frac{1}{2}\cos(\omega t - \beta) + \frac{1}{2}\cos(\omega t + \beta)$ and $\sin\beta\sin\omega t = \frac{1}{2}\cos(\omega t - \beta) - \frac{1}{2}\cos(\omega t + \beta)$. The result is

$$u(t) = u_0 \cos(\omega_n t - \beta) = u_0 \cos\ 2\pi\frac{t}{T_n} - \beta$$

Similarly,

$$u(t) = u_0 \sin(\omega_n t + \alpha)$$

where a new phase angle, α, is introduced,

$$\alpha = \tan^{-1} \frac{u(0)\omega_n}{\dot{u}(0)}$$

These phase angles and the maximum amplitude are related as shown in Fig. 2.3. Although period (or one of the frequencies), amplitude, and phase angle can be used to completely define the free vibration of a SDOF oscillator, the phase angle is of much less practical interest compared to period and amplitude.

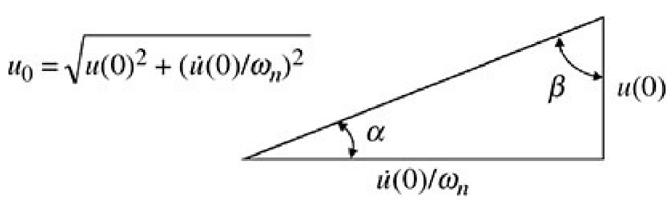

FIGURE 2.3 Relationship between phase angles and amplitude.

Example 1

Given a SDOF system with stiffness k = 250 lb/in, mass m = 0.129 lb · s^2/in, initial amplitude u_0 = 0.15 in, and initial phase angle β = 0°, compare its free vibration response for the following

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conditions:

i. Same amplitude and phase angle but differ in period by a factor of 2, which requires quadrupling the mass (or reducing the stiffness by a factor of 4).
ii. The two solutions have same period and phase angle but differ in amplitude by a factor of 2.
iii. The two solutions have same amplitude (u_0 = 0.15 in) and period but differ in phase angle by 45°.

Solution Determine the period of the system using the given stiffness and mass; k = 250 lb/in and mass m = 0.129 lb · s²/in, from Eq. (2.5),

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.129\ \text{lb}\cdot\text{s}^2/\text{in}}{250\ \text{lb/in}}} = 0.143\ \text{s}$$

Using the alternate equation of motion for the baseline case, $u(t) = u_0 \cos\ 2\pi\dfrac{t}{T_n} - \beta$, we get

$$u(t) = 0.15\ \text{in}\cdot\cos\ 2\pi\frac{t}{0.143\ \text{s}}$$

i. *The equation of motion for a case with double the period (solid line) compared to the baseline (dash line) case is shown in the following graph.*

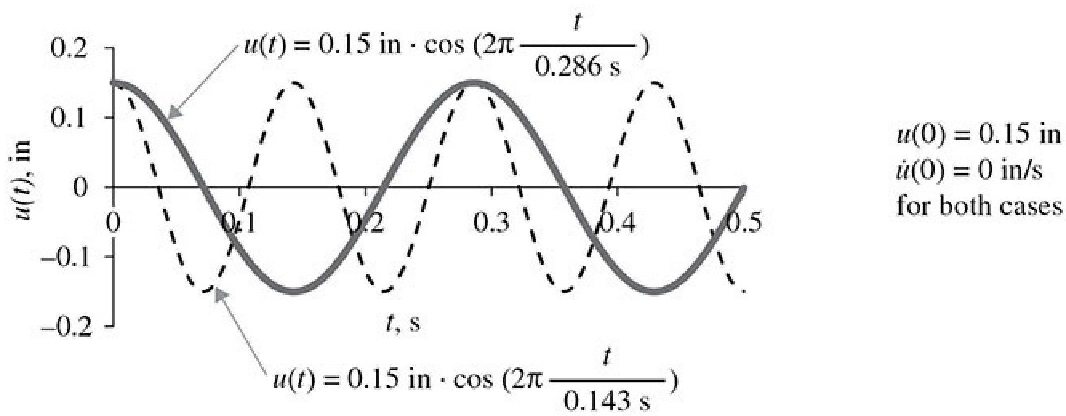

Figure E1.1 Free vibration response for cases with two different periods.

ii. *The equation of motion for a case with double the amplitude (solid line), from u_0 = 0.15 in to u_0 = 0.30 in, compared to the baseline (dash line) case is shown in the following graph.*

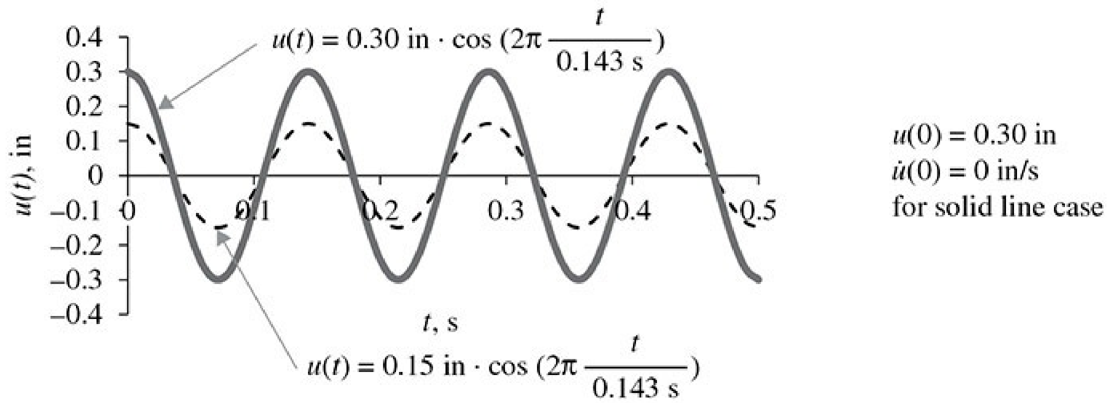

FIGURE E1.2 Free vibration response for cases with two different amplitudes.

iii. *The equation of motion for a case with an increase in the phase angle β from 0° (baseline, dash line) to 45° is shown in the following graph.*

Notice that for this last case, the initial velocity is not zero; that is, the slope of the solid line is not zero as in cases *i* and *ii*. Also, the initial displacement is not equal to the amplitude. ▲

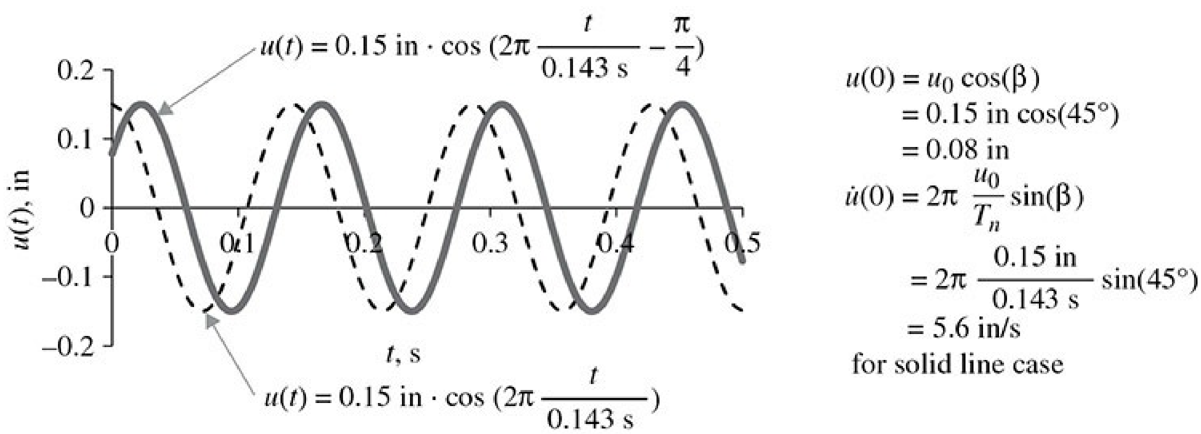

FIGURE E1.3 Free vibration response for cases with two different phase angles.

Example 2

A 2-lb body is attached to the end of a spring as shown in Fig. 2.1 and is pulled 4 in. The body is suddenly released and performs simple harmonic motion. Do the following:

i. Compute the effective stiffness of the spring if it displaces 1 in from its equilibrium position when a 3-lb force is applied.
ii. Determine the natural period of the oscillator.
iii. Determine the amplitude of the motion of the oscillator.
iv. What is the force exerted on the body by the spring just before it is released?

Solution

i. *Equivalent stiffness of the spring is determined assuming a linear force-displacement*

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relationship,

$$k = \frac{F}{} = \frac{3\ \text{lb}}{1\ \text{in}} = 3\ \frac{\text{lb}}{\text{in}}$$

ii. *The natural period*
First, determine the mass using the weight and the acceleration due to gravity,

$$m = \frac{W}{g} = \frac{2\ \text{lb}}{386.4\ \text{in/s}^2} = 0.00518\ \frac{\text{lb}\cdot\text{s}^2}{\text{in}}$$

The natural period can now be computed using Eq. (2.5),

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.00518\ \text{lb}\cdot\text{s}^2/\text{in}}{3\ \text{lb/in}}} = 0.26\ \text{s}$$

This corresponds to frequency, $f_n = 1/T_n = 3.8$ Hz and angular frequency, $\omega_n = 2\pi f = 24$ rad/s.

iii. *Determine the amplitude of displacement. Since the initial velocity is zero, the amplitude is the initial displacement before the body is released, thus, u0 = 4 in.*

iv. *The force exerted on the body by the spring just before it is released can be determined assuming a linear force-displacement relationship with the equivalent stiffness and the initial displacement of 4 in.*

$$F = -ku(0) = -3\ \text{lb/in}(4\ \text{in}) = -12\ \text{lb}\ \blacktriangle$$

2.2 Free Vibration Response of SDOF Systems with Viscous Damping

The undamped system discussed in the last section, once excited, will oscillate indefinitely with constant amplitude at its natural frequency, ω_n. However, we know from experience that a system cannot oscillate indefinitely. Any system undergoing motion always experiences frictional or damping forces that dissipate energy. Through these forces, the mechanical energy of the system (kinetic or potential) is transformed to other forms of energy, such as heat. This behavior is quite complex and difficult to characterize analytically; therefore, it is usually assumed that these forces are proportional to the magnitude of the velocity and opposite to the direction of motion. This type of damping is known as *viscous damping* and is produced in a body restrained by a surrounding viscous fluid. As shown in Fig. 2.4, a viscous damper is included in a frame model, which is idealized into the oscillator model. A free-body diagram of the oscillator model includes a viscous force, $c\dot{u}$, due to damping effects.

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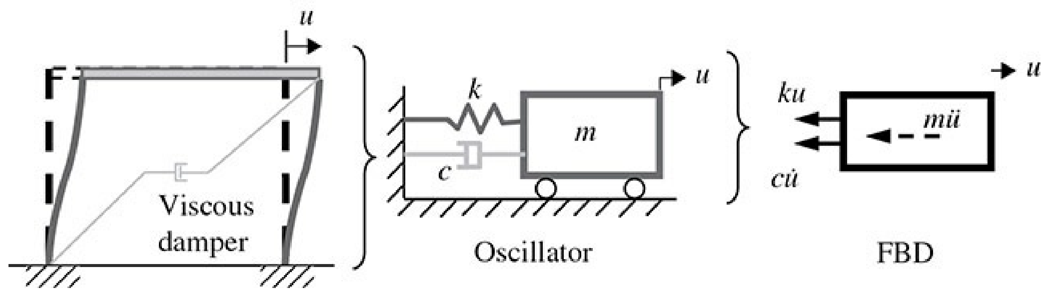

FIGURE 2.4 Idealized portal frame as an oscillator and free-body diagram (FBD).

Horizontal equilibrium of the free-body diagram shown in Fig. 2.4 yields the equation of motion for a damped oscillator,

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{2.8}$$

Here, c is the damping coefficient, which is discussed in more detail in the next section. Again, this is a second order, linear, and homogeneous differential equation with constant coefficients, the solution to which is also of exponential form,

$$u(t) = e^{\rho t}$$

where ρ is an arbitrary constant. Again, this equation represents the horizontal displacement of the mass and can be differentiated with respect to time to obtain the velocity, $\dot{u}(t) = \rho e^{\rho t}$, which can be differentiated to determine the acceleration, $\ddot{u}(t) = \rho^2 e^{\rho t}$. Substituting the equations for displacement, velocity, and acceleration into Eq. (2.8) we get the characteristic equation, $m\rho^2 e^{\rho t} + c\rho e^{\rho t} + ke^{\rho t} = (m\rho^2 + c\rho + k)e^{\rho t} = 0$, which can be used to solve for ρ. Since the exponential function generally is not zero, the quantity in parenthesis must always be zero, $m\rho^2 + c\rho + k = 0$. This is a quadratic equation and can be solved for ρ,

$$\rho = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \Rightarrow \begin{cases} \rho_1 = \dfrac{-c + \sqrt{c^2 - 4mk}}{2m} \\ \rho_2 = \dfrac{-c - \sqrt{c^2 - 4mk}}{2m} \end{cases} \tag{2.9}$$

Since the equation of motion is a second-order differential equation, we need two independent solutions and thus two constants of integration, C_1 and C_2, which are determined from initial conditions (displacement and velocity at time $t = 0$).

$$u(t) = c_1 e^{\rho_1 t} + c_2 e^{\rho_2 t} \tag{2.10}$$

Substituting the ρ parameters into this equation,

$$u(t) = e^{\frac{-c}{2m}t} \;\; C_1 e^{\frac{\sqrt{c^2-4mk}}{2m}t} + C_2 e^{\frac{-\sqrt{c^2-4mk}}{2m}t}$$

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This result leads to three different solutions depending on the quantity under the square root operator (see Fig. 2.5):

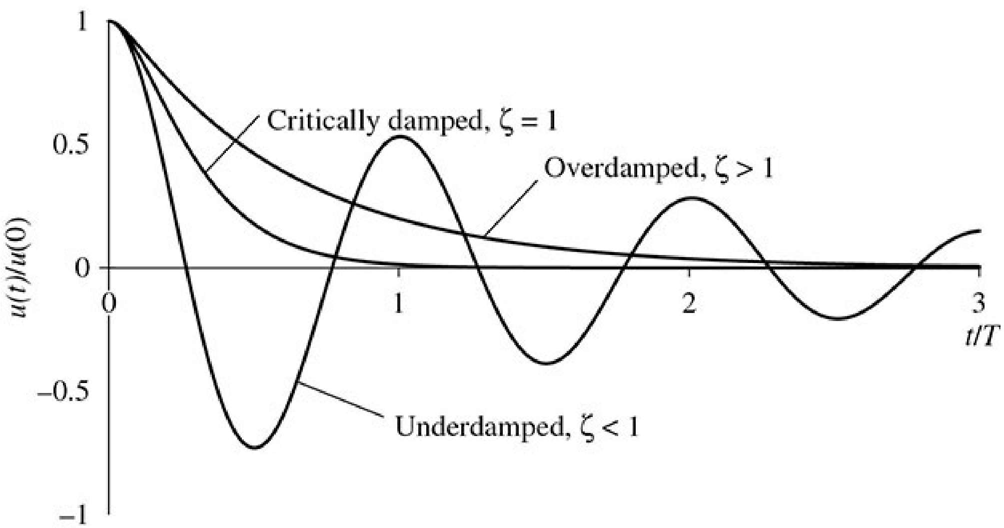

FIGURE 2.5 Free vibration response of damped SDOF system.

i. If $c^2 - 4mk = 0$, the resulting motion is the boundary between oscillatory and nonoscillatory motion. For this case, the system is classified as *critically damped*.
ii. If $c^2 - 4mk > 0$, exponents are real numbers. Like the critically damped case, the motion for this case decays exponentially without vibrating. For this case, the system is classified as *overdamped*.
iii. If $c^2 - 4mk < 0$, exponents are imaginary values and result in an oscillatory motion. For this case, the system is classified as *underdamped*.

Critically damped system, $c^2 - 4mk = 0$, is used to define the critical damping coefficient, c_{cr}:

$$c_{cr} = 2\sqrt{km} \tag{2.11}$$

Which can be written as (since $\omega_n = \sqrt{k/m}$),

$$c_{cr} = 2m\omega_n = \frac{2k}{\omega_n}$$

With the critical damping coefficient, c_{cr}, we now define the *damping ratio* or *damping factor*, which is an expedient way to define damping for practical structural systems and is usually expressed in percentage form,

$$\zeta = \frac{c}{c_{cr}} \tag{2.12}$$

The damping ratio can be used to rewrite the equation of motion as

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$$\ddot{u} + 2\zeta\omega_n \dot{u} + \omega_n^2 u = 0 \quad (2.13)$$

And it can be used to define the three damping states as follows (see Fig. 2.5):

- For overdamped systems $\zeta > 1$
- For critically damped systems $\zeta = 1$
- For underdamped systems $\zeta < 1$

As shown in Fig. 2.5, the first two cases ($\zeta > 1$ and $\zeta = 1$) do not oscillate; this type of motion is known as *aperiodic*. Notice that after the initial displacement, the SDOF system moves back toward the equilibrium position. An underdamped system ($\zeta > 1$) will oscillate as its motion decays back to the equilibrium position. In the following subsections these three cases are discussed in more detail.

2.2.1 Critically Damped System

Although most structural systems are underdamped, we also need to consider the other two types of systems to fully understand damping. For example, as discussed in the last section, critically damped systems are used to define the damping ratio, ζ, which is typically used to describe damping in structural systems.

The solution for a critically damped system yields unknown parameters that result in repeated roots (with $\zeta = 1$),

$$\rho_1 = \rho_2 = -\zeta\omega_n = \omega_n$$

Since repeated roots only yield one independent solution to Eq. (2.8) [or Eq. (2.13)], we must find a second independent solution, which can be obtained by multiplying the first solution by time; that is,

$$u(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t} = (C_1 + C_2 t)e^{-\omega_n t}$$

We now use initial conditions (at time $t = 0$) to solve for constants C_1 and C_2. First, the initial displacement at time $t = 0$, $u(0) = (C_1 + C_2(0))e^{-\omega_n(0)}$ yields $C_1 = u(0)$. Next, differentiating the displacement equation with respect to time we get, $\dot{u}(t) = (-\omega_n C_1 - \omega_n C_2 t + C_2)e^{-\omega_n t}$.

Evaluating this equation at time $t = 0$, $\dot{u}(0) = (-\omega_n C_1 - C_2(0) + C_2)e^{-\zeta\omega_n(0)}$, yields $C_2 = \dot{u}(0) + u(0)\omega_n$. The complete solution is the free vibration response of a critically damped SDOF system,

$$u(t) = \left\{u(0) + \left[\dot{u}(0) + u(0)\omega_n\right]t\right\}e^{-\omega_n t} \quad (2.14)$$

A graph of which is given in Fig. 2.5. A critically damped structure shows motion that is nonoscillatory and decays to zero over time. The decay in the motion is mathematically produced by the exponential part of the solution.

Example 3

The typical suspension for a vehicle consists of four springs and four viscous dampers. If the weight, say 4,000 lb, of the vehicle causes a 4 in static displacement, determine the required damping coefficient of each damper to achieve critical damping.

Solution

i. *Determine the equivalent stiffness of the suspension assuming a linear force displacement relationship,*

$$k = \frac{W}{u_0} = \frac{4{,}000 \text{ lb}}{4 \text{ in}} = 1{,}000 \, \frac{\text{lb}}{\text{in}}$$

ii. *Determine the mass using the weight and the acceleration due to gravity,*

$$m = \frac{W}{g} = (4{,}000 \text{ lb}) / \ 386.4 \, \frac{\text{in}}{\text{s}^2} \ = 10.35 \, \frac{\text{lb} \cdot \text{s}^2}{\text{in}}$$

iii. *The damping coefficient for this case is equal to the critical damping coefficient, from Eqs. (2.11) and (2.12),*

$$c = \zeta(2\sqrt{km}) = 1 \ 2\sqrt{1{,}000 \, \frac{\text{lb}}{\text{in}} \ \ 10.35 \, \frac{\text{lb} \cdot \text{s}^2}{\text{in}}} \ = 203 \, \frac{\text{lb} \cdot \text{s}}{\text{in}}$$

Thus, the damping coefficient of each of the four dampers is 50.9 lb · s/in. ▲

2.2.2 Overdamped System

For an overdamped system, the quantity under the square root operator is always positive ($c^2 - 4mk > 0$), and thus the exponents are real numbers. The solution for this case is given by Eq. (2.10), which in terms of the damping ratio, ζ, and $\bar{\omega} = \omega_n \sqrt{\zeta^2 - 1}$, can be written as

$$u(t) = e^{-\zeta\omega_n t}\left(C_1 e^{\bar{\omega}t} + C_2 e^{-\bar{\omega}t}\right) \tag{2.15}$$

Applying initial conditions (at time t = 0) to solve for constants C_1 and C_2 results in a system of equations in $u(0)$ and $\dot{u}(0)$, the solution to which results in:

$$C_1 = \frac{\dot{u}(0) + (\zeta\omega_n + \bar{\omega})u(0)}{2\bar{\omega}}$$

$$C_2 = -\frac{\dot{u}(0) + (\zeta\omega_n - \bar{\omega})u(0)}{2\bar{\omega}}$$

The complete solution is the free vibration response of an overdamped SDOF system,

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$$u(t) = \frac{e^{-\zeta\omega_n t}}{2\bar{\omega}}\left\{[\dot{u}(0) + (\zeta\omega_n + \bar{\omega})u(0)]e^{\bar{\omega}t} - [\dot{u}(0) + (\zeta\omega_n - \bar{\omega})u(0)]e^{-\bar{\omega}t}\right\} \tag{2.16}$$

A graph of which is given in Fig. 2.5. Just like the critically damped case, the motion for this case does not oscillate and decays exponentially, albeit more slowly than the critically damped case. That is, it takes the overdamped case more time to return to the neutral position.

2.2.3 Underdamped System

First, we write the unknown parameters ρ_1 and ρ_2 in terms of the damping ratio, ζ, $\rho_1, \rho_2 = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$. These parameters are complex since $\zeta < 1$; and thus, in complex number form, $\rho_1, \rho_2 = (-\zeta \pm i\sqrt{1 - \zeta^2})\omega_n$; that is, they are complex conjugates. The solution given by Eq. (2.10) is

$$u(t) = e^{-\zeta\omega_n t}\ \ C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}$$

This equation can be expressed in polar form (in terms of sines and cosines) by making use of Euler's identities (introduced earlier); and since trigonometric functions are real-value solutions to the equation of motion, we can express the general solution in real form as:

$$u(t) = e^{-\zeta\omega_n t}\left[A\sin\left(\omega_n t\sqrt{1-\zeta^2}\right) + B\cos\left(\omega_n t\sqrt{1-\zeta^2}\right)\right] \tag{2.17}$$

where *A* and *B* are arbitrary constants.

While this motion is periodic, it is no longer of uniform frequency. That is, the frequency (and period) is a function of the damping ratio—the *damped frequency*,

$$\omega_D = \omega_n\sqrt{1-\zeta^2} \tag{2.18}$$

And the *damped period*,

$$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{T_n}{\sqrt{1-\zeta^2}} \tag{2.19}$$

From initial conditions (at time $t = 0$) we can solve for constants *A* and *B*. First, the initial displacement $u(0) = e^0\left[B\sin(0) + A\cos(0)\right]$ yields $A = u(0)$. Then, we differentiate the displacement equation with respect to time to get (substituting $\omega_D = \omega_n\sqrt{1-\zeta^2}$),

$$\dot{u}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}[B\sin\omega_D t + A\cos\omega_D t] + e^{-\zeta\omega_n t}[B\omega_D\cos\omega_D t + A\omega_D\sin\omega_D t]$$

Evaluating this equation at time $t = 0$, $\dot{u}(0) = -\zeta\omega_n e^0[B\sin(0) + A\cos(0)] + e^0[B\omega_D\cos(0) + A\omega_D\sin(0)]$, yields $B = \dfrac{\dot{u}(0) + \zeta\omega_n u(0)}{\omega_D}$. The complete solution is the free vibration response of an underdamped SDOF system,

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$$u(t) = e^{-\zeta\omega_n t}\left\{\left[\frac{\dot{u}(0)+\zeta\omega_n u(0)}{\omega_D}\right]\sin\omega_D t + u(0)\cos\omega_D t\right\} \tag{2.20}$$

A graph of which is given in Fig. 2.5 and compared to the undamped case in Fig. 2.6. The damped structure clearly shows motion decay over time. Again, the decay in the motion is mathematically produced by the exponential part of the solution (dashed lines in Fig. 2.6). This envelops the amplitude of the vibration and is tangent to the solution.

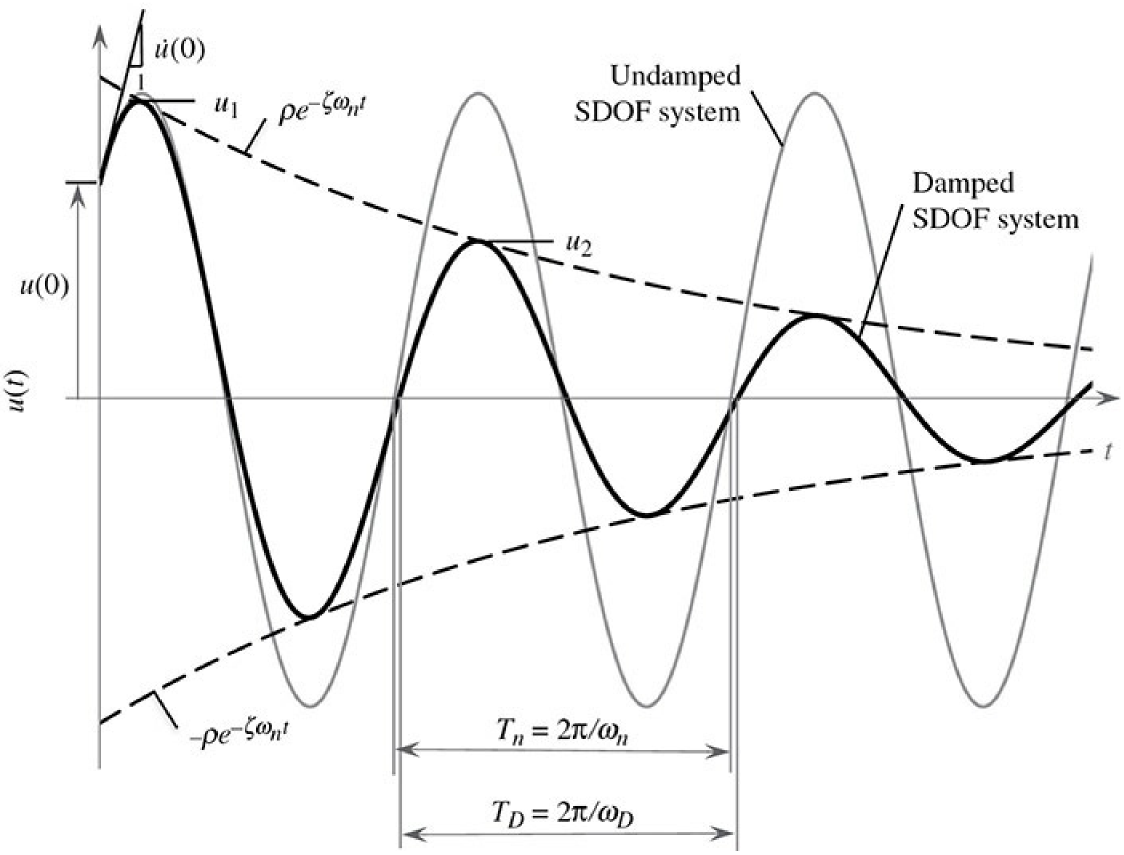

FIGURE 2.6 Comparison of free vibration response of undamped and damped SDOF systems.

Like the undamped free vibration solution, the damped free vibration solution [Eq. (2.20)] can be written in terms of phase angle, ϕ, and amplitude, which decays, $\rho e^{-\zeta\omega_n t}$ (as shown in Fig. 2.6). In terms of sine,

$$u(t) = \rho e^{-\zeta\omega_n t}\sin(\omega_D t + \phi) \tag{2.21}$$

where

$$\rho = \sqrt{u(0)^2 + \left(\frac{\dot{u}(0) + \zeta\omega_n u(0)}{\omega_D}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{u(0)\omega_D}{\dot{u}(0) + \zeta\omega_n u(0)}\right)$$

Similarly, in terms of cosine,

$$u(t) = \rho e^{-\zeta\omega_n t}\cos\left(\omega_D t - \theta\right)$$

where a new phase angle, θ, is introduced,

$$\theta = \tan^{-1}\left(\frac{\dot{u}(0) + \zeta\omega_n u(0)}{u(0)\omega_D}\right)$$

2.2.4 Equivalent Structural Damping Modeled with Viscous Damping

As noted earlier, in addition to attenuating vibration amplitudes, structural damping (modeled as viscous damping) has the effect of lengthening the period from the natural state (or lowering the natural frequency); see Fig. 2.7. Most structural systems exhibit small inherent damping (less than 15% as listed in Table 2.1); and as shown in Fig. 2.7, for this low range of damping, the effects on the period are negligible. Thus, the damped (T_D) and undamped (T_n) periods are practically equal for values of ζ between 0% and 15%, and thus, in practice, T_D is usually taken as T_n (except when supplemental damping is added to a structural system).

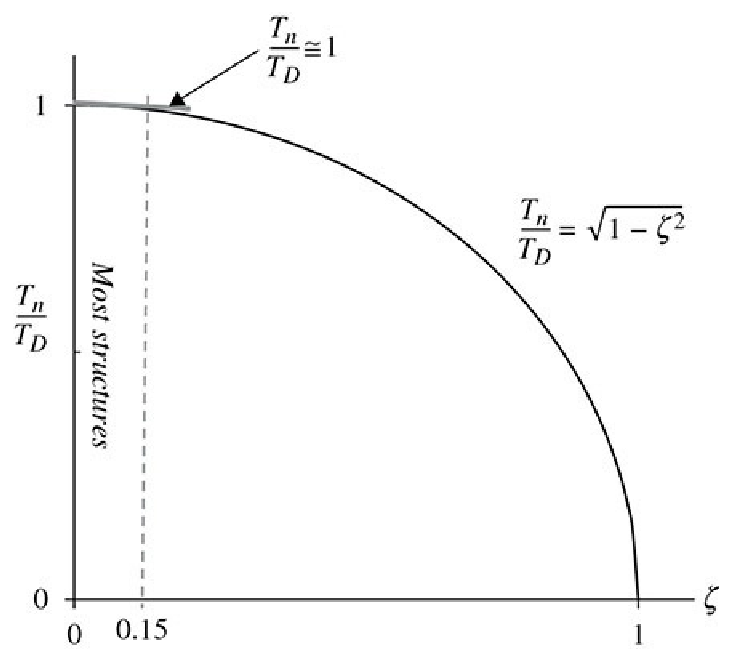

FIGURE 2.7 Effects of damping on vibration period.

Type of Construction	ζ
Steel frame with welded connections and flexible walls	0.02
Steel frame with welded connections and normal floors and exterior cladding	0.05
Steel frame with bolted connections and normal floors and exterior cladding	0.10
Concrete frame with flexible interior walls	0.05
Concrete frame with flexible interior walls and exterior cladding	0.07
Concrete frame with concrete or masonry infill walls	0.10
Concrete or masonry shear walls	0.10
Wood frame and shear walls	0.15

TABLE 2.1 Typical Damping Ratios

Recall that the damping ratio, ζ, characterizes the rate at which oscillations decay from one cycle of motion to the next and has a significant effect on the response of elastic systems (but rather limited beyond that, inelastic response). Current knowledge of damping mechanisms in structural systems is rather limited. It is believed that damping in building systems is the result of the following damping mechanisms: friction in structural elements (friction in bolted and nailed connections and cracking of concrete), intrinsic material damping, friction in nonstructural components and their connections to the structure, and soil-structure interaction (radiation of vibration waves back into the soil and intrinsic damping in the soil). All these damping mechanisms are difficult to characterize; thus, it is difficult to determine structural damping analytically. For this reason, there are a number of references that provide a range of values, a summary for the most common building systems is tabulated in Table 2.1. These values are presented only to illustrate that real structures do not possess inherent damping greater than 15%; when damping is critical in design, a more accurate reference should be consulted. From values listed in Table 2.1, it should also be clear that the damping ratio depends on the type of building construction, not just construction material.

Example 4

A vehicle weighing 4,000 lb with suspension stiffness of 1,000 lb/in is located at midspan on the beam shown below. Neglect the weight of the beam and model the system as a simple oscillator to determine the equivalent mass, equivalent spring constant, and equivalent damping coefficient assuming a damping ratio of 10%. The beam properties are E = 29,000 kip/in^2 and I_x = 1,000 in^4.

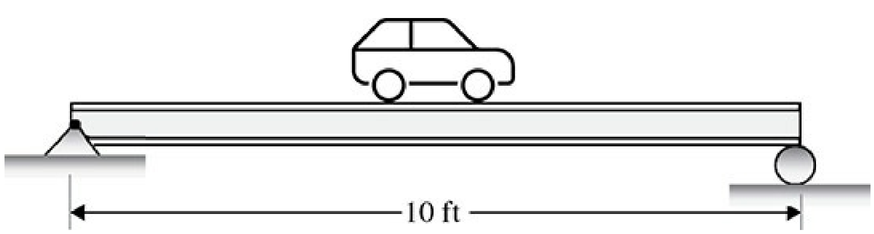

FIGURE E4.1 Simply supported beam model.

Solution

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i. *The equivalent stiffness of the system is a combination of the stiffness of the vehicle suspension and that of the beam using a stiffness model for two springs in series.* First, as shown in Chap. 1, the stiffness of a simply supported beam loaded with a midspan load, *P*, that causes a displacement δ can be calculated using the displacement equation

$$= \frac{PL^3}{48EI}$$

$$k_b = \frac{P}{} = \frac{48EI}{L^3} = \frac{48(29{,}000\ \text{kip/in}^2)1{,}000\ \text{in}^4}{(10\ \text{ft} \cdot 12\ \text{in/ft})^3} = 805\ \text{kip/in}$$

The equivalent stiffness of the system in series is (when the springs are in parallel, they are simply combined),

$$\frac{1}{k_E} = \frac{1}{k} + \frac{1}{k_b} = \frac{1}{1{,}000\ \text{kip/in}} + \frac{1}{805\ \text{kip/in}}$$

$$k_E = 446\ \text{kip/in}$$

ii. *The equivalent mass of the system is based only on the weight of the vehicle since the beam is assumed to be weightless.*

$$m_E = \frac{W}{g} = \frac{4{,}000\ \text{lb}}{386.4\ \text{in/s}^2} = 10.3\ \frac{\text{lb} \cdot \text{s}^2}{\text{in}}$$

iii. *The equivalent damping coefficient can be determined using Eqs. (2.12) and (2.13)*

$$c_E = \zeta(2\sqrt{km}) = 0.1\ \ 2\sqrt{446\ \frac{\text{lb}}{\text{in}}\quad 10.3\ \frac{\text{lb} \cdot \text{s}^2}{\text{in}}} \ = 13.6\ \frac{\text{lb} \cdot \text{s}}{\text{in}}\ \blacktriangle$$

2.2.5 Logarithmic Decrement

The oscillation decay can be used to experimentally determine the inherent damping in a system. The process entails applying an initial displacement and recording the resulting free vibration response. To measure the rate of decay of the amplitude of motion, we use the logarithmic decrement, δ, which is obtained using the ratio of the value of two successive peak displacement amplitudes,

$$u_1 = u(t_1) = \rho e^{-\zeta\omega_n t_1} \sin(\omega_D t_1 + \phi) \text{ and}$$

$$u_2 = u(t_1 + T_D) = u(t_1 + 2\pi/\omega_D)$$

$$= \rho e^{-\zeta\omega_n (t_1 + 2\pi/\omega_D)} \sin\ \ \omega_D(t_1 + 2\pi/\omega_D) + \phi$$

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$$= \rho e^{-\zeta\omega_n(t_1+2\pi/\omega_D)}\sin(\omega_D t_1+\phi)$$

Note that $\sin[\omega_D(t_1+2\pi/\omega_D)+\phi]=\sin[\omega_D t_1+\phi+2\pi]=\sin(\omega_D t_1+\phi)$ since the values of the sine function repeat at intervals of 2π. Taking the ratio of these displacements,

$$\frac{u_1}{u_2}=\frac{\rho e^{-\zeta\omega_n t_1}}{\rho e^{-\zeta\omega_n(t_1+2\pi/\omega_D)}}=e^{\zeta\omega_n(t_1+2\pi/\omega_D)-\zeta\omega_n t_1}=e^{\zeta\omega_n 2\pi/\omega_D}$$

The natural log of both sides of this relationship yields the logarithmic decrement,

$$\delta=\ln\left(\frac{u_1}{u_2}\right)=\zeta\omega_n\frac{2\pi}{\omega_D}=\zeta\omega_n T_D \tag{2.22}$$

Replacing the damped frequency, $\omega_D=\omega_n\sqrt{1-\zeta^2}$ in Eq. (2.22), yields δ in terms of ζ only,

$$\delta=\frac{\zeta\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}}=\frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.23}$$

For small damping ratio values (up to ζ = 0.3), which include all practical structures, the value under the square root operator is close to unity and the logarithmic decrement can be approximated as

$$\delta\approx 2\pi\zeta \tag{2.24}$$

For lightly damped systems where successive oscillation peaks have similar ordinates, values of ζ can more accurately be determined using two peak amplitudes several cycles apart u_i and u_{i+n} (where n is any integer),

$$\ln\frac{u_i}{u_{i+n}}=n\zeta\omega_n T_D=n\delta=\frac{2\pi\zeta n}{\sqrt{1-\zeta^2}} \tag{2.25}$$

For small damping,

$$\ln\frac{u_i}{u_{i+n}}=2\pi\zeta n$$

$$u(t)=\rho e^{-\zeta\omega_n t}\cos(\omega_D t-\theta)$$

Also, the logarithmic decrement can be defined in terms of the acceleration since it is easier to measure acceleration (using an accelerometer) than displacement. First, take the derivative of twice,

$$\dot{u}(t) = \rho e^{-\zeta\omega_n t}\left[-\zeta\omega_n \cos(\omega_D t-\theta) - \omega_D \sin(\omega_D t-\theta)\right]$$

$$\begin{aligned}\ddot{u}(t) = \rho e^{-\zeta\omega_n t}\Big\{&\zeta\omega_n\left[\zeta\omega_n \cos(\omega_D t-\theta) + \omega_D \sin(\omega_D t-\theta)\right] \\ &+ \omega_D\left[\zeta\omega_n \sin(\omega_D t-\theta) - \omega_D \cos(\omega_D t-\theta)\right]\Big\}\end{aligned}$$

$$\ddot{u}(t) = \rho e^{-\zeta\omega_n t}\left[(\zeta\omega_n)^2 \cos(\omega_D t-\theta) + 2\omega_D\zeta\omega_n \sin(\omega_D t-\theta) - \omega_D^2 \cos(\omega_D t-\theta)\right]$$

Again, values of the sine (and cosine) function repeat at intervals of 2π. That is, values of sine and cosine functions are equal when the time is increased by T_D. Taking the natural log of both sides of the ratio of the accelerations at two suggestive peak values yields the logarithmic decrement, δ, in terms of the acceleration, which is identical to Eqs. (2.22) or (2.23), albeit in terms of the acceleration,

$$\delta = \ln\frac{\ddot{u}_1}{\ddot{u}_2} = \zeta\omega_n T_D = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.26}$$

Example 5

A free vibration test is conducted on a cellphone tower, which can be modeled as a SDOF inverted pendulum system. The tower is first pulled with a lateral force of 15 kips that produces a 3 in horizontal displacement. The load is suddenly released, and the ensuing free vibration is recorded. At the end of four complete cycles, the time is 2 seconds, and the amplitude is 1 in. Using this information, compute the damping ratio, natural period, effective stiffness, effective weight, damping coefficient, and number of cycles required for the displacement to decrease to 0.1 in.

FIGURE E5.1 Cellphone tower.

Solution

i. *The damping ratio is determined using $u_i = u_1 = 3$ in at $t = 0$ second and $u_{i+n} = u_5 = 1$ in at $n = 4$, from Eq. (2.25),*

$$\ln \frac{u_1}{u_5} = \ln \frac{3}{1} = n\zeta 2\pi$$

Solving for ζ,

$$\zeta = \frac{1}{4 \times 2\pi} \ln \frac{3}{1} = 0.0437, \text{ or } 4.37\%$$

Since this is much less than 30%, a small damping assumption is applicable.

ii. *The damped period is determined by dividing the elapsed time by the number of cycles,*

$$T_D = 2 \text{ s}/4 \text{ cycles} = 0.5 \text{ s}$$

Small damping leads to a natural period that is approximately equal to the damped period.

$$T_n \cong T_D = 0.5 \text{ s}$$

iii. *Determine the equivalent stiffness of the tower assuming a linear force-displacement relationship,*

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$$k = F/u_0 = 15 \text{ kip}/3 \text{ in} = 5 \text{ kip/in}$$

iv. *Determine the effective weight using the period and the stiffness, from Eq. (2.5),*

$$T_n = 2\pi\sqrt{m/k}$$

$$m = \frac{T_n^2}{4\pi^2}k = \frac{(0.5 \text{ s})^2}{4\pi^2}5\,\frac{\text{kip}}{\text{in}} = 0.0317\,\frac{\text{kip}\cdot\text{s}^2}{\text{in}}$$

$$W = mg = (0.0317 \text{ kip}\cdot\text{s}^2/\text{in})(386.4 \text{ in/s}^2) = 12.23 \text{ kip}$$

v. *To determine the damping coefficient, use Eqs. (2.12) and (2.13)*

$$c = \zeta(2\sqrt{km}) = 0.0437\,(2\sqrt{(5 \text{ kip/in})(0.0317 \text{ kip}\cdot\text{s}^2/\text{in})}) = 0.035 \text{ kip}\cdot\text{s/in}$$

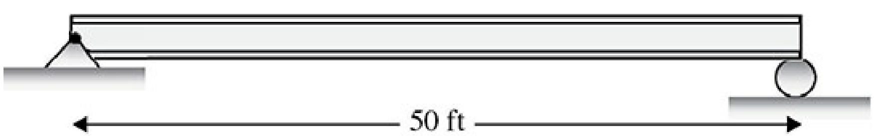
vi. *Use the logarithmic decrement, δ, to determine the number of cycles required for the displacement to diminish to 0.1 in, from Eq. (2.25) for small damping,*

$$\ln\frac{u_i}{u_{i+n}} = 2\pi n\zeta$$

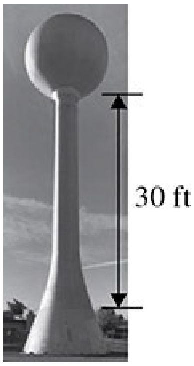
$$n = \frac{1}{2\pi\zeta}\ln\frac{u_i}{u_{i+n}} = \frac{1}{2\pi(0.0437)}\ln\frac{2 \text{ in}}{0.1 \text{ in}} = 10.9 \text{ cycles} \quad \text{or} \quad 11 \text{ cycles} \blacktriangle$$

2.3 Problems

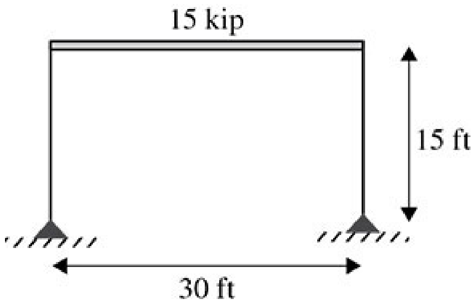
2.1 Given the simply supported beam shown and assuming its 200 kip weight is lumped at midspan, determine the natural frequency and period of the system (E = 29,000 kip/in^2 and I_x = 13,000 in^4).



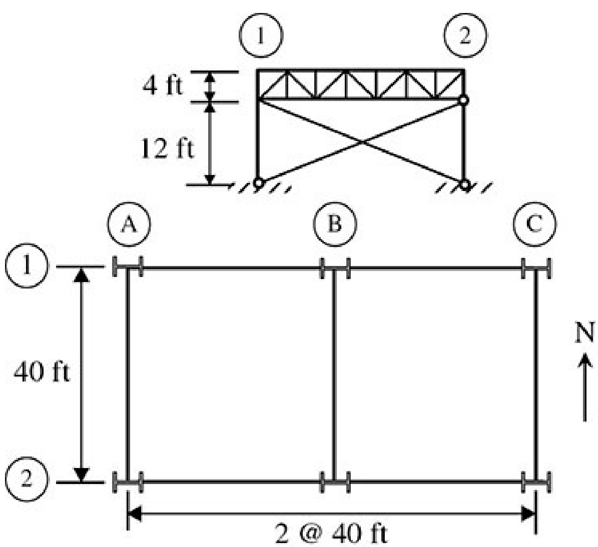
2.2 For the 5 kip water tank shown below, determine the natural frequency and period of the system. Assume the supporting column has I = 10,000 in^4 and $E = 3 \times 10^7$ psi.



2.3 Determine the period for the following steel (E = 29,000 kip/in^2) building frame. The columns have I_x = 75 in^4 and the beam has I_x = 150 in^4.



2.4 The structural system depicted below has a roof weighing 22.5 psf and side sheathing weighing 10 psf. The lateral force resisting system in the North-South direction (one on each side, along column lines A and C) consists of bracing with ½ in steel rods (E = 29,000 ksi). Determine the natural period of the building assuming the total weight is concentrated at the bottom of the roof trusses.



2.5 Determine the natural period for the following structural system.

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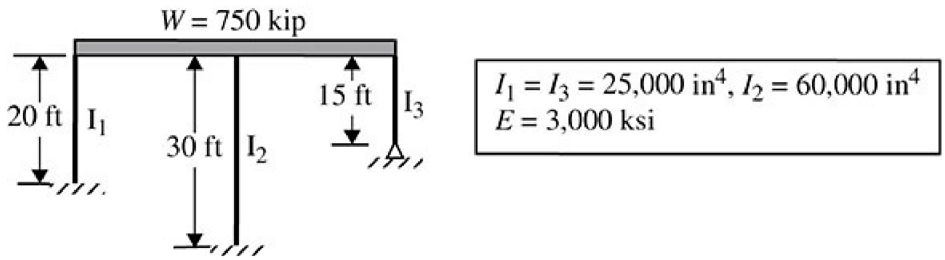
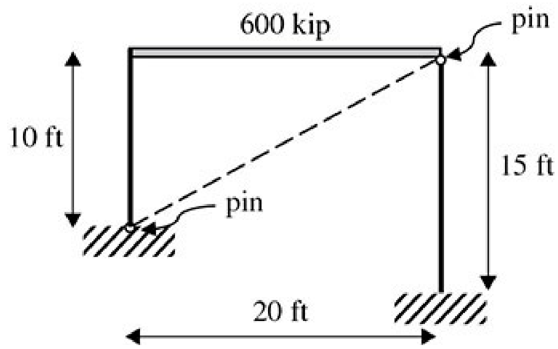
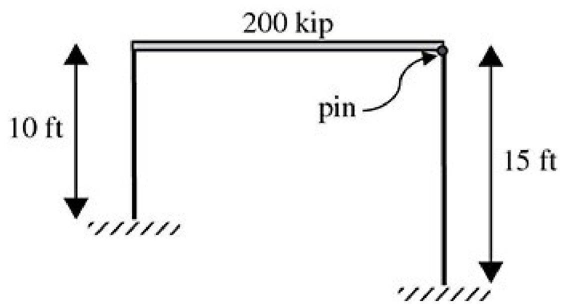


2.6 The following frame has a rigid beam, a ½-in-diameter diagonal steel (E = 29,000 ksi) brace, and two steel columns (I_x = 82.7 in^4). Determine its natural frequency and period.



2.7 Determine the natural frequency and period of the following building frame, which has a rigid beam that is pin connected to one column and rigidly connected to the other as shown; each column has EI = 40,000,000 kip · in^2.



2.8 The tower shown below is in the city of Tequila, Jalisco, Mexico. To determine the dynamic properties of the tower, a cable is attached to the top and pulled with a crane with a force of 20 kip, which causes a horizontal displacement of 1 in. After the cable is suddenly cut, the resulting free vibration is recorded. At the end of 10 complete cycles, the time is 1 second and the amplitude is 0.11 in. From this information, compute (a) undamped natural period, (b) effective stiffness, (c) effective weight, and (d) effective damping coefficient.

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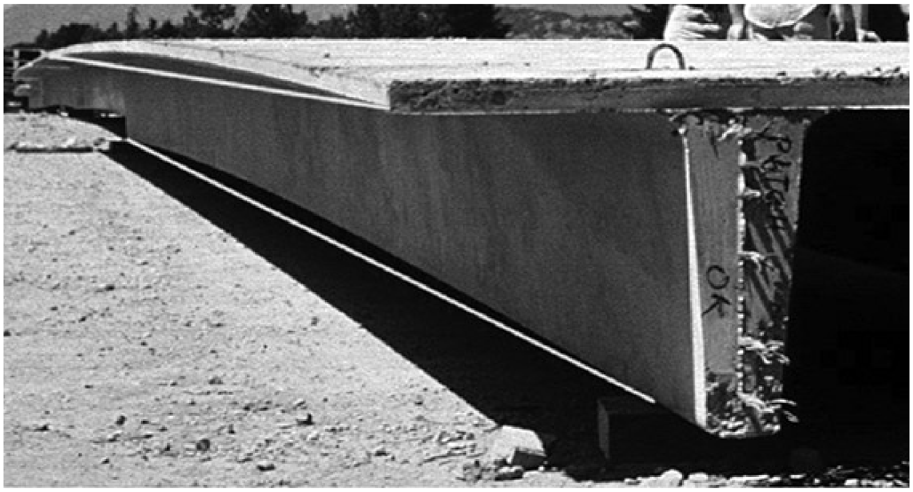
2.9 A wind turbine can be modeled as a concentrated mass atop a weightless column. To determine the dynamic properties of the system, a cable is attached to the turbine and a crane. A lateral force of 200 lbs is applied to the turbine causing a horizontal displacement of 1.0 in. When the cable is suddenly cut, the resulting free vibration is recorded. At the end of four complete cycles, the time is 1.25 seconds, and the amplitude is 0.64 in. From this information, compute the (a) undamped natural period, (b) effective stiffness, (c) effective weight, and (d) effective damping coefficient.

2.10 To determine the dynamic properties of the bridge girder shown, after it has been installed, a 10 kip weight is hoisted with a cable from its midspan, which produces a 2.0 in deflection at midspan. After the cable is suddenly cut, the resulting free vibration is recorded. At the end of five complete cycles, the time is 0.25 second and the amplitude is 0.9 in. From this information, compute (a) undamped natural period, (b) effective stiffness, (c) effective weight, and (d) effective damping coefficient.



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CHAPTER 3

Forced Vibration Response of SDOF Systems —Harmonic Loading

After reading this chapter, you will be able to:

a. Determine the vibration response of an undamped system excited by a harmonic load
b. Describe resonance and beating phenomena
c. Determine the vibration response of a damped system excited by a harmonic load
d. Describe steady-state and transient responses of an oscillator
e. Formulate and solve the equation of motion due to support motion
f. Determine transmissibility for undamped and damped systems
g. Determine response magnification factors for displacement, velocity, and acceleration
h. Estimate damping using response magnification factors
i. Describe vibration isolation

Forced vibration response of a single-degree-of-freedom (SDOF) system is caused by a force in combination with initial position and velocity (unlike free vibration response where only initial conditions mobilized a response of the system). Again, to characterize the response of a SDOF system subjected to a time-dependent force, we first need to formulate the equation of motion for the system by applying equilibrium to a free-body diagram (FBD) of the mass using D'Alembert's principle; here we include the time-dependent force $p(t)$, as illustrated in Fig. 3.1 and described in Chap. 1. Depending on the frequency of the force, it can interrupt the system oscillatory motion (even if no damping is present), or it can amplify the amplitude. For damped cases, the motion from initial conditions is dissipated (transient response), but the system continues to vibrate at the applied forcing frequency (steady-state response).

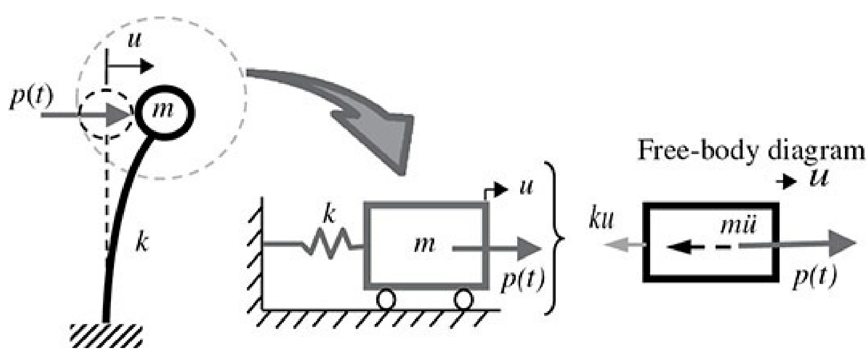

FIGURE 3.1 Free-body diagram of the SDOF model including a time-dependent disturbing force.

In this chapter, we use a periodic, harmonic excitation force, which is a force with magnitudes represented by sine or cosine as functions of time. The formulation derived in this chapter can also be applied to nonharmonic loading cases with the response being obtained using a Fourier method—a superposition of individual responses to the harmonic components of the external excitations.

3.1 Vibration Response of Undamped SDOF Systems Subjected to Harmonic Loading

Consider the portal frame model discussed in Chap. 1 subjected to a time-dependent excitation force, $p(t)$. Following the approach in Sec. 1.5.1, we can derive the equation of motion by applying horizontal equilibrium to the FBD of the oscillator shown in Fig. 3.1 [which is the same as the Eq. (1.2)],

$$\sum F_x = 0; \quad -m\ddot{u} - ku + p(t) = 0 \Rightarrow m\ddot{u} + ku = p(t) \tag{3.1}$$

First, assume a harmonic forcing function of the following form,

$$p(t) = p_0 \sin \omega t \tag{3.2}$$

where

p_0 is the peak magnitude of the force.

ω is the frequency of the force (forcing frequency) in rad/s.

This forcing function produces bound displacements provided that $\omega \neq \omega_n = \sqrt{k/m}$, the resonant condition, which will be discussed in more detail later.

With a forcing function defined, we can solve the equation of motion following standard methods used in the solution of nonhomogeneous differential equations. Equation (3.1) can be rewritten as,

$$m\ddot{u} + ku = p_0 \sin \omega t$$

The general solution to this equation can be expressed as a combination of the particular and complementary solutions,

$$u(t) = u_c(t) + u_p(t) \tag{3.3}$$

where

u_c is the complementary solution, which is the solution to the homogeneous equation (the free vibration solution) given by Eq. (2.3) and repeated here for convenience,

$$u_c(t) = A\cos \omega_n t + B \sin \omega_n t$$

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u_P is the particular solution, which usually takes the same form as the forcing function and in this case, it is assumed to be in phase with the loading; that is,

$$u_p(t) = C\sin\omega t \tag{3.4}$$

where C is the peak value and can be determined by substituting Eq. (3.4) into the original equation of motion [Eq. (3.1)],

$$-m\omega^2 C\sin\omega t + kC\sin\omega t = p_0 \sin\omega t$$

After factoring the sine function that appears in each term, and recognizing that this function is not in general zero, we can rewrite this equation as,

$$-m\omega^2 C + kC = p_0$$

Solving for C,

$$C = \frac{p_0}{k - m\omega^2} = \frac{p_0}{k(1-(m/k)\omega^2)} = \frac{(u_{st})_0}{1-r^2}$$

In the equation above, $(u_{st})_0 = p_0/k$ is the static displacement caused by p_0; and the relationship $m/k = 1/\omega_n^2$ is used to express the equation in terms of a frequency ratio, r, which is the ratio of the forcing frequency, ω to the natural frequency, ω_n, as shown in Eq. (3.5).

$$r = \frac{\omega}{\omega_n} \tag{3.5}$$

We now obtain the total dynamic response of the system by combining the complementary and particular solutions,

$$u(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{(u_{st})_0}{1-r^2}\sin\omega t \tag{3.6}$$

The two constants A and B are arbitrary and can be obtained by evaluating the equation at time t = 0 (initial conditions). Again, the displacement at time t = 0 is u(0); that is,

$$u(0) = A\cos(0) + B\sin(0) + \frac{(u_{st})_0}{1-r^2}\sin(0)$$

which yields $A = u(0)$, the same as for the free vibration case. Next, we differentiate the displacement, Eq. (3.6), with respect to time to obtain the velocity,

$$\dot{u}(t) = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t + \omega\frac{(u_{st})_0}{1-r^2}\cos\omega t \tag{3.7}$$

The velocity at time t = 0 is $\dot{u}(0)$ and yields $B = \frac{\dot{u}(0)}{\omega_n} - \frac{(u_{st})_0}{1-r^2}r$. The complete solution describing the position of the mass of an undamped SDOF system as a function of time is given as,

$$u(t) = u(0)\cos\omega_n t + \left(\frac{\dot{u}(0)}{\omega_n} - \frac{(u_{st})_0}{1-r^2}r\right)\sin\omega_n t + \frac{(u_{st})_0}{1-r^2}\sin\omega t \tag{3.8}$$

In general, the total response can be considered the combination of two distinct response motions as shown in Fig. 3.2 (for r = 0.2 and initial conditions, $u(0) = 0.5p_0/k$ and $\dot{u}(0) = \omega_n p_0/k$). Note that the displacement has been normalized by the equivalent static displacement and the time normalized by the forcing function period.

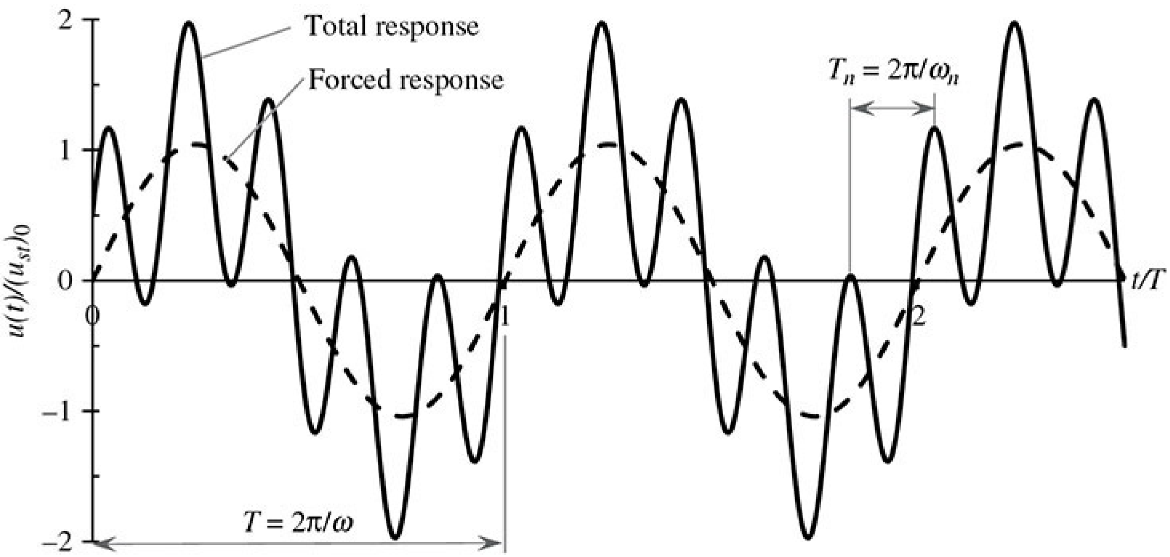

FIGURE 3.2 Forced and total response for r = 0.2, $u(0) = 0.5p_0/k$, and $\dot{u}(0) = \omega_n p_0/k$.

Example 1

Given an undamped SDOF system with stiffness k = 500 lb/in, mass m = 0.129 lb.s^2/ in, initial displacement u (0) = 0 in, and initial velocity $\dot{u}(0)$ = 0 in/s, write the equation of motion and draw, to scale, a graph of the forced and total responses for a harmonic force of $p(t) = 200\ \text{lb}\cdot\sin[(4\ \text{rad/s})t]$.

Solution

i. *Determine natural frequency of the system using the given stiffness and mass; k = 500 lb/in and mass m = 0.129 lb.s^2/in, from Eq. (2.2),*

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500\ \text{lb/in}}{0.129\ \text{lb}\cdot\text{s}^2/\text{in}}} = 62.2\ \text{rad/s}$$

Given the forcing frequency of ω = 4 rad/s, we can get the frequency ratio using Eq. (3.5),

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$$r = \frac{\omega}{\omega_n} = \frac{4 \text{ rad/s}}{62.2 \text{ rad/s}} = 0.0643$$

ii. *The equation of motion is given by Eq. (3.8),*

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n} - \frac{p_0/k}{1-r^2}r \;\; \sin\omega_n t + \frac{p_0/k}{1-r^2}\sin\omega t$$

Substituting the various known parameter values, we get

$$\begin{aligned} u(t) &= \frac{p_0/k}{1-r^2}(\sin\omega t - r\sin\omega_n t) \\ &= \frac{\dfrac{200 \text{ lb}}{500 \text{ lb/in}}}{1-(0.0643)^2}\{\sin[(4 \text{ rad/s})t] - 0.0643\sin[(62.2 \text{ rad/s})t]\} \\ &= 0.402 \text{ in}\{\sin[(4 \text{ rad/s})t] - 0.0643 \sin[(62.2 \text{ rad/s})t]\} \end{aligned}$$

iii. *A graph of this equation or the total response (solid line) is shown plotted along with the steady-state or forced response (dash line) in the following figure:*

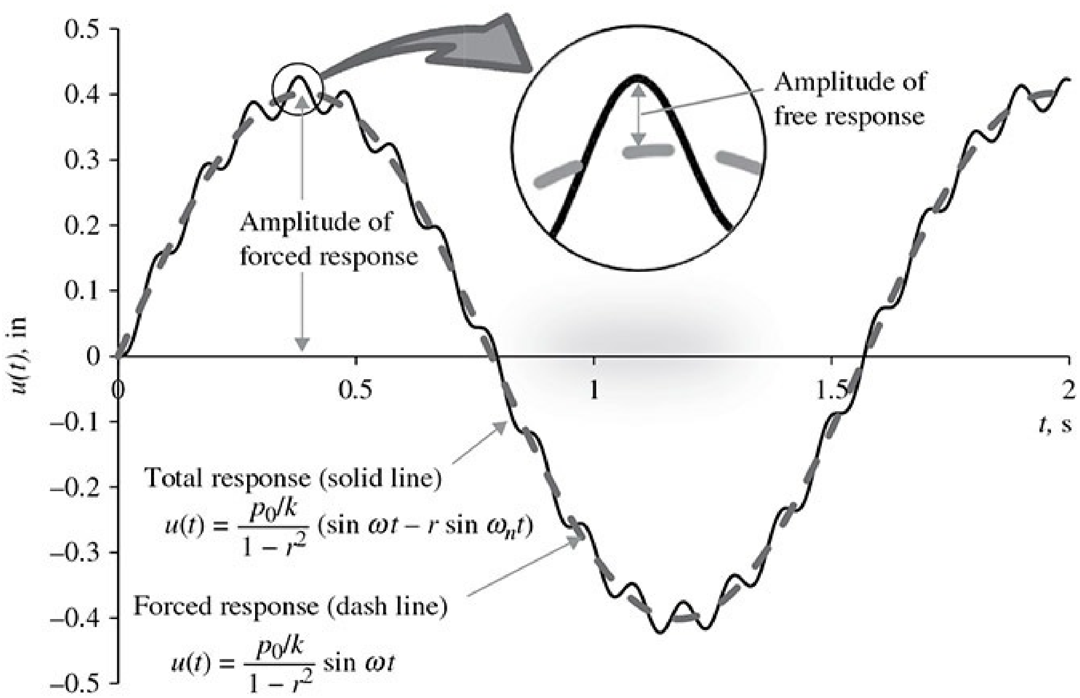

FIGURE E1.1 Forced and total response graphs.

Notice that the maximum amplitude of the total response is not equal to the maximum amplitude of the two separate responses. To obtain the maximum amplitude of the total response, we must find the displacement at the time when the total velocity equals zero.▲

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In general, when the displacement and forcing function amplitudes act in the same direction (i.e., in phase), the response becomes unbounded as ω (forcing function frequency) approaches ω_n (natural circular frequency of the system); that is, as the frequency ratio, r, approaches 1, the total displacement goes to infinity and two important phenomena occur: *beating* and *resonance*.

$$u_p(t) = p_0 \cos \omega t$$

First, consider *beating* when ω is nearly equal to, but slightly less than, ω_n. For this analysis, assume a cosine forcing function. Following the same steps as the preceding paragraphs, we get a total solution of

$$u(t) = \left(u(0) - \frac{(u_{st})_0}{1-r^2}\right)\cos \omega_n t + \left(\frac{\dot{u}(0)}{\omega_n}\right)\sin \omega_n t + \frac{(u_{st})_0}{1-r^2}\cos \omega t \tag{3.9}$$

Now consider a particular case of this equation where initial displacement and velocity are zero,

$$u(t) = \frac{(u_{st})_0}{1-r^2}\left[\cos \omega t - \cos \omega_n t\right] \tag{3.10}$$

The term in brackets can be transformed using identity $\cos A - \cos B = 2\sin\frac{B-A}{2}\sin\frac{A+B}{2}$.

The result is,

$$u(t) = \left[2\frac{(u_{st})_0}{1-r^2}\sin\left(\frac{\omega_n - \omega}{2}t\right)\right]\sin\left(\frac{\omega + \omega_n}{2}t\right) \tag{3.11}$$

Figure 3.3 depicts a graph of Eq. (3.11) (solid line), and clearly shows a rapid, oscillating motion that varies periodically with a period of $T_1 = \frac{4\pi}{\omega + \omega_n}$, but with a slowly varying sinusoidal amplitude (dash line) of $2\frac{(u_{st})_0}{1-r^2}\sin\frac{\omega_n - \omega}{2}t$. This amplitude also varies periodically with a much longer period, $T_b = \frac{4\pi}{\omega_n - \omega}$ called the *beating period*. This phenomenon happens when mechanical equipment is operated from start to a frequency larger than the natural frequency of the supporting system. Note that the ratio of the two periods is the number of oscillations in each beat.

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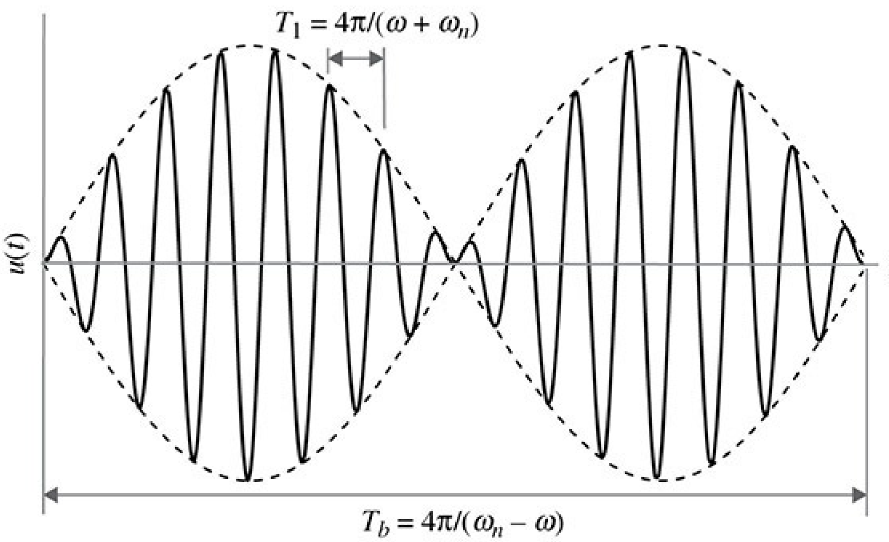

FIGURE 3.3 Beating phenomenon.

As the forcing frequency, ω, moves very close to the natural circular frequency of the system, ω_n, the beating period, T_b, becomes longer and is infinite when $\omega = \omega_n$, and *resonance* occurs. Theoretically, at resonance the displacement amplitude of the system becomes infinite, given enough time. However, real structures do not experience unbound displacement amplitudes because their stiffness decreases beyond the elastic limit, greatly affecting the system characteristics, and because of damping. Still, deformations near resonance can become intolerably large and even dangerous because they can cause structural failure, even for regularly applied small impulses. Also, the first two terms in Eq. (3.8) (the complementary solution) eventually vanish due to damping; these terms represent the *transient* response.

Although the derivation of the equation of motion was based on a small displacement assumption and is no longer valid during resonance, we can use the resulting solution, Eq. (3.11), to show how the displacement amplitude grows unbounded at resonance. First, rewrite Eq. (3.11) in terms of frequencies,

$$u(t) = 2\frac{(u_{st})_0\omega_n^2}{(\omega_n^2 - \omega^2)}\sin\frac{\omega_n - \omega}{2}t \quad \sin\frac{\omega + \omega_n}{2}t$$

$$= 2\frac{(u_{st})_0\omega_n^2}{(\omega_n - \omega)(\omega_n + \omega)}\sin\frac{\omega_n - \omega}{2}t \quad \sin\frac{\omega + \omega_n}{2}t$$

In the limit as ω approaches ω_n, $\omega_n + \omega \cong 2\omega_n$ and let $\omega_n - \omega = \delta$, where δ represents a very small value. Therefore, $\sin\frac{\omega_n - \omega}{2}t = \sin\frac{\delta}{2}t \cong \frac{\delta}{2}t$. We can now write the equation as,

$$u(t) = 2\frac{(u_{st})_0\omega_n^2}{2\delta\omega_n}\frac{\delta}{2}t \ \sin\frac{2\omega}{2}t = \frac{(u_{st})_0\,\omega_n}{2}t \ \sin(\omega t)$$

A plot of the relationship, shown in Fig. 3.4, clearly indicates that the displacement amplitude increases uniformly with time. The rate of growth (dash lines) is given by the quantity in brackets, $\frac{(u_{st})_0 \omega_n}{2} t$. The linear increment between two successive peaks can be obtained by setting $t = T_n = 2\pi/\omega_n$, which results in $(u_{st})_0 \neq$ as shown in Fig. 3.4. This system can operate at resonance for an extended period before the amplitude becomes destructively large; this is why it is acceptable to operate mechanical equipment in route to its operating frequency so long as the machine quickly passes through the resonance frequency.

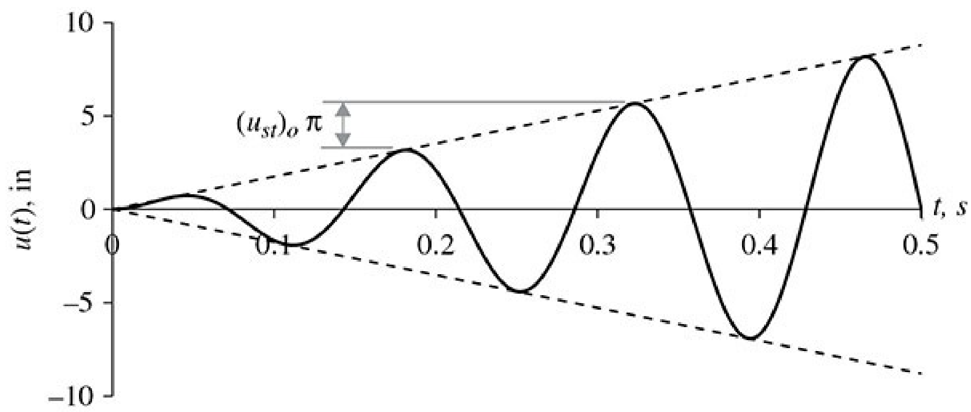

Figure 3.4 Response at resonance.

Further analysis yields additional insight into the response of the oscillator. Recall that the first two terms in Eq. (3.8) are the free vibration (complementary) response and depend primarily on the initial conditions and the natural frequency of the system, ω_n, which results in a period $T_n = 2\pi/\omega_n$ as shown in Fig. 3.2. This portion of the solution is said to be *transient* because in real structures (even with small damping) it eventually vanishes. The last term in Eq. (3.8) is the forced (particular) response, which depends only on the amplitude and frequency of the applied force, ω. This term corresponds to a *steady-state* response because it persists after the transient vibration dissipates and eventually becomes the total solution. To illustrate this, let us rewrite the particular solution to include a phase angle, ϕ

$$u_p(t) = u_0 \sin(\omega t - \phi) \tag{3.12}$$

where

u_0 is the maximum steady-state displacement response.
ϕ is the phase angle of the forcing function.

Comparing Eq. (3.12) to the last term in Eq. (3.8)

$$u_p(t) = u_0 \sin(\omega t - \phi) = \frac{(u_{st})_0}{1 - r^2} \sin \omega t$$

yields

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$$u_0 = \frac{(u_{st})_0}{1-r^2}$$

and

$$\phi = \begin{matrix} 0^\circ & \omega < \omega_n \\ 180^\circ & \omega > \omega_n \end{matrix}$$

The ratio of the maximum steady-state displacement amplitude, u_0, to the equivalent static displacement, $(u_{st})_0$, is known as the deformation response factor and represents the dynamic magnification factor

$$R_d = \frac{u_0}{(u_{st})_0} = \frac{1}{1-r^2} \tag{3.13}$$

Figure 3.5 shows graphs depicting R_d and ϕ as functions of r. First, note that the displacement grows unbounded as r approaches 1, both from the right and the left. This point on the graph, r = 1, corresponds to *resonance*. In actual practice, however, the system would come apart or yield (changing the stiffness, thus changing the natural frequency).

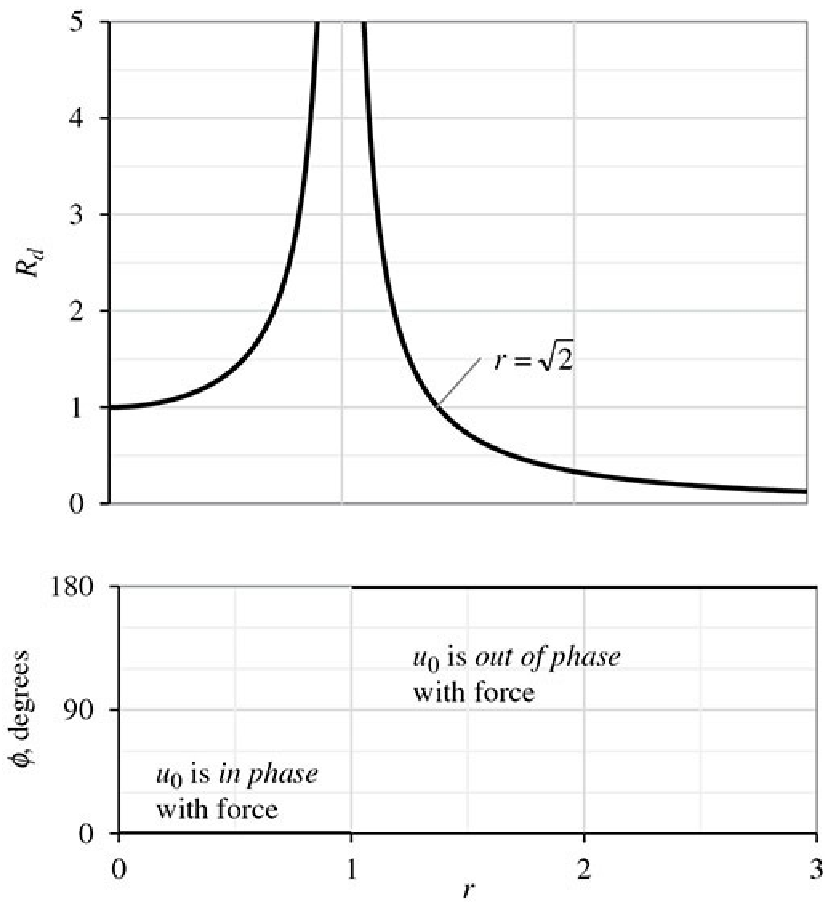

FIGURE 3.5 Deformation response factor, R_d, and phase angle, ϕ, as functions of frequency ratio,

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r.

Also, at this point, the excitation force changes from being vibrating in phase with the applied force to being out of phase; that is, the system displacement and applied force act in the same direction (in phase) for values of $r < 1$; however, for values of $r > 1$, the system displacement acts in the opposite direction of the applied force (out of phase). Furthermore, notice that for small values of r (slowly varying force), R_d is approximately equal to 1; and for large values of r (rapidly varying force), R_d approaches zero, which corresponds to no displacement. An additional point of interest along the abscissa of these graphs is $r = \sqrt{2}$, beyond which all values of R_d are less than or equal to 1; thus, dynamic magnification only occurs for $r < \sqrt{2}$. This implies that the dynamic displacement is less than the static displacement beyond $r = \sqrt{2}$. The point can be obtained by setting $R_d = 1$.

Example 2

Determine the maximum steady-state amplitude of the horizontal motion of the following frame and loading. Assume the steel (E = 29,000 ksi) columns are W10 × 33 (I_x = 171 in^4) and a rigid beam. Also, neglect damping and the mass of the columns.

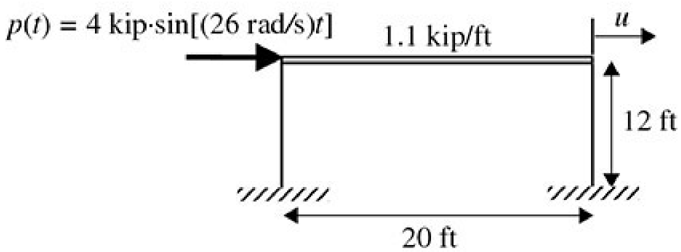

FIGURE E2.1 Portal frame model.

Solution

i. *Determine mass, stiffness, and natural frequency of the SDOF system. This frame can be modeled as a SDOF system with the columns providing the stiffness of the system. The mass and stiffness of the SDOF system are calculated as follows:*
   Mass,

$$m = \frac{W}{g} = \frac{1.1\ \text{kip/ft}(20\ \text{ft})(1{,}000\ \text{lb/kip})}{386.4\ \text{in/s}^2} = 56.9\ \frac{\text{lb}\cdot\text{s}^2}{\text{in}}$$

   Stiffness (two pinned-fixed connected columns),

$$k = \frac{12EI}{L^3} + \frac{12EI}{L^3} = 2\ \frac{12(29{,}000{,}000\ \text{psi})(171\ \text{in}^4)}{(12\ \text{ft} \times 12\ \text{in/ft})^3} = 39{,}858\ \frac{\text{lb}}{\text{in}}$$

   The natural frequency of the SDOF system is

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$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{39{,}858 \text{ lb/in}}{56.9 \text{ lb}\cdot\text{s}^2/\text{in}}} = 26.5 \frac{\text{rad}}{\text{s}}$$

ii. *Determine the maximum steady-state displacement, u_0. This can be determined using the definition of the deformation response factor in Eq. (3.13). First, the equivalent static displacement is calculated using the basic force-displacement relationship for a spring as follows:*

$$(u_{st})_0 = \frac{p_0}{k} = \frac{4{,}000 \cdot \text{lb}}{39{,}858 \text{ lb/in}} = 0.100 \text{ in}$$

$$\omega = 26 \text{ rad/s}$$

Also, given forcing frequency of , the frequency ratio (forcing frequency to natural frequency) can be determined as follows.

$$r = \frac{\omega}{\omega_n} = \frac{26 \text{ rad/s}}{26.5 \text{ rad/s}} = 0.983$$

Now substituting into Eq. (3.13), the maximum steady-state displacement is

$$R_d = \frac{u_0}{(u_{st})_0} = \frac{1}{1 - r^2}$$

$$\xrightarrow{\text{yields}} u_0 = \frac{(u_{st})_0}{1 - r^2} = \frac{0.100 \text{ in}}{1 - (0.983)^2} = 2.92 \text{ in} \blacktriangle$$

Example 3

Wind-driven waves are generated during the transfer of wind energy across the ocean's surface and can be modeled using a sine wave; the top of the wave is called the crest, the bottom is the trough, and the distance between two successive crests (or troughs) is the wavelength. The ocean surface energy (in the form of waves) imparted to an offshore structure depends on the wave period. The most dominant period of a wind-driven wave is approximately 8 seconds.

Offshore structures can be modeled as SDOF systems subject to dynamic loading due to wind-driven wave forces. The natural period of an offshore structure depends on the ocean depth where the system is installed, with larger depths corresponding to higher natural periods. Considering two offshore systems exposed to wind-driven wave forces, one shallow-water structure (~100-ft depth with a natural period of 1 second) and a deep-water structure (~1,000-ft depth with natural period of 5 seconds), determine if their response is dynamic with respect to the dynamic amplification factor, R_d.

Solution Establish the frequency ratio in terms of the periods using Eqs. (2.2) and (3.5),

$$r = \frac{\omega}{\omega_n} = \frac{2\pi/T}{2\pi/T_n} = \frac{T_n}{T}$$

i. *Use this equation to determine frequency ratio for the shallow-water system,*

$$r = \frac{1\ \text{s}}{8\ \text{s}} = 0.125$$

Now determine the dynamic amplification factor with Eq. (3.13),

$$R_d = \frac{1}{1 - r^2} = \frac{1}{1 - (0.125)^2} = 1.016$$

The structural response of the shallow-water system is essentially static, and hydrostatic analysis may be employed.

ii. *Now determine frequency ratio for the deep-water system,*

$$r = \frac{5\ \text{s}}{8\ \text{s}} = 0.625$$

The dynamic amplification factor with Eq. (3.13),

$$R_d = \frac{1}{1 - r^2} = \frac{1}{1 - (0.625)^2} = 1.642$$

The structural response of the deep-water system is clearly dynamic, and hydrodynamic analysis must be employed.

Note that in both of these cases we ignored the effect of damping and the effect of fluid drag, both of which have significant influence on the response of all offshore systems. ▲

3.2 Vibration Response of Damped SDOF Systems Subjected to Harmonic Loading

$$p(t) = p_0 \sin \omega t$$

We now include damping effects to the same SDOF system from the previous section subject to a harmonic excitation force, . With damping included, the equation of motion becomes

$$m\ddot{u} + c\dot{u} + ku = p_0 \sin \omega t \tag{3.14}$$

where

p_0 is the peak magnitude of the force.
ω is the frequency of the force.
c is the damping coefficient.

When damping is nonzero, displacements are bounded, even for the resonant condition. We can solve the equation of motion following standard methods used in the solution of

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nonhomogeneous differential equations. The general solution to this equation is of the same form as that for the undamped case,

$$u(t) = u_c(t) + u_P(t) \tag{3.15}$$

where

u_c is the complementary solution, which is the free vibration response (homogeneous solution) for the underdamped case (since most structures fall under this category) given by Eq. (2.17) which is repeated here for convenience,

$$u_c(t) = e^{-\zeta\omega_n t}[A\cos\omega_D t + B\sin\omega_D t]$$

$$u_P$$

is the particular solution, which usually takes the same form as the forcing function,

$$u_P(t) = C\sin\omega t + D\cos\omega t \tag{3.16}$$

where C and D are arbitrary constants.

$$u_P$$

The constants C and D are determined by taking time derivatives of the particular solution, , and substituting them into the original equation of motion [Eq. (3.14)]. The time derivatives of u_P are shown as follows:

$$\dot{u}_P(t) = C\omega\cos\omega t - D\omega\sin\omega t$$

$$\ddot{u}_P(t) = -C\omega^2\sin\omega t - D\omega^2\cos\omega t$$

Substituting u_P and its time derivatives into Eq. (3.14) and factoring out the trigonometric functions on the left-hand side result in the following:

$$(kC - \omega Dc - m\omega^2 C)\sin\omega t + (kD + \omega Cc - m\omega^2 D)\cos\omega t = p_0\sin\omega t$$

The solution to this equation must be valid for all values of time. Therefore, the coefficient of the sine term must be equal to p_0 and the coefficient of the cosine term must be equal to zero. That is,

$$kD + \omega Cc - m\omega^2 D = 0 \text{ and } kC - \omega Dc - m\omega^2 C = p_0.$$

Substituting $c = 2\zeta\omega_n m$ and $k = \omega_n^2 m$ into these equations,

$$\omega_n^2 mC - 2\zeta\omega_n m\omega D - m\omega^2 C = p_0 \text{ and } \omega_n^2 mD + 2\zeta\omega_n m\omega C - m\omega^2 D = 0$$

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Now divide each equation by $\omega_n^2 m$ or k and substitute $r = \omega/\omega_n$,

$$(1-r^2)C - 2\zeta r D = p_0/k \text{ and } 2\zeta r C + (1-r^2)D = 0$$

Solving for constants *C* and *D*,

$$C = \frac{p_0}{k}\frac{(1-r^2)}{(1-r^2)^2+(2\zeta r)^2} \quad \text{and} \quad D = -\frac{p_0}{k}\frac{2\zeta r}{(1-r^2)^2+(2\zeta r)^2}$$

Then, substituting these constants into the particular solution,

$$u_p(t) = \frac{p_0/k}{(1-r^2)^2+(2\zeta r)^2}((1-r^2)\sin\omega t - 2\zeta r\cos\omega t) \tag{3.17}$$

Combining the complementary, $u_c(t)$, and particular, $u_p(t)$, solutions into Eq. (3.15), we obtain the total dynamic response of the system

$$u(t) = e^{-\zeta\omega_n t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{p_0/k}{(1-r^2)^2+(2\zeta r)^2}((1-r^2)\sin\omega t - 2\zeta r\cos\omega t) \tag{3.18}$$

Again, the two arbitrary constants *A* and *B* can be obtained using the initial (at time *t* = 0) conditions. The displacement at time *t* = 0 is *u*(0), which yields the following when substituted $p_0/k = (u_{st})_0$,

$$A = u(0) + \frac{(u_{st})_0 2\zeta r}{(1-r^2)^2+(2\zeta r)^2}$$

Next, differentiate the displacement equation with respect to time to obtain the velocity,

$$\begin{aligned}\dot{u}(t) = &-\zeta\omega_n e^{-\zeta\omega_n t}[B\sin\omega_D t + A\cos\omega_D t] + e^{-\zeta\omega_n t}[B\omega_D\cos\omega_D t - A\omega_D\sin\omega_D t] \\ &+ \frac{(u_{st})_0}{(1-r^2)^2+(2\zeta r)^2}((1-r^2)\omega\cos\omega t + 2\zeta r\omega\sin\omega t)\end{aligned} \tag{3.19}$$

The velocity at time *t* = 0 is and yields

$$B = \frac{\dot{u}(0) + u(0)\zeta\omega}{\omega_D} + \frac{r\omega_n(u_{st})_0}{\omega_D}\,\frac{2\zeta^2-(1-r^2)}{(1-r^2)^2+(2\zeta r)^2}$$

The complete solution describing the position of the mass of a damped SDOF system as a function of time is given as,

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$$u(t) = e^{-\zeta\omega_n t}\left\{\left[u(0) + \frac{(u_{st})_0 2\zeta r}{(1-r^2)^2 + (2\zeta r)^2}\right]\cos\omega_D t + \left[\frac{\dot{u}(0) + u(0)\zeta\omega}{\omega_D} + \frac{r\omega_n (u_{st})_0}{\omega_D}\left[\frac{2\zeta^2 - (1-r^2)}{(1-r^2)^2 + (2\zeta r)^2}\right]\right]\sin\omega_D t\right\} + \frac{(u_{st})_0}{(1-r^2)^2 + (2\zeta r)^2}((1-r^2)\sin\omega t - 2\zeta r\cos\omega t) \quad (3.20)$$

In general, the total response can be considered the combination of two distinct response motions as shown in Fig. 3.6 (for a frequency ratio, r, of 0.2 and initial conditions, $u(0) = 0$, and $\dot{u}(0) = \omega_n p_0/k$), which is similar to Fig. 3.2 but with damping. Note that the displacement has been normalized by the equivalent static displacement and the time normalized by the forcing function period.

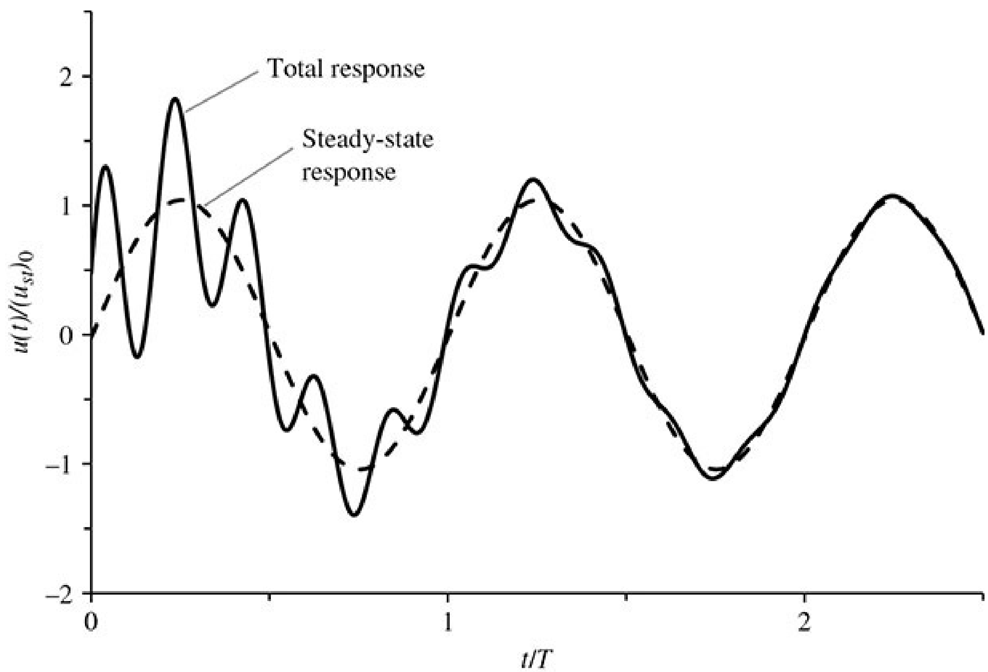

FIGURE 3.6 Harmonic force response for $r = 0.2$, $\zeta = 0.05$, $u(0) = 0$, and.

Example 4

Given a damped SDOF system with stiffness $k = 4{,}000$ lb/ft, mass $m = 4$ lb, damping ratio $\zeta = 0.05$, initial displacement $u\,(0) = 0$ ft, and initial velocity $\dot{u}(0) = 0$ ft/s, graph the transient, steady-state and total responses for a harmonic force of amplitude 200 lb and frequency ratio of (a) 0.25, (b) 0.9, and (c) 2.5.

Solution

i. *Determine natural frequency of the system using the given stiffness and mass, $k = 4{,}000$ lb/ft and mass $m = 4$ lb . s^2/ft, from Eq. (2.2),*

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4{,}000 \text{ lb/ft}}{4 \text{ lb}\cdot\text{s}^2/\text{ft}}} = 31.62 \text{ rad/s}$$

The damped frequency, ω_D, can be determined using Eq. (2.18),

$$\omega_D = \omega_n\sqrt{1-\zeta^2} = 31.62 \text{ rad/s}\sqrt{1-(0.05)^2} = 31.58 \text{ rad/s}$$

$$\dot{u}(0)$$

ii. *The equation of motion is given by Eq. (3.20) and normalized by the equivalent static displacement, with initial displacement u(0) = 0 ft, and initial velocity = 0 ft/s,*

$$\frac{u(t)}{(u_{st})_0} = e^{-\zeta\omega_n t} \quad \frac{2\zeta r}{(1-r^2)^2+(2\zeta r)^2} \cos\omega_D t + \frac{r\omega_n}{\omega_D}\ \frac{2\zeta^2-(1-r^2)}{(1-r^2)^2+(2\zeta r)^2}\ \sin\omega_D t$$
$$+\frac{1}{(1-r^2)^2+(2\zeta r)^2}((1-r^2)\sin\omega t - 2\zeta r\cos\omega t)$$

iii. *A graph of this equation (solid line) along with a graph of the steady-state (dash line) response and transient (dotted line) are shown in the following figures for the various values of r: (*a) $r = 0.25$ and $\omega = r\omega_n = 0.25(31.62 \text{ rad/s}) = 7.90$ rad/s.

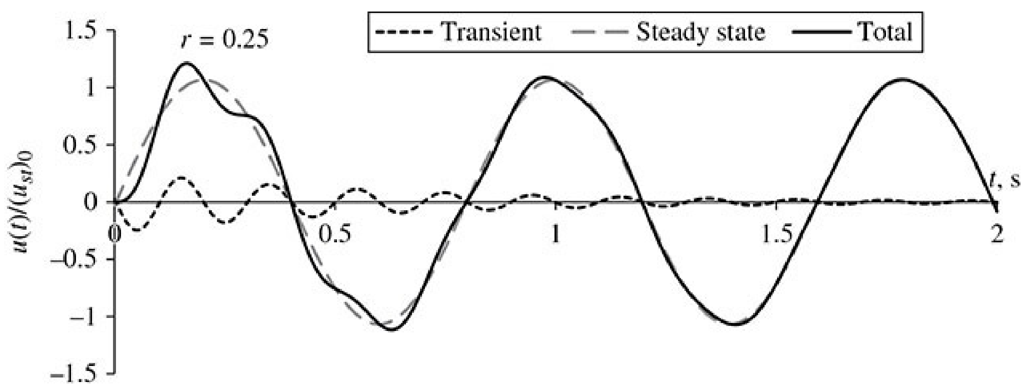

FIGURE E4.1 Response for $r = 0.25$ and $\omega = 7.90$ rad/s.

(b) $r = 0.75$ and $\omega = r\omega_n = 0.75(31.62 \text{ rad/s}) = 23.72$ rad/s.

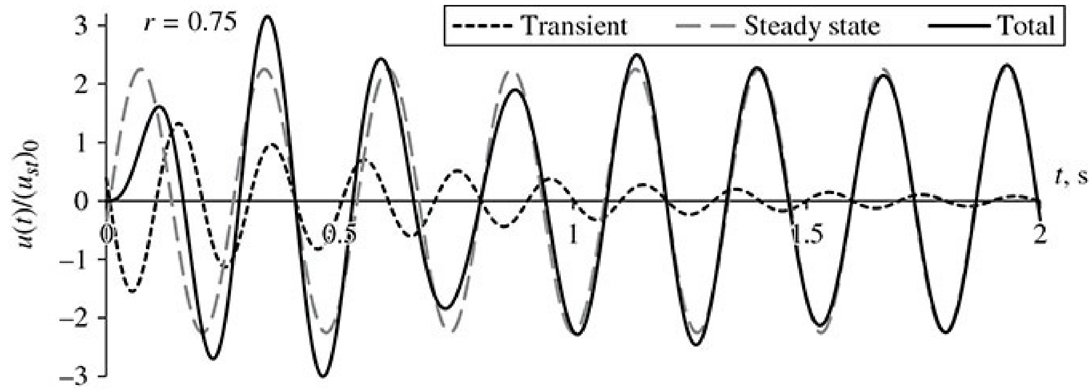

FIGURE E4.2 Response for $r = 0.75$ and $\omega = 23.72$ rad/s.

(c) $r = 2.5$ and $\omega = r\omega_n = 2.5(31.62\text{ rad/s}) = 79.06$ rad/s.

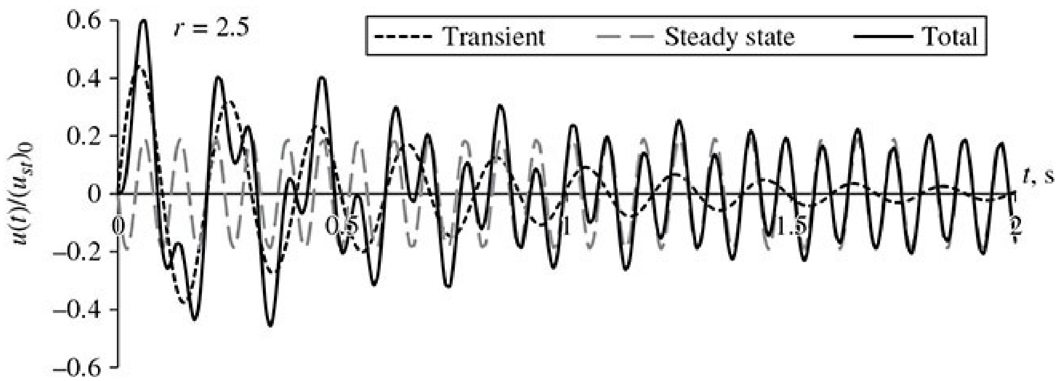

FIGURE E4.3 Response for $r = 2.5$ and $\omega = 79.06$ rad/s.

Notice that even for small damping, the transient response noticeably diminishes in less than 2 seconds, after which the total response is dominated by the steady-state response. Also, the first graph (Fig. E4.1) clearly shows that when r is small (0.25), the response is nearly static; whereas when r approaches 1 (Fig. E4.2), the response is clearly dynamic; and as r increases beyond a certain point, the response is less than the equivalent static response as Fig. E4.3 shows. This is a result of the transient and steady-state responses being out of phase, that is, when the oscillator is moving in one direction, the force is pushing in the opposite direction, lessening the effect of the dynamic force. ▲

It is clear from Example 4 that the transient response attenuates as t goes to infinity. Also, as noted earlier for undamped systems, the motion becomes unbounded as ω approaches ω_n ($r \to 1$), as t increases toward infinity. However, for damped cases [Eq. (3.20)], the resonant response, $u_{\text{res}}(t)$ remains bounded as ω approaches ω_n. Taking the limit of Eq. (3.20) as the frequency ratio, r goes to 1 and assuming initial conditions are zero,

$$u_{\text{res}}(t) = \lim_{r\to 1} u(t) = e^{-\zeta\omega_n t} \quad \frac{(u_{st})_0}{2\zeta}\cos\omega_D t + \frac{\omega_n (u_{st})_0}{2\omega_D}\sin\omega_D t \ + \frac{(u_{st})_0}{2\zeta}\cos\omega t$$

This relationship (just like the undamped case shown in Fig. 3.4) indicates that the displacement amplitude increases uniformly with time, but it is now bounded to a maximum of $\frac{1}{2\zeta}$ times the static displacement, as shown in Fig. 3.7. Now taking the limit of the resonant response as time goes to infinity,

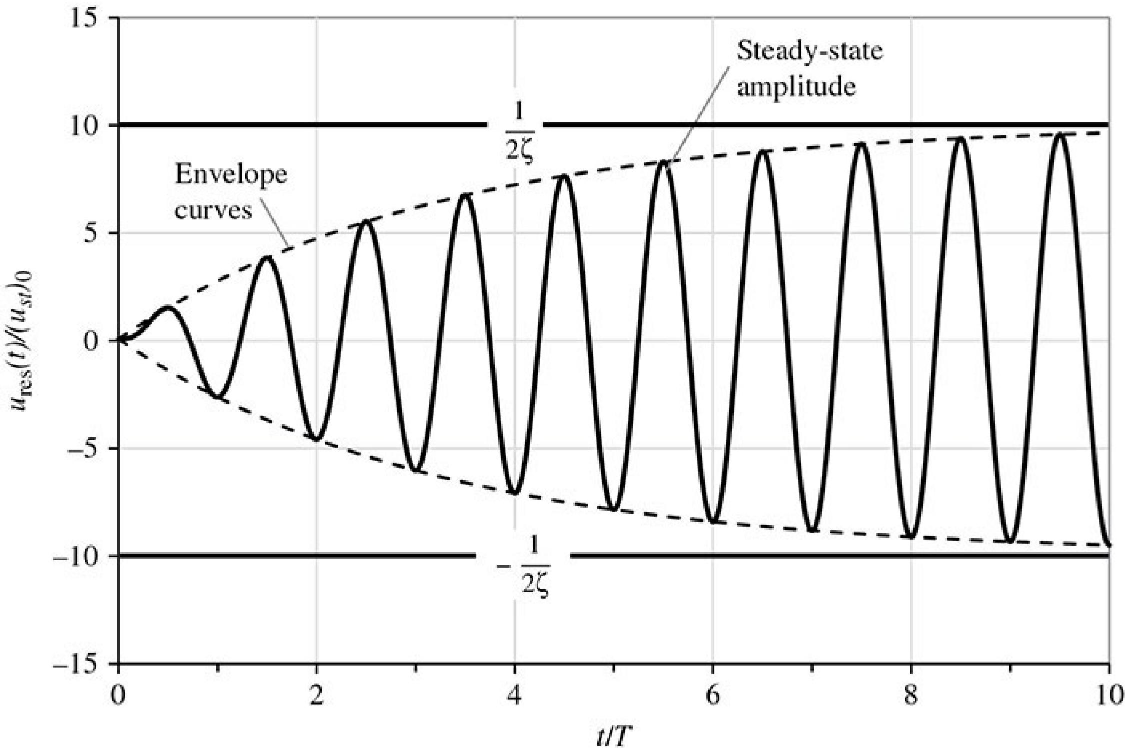

FIGURE 3.7 Resonant response (r = 1) to harmonic force for ζ = 0.05, $u(0)$ = 0, and $\dot{u}(0) = 0$.

$$\lim_{t\to\infty} u_{\text{res}}(t) = \lim_{t\to\infty} e^{-\zeta\omega_n t} \left[\frac{(u_{st})_0}{2\zeta} \cos\omega_D t + \frac{\omega_n (u_{st})_0}{2\omega_D} \sin\omega_D t \right] + \frac{(u_{st})_0}{2\zeta} \cos\omega t = \frac{(u_{st})_0}{2\zeta}$$

That is, the transient (exponential term) response dies out rather quickly, and the response is entirely steady-state and does not exceed $\frac{(u_{st})_0}{2\zeta}$.

An alternative formulation for the steady-state response (like the undamped oscillator) yields additional insight into the structural response of the damped case. We focus on the steady-state response [last term in Eq. (3.20)] because the exponential term (transient response) vanishes rather quickly as time increases. That is, the steady-state response eventually becomes the total solution after the transient vibration vanishes. Rewriting the particular solution in the alternative form that includes a phase angle, ϕ, and maximum steady-state displacement response amplitude, u_0,

$$u_p(t) = u_0 \sin(\omega t - \varphi) \tag{3.21}$$

where

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$$u_0 = \sqrt{C^2 + D^2} = \frac{(u_{st})_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$= \tan^{-1}(-D/C) = \tan^{-1}\ \frac{2\zeta r}{1-r^2}$$

With this relationship, we can again define the deformation response factor (also known as the dynamic magnification factor), R_d, as the ratio of the maximum steady-state displacement, u_0, to the equivalent static displacement, $(u_{st})_0$,

$$R_d = \frac{u_0}{(u_{st})_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\zeta)^2}} \tag{3.22}$$

Note that this ratio is a function of r and ζ only. In order to find maximum values of R_d, we take the derivative of R_d with respect to r in order to determine critical points:

$$\frac{dR_d}{dr} = \frac{2r(1-r^2-2\zeta^2)}{[(1-r^2)^2 + (2r\zeta)^2]^{3/2}} = 0$$

Maximizing R_d results in four conditions of interest:

$$\zeta = \sqrt{0.5}$$

1. $r = 0$ leads to $1 - 2\zeta^2 = 0$, *or*
2. $r = \infty$ corresponds to no displacement
3. $r = 1$ leads to resonance, $R_{d_{res}} = 1/2\zeta$; so, as $\zeta \to 0$, $R_{d_{res}} \to \infty$
4. r other than 0, 1, or infinity, requires that $1 - r^2 - 2\zeta^2 = 0 \Rightarrow r_{peak} = \sqrt{1-2\zeta^2}$

   Substituting this into Eq. (3.21) yields a maximum response of,

$$R_{d\max} = \begin{cases} \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} & \text{for } \zeta < \dfrac{1}{\sqrt{2}} = 0.707 \\ 1 & \text{for } \zeta \ge 1/\sqrt{2} \end{cases}$$

Figure 3.8 shows graphs depicting the deformation response factor, R_d, as a function of frequency ratio, r, and are known as *response spectra*. This family of graphs is the same as that shown in Fig. 3.5 but includes damping. Notice that in this case the displacement does not grow unbounded as r approaches 1. In fact, as damping increases, the displacement at the resonant frequency decreases, and for damping ratios exceeding 70.7%, the dynamic displacement is always less than the static displacement. Again, for small values of r (slowly varying force), R_d is approximately equal to 1; and for large values of r (rapidly varying force), R_d approaches zero, which corresponds to no displacement.

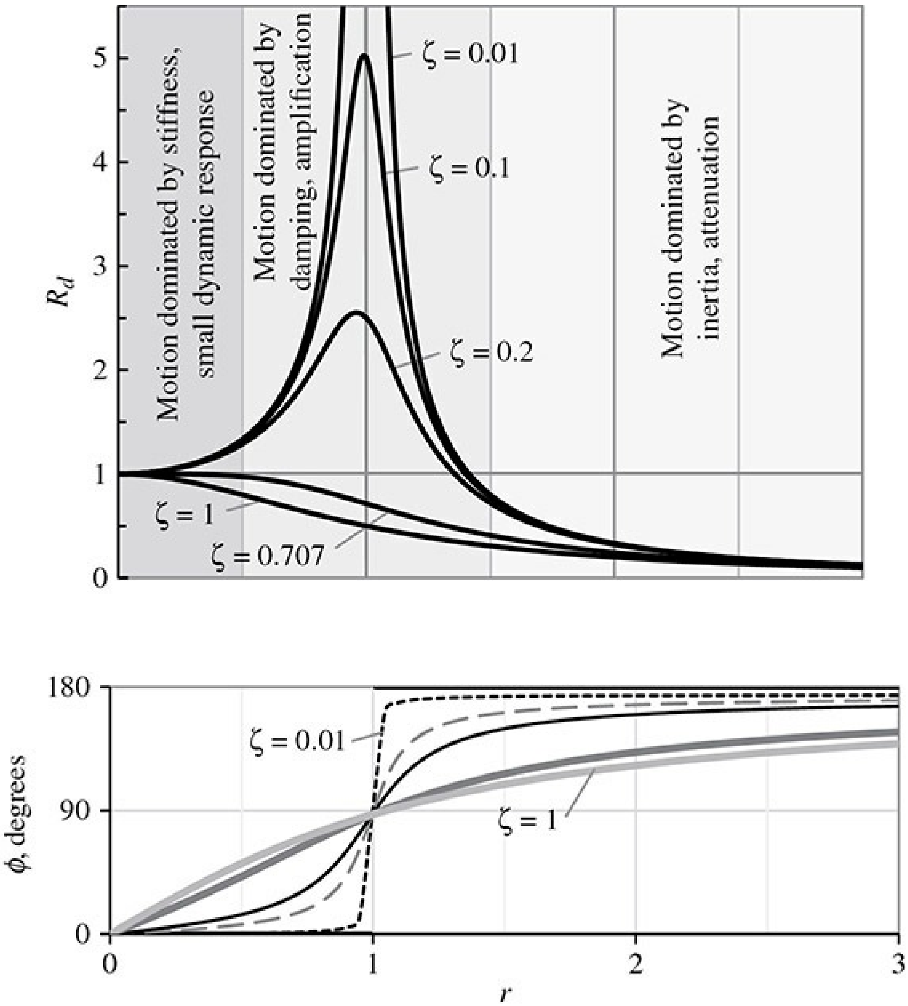

FIGURE 3.8 Damped response factor, R_d, and phase angle, ϕ, versus frequency ratio, r, for various values of damping ratio, ζ.

Also, we can analyze separately the effect of the four forces acting on the FBD of the SDOF oscillator: inertia ($m\ddot{u}$), damping ($c\dot{u}$), spring (ku), and disturbing (p) forces, which when combined yield the equation of motion, Eq. (3.14) (repeated here for convenience),

$$m\ddot{u} + c\dot{u} + ku = p$$

where

$$p = p_0 \sin \omega t$$
$$ku = ku_p = ku_0 \sin(\omega t - \quad) = kR_d p_0/k \sin(\omega t - \quad) = R_d p_0 \sin(\omega t - \quad)$$
$$c\dot{u} = c\dot{u}_p = c\omega u_0 \cos(\omega t - \quad) = R_d p_0/kc\omega \cos(\omega t - \quad) = 2\zeta r R_d p_0 \cos(\omega t - \quad)$$
$$m\ddot{u} = m\ddot{u}_p = -m\omega^2 u_0 \sin(\omega t - \quad) = -m\omega^2 R_d p_0/k \sin(\omega t - \quad) = -r^2 R_d p_0 \sin(\omega t - \quad)$$

From these relationships, it is clear that we can delineate three distinct response regions on the

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response spectrum diagram shown in Fig. 3.8:

i. For $r << 1$ (small values of r), both inertial force, $m\ddot{u}$, and damping force, $c\dot{u}$, are relatively small. Thus, the magnitude of the disturbing force, p, must be balanced by the spring force, ku. This indicates that the motion is dominated by the spring term in the equation of motion.

ii. For $r = 1$, the inertial force, $m\ddot{u}$ is equal to the spring force, ku, but out of phase, balancing each other out. In this case, the magnitude of the disturbing force must be balanced by the damping force, $c\dot{u}$. This indicates that the motion is dominated by the damping term in the equation of motion.

iii. For $r >> 1$ (large values of r), the inertial force, $m\ddot{u}$, is much larger than the spring force, ku, and damping force, $c\dot{u}$; therefore, the magnitude of the disturbing force, p, must be balanced by the inertial force, $m\ddot{u}$. This indicates that the motion is dominated by the inertial term in the equation of motion.

Example 5

Considering the building frame and loading given in Example 2, but with damping of 5%, determine (a) maximum steady-state displacement, u_0, (b) the maximum base shear, and (c) the maximum normal stresses in the columns ($I_x = 171$ in^4 and $S_x = 35$ in^3 for W10 × 33).

Solution The mass, stiffness, natural frequency of the SDOF system, equivalent static displacement, and frequency ratio were determined in Example 2 and are summarized as follows:

Mass,

$$m = 56.9 \text{ lb} \cdot \text{s}^2/\text{in}$$

Stiffness,

$$k = 39,858 \text{ lb/in}$$

The natural frequency of the SDOF system,

$$\omega_n = 26.46 \text{ rad/s}$$

Equivalent static displacement,

$$(u_{st})_0 = 0.100 \text{ in}$$

Frequency ratio,

$$r = 0.9827$$

i. *Determine the maximum steady-state displacement. The maximum steady-state displacement, u_0, can be determined using the deformation response factor in Eq. (3.22),*

$$u_0 = \frac{(u_{st})_0}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = \frac{0.100 \text{ in}}{\sqrt{(1-0.9827^2)^2 + (2(0.9827)(0.05))^2}} = 0.961 \text{ in}$$

Comparing this maximum displacement to the undamped case in Example 2, it is reduced to a third.

ii. *Determine the maximum base shear. Calculate maximum base shear in each column using the force-displacement relationship as discussed in Sec. 1.7.2:*

$$V_{\max} = u_0 \frac{12EI}{L^3} = 0.961 \text{ in} \frac{12(29{,}000 \text{ ksi})(171 \text{ in}^4)}{(12 \text{ ft} \times 12 \text{ in/ft})^3} = 19.2 \text{ kip}$$

iii. *Determine the maximum normal stress due to bending. The maximum bending moment at the base and top of the columns is calculated from equilibrium of the column member as discussed in Sec. 1.7.2,*

$$M_{\max} = u_0 \frac{6EI}{L^2} = 0.964 \text{ in} \frac{6(29{,}000 \text{ ksi})(171 \text{ in}^4)}{(12 \text{ ft} \times 12 \text{ in/ft})^2} = 1{,}379 \text{ kip} \cdot \text{in}$$

Applying the flexure formula from mechanics of materials, the normal stress [Eq. (1.17)] in each column due to the bending moment is determined as,

$$\sigma_{\max} = \frac{M_{\max}}{S_x} = \frac{1{,}379 \text{ kip} \cdot \text{in}}{35 \text{ in}^3} = 39.4 \text{ ksi} \blacktriangle$$

The response magnification factor can also be written in terms of velocity by taking the time derivative of Eq. (3.21) and by taking the time derivative of velocity to get acceleration. First, rewrite Eq. (3.21) as,

$$u_p(t)/(p_0/k) = R_d \sin(\omega t - \varphi)$$

i. To determine the velocity response factor, R_v $r = \omega/\omega_n$, we take the time derivative and substitute definitions of the frequency ratio, $\omega_n = \sqrt{k/m}$, and natural frequency, :

$$\dot{u}_p(t)/(p_0/k) = R_d \omega \cos(\omega t - \varphi) = R_d \omega \frac{\omega_n}{\omega_n} \cos(\omega t - \varphi) = R_d r \sqrt{k/m} \cos(\omega t - \varphi)$$

Normalizing the velocity into a dimensionless quantity, $\dot{u}_p(t)/(p_0/\sqrt{km})$

$$\dot{u}_p(t)/(p_0/\sqrt{km}) = R_d r \cos(\omega t - \quad) = R_v \cos(\omega t - \quad)$$

where $R_v = rR_d$

ii. Similarly, the acceleration response factor, R_a, can be determined as follows:

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$$\ddot{u}_P(t) = -R_d\omega^2 \sin(\omega t - \varphi) = -R_d\omega\left(\frac{\omega_n}{\omega_n}\right)^2 \sin(\omega t - \varphi) = R_d r^2 k/m \sin(\omega t - \varphi)$$

Normalizing the acceleration into a dimensionless quantity, $\ddot{u}_P(t)/(p_0/m)$

$$\ddot{u}_P(t)/(p_0/m) = R_d r^2 \cos(\omega t - \varphi) = R_a \cos(\omega t - \varphi)$$

where $R_a = r^2 R_d$

The relationship between the three response factors is shown below,

$$rR_d = R_v = \frac{R_a}{r}$$

In fact all three factors contain the same information; thus, we can combine the three quantities into a single four-way graph as a function of frequency ratio, r. This can be accomplished by considering separately the relationships between R_d and R_v, and that between R_v and R_a, and taking the logarithms of both sides of each equation, that is,

$$\log R_v = \log r + \log R_d$$

$$\Rightarrow \log R_d = -\log r + \log R_v$$

$$\log R_v = -\log r + \log R_a$$

$$\Rightarrow \log R_a = \log r + \log R_v$$

These relationships represent equations of straight lines: the displacement with a slope of –1 and the acceleration with a slope of +1. That is, lines of constant R_d and R_a are inclined at –45° and +45° with respect to the r axis, respectively. Thus, to construct a four-way graph, we need to plot R_v versus r on vertical and horizontal logarithmic scales, and add logarithmic scales inclined at –45° for R_d and +45° for R_a with respect to the r axis, as shown in Fig. 3.9. So, R_v is read on the vertical scale; R_d is read on the scale at –45°; and R_a is read on the scale at +45°.

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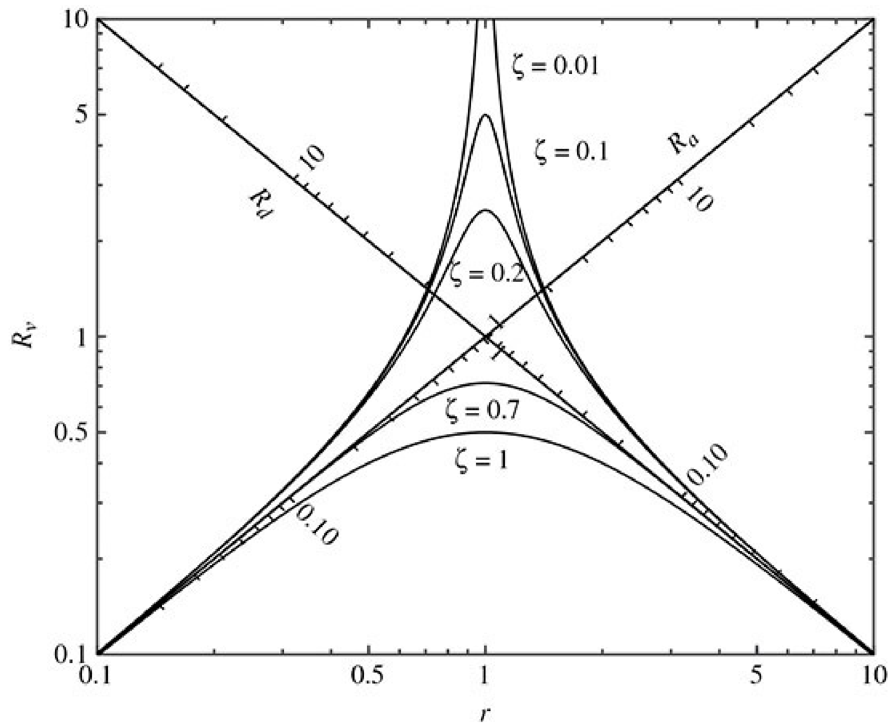

FIGURE 3.9 Four-way logarithmic plot of damped response factors R_d, R_v, and R_a versus frequency ratio, r, for various values of damping ratio, ζ.

Finally, we can also maximize R_v and R_a in terms of r by taking their derivatives with respect to r following a process similar to the process for maximizing R_d. The results are

$$R_{v\max} = \frac{1}{2\zeta}$$

$$R_{a\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

3.3 Vibration Response of SDOF Systems to Support Excitation

Vibration can also be induced in structures by way of their supports. There are a number of cases where time-varying excitations (displacements or accelerations) of the supports (or foundation) of a structure can induce dynamic stresses and deflections. Some examples include foundation accelerations caused by earthquakes, motions of buildings and bridges caused by external excitations such as traffic or wind, support motions of a piece of equipment mounted in a system, etc. The dynamic response of a system to support motion is similar to that of the same system subjected to a time-varying force as discussed in the last section. We can characterize the displacement response of the portal frame depicted in Fig. 3.10, where three different displacements are shown: ground displacement, $u_g(t)$, the relative displacement of the mass with respect to the ground, $u(t)$, and the total displacement with respect to a reference axis, $u^t(t)$.

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Again, we can write the equation of motion for the system by applying equilibrium to the FBD of the oscillator using D'Alembert's principle. For this case, stiffness and damping forces are proportional to the relative displacement, whereas the inertial force is proportional to the total mass displacement.

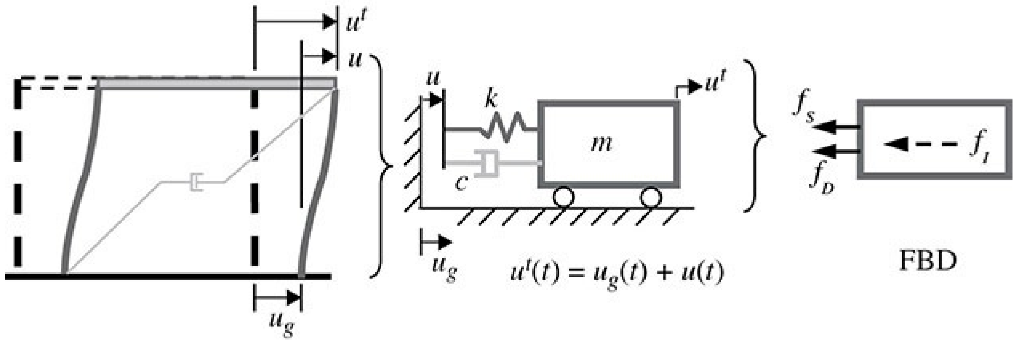

FIGURE 3.10 Idealized SDOF system and free-body diagram for a portal frame.

Horizontal equilibrium of the FBD shown in Fig. 3.10 yields the equation of motion,

$$\sum F_x = 0; \qquad f_I(t) + f_D(t) + f_S(t) = 0$$

where

$f_I(t) = m\ddot{u}^t(t)$ is the inertial force.

$f_D(t) = c\dot{u}(t)$ is the damping force.

$f_S(t) = ku(t)$ is the stiffness force.

So, the equation of motion can be rewritten as,

$$m\ddot{u}^t(t) + c\dot{u}(t) + ku(t) = 0$$

$$\ddot{u}^t(t) = \ddot{u}(t) + \ddot{u}_g(t)$$

This equation can be further stated in terms of the relative and ground motion by substituting the definition of total acceleration as into the equation of motion,

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -m\ddot{u}_g(t) \tag{3.23}$$

Note that the right-hand side is an effective support excitation loading that opposes the sense of ground acceleration and is similar to the time-varying force discussed in Sec. 3.2.

Alternatively, we can rewrite the equation of motion in terms of the total displacement, in which case the right-hand side is the effective loading that depends on displacement and velocity of the support. The resulting response is the total displacement of the mass from a fixed reference point, rather than displacement relative to the moving base.

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$$m\ddot{u}^t + c\dot{u}^t + ku^t = ku_g(t) + c\dot{u}_g(t) \tag{3.24}$$

Example 6

Consider the response of a vehicle traveling at a constant speed, *v*, along a wavy causeway (a bridge passing over water). The vehicle can be crudely modeled as a mass supported by a spring-dashpot system, and the causeway undulation can be characterized as a function of position along the road. For regularly spaced undulations, these deformations will cause a harmonic excitation in the vehicle as it travels over the causeway at a constant speed. Derive the equation of motion for the vehicle.

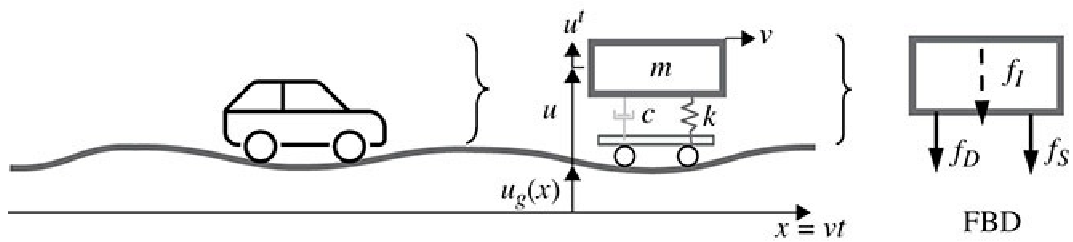

FIGURE E6.1 Idealized SDOF system and free-body diagram for a vehicle riding on a wavy causeway.

Solution Note that the total displacement, u^t, is measured from the static equilibrium position under vehicle weight *mg*. That is, when the vehicle is at rest under its own weight, the at rest displacement produced by the weight causes a balancing spring force. Thus, the spring force consists of two parts: one that balances the at rest weight and one that resists vibrations. The at rest displacement can be ignored provided that all displacements are measured from the position of static equilibrium (as discussed in Chap. 1, Example 2). Also, note that the position of the vehicle from a reference horizontal point at time $t = 0$ is given by the speed times the lapse time, i.e., *vt*.

i. *Use D'Alembert's principle and apply vertical equilibrium of the FBD to get the equation of motion,*

$$\overset{+}{\uparrow} \; \sum F_y = 0; \qquad -f_I - f_D - f_S = 0$$

where $f_I = m\ddot{u}^t(t)$

And since both the dashpot and spring only go through a relative displacement, *u*,

$$f_D = c(\dot{u}^t(t) - \dot{u}_g(x))$$

$$f_S = c(u^t(t) - u_g(x))$$

Thus, the equation of motion can be rewritten as,

$$m\ddot{u}^t(t) + c(\dot{u}^t(t) - \dot{u}_g(x)) + c(u^t(t) - u_g(x)) = 0$$

Since $x = vt$, it becomes the argument in the ground displacement and velocity functions.

$$m\ddot{u}^t(t) + c\dot{u}^t(t) + cu^t(t) = c\dot{u}_g(vt) + ku_g(vt) \blacktriangle$$

Before we consider a general support excitation, let us analyze a system subjected to a periodic, harmonic base excitation (such as the dynamic action of machinery) given by,

$$u_g(t) = u_{g0}\sin\omega t \tag{3.25}$$

where

u_{g0} is the peak amplitude of the ground displacement.
ω is the frequency of the support motion (not the disturbing force frequency, but has the same effect).

Substituting this displacement relation into Eq. (3.24),

$$m\ddot{u}^t + c\dot{u}^t + ku^t = ku_{g0}\sin\omega t + c\omega u_{g0}\cos\omega t \tag{3.26}$$

The two terms on the right-hand side can be combined into an equivalent form using the trigonometric identity $A\sin\omega t + B\cos\omega t = C\sin(\omega t + \alpha)$, with $C = \sqrt{A^2 + B^2}$, and $\tan\alpha = \frac{B}{A}$. Thus,

$$m\ddot{u}^t + c\dot{u}^t + ku^t = F_0\sin(\omega t + \alpha) \tag{3.27}$$

where

$$F_0 = u_{g0}\sqrt{k^2 + (c\omega)^2} = u_{g0}k\sqrt{1 + (2r\zeta)^2}$$

$$\tan\alpha = \frac{c\omega}{k} = 2r\zeta$$

Eq. (3.27) is the differential equation for the oscillator excited by the harmonic force $F_0\sin(\omega t + \alpha)$, the solution of which is analogous to that for Eq. (3.14), the complementary (transient) and particular (steady-state) solutions given by Eq. (3.20). Since the transient response vanishes relatively quickly, even for small amounts of damping, we only consider the steady-state solution, Eq. (3.21), $u_P(t) = \frac{p_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\sin(\omega t - \varphi)$.

$$u^t(t) = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2r\zeta)^2}}\sin(\omega t + \alpha - \varphi)$$

Substituting $F_0 = u_{g0}k\sqrt{1 + (2r\zeta)^2}$ yields

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$$u^t(t) = \frac{u_{g0}\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}\sin(\omega t+\alpha-\varphi) \quad (3.28)$$

This response describes the transmission of support motion to the mass. The quotient of the two square root quantities gives a deformation response factor, similar to the one discussed earlier, but in this context is now known as the transmissibility, T_r , which is discussed in more detail in the next section.

$$T_r = \frac{u_0^t}{u_{g0}} = \frac{\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{u_0^t}{u_{st}} \quad (3.29)$$

where $u_{st} = F_0/k$ is the static displacement, again.

Example 7

Consider the system introduced in Example 6, in which the vehicle weighs 4,000 lb, the damping ratio is 0.4, and the spring stiffness is 1,250 lb/in (which is determined by applying a known load to the vehicle and recording the deflection, k = load/deflection). The causeway profile can be characterized using a sine function with wavelength of girder span of 40 ft and amplitude (midspan displacement due to concrete creep) of 1.2 in. Determine the maximum steady-state vertical response of the vehicle as it travels at 45 mi/h.

Solution

i. *Determine the frequency ratio of the idealized SDOF system.*

Mass,

$$m = W/g = 4{,}000 \text{ lb}/(386.4 \text{ in/s}^2) = 10.35 \text{ lb·s}^2/\text{in}$$

Stiffness,

$$k = 1{,}250 \text{ lb/in}$$

Natural frequency,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1250 \text{ lb/in}}{10.35 \text{ lb·s}^2/\text{in}}} = 11 \text{ rad/s}$$

Excitation frequency can be obtained using the causeway wavelength and vehicle speed by first determining the period of the causeway undulations,

$$T = \frac{\text{wavelength}}{\text{speed}} = \frac{40 \text{ ft}}{45 \text{ mi/h}} \; \frac{1 \text{ mi}}{5{,}280 \text{ ft}} \; \frac{3{,}600 \text{ s}}{1 \text{ h}} = 0.606 \text{ s}$$

$$\omega = 2\pi/T = 2\pi/0.606 \text{ s} = 10.37 \text{ rad/s}$$

So, the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{10.37}{11} = 0.941$$

ii. *Determine the maximum steady-state vertical displacement of the vehicle. Use Eq.* (3.28) *with* u_{g0} = 1.2 in, the amplitude of the road surface undulations,

$$u^t\max = \frac{u_{g0}\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{1.2\sqrt{1+(2\cdot 0.941\cdot 0.4)^2}}{\sqrt{(1-0.941^2)^2+(2\cdot 0.941\cdot 0.4)^2}} = 1.97 \text{ in}$$

The relative displacement experienced by the vehicle suspension (springs and dampers) is then,

$$u_0 = u_0^t - u_{g0} = 1.97 - 1.2 = 0.772 \text{ in}$$

For the case of no damping,

$$u^t\max = \frac{u_{g0}}{1-r^2} = \frac{1.2}{1-0.941^2} = 10.5 \text{ in}$$

This large displacement is clearly beyond the range of the springs and demonstrates the importance of shock absorbers! ▲

3.4 Transmissibility and Vibration Isolation

In Sec. 3.2 we discussed the effect of harmonic force excitations on structural systems. These forces are produced by various external (wind, blasts, ocean waves, etc.) or internal (machines and engines mounted on structural frames) sources. In general, the transmission of these forces to supports (the housing superstructures or foundations) or surroundings is undesirable and should be minimized, which can be accomplished by an isolation system consisting of springs and dampers. This can be classified as *force isolation*.

The inverse problem of vibrations from the surroundings to supporting systems is equally undesirable. This is particularly critical when sensitive equipment is mounted on structures located close to the vibration source, or when the structure itself is subjected to disturbing foundation vibrations such as those from earthquakes or heavy vehicular traffic. This case can be classified as *motion isolation* since the undesirable vibrations are caused by motion rather than force.

Let us consider first transmissibility in support excitation as discussed in Sec. 3.3 and given by Eq. (3.29). A family of graphs of this equation as a function of frequency ratio, r, for various values of damping ratio, ζ, is shown in Fig. 3.11—all dimensionless parameters. Notice that this graph is similar to that shown in Fig. 3.8.

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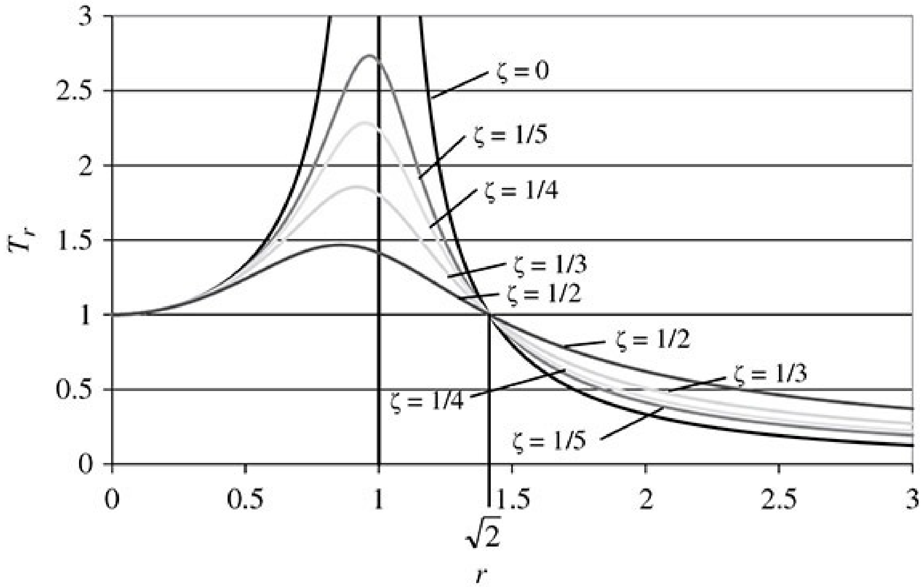

FIGURE 3.11 Transmissibility, T_r, versus frequency ratio, r, for various values of damping ratio, ζ.

Given that all curves pass through $r = \sqrt{2}$, a detailed analysis of the family of graphs leads to two important observations: (a) dynamic amplification of the response and (b) vibration response below the static displacement, which results in vibration isolation.

a. Since $T_r > 1$ for $r < \sqrt{2}$, the dynamic response is always larger than the static response. Thus, there is a dynamic amplification in the response (force or displacement); however, the dynamic response can be attenuated by increasing damping. Also notice that as r approaches zero (slowly varying displacements), $u_0^t \rightarrow u_{g0}$, that is, the mass moves rigidly with the ground motion, both undergoing the same ground acceleration. And most importantly, for $r = 1$, the system experiences *resonance*, which for $\zeta \rightarrow 0, T_r \rightarrow \infty$.

b. For $r > \sqrt{2}$, $T_r < 1$, which means vibration isolation can be achieved. In this case, increasing damping decreases the effectiveness of the motion (or force) vibration isolation. Since increasing damping increases transmissibility in the isolation range ($r > \sqrt{2}$), the most effective vibration absorbers consist of spring elements having little or no damping. This implies a tradeoff when selecting a soft spring to reduce the transmitted force but increases the static displacement or vice versa with a stiff spring. Also, for large values of r (rapidly varying displacements), the system amplitude, $u_o^t = 0$, that is, the mass stays still while the ground beneath it moves; this implies complete isolation.

Following is a discussion of the transmissibility of forces and displacements separately even though they are identical from a practical standpoint since the isolation devices used in the two cases are the same—springs and dashpots.

3.4.1 Transmissibility of Force from the Structure to the Foundation

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The effect of a harmonic force excitation on the foundation of a supporting system can be determined as the magnitude of the force transmitted to the support through the spring (stiffness) and the dashpot (damping) elements. Substituting Eq. (3.21) into the stiffness and damping terms of the equation of motion, we get the force transmitted from the structure to the foundation

$$(f_T)_0 = ku_p + c\dot{u}_p = (u_{st})_0 R_d[k\sin(\omega t - \quad) + c\omega\cos(\omega t - \quad)]$$

Substituting Eq. (3.22) into this relationship, we get the maximum magnitude of the force transmitted to the foundation,

$$(f_T)_0 = (u_{st})_0 R_d \sqrt{k^2 + c^2\omega^2} = \frac{p_0\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} \tag{3.30}$$

The proportion of the amplitude of the excitation (disturbing) force, p_0, transmitted to the foundation is defined as the *transmissibility*,

$$T_r = \frac{(f_T)_0}{p_0} = \frac{\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} \tag{3.31}$$

Example 8

A 1,000 lb air compressor is mounted on four springs each with stiffness of 10,000 lb/in. When the compressor is operated at 1,000 cycle/min, it causes a harmonic disturbing force of 100 lb. First, determine the force transmitted to the base of the air compressor assuming no damping. Then, determine the percent decrease in transmitted force assuming the springs are replaced with a rubber pad of stiffness 40,000 lb/in and damping that yields a damping ratio of 0.1.

Solution

i. *Determine the frequency ratio for the four springs:*

Mass,

$$m = W/g = 1{,}000 \text{ lb}/(386.4 \text{ in/s}^2) = 2.588 \text{ lb}\cdot\text{s}^2/\text{in}$$

Stiffness,

$$k_t = 4k = 4(10{,}000 \text{ lb/in}) = 40{,}000 \text{ lb/in}$$

Natural frequency,

$$\omega_n = \sqrt{\frac{k_t}{m}} = \sqrt{\frac{40{,}000}{2.588}} = 124.3 \text{ rad/s}$$

Also, the forcing frequency is given as 1,000 cycle/min, which in rad/s is

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$$\omega = 1{,}000 \frac{\text{cycle}}{\text{min}} \left(\frac{2\pi \text{ rad}}{\text{cycle}} \right) \left(\frac{1 \text{ min}}{60 \text{ s}} \right) = 104.7 \text{ rad/s}$$

So, the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{104.7}{124.3} = 0.842$$

ii. *Determine the force transmitted to the base: Using Eq. (3.30) with* $\xi = 0$ *we can determine the magnitude of the force transmitted to the base by the springs,*

$$(f_T)_s = \frac{p_0 \sqrt{1 + (2r\xi)^2}}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} = \frac{100 \text{ lb} \sqrt{1 + (0)^2}}{\sqrt{(1 - r^2)^2 + (0)^2}} = \frac{100 \text{ lb}}{1 - (0.842)^2} = 343.7 \text{ lb}$$

iii. *Determine the percent decrease in transmitted force by changing from four springs to a rubber pad support. Again, we use Eq. (3.30) with* $\xi = 0.1$ *to determine the magnitude of the force transmitted to the base. Since the stiffness of the two systems is the same, the frequency ratio, r, is the same for both cases:*

$$(f_T)_r = \frac{p_0 \sqrt{1 + (2r\xi)^2}}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} = \frac{100 \text{ lb} \sqrt{1 + (2 \times 0.842 \times 0.1)^2}}{\sqrt{(1 - 0.842^2)^2 + (2 \times 0.842 \times 0.1)^2}} = 301.6 \text{ lb}$$

The percent decrease in force is

$$\% \downarrow (f_T)_0 = \frac{(f_T)_s - (f_T)_r}{(f_T)_s} \cdot 100 = \frac{343.7 - 301.6}{343.7} \cdot 100 = 12.2\% \ \blacktriangle$$

3.4.2 Transmissibility of Vibration from the Foundation to the Structure

The effect of a harmonic base excitation on a supported system can also be determined as the magnitude of the displacement transmitted to the supported system through the spring (stiffness) and the dashpot (damping) elements. However, a relationship for transmissibility of motion from the foundation to the structure has already been derived and given by Eq. (3.29), which provides the degree of relative isolation defined as the ratio of the amplitude of total motion of the system to the static displacement. Also, transmissibility can be expressed in terms of acceleration since the second derivative of Eq. (3.28) is

$$\ddot{u}^t(t) = \frac{-\omega^2 u_{g0} \sqrt{1 + (2r\zeta)^2}}{\sqrt{(1 - r^2)^2 + (2r\zeta)^2}} \sin(\omega t + \alpha - \varphi)$$

And the second derivative of the disturbing function, Eq. (3.25), is

$$\ddot{u}_g(t) = -\omega^2 u_{g0} \sin \omega t$$

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So, the ratio of the amplitude of these two relationships is also equal to the transmissibility of displacements,

$$T_r = \frac{\ddot{u}_0^t}{\ddot{u}_{g0}} = \frac{\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} \tag{3.32}$$

Displacement transmitted to foundations can also be determined by considering the relative displacement, $u = u^t - u_g$, which can be written as

$$\frac{u_0}{u_{g0}} = \frac{r^2}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} \tag{3.33}$$

Example 9

A 50-lb package is suspended in a box by two springs each having a stiffness k = 250 lb/in and providing total 2% damping (i.e., 0.02 damping ratio). The box is transported on the bed of a truck, which due to the suspension motion experiences vertical harmonic excitations of amplitude $u(t)$ = 1.5 in·sin [(2 rad/s)·t]. The owner of the box has specified that the maximum relative displacement of the package be less than 0.05 in. Does the system satisfy this requirement?

Solution There are two ways to solve this problem:

1. Using transmissibility to determine the total displacement with Eq. (3.29) and then compute the relative displacement using $u_0 = u_0^t - u_{g0}$.
2. Directly determining the relative displacement using Eq. (3.33).

i. *Determine the frequency ratio of the idealized SDOF system.*

Mass,

$$m = W/g = 50 \text{ lb}/(386.4 \text{ in/s}^2) = 0.1294 \text{ lb}\cdot\text{s}^2/\text{in}$$

Stiffness,

$$k_t = 2k = 2(250 \text{ lb/in}) = 500 \text{ lb/in}$$

Natural frequency,

$$\omega_n = \sqrt{\frac{k_t}{m}} = \sqrt{\frac{500}{0.1294}} = 62.16 \text{ rad/s}$$

Forcing frequency (given in the displacement equation) is

$$\omega = 2 \text{ rad/s}$$

So, the frequency ratio is

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$$r = \frac{\omega}{\omega_n} = \frac{2}{62.16} = 0.0322$$

ii. *Determine the relative displacement of the mass.*

1. Use transmissibility with u_{g0} = 1.5 in, amplitude of the system displacement given in the displacement equation,

$$T_r = \frac{u_0^t}{u_{g0}} = \frac{\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

$$\Rightarrow u_0^t = \frac{u_{g0}\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{(1.5)\sqrt{1+(2(0.032)(0.05))^2}}{\sqrt{(1-(0.032)^2)^2+(2(0.032)(0.05))^2}} = 1.50155 \text{ in}$$

The relative displacement is then given by

$$u_0 = u_0^t - u_{g0} = 1.50155 \text{ in} - 1.5 \text{ in} = 0.00155 \text{ in} < 0.05 \text{ in}$$

The displacement limit is satisfied!

2. The second approach utilizes Eq. (3.33),

$$\frac{u_0}{u_{g0}} = \frac{r^2}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

$$u_0 = \frac{u_{g0}r^2}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{(1.5)(0.032)^2}{\sqrt{(1-(0.032)^2)^2+(2(0.032)(0.05))^2}} = 0.00155 \text{ in}$$

Same as above! ▲

Example 10

A 3,500-lb truck sleeper cabin is supported by four springs each with stiffness 750 lb/in and provides total damping that is 10% of critical damping. Assuming a vertical acceleration of 0.1 g at a frequency of 10 Hz during a typical trip, determine the acceleration transmitted to the truck passenger in the cab. Assuming the passenger can only tolerate an acceleration of 0.01 g, suggest a solution assuming no modification to the spring system.

Solution

i. *Determine the acceleration transmitted to the truck driver. Idealize the truck cab as SDOF system and determine the frequency ratio.*

Mass,

$$m = W/g = 3{,}500 \text{ lb}/386.4 \text{ in/s}^2 = 9.058 \text{ lb} \cdot \text{s}^2\text{/in}$$

Stiffness,

$$k_t = 4k = 2(750 \text{ lb/in}) = 3{,}000 \text{ lb/in}$$

Natural frequency,

$$\omega_n = \sqrt{\frac{k_t}{m}} = \sqrt{\frac{3,000}{9.058}} = 18.2 \text{ rad/s}$$

Excitation frequency in rad/s is

$$\omega = 2\pi f = 2\pi(10/\text{s}) = 62.83 \text{ rad/s}$$

So, the frequency ratio is

$$r = \frac{\omega}{\omega_n} = \frac{62.83}{18.2} = 3.452$$

From Eqs. (2.11) and (2.12), we get the damping coefficient,

$$c = 2\zeta\sqrt{km} = 2(0.1)\sqrt{(3,000)(9.058)} = 32.97 \text{ lb}\cdot\text{s/in}$$

Using Eq. (3.32), we can determine acceleration transmissibility with $\ddot{u}_{g0}$ = 0.1 g and ζ = 0.1,

$$T_r = \frac{\ddot{u}_0^t}{\ddot{u}_{g0}} = \frac{\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

$$\Rightarrow \ddot{u}_0^t = \frac{\ddot{u}_{g0}\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{0.1 \text{ g}\sqrt{1+(2(3.45)(0.1))^2}}{\sqrt{(1-(3.45)^2)^2+(2(3.45)(0.1))^2}} = 0.0111 \text{ g}$$

This exceeds the limit of 0.01 g!

ii. *Determine a strategy to reduce acceleration to 0.01 g. Since the cabin suspension (damping and stiffness) system cannot be modified, we can change the weight of the cabin!* New mass with arbitrary weight (3,500 lb plus added weight),

$$m = W_t/g = W_t/386.4 \text{ in/s}^2$$

Natural frequency,

$$\omega_n = \sqrt{\frac{k_t}{m}} = \sqrt{\frac{3,000 \times 386.4}{W_t}} = \frac{1076.7}{\sqrt{W_t}}$$

Frequency ratio,

$$r = \frac{\omega}{\omega_n} = \frac{62.83\sqrt{W_t}}{1076.7} = 0.0584\sqrt{W_t}$$

Damping coefficient,

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{14.39 \text{ lb}\cdot\text{s/in}}{2\sqrt{(3{,}000)(W_t/386.4)}} = 5.92/\sqrt{W_t}$$

Now use Eq. (3.32) with $\ddot{u}_{g0}$ = 0.1 g and $\ddot{u}_0^t$ = 0.01 g

$$T_r = \frac{\ddot{u}_0^t}{\ddot{u}_{g0}} = \frac{0.01 \text{ g}}{0.1 \text{ g}} = 0.1 = \frac{\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

$$= \frac{\sqrt{1+\ 2(0.0584\sqrt{W_t})(5.92/\sqrt{W_t})\ ^2}}{\sqrt{(1-(0.0584\sqrt{W_t})^2)^2+[2(0.0584\sqrt{W_t})(5.92/\sqrt{W_t})]^2}}$$

This yields the following quadratic equation in terms of weight:

$$1.1598\times10^{-7}W_t^2 - 6.8113\times10^{-7}W_t - 1.4620 = 0$$

Solving this equation, we need a total weight W_t = 3856 lb, which means that if the passenger wants to sleep soundly, an additional 356 lb must be added to the sleeper cab! ▲

3.4.3 Force and Motion Vibration Isolation

It is clear from Fig. 3.11 that isolation vibration from force or base motion is only achievable for systems operating at disturbing frequencies above a critical value of $r > \sqrt{2}$. In design, it is convenient to express response above this point as isolation effectiveness, I_E, rather than transmissibility. I_E is defined as

$$I_E = 1 - T_r \tag{3.34}$$

This relationship implies that complete isolation is achieved when I_E = 1, that is, as $r \to \infty$. And no isolation is possible when I_E = 0; that is, when r ″ $\sqrt{2}$.

Returning to the transmissibility relationship, Eq. (3.29), and considering that damping has a detrimental effect, if any at all, in the isolation range ($r > \sqrt{2}$, as shown in Fig. 3.11), it is reasonable to assume that $\zeta = 0 \quad (2r\zeta)^2 = 0$. Thus, Eq. (3.29) becomes

$$T_r = \frac{\sqrt{1+(2r\zeta)^2}}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{1}{\sqrt{(1-r^2)^2}} = \frac{1}{\pm(1-r^2)}$$

Taking the negative root, we get

$$T_r \approx \frac{1}{r^2-1} \qquad I_E = 1 - \frac{1}{r^2-1}$$

Now solving for r^2 in terms of I_E,

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$$r^2 = \frac{2 - I_E}{1 - I_E}$$

In design, typically we need to select a supporting spring system for a reciprocating machine (needing isolation) of known weight and operating frequency. The last relationship can be solved for the stiffness, k , in terms of machine weight, W_m, operating frequency, f, and isolation effectiveness, I_E . That is, substituting $r = f/f_n$, $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, and $m = W_m/g$ into this equation, we get

$$k = 4\pi^2 f^2 \frac{W_m}{g} \frac{1 - I_E}{2 - I_E} \tag{3.35}$$

Example 11

A sensitive, 50-lb instrument that requires insulation from vibration is being installed in a building where a reciprocating machine is in use. The machine causes the building floor to vibrate in a harmonic motion at frequency of 1,000 cycles/min. The equipment is going to be installed on four springs of equal stiffness and negligible damping. Determine the stiffness of each spring if the amplitude of the transmitted vibration is to be limited to less than 15% of the floor vibration.

Solution We first solve this problem using transmissibility and then using Eq. (3.35).

1. After idealizing the system as a SDOF system, use transmissibility with $\zeta = 0$ to determine the required stiffness to limit the vibration transmitted.

$$T_r = \frac{\sqrt{1 + (2r\zeta)^2}}{\sqrt{(1 - r^2)^2 + (2r\zeta)^2}} = \frac{1}{\pm(1 - r^2)}$$

$$\sqrt{2}$$

   i. *Determine the maximum required frequency ratio, r.* Since $T_r < 1$, then $r >$, we take the negative root,

$$T_r = \frac{1}{(r^2 - 1)}$$

   and establish an inequality such that the transmissibility ratio is less than 0.15, that is,

$$\frac{1}{(r^2 - 1)} < 0.15$$

   Solving this inequality for r, we determine that frequency ratio, r, must be greater than 2.77.

   ii. *Determine the required natural cyclical frequency.* Use the frequency ratio in the

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following form,

$$r = \frac{f}{f_n} > 2.77$$

where f = 1,000 cycles/min (1 min/60 s) = 16.67 Hz.
The frequency ratio equation then gives a natural cyclical frequency of,

$$\frac{16.67\ \text{Hz}}{f_n} > 2.77 \qquad f_n < 6.02\ \text{Hz}$$

iii. *Determine the required stiffness to limit the vibration transmitted to less than 15%*. First, we need the mass,

$$m = W/g = 50\ \text{lb}/(386.4\ \text{in/s}^2) = 0.1294\ \text{lb·s}^2/\text{in}$$

We now determine the stiffness,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_t}{m}} < 6.02\,\frac{1}{\text{s}}$$

Solving for the total stiffness, k_t < 185 lb/in. Since the instrument is supported on four springs, the stiffness of each spring must be less than 46.3 lb/in in order to limit the transmitted vibration to less than 15%.

2. We now use Eq. (3.35) to determine k so that $I_E < 1 - T_r = 1 - 0.15 = 0.85$.

$$k = 4\pi^2 f^2 \frac{W_m}{g}\frac{1-I_E}{2-I_E} = 4\pi^2(16.67/\text{s})^2 \frac{(50\ \text{lb})}{386.4\ \text{in/s}^2}\frac{1-0.85}{2-0.85} = 185\ \text{lb/in}$$

We get the same value as Part 1, but more expeditiously. ▲

3.5 Damping Evaluation Using Response to Harmonic Loading

In the last chapter we discussed modeling damping using the highly idealized concept of viscous damping. The various damping mechanisms listed in Sec. 2.2.4 can be divided into several general types, with hysteretic (structural) and Coulomb (friction) damping being the better understood and most widely used to determine more accurate estimates of damping. Hysteretic damping results from the loss of input energy by internal friction mechanisms, such as intrinsic material damping that is primarily due to friction between particles during loading and unloading of the material. Coulomb damping results from the loss of energy by external friction mechanisms, such as friction between structural elements (friction in bolted and nailed connections and friction in cracked concrete) as well as friction in nonstructural components and their connections to the structure.

Since it is difficult to distinguish the source of damping, the magnitude of damping is usually estimated experimentally. In Chap. 2, we discussed using the free vibration decay curve or

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logarithmic decrement, δ, to estimate the damping of a SDOF system [Eq. (2.26)]. In the following subsections, we present two additional methods to estimate damping using harmonic excitation analysis presented in this chapter. To verify results determined using one of the three methods, we can utilize the following estimates for damping for various structural systems: 3% for steel, 5% for concrete, 7% for masonry, and 10% for wood (from *Seismic Design Guidelines for Buildings*. Department of the Army, Navy, and Air Force, Washington, DC, 1988); or values from Table 2.1 when more details are available.

3.5.1 Resonant Amplification Method

Recall that harmonic excitation response can be used to develop a response spectrum (displacement amplitude vs. applied frequency or deformation response factor, R_d, vs. frequency ratio, r, as shown Fig. 3.8). Figure 3.8 clearly shows that damping strongly influences the resonant response amplitude, various properties of which can be used to estimate the damping ratio. One such property is the resonant response (displacement, velocity, or acceleration) amplitude that yields the maximum response amplitude, for example, $(R_d)_{max}$ as depicted in Fig. 3.12. As noted earlier, $(R_d)_{max} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ occurs at a damped frequency of $\omega_D = \omega_n\sqrt{1-\zeta^2}$. We further simplify the analysis by recognizing that the value under the square root operator is close to unity for small damping ratio values, $\zeta < 0.3$, which include most practical structures as noted in the derivation of approximate logarithmic decrement method—Eq. (2.24). That is,

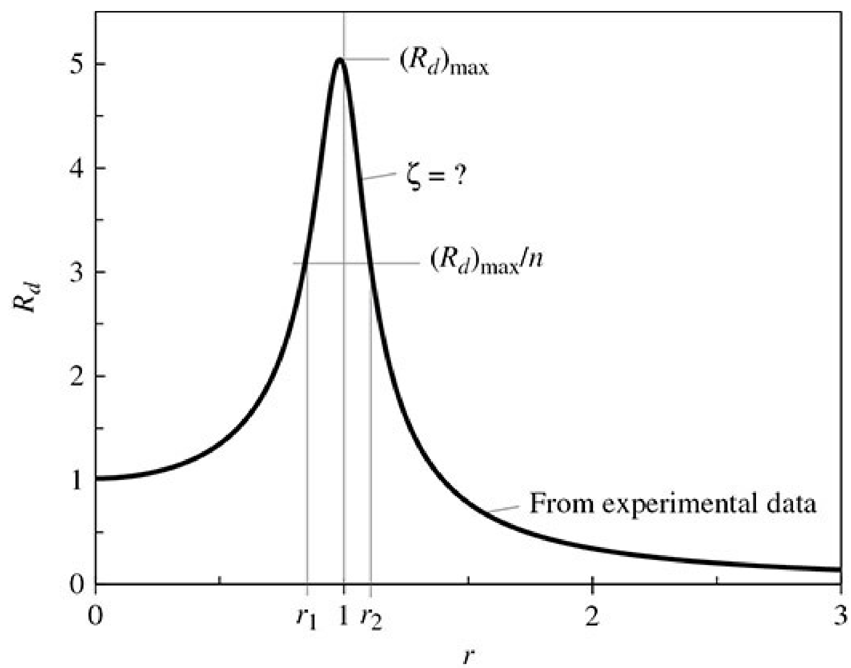

FIGURE 3.12 SDOF oscillator *response spectrum* to harmonic excitation.

$$(R_d)_{max} = R_{d\,res} = \frac{(u_0)_{res}}{(u_{st})_0} = \frac{1}{2\zeta}$$

Or, solving for the damping ratio

$$\zeta = \frac{1}{2R_{d\,\mathrm{res}}} = \frac{(u_{st})_0}{2(u_0)_{\mathrm{res}}} \tag{3.36}$$

This method is relatively simple and can be implemented with simple transducers that can measure displacement or acceleration amplitude. That is, while vibrating the structure over a range of frequencies spanning the resonant frequency, the maximum response amplitude is recorded, $(u_0)_{\mathrm{res}}$. The error introduced in evaluating damping using Eq. (3.36) is insignificant for most ordinary structures. However, evaluating displacements is difficult considering that a large force amplitude needs to be applied to a structure to be able to read displacement values. However, acceleration transducers (known as accelerometers) can read very low-amplitude accelerations, thus we can use the maximum acceleration response ratio, $(R_d)_{\max} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ (or the acceleration transmissibility relationship). That is, we transmit an excitation acceleration, $\ddot{u}_{g0}$, to the structure using a harmonic vibration generator, the acceleration response of which, $\ddot{u}_0^t$, can be read by an accelerometer. Thus, using Eq. (3.32), we can find the damping ratio as follows:

$$T_{r,\mathrm{res}} = \frac{(\ddot{u}_0^t)_{\mathrm{res}}}{(\ddot{u}_{g0})_{\mathrm{res}}} = \frac{\sqrt{1+(2(1)\zeta)^2}}{\sqrt{(1-(1)^2)^2+(2(1)\zeta)^2}} = \frac{\sqrt{1+(2\zeta)^2}}{2\zeta}$$

We again assume that the value under the square root operator in the practical range of interest is close to unity for small damping ratio values, so

$$\zeta = \frac{(\ddot{u}_{g0})_{\mathrm{res}}}{2(\ddot{u}_0^t)_{\mathrm{res}}} \tag{3.37}$$

3.5.2 Half-Power Bandwidth Method

For this method, we also use the maximum response amplitude, $(R_d)_{\max}$ to estimate the damping ratio. However, we use two different frequency ratios (or frequencies) where the response amplitude is reduced to a level of $(R_d)_{\max}/n$, where n is an arbitrary constant. The range of frequency ratio from r_1 to r_2 is called the bandwidth and the values themselves are called the half-power points.

Similar to the previous method, we obtain $(R_d)_{\max}$ from the measured data; we then select values r_1 to r_2 such that the response amplitude, R_d equals $(R_d)_{\max}/n$ as shown in Fig. 3.12. Since n is an arbitrary constant, we choose a convenient value of $\sqrt{2}$. Using Eqs. (3.22) and (3.36), we obtain the values of r_1 to r_2 as follows:

$$R_d = \frac{1}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

and

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$$R_d = \frac{(R_d)_{\max}}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta} = \frac{1}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} \Rightarrow \sqrt{(1-r^2)^2+(2r\zeta)^2} = 2\sqrt{2}\zeta$$

Squaring both sides of this relationship,

$$(1-r^2)^2+(2r\zeta)^2 = 8\zeta^2$$

Finding the roots of the resulting quadratic equation,

$$r^2 = 1-2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}$$

Here we assume that the value under the square root operator in the practical range of interest of ζ is close to unity, so

$$r_1^2 = 1-2\zeta^2-2\zeta$$

$$r_2^2 = 1-2\zeta^2+2\zeta$$

And assuming that because of the small value of $1-2\zeta-2\zeta^2 \approx (1-\zeta)^2 = 1-2\zeta-\zeta^2$, ζ and $1+2\zeta-2\zeta^2 \approx (1+\zeta)^2 = 1+2\zeta+\zeta^2$, so

$$r_1^2 = (1-\zeta)^2 \qquad r_1 = 1-\zeta$$

$$r_2^2 = (1+\zeta)^2 \qquad r_2 = 1+\zeta$$

Adding r_1 and r_2,

$$2 = r_2 + r_1$$

Now subtract r_1 from r_2, and solve for ζ

$$\zeta = \frac{r_2 - r_1}{2}$$

Combining these two relationships, we get,

$$\zeta = \frac{r_2-r_1}{r_2+r_1} = \frac{f_2-f_1}{f_2+f_1} \tag{3.38}$$

where the fs are the forcing frequencies when the amplitude of the response is equal to $1/\sqrt{2}$; since $r = f/f_n$.

Example 12

Consider the cellphone tower from Example 5, in Chap. 2, where we determined the damping

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ratio using logarithmic decrement (a 3 in initial horizontal displacement decreased to 1 in after four complete cycles). We now determine the damping ratio using the two methods discussed in this chapter. For this part of the analysis, we collected data from a frequency response test of the tower. The data were plotted to construct a response spectrum, R_d versus r.

From the plot it was determined that $(R_d)_{max}$ = 13.2, and the response ratios corresponding to half-power points are 8.9 and 9.7. Estimate the amount of damping in the system using the two methods discussed in this chapter and compare them with the results from Chap. 2, Example 5. Also, assuming the tower is made from steel, how does it compare with the value given at the beginning of this section, 3% for steel.

Solution

1. The damping ratio by the free vibration decay method (results from Example 5 in Chap. 2 are repeated here for comparison purposes).

ζ is determined using $u_1 = 3$ in $u_5 = 1$ in at $n = 4$, from Eq. (2.25),

$$\zeta = \frac{1}{2\pi n}\ln\frac{u_i}{u_{i+n}} = \frac{1}{4\times 2\pi}\ln\frac{3}{1} = 0.0437\text{, or } 4.37\%$$

2. The damping ratio by resonant amplification method, Eq. (3.36).

$$\zeta = \frac{1}{2R_{d\,res}} = \frac{1}{2(13.2)} = 0.0379\text{, or } 3.79\%$$

3. The damping ratio by half-power bandwidth method, Eq. (3.38).

$$\zeta = \frac{r_2 - r_1}{r_2 + r_1} = \frac{9.7 - 8.9}{9.7 + 8.9} = 0.0430\text{, or } 4.30\%$$

All three values are relatively close to each other and somewhat near to the approximate steel value. ▲

3.6 Problems

3.1 An undamped SDOF system has a mass of m = 0.3 lb.s^2/in and a spring stiffness of 20 lb/in, and is excited by a harmonic force having an amplitude, F_0 = 25 lb, and an excitation frequency, ω = 10 rad/s. The initial displacement, u (0) = 0.6 in and initial velocity $\dot{u}(0)$ = 0.05 in/s. Determine (a) the frequency ratio, (b) the amplitude of the forced response, (c) the displacement of the mass at time t = 2 seconds, and (d) the velocity of mass at time t = 4 seconds.

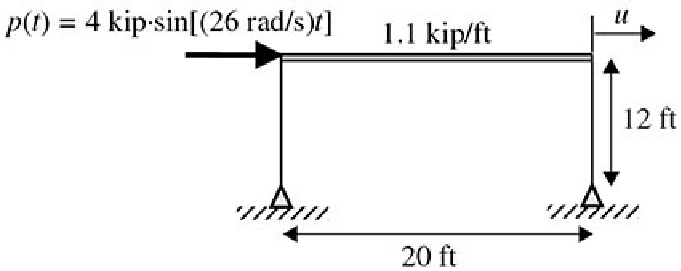
3.2 For the undamped SDOF system given in Prob. 3.1, draw, to scale, the forced response and total response curves. Also, determine how the forced response and total response curves change when the excitation frequency changes to 40 rad/s.

3.3 Derive Eq. (3.9), which is the response of an undamped SDOF system subjected to a harmonic forcing function of $u_p(t) = C\cos\omega t$.

3.4 An undamped SDOF system weighs 50 lb and has a spring stiffness of 250 lb/in. It is excited by a harmonic force having an amplitude p_0 = 200 lb and an excitation frequency of ω = 38.5 rad/s, which results in a beating condition. The initial displacement and velocity are zero. Draw the total response curve and determine the beat period as well as the number of oscillations in each beat.

3.5 Given an 1,100 lb system supported on springs with total stiffnesses of 2,000 lb/in that can be modeled as a SDOF oscillator, determine the displacement amplitude excited at resonance by a harmonic force of 675 lb after 1.5 cycles.

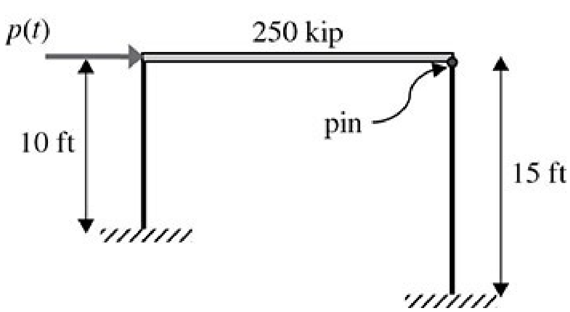
3.6 Determine the maximum steady-state amplitude of the horizontal motion of the following frame and loading. Assume the steel (E = 29,000 ksi) columns are W10 × 33 (I_x = 171 in^4) and a rigid beam. Also, neglect damping and the mass of the columns.



3.7 Given a damped SDOF system with stiffness k = 2,000 lb/ft, mass m = 2 lb.s^2/ft, damping ratio ζ = 0.05, initial displacement u (0) = 0 ft, and initial velocity $\dot{u}(0)$ = 0 ft/s, use MATLAB to graph the transient, steady-state, and total responses for a harmonic force of amplitude 400 lb and frequency ratios of (i) 0.25 and (ii) 2.5.

3.8 Consider the structural steel (E = 29,000 ksi) building frame given in Prob. 3.6, but with damping of 5%, determine (a) maximum steady-state displacement u_0, (b) the maximum base shear, and (c) the maximum normal stresses in the W10 × 33 columns (I_x = 171 in^4 and S_x = 35 in^3).

3.9 Given a structural steel (E = 29,000 ksi) building portal frame excited by a harmonic force having an amplitude, F_0 = 200 lb and an excitation frequency, ω = 5.3 rad/s, determine (a) the maximum displacement u, (b) the maximum base shear, and (c) the maximum normal stresses in the W8 × 21 columns (I_x = 75 in^4 and S_x = 18 in^3). Assume the beam is rigid and damping of 5%.



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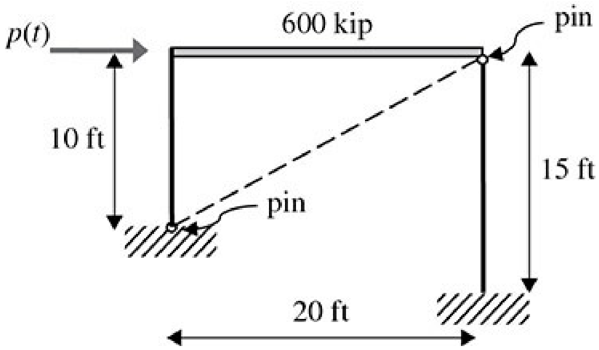
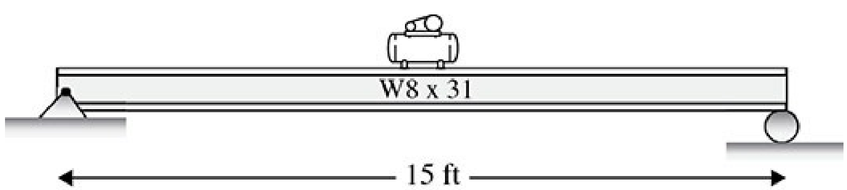
3.10 The following frame has a ½-in-diameter steel (E = 29,000 ksi) rod brace, and two steel columns (E = 29,000 ksi and I_x = 82.7 in^4). Determine the maximum force in the diagonal member and the maximum shear force in each of the columns when the frame is subjected to force $p(t)$ = 900 lb·sin[(8 rad/s)t]. Assume the beam is rigid and damping of 5%.



3.11 Consider the vehicle model system presented in Example 6. Assume the vehicle weighs 4,000 lb, and has a damping ratio of 0.25 and spring stiffness of 1,500 lb/in. A rough road profile can be modeled using a sine function with wavelength of 60 ft and amplitude of 1.2 in. Determine the maximum steady-state vertical response of the vehicle as it travels at 65 mi/h.

3.12 A 1,000 lb air compressor is directly mounted at midspan on the beam shown below (E = 29,000 kip/in^2 and I_x = 110 in^4). When the compressor is operated at 900 r/min, it causes a harmonic disturbing force of 25 lb. First, determine the force transmitted to the supports of the beam assuming no damping. Second, assume rubber pads are placed between the beam and its supports, which produce damping that yields a damping ratio of 0.1. What is the percent decrease in transmitted force? Assume the beam weight is negligible.



3.13 A one-story structure is idealized as a SDOF oscillator with stiffness k = 1,000 lb/in, mass m = 1,000 lb · s^2/in, damping ratio ζ = 0.07. Traffic conditions outside the structure generate support vibrations that can be modeled as harmonic displacement having an amplitude u_0 = 0.92 in and an excitation frequency of ω = 26.5 rad/s. The owner of the structure has specified that maximum relative displacement and velocity of the roof should be less than 0.05 in and 1 in/s, respectively. Does the spring damping system satisfy these requirements?

3.14 The seat for a 200 lb truck driver is supported by four springs, each having stiffness of 100 lb/in and total damping of 5%. During a typical trip, it is estimated that the seat will experience vertical displacements of $u(t)$ = 0.5 in·sin [(4 rad/s)·t]. For the driver to have a comfortable trip, maximum relative displacement and velocity of the seat should be less than 0.05 in and 1 in/s, respectively. Does the spring damping system satisfy these requirements?

3.15 The bridge beam depicted below supports a motor that weighs 1,000 lb at midspan. The motor produces a vertical harmonic force of 25 lb when operated at 900 r/min. Assuming

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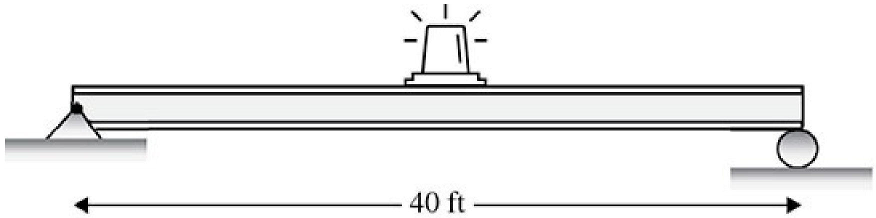
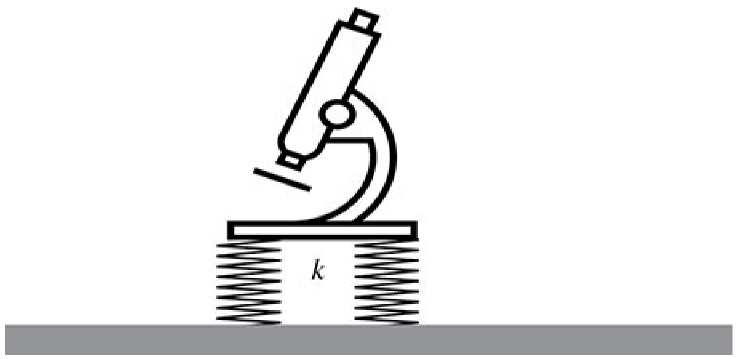
the beam weight is negligible and 5% damping, determine the amplitude of vertical motion and force transmitted to the beam supports after a long period of time (E = 29,000 kip/in^2 and I_x = 10,000 in^4).



3.16 A vibration isolation block weighing 2,000 lb is to be installed in a laboratory on four springs so that the vibration from adjacent factory operations (as high as 1,500 cycles/min) will not disturb certain experiments. Determine the stiffness of system, k, such that the motion of the block is limited to 10% of the floor vibration, neglect damping.

3.17 A sensitive instrument that weighs 250 lb is mounted in a new engineering building in one of the laboratories. The instrument currently experiences a vibration amplitude of 0.4 in when an air compressor runs at a frequency of 35 Hz. Neglecting damping, determine (a) the stiffness necessary for a mount in order to achieve 80% isolation and (b) the resulting vibration amplitude of the instrument.



3.18 A 1,000 lb reciprocating machine causes a vertical harmonic force of 50 lb at the operating frequency of 20 Hz. Design a supporting spring system (select the stiffness) so that the vibration generated in the housing building does not transmit more than 10 lb from the machine to the building.

3.19 An instrument of mass m = 0.5 lb·s^2/in is mounted on a supporting base that experiences a motion $u(t)$ = 0.8 in·sin [(23.5 rad/s)·t]. Design (select values of k and c of the mount) a vibration isolator for the instrument such that its displacement is no large than 0.2 in. Assume that the damping ratio ζ = 0.07.

3.20 A sensitive instrument with weight of 115 lb is to be installed at a location where vertical acceleration is 0.1 g at a frequency of 10 Hz. This instrument is mounted on a rubber pad of stiffness 75 lb/in and damping of 10%. What acceleration is transmitted to the instrument? If the instrument can tolerate only an acceleration of 0.005 g, suggest a solution assuming that the same rubber pad is used.

3.21 A wind turbine can be modeled as a concentrated mass atop a weightless tower to determine the damping of the system. One approach is to use logarithmic decrement; a second

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approach is to use the harmonic loading frequency sweep test to obtain the resonant amplification factor and half-power bandwidth. Estimate the damping ratio using these three methods. For logarithmic decrement, use a 0.9-g initial horizontal acceleration that decreases to 0.08 g after 11 complete cycles. For the other two methods, acceleration response spectrum, R_a versus f with $(R_d)_{\max}$ = 12.8 at a frequency of 3.59 Hz, and the frequencies corresponding to half-power points are 3.44 Hz and 3.74 Hz. Compare the results and assuming the tower is made from steel, how does it compare with the value given in this chapter, 3% for steel.

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CHAPTER 4

Vibration Response of SDOF Systems to General Dynamic Loading

After reading this chapter, you will be able to:

a. Determine the response of a single-degree-of-freedom (SDOF) system to a general load by direct and numerical integration of Duhamel's integral
b. Generate shock spectra for various nonperiodic dynamic loadings
c. Use shock spectra to determine the maximum response of structures
d. Perform direct numerical solution of the equation of motion to solve for the response of a SDOF system subjected to a general dynamic load
e. Use computational algorithms to solve general dynamics loading problems

In Chap. 3, we examined the response of structures subjected to periodic harmonic loadings. This type of loading is important because of its use in determining some of the parameters needed to analyze systems subjected to general dynamic loading. In particular, it can be used to solve general periodic (nonharmonic) loading cases by expanding the load into a Fourier series and using superposition of the response to individual harmonic forcing functions to obtain the total solution, a process that is not covered in this book.

In this chapter, we examine the vibration response of a structure caused by general, nonperiodic loads. These types of loads are modeled with general, nonperiodic forcing functions that vary with time and occur during a time period of a specified length. The response of a SDOF system excited by a general nonperiodic force can be determined using a convolution integral, or various other numerical methods. The convolution integral approach is an analytical method that can be used to obtain a closed-form solution when the forcing function is well defined. Analytical solutions are generally intractable for all but the simplest forcing functions. In general, numerical methods are employed in most general forcing function cases where an analytical solution is difficult or not available. However, analytical solutions can be useful in describing the response of a SDOF system and characterizing the influence of certain parameters on system response.

First, we focus on the use of a convolution integral known as Duhamel's integral to analyze a SDOF system subjected to a general forcing function. Duhamel's integral is based on the response of the system to an impulse, which we define as a force that abruptly jumps from zero to a constant value. For complex analytical functions, we present numerical integration methods for evaluating Duhamel's integral, which could be considered numerical methods in themselves, but are computationally inefficient. We illustrate the use of these numerical methods by solving several nonperiodic, general forcing function examples. These examples are used to construct shock spectra of various dynamic loads. We also describe the process for determining the maximum response of structures using shock spectra. Finally, we investigate more computationally efficient numerical methods to evaluate the response of a SDOF system to various complex loadings, including ground acceleration caused by seismic and traffic vibrations and other ground motions.

In this chapter, we make extensive use of computer software, such as MATLAB because of its efficiency in determining results for most general forcing function problems. MATLAB (MATrix LABoratory) was originally designed as an interactive software system for matrix operations, such as solving systems of linear equations, computing eigenvalues and eigenvectors, etc. Recent versions of the program include extensive graphics capabilities that can be used to easily plot results. It also has an extensive library of functions, including one that performs differentiation and integration as well as convolution integration (the *conv* function), which will be used in this chapter to develop scripts to solve many of the problems encounter in structural dynamics. In latter chapters we will use these MATLAB scripts to solve more complex dynamic problems.

4.1 Response of a SDOF System to an Impulse

In this section we consider an impulse load, which represents the simplest form of a general nonperiodic forcing function and is a constant load, p, applied on a structure over a very short time, Δt. The magnitude of the impulse is the product of p and Δt, that is, the area of the shaded rectangle in Fig. 4.1, $p\Delta t$.

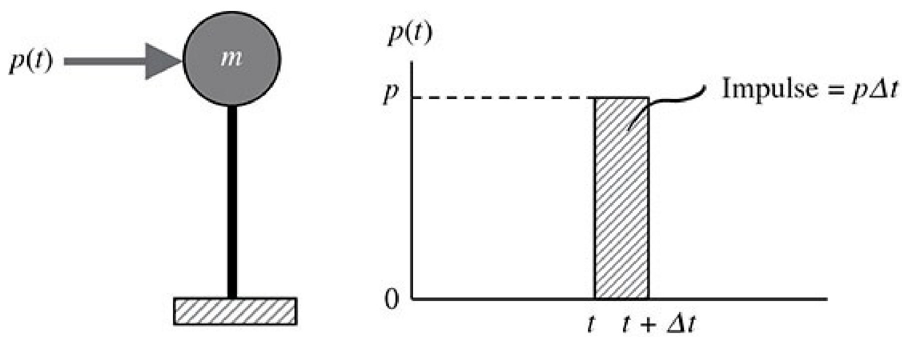

FIGURE 4.1 Impulse loading.

The impulse applied to a SDOF system is directly related to its change in momentum (the product of mass, m, and velocity, $\dot{u}$). Using Newton's second law of motion in terms of the momentum,

$$p(t) = m\frac{d\dot{u}}{dt} \tag{4.1}$$

where

$p(t)$ represents the general loading function.
m is the mass, which remains constant in this case.

Rearranging Eq. (4.1) we get the magnitude of the impulse in terms of the change in momentum

$$m d\dot{u} = p(t) dt \tag{4.2}$$

The incremental form of which leads to the definition of impulse associated with Fig. 4.1,

$$\text{Impulse} = p\Delta t = m\Delta\dot{u} = m(\dot{u}_2 - \dot{u}_1) \tag{4.3}$$

where

- $\dot{u}_1$ represents the velocity before the impulse.
- $\dot{u}_2$ represents the velocity after the impulse.

This result can be used to determine the response of a SDOF system since the impulse acts over an infinitesimally short duration of time (Δt is very small, Δt << 1), and the spring and damper have no effect during the time the impulse is applied because there is little time for the mass to experience a displacement, but the impulse will impart an initial velocity. This implies that if the system is initially at rest, its response to the impulse will be transient. That is, we can model the vibration response of a SDOF system as free vibration (as derived in Chap. 2), with initial conditions given by the effect of the impulse.

Considering an underdamped SDOF system subjected to an impulse at time $t = 0$ as shown in Fig. 4.2, the equation of motion is given by Eq. (2.8) and repeated here for convenience,

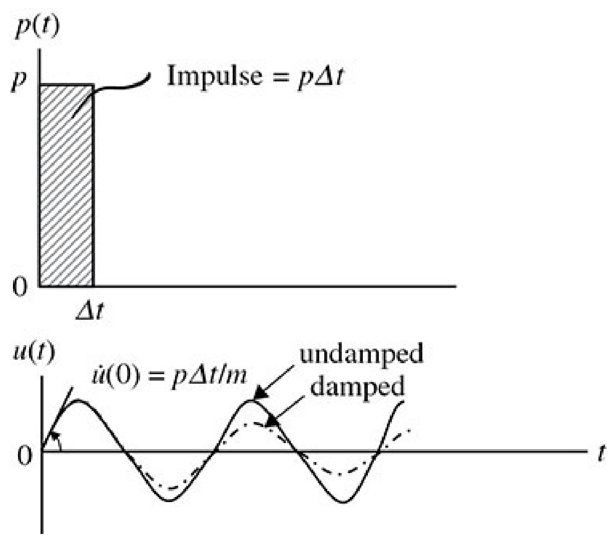

FIGURE 4.2 Displacement response to impulse loading.

$$m\ddot{u} + c\dot{u} + ku = 0$$

where, again, c is the damping coefficient and k the stiffness of the system. The free vibration response of this system is given by Eq. (2.17) and repeated here for convenience.

$$u(t) = e^{-\zeta\omega_n(t)}\left[\left(\frac{\dot{u}(0) + \zeta\omega_n u(0)}{\omega_D}\right)\sin(\omega_D t) + u(0)\cos(\omega_D t)\right] \tag{4.4}$$

The velocity and displacement before the impulse is applied are zero, $\dot{u}_1 = 0$. Therefore, the initial excitation of the SDOF oscillator is produced by the impulse, which we use to define the initial conditions. Since Δt is arbitrary and can be chosen to be infinitesimally short, we define the initial velocity on the mass at time $t = \Delta t$ as $\dot{u}_2$, which represents the velocity right after the impulse based on Eq. (4.3),

$$\dot{u}(0) = \dot{u}_2 = \frac{p\Delta t}{m} \tag{4.5}$$

and initial displacement,

$$u(0) = 0 \tag{4.6}$$

Substituting these initial conditions into Eq. (4.4), we get

$$u(t) = e^{-\zeta\omega_n(t)}\left(\frac{p\Delta t}{m\omega_D}\right)\sin(\omega_D t) \tag{4.7}$$

If we neglect damping, we get the undamped response,

$$u(t) = \frac{p\Delta t}{m\omega_n}\sin(\omega_n t) \tag{4.8}$$

The impulse response function of a damped [Eq. (4.7)] and undamped [Eq. (4.8)] SDOF system is shown in Fig. 4.2.

Now, apply the impulse at an arbitrary time, $t = \tau$ as shown in Fig. 4.3. The displacement is zero until the impulse is applied; thereafter, the displacement is given by Eq. (4.7), but with a time, t, equal to the time elapse before the impulse is applied, $t - \tau$,

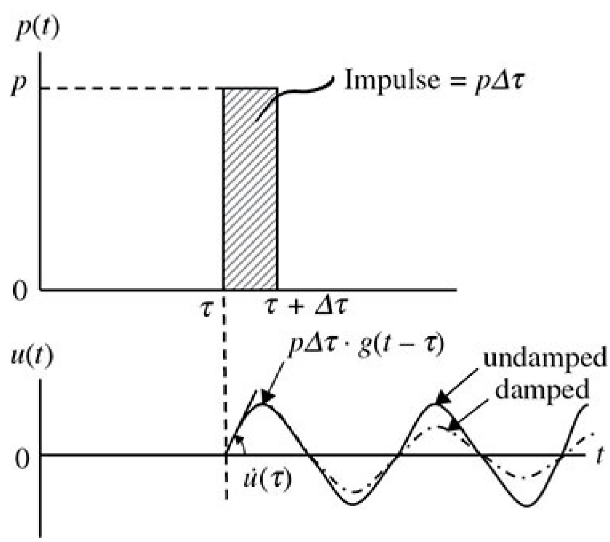

FIGURE 4.3 Displacement response to impulse loading applied at time, $t = \tau$.

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$$u(t) = e^{-\zeta\omega_n(t-\tau)}\left[\left(\frac{p\Delta t}{m\omega_D}\right)\sin\left[\omega_D(t-\tau)\right]\right] \tag{4.9}$$

For this equation, the initial conditions (initial velocity and displacement) are defined using the initial velocity on the mass at time $t = \tau$, which are the same as for the impulse shown in Fig. 4.2, $\dot{u}(\tau) = \frac{p\ \ \tau}{m}$ and $u(\tau) = 0$.

For this case, the undamped response is given by

$$u(t) = \left(\frac{p\Delta\tau}{m\omega_n}\right)\sin\left[\omega_n(t-\tau)\right] \tag{4.10}$$

The impulse response function of a damped [Eq. (4.9)] and undamped [Eq. (4.10)] SDOF system for this case is shown in Fig. 4.3.

If a unit impulse is applied to the system, such that $p\Delta\tau = 1$, then the response of the SDOF system given by Eq. (4.9) becomes

$$g(t-\tau) \equiv \frac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_D}\sin\left[\omega_D(t-\tau)\right] \tag{4.11}$$

where $g(t-\tau)$ is defined as the unit impulse response function. Should the impulse be different than 1, we can multiply Eq. (4.11) by the actual magnitude of the impulse in order to establish the response of the system (damped or undamped cases),

$$u(t) = p\Delta\tau \times g(t-\tau) \tag{4.12}$$

Example 1

Determine the vibration response of the following steel (E = 29,000 ksi) frame when excited by two impacts from two separate projectiles. The first impact creates a force of 4 kip and acts over 0.02 seconds; the second impact comes 0.5 seconds later and has a force magnitude of 3 kip and acts over 0.015 seconds. Assume the columns are W10 × 33 (I_x = 171 in^4), a rigid beam, and 5% damping. Also, neglect the mass of the columns.

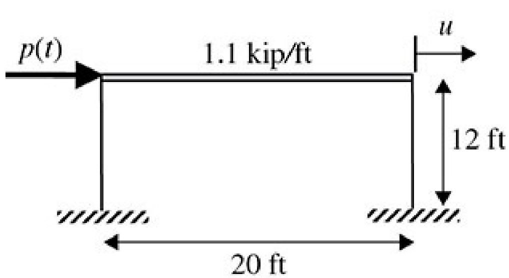

FIGURE E1.1 Portal frame model.

Solution

i. *Determine mass, stiffness, and natural frequency of the SDOF system.* Similar to Example 2 of Chap. 2, the frame can be modeled as a SDOF system since the beam is rigid and thus

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the stiffness of the system is the result of the lateral deformations of the columns.
Mass,

$$m = \frac{W}{g} = \frac{1.1\ \text{kip/ft}(20\ \text{ft})(1{,}000\ \text{lb/kip})}{386.4\ \text{in/s}^2} = 56.9\ \frac{\text{lb}\cdot\text{s}^2}{\text{in}}$$

Stiffness (two fixed-fixed connected columns),

$$k = \frac{12EI}{L^3} + \frac{12EI}{L^3} = 2\ \frac{12(29{,}000{,}000\ \text{psi})(171\ \text{in}^4)}{(12\ \text{ft} \times 12\ \text{in/ft})^3} = 39{,}858\ \frac{\text{lb}}{\text{in}}$$

The natural frequency of the SDOF system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{39{,}858\ \text{lb/in}}{56.9\ \text{lb}\cdot\text{s}^2/\text{in}}} = 26.46\ \frac{\text{rad}}{\text{s}}$$

The damped natural frequency of the SDOF system is

$$\omega_D = \omega_n\sqrt{1-\zeta^2} = 26.46\ \text{rad/s}\sqrt{1-(0.05)^2} = 26.42\ \text{rad/s}$$

ii. *Use superposition to obtain the response of the two impulses.*

$$u(t) = \begin{cases} p_1\Delta\tau_1 \cdot g(t) & \text{for } t \le 0.5\ \text{s} \\ p_1\Delta\tau_1 \cdot g(t) + p_2\Delta\tau_2 \cdot g(t-\tau) & \text{for } t > 0.5\ \text{s} \end{cases}$$

where

$p_1\ \ \tau_1 = 4\ \text{kip}(0.02\ \text{s}) = 80\ \text{lb}\cdot\text{s}$

$p_2\ \ \tau_2 = 3\ \text{kip}(0.015\ \text{s}) = 45\ \text{lb}\cdot\text{s}$

$\tau = 0.5\ \text{s}$

Substituting the equations for *g*'s,

$$u(t) = 80\ \text{lb}\cdot\text{s}\frac{e^{-\zeta\omega_n(t)}}{m\omega_D}\sin(\omega_D t) \qquad \text{for } t \le 0.5\ \text{s}$$

Otherwise,

$$u(t) = 80\ \text{lb}\cdot\text{s}\frac{e^{-\zeta\omega_n(t)}}{m\omega_D}\sin(\omega_D t) + 45\ \text{lb}\cdot\text{s}\frac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_D}\sin\ \ \omega_D(t-\tau)$$

iii. *Substituting the values for ζ, m, ω_n, and ω_D, then plotting the total response of the SDOF system with MATLAB, we get the following graph:*

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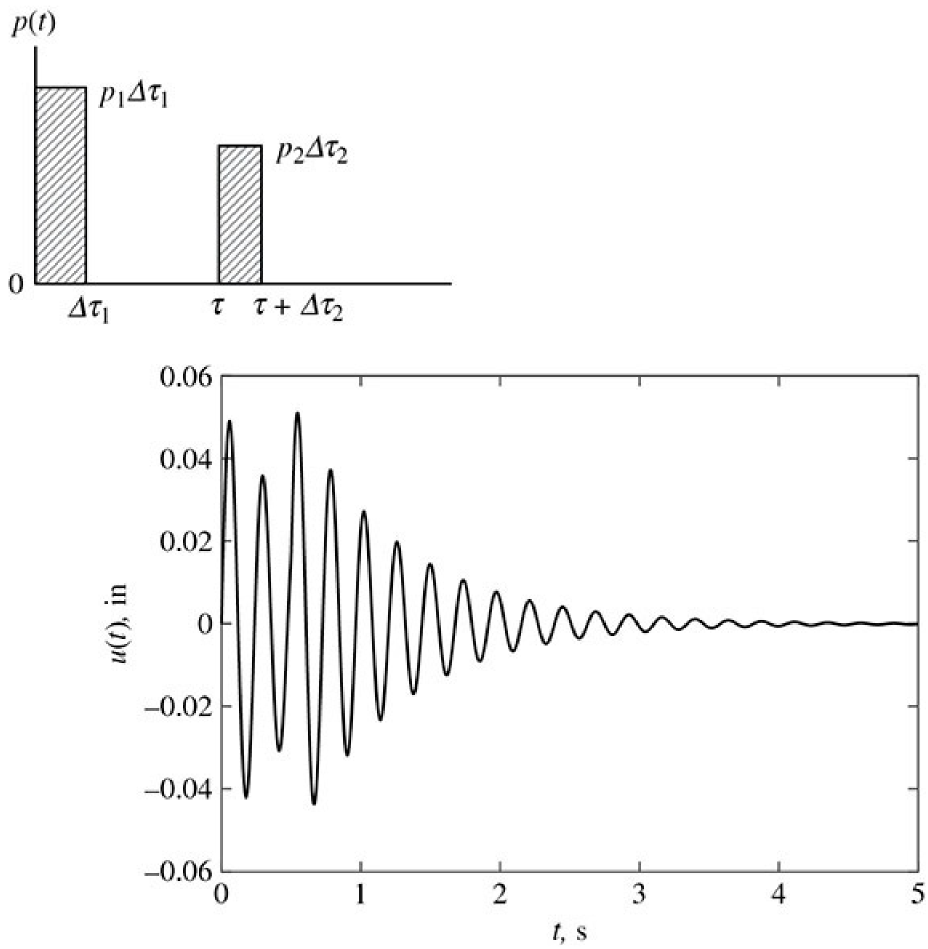

FIGURE E1.2 Total response of a SDOF system to a double impact loading.

The MATLAB script is as follows:

```
% Chapter 4, Example 1
L=12; % column height in ft
E=29000000; % Modulus of Elasticity in lb/in^2
I=171; % moment of inertia in in^4
g=386.4; % acceleration due to gravity in in/sec^2
xi = 0.05; % damping ratio
W=1.1*20*1000; % weight on frame
m = W/g; % mass
k = 2*12*E*I/(L*12)^3; % stiffness in lb/in
wn=sqrt(k/m); % natural frequency
wD=wn*sqrt(1-xi^2); % damped natural frequency
p1deltatmwD=80/m/wD; % lb-sec
p2deltatmwD=45/m/wD; % lb-sec
t = linspace(0,5,1000);
for i = 1:1000
    if t(i) <= 0.5
        ut(i) = p1deltatmwD*exp(-xi*wn*t(i))*sin(wD*t(i));
    else
        ut(i) = p1deltatmwD*exp(-xi*wn*t(i))*sin(wD*t(i))+...
            p2deltatmwD*exp(-xi*wn*(t(i)-0.5))*sin(wD*(t(i)-0.5));
    end
end
%Create plot
plot (t, ut, 'LineWidth',1, 'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t, sec', 'FontAngle','italic');
ylabel ('u(t), in','FontAngle','italic');
```
▲

4.2 General Forcing Function and Duhamel's Integral

Now consider the general loading function, $p(t)$, given in Fig. 4.4, which can be represented by a series of short impulses having varying magnitudes at successive incremental times, one after another. The incremental impulse at some particular time, $t = \tau$, is given by the force, $p(\tau)$, acting on the system for a short period of time, $\Delta\tau$, that is, $p(\tau)\Delta\tau$. Each of these impulses causes an incremental displacement response $\delta u(t)$, with an incremental response at time $t = \tau$ determined using Eq. (4.12) as

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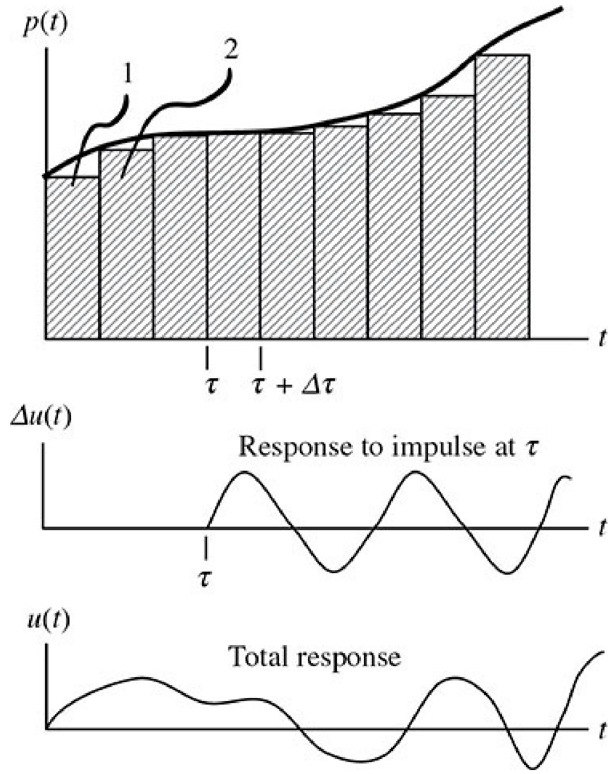

FIGURE 4.4 Discretization of a general loading function and response.

$$\Delta u(t) = \left[p(\tau)\Delta\tau\right] g(t-\tau) \qquad \text{for } t > \tau \tag{4.13}$$

The total displacement response from all the incremental impulses can be determined by summing their respective incremental displacement responses. That is,

$$u(t) = \sum \Delta u(t) \cong \sum \left[p(\tau)\Delta\tau\right] g(t-\tau) \tag{4.14}$$

In the limit, as $\Delta\tau$ becomes infinitesimally small or approaches zero,

$$u(t) = \int_0^t p(\tau) g(t-\tau) d\tau \tag{4.15}$$

Substituting the unit impulse response function, Eq. (4.11) into Eq. (4.15), we get the total response of an underdamped SDOF system at any time t as the superposition of the responses to all the impulses occurring before t. That is,

$$u(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) e^{-\zeta\omega_n(t-\tau)} \left[\sin \omega_D (t-\tau)\right] d\tau \tag{4.16}$$

Equation (4.16) [as well as Eq. (4.15)] is a convolution integral and is known as Duhamel's integral. It gives the response of an underdamped SDOF system to any general excitation. Also, if the forcing function, $p(t)$, cannot be expressed by an analytical equation, which is often the case in many applications, the integral of Eq. (4.16) can be evaluated using numerical integration.

It is important to recognize that Eq. (4.16) assumes the system is at rest before the application of the forcing function, $p(t)$. Also, Eq. (4.16) represents the particular solution of the response, $u_p(t)$, as described in Chap. 3, with the total solution given by Eq. (3.20). When initial displacement, $u(0)$ and initial velocity, $\dot{u}(0)$ are included, the complementary (free vibration response) part, $u_c(t)$, of the solution [Eq. (2.26)] is combined with Eq. (4.16) yielding the total response (including initial conditions) of

$$u(t) = e^{-\zeta\omega_n t}\left[\left(\frac{\dot{u}(0)+\zeta\omega_n u(0)}{\omega_D}\right)\sin\omega_D t + u(0)\cos\omega_D t\right] + \frac{1}{m\omega_D}\int_0^t p(\tau)e^{-\zeta\omega_n(t-\tau)}\left[\sin\omega_D(t-\tau)\right]d\tau \tag{4.17}$$

If damping in the system is negligible, then Eqs. (4.16) and (4.17) can be expressed as follows:

$$u(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\left[\sin\omega_n(t-\tau)\right]d\tau \tag{4.18}$$

and

$$u(t) = \left(\frac{\dot{u}(0)}{\omega_n}\right)\sin\omega_n t + u(0)\cos\omega_n t + \frac{1}{m\omega_n}\int_0^t p(\tau)\left[\sin\omega_n(t-\tau)\right]d\tau \tag{4.19}$$

$$\dot{u}(0)$$

Again, it is important to note that Eq. (4.18) assumes that the system is initially at rest, whereas Eq. (4.19) includes initial displacement, $u(0)$, and initial velocity, .

Example 2

Consider an undamped SDOF system excited by a constant force as shown in Fig. E2.1. Assuming the SDOF system is initially at rest, use Duhamel's integral to determine the displacement response, $u(t)$, of the system. Also, graph the response as a function of two dimensionless quantities: dynamic load factor (DLF) and time expressed as a dimensionless parameter by using the period instead of the circular frequency, t/T_n.

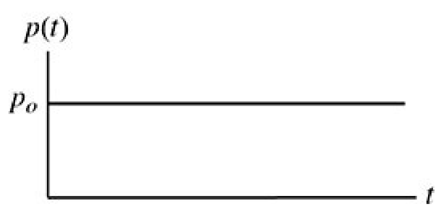

FIGURE E2.1 Step loading function.

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Solution

i. *Identify the forcing function and equation of motion.* The time-dependent forcing function, $p(t)$, can be written as

$$p(t) = p_o$$

This forcing function is a step, dynamically applied constant force because it is abruptly applied. When the force is applied very slowly it induces no vibration and is considered statically applied.

Since the damping is neglected, the equation of motion is given as

$$m\ddot{u}(t) + ku(t) = p_o$$

ii. *Apply Duhamel's integral, Eq. (4.18) since $\zeta = 0$ and the system is initially at rest ($u(0) = 0$ and $\dot{u}(0) = 0$).*

Substituting the forcing function into Eq. (4.18) and integrating results in the response of the system,

$$u(t) = \frac{1}{m\omega_n}\int_0^t p_o \sin\omega_n(t-\tau)d\tau = \frac{p_o}{m\omega_n^2}\cos\omega_n(t-\tau)\Big|_0^t$$

Or,

$$u(t) = \frac{p_o}{m\omega_n^2}\ 1 - \cos\omega_n t$$

Since the stiffness, $k = \omega_n^2 m$, the response is

$$u(t) = \frac{p_o}{k}\ 1 - \cos\omega_n t\ = u_{st}\ 1 - \cos\omega_n t$$

where

u_{st} represents the static displacement of the SDOF system. That is, if the force, p_o, were applied very slowly such that no vibrations were induced, the displacement would be $u_{st} = p_o/k$.

The DLF, which represents the amplification of the displacement above the static displacement, is given as $u(t)/u_{st}$ = $\left[1 - \cos\omega_n t\right]$..

iii. *Plot the response in terms of DLF and a dimensionless time variable.*

Substituting for $\omega_n = 2\pi/T_n$, the following equation can be used to represent the response of the SDOF system as DLF versus t/T_n.

$$\frac{u(t)}{u_{st}} = \mathrm{DLF} = \left[1 - \cos 2\pi\frac{t}{T_n}\right]$$

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Figure E2.2 shows the graph of this equation, the response of SDOF system to a step force.

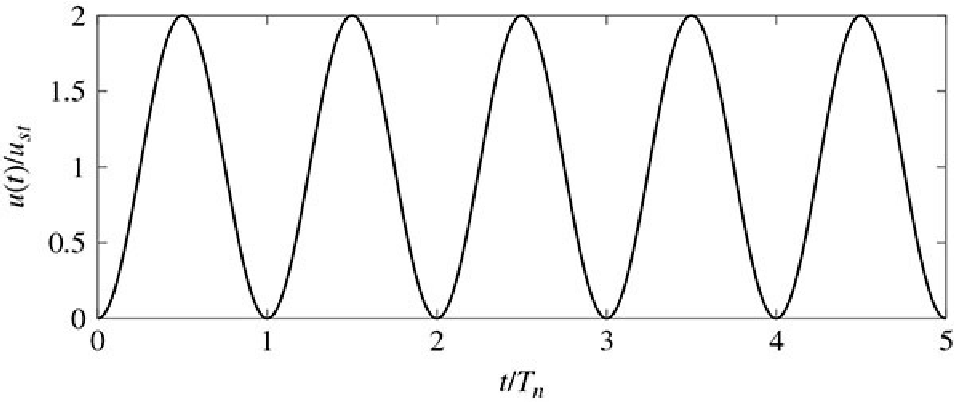

FIGURE E2.2 Response of a SDOF system to the step force, $p(t) = p_0$.

The response of a SDOF system to the step force, $p(t) = p_o$, is similar to the free vibration response of an undamped system except that it is shifted up by the static displacement, u_{st}. The maximum displacement, ($u_{\max}$) of a SDOF system subjected to a suddenly applied force, p_o, is twice the displacement of the same SDOF system when an equivalent static force, p_o is applied very slowly, and it occurs when $t = T_n/2$. This is also true for internal forces, stresses, etc. ▲

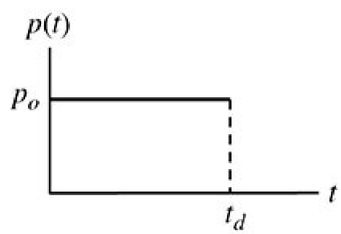

FIGURE E3.1 Rectangular impulse loading function.

Example 3

Consider the undamped SDOF system described in Example 2, but excited by a rectangular impulse, which has constant force, p_o, and is suddenly applied for a specified duration of time, t_d, as shown in Fig. E3.1. Use Duhamel's integral to determine the displacement response, $u(t)$, of the system. Also, graph the response as a function of two dimensionless quantities as described in Example 2.

Solution

i. *Identify the forcing function.* The forcing function, $p(t)$, is given as

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$$p(t) = \begin{cases} p_o & t \le t_d \\ 0 & t > t_d \end{cases}$$

The solution to this type of loading can be obtained using the results of Example 2. The process entails dividing the solution into two parts as shown in Fig. E3.2. The first part of the solution, up to the end of the rectangular forcing function, is equal to the solution given in Example 2; while the second can be obtained as the sum of two-step functions where the first forcing function, $p_1(t)$, with a magnitude of p_o starts at $t = 0$ and the second forcing function, $p_2(t)$, with a magnitude of $-p_o$, starts at $t = t_d$. This is described analytically as

$$p_1(t) = p_o$$

$$p_2(t) = \begin{cases} 0 & t \le t_d \\ -p_o & t > t_d \end{cases}$$

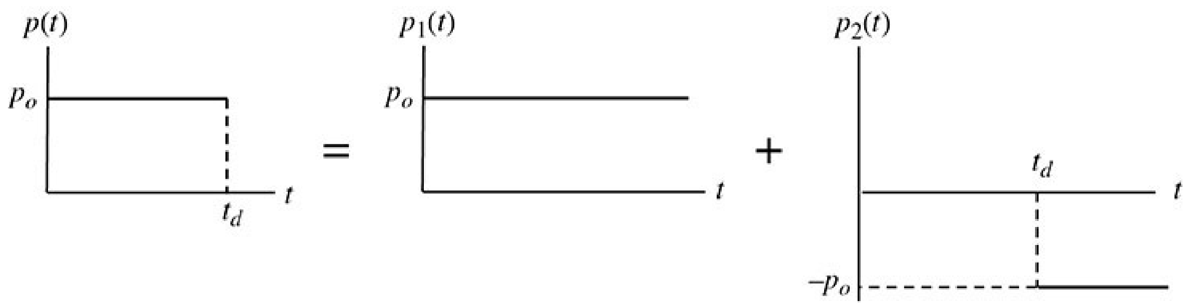

FIGURE E3.2 Rectangular impulse as the sum of two step forcing functions.

ii. *Apply Duhamel's integral (or previous solutions) to obtain the response to forcing functions p1(t) and p2(t).* $u_1(t)$, the response of the SDOF system to the step force, $p_1(t) = p_o$, is identical to the response determined in Example 2, that is,

$$u_1(t) = u_{st}\left[1 - \cos\omega_n t\right]$$

$u_2(t)$, the response of the SDOF system to the second step forcing function, $p_2(t)$, can also be obtained similar to the response determined in Example 2, except that the step force has a magnitude of $-p_o$ and experiences a time delay of $t - t_d$, which can be substituted in the response determined in Example 2 to get

$$u_2(t) = \frac{-p_o}{k}\left\{1 - \cos\left[\omega_n(t - t_d)\right]\right\} = u_{st}\left\{\cos\left[\omega_n(t - t_d)\right] - 1\right\}$$

t ″ t_d We can now obtain the total time history response as two separate response regions; during the pulse when $u_1(t)$, the total response of the system is given by , and after the

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pulse, for $t > t_d$. The total response of the undamped SDOF system is the sum of the two responses, $u_1(t) + u_2(t)$. That is,

$$u(t) = \begin{cases} u_1(t) = u_{st}\left[1 - \cos(\omega_n t)\right] & t \le t_d \\ u_1(t) + u_2(t) = u_{st}\left\{\cos\left[\omega_n(t - t_d)\right] - \cos(\omega_n t)\right\} & t > t_d \end{cases}$$

These relationships can be written in terms of the DLF = $u(t)/u_{st}$, a dimensionless time parameter, t/T_n, and ratio of duration of the loading to the natural period of the structure, t_d/T_n:

$$\text{DLF} = \frac{u(t)}{u_{st}} = \begin{cases} 1 - \cos(2\pi t/T_n) & t \le t_d \\ \cos\left[2\pi(t/T_n - t_d/T_n)\right] - \cos(2\pi t/T_n) & t > t_d \end{cases}$$

iii. *Plot the response of the SDOF system.* Figure E3.3 shows graphs of DLF = $u(t)/u_{st}$ versus t/T_n using these equations for different values of t_d/T_n.

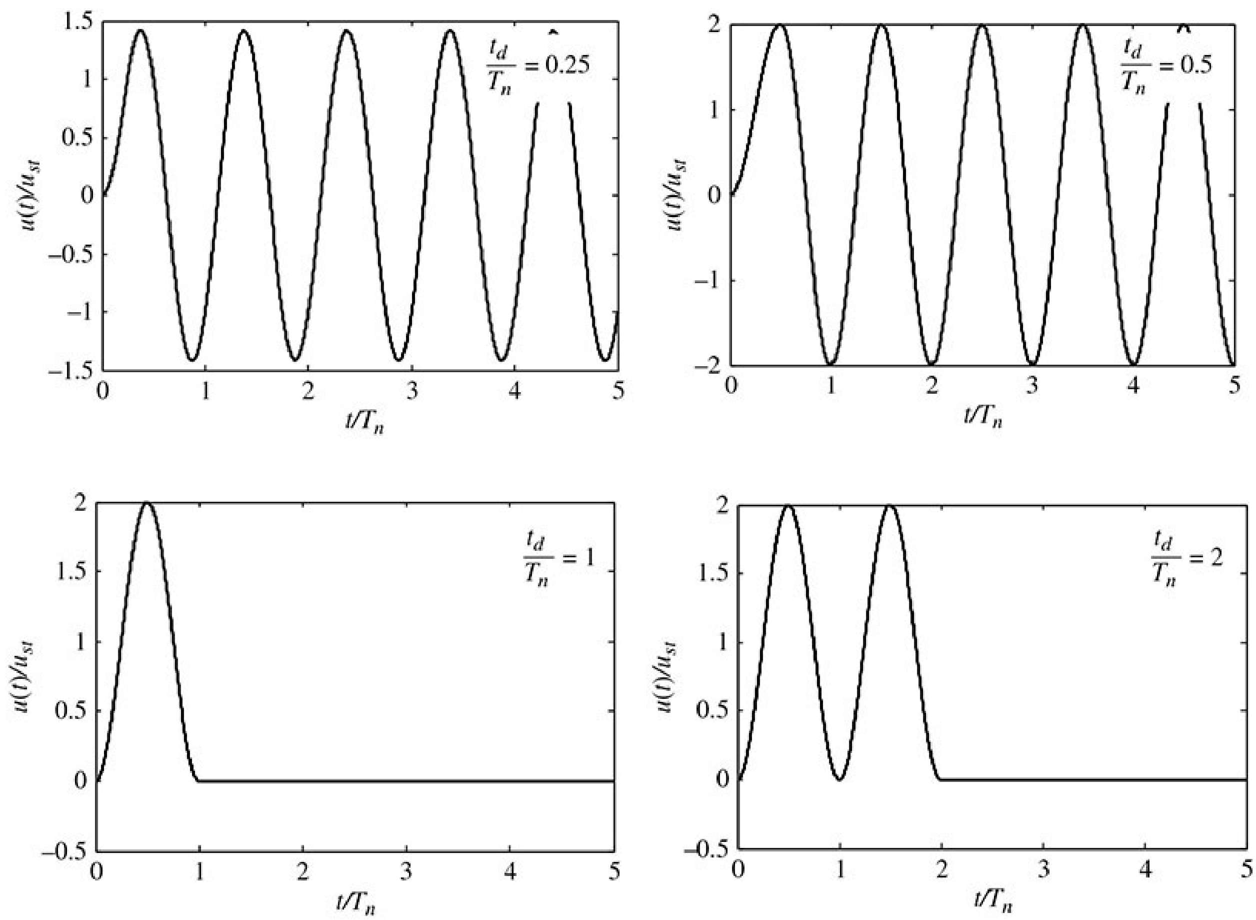

FIGURE E3.3 Response of SDOF system due to a rectangular pulse and varying pulse durations. ▲

4.3 Numerical Evaluation of Duhamel's Integral

For other than simple cases, Duhamel's integral must be evaluated numerically using methods such as the Euler's method, trapezoidal rule, or Simpson's rule, or a computer program such as MATLAB. First, consider a damped system with Duhamel's integral given by Eq. (4.16), repeated here rearranged as follows:

$$u(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_D}\int_0^t p(\tau)e^{\zeta\omega_n\tau}\left[\sin(\omega_D t - \omega_D\tau)\right]d\tau$$

To apply a numerical method, the sine function within the integration must first be transformed using the following trigonometric identity,

$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$$

Rewriting the sine component in the integral as,

$$\sin(\omega_D t - \omega_D\tau) = \sin(\omega_D t)\cos(\omega_D\tau) - \cos(\omega_D t)\sin(\omega_D\tau)$$

Substitution into Eq. (4.16),

$$u(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_D}\int_0^t p(\tau)e^{\zeta\omega_n\tau}\left[\sin(\omega_D t)\cos(\omega_D\tau) - \cos(\omega_D t)\sin(\omega_D\tau)\right]d\tau$$

Next, factor the quantities that are constant in terms of τ,

$$u(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_D}\left\{\sin(\omega_D t)\int_0^t p(\tau)e^{\zeta\omega_n\tau}\left[\cos(\omega_D\tau)\right]d\tau - \cos(\omega_D t)\int_0^t p(\tau)e^{\zeta\omega_n\tau}\left[\sin(\omega_D\tau)\right]d\tau\right\}$$

The resulting relationship with a couple of new parameters is

$$u(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_D}\left[\sin(\omega_D t)A_D(t) - \cos(\omega_D t)B_D(t)\right] \tag{4.20}$$

where

$$A_D(t) = \int_0^t p(\tau)e^{\zeta\omega_n\tau}\left[\cos(\omega_D\tau)\right]d\tau$$

$$B_D(t) = \int_0^t p(\tau)e^{\zeta\omega_n\tau}\left[\sin(\omega_D\tau)\right]d\tau$$

The undamped system, with Duhamel's integral given by Eq. (4.18), can be obtained by setting $\zeta = 0$ in the last three equations and changing the damped frequency to the natural

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frequency:

$$u(t)=\frac{1}{m\omega_n}\left[\sin(\omega_n t)A(t)-\cos(\omega_n t)B(t)\right] \tag{4.21}$$

where

$$A(t)=\int_0^t p(\tau)\left[\cos(\omega_n \tau)\right]d\tau$$

$$B(t)=\int_0^t p(\tau)\left[\sin(\omega_n \tau)\right]d\tau$$

These integral quantities, $A_D(t)$, $B_D(t)$, $A(t)$, and $B(t)$ can be evaluated numerically using numerical integration by Euler's method, trapezoidal rule, Simpson's rule, or MATLAB. The details are covered in the next subsections, but first, write a general function, $\mathrm{I}(\tau)$, and its integral over a range from zero to time, t,

$$H(t)=\int_0^t \mathrm{I}(\tau)d\tau \tag{4.22}$$

4.3.1 Euler's Method

This method is also known as the forward rectangular method and is used to determine the area under the function $\mathrm{I}(t)$ as shown in Fig. 4.5.

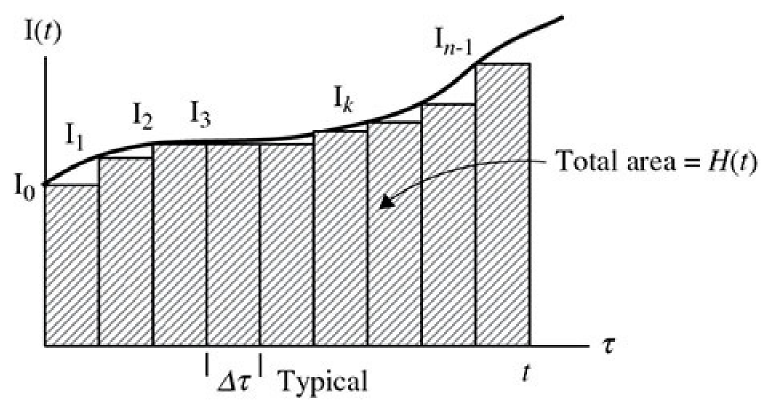

FIGURE 4.5 Forward rectangular discretization of function $\mathrm{I}(t)$.

We can assume any width for each time segment, but to simplify the calculations, $\Delta\tau$ is usually assumed to be uniform. That is,

$$\Delta\tau=\frac{(t-0)}{n}$$

where the numerator is given by the difference in the limits of integration, and n is the number of segments. The area is given by the sum of all the rectangular areas,

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$$H(t)=\int_0^t \mathrm{I}(\tau)d\tau \approx \Delta\tau \mathrm{I}_0+\Delta\tau \mathrm{I}_1+\cdots+\Delta\tau \mathrm{I}_k+\cdots+\Delta\tau \mathrm{I}_{n-1}=\Delta\tau\sum_{k=0}^{n-1}\mathrm{I}(k\Delta\tau) \tag{4.23}$$

where $k\Delta\tau$ is the function evaluated at time $(k\Delta\tau)$. That is,

$\mathrm{I}_0=\mathrm{I}(0)$
$\mathrm{I}_1=\mathrm{I}(\Delta\tau)$
$\mathrm{I}_2=\mathrm{I}(2\Delta\tau)$

4.3.2 Trapezoidal Rule

This method is also known as the linear method and is used to determine the area under the function I(t) as shown in Fig. 4.6. This is more accurate than Euler's method since the trapezoidal areas result in smaller gaps under the function; therefore, it gives the exact area beneath linear functions.

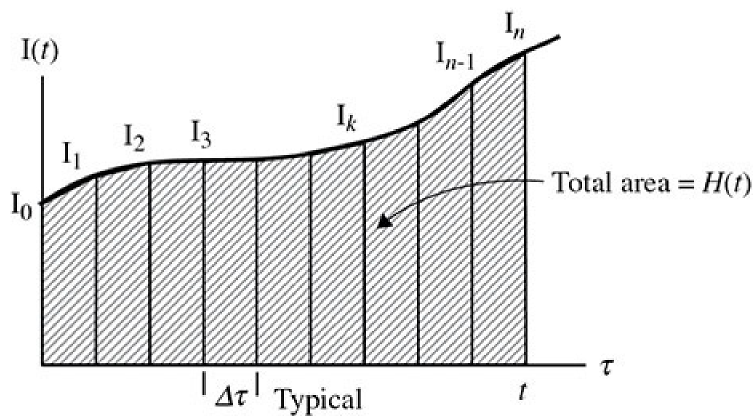

FIGURE 4.6 Linear discretization of function I(t).

For this case, the width of the time segment is uniform and obtained as,

$$\Delta\tau=\frac{(t-0)}{n}$$

where the numerator is given by the difference in limits of integration, and n is the number of segments. The area is given by the sum of all the trapezoidal areas,

$$\begin{aligned}H(t)&=\int_0^t \mathrm{I}(\tau)d\tau \approx \Delta\tau\frac{(\mathrm{I}_0+\mathrm{I}_1)}{2}+\Delta\tau\frac{(\mathrm{I}_1+\mathrm{I}_2)}{2}+\cdots+\Delta\tau\frac{(\mathrm{I}_{k-1}+\mathrm{I}_k)}{2}+\cdots+\Delta\tau\frac{(\mathrm{I}_{n-1}+\mathrm{I}_n)}{2}\\&=\frac{\Delta\tau}{2}\left[\mathrm{I}(0)+2\sum_{k=0}^{n-1}\mathrm{I}(k\Delta\tau)+\mathrm{I}(t)\right]\end{aligned} \tag{4.24}$$

where $k\Delta\tau$ is the function evaluated at time $(k\Delta\tau)$. That is,

$I_0 = I(0)$
$I_1 = I(\Delta\tau)$
$I_2 = I(2\Delta\tau)$
etc.

4.3.3 Simpson's Rule

This method is also known as the parabolic method and is used to determine the area under the function I(t) as shown in Fig. 4.7. This is more accurate than the previous two methods since the Simpson's rule uses intervals topped with parabolas to approximate the area; therefore, it gives the exact area beneath quadratic functions.

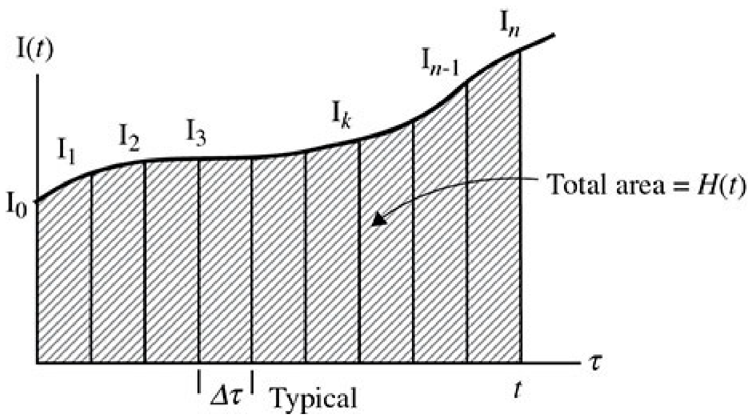

FIGURE 4.7 Parabolic discretization of function I(t).

Again, the width of the time segment is uniform and obtained as,

$$\Delta\tau = \frac{(t-0)}{n}$$

where the numerator is given by the difference in limits of integration, and n is the number of segments and must be an even integer. The area is given by the sum of all the parabolic areas,

$$\begin{aligned} H(t) &= \int_0^t I(\tau)d\tau \approx \frac{\Delta\tau}{3}\left(I_0 + 4I_1 + 2I_2 + \cdots + 4I_{n-1} + I_n\right) \\ &= \frac{\Delta\tau}{3}\left[I(0) + 2\sum_{k=2,4,6,..}^{n-2} I(k\Delta\tau) + 4\sum_{k=1,3,5,..}^{n-1} I(k\Delta\tau) + I(t)\right] \end{aligned} \tag{4.25}$$

where $k\Delta\tau$ is the function evaluated at time $(k\Delta\tau)$. That is,

$I_0 = I(0)$
$I_1 = I(\Delta\tau)$
$I_2 = I(2\Delta\tau)$
etc.

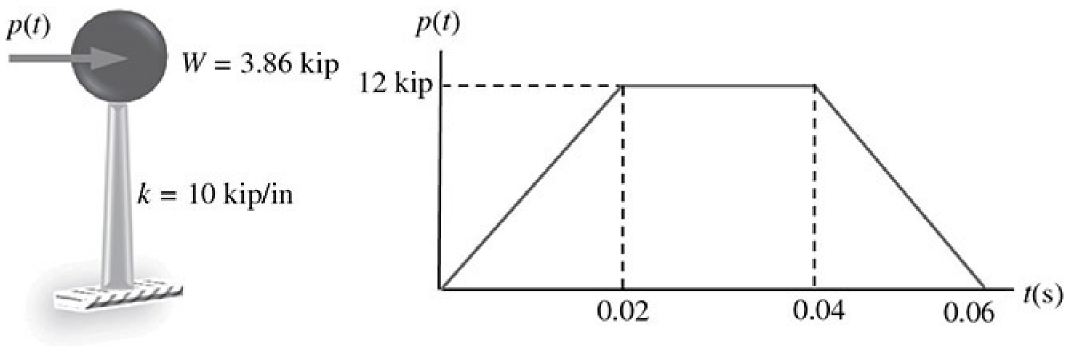

FIGURE E4.1 SDOF system and trapezoidal impulse loading function.

Example 4

Given the following tower subjected to a trapezoidal impulsive excitation, use Euler's method, Trapezoidal rule, and Simpson's rule to determine the dynamic response of the tower. Use a spreadsheet program (e.g., MS Excel) to perform calculations and assume negligible damping.

Solution

i. *Determine natural frequency of the SDOF system.*

Stiffness is given as 10 kip/in

Mass can be calculated as follows:

$$m = \frac{W}{g} = \frac{3.86 \text{ kip}}{386.4 \text{ in/s}^2} = 0.01 \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

So, the natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10 \text{ kip/in}}{0.01 \text{ kip} \cdot \text{s}^2/\text{in}}} = 31.6 \frac{\text{rad}}{\text{s}}$$

Assume time increment, $\Delta\tau = 0.01$ seconds.

ii. *Discretize the trapezoidal impulse loading function, $p(\tau)$.* This is listed in the third column of table below (same for all three methods) and is obtained by evaluating the loading function at the various time increments starting at zero, then ramping up to 6 kip at the first-time increment, etc.

iii. *Displacement response as a function of time calculations:*

Integrands $I_2(\tau) = p(\tau)[\sin(\omega_n \tau)]$ and $I_1(\tau) = p(\tau)[\cos(\omega_n \tau)]$ are listed in the fourth and fifth columns and are the same for all three methods. Integrals $A(t)$ and $B(t)$ are then evaluated using the $H(t)$ function for each of the methods within their respective portions of the table. The last column gives the resulting displacement.

Notice that the maximum response for the Euler and Trapezoidal methods is identical, u_{max} = 1.407 in; however, Simpson's method should yield a slightly more accurate result, u_{max}

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= 1.395 in. As will be shown in Example 6, where we solve this problem using the MATLAB convolution integral solution but discretized more finely (50 time-increments compared to 10), u_{max} = 1.397 in compared to Simpson's 1.395 in. ▲

Spreadsheet Numerical Calculations							
W (kip) = 3.86				k (kip/in) = 10		$\Delta\tau = 0.01$	
m(k-s^2/in) = 0.01				ω_n(rad/s) = 31.6228		$A(t) = \int_0^t p(\tau)[\cos(\omega_n\tau)]\,d\tau$	
$u(t) = \dfrac{1}{m\omega_n}[\sin(\omega_n t)A(t) - \cos(\omega_n t)B(t)]$						$B(t) = \int_0^t p(\tau)[\sin(\omega_n\tau)]\,d\tau$	
Euler's Method	$\boldsymbol{H(t) = \int_0^t \mathrm{I}(\tau)\,d\tau \approx \Delta\tau \sum_{k=0}^{n-1} \mathrm{I}(k\Delta\tau)}$						
$\boldsymbol{t}$**(s)**	$\boldsymbol{\omega_n\tau}$	$\boldsymbol{p(\tau)}$**, kip**	$\mathbf{I_1}(\tau)$	$\mathbf{I_2}(\tau)$	$\boldsymbol{A(t)}$	$\boldsymbol{B}$**(t)**	$\boldsymbol{u(t)}$**, in**
0.00	0.000	0.0	0.000	0.000	0.000	0.000	0.000
0.01	0.316	6.0	5.702	1.866	0.000	0.000	0.000
0.02	0.632	12.0	9.679	7.094	0.057	0.019	0.059
0.03	0.949	12.0	6.993	9.752	0.154	0.090	0.230

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Spreadsheet Numerical Calculations							
t(s)	$\omega_n\tau$	$p(\tau)$, kip	$I_1(\tau)$	$I_2(\tau)$	$A(t)$	$B(t)$	$u(t)$, in
0.04	1.265	12.0	3.614	11.443	0.224	0.187	0.497
0.05	1.581	6.0	–0.062	6.000	0.260	0.302	0.832
0.06	1.897	0.0	0.000	0.000	0.259	0.362	1.143
0.07	2.214	0.0	0.000	0.000	0.259	0.362	1.342
0.08	2.530	0.0	0.000	0.000	0.259	0.362	1.407
0.09	2.846	0.0	0.000	0.000	0.259	0.362	1.333
0.10	3.162	0.0	0.000	0.000	0.259	0.362	1.126
Trapezoidal Rule	$H(t)=\int_0^t I(\tau)\,d\tau \approx \frac{\Delta\tau}{2}\left[I(0)+2\sum_{k=0}^{n-1} I(k\Delta\tau)+I(t)\right]$						
t(s)	$\omega_n\tau$	$p(\tau)$, kip	$I_1(\tau)$	$I_2(\tau)$	$A(t)$	$B(t)$	$u(t)$, in
0.00	0.000	0.0	0.000	0.000	0.000	0.000	0.000
0.01	0.316	6.0	5.702	1.866	0.029	0.009	0.000
0.02	0.632	12.0	9.679	7.094	0.105	0.054	0.059
0.03	0.949	12.0	6.993	9.752	0.189	0.138	0.230
0.04	1.265	12.0	3.614	11.443	0.242	0.244	0.497
0.05	1.581	6.0	–0.062	6.000	0.260	0.332	0.832
0.06	1.897	0.0	0.000	0.000	0.259	0.362	1.143
0.07	2.214	0.0	0.000	0.000	0.259	0.362	1.342
0.08	2.530	0.0	0.000	0.000	0.259	0.362	1.407
0.09	2.846	0.0	0.000	0.000	0.259	0.362	1.333
0.10	3.162	0.0	0.000	0.000	0.259	0.362	1.126
Simpson's Rule	$H(t)=\int_0^t I(\tau)\,d\tau \approx \frac{\Delta\tau}{3}\left[I(0)+2\sum_{k=2,4,6\ldots}^{n-2} I(k\Delta\tau)+4\sum_{k=1,3,5\ldots}^{n-1} I(k\Delta\tau)+I(t)\right]$						
t(s)	$\omega_n\tau$	$p(\tau)$, kip	$I_1(\tau)$	$I_2(\tau)$	$A(t)$	$B(t)$	$u(t)$, in
0	0	0.0	0.000	0.000	0.000	0.000	0.000
0.01	0.316	6.0	5.702	1.866	–	–	–
0.02	0.632	12.0	9.679	7.094	0.108	0.049	0.079
0.03	0.949	12.0	6.993	9.752	–	–	–
0.04	1.265	12.0	3.614	11.443	0.246	0.240	0.512
0.05	1.581	6.0	–0.062	6.000	–	–	–
0.06	1.897	0.0	0.000	0.000	0.257	0.358	1.134
0.07	2.214	0.0	0.000	0.000	–	–	–
0.08	2.530	0.0	0.000	0.000	0.257	0.358	1.395
0.09	2.846	0.0	0.000	0.000	–	–	–
0.10	3.162	0.0	0.000	0.000	0.257	0.358	1.117

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4.3.4 MATLAB

Although we can find closed-form solutions to some general forcing function problems, it is more efficient to determine the results using computer software, such as MATLAB or even MS Excel. In this section we use MATLAB's *conv* function to solve Duhamel's convolution integrals numerically. To facilitate the development of a script to perform the convolution operation, we first rewrite Eq. (4.18) in terms of two functions that are then discretized into dimensionless vectors. The first function includes the shape of the pulse and can be written as a generalized forcing function $p(t) = p_o x(t)$, which is substituted into Eq. (4.18) as $p(\tau) = p_o x(\tau)$,

$$u(t) = \frac{p_o}{m\omega_n}\int_0^t x(\tau)\cdot\sin\left[\omega_n(t-\tau)\right]d\tau \tag{4.26}$$

We also write this equation in terms of a dimensionless time factor, the ratio of time to the natural period of the structure, t/T_n. That is, substituting $\omega_n = 2\pi/T_n$ and $\omega_n = \sqrt{k/m}$ into Eq. (4.26),

$$u(t) = \frac{\omega_n}{\omega_n}\frac{p_o}{m\omega_n}\int_0^t x(\tau)\cdot\sin\left[\omega_n(t-\tau)\right]d\tau = \frac{2\pi}{T_n}\frac{p_o}{m(\sqrt{k/m})^2}\int_0^t x(\tau)\cdot\sin\left[\frac{2\pi}{T_n}(t-\tau)\right]d\tau$$

Or,

$$\frac{u(t)}{u_{st}} = 2\pi\int_0^t x\left(\frac{\tau}{T_n}\right)\cdot\sin\left[2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right]d\tau \tag{4.27}$$

where $u_{st} = \frac{p_o}{k}$. Equation (4.27) can be written in convolution form as

$$\text{DLF} = \frac{u(t)}{u_{st}} = 2\pi\int_0^t x\left(\frac{\tau}{T_n}\right)\cdot h\left[2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right]d\tau \tag{4.28}$$

where

$$h\left[2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right] = \sin\left[2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right]$$

For the damped case, we use Eq. (4.16), which also can be rewritten in terms of the same dimensionless time factor, t/T_n, by substituting $\omega_n = 2\pi/T_n$, $\omega_n = \sqrt{k/m}$, and $\omega_D = \omega_n\sqrt{1-\zeta^2}$:

$$\frac{u(t)}{u_{st}} = \frac{2\pi}{\sqrt{1-\zeta^2}}\int_0^t x\left(\frac{\tau}{T_n}\right)e^{-\zeta 2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)}\sin\left[2\pi\sqrt{1-\zeta^2}\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right]d\tau \tag{4.29}$$

where, again, $u_{st} = \frac{p_o}{k}$. This equation can be written in convolution form as

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$$\text{DLF} = \frac{u(t)}{u_{st}} = \frac{2\pi}{\sqrt{1-\zeta^2}} \int_0^t x\left(\frac{\tau}{T_n}\right) \cdot h\left[2\pi\sqrt{1-\zeta^2}\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)\right] d\tau \tag{4.30}$$

where

$$h\left[2\pi\sqrt{1-\zeta^2}\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)\right] = e^{-\zeta 2\pi\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)} \sin\left[2\pi\sqrt{1-\zeta^2}\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)\right]$$

Example 5

Use Eq. (4.28) and the convolution function in MATLAB to solve for the response of a SDOF system excited by the rectangular impulse shown in Fig. E3.1. Compare the results to the exact solution derived in Example 3 for $t_d/T_n = 1$.

Solution

$$x(t) = \begin{cases} 1 & t \le t_d \\ 0 & t > t_d \end{cases}$$

i. *The MATLAB script is as follows:*

```
clear all % Chapter 4, Example 5
tdT = input ('Enter the value for td/T ratio: '); % Specify td/T
n = 500; % vector with n equally spaced points to represent t/Tn
tT = linspace(0,5,n);% loop over t/Tn to create vector of input
for i = 1:length(tT)
    if tT(i) <= tdT
        p(i) = 1;
    else
        p(i) = 0;
    end
end
% Integrate pulse function to get response
p=p*(tT(2)-tT(1));
h=sin(2*pi*tT);
uust=2*pi*conv(p,h);
%create plot
plot (tT, uust(1:length(tT)), 'LineWidth',1, 'Color',[0 0 1]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t/T_n', 'FontAngle','italic');
ylabel ('u(t)/u_{st}','FontAngle','italic');
```

ii. *Compare the results to the exact solution for td/Tn = 1.* These results and exact solution (the case given in Fig. E3.3) are identical up to $t_d/T_n = 1$, but to get close to the exact solution, we needed over 500 increments. Beyond $t_d/T_n = 1$, the results obtained by this method oscillate a little.

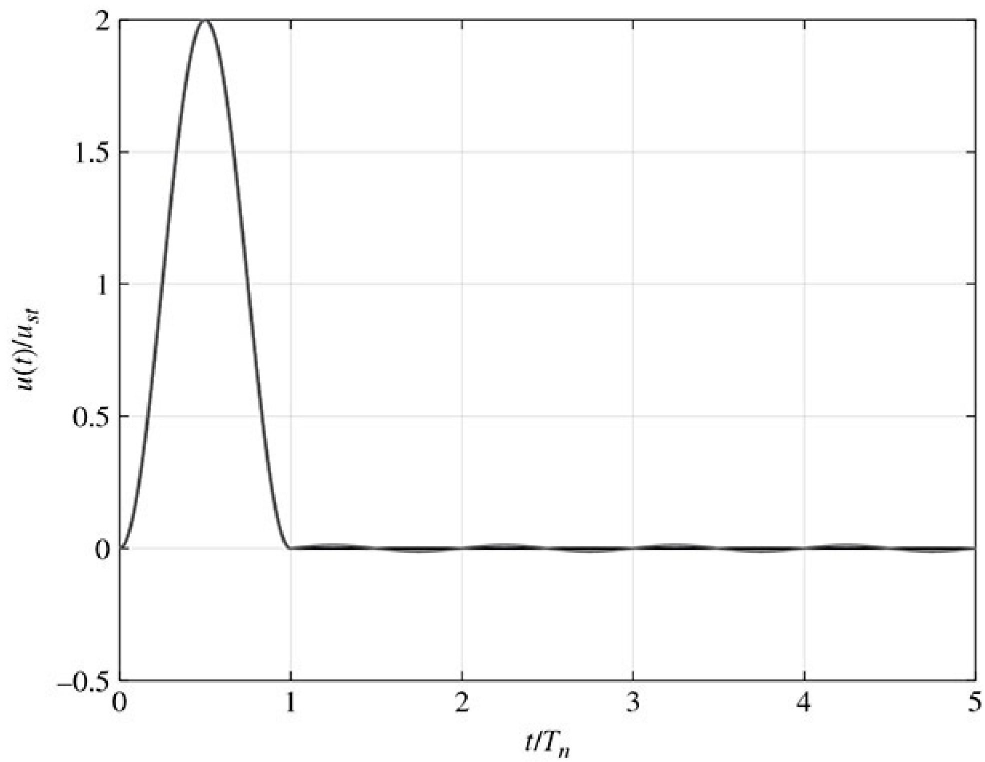

FIGURE E5.1 Response of SDOF system due to a rectangular pulse for $t_d / T_n = 1$. ▲

Example 6

Solve Example 4 using the convolution function in MATLAB and Eq. (4.18).

Solution

i. *Determine natural frequency of the SDOF system.* Stiffness is given as 10 kip/in. Mass can be calculated as follows:

$$m = \frac{W}{g} = \frac{3.86 \text{ kip}(1{,}000 \text{ lb/kip})}{386.4 \text{ in/s}^2} = 10 \frac{\text{lb} \cdot \text{s}^2}{\text{in}}$$

So, the natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000 \text{ lb/in}}{10 \text{ lb} \cdot \text{s}^2/\text{in}}} = 31.6 \frac{\text{rad}}{\text{s}}$$

ii. *Obtain the function x(t):*

$$x(t) = \begin{array}{cc} 50t & t \le 0.02 \text{ s} \\ 1 & 0.02 \text{ s} < t \le 0.04 \text{ s} \\ 3 - 50t & 0.04 \text{ s} < t \le 0.06 \text{ s} \\ 0 & t > 0.06 \text{ s} \end{array}$$

Also, p_o = 12 kip

iii. *Displacement response as a function of time script:*

```
clear all % Chapter 4, Example 6
omega = 31.6; % Natural frequency in rad/sec
po=12000; % Peak force in lb
m=10; % Mass in lb square second per in
dt=0.001; % Time increment for calculations
% loop over t to create vector based on pulse function values
for i = 1:500;
    t(i)=dt*i;
    if t(i) <= 0.02
        p(i) = 50*t(i);
    elseif t(i)<= 0.04
        p(i) = 1;
    elseif t(i)<= 0.06
        p(i) = 3-50*t(i);
    else
        p(i) = 0;
    end
end
p=p*dt;
h=sin(omega*t);
uust=po*conv(p,h)/(m*omega);
uustr = max(abs(uust)) % maximum displacement
%create plot
plot (t, uust(1:500), 'LineWidth',1, 'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t, sec)', 'FontAngle', 'italic');
ylabel ('u(t), in', 'FontAngle', 'italic');
```

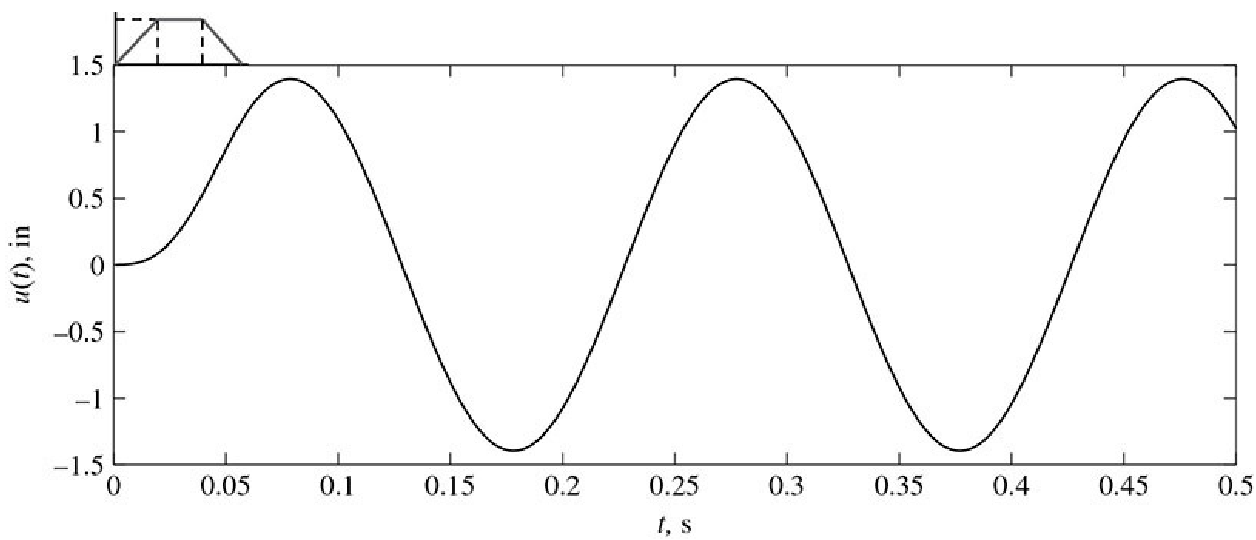

FIGURE E6.1 Undamped response of SDOF system to trapezoidal loading function.

The maximum dynamic displacement is 1.397 in versus the static displacement of p_o/k = 1.2 in.
▲

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Example 7

Solve Example 6, but with 5% damping.

Solution All parameters are the same except we now use Eq. (4.16), with $\omega_n = \sqrt{k/m}$ and $\omega_D = \omega_n\sqrt{1-\zeta^2}$:

$$u(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)\left\{e^{-\zeta\omega_n(t-\tau)}\sin\left[\omega_D(t-\tau)\right]\right\}d\tau$$

i. *Displacement response as a function of time script:*

```
clear all % Chapter 4, Example 7
xi = 0.05; % damping ratio
omega = 31.6; % Natural frequency in rad/sec
omegaD = omega*sqrt(1-xi^2); % damped frequency in rad/sec
po=12000; % Peak force in lb
m=10; % Mass in lb square second per in
dt=0.001; % Time increment for calculations
% loop over t to create vector based on pulse function values
for i = 1:500;
    t(i)=dt*i;
    if t(i) <= 0.02
        p(i) = 50*t(i);
    elseif t(i)<= 0.04
        p(i) = 1;
    elseif t(i)<= 0.06
        p(i) = 3-50*t(i);
    else
        p(i) = 0;
    end
end
p=p*dt;
h=exp(-xi*omega*t).*sin(omegaD*t);
uust=po*conv(p,h)/(m*omegaD);
uustr = max(abs(uust)) % maximum displacement
%create plot
plot (t, uust(1:500), 'LineWidth',1, 'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t, sec)', 'FontAngle', 'italic');
ylabel ('u(t), in', 'FontAngle', 'italic');
```


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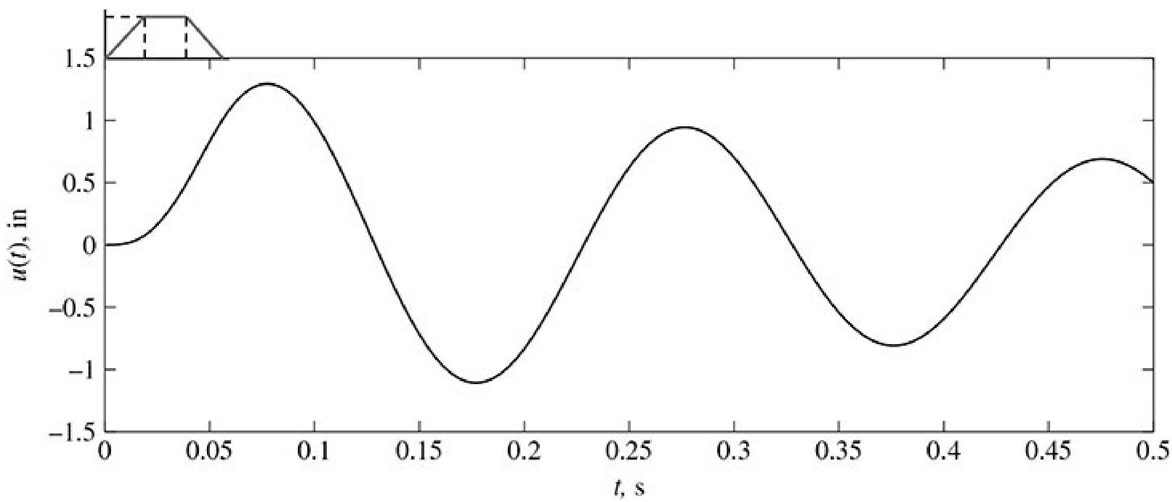

FIGURE E7.1 Damped response of SDOF system to trapezoidal loading function.

The maximum damped displacement is 1.294 in versus the undamped displacement of 1.397 in. ▲

4.4 Response (Shock) Spectra

The results of Example 3 (Fig. E3.3) show that each time we specify a different natural period of the structure relative to the duration of the pulse (t_d/T_n), we obtain a new response, each of which has a maximum value. This is similar to the response spectrum derived in Chap. 3, which is based on frequency ratio, or frequency since the excitation frequency is constant. The envelope of the maximum response as a function of the load duration relative to the period of the structure can be represented graphically as a *shock spectrum*. That is, a response or shock spectrum is a plot of the maximum response (displacement, acceleration, or any other quantity) to a specified load excitation for all possible SDOF systems.

Shock spectra are useful in design because we are only interested in peak values of the response, not the entire response history. The plot includes the relationship among amplitude of the maximum response, the nature of the excitation (shape, duration, and amplitude), and properties of the SDOF system (mass, damping, and stiffness). For example, Fig. 4.8 shows the time history response of three different SDOF oscillators to an excitation $p(t)$; each of the oscillators has a different period, but equal damping ratios. The figure also shows the maximum response as a function of the period (maximum response vs. T plot), which is defined as the response (or shock) spectrum.

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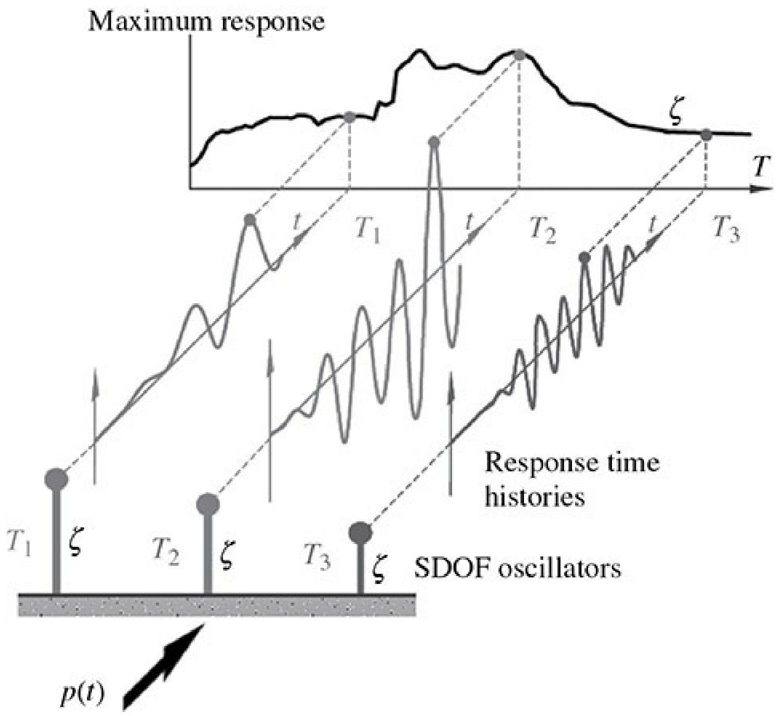

FIGURE 4.8 Definition of a response (shock) spectrum.

To obtain shock spectra, we can solve Duhamel's integral analytically or numerically, both of which were introduced in the last section. Numerical integration can be performed using one of the standard numerical integration technics such as Euler (forward rectangular), Trapezoidal (linear), or Simpson's (parabolic) methods. In this section, we use MATLAB's numerical integration operator *conv*. Following are some examples used to generate the shock spectra. Also, we demonstrate the use of shock spectra in determining the maximum response of structural systems.

Example 8

Use the results from Example 3 to draw the shock spectrum for the rectangular pulse force in Fig. E3.1.

Solution

i. *Use the DLF versus td/Tn results obtained in part (ii) in Example 3; that is,*

$$\mathrm{DLF} = \frac{u(t)}{u_{st}} = \begin{cases} \left[1-\cos 2\pi t/T_n\right] & t \le t_d \\ \left[\cos 2\pi(t/T_n - t_d/T_n) - \cos 2\pi t/T_n\right] & t > t_d \end{cases}$$

The maximum values of the DLF, $\mathrm{DLF}_{\max}$, for various values of, t_d/T_n can be obtained more efficiently using MATLAB. The results of $\mathrm{DLF}_{\max}$ versus t_d/T_n can then be graphed to obtain the shock spectrum of a pulse loading as shown in Fig. E8.1. The script is as follows:

```
clear all % Chapter 4, Example 8
%create two vectors of n length to represent td/Tn and t/Tn
n = 500;
tdT = linspace(0,5,n);
tT = linspace(0,5,n);
% loop over td/Tn and t/Tn to create the shock spectrum
for j = 1:n
for i = 1:n
    if tT(i) <= tdT(j)
        uust(i) = 1 - cos(2*pi*tT(i));
    else
        uust(i) = cos(2*pi*(tT(i)-tdT(j)))- cos(2*pi*tT(i));
    end
end
uustr(j)=max(abs(uust)); % Select the max values from each response
end
%create plot of the shock spectrum
plot (tdT,uustr,'LineWidth',1,'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t_d /T_n', 'FontAngle','italic');
ylabel ('DLF_{max}','FontAngle','italic');
axis([0 5 0 2.2]);
grid on;
```

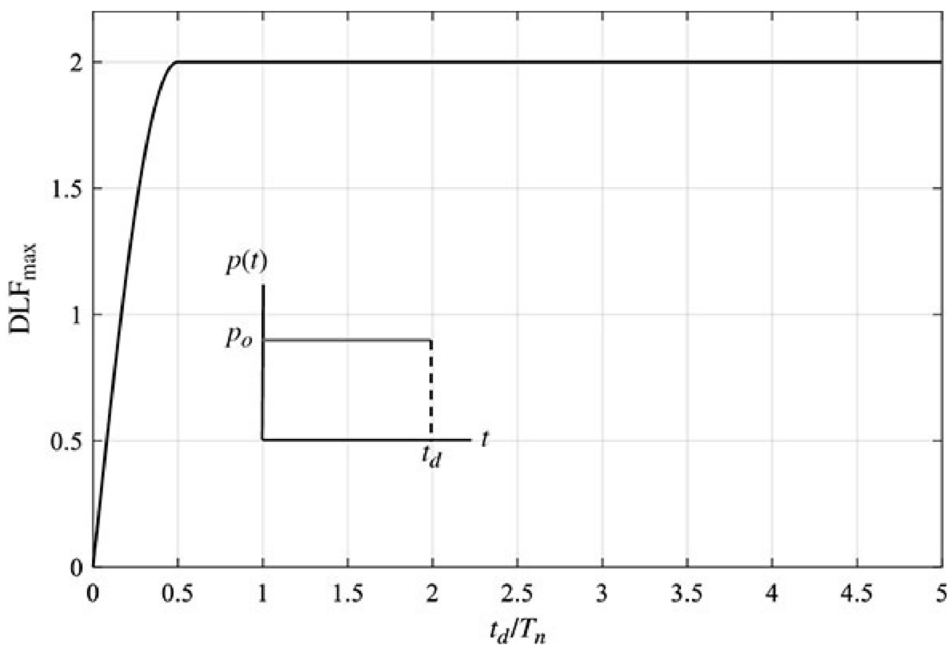

FIGURE E8.1 Shock spectrum of a SDOF system subject to a rectangular impulse load. ▲

Example 9

Draw the shock spectrum for the response of a SDOF system subjected to the symmetric triangular pulse loading function shown in Fig. E9.1.

Solution

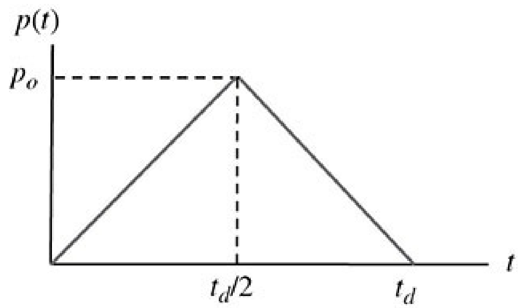

FIGURE E9.1 Symmetric triangular impulse loading function.

$$\text{DLF} = 2\pi\int_0^t x\left(\frac{\tau}{T_n}\right)\cdot h\left[2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right]d\tau$$

where

$$h\left[2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right] = \sin\left[2\pi\left(\frac{t}{T_n}-\frac{\tau}{T_n}\right)\right]$$

i. *Obtain function x(t):*

$$x(t) = \begin{cases} 2t/t_d & t \le t_d/2 \\ 2-2t/t_d & t_d/2 < t \le t_d \\ 0 & t > t_d \end{cases}$$

ii. *Displacement response as a function of time script:*

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```
clear all % Chapter 4, Example 9
% create two vectors of equal n length to represent td/Tn and t/Tn
n = 500;
tdT = linspace(0,5,n);
tT = linspace(0,5,n);
% loop over td/Tn and t/Tn to create the shock spectrum
for j = 1:n
    for i = 1:n
        if tT(i) <= tdT(j)/2
            p(i) = 2*tT(i)/tdT(j);
        elseif tT(i) <= tdT(j)
            p(i) = 2-2*tT(i)/tdT(j);
        else
            p(i) = 0;
        end
    end
    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=sin(2*pi*tT);
    uust=2*pi*conv(p,h);
    uustr(j)=max(abs(uust)); % Select the max values from each response
end
%create plot of the shock spectrum
plot (tdT,uustr,'LineWidth',1,'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t_d /T_n', 'FontAngle','italic');
ylabel ('DLF_{max}','FontAngle','italic');
axis([0 5 0 1.6]);
grid on;
```

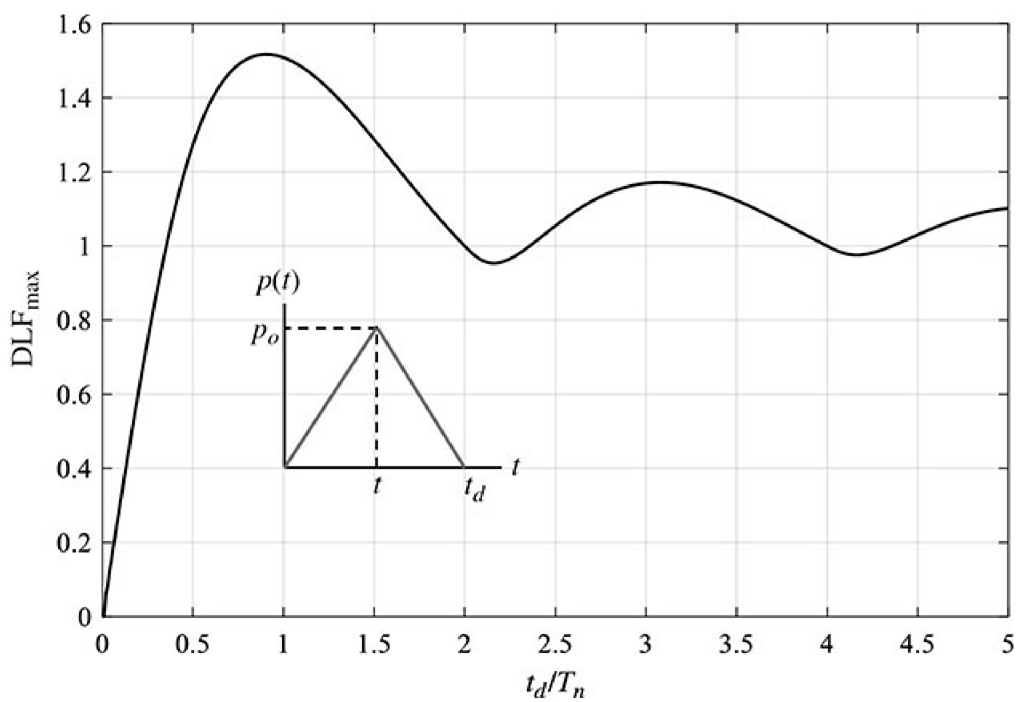

FIGURE E9.2 Shock spectrum of a SDOF system subject to a symmetric triangular impulse load.
▲

Example 10

Draw the shock spectrum for the response of a SDOF system subjected to the triangular pulse loading function shown in Fig. E10.1.

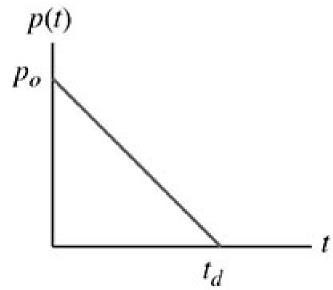

FIGURE E10.1 Triangular impulse loading function.

Solution

i. *Obtain function x(t):*

$$x(t) = \begin{cases} 1 - t/t_d & t \le t_d \\ 0 & t > t_d \end{cases}$$

ii. *Displacement response as a function of time script:*

@seismicisolation

```
clear all % Chapter 4, Example 10
% create two vectors of equal n length to represent td/Tn and t/Tn
n = 500;
tdT = linspace(0,5,n);
tT = linspace(0,5,n);
% loop over td/Tn and t/Tn to create the shock spectrum
for j = 1:n
    for i = 1:n
        if tT(i) <= tdT(j)
            p(i) = 1-tT(i)/tdT(j);
        else
            p(i) = 0;
        end
    end
    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=sin(2*pi*tT);
    uust=2*pi*conv(p,h);
    uustr(j)=max(abs(uust)); % Select the max values from each response
end
%create plot of the shock spectrum
plot (tdT,uustr,'LineWidth',1,'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t_d /T_n', 'FontAngle','italic');
ylabel ('DLF_{max}','FontAngle','italic');
axis([0 5 0 2]);
grid on;
```

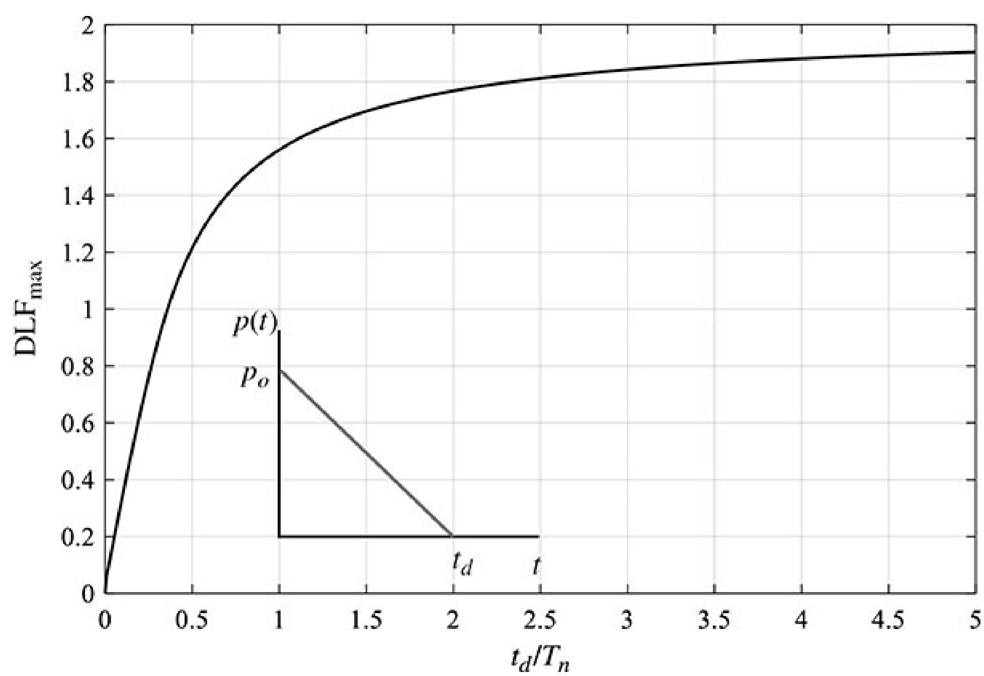

FIGURE E10.2 Shock spectrum of a SDOF system subject to a triangular impulse load. ▲

Example 11

Draw the shock spectrum for the response of a SDOF system subjected to the half-cycle sine pulse force shown in Fig. E11.1. Repeat for a full-cycle sine pulse and compare the two graphs.

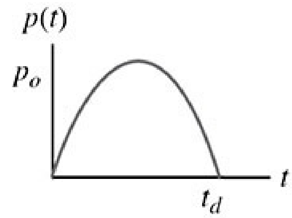

FIGURE E11.1 Half-cycle sine impulse loading function.

Solution

i. *Obtain function x(t):*

Half-cycle case:

$$x(t)=\begin{cases}\sin(\pi t/t_d) & t\le t_d\\ 0 & t>t_d\end{cases}$$

Full-cycle case:

$$x(t)=\begin{cases}\sin(\pi t/t_d) & t\le t_d\\ -\sin(\pi t/t_d) & t_d<t\le 2t_d\\ 0 & t>2t_d\end{cases}$$

ii. *Displacement response for half-cycle sine impulse force as a function of time script:*

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```
clear all % Chapter 4, Example 11, half-cycle sine
n = 500;
tdT = linspace(0,5,n); % vector of n length to represent td/Tn
tT = linspace(0,5,n); % vector of n length to represent t/Tn
% loop over td/Tn and t/Tn to create the shock spectrum
for j = 1:n
    for i = 1:n
        if tT(i) <= tdT(j)
            p(i) = sin(pi*tT(i)/tdT(j));
        else
            p(i) = 0;
        end
    end
    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=sin(2*pi*tT);
    uust=2*pi*conv(p,h);
    uustr(j)=max(abs(uust)); % Select the max values from each response
end
%create plot of the shock spectrum
plot (tdT, uustr,'LineWidth',1,'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t_d /T_n', 'FontAngle','italic');
ylabel ('DLF_{max}','FontAngle','italic');
axis([0 5 0 1.8]);
grid on;
```

iii. *Displacement response for full-cycle sine impulse force as a function of time script:*

```
clear all % Chapter 4, Example 11, full-cycle sine
% create two vectors of equal n length to represent td/Tn and t/Tn
n = 500;
tdT = linspace(0,5,n);
tT = linspace(0,5,n);
% loop over td/Tn and t/Tn to create the shock spectrum
for j = 1:n
    for i = 1:n
        if tT(i) <= 2*tdT(j)
            p(i) = sin(pi*tT(i)/tdT(j));
        else
            p(i) = 0;
        end
end
    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=sin(2*pi*tT);
    uust=2*pi*conv(p,h);
    uustr(j)=max(abs(uust)); % Select the max values from each response
end
%create plot of the shock spectrum
```


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```
plot (tdT, uustr,'LineWidth',1,'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t_d /T_n', 'FontAngle','italic');
ylabel ('DLF_{max}','FontAngle','italic');
axis([0 5 0 3.2]);
grid on;
```

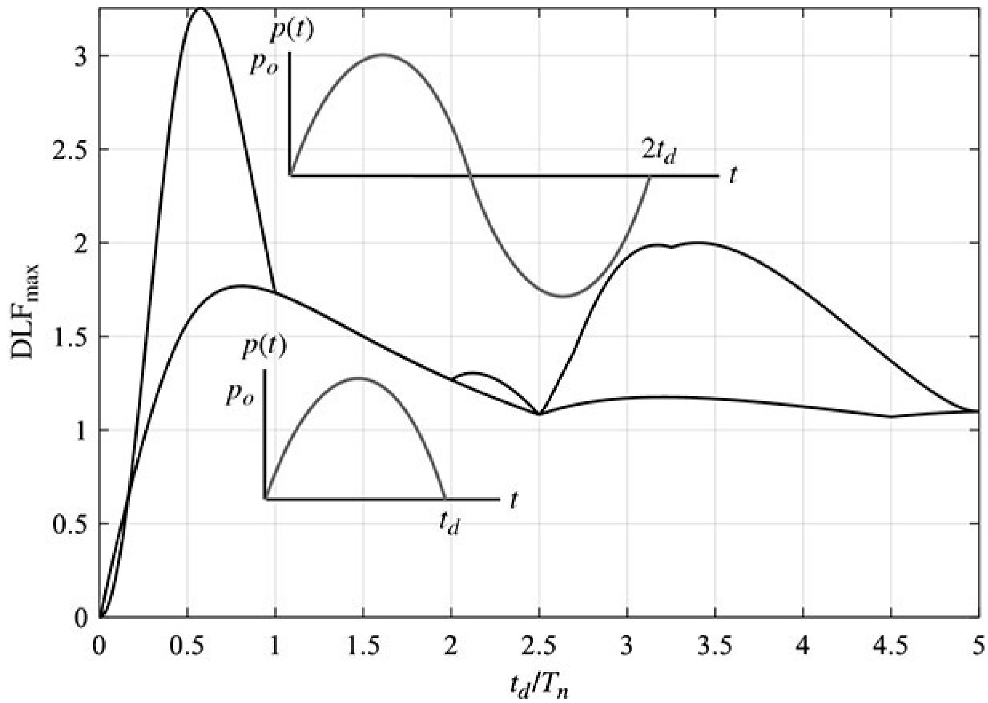

FIGURE E11.2 Shock spectrum of a SDOF system subject to a half- and full-cycle sine impulse load. ▲

Example 12

Draw the shock spectrum for the response of a SDOF system subjected to the half-cycle sine pulse force shown in Fig. E11.1. Repeat for damping ratios of 0.01, 0.1, 0.2, and 0.4.

Solution All parameters are the same except we now use Eq. (4.30):

$$\mathrm{DLF} = \frac{u(t)}{u_{st}} = \frac{2\pi}{\sqrt{1-\zeta^2}} \int_0^t x\left(\frac{\tau}{T_n}\right) \cdot h\left[2\pi\sqrt{1-\zeta^2}\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)\right] d\tau$$

where

$$h\left[2\pi\sqrt{1-\zeta^2}\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)\right] = e^{-\zeta 2\pi\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)} \sin\left[2\pi\sqrt{1-\zeta^2}\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)\right]$$

i. *Displacement response as a function of time script:*

@seismicisolation

```
clear all % Chapter 4, Example 12
xi = [0 0.01 0.1 0.2 0.4]'; % damping ratio
for z = 1:5
n = 500;
tdT = linspace(0,5,n); % vector of n length to represent td/Tn
tT = linspace(0,5,n); % vector of n length to represent t/Tn
% loop over td/Tn and t/Tn to create the shock spectrum
for j = 1:n
    for i = 1:n
        if tT(i) <= tdT(j)
            p(i) = sin(pi*tT(i)/tdT(j));
        else
            p(i) = 0;
        end
    end
    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=exp(-xi(z)*2*pi*tT).*sin(2*pi*sqrt(1-xi(z)^2)*tT);
    uust=(2*pi/sqrt(1-xi(z)^2))*conv(p,h);
    uustr(j)=max(abs(uust)); % Select the max values from each response
end
%create plot of the shock spectrum
plot (tdT, uustr,'LineWidth',1,'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t_d /T_n', 'FontAngle','italic');
ylabel ('DLF_{max}','FontAngle','italic');
axis([0 5 0 1.8]);
grid on;
hold on
end
```

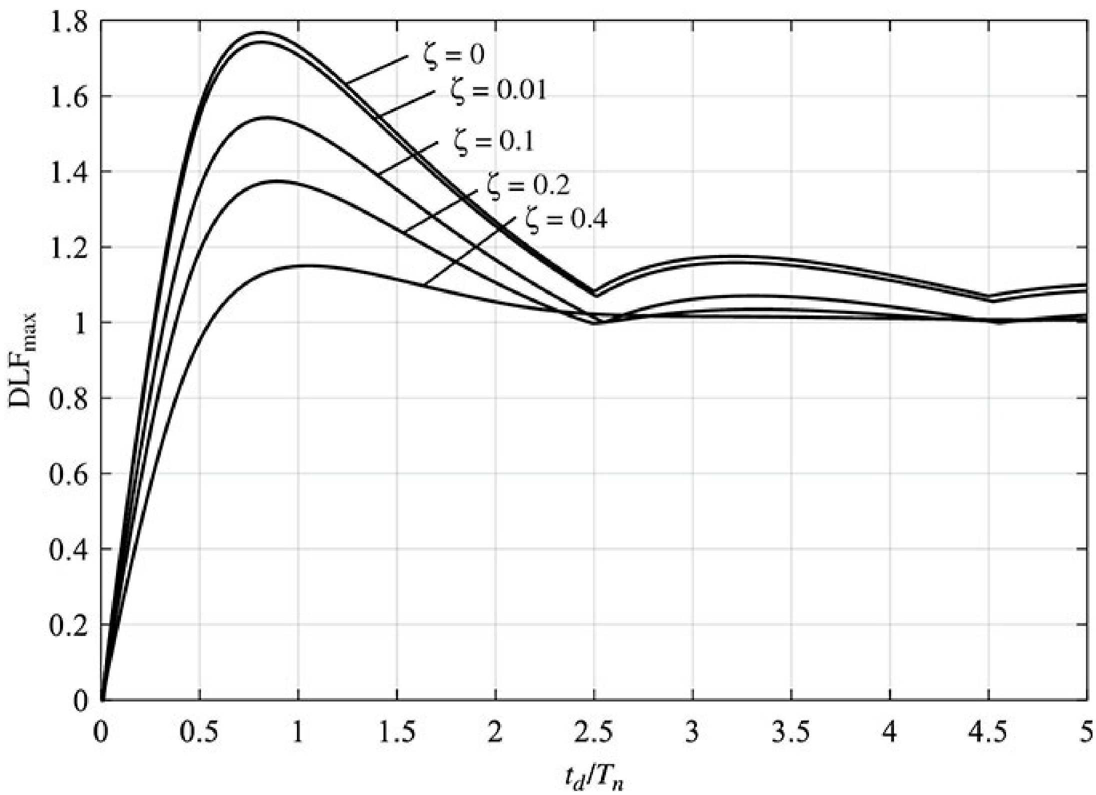

FIGURE E12.1 Response of damped SDOF system to half-cycle sine impulse load. ▲

Example 13

Given the building frame shown in Fig. E13.1, which is subjected to a triangular impulse force (see Fig. E10.1) of amplitude p_o = 5 kip and duration, t_d = 0.5 seconds, determine (i) maximum displacement at the top, (ii) the maximum base shear, and (iii) the maximum bending stresses in the columns. Assume the beam is rigid.

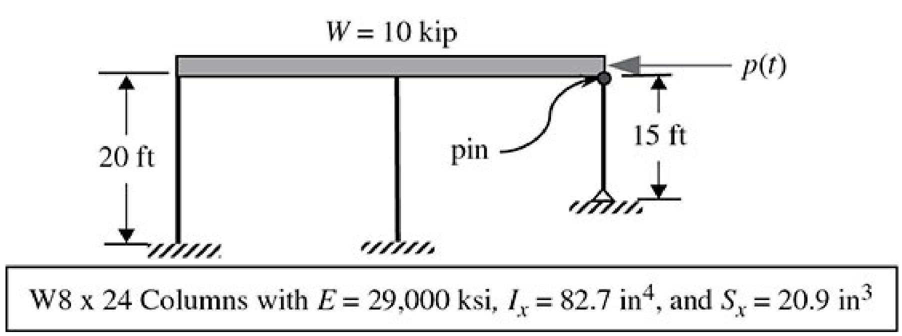

FIGURE E13.1 Building frame geometry and column properties.

Solution

i. *Mass, stiffness, and natural period of the SDOF system.* The building frame can be modeled as a SDOF system, assuming only lateral deformations of the columns. The stiffness of the system is the sum of column lateral stiffnesses. The mass and stiffness of the SDOF system are calculated as follows:

Mass,

$$m = \frac{W}{g} = \frac{10\ \text{kip}(1{,}000\ \text{lb/kip})}{386.4\ \text{in/s}^2} = 25.9\ \frac{\text{lb}\cdot\text{s}^2}{\text{in}}$$

Stiffness (see Chap. 1); the 20 ft columns are fixed-fixed, while the 15 ft column is pinned-pinned, which means that it contributes no stiffness to the frame. Thus,

$$k = 2\frac{12EI}{h_{20}^3} = \frac{24(29{,}000{,}000\ \text{psi})(82.7\ \text{in}^4)}{(20\ \text{ft}\times 12\ \text{in/ft})^3} = 4{,}164\ \frac{\text{lb}}{\text{in}}$$

The natural frequency of the SDOF system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4{,}164}{25.9}} = 12.7\ \frac{\text{rad}}{\text{s}}$$

The natural period of the SDOF system is

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi\ \text{rad}}{12.7\ \text{rad/s}} = 0.5\ \text{s}$$

ii. *Determine the maximum displacement.* The maximum displacement due to the applied triangular pulse forcing function can be determined using the shock spectrum obtained in Fig. E10.2. This figure provides the maximum dynamic load factor, $\text{DLF}_{\max} = u_o/ust$ as a function of the ratio of pulse duration to natural period, *td*/*Tn*. Where the equivalent static displacement, *ust*, can be calculated using the basic force-displacement relationship for the SDOF system as follows:

$$u_{st} = \frac{p_o}{k} = \frac{5\ \text{kip}(1{,}000\ \text{lb/kip})}{4{,}164\ \text{lb/in}} = 1.20\ \text{in}$$

Also, the ratio of pulse duration to natural period can be determined as follows:

$$\frac{t_d}{T_n} = \frac{0.5\ \text{s}}{0.5\ \text{s}} = 1$$

Finally, from the shock spectrum in Fig. E10.2, we can determine $\text{DLF}_{\max} \backsimeq 1.55$. Now solve for $u_{\max}$,

$$u_{\max} = u_o = \text{DLF}_{\max}\cdot u_{st} = 1.55(1.20\ \text{in}) = 1.86\ \text{in}$$

iii. *Maximum base shear*. Since the 15 ft column does not contribute to the lateral load resistance (only gravity load resistance), its shear and bending moment are zero; for the 20 ft columns we can calculate the maximum base shear at the support for each column using the force-displacement relationship as discussed in Sec. 1.7.2:

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$$V_{\max} = u_o \frac{12EI}{L^3} = 1.86 \text{ in} \frac{12(29{,}000{,}000 \text{ psi})(82.7 \text{ in}^4)}{(20 \text{ ft} \times 12 \text{ in/ft})^3} = 3{,}872 \text{ lb}$$

iv. *Maximum normal stress due to bending*. The maximum bending moment at the base and top of each of the 20 ft columns is calculated from equilibrium of the column member as discussed in Sec. 1.7.2,

$$M_{\max} = u_o \frac{6EI}{L^2} = 1.86 \text{ in} \frac{6(29{,}000{,}000 \text{ psi})(82.7 \text{ in}^4)}{(20 \text{ ft} \times 12 \text{ in/ft})^2} = 464{,}671 \text{ lb} \cdot \text{in}$$

Applying the flexure formula from mechanics of materials [Eq. (1.17)], the normal stress in each column due to the bending moment is determined as

$$\sigma_{\max} = \frac{M_{\max}}{S_x} = \frac{464{,}671 \text{ lb} \cdot \text{in}}{20.9 \text{ in}^3} = 22{,}233 \text{ psi} \quad \blacktriangle$$

4.5 Approximate Analysis for Short-Duration Excitation Pulses

When the pulse duration is rather small, the stiffness or damping reactions are not mobilized; thus, the inertial force reacts the entire short-duration impulse. The reason for this is that the material needs time to deform and $u(t) << 1$ for very small t_d. So, the equation of motion can be written as

$$m\ddot{u}(t) = p(t)$$

Or,

$$\ddot{u}(t) = \frac{p(t)}{m}$$

We can obtain the velocity at time t_d by integrating the acceleration over $0 \le t \le t_d$,

$$\dot{u}(t) = \int_0^{t_d} \ddot{u}(t)dt = \frac{1}{m}\int_0^{t_d} p(t)dt = \frac{I}{m}$$

Here we define the integral quantity as the impulse.

We can obtain the displacement response equation in terms of the ratio of pulse duration to natural period, t_d/T_n. First, we rewrite the equation for velocity in a slightly different form,

$$\dot{u}(t) = \int_0^{t_d} \frac{kp_o}{mk} x(t)dt = \omega_n^2 u_{st} t_d \int_0^{t_d} x(t)\frac{dt}{t_d} = \omega_n^2 u_{st} t_d \lambda$$

where λ is dimensionless and is equivalent to the area of a pulse with a unit magnitude and

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duration, see Fig. 4.9. We can now determine the displacement response from the solution to an undamped system starting from rest, but with an initial velocity given by this velocity since the load duration is very short; that is, we use the homogeneous solution, Eq. (2.4) with $u(0) = 0$ and $\dot{u}(0) = \omega_n^2 u_{st} t_d \lambda$,

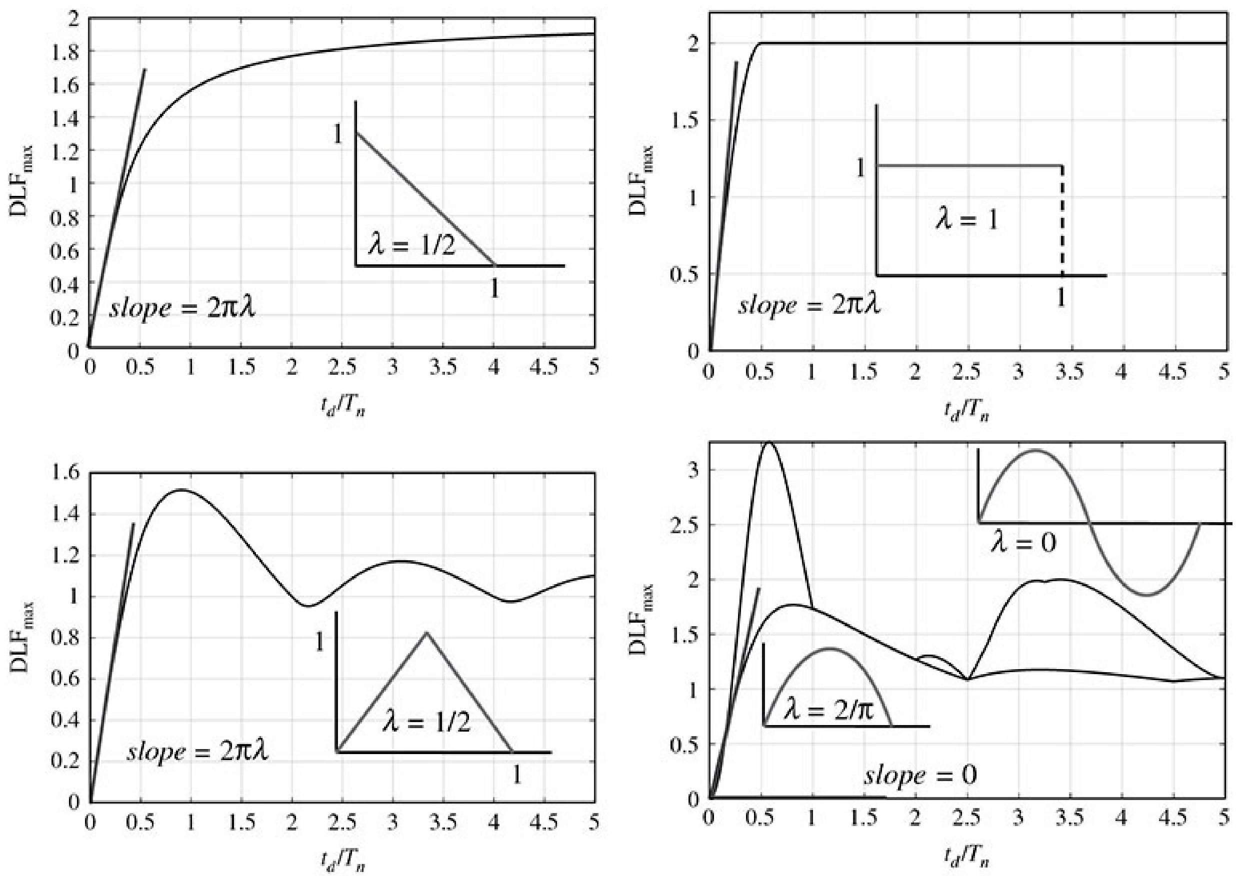

FIGURE 4.9 Initial slope, $2\pi\lambda$ of shock spectra and dimensionless parameter λ.

$$u(t) = \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t = \frac{\omega_n^2 u_{st} t_d \lambda}{\omega_n}\sin\omega_n t = 2\pi\lambda u_{st}\frac{t_d}{T_n}\sin\omega_n t$$

The maximum displacement is given when the sine function is equal to 1,

$$u_{\max} = 2\pi\lambda u_{st}\frac{t_d}{T_n} \tag{4.31}$$

This can also be written in terms of the impulse as

$$u_{\max} = \frac{\dot{u}(0)}{\omega_n} = \frac{I}{\sqrt{km}} \tag{4.32}$$

The maximum dynamic load factor can be obtained from Eq. (4.31),

$$\text{DLF}_{\max} = \frac{u_{\max}}{u_{st}} = 2\pi\lambda\frac{t_d}{T_n} \tag{4.33}$$

This is the response spectrum for a short-duration pulse and is linear in t_d/T_n with a slope of $2\pi\lambda$. Notice this slope, $2\pi\lambda$, is the initial slope of all the shock spectra derived in the last section, as shown in Fig. 4.9. From the graphs it is clear that this approximation yields reasonably accurate results for $t_d/T_n < 0.25$. Also, the response appears to be independent of the shape and controlled only by the pulse area.

One particular application of this approximate method is pressure loading from blast (intentional or accidental), see Fig. 4.10. Considering Eq. (4.31), we can estimate the effect of a blast load by assuming that $\lambda = 1$, (but we know it is less since the area is smaller than a rectangle). The maximum response is

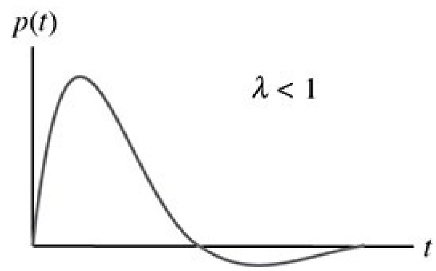

FIGURE 4.10 Blast-induced pressure wave striking a structure.

$$u_{\max} = 2\pi\frac{t_d}{T_n}u_{st}$$

Therefore, depending on the magnitude of t_d/T_n, the effect of the blast load can be smaller than the static effect of the maximum pressure, which can still be very large depending on the location of the blast relative to the structure. To reduce the effect of the blast load we can create a longer defensible space, or we can increase the period by changing the system properties. Rewriting Eq. (4.31) again with $T_n = 2\pi\sqrt{m/k}$,

$$u_{\max} = t_d\sqrt{k/m}\,u_{st}$$

So, we can increase m or decrease k to reduce the effect of the blast pressure.

We can also modify the impulse approximation to define the maximum load on the system, rather than assuming a rectangular pulse with $\lambda = 1$,

$$p_{\max} = \omega_n I = I\sqrt{k/m} \tag{4.34}$$

As noted earlier, these estimates of response seem to be relatively accurate for $t_d/T_n < 0.25$.

Example 14

A 100 kip water tank is supported by an 80 ft high tower and is subjected to an air blast load

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shown in Fig. E14.1. Determine the maximum displacement of the water tank, and the base shear and bending moment at the base of the tower. Assume a stiffness of 8.2 kip/in.

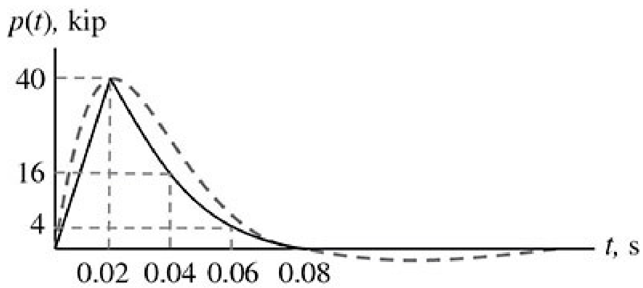

FIGURE E14.1 Approximate air blast over pressure striking a water tank.

Solution

i. *Mass, stiffness, and natural period of the SDOF system*. The tower can be modeled as a SDOF system with given stiffness of 8.2 kip/in and mass,

$$m = \frac{W}{g} = \frac{100 \text{ kip}}{386.4 \text{ in/s}^2} = 0.259 \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

The natural frequency of the SDOF system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8.2}{0.259}} = 5.63 \frac{\text{rad}}{\text{s}}$$

The natural period of the SDOF system is

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi \text{ rad}}{5.63 \text{ rad/s}} = 1.12 \text{ s}$$

ii. *Determine the impulse*. The pulse duration is shown in Fig. E14.1, t_d = 0.08 seconds. So,

$$t_d/T_n = 0.08 \text{ s}/1.12 \text{ s} = 0.071 \ll 0.25$$

Therefore, the forcing function can be treated as an impulse of magnitude I. This means that for such a short-duration load, neither stiffness nor damping forces would be mobilized.

The impulse can be determined by finding the area under the blast load function, which in this case must be obtained using numerical integration; let us use the Trapezoidal rule.

$$I = \int_0^{t_d} p(t)dt = \frac{0.02}{2}[0 + 2(40) + 2(16) + 2(4) + 0] = 1.2 \text{ kip} \cdot \text{s}$$

iii. *Determine the maximum displacement*. We can determine the maximum displacement due to the applied blast pulse using Eq. (4.32),

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$$u_{max} = \frac{I}{\sqrt{km}} = \frac{1.2 \text{ kip}\cdot\text{s}}{\sqrt{8.2 \text{ kip/in}(0.259 \text{ kip}\cdot\text{s}^2/\text{in})}} = 0.822 \text{ in}$$

iv. *Determine the maximum base shear and bending moment*. First, we determine the maximum equivalent static force using Eq. (4.34), which can be obtained as

$$p_{max} = u_{max}k = \frac{Ik}{\sqrt{km}} = I\sqrt{\frac{k}{m}} = I\omega_n = 1.2 \text{ kip}\cdot\text{s}\left(5.63\,\frac{\text{rad}}{\text{s}}\right) = 6.75 \text{ kip}$$

We can now obtain the resulting shear and bending moment diagrams as follows:

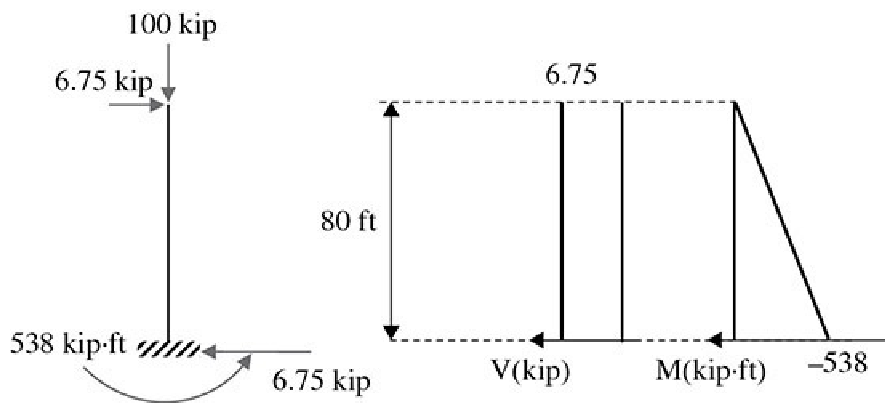

FIGURE E14.2 Shear force and bending moment diagrams for tower. ▲

4.6 Response to Ground Motion

The most important application of the dynamic response to ground motion is in the design of structures subjected to seismic loading. This type of analysis is also used for the analysis of buildings, and components housed in buildings, subjected to ground excitations from vehicular and other excitation vibrations. In all these cases, we can start with the equation of motion for ground excitation derived in Chap. 3 [Eq. (3.23)], rewritten here for convenience:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -m\ddot{u}_g(t)$$

The total response of an underdamped SDOF system including initial conditions subjected to a ground excitation, $\ddot{u}_g(t)$, can be determined using Eq. (4.17) with a new forcing function of

$$p(t) = -m\ddot{u}_g(t) \tag{4.35}$$

When the system starts from rest, we can determine the response due to a ground excitation for an underdamped system in terms of the relative displacement of the mass with respect to the ground by substituting Eq. (4.35) into Duhamel's integral [Eq. (4.16)]. That is,

$$u(t) = -\frac{1}{\omega_D}\int_0^t \ddot{u}_g(t)e^{-\zeta\omega_n(t-\tau)}\left[\sin\omega_D(t-\tau)\right]d\tau \tag{4.36}$$

For ground motion, the equation for DLF can be written in terms of the relative displacement u_o/u_{st} or the total acceleration $\ddot{u}_o^t/\ddot{u}_{go}$. First, we can normalize time with respect to the natural period as a dimensionless time parameter, t/T_n. Then, we can rewrite the equivalent static displacement, $u_{st} = m\ddot{u}_{go}/k$, assuming p_o is equivalent to $m\ddot{u}_{go}$, where $\ddot{u}_{go}$ is the peak value of the ground acceleration, $\ddot{u}_g$. This assumption yields $u_{st} = m\ddot{u}_{go}/k = \ddot{u}_{go}/\omega_n^2$, since $k = \omega_n^2 m$. The DLF can then be written as

$$\text{DLF} = \frac{u_o}{u_{st}} = \frac{\omega_n^2 u_o}{\ddot{u}_{go}} = \frac{\ddot{u}_o^t}{\ddot{u}_{go}} \tag{4.37}$$

where $\ddot{u}_o^t$ is the maximum total acceleration and is related to the maximum relative displacement by $\omega_n^2 u_o$. That is, ignoring damping in the equation of motion Eq. (3.23) ($m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -m\ddot{u}_g(t)$) and using $\ddot{u}^t(t) = \ddot{u}(t) + \ddot{u}_g(t)$, we get

$$m\ddot{u}^t(t) = ku(t)$$

Or,

$$\ddot{u}^t(t) = \omega_n^2 u(t) \tag{4.38}$$

From Eq. (4.28) we determine the convolution integral component of Eq. (4.36),

$$\text{DLF} = \frac{\ddot{u}^t}{\ddot{u}_{go}} = 2\pi\int_0^t x\left(\frac{\tau}{T_n}\right)\cdot h\left[2\pi\left(\frac{t}{T_n} - \frac{\tau}{T_n}\right)\right]d\tau \tag{4.39}$$

where $x(\xi)$ is a function that describes the applied displacement (or acceleration) history function. This implies that all of the shock spectra derived thus far are applicable to ground excitation forcing functions as well.

However, generating a shock spectrum using the *conv* function in MATLAB is relatively simple. For instance, in the following example, we use the 2021 Haiti earthquake ground acceleration to generate relative displacement and total acceleration spectra. Although we have used the terms shock and response interchangeably thus far, *response spectrum* is customarily reserved for shock spectra dealing with seismic response. As noted earlier, a response spectrum is a plot of the maximum response as a function of the SDOF system period.

Example 15

Consider the response of the vehicle from Chap. 3, Example 7, but now running over a speed bump as shown in Fig. E15.1. The vehicle weighs 4 kip and can be modeled as a mass supported by a spring-dashpot suspension system with stiffness of 0.8 kip/in and damping of 40%. The speed bump undulation (3 ft long and 6 in high) is approximately a half-cycle sine function shape. Determine the maximum vertical acceleration of the vehicle when it travels at constant

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speeds of 5 and 10 mi/h.

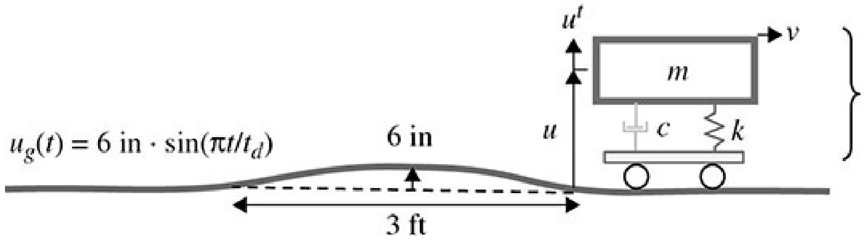

FIGURE E15.1 Idealized SDOF system for a vehicle running over a speed bump.

Solution

i. *Determine the frequency and period of the idealized SDOF system.*
Mass,

$$m = W/g = 4{,}000 \text{ lb}/(386.4 \text{ in/s}^2) = 10.35 \text{ lb} \cdot \text{s}^2/\text{in}$$

Stiffness,

$$k = 1{,}250 \text{ lb/in}$$

Natural frequency,

$$\omega_n = \sqrt{k/m} = \sqrt{1{,}250/10.35} = 11 \text{ rad/s}$$

Period,

$$T_n = 2\pi/\omega_n = 2\pi/11 \text{ rad/s} = 0.571 \text{ s}$$

ii. *Determine the excitation parameters.* Pulse duration for the two constant speeds—pulse length divided by speed,

$$t_{d5} = \frac{3 \text{ ft}}{5 \text{ mi/h}}\left(\frac{1 \text{ mi}}{5{,}280 \text{ ft}}\right)\left(\frac{3{,}600 \text{ s}}{1 \text{ h}}\right) = 0.409 \text{ s}$$

$$t_{d10} = \frac{3 \text{ ft}}{10 \text{ mi/h}}\left(\frac{1 \text{ mi}}{5{,}280 \text{ ft}}\right)\left(\frac{3{,}600 \text{ s}}{1 \text{ h}}\right) = 0.205 \text{ s}$$

Pulse duration to period ratios,

$$\frac{t_{d5}}{T_n} = \frac{0.409 \text{ s}}{0.571 \text{ s}} = 0.716$$

$$\frac{t_{d10}}{T_n} = \frac{0.205 \text{ s}}{0.571 \text{ s}} = 0.358$$

The ground excitation equation acceleration, (take the derivative of the given ground

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displacement equation twice),

$$\ddot{u}_g(t) = \frac{d^2u_g(t)}{dt^2} = -\frac{\pi^2}{t_d^2} 6 \text{ in} \cdot \sin(\pi t/t_d)$$

So,

$$\ddot{u}_{g_{\max}} = \frac{\pi^2}{t_d^2} 6 \text{ in}$$

iii. *Determine the maximum dynamic load factors for the two speeds using Fig. E12.1 and damping of 0.4.*

$$\frac{t_{d5}}{T_n} = 0.716 \rightarrow \text{DLF}_{\max 5} = 1.5$$

$$\frac{t_{d10}}{T_n} = 0.358 \rightarrow \text{DLF}_{\max 10} = 0.6$$

iv. *Determine the maximum steady-state vertical relative displacement of the vehicle.* Use the middle of the Eq. (4.37),

$$\text{DLF}_{\max} = \frac{\omega_n^2 u_o}{\ddot{u}_{go}} \Rightarrow u_o = \frac{\text{DLF}_{\max}\ddot{u}_{go}}{\omega_n^2} = \frac{\text{DLF}_{\max}\frac{\pi^2}{t_d^2}6 \text{ in}}{4\pi^2/T_n^2} = 1.5 \text{ in}\left(\frac{T_n}{t_d}\right)^2 \text{DLF}_{\max}$$

$$u_{o5} = 1.5 \text{ in}(1/0.716)^2(1.5) = 4.39 \text{ in}$$

$$u_{o10} = 1.5 \text{ in}(1/0.358)^2(0.6) = 7.02 \text{ in}$$

This is only approximate since we have damping and part of the formulation assumes no damping, Eq. (4.38). Also, these are large displacements and are clearly beyond the range of the springs and demonstrates the importance of good shock absorbers! Also, we could easily find the force in the system by multiplying this by the stiffness.

v. *Determine the maximum total acceleration.* Use the last of Eq. (4.37),

$$\text{DLF}_{\max} = \frac{\ddot{u}_o^t}{\ddot{u}_{go}} \Rightarrow \ddot{u}_o^t = \text{DLF}_{\max}\ddot{u}_{go} = \text{DLF}_{\max} 6 \text{ in}\frac{\pi^2}{t_d^2}$$

$$\ddot{u}_{o5}^t = 1.5(6 \text{ in})\frac{\pi^2}{(0.409 \text{ s})^2} = 531 \text{ in/s}^2 = 1.37 \text{ g}$$

$$\ddot{u}_{o10}^t = 0.6(6 \text{ in})\frac{\pi^2}{(0.205 \text{ s})^2} = 845 \text{ in/s}^2 = 2.19 \text{ g}$$

Since both accelerations are larger than 1 g, the vehicle would fly up into the air! ▲

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Example 16

Draw the response of the frame shown in Fig. E13.1 when subjected to the ground motion of the Ease-West component of the horizontal ground acceleration recorded at the US Embassy in Port au Prince, Haiti during the Petit Trou de Nippes Earthquake of August 14, 2021, from here on referred to as the 2021 Haiti earthquake. Assume 2% damping. The input ground acceleration was obtained from https://www.strongmotioncenter.org/cgi-bin/CESMD/archive.pl and is shown graphically in Fig. E16.1 (the data are given in units of cm/s^2, and was converted to units of g by dividing each point by 980.8 cm/s^2). Also, obtain the displacement response spectrum for periods of up to 5 seconds.

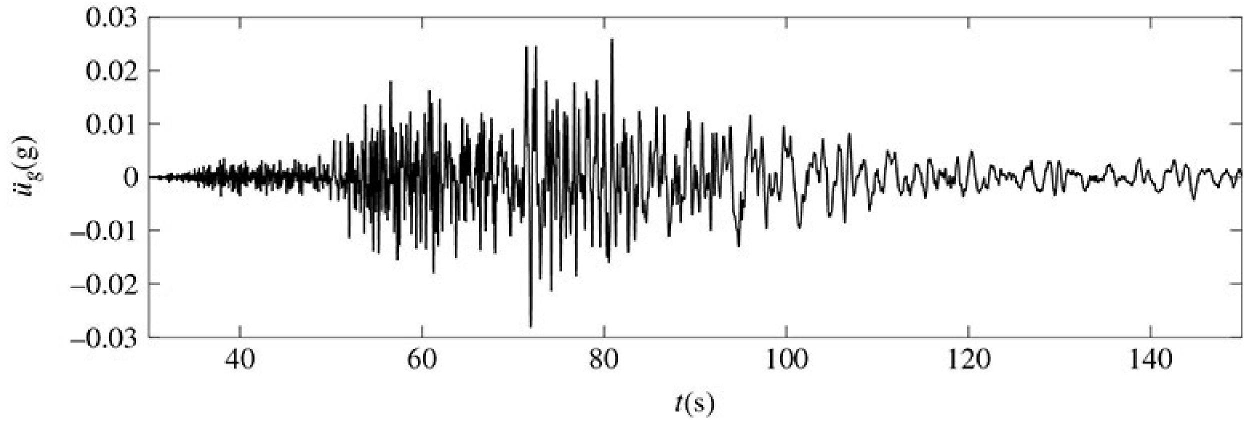

FIGURE E16.1 Ground motion acceleration time history for 2021 Haiti earthquake.

FIGURE E16.2 Epicenter and location of the recoding station in Port au Prince.

Solution Seismic ground accelerations cannot be described analytically and thus the response of the SDOF system subjected to a ground acceleration is evaluated using numerical methods. Numerical integration using the *conv* function in MATLAB, or other numerical integration methods, can be difficult to implement for complex cases and are not applicable in the case of nonlinear behavior. Therefore, seismic ground excitation loadings are typically treated using direct integration of the equation of motion. In the next section, we cover three direct integration methods, but first, we demonstrate the use of MATLAB *conv* function to integrate seismic ground excitation produced by the 2021 Haiti earthquake.

i. *The natural frequency of the frame, ω_n, was obtained in Example 13 as* $12.7\ rad/s$. Using the *conv* function introduced in Sec. 4.2, we can write the following script to perform the analysis:

```
clear all % Chapter 4, Example 16
dam = 0.02; % given damping of 2%
wn = 12.7; % rad/sec, determined in Example 13
load ('Haiti2021.mat', '-ascii'); % load Haiti 2021 data
N = length(Haiti2021); % number of points in the ground acceleration file
acc = Haiti2021/980.8; % ground acceleration data g in mm/sec^2
DT = 0.005; % sampling rate
% create the time vector using the sampling rate
for i=N:-1:1
  t(i)=i*DT;
end
% Integrate the ground acceleration using convolution
p=acc*DT;
h=exp(-dam*wn*t).*sin(wn*t);
u=conv(p,h)*386.4/wn;
uo=max(abs(u)) % determine maximum displacement
% create a new time vector adjusted to the conv length to graph response
for j=1:2*N-1
  t2(j)=j*DT/2;
end
%create plot
plot (t2, u, 'LineWidth',1, 'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('t(sec)','FontAngle','italic');
ylabel ('u(in)','FontAngle', 'italic');
axis([20 60 -.2 .2]);
grid on
```

This gives a maximum displacement of 0.154 in, and the following graphical response history results. It is important to note that this response is for a building frame in Port au Prince, not near the epicenter of the event.

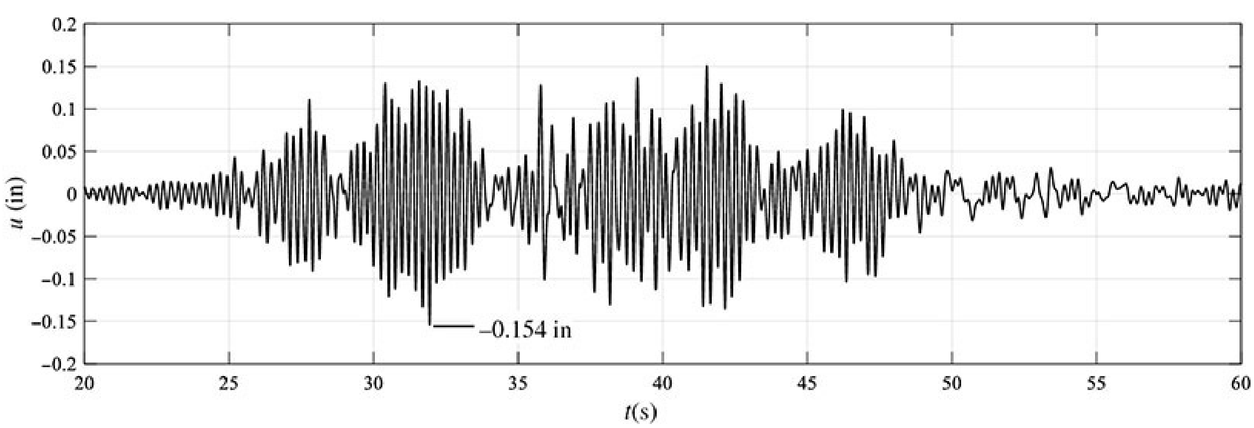

FIGURE E16.3 Displacement response of the SDOF building frame due to 2021 Haiti earthquake.

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ii. *We can modify the above script by obtaining the maximum displacement response for various values of the natural period.* The MATLAB *conv* function is comparatively slow and requires more increments of the period to obtain the results compared to the direct integration methods discussed in the next section. The displacement response as a function of natural period script is:

```
clear all % Chapter 4, Example 16, part 2
dam = 0.02; % damping; can changed to generate different spectra
load ('Haiti2021.mat', '-ascii'); % load Haiti 2021 data
N = length(Haiti2021); % number of points in the ground acceleration file
acc = Haiti2021/980.8; % ground acceleration data
DT = 0.005; % sampling rate
for i=N:-1:1
  t(i)=i*DT;
end
for j = 2000:-1:1
    period(j)=j*DT/2; % natural period array
    wn(j)=2*pi/period(j); % natural frequency array
    p=acc*DT;
    h=exp(-dam*wn(j)*t).*sin(wn(j)*t);
    u=conv(p,h);
    uustr(j)=max(abs(u))/wn(j); % maximum displacement
end
%create plot
plot (period, uustr*386.4, 'LineWidth',1, 'Color',[0 0 0]);
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel ('T_n(sec)','FontAngle','italic');
ylabel ('u(in)','FontAngle','italic');
```

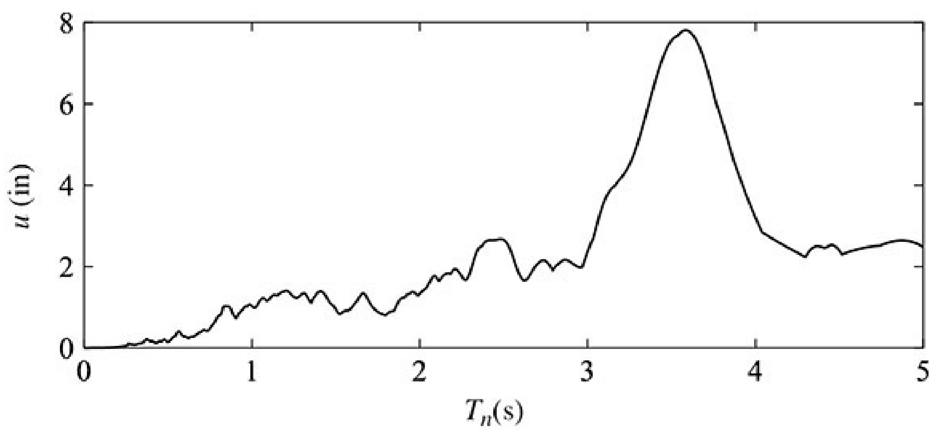

FIGURE E16.4 Displacement response spectrum for SDOF system using MATLAB *conv* function. ▲

4.7 Direct Integration Methods

Numerical integration of the convolution (Duhamel's) integral is computationally inefficient for complex loading functions and not applicable for a nonlinear response. Therefore, both of these cases are generally handled using time-stepping numerical methods for direct integration of the

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differential equation of motion. First, to derive any direct integration numerical algorithm, we divide the excitation function into N equal intervals of δt, and assume a SDOF system characterized by natural frequency, ω_n, and damping ratio, ζ.

In general, time-stepping methods are divided into two categories: *explicit* and *implicit*. Explicit methods use the state of the system at the current time to calculate the state of the system at the next time interval; while implicit methods use relationships involving both current and later states of the system to find a solution. Consequently, implicit methods require iteration, making them more computationally intensive. However, implicit methods are more practical in cases where the use of an explicit method requires impractically small-time steps, δt, to keep the error in the result bounded. Thus, for a desired level of accuracy, implicit methods can take much less computational time because they can use longer time steps. Summaries of three commonly applied methods follow.

4.7.1 Nigam–Jennings Algorithm (Explicit)

For each interval, δt, the response is calculated using the conditions at the beginning of the time interval—initial conditions. These initial conditions are given by the displacement and velocity at the end of the preceding time interval. We also assume the forcing function is given by an equivalent acceleration [load, $a(\tau) = -p(\tau)/\mathrm{m}$ or base acceleration $a(\tau) = \ddot{u}_g(\tau)$] to be linear and piecewise continuous as shown in Fig. 4.11,

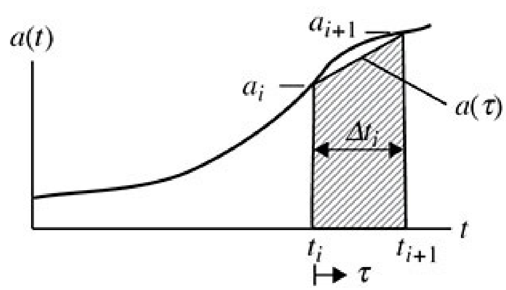

FIGURE 4.11 Discretization of excitation time history function.

$$a(\tau) = a_i + \left(\frac{\Delta a_i}{\Delta t}\right)\tau, \quad t_i \le \tau \le t_{i+1} \tag{4.40}$$

where

$$\Delta a_i = a_{i+1} - a_i$$
$$t_i = i \cdot \Delta t$$

and

$$\Delta t = t_{i+1} - t_i$$

for i = 1, 2, 3, …, N

The equation of motion for the time interval can be written as

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$$\ddot{u} + 2\omega_n \zeta \dot{u} + \omega_n^2 u = -a_i - \left(\frac{\Delta a_i}{\Delta t}\right)\tau, \quad t_i \le \tau \le t_{i+1} \tag{4.41}$$

The solution to this linear differential equation with constant coefficients is given by complementary and particular components [similar to Eq. (3.15)],

$$u(t) = u_c(t) + u_p(t) \tag{4.42}$$

where over the interval, δt, the complementary solution [similar to Eq. (2.17)] is

$$u_c(\tau) = e^{-\zeta\omega_n(\tau - t_i)}\left(C_i \cos\omega_D(\tau - t_i) + D_i \sin\omega_D(\tau - t_i)\right)$$

and over the same interval, the particular solution is

$$u_p(\tau) = B_i + A_i(\tau - t_i)$$

where A_i and B_i are constants of integration, determined by substituting this equation back into the equation of motion. After applying boundary conditions, the formulas to calculate the displacement, velocity, and acceleration at time step $t_i+_1 = t_i + \delta t$ are:

$$u_{i+1} = a_{11}u_i + a_{12}\dot{u}_i + b_{11}a_i + b_{12}a_{i+1} \tag{4.43}$$

$$\dot{u}_{i+1} = a_{21}u_i + a_{22}\dot{u}_i + b_{21}a_i + b_{22}a_{i+1} \tag{4.44}$$

$$\ddot{u}_{i+1} = -\omega_n^2 u_{i+1} - 2\zeta\omega_n \dot{u}_{i+1} - a_{i+1} \tag{4.45}$$

respectively, where the coefficients of the displacements and velocities in the first two equations only need to be computed once and are given as

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$$a_{11} = e^{-\zeta\omega_n \Delta t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D\Delta t + \cos\omega_D\Delta t\right)$$

$$a_{12} = e^{-\zeta\omega_n \Delta t}\frac{\sin\omega_D\Delta t}{\omega_D}$$

$$a_{21} = -\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n \Delta t}\sin\omega_D\Delta t$$

$$a_{22} = e^{-\zeta\omega_n \Delta t}\left(\cos\omega_D\Delta t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D\Delta t\right)$$

$$b_{11} = e^{-\zeta\omega_n \Delta t}\left[\left(\frac{2\zeta^2-1}{\omega_n^2\Delta t} + \frac{\zeta}{\omega_n}\right)\frac{\sin\omega_D\Delta t}{\omega_D} - \left(\frac{2\zeta}{\omega_n^3\Delta t} + \frac{1}{\omega_n^2}\right)\cos\omega_D\Delta t\right] - \frac{2\zeta}{\omega_n^3\Delta t}$$

$$b_{12} = -e^{-\zeta\omega_n \Delta t}\left[\left(\frac{2\zeta^2-1}{\omega_n^2\Delta t}\right)\frac{\sin\omega_D\Delta t}{\omega_D} + \frac{2\zeta}{\omega_n^3\Delta t}\cos\omega_D\Delta t\right] - \frac{1}{\omega_n^2} + \frac{2\zeta}{\omega_n^3\Delta t}$$

$$b_{21} = e^{-\zeta\omega_n \Delta t}\left[\left(\frac{2\zeta^2-1}{\omega_n^2\Delta t} + \frac{\zeta}{\omega_n}\right)\left(\cos\omega_D\Delta t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D\Delta t\right)\right.$$
$$\left. - \left(\frac{2\zeta}{\omega_n^3\Delta t} + \frac{1}{\omega_n^2}\right)\left(\omega_D\sin\omega_D\Delta t + \zeta\omega_n\cos\omega_D\Delta t\right)\right] + \frac{1}{\omega_n^2\Delta t}$$

$$b_{22} = \frac{e^{-\zeta\omega_n \Delta t}}{\omega_n^2\Delta t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D\Delta t + \cos\omega_D\Delta t\right) - \frac{1}{\omega_n^2\Delta t}$$

Example 17

Write a MATLAB script for the Nigam–Jennings algorithm; then use the script to draw displacement, velocity, and acceleration response spectra for the response of the SDOF system subjected to the 2021 Haiti earthquake ground acceleration presented in Example 16.

Solution In this example, we write a MATLAB script based on Eqs. (4.43) to (4.45). The maximum displacement, velocity, and acceleration response as a function of natural period script is as follows:

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```
clear all % Chapter 4, Example 17
d(1) = 0;% initial displacement
v(1) = 0;% initial velocity
dam = 0.02; % damping; can changed to generate different response spectra
DT = 0.005; % sampling rate
load ('Haiti2021.mat', '-ascii'); % load El Centro data
N = length(Haiti2021); % number of points in the ground acceleration file
acc = Haiti2021/980.8; % ground acceleration data
% generate the data to graph a response spectrum by changing period
for j = 1000:-1:1
period(j) = j*DT; % natural period
wd = 2*pi/period(j)*sqrt(1-dam^2); % damped frequency
wn = 2*pi/period(j); % natural frequency
%%% compute values of the response and choose the maximum %%%
% first calculate the elements of the matrices A and B
a11=exp(-dam*wn*DT)*(dam/sqrt(1-dam^2)*sin(wd*DT)+cos(wd*DT));
a12=exp(-dam*wn*DT)/wd*sin(wd*DT);
a21=-wn/sqrt(1-dam^2)*exp(-dam*wn*DT)*sin(wd*DT);
a22=exp(-dam*wn*DT)*(cos(wd*DT)-dam/sqrt(1-dam^2)*sin(wd*DT));
b11=exp(-dam*wn*DT)*(((2*dam^2-1)/(wn^2*DT)+dam/wn)*sin(wd*DT)/wd+...
 (2*dam/(wn^3*DT)+1/wn^2)*cos(wd*DT))-2*dam/(wn^3*DT);
b12=-exp(-dam*wn*DT)*((2*dam^2-1)/(wn^2*DT)*sin(wd*DT)/wd+...
 2*dam/(wn^3*DT)*cos(wd*DT))-1/wn^2+2*dam/(wn^3*DT);
b21=exp(-dam*wn*DT)*(((2*dam^2-1)/(wn^2*DT)+dam/wn)*(cos(wd*DT)-...
 dam/sqrt(1-dam^2)*sin(wd*DT))-(2*dam/(wn^3*DT)+1/wn^2)*(wd*sin(wd*DT)+...
 dam*wn*cos(wd*DT)))+1/(wn^2*DT);
b22=(-1+exp(-dam*wn*DT)*(dam/sqrt(1-dam^2)*sin(wd*DT)+cos(wd*DT)))/(wn^2*DT);
%loop to find response
for i = 1:N-1
     d(i+1) = a11*d(i)+a12*v(i)+b11*acc(i)+b12*acc(i+1);
     v(i+1) = a21*d(i)+a22*v(i)+b21*acc(i)+b22*acc(i+1);
     Responseaccelation(i) = -2*wn*dam*v(i+1)-(wn^2)*d(i+1);
end
% compute the value of the largest response
max_acc(j) = max(abs(Responseaccelation));
max_vel(j) = max(abs(v));
max_dis(j) = max(abs(d));
end
%create plots
subplot(3,1,1), plot(period,max_dis*386.4)
set(gca,'FontSize',12,'FontName','Times New Roman')
ylabel('Displacement(in)','FontAngle','italic')
subplot(3,1,2), plot(period,max_vel*386.4)
set(gca,'FontSize',12,'FontName','Times New Roman')

ylabel('Velocity(in/s)','FontAngle','italic')
subplot(3,1,3), plot(period,max_acc)
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel('T_n(sec)','FontAngle', 'italic')
ylabel('Acceleration(g)','FontAngle','italic')
```

The results for the displacement response spectrum obtained using this method and the *conv*

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function in MATLAB, shown in Fig. E16.4, are virtually identical.

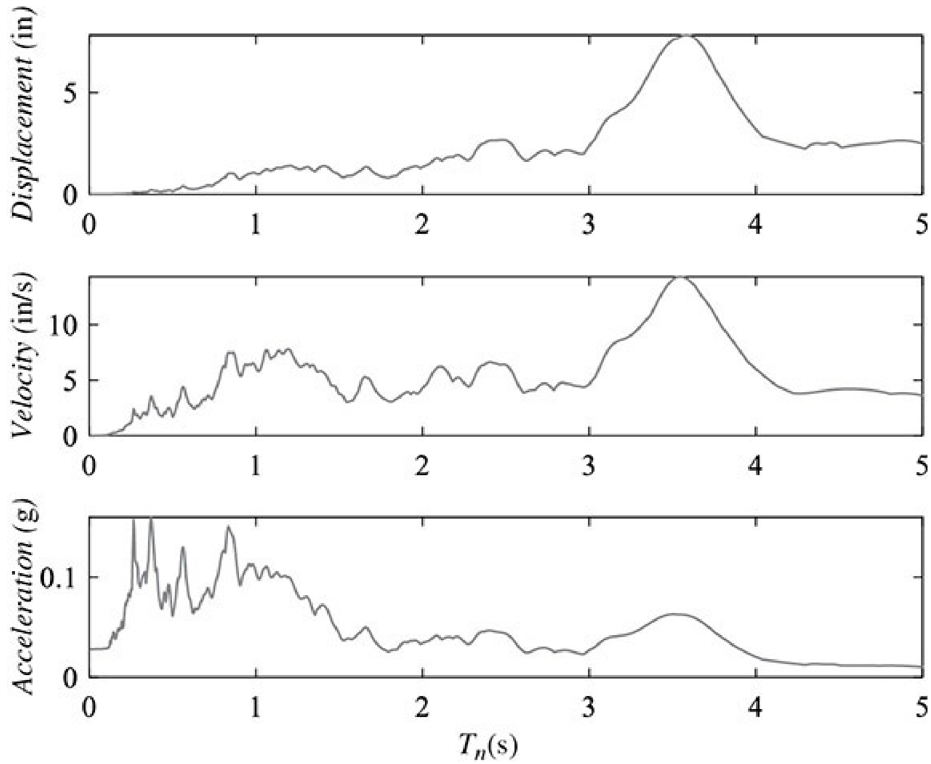

FIGURE E17.1 Displacement, velocity, and acceleration response spectra for SDOF system subjected to the Haiti 2021 earthquake.

The acceleration results clearly show that stiffer systems (those with short natural periods) tend to attract significantly more inertial force than those with longer natural periods. However, until recently the main threat to the Caribbean was from hurricanes not earthquakes; so, building construction tended to be stiffer in order to carry the large wind forces generated by hurricanes. This recent earthquake and direct impact by tropical storm Grace, two days after, clearly shows the need to create resilient buildings considering both wind and seismic loading, a considerable challenge that requires a careful balance between stiffness and flexibility in building construction. ▲

4.7.2 Central Difference Method (Explicit)

This method is based on a finite difference approximation of the time derivatives of the displacement in the equation of motion over a constant time interval, δt. Also, the excitation forcing function is assumed to be linear and piecewise continuous, with approximate time derivatives for time step i of duration δt shown in Fig. 4.12. Substituting the velocity and acceleration listed in Fig. 4.12 into the equation of motion, we get equation of motion for the time interval $2\delta t$, in terms of the ground acceleration,

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$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t}$$

$$\ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}$$

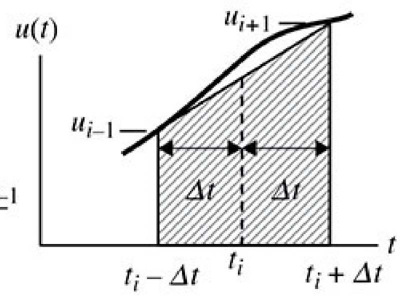

FIGURE 4.12 Finite difference approximation of velocity and acceleration.

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} + 2\omega_n \zeta \frac{u_{i+1} - u_{i-1}}{2\Delta t} + \omega_n^2 u_i = -\ddot{u}_{gi} \tag{4.46}$$

where $\ddot{u}_{gi}$ is the ground acceleration at time step i. For a loading function, we can substitute $\ddot{u}_{gi} = -p_i/m$. Also, u_i and u_{i-1} are known from the preceding step. Rearranging Eq. (4.32) to have these known quantities on the right,

$$\left(\frac{1}{(\Delta t)^2} + \frac{\omega_n \zeta}{\Delta t}\right) u_{i+1} = -\ddot{u}_{gi} - \left(\frac{1}{(\Delta t)^2} - \frac{\omega_n \zeta}{\Delta t}\right) u_{i-1} - \left(\omega_n^2 - \frac{2}{(\Delta t)^2}\right) u_i \tag{4.47}$$

or,

$$\hat{k} u_{i+1} = \hat{a}_i \tag{4.48}$$

where

$$\hat{k} = \frac{1}{(\Delta t)^2} + \frac{\omega_n \zeta}{\Delta t}$$

$$\hat{a}_i = -\ddot{u}_{gi} - \left(\frac{1}{(\Delta t)^2} - \frac{\omega_n \zeta}{\Delta t}\right) u_{i-1} - \left(\omega_n^2 - \frac{2}{(\Delta t)^2}\right) u_i = -\ddot{u}_{gi} - a u_{i-1} - b u_i$$

Determining the initial velocity, $\dot{u}_0$, and displacement, u_0, from the specified initial conditions, we can use the following algorithm for the central difference method:

1. Initial calculations based on known quantities:

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$$\ddot{u}_0 = -\ddot{u}_{g0} - 2\omega_n \zeta \dot{u}_0 - \omega_n^2 u_0$$

$$u_{-1} = u_0 - \Delta t \cdot \dot{u}_0 + \frac{(\Delta t)^2}{2} \ddot{u}_0$$

$$\hat{k} = \frac{1}{(\Delta t)^2} + \frac{\omega_n \zeta}{\Delta t}$$

$$a = \frac{1}{(\Delta t)^2} - \frac{\omega_n \zeta}{\Delta t}$$

$$b = \omega_n^2 - \frac{2}{(\Delta t)^2}$$

2. Calculations for step *i*, where *i* = 0, 1, 2, 3, …

$$\hat{a}_i = -\ddot{u}_{gi} - a u_{i-1} - b u_i$$

$$u_{i+1} = \frac{\hat{a}_i}{\hat{k}}$$

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t}$$

$$\ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}$$

This method is conditionally stable; it requires the following time step length for stability.

$$\Delta t < T_n / \pi \tag{4.49}$$

4.7.3 Newmark's Beta Method for Linear Systems (Implicit)

This is a general method that approximates the variation of the acceleration response with each time interval. In this method, the response can be calculated using nonconstant time intervals, δt_i, following approximations of the time derivatives of the displacement in the equation of motion as

$$\dot{u}_{i+1} = \dot{u}_i + \left[(1-\gamma)\Delta t\right]\ddot{u}_i + \left[\gamma \Delta t\right]\ddot{u}_{i+1} \tag{4.50}$$

$$u_{i+1} = u_i + \Delta t \cdot \dot{u}_i + \left[(0.5-\beta)(\Delta t)^2\right]\ddot{u}_i + \left[\beta (\Delta t)^2\right]\ddot{u}_{i+1} \tag{4.51}$$

where parameters β and γ define the variation of the acceleration over nonconstant time intervals, δt_i. The values of these parameters must be chosen carefully in order to obtain stable results; this includes β = ½ and γ = ¼ (constant average acceleration), and $\beta = \frac{1}{2}$ and $\gamma = \frac{1}{6}$ (linear acceleration).

After obtaining the initial velocity, $\dot{u}_0$, and displacement, u_0, from the specified initial conditions, we can use the following algorithm for the Newmark Beta method. Again, for a

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loading function, we can substitute $\ddot{u}_{gi} = -p_i/m$.

1. Initial calculations based on known quantities and time intervals, δt_i:

$$\ddot{u}_0 = -\ddot{u}_{g0} - 2\omega_n\zeta\dot{u}_0 - \omega_n^2 u_0$$

$$a_1 = \frac{1}{\beta(\Delta t)^2} + \frac{2\omega_n\zeta\gamma}{\beta\cdot\Delta t}$$

$$a_2 = \frac{1}{\beta\cdot\Delta t} + 2\omega_n\zeta\left(\frac{\gamma}{\beta} - 1\right)$$

$$a_3 = \frac{1}{\beta} - 1 + 2\omega_n\zeta\left(\frac{\gamma}{2\beta} - 1\right)\Delta t$$

$$\hat{k} = \omega_n^2 + a_1$$

2. Calculations for step i, where $i = 0, 1, 2, 3, \ldots$

$$\hat{a}_{i+1} = -\ddot{u}_{gi+1} + a_1 u_i + a_2\dot{u}_i + a_3\ddot{u}_i$$

$$u_{i+1} = \frac{\hat{a}_{i+1}}{\hat{k}}$$

$$\dot{u}_{i+1} = \frac{\gamma}{\beta\cdot\Delta t}(u_{i+1} - u_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i$$

$$\ddot{u}_{i+1} = \frac{(u_{i+1} - u_i)}{\beta(\Delta t)^2} - \frac{\dot{u}_i}{\beta\cdot\Delta t} - \left(\frac{1}{2\beta} - 1\right)\ddot{u}_i$$

This method is also conditionally stable; it requires the following time step for stability.

$$\Delta t \le T_n/\pi\sqrt{2(\gamma - 2\beta)} \tag{4.52}$$

which for β = ½ and γ = ¼ is unconditionally stable.

This particular method is useful in the analysis of nonlinear systems; however, details of the algorithm are slightly different than those presented here and are beyond the scope of this book. For complete details of the analysis of nonlinear systems, see the textbook by Chopra, *Dynamics* of *Structures*: *Theory and Applications to Earthquake Engineering*.

Example 18

Given the tower and trapezoidal impulsive excitation from Fig. E4.1 (repeated here for convenience), use the direct integration methods presented in this section to determine the dynamic response of the tower. Use MATLAB to perfume the calculations and assume negligible damping.

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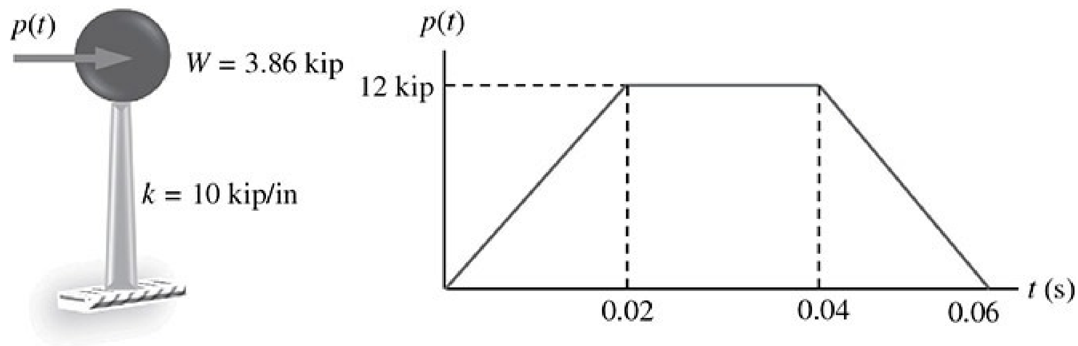

FIGURE E18.1 SDOF sys>tem and trapezoidal impulse loading function (repeated here for convenience).

Solution In this example, we write a MATLAB script based on Eqs. (4.43) to (4.45). The maximum displacement, velocity, and acceleration response as a function of natural period script is given below:

```
clear all % Chapter 4, Example 18
u0 = 0;% initial displacement
v0 = 0;% initial velocity
```

```
dam = 0.2; % damping; can changed to generate different response spectra
dt = 0.005; % sampling rate
W=38.6; % weight in kips
g=386; % gravity in in/sec^2
k = 100; % stiffness kip/in
m = W/g; % mass
wn = sqrt(k/m); % frequency in rad/sec
wd = wn*sqrt(1-dam^2); % damped frequency
p =10*[0 3 6 9 12 12 12 12 12 9 6 3 0 0 0 0 0 0 0 0 0]';% force in kips
t =[0 .005 .01 .015 .02 .025 .03 .035 .04 .045 .05 .055 .06 .065 .07...
    .075 .08 .085 .09 .095 .1]';
t2 =[-.005 0 .005 .01 .015 .02 .025 .03 .035 .04 .045 .05 .055 .06 .065...
    .07 .075 .08 .085 .09 .095 .1]';
N = length(p); % number of points in the force file
acc = -p/m; % force excitation data
method = input ('Enter 1 for Nigam-Jennings, 2 for central difference, 3 for
Newmark gamma 1/4 and 4 for Newmark 1/6 methods: '); -
%%%%%%% Nigam-Jennings Method%%%%%%%
if method == 1
%%% compute values of the response
DT = dt;% sampling rate
u(1) = u0;% initial displacement
v(1) = v0;% initial velocity
% first calculate the elements of the matrices A and B
a11=exp(-dam*wn*DT)*(dam/sqrt(1-dam^2)*sin(wd*DT)+cos(wd*DT));
a12=exp(-dam*wn*DT)/wd*sin(wd*DT);
a21=-wn/sqrt(1-dam^2)*exp(-dam*wn*DT)*sin(wd*DT);
a22=exp(-dam*wn*DT)*(cos(wd*DT)-dam/sqrt(1-dam^2)*sin(wd*DT));
b11=exp(-dam*wn*DT)*(((2*dam^2-1)/(wn^2*DT)+dam/wn)*sin(wd*DT)/wd+...
 (2*dam/(wn^3*DT)+1/wn^2)*cos(wd*DT))-2*dam/(wn^3*DT);
b12=-exp(-dam*wn*DT)*((2*dam^2-1)/(wn^2*DT)*sin(wd*DT)/wd+...
 2*dam/(wn^3*DT)*cos(wd*DT))-1/wn^2+2*dam/(wn^3*DT);
b21=exp(-dam*wn*DT)*(((2*dam^2-1)/(wn^2*DT)+dam/wn)*(cos(wd*DT)-...
 dam/sqrt(1-dam^2)*sin(wd*DT))-(2*dam/(wn^3*DT)+1/wn^2)*(wd*sin(wd*DT)+...
 dam*wn*cos(wd*DT)))+1/(wn^2*DT);
b22=(-1+exp(-dam*wn*DT)*(dam/sqrt(1-dam^2)*sin(wd*DT)+cos(wd*DT)))/(wn^2*DT);
%loop to find response
for i = 1:N-1
     u(i+1) = a11*u(i)+a12*v(i)+b11*acc(i)+b12*acc(i+1);
     v(i+1) = a21*u(i)+a22*v(i)+b21*acc(i)+b22*acc(i+1);
     a(i+1) = -2*wn*dam*v(i+1)-(wn^2)*u(i+1)-acc(i+1);
end
%%%%%%% Central Difference Method%%%%%%%
elseif method == 2
% compute values of the response
a(1) = -acc(1) - 2*wn*dam*v0 - wn^2*u0;
uneg1 = u0 - dt*v0 + dt^2*a(1)/2;
khat_m = 1/dt^2 + dam*wn/dt;
a_m = 1/dt^2 - dam*wn/dt;
b_m = wn^2 - 2/dt^2;
% compute new values of displacement, velocity, and acceleration
% use i = 0 to get u(1)
ahat0 =  -acc(1) - a_m*uneg1 - b_m*u0;
u(1) = ahat0/khat_m;
for i = 1:N;
```


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```
    if i == 1;
        ahat(i) =  -acc(i) - a_m*uneg1 - b_m*u(i);
        u(i+1) = ahat(i)/khat_m;
        v(i) = (u(i+1) - uneg1)/(2*dt);
        a(i) = (u(i+1) - 2*u(i) + uneg1)/dt^2;
    else
        ahat(i) = -acc(i) - a_m*u(i-1)- b_m*u(i);
        u(i+1) = ahat(i)/khat_m;
        v(i) = ( u(i+1) - u(i-1) ) / (2*dt);
        a(i) = (u(i+1) - 2*u(i) + u(i-1))/dt^2;
    end
end
u=u(1:21)
%%%%%%% Newmark Beta (1/4) Method %%%%%%%
elseif method == 3
% initial calculations
beta = 1/2; % constant average acceleration parameter
gamma = 1/4; % constant average acceleration parameter
a(1) = -acc(1) - 2*wn*dam*v0 - wn^2*u0;
a_1 = 1/(beta*dt^2)+2*dam*wn*gamma/(beta*dt);
a_2 = 1/(beta*dt) + 2*dam*wn*(gamma/beta-1);
a_3 = 1/(2*beta) - 1 + 2*dam*wn*(gamma/(2*beta)-1)*dt;
khat_m = wn^2 + a_1;
% compute new values of displacement, velocity, and acceleration
% use i = 0 to get u(1)
ahat(1) =  -acc(1) + a_1*u0 + a_2*v0 + a_3*a(1);
u(1) = ahat(1)/khat_m;
v(1) = gamma*(u(1)-u0)/(beta*dt)+(1-gamma/beta)*v0+dt*a(1)*(1-gamma/
(2*beta));
a(1) = (u(1)-u0)/(beta*dt^2)-v0/(beta*dt)-(1/(2*beta)-1)*a(1);
for i = 1:N-1;
    ahat(i+1) = -acc(i+1) + a_1*u(i) + a_2*v(i) + a_3*a(i);
    u(i+1) = ahat(i+1)/khat_m;
    v(i+1) = gamma/(beta*dt)*(u(i+1)-u(i))+(1-gamma/beta)*v(i)+dt*(1-gamma/
(2*beta))*a(i);
    a(i+1) = (u(i+1)-u(i))/(beta*dt^2)-v(i)/(beta*dt)-(1/(2*beta)-1)*a(i);
end
%%%%%%% Newmark Beta (1/6 Method %%%%%%%
else
% initial calculations
beta = 1/2; % constant average acceleration parameter
gamma = 1/6; % constant average acceleration parameter
a(1) = -acc(1) - 2*wn*dam*v0 - wn^2*u0;
a_1 = 1/(beta*dt^2)+2*dam*wn*gamma/(beta*dt);
a_2 = 1/(beta*dt) + 2*dam*wn*(gamma/beta-1);
a_3 = 1/(2*beta) - 1 + 2*dam*wn*(gamma/(2*beta)-1)*dt;
khat_m = wn^2 + a_1;
% compute new values of displacment, velocity, and acceleration
% use i = 0 to get u(1)
ahat(1) =  -acc(1) + a_1*u0 + a_2*v0 + a_3*a(1);
u(1) = ahat(1)/khat_m;
v(1) = gamma*(u(1)-u0)/(beta*dt)+(1-gamma/beta)*v0+dt*a(1)*(1-gamma/
(2*beta));
a(1) = (u(1)-u0)/(beta*dt^2)-v0/(beta*dt)-(1/(2*beta)-1)*a(1);
```


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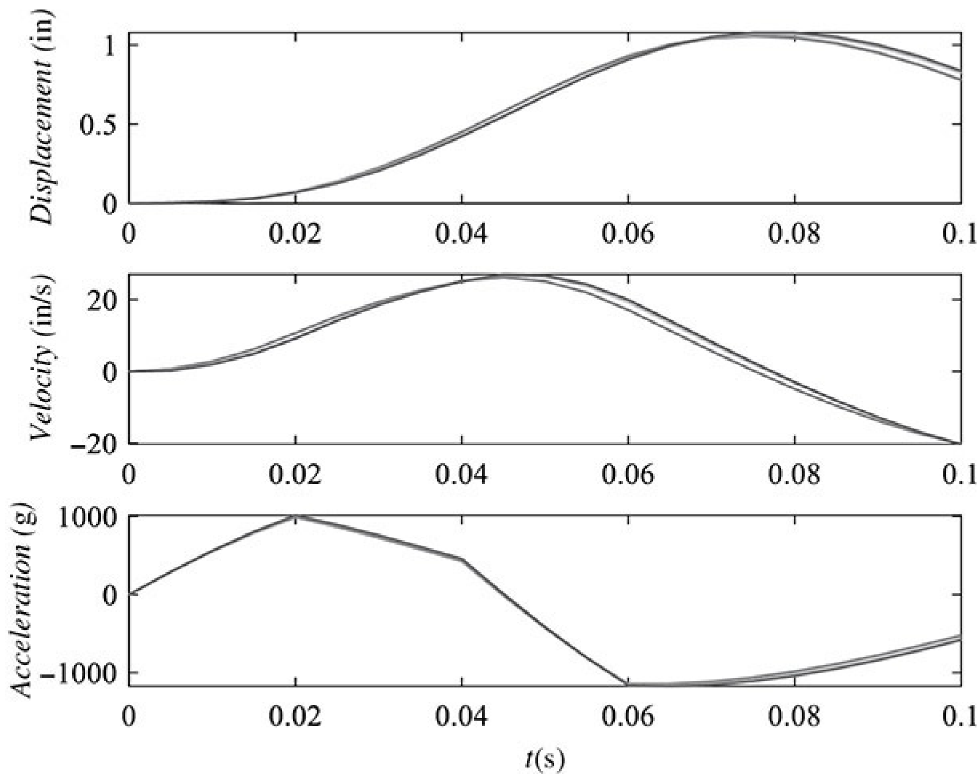
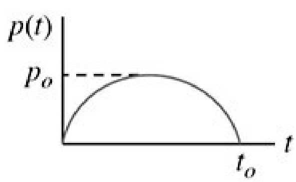


FIGURE E18.2 Displacement, velocity, and acceleration response for SDOF system subjected to trapezoidal load shown in Fig. E4.1.

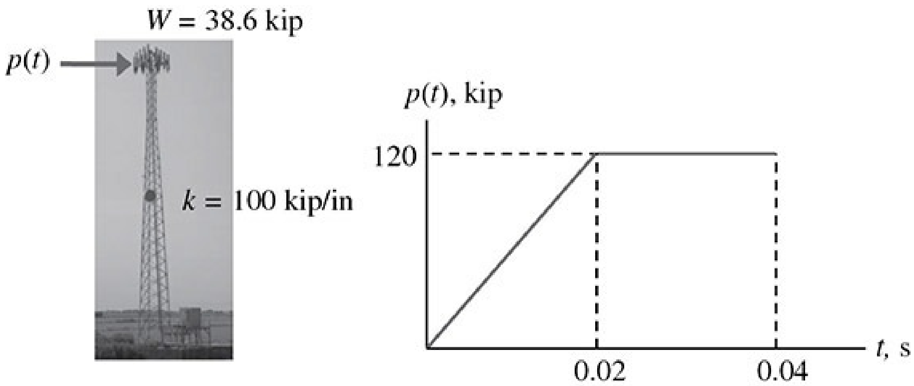
The results for the displacement and velocity response obtained using all three methods are virtually identical. ▲

4.8 Problems

4.1 Use Duhamel's integral to determine the response of an undamped SDOF system subjected to a half-sine impulse that is applied from $0 \le t \le t_d$. Assume the SDOF system is initially at rest and $t_d/T \neq 0.5$. Hint, this can be solved by using superposition of two sinusoidal excitations.

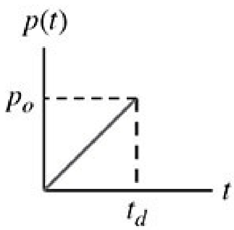


4.2 Use Duhamel's integral to determine the dynamic response of a tower subjected to the load shown.



4.3 Write a MATLAB script using the *conv* function to plot the shock spectrum for the rectangular pulse in Example 3.

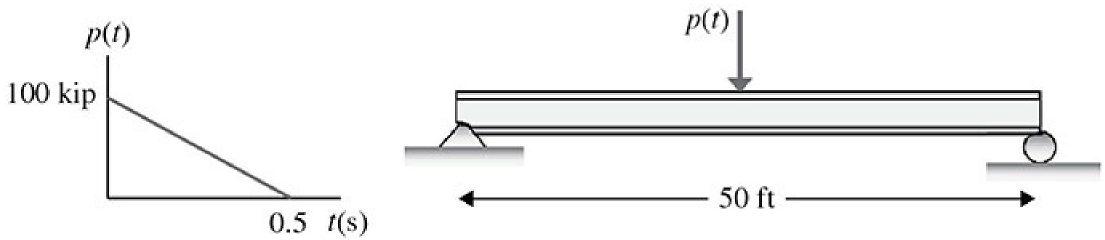
4.4 Draw the shock spectrum for a SDOF system for the triangular impulse loading shown.



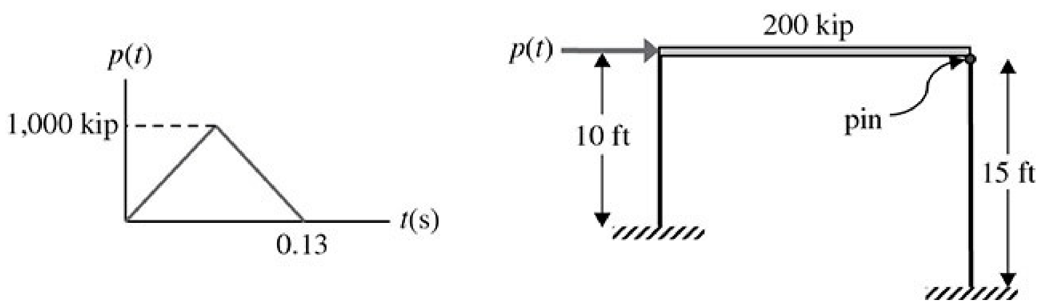
4.5 Assuming that Burn's tower (a water tank weighing 4,200 kip) on the campus of University of the Pacific is supported on a 120-ft-tall cantilever tower with stiffness of 2,000 kip/in, determine the design values for lateral deformation and base shear for a symmetric triangular impulse (Fig. E9.1) of amplitude 100 kip and duration t_d = 0.1 seconds.

4.6 Now assume Burn's tower has 5% damping and is subjected to ground acceleration due the 2021 Haiti earthquake, Examples 16 and 17. Determine the design values for lateral deformation and base shear.

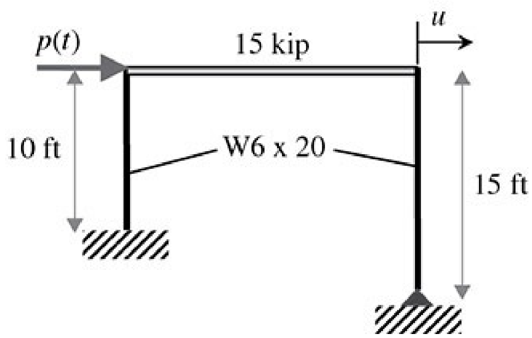
4.7 The bridge beam depicted below supports a rigid deck that weighs 200 kip. The beam is subjected to the blast load described by the triangular impulse loading shown. Determine the maximum stress in the beam (E = 29,000 kip/in^2, I_x = 13,000 in^4, and S_x = 830 in^3).



4.8 Consider the following building frame with a rigid beam that is pinned connected to one column and rigidly connected to the other as shown; each column has EI = 40,000,000 kip · in^2. The frame is subjected to the blast load shown at the beam level. Determine maximum stress in each column given S_x = 2000 in^3.



4.9 Given the building frame shown below, which is subjected to a triangular impulse force (see Fig. E10.1) of amplitude p_o = 2 kip and duration, t_d = 0.6 seconds, determine (i) maximum displacement at the top, (ii) the maximum base shear, and (iii) the maximum bending stresses in the columns (I_x = 41.4 in^4 and S_x = 13.4 in^3 for W6 × 20). Assume the beam is rigid.



4.10 Use the MATLAB script provided in this chapter to draw response spectra for the displacement, velocity, and acceleration for a SDOF system subjected to the 2021 Haiti Earthquake ground acceleration for 2%, 5%, and 10% damping.

4.11 The building frame shown is subjected to the 2012 Haiti earthquake; determine the total stiffness, structural period, deflection of the beam, base shear, and bending stresses in each of the columns. The roof weighs 10 kip and the substructure has 5% damping, modulus of elasticity, E = 29,000 ksi, and the other geometric properties shown.

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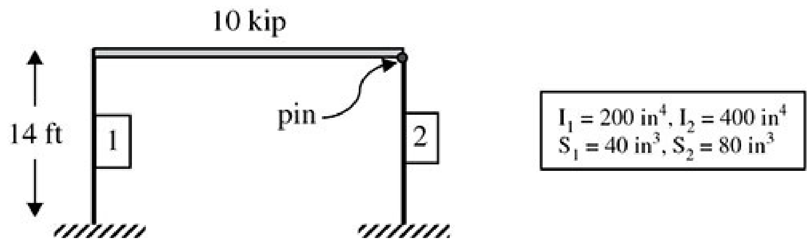
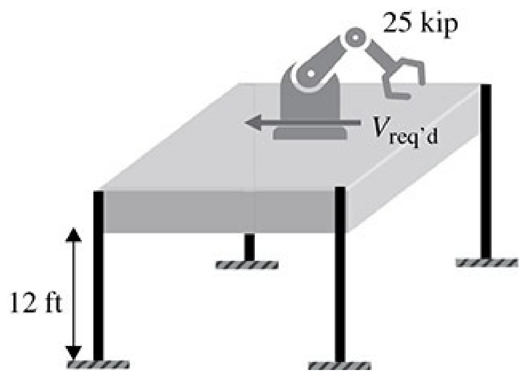
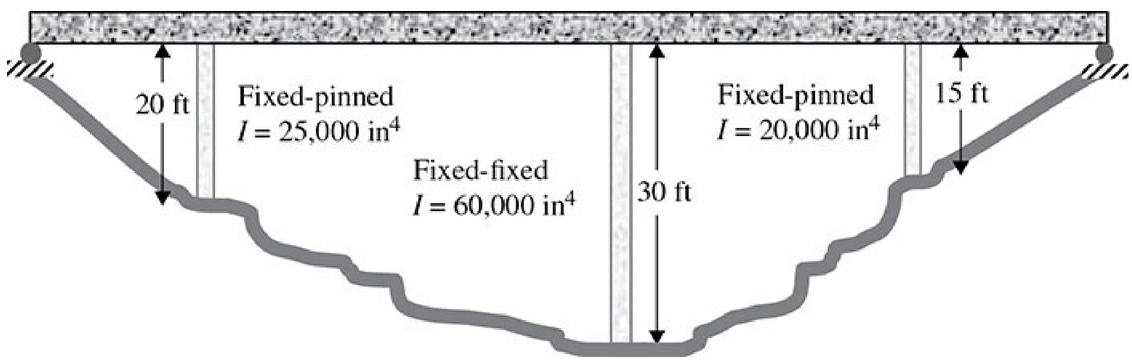


4.12 A 25 kip robotic arm will be installed on top of the frame shown, which consists of a rigid slab and four columns with $EI_x = 125{,}000$ kip · in^2. The frame is located in Port au Prince, Haiti near the US Embassy; determine approximately the magnitude of the shear force ($V_{\text{req'd}}$) needed to design the equipment anchorage. Assume damping is 5%.



4.13 The reinforced concrete bridge structure shown is subjected to the 2021 Haiti earthquake; determine the total stiffness, structural period, deflection of the deck, and base shear. The deck weighs 750 kip and the substructure has 2% damping, modulus of elasticity, $E = 3{,}000$ ksi, and the other geometric properties shown.



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CHAPTER 5

Vibration of Generalized SDOF Systems with Distributed Mass and Distributed Stiffness

After reading this chapter, you will be able to:

a. Develop the generalized single-degree-of-freedom (SDOF) equation of motion for systems with both distributed stiffness and mass
b. Determine generalized properties, including natural period and participation factor for generalized shear building systems
c. Use response spectra and participation factors to calculate maximum base shear and overturning moment response of generalized shear building systems
d. Determine generalized properties, including natural period and participation factor for generalized distributed mass/stiffness systems
e. Use response spectra and participation factors to calculate maximum base shear and overturning moment response of generalized distributed mass/elasticity systems

In Sec. 1.6 we discussed the three possible combinations of distributed mass and distributed stiffness: localized mass, but distributed stiffness; localized stiffness, but distributed mass; and both distributed stiffness and mass. For the last case, mass and stiffness can be treated as continuous or can be distributed discretely at several points. In Sec. 1.6 we also described a process that lumps the distributed mass and stiffness, which effectively allowed us to model a system as SDOF having both localized mass and stiffness. While the result of this analysis can be sufficiently accurate for many practical systems, there are numerous other cases where localized mass and stiffness must be considered simultaneously, such as multistory buildings and towers.

To properly capture the complete dynamic response of multistory buildings, they should be treated as multi-degree-of-freedom (MDOF) systems where the mass and stiffness are distributed discretely at several points requiring simultaneous manipulation of multiple algebraic equations. Towers should be modeled as continuous systems with infinite number of degrees of freedom where the distributed mass and stiffness are modeled as functions of position, which requires formulating the equation of motion in terms of partial differential equations in both position and time. However, even these complex systems can be modeled as generalized SDOF systems. While this analysis approach is not as precise as modeling a system as MDOF or with infinite number of degrees of freedom, the result is more accurate than approximating the entire system as a simple SDOF case.

The generalized SDOF system analysis presented in this chapter is based on replacing the required partial differential equation in time and space with an approximate differential equation in time only. The space variation is tracked using an approximate equation of the mode of

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vibration for the system. The process entails selecting a point of interest along the distributed system and tracking its motion with respect to time using a SDOF analysis. To obtain the response of all other points, we scale the magnitude of the point in question based on the approximate shape of the mode of vibration. The solution to the differential equation representing the generalized SDOF system follows the standard procedures presented in previous chapters; and we can apply the solution techniques discussed thus far, including the response spectrum analysis presented in Chap. 4.

The formulation of the equation of motion of a generalized SDOF system entails condensing all degrees of freedom of the system into one DOF. In the following sections we present analysis procedures for a multistory shear building frame (discrete system) and a cantilever tower with distributed properties (continuous system), both subjected to force and ground motion excitations. In each case, we assume axial deformations to be negligible during the dynamic excitation, which leaves the lateral translation (or deflection) as the only possible deformation of the system.

5.1 Discrete System Analysis (Shear Buildings)

Multistory building systems can be modeled as having a discrete number of degrees of freedom, the equation of motion for which can be formulated by assuming the building distributed mass to be concentrated as lumped masses at various levels, as shown in Fig. 5.1. See Sec. 1.6.1 for a discussion on how to determine the lumped mass at each level. Also shown in Fig. 5.1 is an equivalent generalized SDOF system. This requires that the distributed dynamic properties (mass, damping, and stiffness) be replaced with lumped properties at the point being tracked, the top of the building. For discrete MDOF systems, the process involves combining the discrete properties associated with each DOF into a set of generalized properties. These generalized properties are most useful in applying the results from a SDOF system analysis to a MDOF system analysis modeled as a generalized SDOF case.

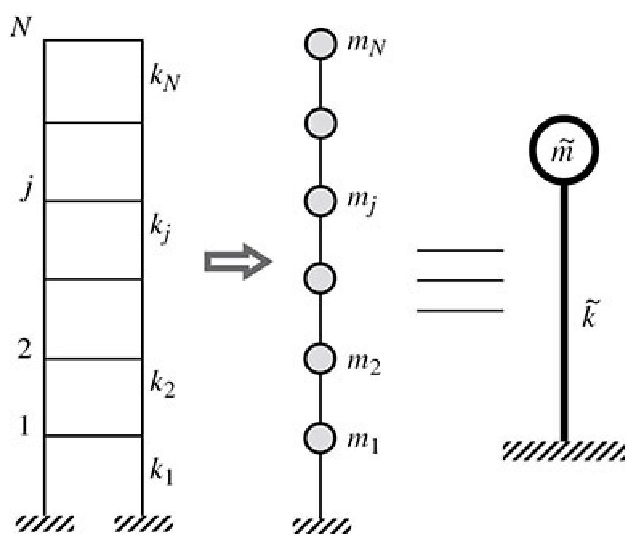

FIGURE 5.1 MDOF building and equivalent generalized SDOF system.

5.1.1 Forced Vibration Response of Generalized SDOF Discrete Systems

The actual motion of the shear building shown in Fig. 5.1 subjected to a force excitation is a function of position and time. The time effect is only tracked at the top of the building using $z(t)$; while the position of the entire shear building is followed using an assumed shape vector, ψ_j, which is typically determined by evaluating a shape function at each level. The position of any DOF or floor level can be obtained by determining the motion of the top of the building, $z(t)$, and scaling it to the other levels using a suitable shape vector, ψ_j; as shown in Fig. 5.2. That is, at an arbitrary level j the displacement is

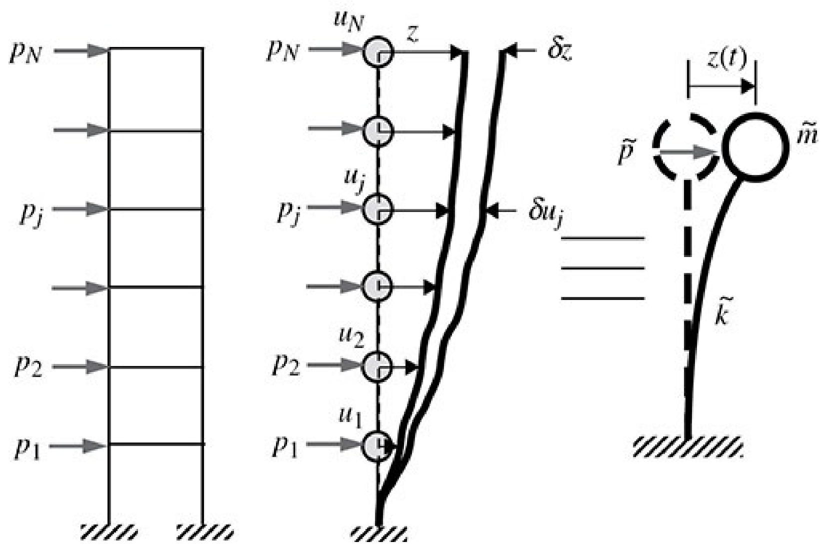

FIGURE 5.2 MDOF building and equivalent generalized SDOF system subjected to lateral forces, $p_i(t)$.

$$u_j(t) = \psi_j \cdot z(t) \tag{5.1}$$

where $j = 1, 2, \cdots N$ number of stories.

The motion at the top of the building, $z(t)$, can be obtained using a generalized equation of motion for the equivalent SDOF system shown in Fig. 5.2. This equation can be obtained using D'Alembert's principle (introduced in Chap. 1), either with direct equilibrium (Sec. 1.5.1) or the principle of virtual work (Sec. 1.5.2). The resulting equation is of the same form as the one that characterizes the vibration of a simple SDOF system; thus, all analysis procedures covered so far are applicable to this case.

The most important step in the analysis process is selecting a suitable shape function, $\psi(x)$, that satisfies the geometric boundary conditions of the system. This function is then used to obtain the shape vector, ψ_j. The accuracy of the results depends on how close the assumed shape function approximates the actual shape of the deformed vibrating frame. Also, the function $\psi(x)$ must be normalized at the top of the building such that $\psi(H) = 1$, where H is the height of the building. Figure 5.3 presents several suitable shape functions based on building elevation aspect ratio.

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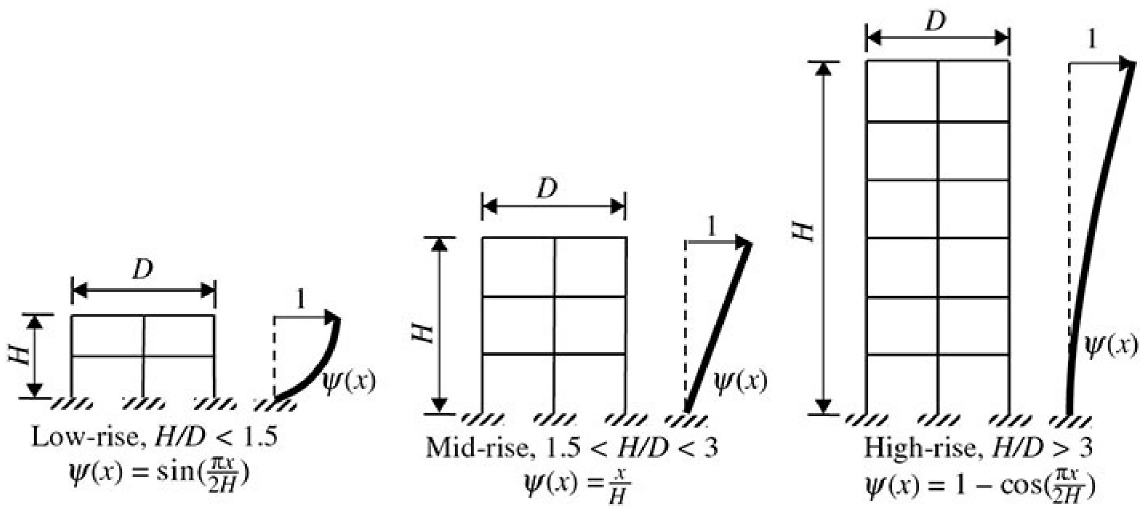

FIGURE 5.3 Suitable shape functions based on building elevation aspect ratio.

We can now use the principle of virtual work (Sec. 1.5.2) to derive the generalized equation of motion. Again, the principle states that the virtual work of all internal forces, δW_I, equals the virtual work of all external forces, δW_E, for a given virtual displacement, δu. That is, [Eq. (1.4), repeated here for convenience].

$$\delta W_I = \delta W_E \tag{5.2}$$

where δW_I includes the contribution from stiffness and damping forces:

$$\delta W_I = \delta W_{\text{stiffness}} + \delta W_{\text{damping}} \tag{5.3}$$

And δW_E includes the contribution from inertial forces of each mass and the applied force:

$$\delta W_E = \delta W_{\text{inertial}} + \delta W_{\text{applied force}} \tag{5.4}$$

The virtual work of each of these groups of forces is given by summation of the product of the magnitude of the force, p_j, and the corresponding virtual displacement, δu_j, as shown in Fig. 5.2. The stiffness and damping forces can be viewed as effective shear forces in each story, while the inertial and applied forces can be regarded as effective lateral loads applied at each level. The stiffness forces are functions of *story drift* (relative displacement from one floor to the next, $u_j - u_{j-1}$); the damping forces are functions of relative velocity from one floor to the next; and the inertial forces are functions of the total acceleration at each level.

First, determine the contribution of the stiffness forces to the virtual work equation. The stiffness force at level *j*, f_{sj}, is equal to the corresponding story stiffness times story drift,

$$f_{sj} = k_j(u_j - u_{j-1}) \tag{5.5}$$

where k_j is the sum of all the lateral stiffness systems (columns, shear walls, bracing, etc.) in the direction in question at story *j*. Substituting Eq. (5.1) evaluated at stories *j* and *j* – 1 into Eq.

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(5.5),

$$f_{sj} = k_j(z(t) \cdot \psi_j - z(t) \cdot \psi_{j-1}) \tag{5.6}$$

We can factor out $z(t)$ and replace the shape vector values with the change in values between two stories ($\psi_j = \psi_j - \psi_{j-1}$),

$$f_{sj} = k_j \cdot z(t) \cdot \Delta\psi_j \tag{5.7}$$

The virtual work of this force is

$$\delta W_{\text{stiffenss}\, j} = f_{sj}(\delta u_j - \delta u_{j-1}) \tag{5.8}$$

where δu_j represents the internal virtual displacement of the structure. Substituting the value of f_{sj} from Eq. (5.7) and virtual values of Eq. (5.1) into Eq. (5.8) results in the internal virtual work due to stiffness,

$$\delta W_{\text{stiffenss}\, j} = k_j \cdot z(t) \cdot \Delta\psi_j(\delta z \psi_j - \delta z \psi_{j-1}) \tag{5.9}$$

We can factor out δz and again substitute $\psi_j = \psi_j - \psi_{j-1}$,

$$\delta W_{\text{stiffenss}\, j} = \delta z \cdot z(t) \cdot k_j \cdot \Delta\psi_j^2 \tag{5.10}$$

We can then add the contribution of all N stories to obtain the total internal work from the stiffness forces in each story,

$$\delta W_{\text{stiffness}} = \delta z \cdot z(t) \sum_{j=1}^{N} k_j \cdot \Delta\psi_j^2 \tag{5.11}$$

Similarly, we can determine the contribution of the damping forces to the virtual work equation. The damping force at level j, f_{dj}, is equal to the corresponding story damping times story velocity,

$$f_{dj} = c_j(\dot{u}_j - \dot{u}_{j-1}) \tag{5.12}$$

where c_j is the sum of all the damping systems in the direction in question at story j. Substituting Eq. (5.1) evaluated at stories j and j – 1 into this equation,

$$f_{dj} = c_j\left(\dot{z}(t)\psi_j - \dot{z}(t)\psi_{j-1}\right) \tag{5.13}$$

We can factor out the velocity and again substitute $\psi_j = \psi_j - \psi_{j-1}$,

$$f_{dj} = c_j \dot{z}(t) \cdot \Delta\psi_j \tag{5.14}$$

The virtual work of this force is

$$\delta W_{\text{damping}\,j} = f_{dj}(\delta u_j - \delta u_{j-1}) \tag{5.15}$$

Substituting value of f_{dj} from Eq. (5.14) and virtual values of Eq. (5.1) into Eq. (5.15) results in the internal virtual work due to damping,

$$\delta W_{\text{damping}\,j} = c_j \cdot \dot{z}(t) \cdot \Delta\psi_j (\delta z \psi_j - \delta z \psi_{j-1}) \tag{5.16}$$

We can factor out δz and again substitute $\psi_j = \psi_j - \psi_{j-1}$,

$$\delta W_{\text{damping}\,j} = \delta z \cdot \dot{z}(t) \cdot c_j \cdot \Delta\psi_j^2 \tag{5.17}$$

We can then add the contribution of all N stories to obtain the total internal work from the damping forces in each story,

$$\delta W_{\text{damping}} = \delta z \cdot \dot{z}(t) \sum_{j=1}^{N} c_j \cdot \Delta\psi_j^2 \tag{5.18}$$

Substituting Eqs. (5.12) and (5.18) into Eq. (5.3) we get the total internal virtual work,

$$\delta W_I = \delta z \cdot z(t) \sum_{j=1}^{N} k_j \cdot \Delta\psi_j^2 + \delta z \cdot \dot{z}(t) \sum_{j=1}^{N} c_j \cdot \Delta\psi_j^2 \tag{5.19}$$

Finally, we determine the contribution of the external forces to the virtual work equation. First, the inertial force at level j, f_{Ij}, is equal to the corresponding mass times story acceleration,

$$f_{Ij} = m_j \cdot \ddot{u}_j \tag{5.20}$$

Substituting the second derivative with respect to time of Eq. (5.1) into this equation,

$$f_{Ij} = m_j \left(\ddot{z}(t) \cdot \psi_j\right) \tag{5.21}$$

The external virtual work of the inertial force is

$$\delta W_{\text{inertial}\,j} = -f_{Ij} \delta u_j \tag{5.22}$$

This is negative because the sense of the force is opposite to the direction of the virtual displacement. Substituting the value of f_{Ij} from Eq. (5.21) and virtual values of Eq. (5.1),

$$\delta W_{\text{inertial}\,j} = -m_j \left(\ddot{z}(t) \cdot \psi_j\right) \delta z \cdot \psi_j \tag{5.23}$$

The external virtual work of the applied force at level j, p_j, is

$$\delta W_{\text{applied force}\,j} = p_j \cdot \delta u_j \tag{5.24}$$

Substituting the virtual values of Eq. (5.1),

$$\delta W_{\text{applied force}\,j} = p_j \cdot \delta z \cdot \psi_j \tag{5.25}$$

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We can now sum the contribution of all N stories to obtain the total external work from the inertial and applied forces at each level j,

$$\delta W_E = \delta z \sum_{j=1}^{N} p_j \cdot \psi_j - \delta z \cdot \ddot{z}(t) \sum_{j=1}^{N} m_j \cdot \psi_j^2 \tag{5.26}$$

We can now apply the principle of virtual work [Eq. (5.2)] by setting the internal virtual work, Eq. (5.19) equal to the external virtual work, Eq. (5.26),

$$\delta z \cdot z(t) \sum_{j=1}^{N} k_j \cdot \Delta\psi_j^2 + \delta z \cdot \dot{z}(t) \sum_{j=1}^{N} c_j \cdot \Delta\psi_j^2 = \delta z \sum_{j=1}^{N} p_j \cdot \psi_j - \delta z \cdot \ddot{z}(t) \sum_{j=1}^{N} m_j \cdot \psi_j^2 \tag{5.27}$$

After dividing both sides by the virtual displacement, δz, since it appears in each term, and rearranging the result, we get,

$$\ddot{z}(t) \sum_{j=1}^{N} m_j \cdot \psi_j^2 + \dot{z}(t) \sum_{j=1}^{N} c_j \cdot \Delta\psi_j^2 + z(t) \sum_{j=1}^{N} k_j \cdot \Delta\psi_j^2 = \sum_{j=1}^{N} p_j \cdot \psi_j \tag{5.28}$$

This is the equation of motion of a generalized SDOF system, which can be written in the same form as the equation of motion of a simple SDOF system,

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t) \tag{5.29}$$

where

$\tilde{m} = \sum_{j=1}^{N} m_j \cdot \psi_j^2$ is the generalized mass.

$\tilde{k} = \sum_{i=1}^{N} k_j \cdot (\ \psi_j)^2$ is the generalized stiffness.

$\tilde{c} = \sum_{i=1}^{N} c_j \cdot (\ \psi_j)^2$ is the generalized damping.

$\tilde{p} = \sum_{i=1}^{N} p_j \cdot \psi_j$ is the generalized force.

N in the summation is the number of stories in the building.

$\Delta\psi_j$ is the relative value of the shape function of two consecutive levels, that is, $\psi_j = \psi_j - \psi_{j-1}$.

m_j is the mass at level j.

k_j is the stiffness for the jth story.

c_j is the damping for the jth story.

We can now write Eq. (5.29) into the form presented in Chap. 2,

$$\ddot{z} + 2\zeta\tilde{\omega}_n\dot{z} + \tilde{\omega}_n^2 z = \tilde{p}(t)/\tilde{m} \tag{5.30}$$

where the generalized natural frequency is given as

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$$\tilde{\omega}_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} \tag{5.31}$$

the generalized natural period is given by

$$\tilde{T}_n = 2\pi\sqrt{\frac{\tilde{m}}{\tilde{k}}} \tag{5.32}$$

ζ is an estimate of the damping ratio.

The dynamic response caused by any dynamic force excitation can be determined using the SDOF system analysis discussed in Chap. 3. However, the overall closed-form solution describing the position of the masses generally requires a complete response history analysis, which is only feasible for relatively well-defined, simple forcing function cases, such as harmonic loading given by Eq. (3.20). For purposes of design and analysis, we are usually interested in the maximum response (displacement or acceleration) of the various masses. The maximum response can be obtained from the dynamic magnification factor, R_d, given by Eq. (3.22) in Chap. 3; or we can use the shock spectrum analysis method discussed in Chap. 4, which is similarly based on a dynamic load factor (DLF). Recall that in both cases, DLF is a dimensionless variable, which for the analysis presented in this chapter, DLF = z/z_{st}, where $z_{st} = \tilde{p}_o/\tilde{k}$ is the equivalent static displacement at the top of the building. Thus, the solution to the equation of motion [Eq. (5.29)] based on a dynamic load factor analysis gives the maximum displacement at the top of the building as (we can also substitute R_d for DLF in the following equations),

$$z = \text{DLF} \cdot z_{st} \tag{5.33}$$

These maximum displacements can then be distributed to other floors of the building using the shape vector, ψ_j, [Eq. (5.1)] as shown in Fig. 5.2,

$$u_{jo} = \psi_j \cdot z = \psi_j \cdot \text{DLF} \cdot z_{st} \tag{5.34}$$

where the equivalent static displacement is

$$z_{st} = \frac{\tilde{p}_o}{\tilde{k}} = \frac{\sum_{i=1}^{N} p_{oj} \cdot \psi_j}{\sum_{i=1}^{N} k_j \cdot (\Delta\psi_j)^2} = \frac{\sum_{i=1}^{N} p_j \cdot \psi_j}{(\tilde{\omega})^2 \sum_{j=1}^{N} m_j \cdot \psi_j^2} \tag{5.35}$$

The second equation is the result of $k = \omega_n^2 m$. This ratio is sometimes used to represent the *modal participation factor* and will be discussed in more detail in the next section and in Chap. 7. The displacement can also be written in terms of the acceleration, $\ddot{z}$, as

$$u_{jo} = \psi_j \frac{\ddot{z}}{\tilde{\omega}_n^2} \tag{5.36}$$

Since the acceleration is proportional to the displacement, that is, $\ddot{z} = \tilde{\omega}_n^2 z$.

The equivalent static forces associated with these floor displacements can be obtained using the product of the maximum floor displacement and associated stiffness (which is a more difficult task and only applies to linear elastic cases) or the product of the associated story mass and maximum story acceleration (the inertial force from Newton's second law), that is,

$$f_{jo} = m_j \cdot \psi_j \cdot \ddot{z} = m_j \cdot \psi_j \cdot \tilde{\omega}_n^2 \cdot z = \tilde{\omega}_n^2 \cdot m_j \cdot u_{jo} \tag{5.37}$$

Also, as discussed in Sec. 1.7, with these forces we can conduct a static structural analysis to determine element forces (bending moment, shear force, and axial force) and stresses needed for design of structural elements; no additional dynamic analysis is necessary. Figure 5.4 shows these equivalent static forces along with the base shear and overturning moment. The internal story shear force, $V_{\max\,i}$, and internal story moment, $M_{\max\,i}$, at an arbitrary level i can be obtained by applying static equilibrium to the right-hand side free-body diagram,

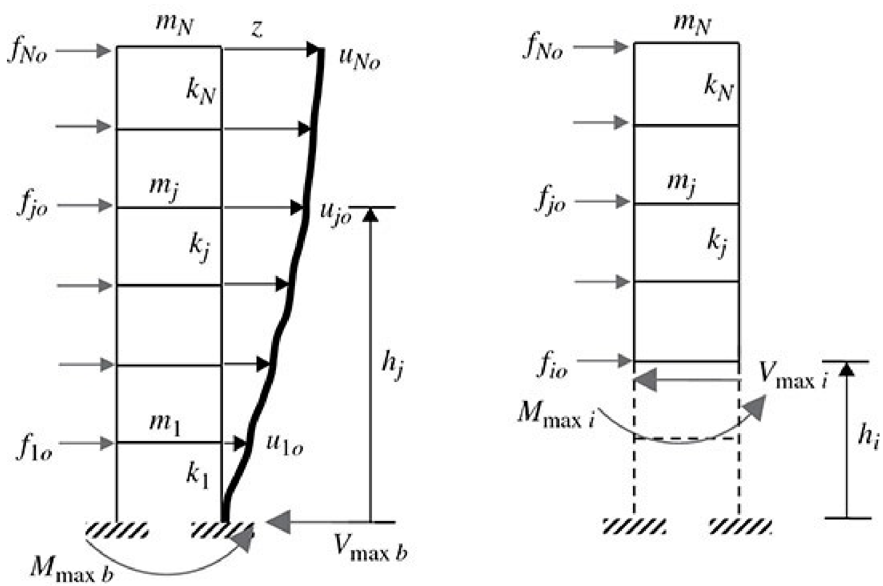

FIGURE 5.4 Maximum dynamic displacements and associated equivalent static forces.

$$V_{\max i} = \sum_{j=i}^{N} f_{jo} \tag{5.38}$$

$$M_{\max i} = \sum_{j=i}^{N} (h_j - h_i) f_{jo} \tag{5.39}$$

Setting i equal to 1 in Eqs. (5.38) and (5.39) results in the base shear force, $V_{\max\,b}$, and overturning moment, $M_{\max\,b}$,

$$V_{\max b} = \sum_{j=1}^{N} f_{jo} \tag{5.40}$$

$$M_{\max b} = \sum_{j=1}^{N} h_j f_{oj} \tag{5.41}$$

In the derivation of Eqs. (5.39) and (5.41) it is assumed that the floor weights are directly in the center of the building frame, thus, making no contribution to the moment equilibrium equation. However, for buildings with eccentric weights, a full moment static equilibrium analysis should be conducted to account for the contribution of these eccentric forces in the overturning moment.

5.1.2 Analysis Summary of Generalized SDOF Systems Forced Vibration Response

The following is a brief step-by-step procedure to estimate the maximum response of a building structure modeled as a generalized SDOF system subjected to general forcing functions at various levels:

1. Determine the mass at each floor in vector form, $\{m\}$; this can be done using the given floor weights.
2. Determine the stiffness at each floor in vector form, $\{k\}$; this can be done using column material and geometric properties.
3. Select an appropriate damping ratio.
4. Determine a vector of all the applied force magnitudes.
5. Obtain the shape vector, ψ_j, from a suitable shape function, $\psi(x)$, that satisfies the geometric boundary conditions of the system.
6. Obtain the relative change in mode shape between floors, $\Delta\psi_j$.
7. Determine the generalized properties; see Eq. (5.29).
8. Determine the generalized natural frequency and period using Eqs. (5.31) and (5.32), respectively.
9. Determine the pulse duration over period.
10. Determine the equivalent static displacement, $\frac{\tilde{p}}{\tilde{k}}$, Eq. (5.35).
11. Determine the dynamic load factor, R_d, in Chap. 3 or DLF in Chap. 4.
12. Determine the maximum displacement at the top of the building, Eq. (5.33).
13. Distribute the generalized displacements to each level, Eq. (5.34).
14. Determine the floor forces, Eq. (5.37).
15. Determine the floor shear and base shear forces, Eqs. (5.38) and (5.40).
16. Determine the floor overturning and base overturning moment using static equilibrium or Eqs. (5.39) and (5.41).

Example 1

Consider the following three-story building frame subjected to air blast loading modeled as shown for each floor of the building. Estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments. Assume a uniform damping ratio of 5%. Also, assume beams are rigid and a shape function appropriate for a mid-rise building. Each story has a stiffness k = 326.3 kip/in.

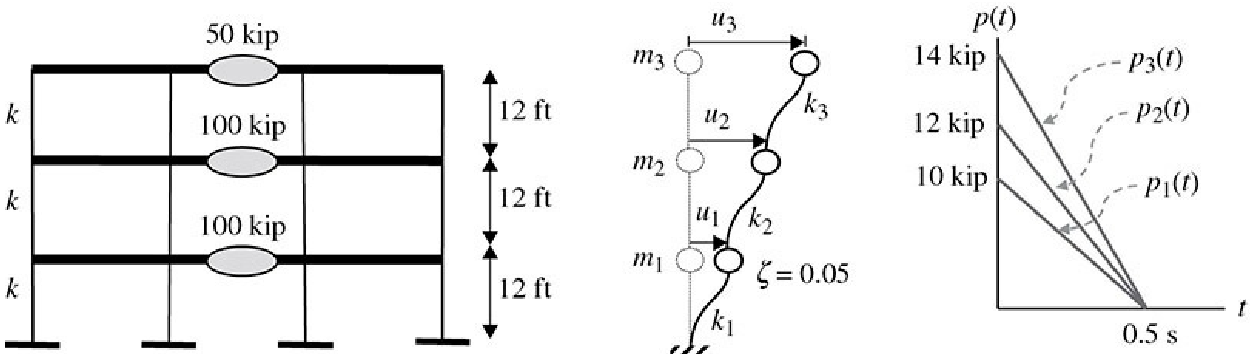

FIGURE E1.1 Building frame schematic and loading.

Solution

i. *Determine the mass and stiffness vectors.*

$$\{m_j\} = \begin{Bmatrix} m \\ m \\ m/2 \end{Bmatrix} = \begin{Bmatrix} 100\text{ kip} \\ 100\text{ kip} \\ 50\text{ kip} \end{Bmatrix} / 386.4\text{ in/s}^2 = \begin{Bmatrix} 0.2588 \\ 0.2588 \\ 0.1294 \end{Bmatrix} \text{kip}\cdot\text{s}^2/\text{in}$$

$$\{k_j\} = \begin{Bmatrix} k \\ k \\ k \end{Bmatrix} = \begin{Bmatrix} 326.3 \\ 326.3 \\ 326.3 \end{Bmatrix} \text{kip/in}$$

ii. *Determine the shape vector, ψ_j.* This can be obtained using Fig. 5.3 for a mid-rise building function, $\psi_j = x/H$, where H = 36 ft.

Level, j	x (ft)	$\psi(x)$	ψ_j
3	36	$36/H$	1
2	24	$24/H$	2/3
1	12	$12/H$	1/3

ψ_j in vector form,

$$\psi_j = \begin{Bmatrix} 1/3 \\ 2/3 \\ 1 \end{Bmatrix}$$

iii. *Determine the generalized properties.* The generalized mass, stiffness, and force of the generalized SDOF system are calculated as follows:

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Level, j	m_j	k_j	ψ_j	$\Delta\psi_j$	$p_{oj}\cdot\psi_j$	$m_j\cdot\psi_j^2$	$k_j\cdot(\Delta\psi_j)^2$
3	$m/2$	k	1	1/3	14 kip·(1)	$m/2\cdot(1)^2$	$k\cdot(1/3)^2$
2	m	k	2/3	1/3	12 kip·(2/3)	$m\cdot(2/3)^2$	$k\cdot(1/3)^2$
1	m	k	1/3	1/3	10 kip·(1/3)	$m\cdot(1/3)^2$	$k\cdot(1/3)^2$
				$\sum_{j=1}^{3}=$	25.33 kip	$\frac{19}{18}m$	$\frac{1}{3}k$

Substituting the given values for mass and stiffness, the following properties of the generalized SDOF are calculated:
Generalized mass,

$$\tilde{m}=\sum_{j=1}^{3}m_j\cdot\psi_j^2=\frac{19}{18}m=\frac{19}{18}\frac{W}{g}=\frac{19}{18}\cdot\frac{100\text{ kip}(1{,}000\text{ lb/kip})}{386.4\text{ in/s}^2}=273.2\,\frac{\text{lb}\cdot\text{s}^2}{\text{in}}$$

Generalized stiffness,

$$\tilde{k}=\sum_{j=1}^{3}k_j\cdot(\Delta\psi_j)^2=\frac{1}{3}k=\frac{1}{3}326.3\,\frac{\text{kip}}{\text{in}}\left(1{,}000\,\frac{\text{lb}}{\text{kip}}\right)=108{,}767\,\frac{\text{lb}}{\text{in}}$$

Generalized force,

$$\tilde{p}=\sum_{j=1}^{3}p_{oj}\cdot\psi_j=25.33\text{ kip}$$

iv. *Determine the generalized natural period, frequency, and equivalent static displacement.*
Natural period,

$$\tilde{T}_n=2\pi\sqrt{\frac{\tilde{m}}{\tilde{k}}}=2\pi\sqrt{\frac{273.2\text{ lb}\cdot\text{s}^2/\text{in}}{108{,}767\text{ lb/in}}}=0.315\text{ s}$$

Ratio of the pulse duration, t_d = 0.5 second to this period:

$$tdT=\frac{t_d}{T_n}=\frac{0.5\text{ s}}{0.315\text{ s}}=1.587$$

Natural frequency,

$$\tilde{\omega}_n=\sqrt{\frac{\tilde{k}}{\tilde{m}}}=\sqrt{\frac{108{,}767\text{ lb/in}}{273.2\text{ lb}\cdot\text{s}^2/\text{in}}}=19.95\,\frac{\text{rad}}{\text{s}}$$

Equivalent static displacement,

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$$z_{st} = \frac{\tilde{p}_o}{\tilde{k}} = \frac{25{,}333 \text{ lb}}{108{,}767 \text{ lb/in}} = 0.233 \text{ in}$$

v. *Determine maximum floor displacements.* Determine the maximum displacement at the top of the building, $z(t)$, using the dynamic load factor (from Duhamel's integral or Chap. 4, Example 12, but with a triangular load and variable damping). Implement the solution using a MATLAB script.

$$z_{\max} = \frac{\tilde{p}_o}{\tilde{k}} \cdot \text{DLF}_{\max}$$

Maximum dynamic load factor from the script:

$$\text{DLF}_{\max} = 1.584$$

Maximum displacement at the top of the building:

$$z_{\max} = \frac{\tilde{p}_o}{\tilde{k}} \cdot \text{DLF}_{\max} = 0.233 \text{ in}(1.584) = 0.369 \text{ in}$$

We can now determine the maximum floor displacements using Eq. (5.34),

$$u_{jo} = \psi_j \cdot z_{\max} = \begin{Bmatrix} 1/3 \\ 2/3 \\ 1 \end{Bmatrix} (0.369 \text{ in}) = \begin{Bmatrix} 0.123 \\ 0.246 \\ 0.369 \end{Bmatrix} \text{in}$$

vi. *Determine the equivalent static story forces.* Determine the equivalent static story forces using Eq. (5.37),

$$f_{jo} = \tilde{\omega}_n^2 \cdot m_j \cdot u_{jo}$$
$$f_{3o} = \tilde{\omega}_n^2 \cdot m_3 \cdot u_{3o} = (19.95 \text{ rad/s})^2 (0.1294 \text{ kip} \cdot \text{s}^2/\text{in})(0.369 \text{ in}) = 19 \text{ kip}$$
$$f_{2o} = \tilde{\omega}_n^2 \cdot m_2 \cdot u_{2o} = (19.95 \text{ rad/s})^2 (0.2588 \text{ kip} \cdot \text{s}^2/\text{in})(0.246 \text{ in}) = 25.3 \text{ kip}$$
$$f_{1o} = \tilde{\omega}_n^2 \cdot m_1 \cdot u_{1o} = (19.95 \text{ rad/s})^2 (0.2588 \text{ kip} \cdot \text{s}^2/\text{in})(0.123 \text{ in}) = 12.7 \text{ kip}$$

vii. *Determine maximum shear forces at each level and the base shear.* Story shear forces based on static equilibrium at each level, or Eq. (5.38):

$$\{V\} = \begin{Bmatrix} 19 \\ 19+25.3 \\ 19+25.3+12.7 \end{Bmatrix} \text{kip} = \begin{Bmatrix} 19 \\ 44.3 \\ 57 \end{Bmatrix} \text{kip}$$

Determine the base shear force using Eq. (5.40), or the shear at the bottom

$$V_{\max b} = \sum_{j=1}^{3} f_{jo} = 19\text{ kip} + 25.3\text{ kip} + 12.7\text{ kip} = 57\text{ kip}$$

viii. *Estimate the maximum overturning moment at each level and the base.* Story overturning moments using static equilibrium at each level, or Eq. (5.39):

$$\{M_1\} = \left\{ \begin{array}{c} 19(12) \\ 19(24) + 25.3(12) \\ 19(36) + 25.3(24) + 12.7(12) \end{array} \right\} \text{kip} \cdot \text{ft} = \left\{ \begin{array}{c} 228 \\ 760 \\ 1,444 \end{array} \right\} \text{kip} \cdot \text{ft}$$

Determine the overturning moment using Eq. (5.41),

$$M_{\max b} = \sum_{j=1}^{3} h_j f_{oj} = 19(36) + 25.3(24) + 12.7(12) = 1,444\text{ kip} \cdot \text{ft}$$

Alternatively, we can draw shear force and bending moment diagrams by treating the building as a cantilever beam in order to obtain the internal story shear forces and moments shown in Fig. E1.2. This diagram also summarizes the results of the analysis.

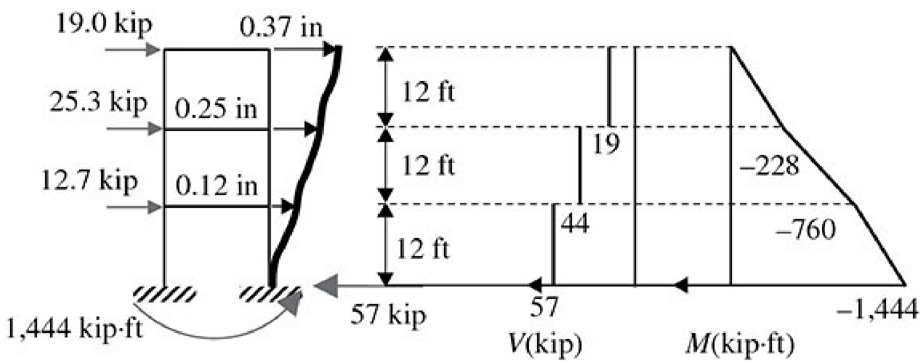

FIGURE E1.2 Lateral forces, internal shear, and internal moment diagram.

ix. *MATLAB script to perform all the operations and results.* The Duhamel's integral algorithm including damping; see Chap. 4, Example 12:

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```
clear all % Chapter 5, Example 1
m = [100;100;50]/386.4; % mass vector in kip-sec2/in
k = [1;1;1]*326.3; % stiffness vector in kip/in
xi = 0.05; % damping ratio
p = [10; 12; 14];% force vector, kip
phi = [1/3;2/3;1]; % assumed mode shape
dphi = [1/3;1/3;1/3]; % delta phi
mt = sum(phi.^2.*m) % generalized mass
kt = sum(dphi.^2.*k)% generalized stiffness
pt = sum(phi.*p)% generalized force
omegan = sqrt(kt/mt) % determine frequency
period = 2*pi./omegan % determine the period from omega, sec
td = 0.5; % pulse duration, sec
tdT = td/period % td/Tn
stat_displ = pt/kt % equivalent static displacement
%%%% Determine the maximum dynamic load factor, DLFmax
n = 500; % first setup the pulse function in t/Tn
tT = linspace(0,5,n);
for i = 1:n
     if tT(i) <= tdT
         p(i) = 1-tT(i)/tdT;
     else
         p(i) = 0;
     end
end
% Use convolution to integrate pulse function to get DLFmax
dt=tT(2)-tT(1);
p=p*dt;
h=exp(-xi*2*pi*tT).*sin(2*pi*sqrt(1-xi^2)*tT);
uust=(2*pi/sqrt(1-xi^2))*conv(p,h);
DLFmax=max(abs(uust)) % Select the max value
% Top floor displacement
z_max = DLFmax*stat_displ
% Distribute z_max to get other floor displacements at each level
ui_max = phi.*z_max
% Use mass times acceleration to get floor forces at each level
f=omegan^2*m.*ui_max(:)
% Add floor forces to get base shear
V=sum(f(:))
% Sum moments at the base to get overturning moment
heights=[12;24;36];
OTM = f'*heights
```

```
z_max = 0.3688
ui_max =
    0.1229
    0.2459
    0.3688
f =
    12.6677
    25.3355
    19.0016
V = 57.0048
OTM = 1.4441e+03
```
▲

5.1.3 Support Excitation Vibration Response of Generalized SDOF Discrete Systems

Again, for support excitation of the shear building shown in Fig. 5.1, we can obtain the response at the top of the building, $z(t)$, and scale it to the other levels using Eq. (5.1). The formulation for obtaining the motion at the top of the building, $z(t)$, can once again be obtained using a generalized equation of motion for the equivalent SDOF system, but one where the base undergoes ground motion as shown in Fig. 5.5. The equation is obtained using D'Alembert's principle and the principle of virtual work. The resulting equation is of the same form as the one that characterizes the vibration of a simple SDOF system; thus, all analysis procedures covered so far are applicable to this case. And similar to the previous case, the most important step is selecting a suitable shape function, $\psi(x)$, which can be obtained from Fig. 5.3.

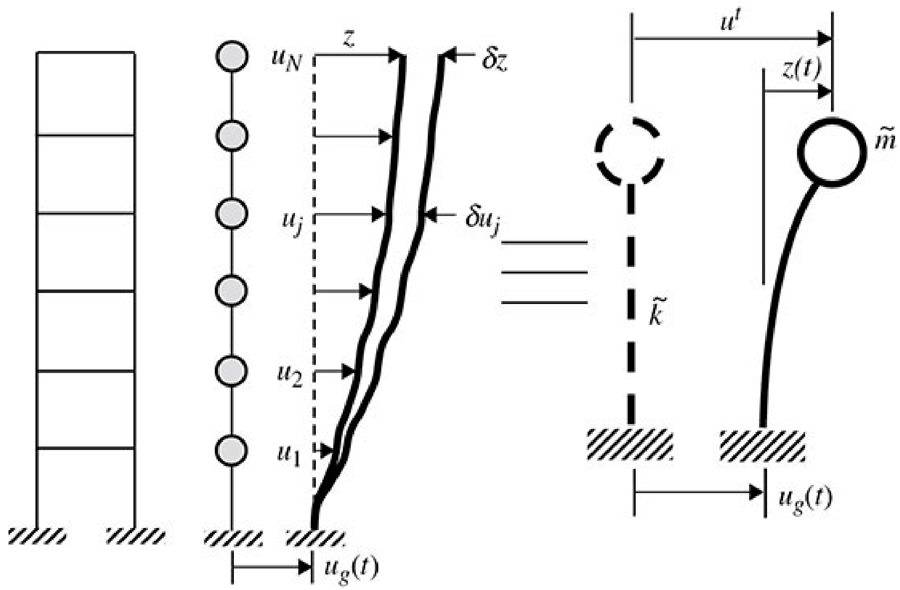

FIGURE 5.5 MDOF building and equivalent generalized SDOF system subjected to support motion.

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To account for the ground excitation loading in the formulation of the equation of motion, we also need the total displacement at level *j*, which is a combination of the relative displacement, $u_j(t)$, and the ground level lateral displacement, $u_g(t)$,

$$u_j^t(t) = u_j(t) + u_g(t) \tag{5.42}$$

Again, we use the principle of virtual work to derive the generalized equation of motion, Eqs. (5.2) to (5.4). Both internal and external virtual work are given by the summation of the product of the magnitude of the force and the corresponding virtual displacement. Again, the internal forces include the stiffness and damping forces in each story and the external forces include inertial lateral forces applied at each level. The stiffness forces are functions of *story drift* (relative displacement from one floor to the next) and their virtual work is given by Eq. (5.11). The damping forces are functions of relative velocity from one floor to the next and their virtual work is given by Eq. (5.18). The inertial forces are functions of the total acceleration at each level and their virtual work is different than for a forced excitation.

The inertial force at level *j*, f_{Ij}, is equal to the corresponding mass times story total acceleration,

$$f_{Ij} = m_j \ddot{u}_j^t \tag{5.43}$$

Substituting the second derivative with respect to time of Eq. (5.42) into this equation,

$$f_{Ij} = m_j \left(\ddot{u}_g + \ddot{u}_j\right) \tag{5.44}$$

Now taking the second derivative with respect to time of Eq. (5.1) evaluated at story *j* and substituting into Eq. (5.44),

$$f_{Ij} = m_j \left(\ddot{u}_g + \ddot{z}(t)\psi_j\right) \tag{5.45}$$

The external virtual work of the inertial force is

$$\delta W_{\text{inertial}\, j} = -f_{Ij}\delta u_j \tag{5.46}$$

This is negative because the sense of the force is opposite the direction of the virtual displacement. Substituting the value of f_{Ij}, from Eq. (5.45) and virtual values of Eq. (5.1),

$$\delta W_{\text{inertial}\, j} = -m_j \left(\ddot{u}_g + \ddot{z}(t)\psi_j\right)\delta z \psi_j \tag{5.47}$$

We can now sum the contribution of all *N* stories to obtain the total external work from the inertial forces at each level *j*,

$$\delta W_E = -\delta z \cdot \ddot{u}_g \sum_{j=1}^{N} m_j \psi_j - \delta z \cdot \ddot{z}(t) \sum_{j=1}^{N} m_j \psi_j^2 \tag{5.48}$$

We can now apply the principle of virtual work [Eq. (5.2)] by setting the internal virtual work, Eq. (5.19) equal to the external virtual work, Eq. (5.48),

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$$\delta z \cdot z(t)\sum_{j=1}^{N} k_j \cdot \Delta\psi_j^2 + \delta z \cdot \dot{z}(t)\sum_{j=1}^{N} c_j \cdot \Delta\psi_j^2 = -\delta z \cdot \ddot{u}_g \sum_{j=1}^{N} m_j \psi_j - \delta z \cdot \ddot{z}(t)\sum_{j=1}^{N} m_j \psi_j^2 \tag{5.49}$$

After eliminating the virtual displacement, δz, since it appears in each term, and rearranging this equation, we get,

$$z(t)\sum_{j=1}^{N} k_j \cdot \Delta\psi_j^2 + \dot{z}(t)\sum_{j=1}^{N} c_j \cdot \Delta\psi_j^2 + \ddot{z}(t)\sum_{j=1}^{N} m_j \psi_j^2 = -\ddot{u}_g \sum_{j=1}^{N} m_j \psi_j \tag{5.50}$$

This is the equation of motion of a generalized SDOF system, which can be written in the same form as the equation of motion of a simple SDOF system,

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = -\tilde{L}\ddot{u}_g(t) \tag{5.51}$$

where

$\tilde{L} = \sum_{i=1}^{N} m_j \cdot \psi_j$ is the generalized force.

$\tilde{m} = \sum_{j=1}^{N} m_j \cdot \psi_j^2$ is the generalized mass.

$\tilde{k} = \sum_{i=1}^{N} k_j \cdot (\Delta\psi_j)^2$ is the generalized stiffness.

$\tilde{c} = \sum_{i=1}^{N} c_j \cdot (\Delta\psi_j)^2$ is the generalized damping.

N in the summation is the number of stories in the building.

$\Delta\psi_j$ is the relative value of the shape function of two consecutive levels, that is, $\Delta\psi_j = \psi_j - \psi_{j-1}$

m_j is the mass at level j.

k_j is the stiffness for the jth story.

c_j is the damping for the jth story.

Rearranging the generalized SDOF system equation of motion, Eq. (5.51) into the form presented in Chap. 2 we get,

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\tilde{\Gamma}\ddot{u}_g(t) \tag{5.52}$$

where the generalized natural frequency and period are given by Eqs. (5.31) and (5.32), respectively, and ζ is an estimate of the damping ratio. And the generalized participation factor is given as

$$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} \tag{5.53}$$

This is an important parameter, and we will discuss it further in Chap. 7.

The dynamic response caused by any dynamic base excitation can be determined using the SDOF system analysis discussed in Chap. 3 by replacing the original ground acceleration,

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$\ddot{u}_g(t)$, with the product of $\ddot{u}_g(t)$ and the participation factor, $\tilde{\Gamma}$. However, an overall closed-form solution describing the position of the masses generally requires a complete response history analysis, which is only feasible for relatively well-defined, simple base motion function cases, such as harmonic motion. However, for purposes of design and analysis, we are primarily interested in the maximum response (displacement or acceleration) of the various masses. Thus, the solution to Eq. (5.52) can be given in terms of relative displacement or total acceleration using the definition of DLF [Eq. (4.37), repeated here for convenience]:

$$\mathrm{DLF} = \frac{u_o}{u_{st}} = \frac{\tilde{\omega}_n^2 u_o}{\ddot{u}_{go}} = \frac{\ddot{u}_o^t}{\ddot{u}_{go}} \tag{4.37}$$

The relative modal displacement and acceleration are

$$z_o = \tilde{\Gamma} \cdot \frac{\ddot{u}_{go}}{\tilde{\omega}_n^2} \cdot \mathrm{DLF} \tag{5.54}$$

$$\ddot{z}_o^t = \tilde{\Gamma} \cdot \mathrm{DLF} \cdot \ddot{u}_{go} \tag{5.55}$$

where z_o represents the maximum displacement at the top of the building (relative modal displacement) as shown in Fig. 5.5. Or in the case of a response spectrum analysis, z_o can be expressed in term of maximum (spectral) displacement, D (the ordinates of the response spectrum corresponding to the period $\tilde{T}_n$ and an appropriate damping ratio),

$$z_o = \tilde{\Gamma} \cdot D \tag{5.56}$$

These maximum displacements can be distributed to the other floors using the shape vector, ψ_j, [Eq. (5.1)] as shown in Fig. 5.5,

$$u_{jo} = \psi_j \cdot z_o = \psi_j \cdot \tilde{\Gamma} \cdot \frac{\ddot{u}_{go}}{\tilde{\omega}_n^2} \cdot \mathrm{DLF} = \psi_j \cdot \tilde{\Gamma} \cdot D \tag{5.57}$$

This can also be written in terms of the spectral acceleration, A, as

$$u_{jo} = \psi_j \cdot \tilde{\Gamma} \cdot \frac{\ddot{u}_{go}}{\tilde{\omega}_n^2} \cdot \mathrm{DLF} = \psi_j \frac{\ddot{z}_o^t}{\tilde{\omega}_n^2} = \psi_j \frac{\tilde{\Gamma}}{\tilde{\omega}_n^2} A \tag{5.58}$$

Since the acceleration is proportional to the displacement, that is, $A = \tilde{\omega}_n^2 D$.

The equivalent static forces associated with these floor displacements can be obtained using the product of the maximum relative floor displacement and associated stiffness, or the product of the associated story mass and maximum total story acceleration. Again, we use the latter, Newton's second law, such that,

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$$
\begin{aligned}
f_{jo} &= m_j \cdot \psi_j \cdot \ddot{z}_o^t = m_j \cdot \psi_j \cdot \Gamma \cdot \mathrm{DLF} \cdot \ddot{u}_{go} = \tilde{\Gamma} \cdot m_j \cdot \psi_j \cdot A \\
&= \tilde{\Gamma} \cdot m_j \cdot \psi_j \cdot \tilde{\omega}_n^2 \cdot D = m_j \cdot \tilde{\omega}_n^2 \cdot u_{oj}
\end{aligned} \tag{5.59}
$$

Also, as discussed in Secs. 1.7 and Sec. 5.1.2, with these forces we can conduct a static structural analysis to determine element forces (bending moment, shear force, and axial force) and stresses needed to design the structural elements; no additional dynamic analysis is necessary. Figure 5.4 shows these equivalent static forces along with the base shear and overturning moment. The internal story shear force, $V_{\max\ i}$, and internal story moment, $M_{\max\ i}$, at an arbitrary level *i* can be obtained by Eqs. (5.38) and (5.39), respectively. Or we can obtain the base shear force, $V_{\max\ b}$, and overturning moment, $M_{\max\ b}$, using Eqs. (5.40) and (5.41), respectively.

5.1.4 Analysis Summary of Support Excitation Vibration Response of Generalized SDOF Systems

The following is a brief step-by-step procedure to estimate the maximum response of a building structure modeled as a generalized SDOF system subjected to support excitation:

1. Determine the mass at each floor in vector form, $\{m\}$; this can be done using the given floor weights.
2. Determine the stiffness at each floor in vector form, $\{k\}$; this can be done using column material and geometric properties.
3. Select an appropriate damping ratio.
4. Obtain the shape vector, ψ_j, from a suitable shape function, $\psi(x)$, that satisfies the geometric boundary conditions of the system.
5. Obtain the relative change in mode shape between floors, $\Delta\psi_j$.
6. Determine the generalized properties; see Eq. (5.51).
7. Determine the generalized natural frequency and period using Eqs. (5.31) and (5.32), respectively.
8. Determine the generalized participation factor using Eq. (5.53).
9. Determine the dynamic load factor, DLF, from the appropriate response spectrum.
10. Determine the maximum displacement at the top of the building using Eq. (5.54) or (5.56).
11. Distribute the generalized displacements to each level using Eq. (5.57) or (5.58).
12. Determine the floor forces, Eq. (5.59).
13. Determine the base shear forces, Eqs. (5.38) and (5.40).
14. Determine the overturning moment using static equilibrium or Eqs. (5.39) and (5.41).

Example 2

Consider the three-story building frame in Example 1 (see Fig. E1.1) subjected to a ground acceleration due to the 2021 Haiti earthquake ground acceleration (presented in Chap. 4, Example 16). Determine (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments. Assume uniform damping of 2%, beams are rigid, each story has a stiffness k = 326.3 kip/in, and a shape function appropriate for a mid-rise building.

Solution From Example 1, we have the following parameters:

Shape vector, ψ_j, and generalized mass, stiffness, and force properties:

Level, j	m_j	k_j	ψ_j	$\Delta\psi_j$	$m_j \cdot \psi_j$	$m_j \cdot \psi_j^2$	$k_j \cdot (\Delta\psi_j)^2$
3	$m/2$	k	1	1/3	$m/2 \cdot (1)$	$m/2 \cdot (1)^2$	$k \cdot (1/3)^2$
2	m	k	2/3	1/3	$m \cdot (2/3)$	$m \cdot (2/3)^2$	$k \cdot (1/3)^2$
1	m	k	1/3	1/3	$m \cdot (1/3)$	$m \cdot (1/3)^2$	$k \cdot (1/3)^2$
$\sum_{j=1}^{3} =$					$1.5m$	$\frac{19}{18}m$	$\frac{1}{3}k$

Substituting the given values for mass and stiffness, the following properties of the generalized SDOF are calculated:

Generalized mass,

$$\tilde{m} = \sum_{j=1}^{3} m_j \cdot \psi_j^2 = \frac{19}{18}m = \frac{19}{18}\frac{W}{g} = \frac{19}{18}\frac{100{,}000 \text{ lb}}{386.4 \text{ in/s}^2} = 273.2 \frac{\text{lb} \cdot \text{s}^2}{\text{in}}$$

Generalized stiffness,

$$\tilde{k} = \sum_{j=1}^{3} k_j \cdot (\Delta\psi_j)^2 = \frac{1}{3}k = \frac{1}{3} 326.3 \frac{\text{kip}}{\text{in}}\left(1{,}000 \frac{\text{lb}}{\text{kip}}\right) = 108{,}767 \frac{\text{lb}}{\text{in}}$$

Natural period,

$$\tilde{T}_n = 2\pi\sqrt{\frac{\tilde{m}}{\tilde{k}}} = 2\pi\sqrt{\frac{273.2 \text{ lb} \cdot \text{s}^2/\text{in}}{108{,}767 \text{ lb/in}}} = 0.315 \text{ s}$$

Natural frequency,

$$\tilde{\omega}_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{108{,}767 \text{ lb/in}}{273.2 \text{ lb} \cdot \text{s}^2/\text{in}}} = 19.95 \frac{\text{rad}}{\text{s}}$$

i. *Determine the generalized force, which is different than the one used in Example 1, and participation factor of the generalized SDOF system.*

Generalized force,

$$\tilde{L} = \sum_{j=1}^{3} m_j \cdot \psi_j = 1.5m = 1.5\frac{W}{g} = 1.5\frac{100{,}000 \text{ lb}}{386.4 \text{ in/s}^2} = 388.2 \frac{\text{lb} \cdot \text{s}^2}{\text{in}}$$

Participation factor,

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$$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = \frac{388.2\ \mathrm{lb \cdot s^2/in}}{273.2\ \mathrm{lb \cdot s^2/in}} = 1.421$$

ii. *Determine maximum floor displacements.* First, determine the maximum displacement at the top of the building using Eq. (5.56), $z_o = \tilde{\Gamma} \cdot D$; where D is obtained from the response spectra in Chap. 4, Fig. E16.4 or the MATLAB code; then use Eq. (5.57) to determine the floor displacements at each level.

$$\tilde{T}_n = 0.315\ \mathrm{s}$$

Spectral displacement for period, (obtained with MATLAB code):

$$D = 0.0945\ \mathrm{in}$$

Maximum displacement response at the top of the building:

$$z_o = \tilde{\Gamma} \cdot D = (1.421)(0.0945\ \mathrm{in}) = 0.134\ \mathrm{in}$$

We can now determine the maximum floor displacements using Eq. (5.57),

$$u_{jo} = \psi_j z_o = \begin{Bmatrix} 1/3 \\ 2/3 \\ 1 \end{Bmatrix} (0.134\ \mathrm{in}) = \begin{Bmatrix} 0.0448 \\ 0.0895 \\ 0.134 \end{Bmatrix} \mathrm{in}$$

iii. *Determine the equivalent static story forces.* Determine the equivalent static story forces using Eq. (5.59),

$$\begin{aligned}
f_{jo} &= m_j \cdot \tilde{\omega}_n^2 \cdot u_{oj} \\
f_{3o} &= m_j \cdot \tilde{\omega}_n^2 \cdot u_{oj} = (50\ \mathrm{kip}/386.4\ \mathrm{in/s^2})(19.95\ \mathrm{rad/s})^2(0.134\ \mathrm{in}) = 6.92\ \mathrm{kip} \\
f_{2o} &= m_j \cdot \tilde{\omega}_n^2 \cdot u_{oj} = (100\ \mathrm{kip}/386.4\ \mathrm{in/s^2})(19.95\ \mathrm{rad/s})^2(0.0895\ \mathrm{in}) = 9.22\ \mathrm{kip} \\
f_{1o} &= m_j \cdot \tilde{\omega}_n^2 \cdot u_{oj} = (100\ \mathrm{kip}/386.4\ \mathrm{in/s^2})(19.95\ \mathrm{rad/s})^2(0.0448\ \mathrm{in}) = 4.61\ \mathrm{kip}
\end{aligned}$$

iv. *Determine base shear and overturning moment.* Determine the base shear force using Eq. (5.40),

$$V_{\max b} = \sum_{j=1}^{3} f_{jo} = 4.61\ \mathrm{kip} + 9.22\ \mathrm{kip} + 6.92\ \mathrm{kip} = 20.8\ \mathrm{kip}$$

Determine the overturning moment using Eq. (5.41),

$$M_{\max b} = \sum_{j=1}^{3} h_j f_{oj} = 12\ \mathrm{ft} \cdot 4.6\ \mathrm{kip} + 24\ \mathrm{ft} \cdot 9.2\ \mathrm{kip} + 36\ \mathrm{ft} \cdot 6.9\ \mathrm{kip} = 526\ \mathrm{kip \cdot ft}$$

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We can also obtain the internal story shear forces and moments at each level using Eqs. (5.38) and (5.39), respectively. Alternatively, we can draw shear force and bending moment diagrams by treating the building as a cantilever beam in order to obtain the internal story shear forces and moments shown in Fig. E2.1. This diagram also summarizes the results of the analysis.

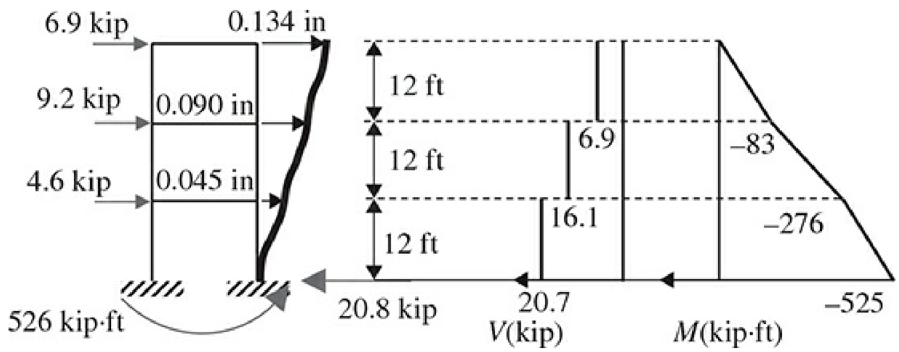

FIGURE E2.1 Lateral forces, internal shear, and internal moment diagram.

v. *We can write a MATLAB script to perform all the operation as follows.*

```
clear all % Chapter 5, Example 2
m = [100;100;50]/386.4; % mass vector in kip-sec2/in
k = [1;1;1]*326.3; % stiffness vector in kip/in
dam = 0.02; % damping ratio
phi = [1/3;2/3;1]; % assumed mode shape
dphi = [1/3;1/3;1/3]; % delta phi
mt = sum(phi.^2.*m) % generalized mass
kt = sum(dphi.^2.*k)% generalized stiffness
lt= sum(phi.*m); % generalized force
par_fac = lt./mt % participation factor
```


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```
wn = sqrt(kt/mt) % determine frequency
period = 2*pi/wn % determine the period from omega, sec
load ('Haiti2021.mat', '-ascii'); % load Haiti 2021 data
N = length(Haiti2021); % number of points in ground acceleration
file
acc = Haiti2021/980.8; % ground acceleration data as a function of g
DT = 0.005; % sampling rate
for i=N:-1:1
  t(i)=i*DT;
end
p=acc*DT;
h=exp(-dam*wn*t).*sin(wn*t);
u=conv(p,h);
Dmax=max(abs(u))/wn*386.4 % maximum displacement
% Top floor displacement
z_max = par_fac*Dmax
% Distribute z_max to get other floor displacements at each level
ui_max = phi.*z_max
% Use mass times acceleration to get floor forces at each level
f=wn^2*m.*ui_max(:)
% Add floor forces to get base shear
V=sum(f(:))
% Sum moments at the base to get overturning moment
heights=[12;24;36];
OTM = f'*heights
```

The results of the script are:

mt = 0.2732

kt = 108.7667

par_fac = 1.4211

wn = 19.9538

period = 0.3149

Dmax = 0.0945

z_max = 0.1343

ui_max =

    0.0448
    0.0895
    0.1343

f =

    4.6124
    9.2248
    6.9186

V = 20.7559

OTM = 525.8156 ▲

5.2 Continuous Systems Analysis

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The equation of motion for beam systems with distributed mass, $m(x)$, and distributed flexural stiffness, $EI(x)$, can be formulated using the cantilever tower shown in Fig. 5.6. The deformed shape of the system along with the virtual deformation used in the formulation of the equation of motion are depicted in the figure. In this case, the deformation of the tower centerline along its length is $u(x,t)$, which can be decoupled into two functions: one in terms of position, $\psi(x)$, and the other in terms of time, $z(t)$; the latter representing the dynamic displacement at the top of the tower. Also shown in the figure is an equivalent generalized SDOF system.

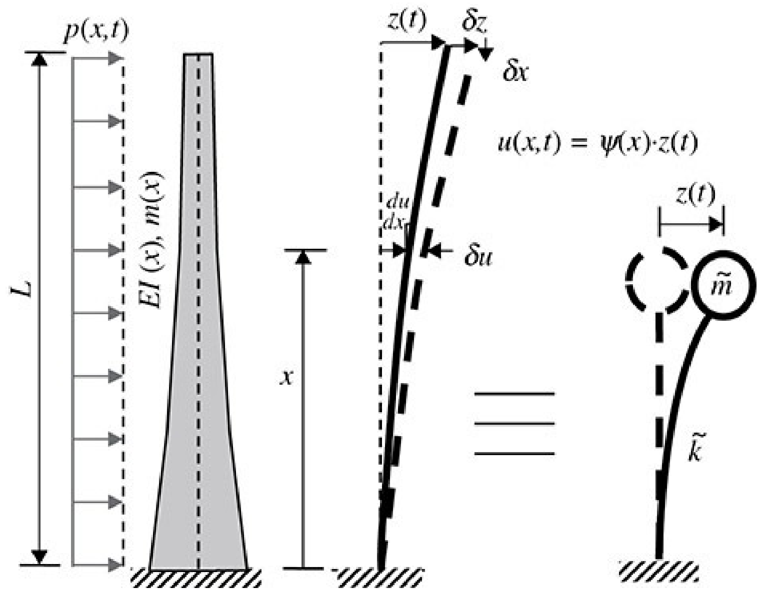

FIGURE 5.6 Tower with distributed properties and equivalent generalized SDOF system.

5.2.1 Forced Vibration Response of Generalized SDOF Continuous Systems

The equation of motion for a generalized SDOF continuous system subjected to a time-dependent input force excitation can be formulated in terms $z(t)$ only, provided we can find an appropriate shape of the deformed system as it vibrates, $\psi(x)$. The generalized equation of motion in terms of $z(t)$ can again be derived using the principle of virtual work as described in the last section; the resulting second-order ordinary differential equation is of the same form as Eq. (5.29) (rewritten below for convenience), except that the generalized properties are now computed using an integral form:

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t)$$

In the continuous system, integration is used to determine the generalized properties as follows:

- $\tilde{m} = \int_0^L m(x)[\psi(x)]^2\,dx$ is the generalized mass.
- $\tilde{k} = \int_0^L EI(x)[\psi\;(x)]^2\,dx$ is the generalized stiffness.
- $\tilde{p} = \int_0^L p(x,t)\psi(x)dx$ is the generalized force.
- $\tilde{c}$ is the generalized damping, which is not explicitly determined; it is estimated using the

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damping ratio.

L is the span or in the case of Fig. 5.6, tower height.

$m(x)$ is the mass per unit length or height at an arbitrary length or height x.

$EI(x)$ is the flexural stiffness per unit height at an arbitrary height x.

$\psi''(x) = d^2\psi/dx^2$, the second derivative of $\psi(x)$ with respect to x.

$p(x)$ is the force per unit height at an arbitrary height x.

We can rearrange this equation and obtain Eq. (5.30) (rewritten here for convenience),

$$\ddot{z} + 2\zeta\tilde{\omega}_n\dot{z} + \tilde{\omega}_n^2 z = \tilde{p}(t)/\tilde{m}$$

where the frequency and period are given by Eqs. (5.31) and (5.32), respectively, and again, ζ is an estimate of the damping ratio.

Again, this generalized equation of motion is the same as the one that characterizes the vibration of the SDOF oscillator introduced in Chap. 3; thus, all analysis procedures described previously can be used to determine the response $z(t)$. With $z(t)$ and the shape function $\psi(x)$, we can determine a function of the shape into which the tower vibrates, the displacement at an arbitrary length or height x,

$$u(x,t) = \psi(x)z(t) \tag{5.60}$$

The accuracy of the results in this case depends on how close the assumed shape function tracks the actual shape of the deformed system. A satisfactory approximation for the shape function, $\psi(x)$, must satisfy the geometric boundary conditions of the system.

The remainder of this section is focused on ***cantilever beams***, which can be used to model towers. The shape function $\psi(x)$ for these types of systems must include zero ground lateral displacement and zero rotation at the fixed support; that is, $u(0) = 0$ and $u'(0) = 0$, where the prime notation indicates differentiation with respect to x, $u'(x) = du/dx$. Also, since z is only a function of t, these boundary conditions reduce to $\psi(0) = 0$ and $\psi'(0) = 0$. Furthermore, the function $\psi(x)$ must be normalized at the top of the tower such that $\psi(L) = 1$, where L is the height of the tower.

This analysis can be used to determine internal stress resultants in the tower, which can then be used to determine the stresses. The analysis procedure can be used with a variety of cantilever systems subjected to different time-dependent loads. As in the shear building, here we assume a tower subjected to a time-dependent force first and then a ground lateral displacement of $u_g(t)$. For a time-dependent force, we can obtain the maximum response from the dynamic magnification factor, R_d, given by Eq. (3.22); or we can use the shock spectrum analysis method discussed in Chap. 4, which is similarly based on a DLF. Recall that in both cases DLF = z/z_{st}, where $z_{st} = p_o/\tilde{k}$ is the equivalent static displacement at the top of the tower. Thus, Eq. (5.33) gives the solution to the equation of motion [Eq. (5.29)] based on a dynamic load factor analysis (repeated here for convenience),

$$z = \mathrm{DLF} \cdot z_{st}$$

These maximum dynamic displacements along the entire tower can be obtained using Eq.

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(5.60):

$$u_o(x) = z \cdot \psi(x) = \psi(x) \cdot \text{DLF} \cdot z_{st} \tag{5.61}$$

where, the equivalent static displacement is

$$z_{st} = \frac{p_o}{\tilde{k}} = \frac{p_o}{\int_0^L EI(x)[\psi''(x)]^2 dx} = \frac{p_o}{(\tilde{\omega})^2 \int_0^L m(x)[\psi(x)]^2 dx} \tag{5.62}$$

The second equation is the result of $k = \omega_n^2 m$. The displacement can also be written in terms of the acceleration, $\ddot{z}$, as

$$u_o(x) = \frac{\ddot{z}}{\omega_n^2} \cdot \psi(x) \tag{5.63}$$

Since the acceleration is proportional to the displacement, that is, $\ddot{z} = \tilde{\omega}_n^2 z$.

The equivalent static distributed force associated with this displacement function can be obtained using the product of the maximum displacement and associated flexural stiffness, or the product of the mass and maximum total acceleration, that is,

$$f_o(x,t) = m(x) \cdot \psi(x) \cdot \ddot{z} = m(x) \cdot \psi(x) \cdot \tilde{\omega}_n^2 \cdot z = \tilde{\omega}_n^2 \cdot m(x) \cdot u_o(x) \tag{5.64}$$

Also, as discussed in Sec. 1.7, with these forces we can conduct a static structural analysis to determine element forces (bending moment, shear force, and axial force) and stresses needed for design of structural elements; no additional dynamic analysis is necessary. The internal shear force, $V_{\max}$, and internal story moment, $M_{\max}$, at an arbitrary height, x, can be obtained by applying static equilibrium,

$$f_o(x,t) = m(x) \cdot \psi(x) \cdot \ddot{z} = m(x) \cdot \psi(x) \cdot \tilde{\omega}_n^2 \cdot z = \tilde{\omega}_n^2 \cdot m(x) \cdot u_o(x) \tag{5.64}$$

$$V_{\max}(x) = \int_x^L f_o(\xi) d\xi \tag{5.65}$$

where ξ ranges from x to L.

Setting x equal to 0 in Eqs. (5.65) and (5.66) gives the base shear force, $V_{\max\ b}$, and overturning moment, $M_{\max\ b}$, respectively:

$$M_{\max}(x) = \int_x^L (\xi - x) f_o(\xi) d\xi \tag{5.66}$$

$$V_{\max b} = \int_0^L f_o(x) dx \tag{5.67}$$

Example 3

Assuming that Burn's tower (a 120 ft tall cantilever tower weighing 9 kip/ft) on the campus of

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University of the Pacific has a flexural rigidity, $EI = 4 \times 10^9$ kip · ft^2, determine the equation of motion and displacement time history of the entire tower for the wind gust impulse load shown, which is assumed to act uniformly along the entire height of the tower. Also, assume negligible damping and a shape function of $\psi(x) = 1 - \cos(\pi x/240 \text{ ft})$.

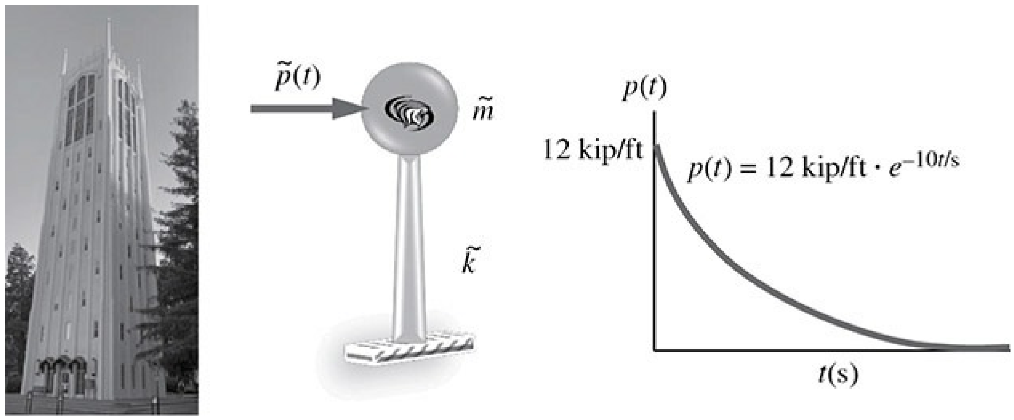

FIGURE E3.1 Burn's tower generalized SDOF system and exponential impulse loading function.

Solution

i. *Determine the mass and stiffness properties of the tower.* The properties per unit length at an arbitrary height, x, are:
Mass at x per unit foot of tower,

$$m(x) = \text{weight}(x)/g = 9 \text{ kip/ft}/32.2 \text{ ft/s}^2 = 0.280 \text{ (kip} \cdot \text{s}^2\text{/ft)/ft}$$

Flexural stiffness at x,

$$EI(x) = 4 \times 10^9 \text{ kip} \cdot \text{ft}^2$$

ii. *Determine the generalized properties.* The generalized mass, stiffness, and force of the generalized SDOF system are calculated as follows:
Generalized mass,

$$\tilde{m} = \int_0^L m(x)[\psi(x)]^2 dx = m\int_0^L \left[1 - \cos\left(\frac{\pi x}{2L}\right)\right]^2 dx = 0.227\, mL$$
$$= 0.227(0.280 \text{ (kip} \cdot \text{s}^2/\text{ft)/ft})(120 \text{ ft}) = 7.61 \text{ kip} \cdot \text{s}^2/\text{ft}$$

Generalized stiffness,

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$$\psi''(x) = \frac{d^2\psi(x)}{dx^2} = \left(\frac{\pi}{4L^2}\right)^2 \cos^2\left(\frac{\pi x}{2L}\right)$$

$$\tilde{k} = \int_0^L EI(x)[\psi''(x)]^2 dx = EI\int_0^L \left[\left(\frac{\pi}{4L^2}\right)^2 \cos^2\left(\frac{\pi x}{2L}\right)\right]^2 dx = \frac{3.04EI}{L^3}$$
$$= 3.04(4\times 10^9 \text{ kip}\cdot\text{ft}^2)/(120 \text{ ft})^3 = 7{,}037 \text{ kip/ft}$$

Generalized force,

$$\tilde{p} = \int_0^L p(x,t)\psi(x)dx = p(t)\int_0^L (1)\left[1-\cos\left(\frac{\pi x}{2L}\right)\right]dx = 0.363Lp_o(t) = 523 \text{ kip}\cdot e^{-10t/\text{s}}$$

iii. *Determine the period, frequency, and equation of motion of the generalized SDOF system.*
Natural period,

$$\tilde{T}_n = 2\pi\sqrt{\frac{\tilde{m}}{\tilde{k}}} = 2\pi\sqrt{\frac{7.61 \text{ kip}\cdot\text{s}^2/\text{ft}}{7{,}037 \text{ kip/ft}}} = 0.207 \text{ s}$$

Natural frequency,

$$\tilde{\omega}_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{7{,}037 \text{ kip/ft}}{7.61 \text{ kip}\cdot\text{s}^2/\text{ft}}} = 30.4\frac{\text{rad}}{\text{s}}$$

The equation of motion is written below [Eq. (5.29)],

$$7.61 \text{ kip}\cdot\text{s}^2/\text{ft}\cdot\ddot{z} + 7{,}037 \text{ kip/ft}\cdot z = 523 \text{ kip}\cdot e^{-10t/\text{s}}$$

iv. *Determine displacement time history of the tower.* The displacement, $z(t)$, can be obtained using Duhamel's integral [use Eq. (4.18) assuming the systems starts from rest]:

$$z(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)[\sin\omega_n(t-\tau)]d\tau$$

Or,

$$z(t) = \frac{523 \text{ kip}}{\tilde{m}\tilde{\omega}_n}\int_0^t e^{-10\tau/\text{s}}[\sin\tilde{\omega}_n(t-\tau)]d\tau$$

This can be expanded using the trigonometric identity $\sin\tilde{\omega}_n(t-\tau) = \sin\tilde{\omega}_n t\cdot \cos\tilde{\omega}_n\tau - \cos\tilde{\omega}_n t\cdot\sin\tilde{\omega}_n\tau$ as follows:

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$$z(t)=\frac{523\text{ kip}}{\tilde{m}\tilde{\omega}_n}\left[\sin\tilde{\omega}_n t\int_0^t e^{-10\tau/s}\cos\tilde{\omega}_n\tau\cdot d\tau-\cos\tilde{\omega}_n t\int_0^t e^{-10\tau/s}\sin\tilde{\omega}_n\tau\cdot d\tau\right]$$

After integrating by parts and rearranging the various terms, the result is given by

$$z(t)=\frac{523\text{ kip}}{\tilde{k}\left[1+\left(\frac{10/\text{s}}{\tilde{\omega}_n}\right)^2\right]}\left(\frac{10/\text{s}}{\tilde{\omega}_n}\sin\tilde{\omega}_n t-\cos\tilde{\omega}_n t+e^{-10t/s}\right)$$

After substituting values for $\tilde{k}$ and $\tilde{\omega}_n$, we get the displacement time history at the top of the tower.

$$z(t)=0.805\text{ in}(0.329\sin 30.4t-\cos 30.4t+e^{-10t/s})$$

The distributed displacement time history can be obtained using Eq. (5.60),

$$u(x,t)=\psi(x)z(t)=0.805\text{ in}(0.329\sin 30.4t-\cos 30.4t+e^{-10t/s})\left[1-\cos\left(\frac{\pi x}{2L}\right)\right]\blacktriangle$$

5.2.2 Support Excitation Vibration Response of Generalized SDOF Continuous Systems

The formulation for a time-dependent ground excitation of a cantilever tower is similar to the one for the shear building under the same conditions. Again, the displacement at the top of the tower, $z(t)$, with a shape function, $\psi(x)$, is used to distribute to the rest of the tower the effect of the time response. Again, to account for the ground excitation loading in the formulation of the equation of motion, we use the total lateral displacement at a height x, which is a combination of the relative displacement, $u(x,t)$, and the ground level lateral displacement, $u_g(t)$; see Fig. 5.7,

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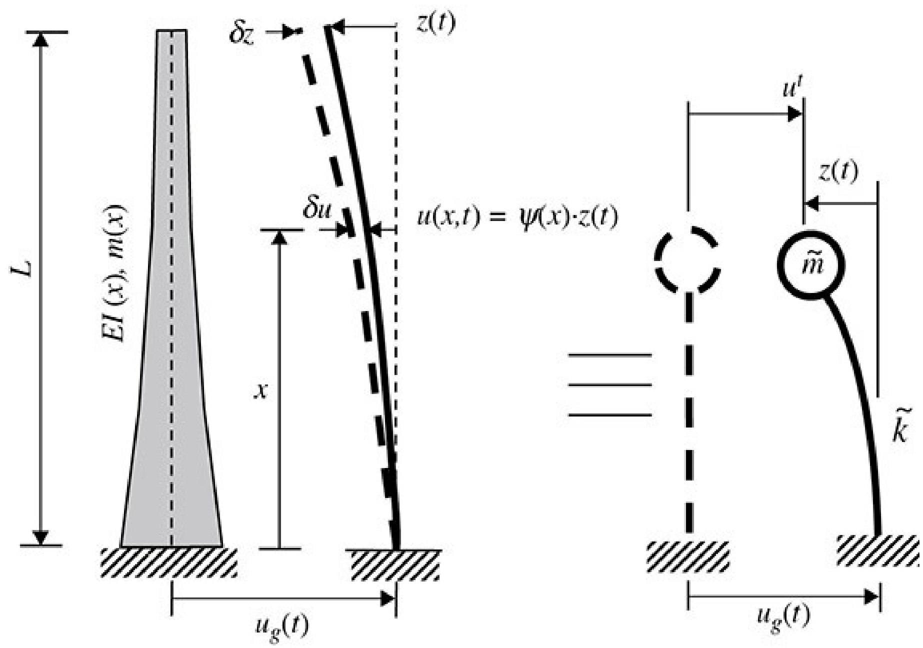

FIGURE 5.7 Tower with distributed properties and equivalent generalized SDOF system.

$$u^t(x,t) = u(x,t) + u_g(t) \tag{5.69}$$

The resulting second-order ordinary differential equation in time for $z(t)$ is of the same form as Eq. (5.51) for shear buildings subjected to ground excitations (rewritten below for convenience), except that the generalized properties are now computed using an integral form,

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = -\tilde{L}\ddot{u}_g(t)$$

where

$\tilde{L} = \int_0^L m(x)\psi(x)\,dx$ is the generalized force.

$\tilde{m} = \int_0^L m(x)[\psi(x)]^2\,dx$ is the generalized mass.

$\tilde{k} = \int_0^L EI(x)[\psi''(x)]^2\,dx$ is the generalized stiffness.

$\tilde{p} = \int_0^L p(x,t)\psi(x)\,dx$ is the generalized force.

$\tilde{c}$ is the generalized damping, which is not explicitly determined; it is estimated using the damping ratio.

L is the span or in the case of Fig. 5.7, tower height.

$m(x)$ is the mass per unit length or height at an arbitrary length or height x.

$EI(x)$ is the flexural stiffness per unit height at an arbitrary height x.

$\psi\ (x) = d^2\psi/dx^2$, the second derivative of $\psi\ (x)$ with respect to x.

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Recall that we rearrange Eq. (5.51) to obtain Eq. (5.52) (rewritten here for convenience),

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\tilde{\Gamma}\ddot{u}_g(t)$$

where the generalized natural frequency and period are given by Eqs. (5.31) and (5.32), respectively, and again ζ is an estimate of the damping ratio. The generalized participation factor is again given by Eq. (5.53) (rewritten here for convenience),

$$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}}$$

The dynamic response caused by any dynamic base excitation can be determined using the SDOF system analysis discussed in Chap. 3 by replacing the original ground acceleration, $u_g(t)$, with the product of $u_g(t)$ and the participation factor, $\tilde{\Gamma}$. However, an overall closed-form solution tower shape generally requires a complete response history analysis, which is only feasible for relatively well-defined, simple base motion function cases, such as harmonic excitation. For purposes of design and analysis, we are primarily interested in the maximum response (displacement or acceleration). Thus, the solution to Eq. (5.52) can be given in terms of relative displacement or total acceleration using the definition of DLF given by Eq. (4.37), repeated here for convenience:

$$\text{DLF} = \frac{u_o}{u_{st}} = \frac{\tilde{\omega}_n^2 u_o}{\ddot{u}_{go}} = \frac{\ddot{u}_o^t}{\ddot{u}_{go}}$$

The relative modal displacement and acceleration are given by Eqs. (5.54) and (5.55), repeated here for convenience:

$$z_o = \tilde{\Gamma} \cdot \frac{\ddot{u}_{go}}{\tilde{\omega}_n^2} \cdot \text{DLF}$$

$$\ddot{z}_o^t = \tilde{\Gamma} \cdot \text{DLF} \cdot \ddot{u}_{go}$$

where z_o represents the maximum displacements at the top of the tower (relative modal displacement) as shown in Fig. 5.7. Or in the case of response spectrum analysis, z_o can be express in terms of maximum (spectral) displacement, D (the ordinates of the response spectrum corresponding to the period $\tilde{T}_n$ and an appropriate damping ratio), given by Eq. (5.56), and repeated here for convenience,

$$z_o = \tilde{\Gamma} \cdot D \tag{5.56}$$

The maximum dynamic displacement along the entire tower can be obtained using Eq. (5.60),

$$u_o(x) = \psi(x) \cdot z = \psi(x) \cdot \tilde{\Gamma} \cdot \frac{\ddot{u}_{go}}{\tilde{\omega}_n^2} \cdot \text{DLF} = \psi(x) \cdot \tilde{\Gamma} \cdot D \tag{5.70}$$

This can also be written in terms of the total acceleration, $\ddot{z}_o^t$, or spectral acceleration, A, as

$$u_o(x) = \psi(x) \cdot \tilde{\Gamma} \cdot \text{DLF} \cdot \ddot{u}_{go} = \psi(x) \frac{\tilde{\Gamma}}{\tilde{\omega}_n^2} A \tag{5.71}$$

Since the acceleration is proportional to the displacement, that is, $A = \tilde{\omega}_n^2 D$.

The equivalent static distributed force associated with this displacement function can be obtained using the product of the maximum displacement and associated flexural stiffness, or the product of the mass and maximum total acceleration, which is given by Eq. (5.64), repeated here for convenience,

$$f_o(x,t) = m(x) \cdot \psi(x) \cdot \ddot{z} = m(x) \cdot \psi(x) \cdot \tilde{\omega}_n^2 \cdot z = \tilde{\omega}_n^2 \cdot m(x) \cdot u_o(x)$$

Also, as discussed in Sec. 1.7, with these forces we can conduct a static structural analysis to determine element forces (bending moment, shear force, and axial force) and stresses needed for design of structural elements; no additional dynamic analysis is necessary. The internal shear force, $V_{\max}$, and internal story moment, $M_{\max}$, at an arbitrary height, x, can be obtained by applying static equilibrium,

$$V_{\max}(x) = \int_x^L f_o(\xi) d\xi = \tilde{\Gamma} A \int_x^L m(\xi) \psi(\xi) d\xi \tag{5.72}$$

$$M_{\max}(x) = \int_x^L (\xi - x) f_o(\xi) d\xi = \tilde{\Gamma} A \int_x^L (\xi - x) m(\xi) \psi(\xi) d\xi \tag{5.73}$$

where ξ ranges from x to L.

Setting x equal to 0 in Eqs. (5.72) and (5.73) gives the base shear force, $V_{\max\, b}$, and overturning moment, $M_{\max\, b}$, respectively,

$$V_{\max b} = \int_0^L f_o(x) dx = \tilde{\Gamma} A \int_0^L m(x) \psi(x) dx = \tilde{\Gamma} A \tilde{L} \tag{5.74}$$

$$M_{\max b} = \int_0^L x f_o(x) dx = \tilde{\Gamma} A \int_0^L x m(x) \psi(x) dx \tag{5.75}$$

Example 4

Consider the tower given in Fig. E4.1 subjected to a ground acceleration due to the 2021 Haiti earthquake (see Chap. 4, Example 16). Determine (a) peak displacement, (b) maximum base shear, and (c) maximum floor overturning moments. Assume diameter, d = 3 ft; thickness = 4 in; modulus of elasticity, E_c = 3,600 ksi; concrete weight, γ_c = 150 pcf; damping, ζ = 5%; and the shape function shown below.

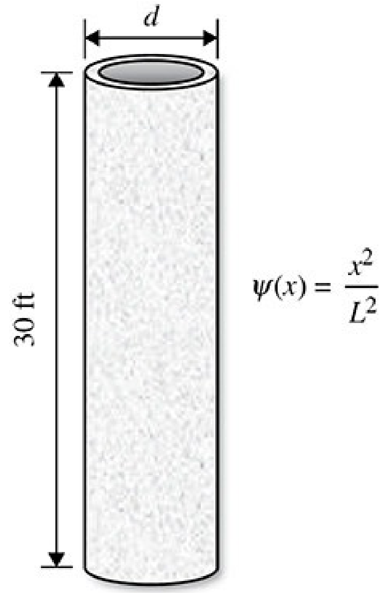

FIGURE E4.1 Schematic of reinforced concrete chimney.

Solution

i. *Determine the properties of the chimney.* The properties per unit length at an arbitrary height, x are:
Cross-sectional area at x,

$$\text{Area}(x) = \pi(R_o)^2 - \pi(R_i)^2 = \pi[(1.5\text{ ft})^2 - (1.167\text{ ft})^2] = 2.79\text{ ft}^2$$

Mass at x per unit foot of tower,

$$m(x) = \text{Area}(x) \cdot \gamma_c/g = 2.79\text{ ft}^2(150\text{ pcf})/32.2\text{ ft/s}^2 = 13.0\ (\text{lb} \cdot \text{s}^2/\text{ft})/\text{ft}$$

Moment of inertia at x,

$$I(x) = \pi/4 \cdot (R_o)^4 - \pi/4 \cdot (R_i)^4 = \pi/4[(1.5\text{ ft})^4 - (1.167\text{ ft})^4] = 2.52\text{ ft}^4$$

Flexural stiffness at x,

$$E_c\, I(x) = 3{,}600\text{ ksi}(1{,}000\text{ lb/kip})(12\text{ in/ft})^2(2.52\text{ ft}^4) = 1.307 \times 10^9\text{ lb} \cdot \text{ft}^2$$

ii. *Determine the generalized mass, stiffness, and force.*
Generalized mass,

$$\tilde{m} = \int_0^L m(x)[\psi(x)]^2 dx = 13\frac{\text{lb} \cdot \text{s}^2}{\text{ft}^2}\int_0^{30\text{ ft}}\left[\frac{x^2}{(30\text{ ft})^2}\right]^2 dx = 78.1\frac{\text{lb} \cdot \text{s}^2}{\text{ft}}$$

Generalized stiffness,

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$$\psi''(x) = \frac{d^2\psi(x)}{dx^2} = \frac{2}{(30\text{ ft})^2}$$

$$\tilde{k} = \int_0^L EI(x)[\psi''(x)]^2 dx = 1.307\times 10^9 \text{ lb}\cdot\text{ft}^2 \int_0^{30\text{ ft}} \left[\frac{2}{(30\text{ ft})^2}\right]^2 dx = 1.94\times 10^5 \ \frac{\text{lb}}{\text{ft}}$$

Generalized force,

$$\tilde{L} = \int_0^L m(x)\psi(x)dx = \frac{13\dfrac{\text{lb}\cdot\text{s}^2}{\text{ft}^2}}{(30\text{ ft})^2}\int_0^{30\text{ ft}} x^2 dx = 130\ \frac{\text{lb}\cdot\text{s}^2}{\text{ft}}$$

iii. *Determine the natural period, frequency, and participation factor of the generalized SDOF system.*
Natural period,

$$T_n = 2\pi\sqrt{\frac{\tilde{m}}{\tilde{k}}} = 2\pi\sqrt{\frac{78.1\text{ lb}\cdot\text{s}^2/\text{ft}}{1.94\times 10^5\text{ lb/ft}}} = 0.126\text{ s}$$

Natural frequency,

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{1.94\times 10^5\text{ lb/ft}}{78.1\text{ lb}\cdot\text{s}^2/\text{ft}}} = 49.8\ \frac{\text{rad}}{\text{s}}$$

Participation factor,

$$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = \frac{130\text{ lb}\cdot\text{s}^2/\text{ft}}{78.1\text{ lb}\cdot\text{s}^2/\text{ft}} = 1.667$$

iv. *Determine maximum displacement at the top of the tower.* We can use Eq. (5.56), $z_o = \tilde{\Gamma}\cdot D$, where D is obtained from the response spectra in Chap. 4, Fig. E16.4 or the MATLAB code. Spectral displacement for period, $\tilde{T}_n = 0.126$ second (obtained with MATLAB code):

$$D = 0.0050\text{ in}$$

We can now determine the maximum displacement at the top of the tower,

$$u_o = z_o = \tilde{\Gamma}\cdot D = (1.667)(0.0050\text{ in}) = 0.0083\text{ in}$$

v. *Determine base shear and overturning moment.* First, we calculate the spectral acceleration,

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$$A = \tilde{\omega}_n^2 D = (49.8\ \text{rad/s})^2(0.0050\ \text{in}) = 12.4\ \text{in/s}^2$$

Determine the base shear force using Eq. (5.74),

$$V_{\max b} = \tilde{\Gamma} A \tilde{L} = 1.667\left(\frac{12.4\ \text{in}}{\text{s}^2}\right)\left(130\frac{\text{lb}\cdot\text{s}^2}{\text{ft}}\right)(\text{ft}/12\ \text{in}) = 224\ \text{lb}$$

Determine the overturning moment using Eq. (5.75),

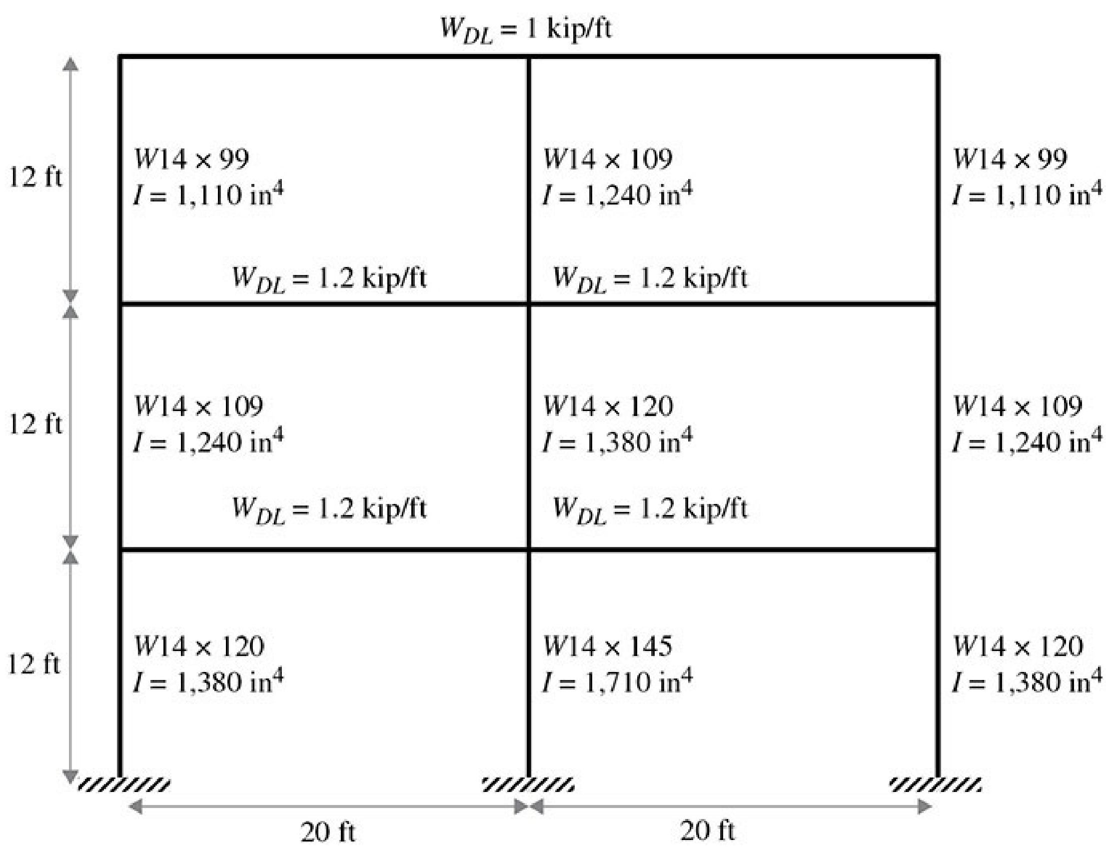
$$M_{\max b} = \tilde{\Gamma} A \int_0^L x m(x)\psi(x)dx = 1.667(12.4\ \text{in/s}^2)\frac{13\frac{\text{lb}\cdot\text{s}^2}{\text{ft}^2}}{(30\ \text{ft})^2}\int_0^{30\ \text{ft}} x^3 dx = 60.5\ \text{kip}\cdot\text{in} \quad \blacktriangle$$

5.3 Problems

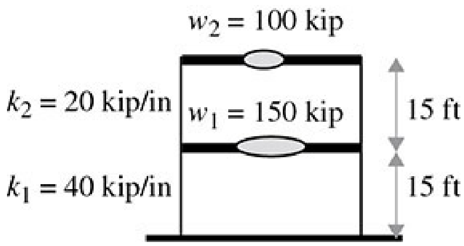
5.1 Solve Example 1 with a shape function given for a low-rise building.

5.2 Solve Example 1 with a shape function of $\psi(x) = (x/H)^2$, where H is the building height.

5.3 Given the three-story building shown below with rigid beams and flexible steel (E = 29,000 ksi) columns, and damping ratio of 5%, use a generalized SDOF analysis (assume a linear shape function) to determine the floor displacements and story shears at the floors due to air blast loading shown in Fig. E1.1.



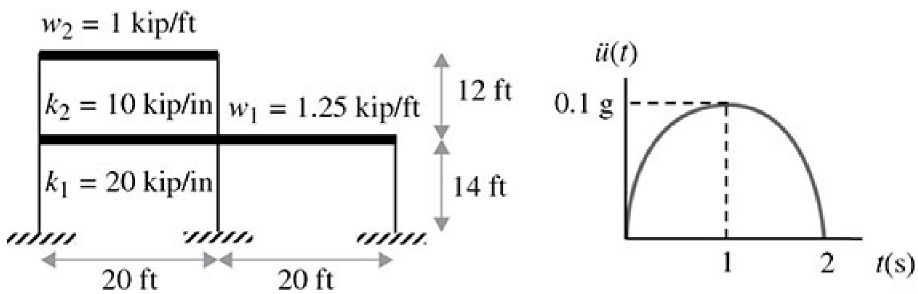
5.4 Use the generalized SDOF analysis for the following building frame to determine the story displacements, story forces, base shear, and overturning moment. Assume damping of 5% and a low-rise building shape function. Also, assume zero initial conditions and the harmonic floor loads given as, $p_1(t) = 4$ kip · sin[(26 rad/s)t] and $p_2(t) = 5$ kip · sin[(26 rad/s)t].



5.5 Solve Example 2 with a shape function given for a low-rise building.

5.6 Solve Example 2 with a shape function of $\psi(x) = (x/H)^2$, where H is the building height.

5.7 Use the generalized SDOF analysis for the following building frame to determine the story displacements, story forces, base shear, and overturning moment due to a ground acceleration caused by a large vehicle passing by, which can be modeled as a half-cycle pulse shown (Chap. 4, Fig. E12.1 gives the shock spectrum). Assume damping of 5% and a shape function of $\psi(x) = \sin(\pi x/2H)$, where H is the building height.

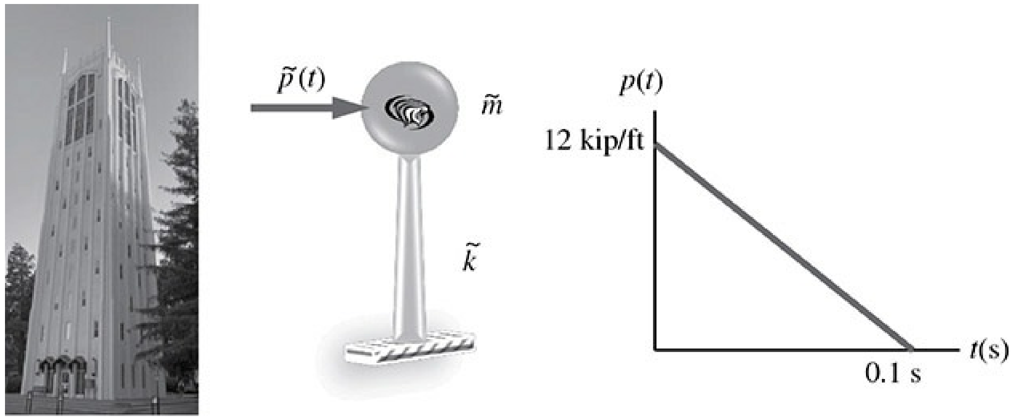


5.8 Given a five-story building frame, with story heights of 12 ft, subjected to a ground acceleration due to the 2021 Haiti earthquake presented in Chap. 4, Example 16, use generalized SDOF analysis to determine (a) peak displacements, (b) maximum base shear, and (c) maximum floor overturning moments. $k = 326.3$ kip/in, $w_5 = 50$ kip, and the weights on the other four floors equal 100 kip. Assume rigid beams, damping of 5%, and a shape function given for a mid-rise building.

5.9 Solve Prob. 5.8 with a shape function given for a high-rise building.

5.10 Solve Example 3 with a shape function of $\psi(x) = 1.5\,(x/L)^2 - 0.5\,(x/L)^3$.

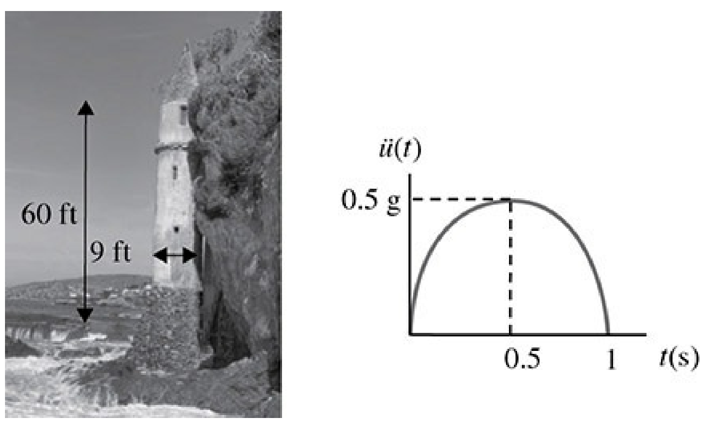
5.11 Assuming that Burn's tower (a 120 ft tall cantilever tower weighing 9 kip/ft) on the campus of University of the Pacific with flexural rigidity, $EI = 4 \times 10^9$ kip · ft^2, determine the equation of motion and displacement time history of the entire tower for the wind gust load shown, which is assumed to act uniformly along the entire heigh of the tower. Also, assume negligible damping and a shape function of $\psi(x) = 1 - \cos(\pi x/240 \text{ ft})$.



5.12 Solve Example 4 with a shape function of ψ (x) = 1.5 (x/L)^2 – 0.5 (x/L)^3.

5.13 Solve Example 4 with a shape function of ψ (x) = 1 – cos ($\pi x/2L$).

5.14 Consider the 60 ft Pirate Tower in Laguna Beach, CA (a concrete cylinder containing a staircase to access Victoria beach from the top of the cliffs) and use the generalized SDOF analysis to determine the top displacement, base shear, and overturning moment due to a ground acceleration caused by breaking ocean waves that can be modeled as the half-cycle pulse shown (Chap. 4, Fig. E12.1 gives the shock spectrum). Assume a shape function of ψ (x) = 1.5 (x/L)^2 – 0.5 (x/L)^3, where L is the tower height. Also, the diameter of the tower, d = 9 ft; it's wall thickness = 1 ft; modulus of elasticity, E_c = 3,600 ksi; and concrete weight, γ_c = 150 pcf.



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CHAPTER 6

Vibration of Multi-Degree-of-Freedom Systems

After reading this chapter, you will be able to:

a. Develop the equation of motion for a multi-degree-of-freedom (MDOF) system
b. Calculate frequencies and periods for a MDOF system
c. Evaluate the response of a MDOF system for a given initial displacement and velocity excitation
d. Determine the flexibility matrix for beams
e. Determine damping ratios using Rayleigh damping parameters

In Chap. 5 we modeled shear buildings, which are MDOF systems, as generalized SDOF systems to obtain the dynamic response caused by dynamic excitations. This analysis procedure is convenient for structures that can assume a unique shape during vibration. However, practical structures can assume several different shapes (mode shapes) during their vibration (see Fig. 6.1) with each mode contributing to the total response of the structure. Also, each mode shape shown in Fig. 6.1 has a unique associated natural frequency/period, with the first (or fundamental) mode of vibration defined as having the lowest frequency, or the longest period. Other modes have larger frequencies (or shorter periods) and are known as higher modes, or higher harmonics. Depending on the frequency content of the time-dependent excitation, different mode shapes can contribute differently to the total response of the system; typically, the first or fundamental mode has the greatest influence on the response, but because of resonance, other modes can dominate the response at higher harmonics.

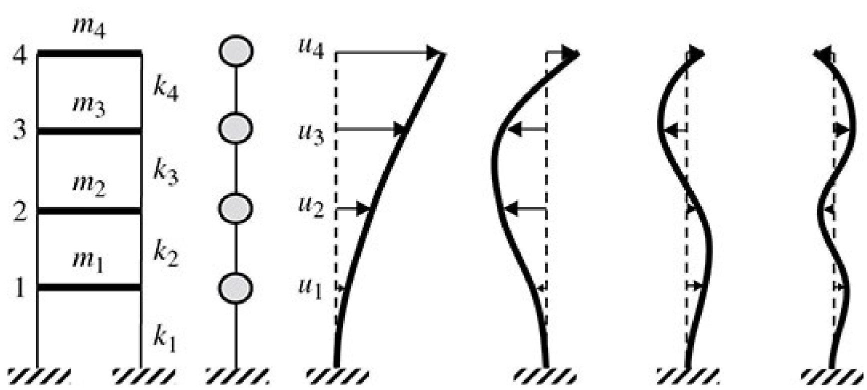

FIGURE 6.1 Idealized-four story shear building system and mode shapes.

Since individual modes have unique periods, each mode can be represented by a generalized SDOF following the process for shear buildings covered in Chap. 5. Also, the shape of each mode can be normalized to have a reference unit amplitude value at any desired specified location; for shear buildings, the amplitude at the top of the structure is usually selected as the reference unit amplitude. The actual amplitude can then be obtained from a dynamic analysis following a procedure similar to that used in the generalized SDOF analysis.

Once we obtain equations or vectors representing mode shapes and associated periods, the maximum response for individual modes can be obtained by analyzing each mode as a distinct generalized SDOF system. However, it is not likely that the maximum response of all modes will occur simultaneously at the same level; therefore, the maximum responses from all generalized SDOF cases are combined statistically in order to obtain the total system response. A number of procedures have been developed to combine the maximum results; two methods, which are discussed in Section 7.1.2, are the square root of the sum of the squares (SRSS) and the complete quadratic combination (CQC) methods.

For discrete systems, all modes can be modeled simultaneously using matrix methods; this is known as modal analysis. For example, the frame shown in Fig. 6.1 has four degrees of freedom and thus four mode shapes. As discussed earlier in the analysis of generalized SDOF systems, each mode shape is associated with a differential equation of motion, which in MDOF systems are generally dependent; thus, Fig. 6.1 consists of four dependent differential equations. In order to solve this dependent system of equations using the methods covered in earlier chapters, we can decouple them into four independent equations.

6.1 Generalized Eigenvalue Problem

First, we need to obtain the mode shapes and associated periods using an eigen matrix operation; the eigenvalues are related to the periods and the corresponding eigenvectors give the mode shapes. The standard eigen matrix operation can be viewed as a linear transformation that maps vectors into multiples of themselves, $\mathbf{Ax} = \lambda\mathbf{x}$. The common typographic convention for representing vectors and matrices is upright boldface type, lower case for vectors, as in **x** and upper case for matrices, as in **A**. Vibration analysis, however, requires the solution of the more general eigenvalue problem, $\mathbf{Ax} = \lambda\mathbf{Bx}$, which can be converted to a standard case by multiplying both sides by the inverse of the matrix **B**, that is, $\mathbf{B}^{-1}\mathbf{Ax} = \lambda\mathbf{x}$. In the analyses presented here, we use curly brackets for vectors and square brackets for matrices rather than boldface type notation, that is,

$$[A]\{x\} = \lambda\{x\} \tag{6.1}$$

where

λ is the scalar proportionality constant.
$[A]$ is a square matrix with size N × N.
$\{x\}$ must be conformable to $[A]$, that is of N × 1 size.

This operation yields N values of λ. Equation (6.1) can also be rewritten as

$$([A] - \lambda[I])\{x\} = \{0\} \tag{6.2}$$

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where $[I]$ is an identify matrix of the same size as $[A]$, N × N

The nonzero solutions to this equation require that λ be chosen so that

$$\det([A] - \lambda[I]) = 0 \tag{6.3}$$

The values of λ that satisfy this relationship correspond to eigenvalues of matrix $[A]$; each λ yields an eigenvector $\{x\}$ solution of Eq. (6.1) to within a multiplicative constant. That is, scaling the eigenvector by any constant factor will still satisfy Eq. (6.1). Consequently, each eigenvector is normalized with respect to one of its elements, usually the first element.

Hand calculations of the eigen problem solution are practically intractable for systems with more than two degrees of freedom. Thus, several numerical methods for solving the eigenvalue problem have been developed, none of which are presented in this book; instead, we use the *eig* MATLAB operator. For students wishing to study these methods, we encourage them to consult any linear algebra textbook. The process of determining eigenvectors (here denoted as *phi*) and the eigenvalues (here denoted as *lambda*) in MATLAB using the *eig* operator is simple; the standard eigenvalue problem in MATLAB syntax is

```
[phi, lambda] = eig(A)
```
(6.4)

whereas the generalized eigenvalue problem is

```
[phi, lambda] = eig(A,B)
```
(6.5)

which does not require conversion to the standard problem ($\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$).

In the sections that follow, we use the generalized eigenvalue problem to obtain the vibration response of structural systems. This solution will then be extended to encompass the calculation of the maximum shear and moment response of MDOF systems.

6.2 Undamped Equations of Motion for MDOF System

To formulate the equation of motion for a MDOF system, we first use the free vibration response of the oscillator system shown in Fig. 6.2, which represents a multistory shear building.

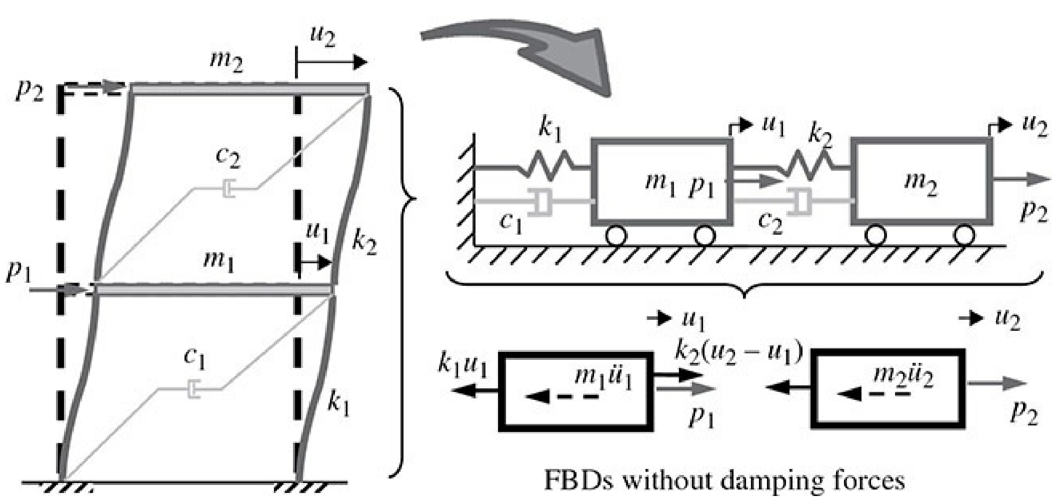

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Figure 6.2 Idealized shear building as oscillators and free-body diagrams (FBDs).

The equations of motion can be derived from equilibrium of the free-body diagrams (FBDs) using D'Alembert's principle. Applying horizontal equilibrium to each oscillator FBD shown in Fig. 6.2 yields two equations of motion.

Equilibrium of mass 1:

$$\overset{+}{\rightarrow}\ \sum F_x = 0;\quad -m_1\ddot{u}_1 - k_1u_1 + k_2(u_2 - u_1) + p_1 = 0 \Rightarrow m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = p_1 \tag{6.6}$$

where

m_1 is the mass of the first floor, which can be determined from the weights on that floor.

k_1 and k_2 are the story stiffnesses of the first and second levels, respectively, which can be determined by adding the lateral stiffnesses of all columns in each story. Each story can be treated as a portal frame as discussed in Chap. 1; thus, for a story of height h, column flexural stiffness EI, and assuming rigid floor diaphragms (rigid frame beams), the lateral stiffness of a column is $12EI/h^3$ and the stiffness of an arbitrary story j is $k_j = \sum_{\text{columns}} 12EI/h^3$.

p_1 is the lateral force applied on the first floor.

Equilibrium of mass 2:

$$\overset{+}{\rightarrow}\ \sum F_x = 0;\quad -m_2\ddot{u}_2 - k_2(u_2 - u_1) + p_2 = 0 \Rightarrow m_2\ddot{u}_2 - k_2u_1 + k_2u_2 = p_2 \tag{6.7}$$

where

m_2 is the mass of the second floor.

p_2 is the lateral force applied on the second floor.

Equations (6.6) and (6.7) are coupled second-order, linear, and homogeneous differential equations with constant coefficients, and can be rewritten in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

In general, the equations of motion are given by

$$[m]\{\ddot{u}\} + [k]\{u\} = \{p\} \tag{6.8}$$

where

$[m]$ is the mass matrix, which is a diagonal matrix (nonzero elements only along the diagonal).

$[k]$ is the stiffness matrix; this can alternatively be obtained by directly finding each coefficient k_{ij} that represents a force needed at level i to hold a unit displacement at level j and holding zero displacements at all other levels.

$\{p\}$ is the force vector.
$\{u\}$ is the displacement vector.
$\{\ddot{u}\}$ is the acceleration vector.

The solution to the homogeneous portion of Eq. (6.8) yields the natural periods (or frequencies) and the corresponding vibration mode shapes. From this equation, it is clear that an Nth degree of freedom system results in N dependent equations, or N by N mass and stiffness matrices.

The shear building depicted in Fig. 6.2 can be treated directly as a two-degree-of-freedom system as shown in Fig. 6.3; this can easily be expanded into an N-degree-of-freedom system. For the case of base motion, we need to establish a system of equations of motion for the model in terms of relative displacements for stiffness forces and total displacements for inertial forces. These equations can more easily be determined by applying equilibrium to the FBDs of the masses shown in Fig. 6.3. Notice that we have not included vertical forces or internal moments in the column stems because we only need to apply horizontal equilibrium to obtain the equations of motion.

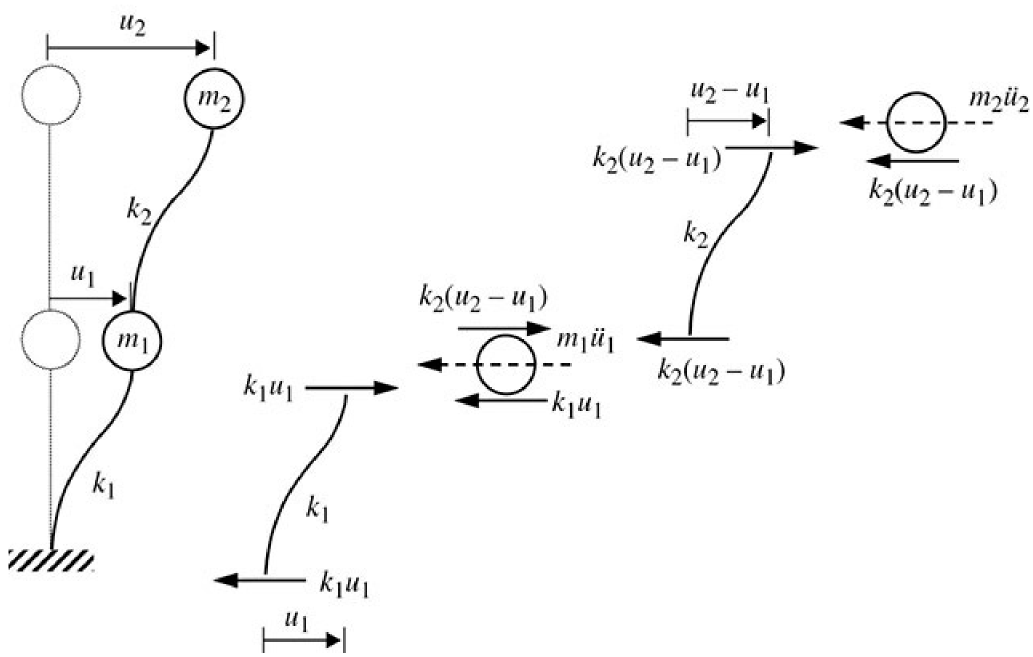

FIGURE 6.3 Two-degree-of-freedom system and free-body diagram of the two masses.

6.2.1 Periods and Mode Shapes for a MDOF System

The free vibration response, $\{p\} = \{0\}$ yields the natural periods or frequencies (similar to SDOF analysis presented in Chap. 2) and the corresponding mode shapes for the structure. Since we have second-order, linear, and homogeneous differential equations with constant coefficients (as in the SDOF case), we assume the resulting displacement can be described by the following simple harmonic equation:

$$\{u\} = \{a\}\sin(\omega_n t - \alpha) \tag{6.9}$$

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where

$\omega_n = \sqrt{k/m}$ is the natural circular frequency with units of radians per second (rad/s), which can be related to the period as $T_n = 2\pi/\omega_n$.

α is the phase shift.

$\{a\}$ is an unknown eigenvector associated with a mode of vibration.

Taking the first and second derivatives of Eq. (6.9) with respect to time.

$$\{\dot{u}\} = \{a\}\omega_n \cos(\omega_n t - \alpha) \tag{6.10}$$

$$\{\ddot{u}\} = -\{a\}\omega_n^2 \sin(\omega_n t - \alpha) = -\omega_n^2\{u\} \tag{6.11}$$

Substituting Eqs. (6.9) and (6.11) into Eq. (6.8) yields

$$[m]\left(-\{a\}\omega_n^2 \sin(\omega_n t - \alpha)\right) + [k]\{\{a\}\sin(\omega_n t - \alpha)\} = \{0\}$$

Or,

$$[m]\left(-\omega_n^2\{a\}\right) + [k]\{a\} = \{0\}$$

Factoring out the vector $\{a\}$ we get

$$[[k] - \omega_n^2[m]]\{a\} = \{0\} \tag{6.12}$$

This homogeneous system of equations has N unknown displacement vectors and N unknown parameters, ω_n^2, and requires a generalized eigen solution. Notice that $\{a\} = \{0\}$ is the trivial solution, so the nontrivial solution (one that gives nonzero values for $\{a\}$) requires,

$$\det\left([k] - \omega_n^2[m]\right) = 0 \tag{6.13}$$

This is the characteristic (or eigen) equation of the system, also known as the frequency equation because expanding the determinate results in an Nth degree polynomial in terms of ω_n^2. That is, Eq. (6.13), in general, should be satisfied for N values of ω_n^2. We can then solve the homogeneous system of equations [Eq. (6.12)] for $\{a\}_1$, $\{a\}_2$, … $\{a\}_N$ for each value of ω_n^2, totaling N modes of vibration.

As noted earlier, Eq. (6.13) can easily be solved with MATLAB using Eq. (6.5), where matrix $[A]$ represents the stiffness matrix $[k]$ and matrix $[B]$ represents the mass matrix $[m]$, where mode shapes (eigenvectors) are denoted by *phi* and the square of the frequencies (eigenvalues) by *lambda*:

```
[phi, lambda] = eig(k,m)
```
(6.14)

For illustration purposes, let us conduct the analysis of the two-degree-of-freedom case in Fig. 6.2 using hand calculations. The displacement response vector is given by Eq. (6.9), which

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for two degrees of freedom is

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \sin(\omega_n t - \alpha) \tag{6.15}$$

Taking two derivatives of Eq. (6.15) with respect to time and substituting into Eq. (6.8) gives the following equation:

$$-\omega_n^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.16}$$

which can be rewritten as

$$\begin{bmatrix} k_1 + k_2 - m_1\omega_n^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_n^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.17}$$

As discussed above, the nontrivial solution requires that the determinate of the matrix be zero

$$\det \begin{bmatrix} k_1 + k_2 - m_1\omega_n^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_n^2 \end{bmatrix} = 0 \tag{6.18}$$

Expanding this determinate results in a second-order polynomial in ω_n^2,

$$m_1 m_2 \omega_n^4 - ((k_1 + k_2)m_2 + m_1 k_2)\omega_n^2 + k_1 k_2 = 0 \tag{6.19}$$

This polynomial is known as the characteristic equation, which is generally satisfied for *N* values of ω_n^2 . The corresponding mode shapes are determined by solving Eq. (6.17) in terms of arbitrary constants.

For the two DOF case, we can use the quadratic formula, $\lambda = (-b \pm \sqrt{b^2 - 4ac})/2a$, to find the ω_n^2 roots of Eq. (6.19), which are the eigenvalues (characteristic values):

$$\lambda_1 = \omega_{n1}^2 = \frac{((k_1 + k_2)m_2 + m_1 k_2) - \sqrt{((k_1 + k_2)m_2 + m_1 k_2)^2 - 4m_1 m_2 k_1 k_2}}{2m_1 m_2} \tag{6.20}$$

and

$$\lambda_2 = \omega_{n2}^2 = \frac{((k_1 + k_2)m_2 + m_1 k_2) + \sqrt{((k_1 + k_2)m_2 + m_1 k_2)^2 - 4m_1 m_2 k_1 k_2}}{2m_1 m_2} \tag{6.21}$$

The frequencies are

$$\omega_{n1} = \sqrt{\lambda_1} \text{ and } \omega_{n2} = \sqrt{\lambda_2} \tag{6.22}$$

Frequencies are usually listed in ascending order, that is, from smallest to largest.

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Each of these eigenvalues [Eqs. (6.20) and (6.21)] is associated with a distinct eigenvector (or mode shape). The individual values of the eigenvector elements are indeterminate since the equation is singular, which means that no unique set of individual values satisfies Eq. (6.17). In general, there are an infinite number of solutions for each eigenvector $\{a\}_N$. However, it is possible to obtain the relative values of each element with respect to one of the elements; it is customary to describe the normal modes by assigning a unit value to one of the eigenvector elements (amplitudes) and solving for the values of the other amplitudes relative to that value.

Substituting $\lambda_1 = \omega_{n1}^2$ into the equations of motion, Eq. (6.17), and solving for a_{11} and a_{21} (two-degree-of-freedom case) yields

$$\begin{bmatrix} k_1 + k_2 - m_1\lambda_1 & -k_2 \\ -k_2 & k_2 - m_2\lambda_1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Setting $a_{21} = 1$, we get

$$\begin{bmatrix} k_1 + k_2 - m_1\lambda_1 & -k_2 \\ -k_2 & k_2 - m_2\lambda_1 \end{bmatrix} \begin{Bmatrix} a_1 \\ 1 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solving for a_{11}, the resulting eigenvector (mode shape) from the second equation is

$$\begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} \dfrac{k_2 - m_2\lambda_1}{k_2} \\ 1 \end{Bmatrix}_1 \tag{6.23}$$

Similarly, we can obtain the second eigenvector (mode shape) as

$$\begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} \dfrac{k_2 - m_2\lambda_2}{k_2} \\ 1 \end{Bmatrix}_2 \tag{6.24}$$

These eigenvectors are referred to as *normal modes* because they are normalized so that the displacements of the mass at the top of the structure are equal to 1. The resulting eigenvectors describe the mode shapes and are defined as $\{\phi_j\}$. The shape of each mode is preserved regardless of the excitation causing the vibration.

We now obtain the total solution of the system by combining the two modes using Eq. (6.9),

$$\begin{aligned} u_1 &= \varphi_{11}q_1 + \varphi_{12}q_2 \\ u_2 &= \varphi_{21}q_1 + \varphi_{22}q_2 \end{aligned} \tag{6.25}$$

where we now assume a new set of harmonic displacement functions, q_j, known as generalized (or modal) coordinates, which yield the same result as Eq. (6.15), but with arbitrary constants A_j and B_j,

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$$q_1 = A_1 \cos\omega_{n1} t + B_1 \sin\omega_{n1} t$$
$$q_2 = A_2 \cos\omega_{n2} t + B_2 \sin\omega_{n2} t \quad (6.26)$$

These define the response in relative magnitudes of the modes. Rewriting Eq. (6.25) in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \text{ or } \{u\} = [\Phi]\{q\} \quad (6.27)$$

The actual amplitudes of the motion can now be found from initial conditions: initial displacement and velocity of each mass.

Example 1

Consider the two-degree-of-freedom system given in Fig. 6.2 assuming all masses are equal, $m_1 = m_2 = m$ and all story stiffness are equal, $k_1 = k_2 = k$, determine (a) the periods and modal matrix (normalized mode shapes) and (b) the displacement response assuming initial displacements $u_1(0) = d_1$, $u_2(0) = d_2$, and initial velocities are zero.

Solution

i. *Determine the equations of motion.* These can be obtained following the process shown in Fig. 6.2 by setting $m_1 = m_2 = m$, $k_1 = k_2 = k$ and applying equilibrium to the FBDs of each mass, resulting in equations similar to Eqs. (6.6) and (6.7). The two equations are combined into a matrix equation, Eq. (6.8):

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

ii. *Determine the periods and modal matrix.* The natural frequencies and mode shapes can be determined by solving an eigenvalue problem; eigenvalues yield the natural frequencies and eigenvectors the mode shapes. Also, the periods are obtained using the natural frequencies. The frequency equation resulting from the expansion of the eigen equation is

$$m^2\omega_n^4 - 3km\,\omega_n^2 + k^2 = 0$$

Using the quadratic formula we can find the roots of this equation, which are the eigenvalues:

$$\omega_n^2 = \frac{3km \pm \sqrt{(3km)^2 - 4m^2k^2}}{2m^2} = \frac{3 \pm \sqrt{5}}{2}\frac{k}{m}$$

The frequencies are (from smallest to largest)

$$\omega_{n1} = \sqrt{\frac{3-\sqrt{5}}{2}\frac{k}{m}} = 0.618\sqrt{\frac{k}{m}} \text{ and } \omega_{n2} = \sqrt{\frac{3+\sqrt{5}}{2}\frac{k}{m}} = 1.618\sqrt{\frac{k}{m}}$$

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The natural periods then can be obtained using $T_{nj} = \frac{2\pi}{\omega_{nj}}$.

To obtain mode shapes, we substitute each eigenvalue (ω_{n1}^2 first) into Eq. (6.17) and set a_2 = 1,

$$\begin{bmatrix} 2k-(3-\sqrt{5})k/2 & -k \\ -k & k-(3-\sqrt{5})k/2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ or}$$

$$k\begin{bmatrix} 1.618 & -1 \\ -1 & 0.618 \end{bmatrix}\begin{Bmatrix} a_1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solving for a_1, using the top row, the resulting eigenvector (mode shape) is

$$\begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}$$

Note that the same information is obtained from both top and bottom rows. That is, the displacement of the lower mass, m_1, is always 0.618 times the displacement of the upper mass, m_2, when the system vibrates in the first mode. Similarly, we can obtain the second eigenvector (mode shape) as

$$\begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix}$$

The modal matrix is

$$[\Phi] = \begin{bmatrix} 0.618 & -1.618 \\ 1 & 1 \end{bmatrix}$$

These two mode shapes represent two possible simple harmonic motions of the frame structure and are known as natural modes of vibration since the masses move in phase at the same frequency. The mode associated with the lowest frequency, Mode 1, is known as fundamental mode, while the modes associated with higher values of frequency are known as harmonic or higher harmonic. A summary is shown in Fig. E1.1.

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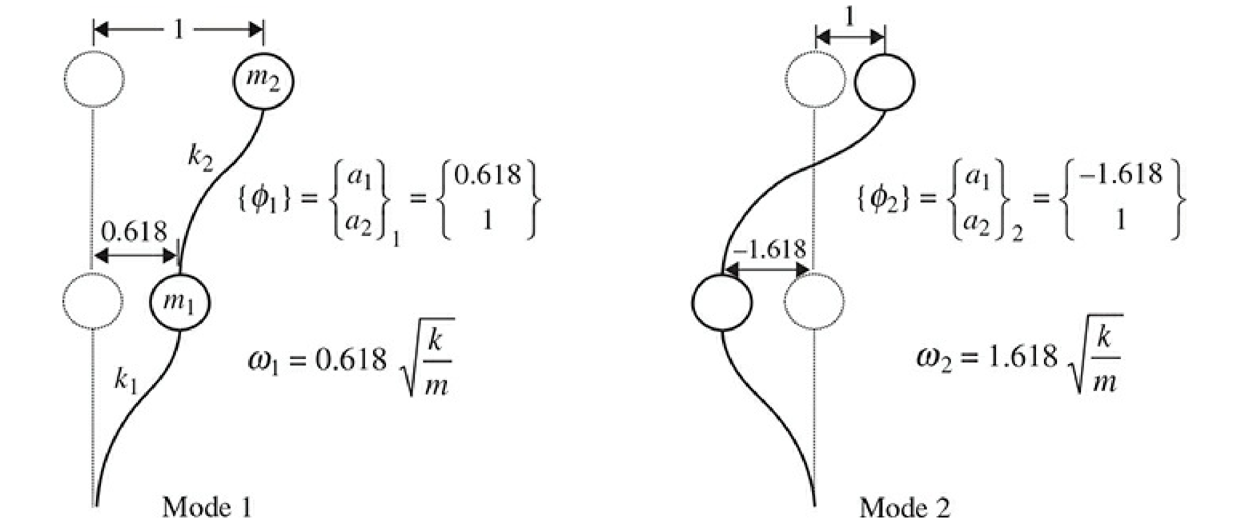

FIGURE E1.1 Mode shapes and associated frequencies.

iii. *Determine the displacement response.* The total displacement response of the system is given by the superposition of the modal harmonic vibrations, Eqs. (6.25) and (6.26),

$$u_1 = \varphi_{11}[A_1 \cos\omega_{n1}t + B_1 \sin\omega_{n1}t] + \varphi_{12}[A_2 \cos\omega_{n2}t + B_2 \sin\omega_{n2}t]$$
$$u_2 = \varphi_{21}[A_1 \cos\omega_{n1}t + B_1 \sin\omega_{n1}t] + \varphi_{22}[A_2 \cos\omega_{n2}t + B_2 \sin\omega_{n2}t]$$

Differentiating these equations with respect to time to obtain the velocities,

$$\dot{u}_1 = \omega_{n1}\varphi_{11}[-A_1 \sin\omega_{n1}t + B_1 \cos\omega_{n1}t] + \omega_{n2}\varphi_{12}[-A_2 \sin\omega_{n2}t + B_2 \cos\omega_{n2}t]$$
$$\dot{u}_2 = \omega_{n1}\varphi_{21}[-A_1 \sin\omega_{n1}t + B_1 \cos\omega_{n1}t] + \omega_{n2}\varphi_{22}[-A_2 \sin\omega_{n2}t + B_2 \cos\omega_{n2}t]$$

The four initial conditions are

$$u_1(0) = d_1$$

$$u_2(0) = d_2$$

$$\dot{u}_1(0) = 0$$

$$\dot{u}_2(0) = 0$$

Apply these conditions,

$$u_1(0) = \varphi_{11}A_1 + \varphi_{12}A_2 = d_1$$
$$u_2(0) = \varphi_{21}A_1 + \varphi_{22}A_2 = d_2$$
$$\dot{u}_1(0) = \omega_{n1}\varphi_{11}B_1 + \omega_{n2}\varphi_{12}B_2 = 0$$
$$\dot{u}_2(0) = \omega_{n1}\varphi_{21}B_1 + \omega_{n2}\varphi_{22}B_2 = 0$$

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These equations yield the arbitrary constants,

$$A_1 = \frac{d_1 - \varphi_{12} A_2}{\varphi_{11}}$$

$$A_2 = \frac{d_2 \varphi_{11} - d_1 \varphi_{21}}{\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}}$$

$$B_1 = B_2 = 0$$

Assume k = 20 kip/in and m = 0.4 kip · s^2/in and graph the response if the system is excited by an initial displacement that corresponds to the first mode, $d_1 = \varphi_{11} = 0.618$ and $d_2 = \varphi_{21} = 1$ in, $A_1 = 1$ in and $A_2 = 0$:

$$u_1 = \varphi_{11}[A_1 \cos \omega_{n1} t] = 0.618 \text{ in}[\cos(0.618\ t\sqrt{k/m})]$$

$$u_2 = \varphi_{21}[A_1 \cos \omega_{n1} t] = 1 \text{ in}[\cos(0.618\ t\sqrt{k/m})]$$

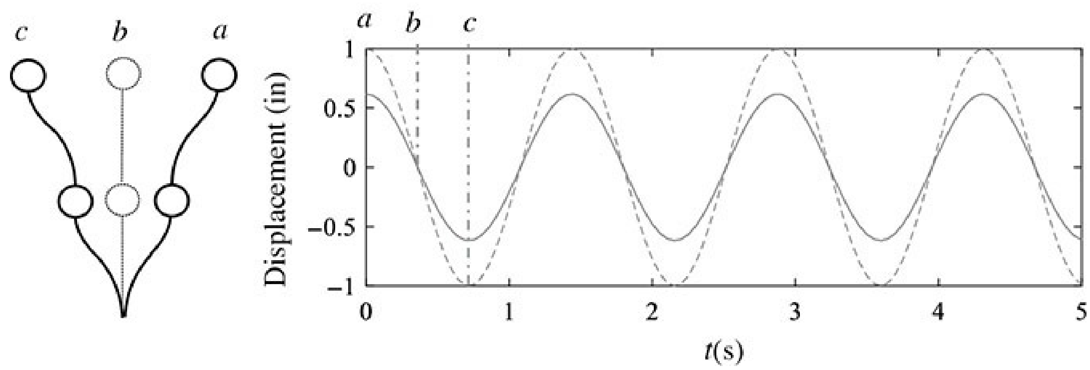

FIGURE E1.2 Displacement caused by first mode; u_1 is solid line and u_2 is dashed line.

Now assume the same system is excited by an initial displacement that corresponds to the second mode, $d_1 = \ _{12} = -1.618$ in and $d_2 = \ _{22} = 1$ in, $A_1 = 0$ and $A_2 = 1$ in:

$$u_1 = \varphi_{12}[A_2 \cos \omega_{n2} t] = -1.618 \text{ in}[\cos(1.618\ t\sqrt{k/m})]$$

$$u_2 = \varphi_{22}[A_2 \cos \omega_{n2} t] = 1 \text{ in } [\cos(1.618\ t\sqrt{k/m})]$$

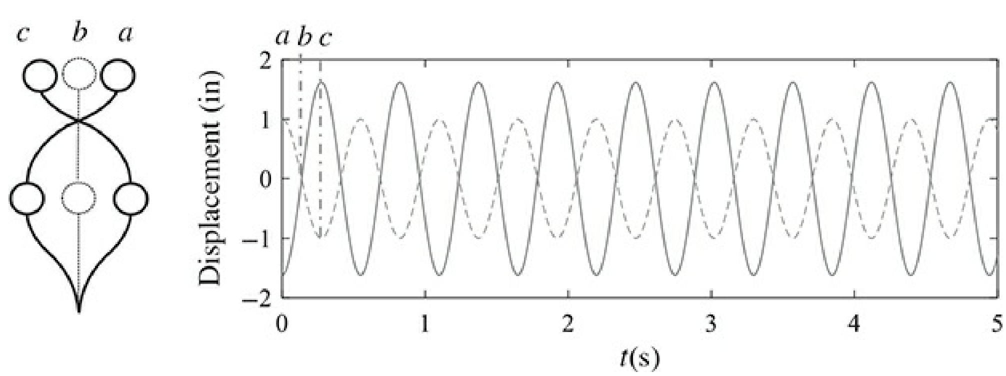

Figure E1.3 Displacement caused by second mode; u_1 is solid line and u_2 is dashed line. ▲

6.2.2 Orthogonality of Mode Shapes (Eigenvectors)

Regardless of how the eigenvectors are normalized, the resulting mode shapes are orthogonal; that is, for two distinct mode shapes of the structure when $i \neq j$,

$$\{\varphi_i\}^T\{\varphi_j\} = 0 \tag{6.28}$$

This means that eigenvectors are perpendicular to each other when they are defined in N-dimensional space. Also, since amplitudes of the eigenvectors are relative, we can scale them such that their magnitude is unity; such a vector, $\{\psi_i\}$, is said to be normal.

$$\{\psi_i\}^T\{\psi_i\} = 1 \tag{6.29}$$

where the superscript T denotes the transpose matrix operation and

$$\{\psi_i\} = \frac{\{\varphi_i\}}{\sqrt{\{\varphi_i\}^T\{\varphi_i\}}}$$

That is,

$$\{\psi_i\}^T\{\psi_i\} = \frac{\{\varphi_i\}^T}{\sqrt{\{\varphi_i\}^T\{\varphi_i\}}}\frac{\{\varphi_i\}}{\sqrt{\{\varphi_i\}^T\{\varphi_i\}}} = \frac{\{\varphi_i\}^T\{\varphi_i\}}{\{\varphi_i\}^T\{\varphi_i\}} = 1$$

If we assemble the normal mode shapes, Eq. (6.29), into a *normal modal matrix*, $[\Psi]$, where the ith column of the matrix corresponds to the ith normal mode shape. The orthogonality condition results in the important condition of the inverse of the matrix being equal to its transpose, $[\Psi]^{-1} = [\Psi]^T$.

One of the most important consequences of this mode shape property is that they are orthogonal with respect to both mass and stiffness matrices. That is, when $i \neq j$

$$\{\varphi_i\}^T[m]\{\varphi_j\} = 0 \text{ and } \{\varphi_i\}^T[k]\{\varphi_j\} = 0 \tag{6.30}$$

when $i = j$, the operations result in diagonal elements for the generalized mass and stiffness matrices:

$$\{\varphi_i\}^T[m]\{\varphi_i\} = M_i \text{ and } \{\varphi_i\}^T[k]\{\varphi_i\} = K_i \tag{6.31}$$

Note that both results are positive definite, $\{v\}^T[A]\{v\} > 0$. These conditions are essential in the analysis of MDOF systems, where we use eigenvectors as mode shapes, which are, in turn, used to determine the overall response of the systems by employing superposition of response contributed by each mode; this analysis is known as *modal analysis* and is a powerful approach for obtaining solutions to MDOF problems.

We can also arrange the mode shapes (say N of them) into a matrix with the ith column of the matrix corresponding to the ith mode shape, $[\Phi]$, known as the *modal matrix*,

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$$[\Phi]_{N\times N} = [\{\varphi_1\}\{\varphi_2\}\cdots\{\varphi_i\}\cdots\{\varphi_N\}] \tag{6.32}$$

Again, since the amplitudes of the eigenvectors are relative, we can scale them such that their magnitude is given by

$$\{\psi_i\} = \frac{\{\varphi_i\}}{\sqrt{\{\varphi_i\}^T[m]\{\varphi_i\}}}$$

These modes can be arranged into a new matrix form,

$$[\Psi] = [\{\psi_1\}\{\psi_2\}\cdots\{\psi_i\}\cdots\{\psi_N\}]$$

This more general expression of the orthogonality condition leads to

$$[\Psi]^T[m][\Psi] = [I] \text{ and } [\Psi]^T[k][\Psi] = \left[\omega_{ni}^2\right] \tag{6.33}$$

where $[I]$ is the identity matrix and ω_{ni}^2 is a diagonal matrix of the square of the mode shape natural frequencies.

Recall that the equation of motion in free vibration, Eq. (6.12), can be written as

$$[k]\{a\} = \omega_n^2[m]\{a\}$$

Or in terms of mode shape *i*,

$$[k]\{\varphi_i\} = \omega_{ni}^2[m]\{\varphi_i\}$$

Premultiplying this equation by the transpose of the mode shape and solving for the natural frequency for mode shape *i*,

$$\omega_{ni}^2 = \frac{\{\varphi_i\}^T[k]\{\varphi_i\}}{\{\varphi_i\}^T[m]\{\varphi_i\}} \tag{6.34}$$

This is known as Rayleigh’s quotient and can be written in matrix form as

$$\left[\omega_n^2\right] = \frac{[\Phi]^T[k][\Phi]}{[\Phi]^T[m][\Phi]} \tag{6.35}$$

where ω_n^2 is a diagonal matrix containing the natural frequencies. Or in terms of the normal modal matrix, the second of Eq. (6.32),

$$\omega_{ni}^2 = [\Psi]^T[k][\Psi]$$

Example 2

Verify that the mode shapes obtained in Example 1 are orthogonal, that is, prove Eqs. (6.28) and (6.29) using the results of Example 1. Also show that the mode shapes are orthogonal with respect to the mass and stiffness matrices, that is, verify Eq. (6.30) from the results of Example 1.

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Solution

i. *Orthogonality conditions, Eqs. (6.28) and (6.29).* Use mode shapes to carry out the operation $\{\varphi_1\}^T\{\varphi_2\} = 0$,

$$\{\varphi_1\}^T\{\varphi_2\} = \{0.618 \quad 1\}\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} = (0.618)(-1.618) + (1)(1) = 0$$

Calculate normal mode shape 1:

$$\psi_1 = \frac{\{\varphi_1\}}{\sqrt{\{\varphi_1\}^T\{\varphi_1\}}} = \frac{\begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}}{\sqrt{\{0.618 \quad 1\}\begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}}} = \frac{\begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}}{1.176} = \begin{Bmatrix} 0.526 \\ 0.851 \end{Bmatrix}$$

Calculate normal mode shape 2:

$$\psi_2 = \frac{\{\varphi_2\}}{\sqrt{\{\varphi_2\}^T\{\varphi_2\}}} = \frac{\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix}}{\sqrt{\{-1.618 \quad 1\}\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix}}} = \frac{\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix}}{1.902} = \begin{Bmatrix} -0.851 \\ 0.526 \end{Bmatrix}$$

Use mode shapes to carry out the operation $\{\psi_i\}^T\{\psi_i\} = 1$,

$$\{\psi_1\}^T\{\psi_1\} = \{0.526 \quad 0.851\}\begin{Bmatrix} 0.526 \\ 0.851 \end{Bmatrix} = 0.276 + 0.724 = 1$$

$$\{\psi_2\}^T\{\psi_2\} = \{-0.851 \quad 0.526\}\begin{Bmatrix} -0.851 \\ 0.526 \end{Bmatrix} = 0.724 + 0.276 = 1$$

ii. *Now prove Eq. (6.30),*

$$\{\varphi_1\}^T[k]\{\varphi_2\} = \{0.618 \quad 1\}\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} = \{0.236k \quad 0.382k\}\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} = 0$$

$$\{\varphi_1\}^T[m]\{\varphi_2\} = \{0.618 \quad 1\}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} = \{0.618m \quad m\}\begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} = 0 \quad \blacktriangle$$

6.2.3 Modal Superposition Analysis of Free Vibration Response

As noted earlier, we can use orthogonality of the mode shapes to uncouple the equations of motion for MDOF systems into N-DOFs. Modal analysis is applicable to both free vibration and forced vibration problems. The *modal matrix* $[\Phi]$, Eq. (6.32), is a coordinate transformation matrix that maps the original (or physical) coordinates, $\{u\}$, of each mass to the *principal (or*

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modal) coordinates, $\{q\}$, Eq. (6.27), repeated here for convenience. This is a combination of the contribution to the total response from all the mode shapes,

$$\{u\} = [\Phi]\{q\}$$

We can write this in element form as follows:

$$u_i(t) = \sum_{j=1}^{N} \varphi_{ij} \cdot q_j(t) \tag{6.36}$$

where

$i = 1, 2, \cdots N$.
$u_i(t)$ is the displacement at level i.
$q_j(t)$ is the generalized modal coordinate j.
φ_{ij} is a component of the modal matrix $[\Phi]$.

Substituting the transformation relationship, Eq. (6.27), into the equations of motion, Eq. (6.8) with $\{p\} = \{0\}$, and premultiplying both sides of the equation by $[\Phi]_T$ yields

$$[\Phi]^T[m][\Phi]\{\ddot{q}\} + [\Phi]^T[k][\Phi]\{q\} = \{0\} \;\text{ or }\; [M]\{\ddot{q}\} + [K]\{q\} = \{0\} \tag{6.37}$$

As noted earlier, because of orthogonality of the modal matrix [Eq. (6.31)], the *modal mass*, $[M]$, and *modal stiffness*, $[K]$, are diagonal (i.e., all off-diagonal elements are zero). The diagonal elements of these matrices are given by

$$M_i = \{\varphi_i\}^T[m]\{\varphi_i\} \;\text{ and }\; K_i = \{\varphi_i\}^T[k]\{\varphi_i\}$$

Notice that Eq. (6.37) is now uncoupled and obtaining the modal coordinates can be accomplished by independently solving each equation. That is,

$$M_i\ddot{q}_i + K_i q_i = 0 \tag{6.38}$$

where

$$M_i = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_N \end{bmatrix}$$

$$K_i = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_N \end{bmatrix}$$

If each equation is divided by the associated modal mass, [or we use the normal modal matrix, Eq. (6.33) in the equations of motion], the equations can be rewritten as

$$\ddot{q}_i + \omega_{ni}^2 q_i = 0 \tag{6.39}$$

So, each equation has the same form as the equation of motion for the SDOF system we described in detail in Chap. 2. This implies that we can solve each of Eq. (6.39) for the various loadings or forcing functions examined previously, including response spectrum analysis. Then, the solution to the original equations of motion can be obtained using Eq. (6.27), superposition of the modal coordinates. Also, for systems with large numbers of degrees-of-freedom problems, we can obtain a reasonable approximation of the response by only including the first few mode shapes; this will be discussed further in Chap. 7.

Example 3

Given the following two-story building frame with k = 20 kip/in and m = 0.4 kip · s^2/in, determine (a) mass and stiffness matrices, (b) periods and modal matrix, and (c) the stiffness and modal matrices. Also write a MATLAB script to perform the computations.

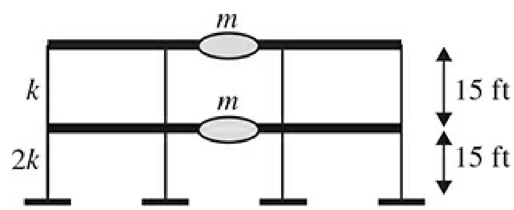

FIGURE E3.1 Schematic of two-story building frame.

Solution

i. *Determine the mass and stiffness matrices.* The mass and stiffness matrices can be obtained using Eq. (6.8) with $m_1 = m_2 = m$ = 0.4 kip · s^2/in, $k_1 = 2k$ = 40 kip/in, and $k_2 = k$ = 20 kip/in.

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \text{kip} \cdot \text{s}^2/\text{in}$$

$$[k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \text{kip/in}$$

ii. *Determine the periods and modal matrix.* The natural frequencies and mode shapes can be determined by solving for the eigenvalues (square of natural frequencies). Or for the two-degree-of-freedom system in this example, we can use Eqs. (6.20) and (6.21) to obtain the eigenvalues. The periods are obtained using the natural frequencies. First, eigenvalues are

$$\omega_n^2 = \frac{4km \pm \sqrt{(4km)^2 - 8m^2k^2}}{2m^2} = \frac{4 \pm 2\sqrt{2}}{2}\frac{k}{m} = (2 \pm \sqrt{2})\,\frac{20\ \text{kip/in}}{0.4\ \text{kip} \cdot \text{s}^2/\text{in}}$$

The frequencies (from smallest to largest) are

$$\omega_{n1} = 5.41 \text{ rad/s and } \omega_{n2} = 13.1 \text{ rad/s}$$

The natural periods, obtained using $T_{nj} = \dfrac{2\pi}{\omega_{nj}}$,

$$T_{n1} = 1.16 \text{ s and } T_{n2} = 0.48 \text{ s}$$

To obtain mode shapes, we can use Eqs. (6.23) and (6.24) or we substitute each eigenvalue (ω_{n1}^2 first) into Eq. (6.17) and set $a_2 = 1$,

$$\begin{bmatrix} 3k-(2-\sqrt{2})k & -k \\ -k & k-(2-\sqrt{2})k \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ or } \begin{bmatrix} 2.414 & -1 \\ -1 & 0.414 \end{bmatrix} \begin{Bmatrix} a_1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solving for a_1, the resulting eigenvector (mode shape) is

$$\begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.414 \\ 1 \end{Bmatrix}$$

Similarly, we can obtain the second eigenvector (mode shape) as

$$\begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.414 \\ 1 \end{Bmatrix}$$

The modal matrix is

$$[\Phi] = \begin{bmatrix} 0.414 & -2.414 \\ 1 & 1 \end{bmatrix}$$

iii. *Determine the modal mass and stiffness matrices.* Use Eq. (6.31) to determine the elements of the modal mass and stiffness matrices.

Modal mass matrix,

$$[M] = [\Phi]^T[m][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.414 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.414 & -2.414 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.469 & 0 \\ 0 & 2.73 \end{bmatrix} \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

Modal stiffness matrix,

$$[K] = [\Phi]^T[k][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.414 & 1 \end{bmatrix} \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} 0.414 & -2.414 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13.7 & 0 \\ 0 & 466.3 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

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iv. *MATLAB script to perform all the operations, the most important of the operations being the eig operator in Eq. (6.5):*

```
clear all % Chapter 6, Example 3
m = [0.4 0; 0 0.4]; % mass matrix
k = [60 -20; -20 20]; % stiffness matrix
[phi,lam]=eig(k,m); % compute eigenvalues and eigenvectors
omegas=sqrt(lam) % determine and show frequencies from eigenvalues
periods = 2*pi*diag(inv(omegas)) % determine the periods from omegas
% Mode shapes by normalizing eigenvectors to get top displ equal to 1
[N_rows, N_cols] = size(phi); % finds N, the size of the matrices
for i = 1:N_cols; % loops over the N modes to normalize them
    norm_phi(:,i) = phi(:,i)./phi(N_rows,i);
end
norm_phi % display normalized modal matrix
% Determine the modal mass and stiffness matrices
M = norm_phi'*m*norm_phi
K = norm_phi'*k*norm_phi
```

The results of this script are the same as those of parts (i) through (iii):

```
omegas =
  5.4120        0
       0  13.0656
periods =
  1.1610
  0.4809
norm_phi =
  0.4142  -2.4142
  1.0000   1.0000
M =
  0.4686   0.0000
  0.0000   2.7314
K =
  13.7258   0.0000
        0 466.2742
```

▲

Example 4

Consider the two-degree-of-freedom system given in Example 1 and determine the decoupled equations of motion in terms of modal coordinates.

Solution

i. *Determine the modal mass and stiffness matrices.* Use Eq. (6.31) to determine the elements of the modal mass and stiffness matrices.

Modal mass matrix, mass transformation coefficients, M_1 and M_2

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} = \begin{bmatrix} 0.618 & 1 \\ -1.618 & 1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 0.618 & -1.618 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1.382m & 0 \\ 0 & 3.618m \end{bmatrix}$$

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Modal stiffness matrix, stiffness coefficients, K_1 and K_2,

$$\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} = \begin{bmatrix} 0.618 & 1 \\ -1.618 & 1 \end{bmatrix} \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 0.618 & -1.618 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.528k & 0 \\ 0 & 9.472k \end{bmatrix}$$

Notice that this also proves the orthogonality condition $(\{\varphi_1\}^T[m]\{\varphi_2\} = \{\varphi_2\}^T[m]\{\varphi_1\} = \{0\}$ and $\{\varphi_1\}^T[k]\{\varphi_2\} = \{\varphi_2\}^T[k]\{\varphi_1\} = \{0\})$ since the off-diagonal elements of the two matrices are zero.

ii. *Determine the equations of motion in terms of the modal coordinates.* Using Eq. (6.38) we get the uncoupled equations of motion,

$$1.382 \cdot m \cdot \ddot{q}_1 + 0.528 \cdot k \cdot q_1 = 0$$
$$3.618 \cdot m \cdot \ddot{q}_2 + 9.472 \cdot k \cdot q_2 = 0 \quad \blacktriangle$$

Example 5

Consider the simplified model of the automobile shown in Fig. E5.1 and determine (a) the mass and stiffness matrices, (b) frequencies, and (c) modal matrix (normalized mode shapes). Assume $k_1 = k$, $k_2 = 2k$, and a total mass of the vehicle of m.

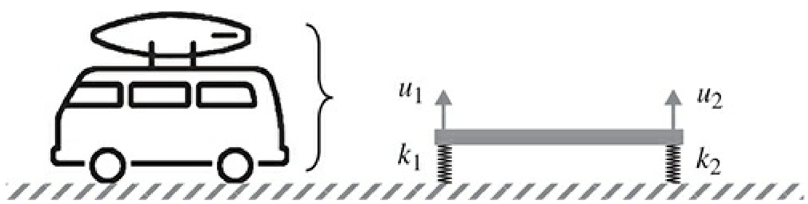

FIGURE E5.1 Automobile and simplified two-degree-of-freedom model.

Solution

i. *Determine the stiffness matrix.* Apply a unit displacement to each of the two DOFs as shown in Fig. E5.2; note that the forces required to hold the displaced shape are the stiffness influence coefficients; that is, each column of the stiffness matrix is a force system required to hold a unit displacement at each DOF.

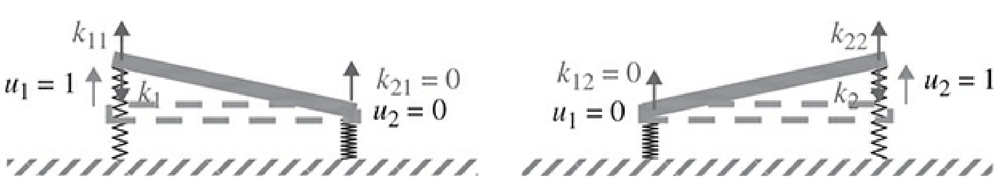

FIGURE E5.2 Stiffness influence coefficients.

From equilibrium of each case in Fig. E5.2, we get

$$u_1 = 1 \text{ and } u_2 = 0 \rightarrow \; k_{11} = k_1 = k \text{ and } k_{21} = 0$$

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$$u_1 = 0 \text{ and } u_2 = 1 \rightarrow \; k_{12} = 0 \text{ and } k_{22} = k_2 = 2k$$

In matrix form,

$$[k] = \begin{bmatrix} k & 0 \\ 0 & 2k \end{bmatrix}$$

Notice that there is no coupling since this matrix is diagonal.

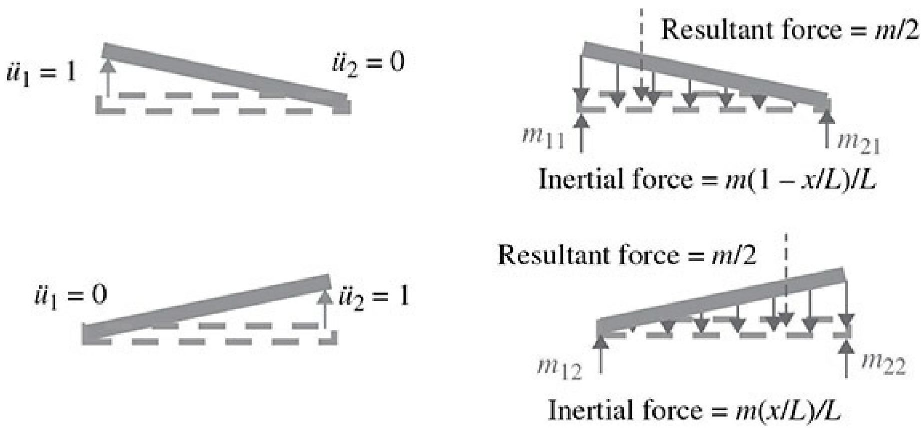

FIGURE E5.3 Mass influence coefficients.

ii. *Determine the mass matrix.* Apply a unit acceleration to each of the two DOFs to determine the distribution of acceleration and the associated inertial forces as shown in Fig. E5.3; note that the forces required to hold the distribution of inertial, forces are the mass influence coefficients; that is, each column of the mass matrix is a force system required to hold a unit acceleration at each DOF.
Equilibrium of the top FBD in Fig. E5.3,

$$\sum F_y = 0;\, m_{11} + m_{21} - m/2 = 0 \Rightarrow m_{11} + m_{21} = m/2$$

$$\sum M_1 = 0; -L/3(m/2) + m_{21}L = 0 \Rightarrow m_{21} = m/6$$

So,

$$m_{11} = m/3$$

Equilibrium of the bottom FBD in Fig. E5.3,

$$\sum F_y = 0;\, m_{12} + m_{22} - m/2 = 0 \Rightarrow m_{12} + m_{22} = m/2$$

$$\sum M_1 = 0; -2L/3(m/2) + m_{22}L = 0 \Rightarrow m_{22} = m/3$$

So,

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$$m_{12} = m/6$$

In matrix form,

$$[m] = \frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Notice that there is coupling since this matrix is not diagonal. The equation of motion can then be written as

$$\frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + k\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

iii. *Determine the natural frequencies.* The natural frequencies can be determined by solving an eigenvalue problem, where the eigenvalues yield the natural frequencies,

$$\det\left([k] - \omega_n^2[m]\right) = 0$$

$$\det\begin{bmatrix} k - m\omega_n^2/3 & -m\omega_n^2/6 \\ -m\omega_n^2/6 & 2k - m\omega_n^2/3 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The frequency equation resulting from the expansion of this eigen equation is

$$m^2\omega_n^4 - 12km\omega_n^2 + 24k^2 = 0$$

Using the quadratic formula we can find the roots of this equation, which are the eigenvalues:

$$\omega_n^2 = \frac{12km \pm \sqrt{(12km)^2 - 96m^2k^2}}{2m^2} = 6 \pm 2\sqrt{3}\,\frac{k}{m}$$

The frequencies are (from smallest to largest)

$$\omega_{n1} = \sqrt{6 - 2\sqrt{3}\,\frac{k}{m}} = 1.592\,\sqrt{\frac{k}{m}} \text{ and } \omega_{n2} = \sqrt{6 + 2\sqrt{3}\,\frac{k}{m}} = 3.076\sqrt{\frac{k}{m}}$$

iv. *Determine the modal matrix.* The mode shapes can now be obtained by substituting each eigenvalue (ω_{n1}^2 first) into the following equation and setting $a_1 = 1$,

$$\begin{bmatrix} k - m\omega_n^2/3 & -m\omega_n^2/6 \\ -m\omega_n^2/6 & 2k - m\omega_n^2/3 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k\begin{bmatrix} 0.155 & -0.423 \\ -0.423 & 1.155 \end{bmatrix}\begin{Bmatrix} 1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

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Solving for a_1, using the top row, the resulting eigenvector (mode shape) is

$$\begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 0.366 \end{Bmatrix}$$

Note that the same information is obtained from both top and bottom rows. That is, the displacement of the rear spring is always 0.366 times the displacement of the front spring when the system vibrates in the first mode. Using the same process, we can obtain the second eigenvector (mode shape) as

$$\begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -1.366 \end{Bmatrix}$$

The modal matrix is

$$[\Phi] = \begin{bmatrix} 1 & 1 \\ 0.366 & -1.366 \end{bmatrix}$$

These two mode shapes represent two possible simple harmonic motions of the automobile suspension and are known as natural modes of vibration since the vehicle moves in phase at the same frequency for each mode. A summary is shown in Fig. E5.4.

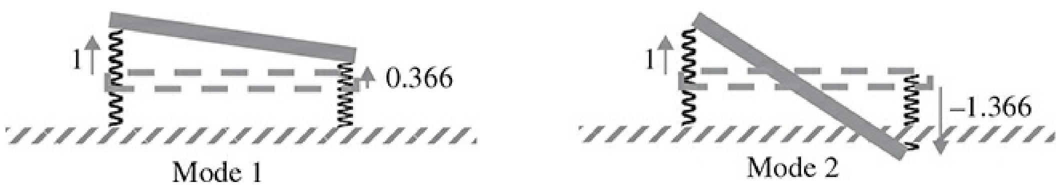

FIGURE E5.4 Mode shapes.

This example illustrates a case where the mass matrix is not diagonal. In general, therefore, mass and stiffness matrices can include off-diagonal elements. ▲

Example 6

Consider the free vibration response of a simply supported beam modeled as a three-degree-of-freedom system with masses lumped at the quarter points and determine (a) the mass matrix, (b) the flexibility matrix, (c) the stiffness matrix from the flexibility matrix, and (d) frequencies and modal matrix (normalized mode shapes). The beam is a W24x370 with E = 29,000 kip/in^2, I_x = 13,400 in^4, and weight = 370 lb/ft.

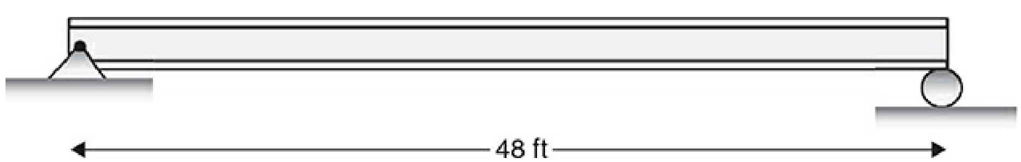

FIGURE E6.1 Simply supported beam model.

Solution

i. *Determine the mass matrix.* We subdivide the beam into four 12 ft segments, with each segment contributing half of its length to each of the three masses. Thus, the tributary length for each mass is 12 ft. So, the masses have equal magnitude of m = 370 lb/ft (12 ft)/386.4 in/s^2 = 11.49 lb · s^2/in. Notice that the portion of mass between supports and exterior masses, the support tributary mass, does not participate in the analysis.

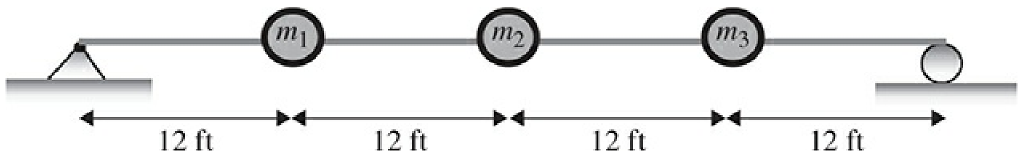

FIGURE E6.2 Lumped masses

Applying a unit acceleration to each of the masses only causes an acceleration and an associated inertial force in each mass. Thus, the force required to hold the inertial forces are the mass influence coefficients; that is, each column of the mass matrix is a force system required to hold a unit acceleration at each DOF.

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 11.49 & 0 & 0 \\ 0 & 11.49 & 0 \\ 0 & 0 & 11.49 \end{bmatrix} \frac{\text{lb} \cdot \text{s}^2}{\text{in}}$$

ii. *Determine the flexibility matrix.* Application of a unit force to each of the three DOFs separately results in deflections of the masses, which are the flexibility influence coefficients as shown in Fig. E6.3. For example, applying the unit load to the first DOF causes displacements of the DOFs, which collectively correspond to the first column of the flexibility matrix. Subsequent DOFs can be loaded to obtain the remaining columns of the flexibility matrix. Also, since the flexibility matrix is equal to the inverse of the stiffness matrix and the stiffness matrix is symmetric, then the flexibility matrix is also symmetric. Thus, only the diagonal and upper or lower influence elements need to be computed.

From structural analysis, the beam displacement, y (or elastic curve), caused by a unit load placed at an arbitrary distance, a_1, from the first support is

$$y = \frac{bx}{6LEI}(L^2 - b^2 - x^2) \text{ for } 0 \le x \le a_1$$

where

L is the span of the beam.

b is $L - a_1$.

EI is the flexural stiffness.

x is the location of y.

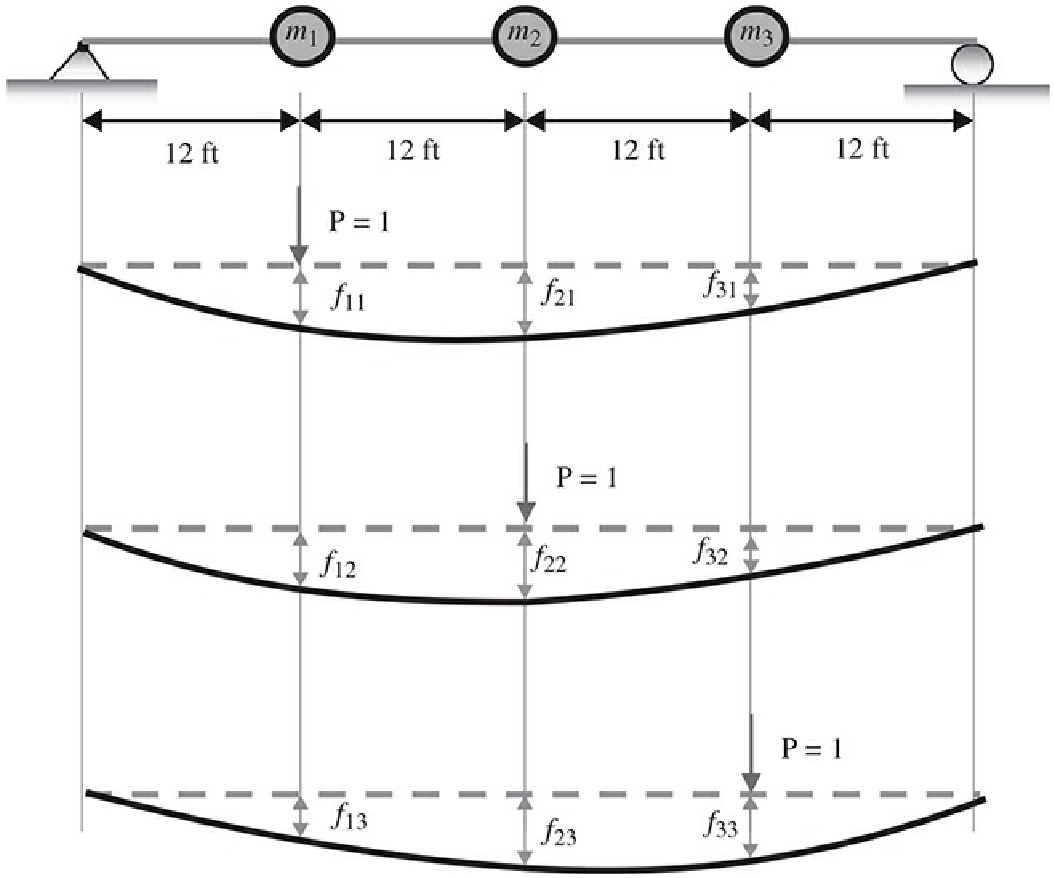

FIGURE E6.3 Flexibility influence coefficients.

So, from the symmetry of the deflected shapes when the unit load is placed at L/4 or 3L/4 in Fig. E6.3,

$$f_{11} = f_{33} = y\left(x = \frac{L}{4}, b = \frac{3L}{4}\right) = \frac{(3L/4)(L/4)}{6LEI}\left[L^2 - \left(\frac{3L}{4}\right)^2 - \left(\frac{L}{4}\right)^2\right] = \frac{9L^3}{768EI}$$

$$f_{31} = f_{13} = y\left(x = \frac{L}{4}, b = \frac{L}{4}\right) = \frac{(L/4)(L/4)}{6LEI}\left[L^2 - \left(\frac{L}{4}\right)^2 - \left(\frac{L}{4}\right)^2\right] = \frac{7L^3}{768EI}$$

When the unit load is placed at midspan, y is

$$y = \frac{(3L^2x - 4x^3)}{48EI} \text{ for } 0 \le x \le L/2$$

So,

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$$f_{12} = f_{21} = f_{32} = f_{23} = y\left(x = \frac{L}{4}\right) = \frac{\left(3L^2(L/4) - 4(L/4)^3\right)}{48EI} = \frac{11L^3}{768EI}$$

$$f_{22} = y\left(x = \frac{L}{2}\right) = \frac{(3L^2(L/2) - 4(L/2)^3)}{48EI} = \frac{16L^3}{768EI}$$

The flexibility matrix,

$$[f] = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} \frac{L^3}{768EI} = \begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} 6.4 \times 10^{-4} \text{ in/kip}$$

iii. *Determine the stiffness matrix.* We can now obtain the stiffness matrix by inverting the flexibility matrix,

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} = [f]^{-1} = \begin{bmatrix} 23 & -22 & 9 \\ -22 & 32 & -22 \\ 9 & -22 & 23 \end{bmatrix} \frac{768EI}{28L^3}$$

$$= \begin{bmatrix} 23 & -22 & 9 \\ -22 & 32 & -22 \\ 9 & -22 & 23 \end{bmatrix} 55{,}770 \, \frac{\text{lb}}{\text{in}}$$

iv. *Determine the natural frequencies and modal matrix.* The natural frequencies can be determined by solving an eigenvalue problem, where the eigenvalues yield the natural frequencies,

$$\det\left([k] - \omega_n^2 [m]\right) = 0$$

We can also write the eigenvalue problem in terms of the flexibility matrix as,

$$\det\left([m][f] - [I]/\omega_n^2\right) = 0$$

Here the eigenvalues are $1/\omega_n^2$.

We can obtain both frequencies and mode shapes using the MATLAB script given in Example 3:

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```
clear all % Chapter 6, Example 6
m = [1 0 0; 0 1 0; 0 0 1]*11.49; % mass matrix in lb-in/sec^2
k = [23 -22 9;-22 32 -22; 9 -22 23]*55770; % stiffness matrix in lb/in
[phi,lam]=eig(k,m); % compute eigenvalues and eigenvectors
omegas=sqrt(lam) % determine and show frequencies from eigenvalues
periods = 2*pi*diag(inv(omegas)) % determine the periods from omegas
% Mode shapes by normalizing eigenvectors to get top displ equal to 1
[N_rows, N_cols] = size(phi); % finds N, the size of the matrices
for i = 1:N_cols; % loops over the N modes to normalize them
    norm_phi(:,i) = phi(:,i)./phi(1,i);
end
norm_phi % display normalized modal matrix
% Determine the modal mass and stiffness matrices
M = norm_phi'*m*norm_phi
K = norm_phi'*k*norm_phi
```

The results of the script are

```
  omegas =
     65.6260         0         0
           0  260.6780         0
           0         0  553.4759
  periods =
      0.0957
      0.0241
      0.0114
  norm_phi =
      1.0000    1.0000    1.0000
      1.4142    0.0000   -1.4142
      1.0000   -1.0000    1.0000
  M =
     45.9600   -0.0000    0.0000
     -0.0000   22.9800   -0.0000
      0.0000   -0.0000   45.9600
  K =
     1.0e+07 *
      0.0198   -0.0000   -0.0000
     -0.0000    0.1562   -0.0000
     -0.0000         0    1.4079
```

▲

6.3 Free Vibration Response of MDOF Systems with Viscous Damping

As noted in Chap. 2, inherent damping in structures is quite small and practically does not affect the values of frequencies or mode shapes for most systems, as shown in Fig. 2.7. However, the response of some damped structures in the resonant region can be significant. Thus, we can add damping to Eq. (6.3) by including a damping matrix that produces damping forces proportional to the velocity of the various degrees of freedom,

$$\text{damping force} = [c]\{\dot{u}\}$$

The matrix equation of motion for free vibration of a damped MDOF system can then be written as

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{0\} \tag{6.40}$$

Typically, this equation cannot be diagonalized because the undamped free vibration mode shapes are not orthogonal with respect to the damping matrix (unlike the mass and stiffness matrices where this mode shape property yielded diagonal modal mass and stiffness matrices). And thus, generally Eq. (6.40) cannot be uncoupled. For example, in some cases of elastic analysis of MDOF systems damping effects might be significant; such cases are classified as *nonclassical damping* because the damping matrix cannot be diagonalized. Solving nonclassical MDOF damping problems is similar to the SDOF case covered in Chap. 2 but requires significantly more computational effort. Since Eq. (6.40) is a system of second-order, linear, and homogeneous differential equations with constant coefficients, we can assume a solution of exponential form, similar to that presented in Chap. 2 but in matrix form,

$$\{u_j(t)\}=\{\varphi_{Dj}\}e^{i\omega_{Dnj}t} \tag{6.41}$$

where ω_{Dnj} is the damped frequency, and $\{\varphi_{Dj}\}$ is the corresponding damped mode shape for DOF *j*, $i=\sqrt{-1}$ in this equation.

We can differentiate Eq. (6.41) with respect to time to obtain the velocity, which then can be differentiated to determine the acceleration. Substituting the equations for displacement, velocity, and acceleration into Eq. (6.40) we get

$$\left\langle -\omega_{Dnj}^2[m]+i\omega_{Dnj}[c]+[k]\right\rangle\{\varphi_{Dj}\}e^{i\omega_{Dnj}t}=\{0\} \tag{6.42}$$

Since the exponential function is usually nonzero, $\left\langle -\omega_{Dnj}^2[m]+i\omega_{Dnj}[c]+[k]\right\rangle\{\ _{Dj}\}=0$, which upon substituting $S_j=i\omega_{Dnj}$ and $S_j^2=-\omega_{Dnj}^2$ we get a quadratic equation in S_j,

$$\left\langle S_j^2[m]+S_j[c]+[k]\right\rangle\{\varphi_{Dj}\}=0$$

This gives nontrivial solutions for $\{\varphi_{Dj}\}$ only if

$$\det\left\langle S_j^2[m]+S_j[c]+[k]\right\rangle=0$$

This is a quadratic eigenvalue problem, and its solution yields damped natural frequencies and mode shapes, values of which can be complex numbers. This solution process is impractical for most real problems, in part, because constructing the damping matrix for such problems requires conducting detailed experiments to obtain the various damping matrix elements. For this reason, this approach is rarely used, instead the damping matrix is assumed to be a linear combination of the mass and stiffness matrices, which along with the undamped eigenvalues and eigenvectors, yields a damping matrix that is orthogonal with respect to the undamped mode shapes. Such cases are classified as *classical damping* because the damping matrix can be diagonalized and

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the equations of motion can be uncoupled. This is known as Rayleigh damping and is presented in the next subsection.

6.3.1 Rayleigh Damping for MDOF Systems

As noted above, for most practical applications we can assume classical damping where we can uncouple the equations of motion by diagonalizing the damping matrix using the undamped mode shapes. In the Rayleigh method, the damping matrix is assumed to be a linear combination of the mass and stiffness matrices as follows:

$$[c] = a_0[m] + a_1[k] \tag{6.43}$$

where a_0 (1/s) and a_1 (s) are arbitrary constants.

Substituting modal coordinates, Eq. (6.27) for physical coordinates in Eq. (6.40), and premultiplying by the transpose of the modal matrix,

$$[\Phi]^T \left\langle [m][\Phi]\{\ddot{q}\} + \left(a_0[m] + a_1[k]\right)[\Phi]\{\dot{q}\} + [k][\Phi]\{q\} \right\rangle = 0$$

Or the equation of motion in modal coordinates can now be expressed as

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = 0 \tag{6.44}$$

where $[C]$ is a diagonal *modal damping matrix* with modal damping coefficients,

$$\begin{aligned} [C] &= a_0[M] + a_1[K] \\ [\Phi]^T[c][\Phi] &= a_0[\Phi]^T[m][\Phi] + a_1[\Phi]^T[k][\Phi] \end{aligned} \tag{6.45}$$

Thus, Eq. (6.44) can now be written as N uncoupled differential equations,

$$M_j\ddot{q}_j + C_j\dot{q}_j + K_jq_j = 0 \tag{6.46}$$

where

$$C_j = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_N \end{bmatrix}$$

Equations (6.46), which are in the same form as the equation of motion for SDOF systems, can also be written in component form analogously to the result in Chap. 2, Eq. (2.13) for the SDOF system,

$$\ddot{q}_j + 2\zeta_j\omega_{nj}\dot{q}_j + \omega_{nj}^2 q_j = 0 \tag{6.47}$$

where ζ_j is defined as

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$$\zeta_j = \frac{C_j}{2M_j\omega_{nj}}$$

Similar to the SDOF case presented in Chap. 2, Eq. (2.20), the solutions to Eq. (6.47) depend on ζ_j. The underdamped free vibration solution is shown below:

$$q_j(t) = e^{-\zeta_j\omega_{nj}t}\left\{\left[\frac{\dot{q}_j(0)+\zeta_j\omega_{nj}q_j(0)}{\omega_{Dnj}}\right]\sin\omega_{Dnj}t + q_j(0)\cos\omega_{Dnj}t\right\}$$

Here, ζ_j can be interpreted as the *modal damping ratio* associated with mode shape *j*. That is,

$$\frac{C_j}{M_j} = 2\zeta_j\omega_{nj} = a_0\frac{M_j}{M_j} + a_1\frac{K_j}{M_j} = a_0 + a_1\omega_{nj}^2$$

From which we can obtain ζ_j as

$$\zeta_j = \frac{a_0}{2\omega_{nj}} + \frac{a_1\omega_{nj}}{2} \tag{6.48}$$

Notice that the mass-proportional damping (first term) is inversely proportional to the frequency, whereas the stiffness-proportional damping (second term) is directly proportional to the frequency. The second term increases damping linearly with frequency leading to unrealistically high damping ratios, particularly in the higher modes of large MDOF systems. Fortunately, higher modes usually do not contribute significantly to the overall response of the system. A reasonable assumption is to use modal damping ratios for the first two modes, ζ_1 and ζ_2, and solve for a_0 and a_1. Then, damping factors for higher modes are assigned automatically. So,

$$\zeta_1 = \frac{a_0}{2\omega_{n1}} + \frac{a_1\omega_{n1}}{2} \text{ and } \zeta_2 = \frac{a_0}{2\omega_{n2}} + \frac{a_1\omega_{n2}}{2}$$

Thus,

$$a_0 = \frac{2\omega_{n1}\omega_{n2}(\zeta_2\omega_{n1} - \zeta_1\omega_{n2})}{\omega_{n1}^2 - \omega_{n2}^2} \text{ and } a_1 = \frac{2(\zeta_1\omega_{n1} - \zeta_2\omega_{n2})}{\omega_{n1}^2 - \omega_{n2}^2} \tag{6.49}$$

Example 7

Given the following three-story building frame with damping ratio of 5% in the first and third modes, assuming Rayleigh damping determine (a) natural frequencies and mode shapes, (b) damping ratio for the second mode, and (c) modal mass, stiffness, and damping matrices. Assume beams are rigid and each story has a stiffness k = 326.3 kip/in.

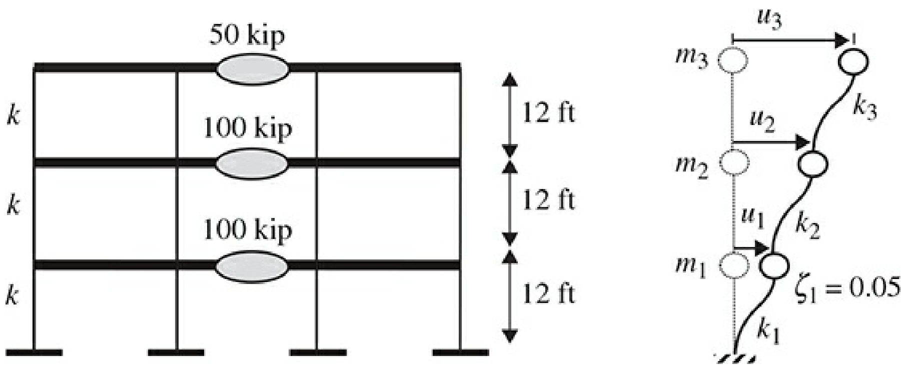

FIGURE E7.1 Building frame schematic (left) and idealized MDOF structural model (right).

Solution

i. *Determine the mass and stiffness matrices, frequencies, and shape vectors.* The equations of motion for this case can be determined using D'Alembert's principle by applying horizontal equilibrium to the FBDs of the following three masses.

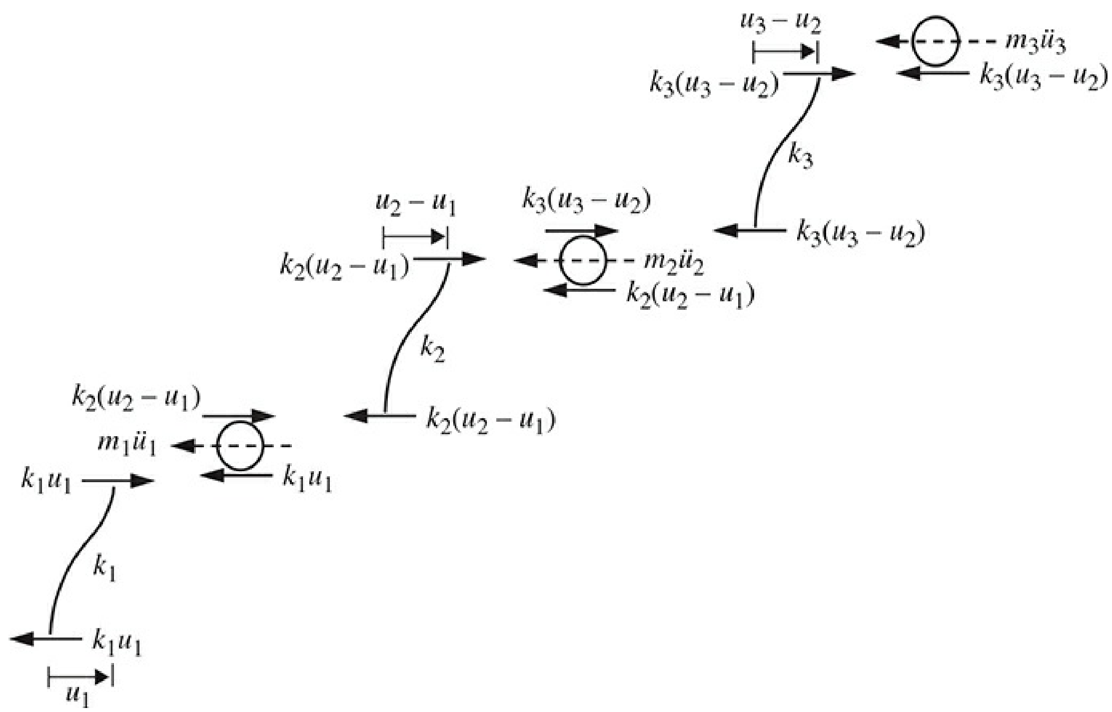

FIGURE E7.2 Free body diagrams for each mass used to determine equations of motion.

The three equations of motion base on horizontal equilibrium, $\sum F_x = 0$, are as follows:

$$-m_1\ddot{u}_1 - k_1u_1 + k_2(u_2 - u_1) = 0 \Rightarrow m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = 0$$
$$-m_2\ddot{u}_2 - k_2(u_2 - u_1) + k_3(u_3 - u_2) = 0 \Rightarrow m_2\ddot{u}_2 - k_2u_1 + (k_2 + k_3)u_2 - k_3u_3 = 0$$
$$-m_3\ddot{u}_3 - k_3(u_3 - u_2) = 0 \Rightarrow m_3\ddot{u}_3 - k_3u_2 + k_3u_3 = 0$$

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From these three dependent equations of motion, we can determine the mass and stiffness matrices by substituting $k_1 = k_2 = k_3 = k = 326.3$ kip/in and $m_1 = m_2 = m = 100$ kip/g and $m_3 = 50$ kip/g,

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 100\text{ kip} & 0 & 0 \\ 0 & 100\text{ kip} & 0 \\ 0 & 0 & 50\text{ kip} \end{bmatrix} / 386.4\,\frac{\text{in}}{\text{s}^2}$$

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot 326.3\,\frac{\text{kip}}{\text{in}}$$

$$[k] = \begin{matrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{matrix} = \begin{matrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{matrix} \cdot 326.3\,\frac{\text{kip}}{\text{in}}$$

We use MATLAB to solve the eigenvalue problem:

```
clear all % clears any previously defined variables
m = [100 0 0; 0 100 0; 0 0 50]/386.4 % mass matrix in kip-sec2/in
k = [2 -1 0; -1 2 -1; 0 -1 1]*326.3 % stiffness matrix in kip/in
[phi,lam]=eig(k,m); % compute eigenvalues and eigenvectors
omegas=sqrt(lam) % determine and show frequencies from eigenvalues
% mode shapes by normalizing eigenvectors to get top displ equal to 1
[N_rows, N_cols] = size(phi); % finds N, the size of the matrices
for i = 1:N_cols; % loops over the N modes to normalize them
    norm_phi(:,i) = phi(:,i)./phi(N_rows,i);
end
norm_phi % display normalized modal matrix
```

The results of the script are

```
omegas =
 18.3803       0        0
       0 50.2160        0
       0       0  68.5963
norm_phi =
  0.5000 -1.0000   0.5000
  0.8660  0.0000  -0.8660
  1.0000  1.0000   1.0000
```

ii. *Determine the damping ratio for the second mode.* With the eigenvalues and eigenvectors, we first determine a_0 and a_1 using the two modes given, ζ_1 and ζ_3; from Eq. (6.49),

$$a_0 = \frac{2\omega_{n1}\omega_{n3}\left(\zeta_3\omega_{n1} - \zeta_1\omega_{n3}\right)}{\omega_{n1}^2 - \omega_{n3}^2} = \frac{2(18.38)(68.6)(0.05(18.38) - 0.05(68.6))}{(18.38)^2 - (68.6)^2} = 1.45/\text{s}$$

$$a_1 = \frac{2(\zeta_1\omega_{n1} - \zeta_3\omega_{n3})}{\omega_{n1}^2 - \omega_{n3}^2} = \frac{2(0.05(18.38) - 0.05(68.6))}{(18.38)^2 - (68.6)^2} = 0.0011 \text{ s}$$

We can now estimate damping ratio for the second mode using Eq. (6.48),

$$\zeta_2 = \frac{a_0}{2\omega_{n2}} + \frac{a_1\omega_{n2}}{2} = \frac{1.45}{2(50.22)} + \frac{(0.0011)(50.22)}{2} = 0.042 \text{ or } 4.2\%$$

Notice that it is not necessary to use the first two modes to determine a_0 and a_1, we can use any two modes and then compute the damping ratio for the remaining vibration modes with the same values of a_0 and a_1.

iii. *Determine the modal mass, stiffness, and damping matrices.* Use MATLAB with `M = norm_phi'*m*norm_phi` for the modal mass, `K = norm_phi'*k*norm_phi` for the modal stiffness, and `C = omegas.*M.*z`.

```
M =
     0.3882    0.0000   -0.0000
     0.0000    0.3882    0.0000
    -0.0000    0.0000    0.3882
K =   1.0e+03 *
     0.1311   -0.0000   -0.0000
          0    0.9789         0
    -0.0000    0.0000    1.8267
C =
     0.3568         0         0
          0    0.8441         0
          0         0    1.3315
```
▲

For many practical MDOF structural dynamics problems, this process is not necessary since higher modes will not contribute significantly to the displacement response. For such cases, we can assume damping to be constant for all mode shapes. Thus, we can use the analysis developed for the undamped case to solve for damped frequencies and mode shapes, and use the uncoupled equations of motion to solve for the response in modal coordinates,

$$\ddot{q}_j + 2\zeta\omega_{nj}\dot{q}_j + \omega_n^2 q_j = 0 \tag{6.50}$$

Finally, we can linearly combine these results using the mode shapes to obtain the overall response in physical coordinates with Eq. (6.27), repeated here for convenience.

$$\{u\} = [\Phi]\{q\}$$

6.4 Problems

6.1 Verify that the mode shapes obtained in Example 5 are orthogonal, that is, prove Eqs. (6.28) and (6.29) using the results of Example 5.

6.2 Using the definition of stiffness and mass influence coefficients formulate the equation of motion for the frame shown below. Assume the beams are rigid.

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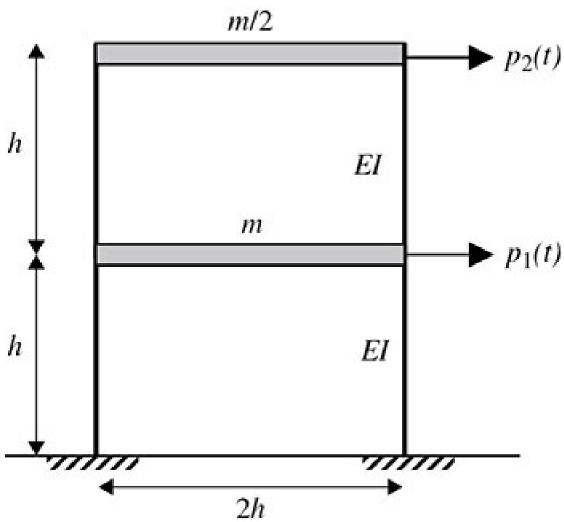
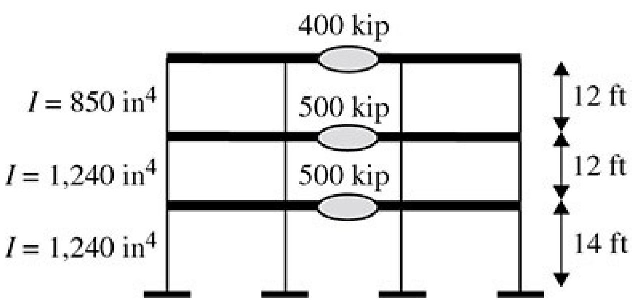
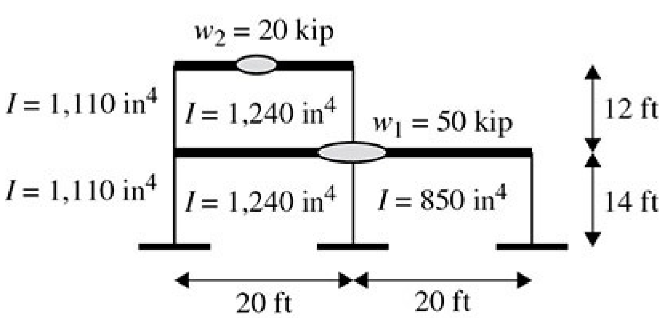


6.3 Consider the two-degree-of-freedom system given in Prob. 6.2, determine (a) periods and modal matrix (normalized mode shapes) and (b) the displacement response assuming initial displacements $u_1(0) = d_1$, $u_2(0) = d_2$ and initial velocities equal to zero.

6.4 Formulate the equation of motion (in matrix form) using D'Alembert's principle for the given three-story building, which has rigid beams and flexible steel (E = 29,000 ksi) columns with total moment of inertia for each floor as shown.



6.5 Formulate the equation of motion (in matrix form) using D'Alembert's principle for the given two-story building, which has rigid beams and flexible steel (E = 29,000 ksi) columns with moment of inertia for each column as shown.



6.6 Formulate the equation of motion (in matrix form) using D'Alembert's principle for the given three-story building, which has rigid beams and flexible steel (E = 29,000 ksi) columns with column sections as shown.

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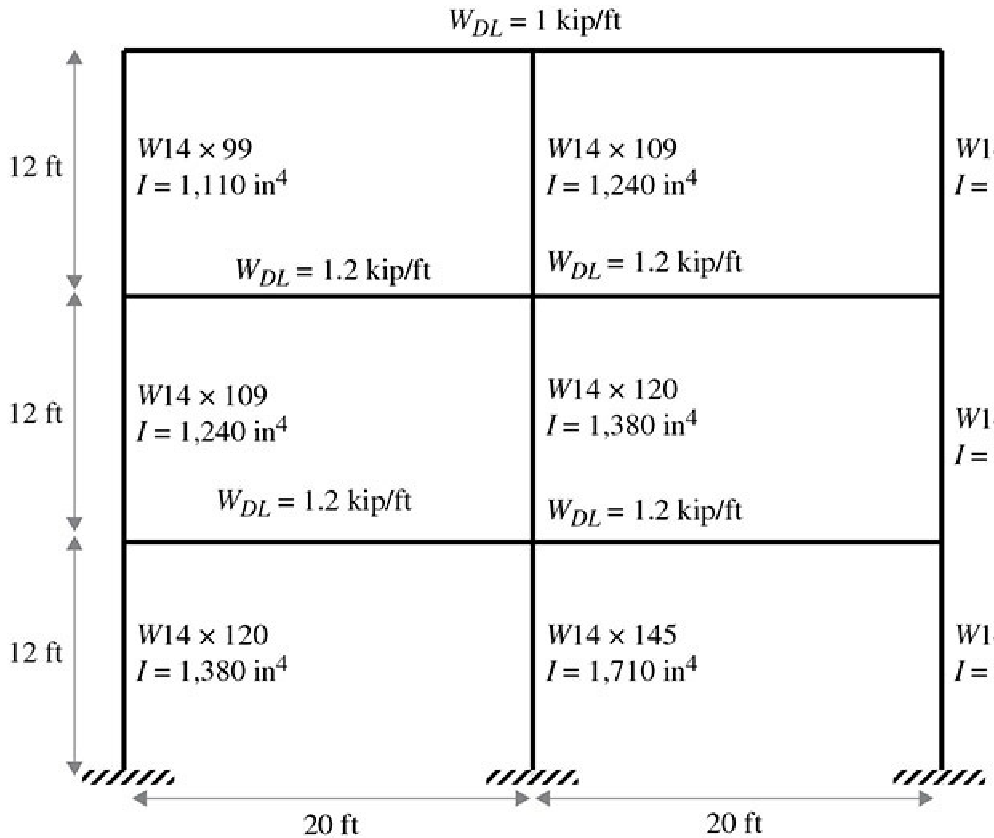
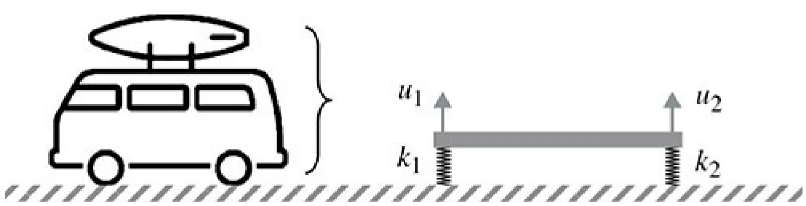


6.7 Consider the simplified model of the automobile shown below and determine (a) the mass and stiffness matrices, (b) frequencies, and (c) modal matrix (normalized mode shapes). Assume $k_1 = 3k$, $k_2 = 2k$, and a total mass of the vehicle of m.



6.8 The following two-story building is supported with steel frames having 30 ft bays and has the floor load shown plus wall load of 20 psf. Formulate the equation of motion (in matrix form) using D'Alembert's principle and determine the natural frequencies, periods, and modal mass and stiffness matrices.

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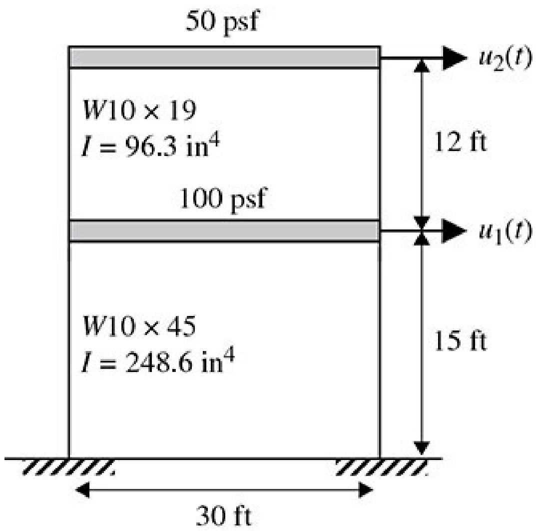
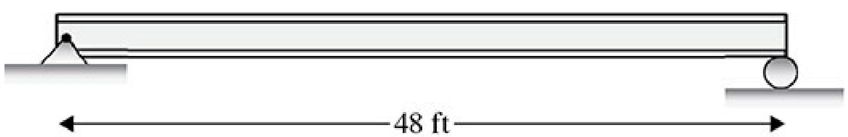


6.9 Given a five-story building with equal story masses, *m*, and stiffnesses, *k*, formulate the equation of motion (in matrix form) using D'Alembert's principle and determine the natural frequencies, periods, and modal mass and stiffness matrices using MATLAB.

6.10 Consider the free vibration response of a simply supported beam modeled as a two-degree-of-freedom system with masses lumped at the third points and determine (a) the mass matrix, (b) the flexibility matrix, (c) the stiffness matrix from the flexibility matrix, and (d) frequencies and modal matrix (normalized mode shapes). The beam is a W24x370 with E = 29,000 ksi, I_x = 13,400 in^4, and weight = 370 lb/ft.



6.11 Given a three-story building frame with damping ratio of 5% in the first and third modes, assuming Rayleigh damping determine (a) damping ratio for the second mode, and (b) modal damping matrix. Assume the given mode shapes and frequencies, floor weights of 120 kip for the first floor, and 80 kip for the second floor and roof.

$$[\Phi] = \begin{bmatrix} 1.68 & -1.208 & -0.714 \\ 1.22 & 0.704 & 1.697 \\ 0.572 & 1.385 & -0.984 \end{bmatrix} \quad \text{and} \quad \{\omega\} = \begin{Bmatrix} 8.77 \\ 25.18 \\ 48.13 \end{Bmatrix} \text{rad/s}$$

6.12 Given a five-story building frame with damping ratio of 5% in the first and third modes, assuming Rayleigh damping determine (a) natural frequencies and mode shapes, (b) damping ratio for the other three modes, and (c) modal mass, stiffness, and damping matrices. Assume beams are rigid and each story has a stiffness k = 326.3 kip/in, the fifth floor has a weight of 50 kip, and the weight on each of the other four floors is 100 kip.

6.13 Given a five-story building frame with a uniform damping ratio of 5%, estimate the modal damping matrix. Assume beams are rigid and each story has a stiffness k = 326.3 kip/in, the fifth floor has a weight of 50 kip, and the weight on each of the other four floors is 100 kip.

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CHAPTER 7

Forced Vibration of MDOF Systems

After reading this chapter, you will be able to:

a. Determine the response of undamped multi-degree-of-freedom (MDOF) systems excited by harmonic loads
b. Determine the response of undamped MDOF systems excited by impulsive loads
c. Determine the response of undamped MDOF systems excited by general loads
d. Determine the response of damped MDOF systems excited by harmonic loads
e. Determine the response of damped MDOF systems excited by impulsive loads
f. Determine the response of damped MDOF systems excited by general loads
g. Formulate and solve the equation of motion for effect of support motion on MDOF systems

In Chap. 6, we considered the free vibration response caused by initial conditions (initial displacement and velocity) on a MDOF system. In this chapter, we focus on the response caused by external excitations, both applied forces to the various degrees of freedom and support motion. The combination of external excitations and initial displacement and velocity can be obtained following a similar process to the one presented in Chap. 3. After the equations of motion of a system are decoupled as shown in Chap. 6, forced vibration response of MDOF systems is analogous to that of single-degree-of-freedom (SDOF) systems, except now we combine results from various mode shapes to acquire the overall response. To get a reasonably accurate result, we do not have to include the effect of every mode shape; this is particularly expeditious for large-degree-of-freedom cases. In fact, the first mode shape tends to contribute more than half of the response for most practical cases.

To characterize the response of MDOF systems subjected to a time-dependent excitation, we first formulate the equation of motion for the various degrees of freedom using D’Alembert’s principle, which is similar to the process described in Chap. 6 (as shown in Fig. 6.2), but equations of equilibrium now include time-dependent forces, $p^i(t)$. Depending on the frequency of the force, it can interrupt the oscillatory motion of the system (even if no damping is present), or it can amplify the amplitude. For damped cases, the motion from the initial conditions is dissipated (transient response), but the system continues to vibrate at the applied forcing frequency (steady-state response) for periodic forces. For impulsive, nonperiodic forces the response of damped systems eventually ceases. In this chapter, we consider several different excitations, from harmonic to impulsive to general, similar to the different excitations considered in Chaps. 3 and 4.

7.1 Forced Vibration Response of Undamped MDOF Systems

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Consider the frame model discussed in Chap. 6, Fig. 6.2 (repeated here as Fig. 7.1 for convenience). Recall that the equation of motion for a MDOF system was formulated using D'Alembert's principle and equilibrium of the two free-body diagrams (FBDs), including the time-dependent forces, $p_i(t)$. Horizontal equilibrium yields two equations of motion, which were given in Chap. 6 as Eqs. (6.6) and (6.7),

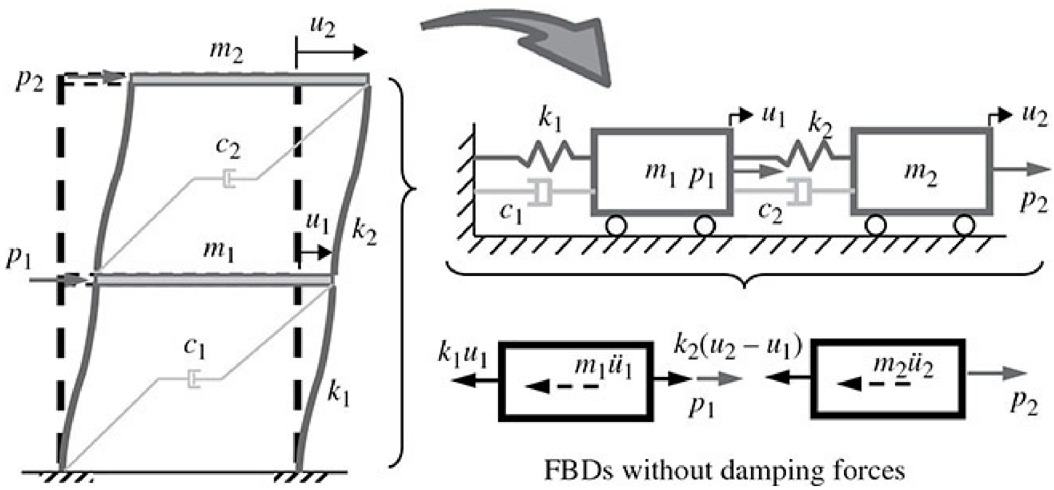

FIGURE 7.1 Idealized shear building as oscillators and FBDs.

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = p_1 \qquad (7.1)$$

$$m_2\ddot{u}_2 - k_2u_1 + k_2u_2 = p_2$$

where

m^1 and m^2 are the floor masses, which can be determined from the weights on the corresponding floor.
k^1 and k^2 are the story stiffnesses, which can be determined by adding the lateral stiffnesses of all columns within the corresponding story. Each story can be treated as a portal frame as discussed in Chap. 1; thus, for a story of height h, column flexural stiffness EI, and assuming rigid floor diaphragms (rigid frame beams), the lateral stiffness of a column is $12EI/h^3$ and the stiffness of an arbitrary story j is $k_j = \sum_{\text{columns}} 12EI/h^3$.

Equations (7.1) are coupled and can be rewritten in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

In general, the equations of motion are written as

$$[m]\{\ddot{u}\} + [k]\{u\} = \{p\} \qquad (7.2)$$

where

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$[m]$ is the mass matrix; this is usually diagonal for systems with localized mass, but as shown in Chap. 6, Example 5 can also be fully populated for cases with distributed mass, in which case, m_{ij} represents the force at coordinate i caused by a unit acceleration at coordinate j.

$[k]$ is the stiffness matrix; this can alternatively be obtained by directly finding each coefficient k_{ij} that represents a force needed at level i to hold a unit displacement at level j while holding the displacements of all other DOFs equal zero.

$\{p\}$ is the force vector.

$\{u\}$ is the physical displacement vector.

$\{\ddot{u}\}$ is the corresponding acceleration vector.

Again, an Nth-degree-of-freedom (N-DOF) system results in N dependent equations [Eq. (7.2)]. These can be solved directly, or as shown in Chap. 6, can be decoupled using the generalized (or modal) coordinates, q^j , by assuming the resulting displacements can be described by harmonic relationships, Eq. (6.9). The process requires a free vibration solution of Eq. (7.2), which yields the natural frequencies and the corresponding vibration mode shapes; see Chap. 6. This modal analysis maps the physical coordinates to the modal coordinates, in the process decoupling the differential equations of motion. That is, we express the solution in terms of the normal modes multiplied by some factor determining the contribution of each mode [Eq. (6.27)],

$$\{u(t)\} = [\Phi]\{q(t)\} \tag{7.3}$$

Or,

$$\begin{aligned} u_1(t) &= \varphi_{11}q_1(t) + \cdots + \varphi_{1N}q_N(t) \\ u_2(t) &= \varphi_{21}q_1(t) + \cdots + \varphi_{2N}q_N(t) \\ &\vdots \\ u_N(\mathrm{t}) &= \varphi_{N1}q_1(\mathrm{t}) + \cdots + \varphi_{NN}q_N(\mathrm{t}) \end{aligned} \tag{7.4}$$

We can also express the acceleration vectors in terms of the generalized coordinates,

$$\{\ddot{u}(t)\} = [\Phi]\{\ddot{q}(t)\} \tag{7.5}$$

So, Eq. (7.2), can be written as

$$[m][\Phi][\{\ddot{q}(t)\} + [k][\Phi]\{q(t)\} = \{p\} \tag{7.6}$$

Premultiplying by the transpose of the modal matrix,

$$[\Phi]^T[m][\Phi]\{\ddot{q}(t)\} + [\Phi]^T[k][\Phi]\{q(t)\} = [\Phi]^T\{p\} \tag{7.7}$$

This operation uncouples the equations of motion because of the mass and stiffness orthogonality relationships,

$$[M]\{\ddot{q}(t)\} + [K]\{q(t)\} = \{P(t)\} \tag{7.8}$$

where the modal forces are defined as

$$\{P(t)\} = [\Phi]^T \{p(t)\} \tag{7.9}$$

Writing Eqs. (7.8) and (7.9) for the *i*th degree of freedom,

$$M_i \ddot{q}_i(t) + K_i q_i(t) = P_i(t) \tag{7.10}$$

and

$$P_i(t) = \{\varphi_i\}^T \{p(t)\} \tag{7.11}$$

7.1.1 Displacements, Nodal Forces, Base Shears, and Overturning Moments

Equation (7.10) essentially represents *N* uncoupled SDOF equations, which can be solved for $\{q(t)\}$ using one of the methods presented in Chaps. 3 and 4. The response is in modal coordinates, which must then be mapped to physical coordinates to obtain the total response, $\{u(t)\} = [\Phi]\{q(t)\}$, [Eq. (7.3)], as discussed in Sec. 6.2.3. Once the dynamic displacement, $\{u(t)\}$, is known at any time, *t*, the nodal force response can be obtained using one of two methods (as discussed in Sec. 1.7):

a. Directly determining the internal element forces using structural analysis.
b. First obtaining equivalent static nodal forces that can then be used to conduct a static structural analysis of the system.

Here, we focus on the second approach because it is relatively straightforward, and no additional dynamic analysis is required. The analysis entails determining external forces, $\{f_i(t)\}_j$, [or in element form, $f_{ij}(t)$] at each node *i* that produce displacements $\{u_i(t)\}_j$ [or in element form, $u_{ij}(t)$] in the stiffness components of the structure for each mode *j*. Recall that each column of the stiffness matrix represents the nodal forces required to hold a nodal displacement equal to 1. In component form,

$$\{f_i(t)\}_j = [k]\{u_i(t)\}_j \tag{7.12}$$

where

$j = 1, 2, \cdots N$ mode shapes.

$\{u_i(t)\}_j$ is the physical displacement at node or level *i* for mode *j*.

$\{f_i(t)\}_j$ is the equivalent static force at node or level *i* for mode *j*.

Or we can establish a matrix of nodal forces with respect to a matrix of nodal displacements:

$$[f(t)] = [k][u(t)] \tag{7.13}$$

Equation (7.13) only applies to linear elastic cases. For nonlinear cases, we could use the inertial force, or Newton's second law. That is, the equivalent static forces associated with nodal accelerations:

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$$[f(t)] = [m][\ddot{u}(t)] \tag{7.14}$$

Obtaining an overall closed-form solution in physical coordinates generally requires a complete response history analysis, which is only feasible for relatively simple cases of forcing functions that are well defined, such as harmonic. However, for design and analysis purposes we are primarily interested in the maximum response of the system at the various nodes. In such cases, we can use all the solutions presented in previous chapters, including the response spectra analysis presented in Chap. 4. Recall that a response spectrum gives the maximum response (usually the displacement) in the form of a dimensionless time variable, known as the dynamic load factor [DLF = $q(t)/q_{st}$], where $q_{st} = p_o/k$ is the equivalent static displacement and can be interpreted as the displacement caused when the amplitude of the force, p_o, is applied very slowly.

The solution to Eq. (7.10) based on the DLF of the *i*th mode as a function of time is given as

$$q_i(t) = P_i/K_i \cdot \mathrm{DLF}_i(t) \tag{7.15}$$

where $q_i(t)$ is the modal response of mode shape *i*, and the equivalent static displacement for the *i*th mode is

$$\frac{P_i}{K_i} = \frac{\{\varphi_i\}^T\{p\}}{\{\varphi_i\}^T[k]\{\varphi_i\}} = \frac{\{\varphi_i\}^T\{p\}}{(\omega_i)^2\{\varphi_i\}^T[m]\{\varphi_i\}} \tag{7.16}$$

The denominator of Eq. (7.16) second equation is the result of $k = \omega_n^2 m$. Equation (7.16) is sometimes used to represent the *modal participation factor* and can be interpreted as the relative contribution of each mode to the total response. However, it is not strictly a measure of the contribution of a mode to a response quantity because it depends on how the mode shapes are normalized as well as the frequency content of the forcing function relative to the modal frequency. For example, if we normalize the eigenvectors to have unit displacement at the first level (rather than the top level assumed here), the resulting modal parameter will change since the occurrence of $\{\varphi_i\}$ in the numerator and denominator is not proportional. Also, if a forcing function frequency is comparable to a particular modal frequency from one of the higher modes, the response can be dominated by that higher mode, because of resonance.

We can now obtain the maximum displacement and nodal force responses for each mode, *j*, in terms of the maximum dynamic load factor, $\mathrm{DLF}^{\max}$,

$$\{u_{\max\,i}\}_j = \{\varphi_{ij}\} \cdot q_{\max\,j} = \{\varphi_{ij}\} \cdot P_j/K_j \cdot \mathrm{DLF}_{\max\,j} \tag{7.17}$$

and

$$\{f_{\max\,i}\}_j = [k]\{u_{\max\,i}\}_j = [k]\{\varphi_{ij}\} \cdot q_{\max\,j} = [k]\{\varphi_{ij}\} \cdot P_j/K_j \cdot \mathrm{DLF}_{\max\,j} \tag{7.18}$$

Equation (7.18) can be written in matrix form as $[f] = [k][u]$, Eq. (7.13), where the columns of [*f*] are the vectors $\{f_{\max\,i}\}_j$ and the columns of [*u*] are the vectors $\{u_{\max\,i}\}_j$.

Also, as discussed in Secs. 1.7 and 5.1.2, with these forces we can conduct a static structural analysis to determine element forces (bending moment, shear force, and axial force), and stresses needed for design of structural elements; no additional dynamic analysis is necessary. Figure 5.4

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shows equivalent static forces along with the base shear and overturning moment for a generalized SDOF case, which is applicable to each mode, *j*. The internal story shear force, $V^{max}{}_{i}$, and internal story moment, $M^{max}{}_{i}$, at an arbitrary level *i*, for each mode *j*, can be obtained by applying static equilibrium and are given by Eqs. (5.38) and (5.39), respectively. Recall that by setting *i* equal to 1 in Eqs. (5.38) and (5.39) we get the base shear force, $V_{max}{}^{b}$, and overturning moment, $M_{max\ b}$, Eqs. (5.40) and (5.41), respectively.

That is, the *base shear*, V_{bj}, for mode *j* is

$$V_{bj} = f_{1j} + f_{2j} + f_{3j} + \cdots = \sum_{i=1}^{N} f_{ij} \tag{7.19}$$

The base shear from all modes in matrix form is

$$\{V_b\} = [f]^T\{1\} = [u_o]^T[k]^T\{1\} = [\Phi]^T[q_o][k]\{1\} \tag{7.20}$$

Since the stiffness matrix is symmetric, $[k]^T = [k]$

The *overturning moment*, M_{OTj}. For mode *j*

$$M_{OTj} = \sum_{i=1}^{N} h_i \cdot f_{ij} \tag{7.21}$$

The overturning moment from all modes in matrix form is

$$\{M_{OT}\} = [f]^T\{\text{heights}\} \tag{7.22}$$

where $\{\text{heights}\}$ is a column vector of the heights of the various floors.

Recall that in the derivation of Eqs. (5.39), (5.41), or (7.22) it is assumed that the floor weights are directly in the center of the building frame, thus, making no contribution to the moment equilibrium equation. However, for buildings with eccentric weights, a full moment static equilibrium analysis should be conducted to account for the contribution of these eccentric forces in the overturning moment.

There are cases where the majority of the response is contained in the first few modes. For example, most seismic forces are generated by the first few modes, whereas other loadings can excite many more modes depending on their excitation frequency content. For such cases, we only have to superimpose the response from a subset of the total *N* mode shapes. In most practical cases, when *N* is large, only the first few modes are included in the analysis; this requires a rational approach in order to obtain reasonable accuracy. There are a number of approaches used to quantify the contribution of the various mode shapes, including modal participation factors and modal contribution factors, which will be discussed later in this chapter. Once an acceptable number of modes has been established, we can estimate the maximum total response using one of the combination rules presented in Sec. 7.1.2.

7.1.2 Combining Maxima Response Values

The total response is difficult to obtain using superposition of modal maxima because, in general, these maxima do not occur simultaneously. A simple absolute summation of the maxima, where

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absolute values of the modal maxima are added directly, is not used since modal maxima generally occur at different times during the response history. This method gives an upper bound to the total response and can be represented in terms of generic maximum modal values, R_j. R_j is intended to represent maximum values due to vibration mode j for any response parameter, such as displacement, acceleration, nodal forces (lateral story forces in shear buildings), etc.

$$R_{\max} \le \sum_{j=1}^{N} \left| R_j \right| \tag{7.23}$$

Other more accurate combination rules have been developed to obtain the total response. These rules are based on random vibration theory to estimate the average maximum response. The most popular rules include the square root of the sum of the squares (SRSS) and the complete quadratic combination (CQC), the latter being more complicated but yielding more accurate results for a wider range of mode shapes.

- The total response for the SRSS rule,

$$R_{\max} \approx \sqrt{\sum_{j=1}^{N} R_j^2} \tag{7.24}$$

  This provides a less conservative estimate of the maximum response and provides results within approximately 10% of the actual maximum response for structures with well separated frequencies.

- The total response for the CQC rule,

$$R_{\max} \approx \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} R_i \rho_{ij} R_j} \tag{7.25}$$

  where

  ρ_{ij} is the cross-modal coefficient that varies from 0 to 1 (1 for the case of $i = j$),

$$\rho_{ij} = \frac{8\sqrt{\zeta_i \zeta_j}\,(\zeta_i + \beta_{ij}\zeta_j)\beta_{ij}^{1.5}}{(1+\beta_{ij}^2)^2 + 4\zeta_i\zeta_j\beta_{ij}(1+\beta_{ij})^2 + 4(\zeta_i^2 + \zeta_j^2)\beta_{ij}^2} \tag{7.26}$$

  $\beta_{ij} = \dfrac{\omega_j}{\omega_i}$, modal frequencies of two distinct mode shapes, and

  $\zeta = \dfrac{c}{c_{cr}}$, [damping ratio given by Eq. (2.12)], so, ζ_i and ζ_j are damping ratios of two distinct mode shapes.

Assuming modal damping factors are equal for all modes, Eq. (7.26) reduces to

$$\rho_{ij} = \frac{8\zeta^2(1+\beta_{ij})\beta_{ij}^{1.5}}{(1+\beta_{ij}^2)^2 + 4\beta_{ij}\zeta^2(1+\beta_{ij})^2} \tag{7.27}$$

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When the modal frequencies are well separated, this matrix tends to the identity matrix and the CQC rule approaches the SRSS rule results. Also, for undamped structures, this relationship is the same as the SRSS method.

As discussed in Sec. 1.7, and demonstrated in Chap. 4, Example 13, with the maximum dynamic displacement (or acceleration), we can conduct a static structural analysis to determine element forces (bending moment, shear force, and axial force), and stresses needed for design; no additional dynamic analysis is necessary. The maximum element force or stress results from each mode shape can be combined using either SRSS or CQC.

7.1.3 Harmonic Forcing Function Response

First, assume a harmonic forcing function vector of the following form,

$$\{p_i(t)\} = \{p_{0i}\}\sin\omega t \tag{7.28}$$

where

p_{0i} is the peak magnitude of the force at the i^{th} DOF.

ω is the forcing frequency in rad/s.

These forcing functions produce bound displacements provided that $\omega \neq \omega_{ni}$, the resonant condition. Notice that the forces' amplitude changes, but not their frequencies. This assumption allows us to apply the solution derived in Chap. 3. For cases where the amplitudes and frequencies change, we can use the Duhamel's integral analysis discussed in the next section.

With a forcing function defined, Eq. (7.28), we can solve the equations of motion following the methods described in Chap. 3. Equation (7.10) can be rewritten as

$$M_i\ddot{q}_i(t) + K_i q_i(t) = P_i = \{\varphi_i\}^T\{p_{0i}\}\sin\omega t \tag{7.29}$$

Since these are a series of SDOF cases, the general solution to each of these equations can be expressed as combinations of particular and complementary solutions,

$$q_i(t) = q_{ci}(t) + q_{Pi}(t) \tag{7.30}$$

We can use Eq. (3.6) for each DOF of Eq. (7.30), where the arbitrary constants are obtained by evaluating the equations at time $t = 0$ (initial conditions). The complete solution describing the position of the ith DOF as a function of time is in the same form as Eq. (3.8) as follows:

$$q_i(t) = q_i(0)\cos\omega_{ni}t + \left(\frac{\dot{q}_i(0)}{\omega_{ni}} - \frac{P_{oi}/K_i}{1-r_i^2}r_i\right)\sin\omega_{ni}t + \frac{P_{oi}/K_i}{1-r_i^2}\sin\omega t \tag{7.31}$$

where $q_i(0)$ and $\dot{q}_i(0)$ are the initial conditions associated with the ith modal coordinate, which can be obtained by using the transformation equation [Eq. (7.3)] to relate modal and physical coordinates,

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$$u_i(0) = \{\varphi_i\} \cdot q_i(0) \text{ and } \dot{u}_i(0) = \{\varphi_i\} \cdot \dot{q}_i(0)$$

$$r_i = \omega / \omega_{ni}$$

And

$$P_{oi} = \{\varphi_i\}^T \{p_{0i}\}$$

All other parameters have been previously defined. We can now obtain the maximum response by superposing the response via Eq. (7.3) or we can estimate the maximum response using one of the combination methods discussed in Sec. 7.1.2. To facilitate the maximum response analysis, we focus on the *steady-state* response [last term in Eq. (7.31)]because it persists after the transient vibration [the first two terms in Eq. (7.31)] dissipates and eventually the *steady-state* response becomes the total solution. The maximum steady-state displacement amplitudes are then given by

$$u_i = \sum_{j=1}^{N} \varphi_{ij} \cdot \frac{P_{oj}/K_j}{1-r_j^2} \tag{7.32}$$

Example 1

Given the following two-story building frame with $k_1 = 2k$ and $k_2 = k$, where $k = 20$ kip/in, and $m_1 = m_2 = 0.4$ kip · s^2/in, determine (a) mass and stiffness matrices, (b) natural frequencies and modal matrices, (c) modal mass and stiffness matrices, and (d) modal force vector and steady-state displacement response. Also, graph the steady-state displacement response time-history. Use the MATLAB script developed in Chap. 3, Example 3, to perform the computations. The two loads are harmonic and are given as, $p^1(t) = 4$ kip · sin[(10 rad/s)t] and $p^2(t) = 5$ kip · sin[(10 rad/s)t]. Assume initial displacements and velocities are zero.

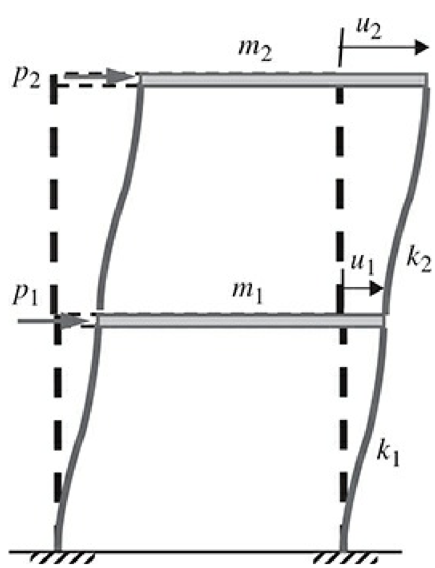

FIGURE E1.1 Schematic of two-story building frame with harmonic loading.

Solution

i. *Determine the mass and stiffness matrices.* The mass and stiffness matrices can be obtained using Eq. (6.8) with $m^1 = m^2 = m = 0.4$ kip · s^2/in, $k^1 = 2k = 40$ kip/in, and $k^2 = k = 20$ kip/in.

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

$$[k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

ii. *Determine the natural frequencies and modal matrix.* The natural frequencies and mode shapes can be determined by solving the eigenvalue problem, which results in the following natural frequencies and mode shapes. First, the natural frequencies are the square root of the eigenvalues: $\omega_{n1} = 5.41 \text{ rad/s}$ and $\omega_{n2} = 13.1 \text{ rad/s}$. The frequency ratios using Eq. (3.5), r_i:

$$r_1 = \frac{\omega_1}{\omega_{n1}} = \frac{10 \text{ rad/s}}{5.41 \text{ rad/s}} = 1.85$$

$$r_2 = \frac{\omega_2}{\omega_{n2}} = \frac{10 \text{ rad/s}}{13.1 \text{ rad/s}} = 0.765$$

The mode shapes are given by the normalized eigenvectors and are written as the modal matrix,

$$[\Phi] = \begin{bmatrix} 0.414 & -2.414 \\ 1 & 1 \end{bmatrix}$$

iii. *Determine the modal mass and stiffness matrices.* Use Eq. (6.31) to determine the elements of the modal mass and stiffness matrices.
Modal mass matrix,

$$[M] = [\Phi]^T [m][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.469 & 0 \\ 0 & 2.73 \end{bmatrix} \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

Modal stiffness matrix,

$$[K] = [\Phi]^T [k][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13.7 & 0 \\ 0 & 466 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

iv. *Determine the modal force vector and the steady-state displacement response.* Use Eq. (7.7) to determine the modal force vector, $\{P\} = [\Phi]^T \{p\}$

$$\{P\} = [\Phi]^T \{p\} = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{Bmatrix} 4 \sin(10t) \\ 5 \sin(10t) \end{Bmatrix} = \begin{Bmatrix} 6.66 \text{ kip} \\ -4.66 \text{ kip} \end{Bmatrix} \sin(10t) = \{P_o\} \sin(10t)$$

The values for the steady-state response in modal coordinates, $q_i(t)$,

$$q_1(t) = \frac{P_{o1}/K_1}{1-r_1^2}\sin(10t) = \frac{6.66/13.7}{1-(1.85)^2}\sin(10t) = -0.201\sin 10t$$

$$q_2(t) = \frac{P_{o2}/K_2}{1-r_2^2}\sin(10t) = \frac{-4.66/466}{1-(0.765)^2}\sin(10t) = -0.0241\sin 10t$$

$$u(t)$$

Finally, determine the values for the steady-state displacement response in physical coordinates using superposition,

$$\{u(t)\} = [\Phi]\{q(t)\} = \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} -0.201\sin 10t \\ -0.0241\sin 10t \end{Bmatrix} = \begin{Bmatrix} -0.025\sin 10t \\ -0.225\sin 10t \end{Bmatrix}$$

v. *Plot the total steady-state displacement response.* The solid line represents the first floor and dashed line is the top floor.

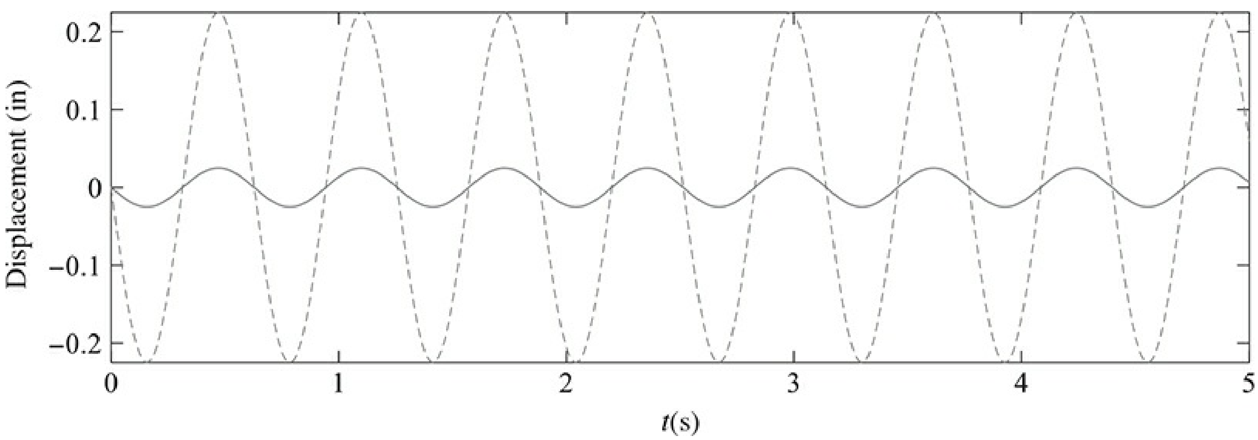

FIGURE E1.2 Steady-state displacement response, dashed line is the top floor and solid line is the first floor

vi. *Use MATLAB script to perform all the operations, the most important of the operations being the eig operator in Eq. (6.5):*

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```
clear all % Chapter 7, Example 1
m = [1 0; 0 1]*0.4; % mass matrix, kip-sec^2/in
k = [3 -1; -1 1]*20; % stiffness matrix, kip/in
syms t; % define time as a symbolic variable
p = [4; 5]*sin(10*t); % force vector, kip
[phi,lam] = eig(k,m); % compute eigenvalues and eigenvectors
omegasn = diag(sqrt(lam)) % determine frequencies from eigenvalues
omegas = [10; 10]; % forcing frequencies, rad/sec
ri = omegas./omegasn % frequency ratios
[N_rows, N_cols] = size(phi); % finds N, the size of the matrices
norm_phi = phi./phi(N_cols,:)% Mode shapes normalized to top displ of 1
M = diag(norm_phi'*m*norm_phi) % display modal mass matrix
K = diag(norm_phi'*k*norm_phi) % display modal stiffness matrix
P = vpa(norm_phi'*p, 4) % display modal force matrix
q = vpa(P./K./(1-ri.^2), 4)% Find the normal coordinates
u = vpa(norm_phi*q, 4) % Find the physical coordinates
% Graph the results
subplot(1,1,1), fplot(t,u(1))
set(gca,'FontSize',12,'FontName','Times New Roman')
ylabel('Displacement(in)','FontAngle','italic')
hold
subplot(1,1,1), fplot(t,u(2), 'LineStyle','--')
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel('t(sec)','FontAngle', 'italic')
```

The results of this script are the same as those of parts (i) through (iii):

```
omegasn =
    5.4120
   13.0656
ri =
    1.8478
    0.7654
```

```
norm_phi =
    0.4142   -2.4142
    1.0000    1.0000
M =
    0.4686
    2.7314
K =
   13.7258
  466.2742
 P =
  6.657*sin(10.0*t)
-4.657*sin(10.0*t)
  q =
  -0.2009*sin(10.0*t)
-0.02411*sin(10.0*t)
  u =
 -0.025*sin(10.0*t)
-0.225*sin(10.0*t)
```
▲

7.1.4 General Forcing Function Response

For this case, consider nonperiodic forcing functions such as those covered in Chap. 4. After defining a forcing function, we can solve the equations of motion following the methods described in Chap. 4. The equation of motion in modal coordinates is given by Eq. (7.10), the general solution for which can be expressed as combinations of particular and complementary solutions, $q_i(t) = q_{ci}(t) + q_{Pi}(t)$, with the actual solution given by Eq. (4.19) for each of these equations. That is, the complete solution for the *i*th DOF in modal coordinates as a function of time is given by the following:

$$q_i(t) = \frac{q_i(0)}{\omega_{ni}} \sin \omega_{ni} + q_i(0) \cos \omega_{ni} t + \frac{1}{M_i \omega_{ni}} \int_0^t P_i(\tau)[\sin \omega_{ni}(t-\tau)] d\tau \tag{7.33}$$

where $q_i(0)$ and $\dot{q}_i(0)$ are the initial conditions associated with the *i*th modal coordinate, which can be obtained by using the transformation equation that relates modal and physical coordinates,

$$u_i(0) = \{\varphi_i\} \cdot q_i(0) \text{ and } \dot{u}_i(0) = \{\varphi_i\} \cdot \dot{q}_i(0)$$

All other parameters have been previously defined.

Also, as discussed in Chap. 3, the first two terms in Eq. (7.33) are *transient* and eventually vanish in real structures (even with small damping). The last term, Duhamel's integral, corresponds to the *steady-state* response because it persists after the transient vibration dissipates and eventually becomes the total solution. Duhamel's integral can be evaluated using the procedures described in Chap. 4. Using the dynamic load factor, DLF, we can obtain the position of the *i*th DOF as a function of time using Eq. (7.17). Again, estimates of the maximum steady-state displacement amplitudes can be obtained using response spectra rather than conducting a detailed solution. Recall that the response spectra are graphs of the maximum response, usually given by the maximum dynamic load factor, DLF_{max}, as a function of frequency (or period). The

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maximum displacement response vector for each mode, j, is given by Eq. (7.18). We can then estimate the maximum overall response using one of the combination rules presented in Sec. 7.1.2.

Example 2

Consider the two-story building frame introduced in Example 1, with $k_1 = 2k = 40$ kip/in, $k_1 = k = 20$ kip/in, and $m_1 = m_2 = 0.4$ kip · s^2/in but subjected to two identical triangular pulses shown [$p_1(t) = p_2(t) = p(t)$] and zero initial displacement and velocity. Estimate the maximum displacement response of each level using the SRSS combination rule. Also, write a MATLAB script to perform the computations.

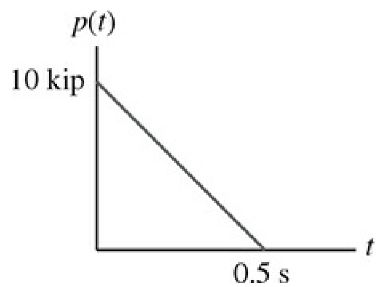

FIGURE E2.1 Pulse loading.

Solution

i. *From Example 1, we have the mass and stiffness matrices.* As well as the eigen solution, which yielded the natural frequencies and modal matrix.

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

$$[k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

Natural frequencies:

$$\omega_{n1} = 5.41 \text{ rad/s}$$

$$\omega_{n2} = 13.1 \text{ rad/s}$$

Periods:

$$T_1 = \frac{2\pi}{\omega_{n1}} = \frac{2\pi}{5.41 \text{ rad/s}} = 1.16 \text{ s}$$

$$T_2 = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{13.1 \text{ rad/s}} = 0.48 \text{ s}$$

Ratio of the pulse duration, $t_d = 0.5$ s to periods, T_i:

$$tdT_1 = \frac{t_d}{T_1} = \frac{0.5\text{ s}}{1.16\text{ s}} = 0.43$$

$$tdT_2 = \frac{t_d}{T_2} = \frac{0.5\text{ s}}{0.48\text{ s}} = 1.04$$

Mode shapes, given by the normalized eigenvectors:

$$[\Phi] = \begin{bmatrix} 0.414 & -2.414 \\ 1 & 1 \end{bmatrix}$$

ii. *Determine the modal mass and stiffness matrices using Eq. (6.31) (from Example 1).*
Modal mass matrix:

$$[M] = [\Phi]^T[m][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix}\begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.469 & 0 \\ 0 & 2.73 \end{bmatrix}\frac{\text{kip}\cdot\text{s}^2}{\text{in}}$$

Modal stiffness matrix:

$$[K] = [\Phi]^T[k][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix}\begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix}\begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13.7 & 0 \\ 0 & 466 \end{bmatrix}\frac{\text{kip}}{\text{in}}$$

iii. *Determine the modal force vector using Eq. (7.9).*

$$\{P\} = [\Phi]^T\{p\} = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix}\begin{Bmatrix} 10\text{ kip} \\ 10\text{ kip} \end{Bmatrix} = \begin{Bmatrix} 14.14 \\ -14.14 \end{Bmatrix}\text{kip}$$

iv. *Determine the equivalent static displacement* $\frac{P_i}{K_i}$ *using Eq. (7.17).*

$$\frac{P_1}{K_1} = \frac{14.14}{13.7} = 1.03\text{ in} \quad \text{and} \quad \frac{P_2}{K_2} = \frac{-14.14}{466} = -0.030\text{ in}$$

Since these are sometimes used as the modal contribution factors, they clearly show that the first mode contributes the majority of the response, as will be shown in the next two steps.

v. *Determine the maximum values for the steady-state response in normal coordinates,* $q_{max\,i}$, *using the dynamic load factor (from Duhamel's integral or Chap. 4 Fig. E10.2).*

$$q_{\max i} = \frac{P_i}{K_i}\cdot \text{DLF}_{\max i}$$

Maximum dynamic load factors:

$$\text{DLF}_{\max 1} = 1.121 \text{ and } \text{DLF}_{\max 2} = 1.575$$

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So,

$$q_{\max 1} = \frac{P_1}{K_1} \cdot \mathrm{DLF}_{\max 1} = 1.03(1.121) = 1.155 \text{ in}$$

$$q_{\max 2} = \frac{P_2}{K_2} \cdot \mathrm{DLF}_{\max 2} = -0.03(1.575) = -0.047 \text{ in}$$

vi. *Determine the maximum values for the steady-state response in physical coordinates, $u_{max\,i}$ using Eq. (7.18).*

$$\{u_{\max}\}_j = \{\varphi_{ij}\} \cdot q_{\max j}$$

$$\{u_{\max}\}_1 = \{\varphi_{i1}\} \cdot q_{\max 1} = \begin{Bmatrix} 0.414 \\ 1 \end{Bmatrix} \cdot (1.155) = \begin{Bmatrix} 0.478 \\ 1.155 \end{Bmatrix} \text{ in}$$

$$\{u_{\max}\}_2 = \{\varphi_{i2}\} \cdot q_{\max 2} = \begin{Bmatrix} -2.414 \\ 1 \end{Bmatrix} \cdot (-0.0478) = \begin{Bmatrix} 0.115 \\ -0.0478 \end{Bmatrix} \text{ in}$$

vii. *Estimate the maximum displacement response of each level using the SRSS combination rule, Eq. (7.24). For the ith level, we have*

$$u_{\mathrm{SRSSmax}\,i} \approx \sqrt{\sum_{j=1}^{N} (u_{\max i})_j^2} = \sqrt{(u_{\max i})_1^2 + (u_{\max i})_2^2}$$

Level 1:

$$u_{\mathrm{SRSSmax}\,1} \approx \sqrt{(u_{\max 1})_1^2 + (u_{\max 1})_2^2} = \sqrt{(0.478)^2 + (0.115)^2} = 0.492 \text{ in}$$

Level 2:

$$u_{\mathrm{SRSSmax}\,2} \approx \sqrt{(u_{\max 2})_1^2 + (u_{\max 2})_2^2} = \sqrt{(1.155)^2 + (-0.0478)^2} = 1.156 \text{ in}$$

viii. *Use MATLAB script to perform all the operations, the most important of the operations being the eig operator in Eq. (6.5).* It also uses the Duhamel's integral algorithm used to develop the response spectrum for the triangular pulse, Chap. 4, Example 10:

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```
clear all % Chapter 7, Example 2
m = [1 0; 0 1]*0.4; % mass matrix, kip-sec^2/in
k = [3 -1; -1 1]*20; % stiffness matrix, kip/in
p = [10; 10];% force vector, kip
[phi,lam] = eig(k,m); % compute eigenvalues and eigenvectors
omegasn = diag(sqrt(lam)) % determine frequencies from eigenvalues
periods = 2*pi./omegasn % determine the periods from omegas, sec
td = [0.5; 0.5]; % pulse duration, sec
tdT = td./periods % td/Tn
[N_rows, N_cols] = size(m); % finds N, the size of the matrices
norm_phi = phi./phi(N_cols,:)% Mode shapes normalized to top displ of 1
M = diag(norm_phi'*m*norm_phi) % display modal mass matrix
K = diag(norm_phi'*k*norm_phi) % display modal stiffness matrix
P = norm_phi'*p % display modal force matrix
par_fac = P./K % participation factor or q2= P./M./omegasn.^2
%%%% Maximum displacements at each level for each mode shape
n = 500;
tT = linspace(0,5,n);
for j = 1:N_cols;

% loop over td/Tn and t/Tn to find DLFmax
    for i = 1:n
        if tT(i) <= tdT(j)
            p(i) = 1-tT(i)/tdT(j);
        else
            p(i) = 0;
        end
    end
    % Integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=sin(2*pi*tT);
    uust=2*pi*conv(p,h);
    DLFmax=max(abs(uust)) % Select the max values from each response
    ui_max(:,j) = DLFmax*norm_phi(:,j)*par_fac(j);
end
ui_max % display displs.; each column corresponds to a mode shape
% Use the SRSS rule to get max displacements at each level
u_maxsrss=sqrt(sum((ui_max).^2,2)) %.^ operation squares each element
```

The results of this script:

```
omegasn =
    5.4120
   13.0656
periods =
    1.1610
    0.4809
tdT =
    0.4307
    1.0397
norm_phi =
    0.4142   -2.4142
    1.0000    1.0000
M =
    0.4686
    2.7314
K =
   13.7258
  466.2742
P =
   14.1421
  -14.1421
par_fac =
    1.0303
   -0.0303
DLFmax =
    1.1210
    1.5747
ui_max =
    0.4784    0.1153
    1.1550   -0.0478
u_maxsrss =
    0.4921
    1.1560
```

▲

7.2 Forced Vibration Response of MDOF Systems with Viscous Damping

As noted in Chap. 2, inherent damping in structures is quite small and generally does not affect the values of frequencies or mode shapes for most practical systems, except in the resonant region, where it can be significant. Therefore, we include damping in Eq. (7.2) as follows:

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{p\} \tag{7.34}$$

In general, this equation cannot be uncoupled (or diagonalized) because the undamped free

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vibration mode shapes are not orthogonal with respect to the damping matrix (unlike the mass and stiffness matrices where this mode shape property yielded diagonal modal mass and stiffness matrices). However, for classical (or Rayleigh) damping, we can write the damping matrix in terms of the mass and stiffness matrices, both of which can be diagonalized [see Eq. (6.43)]. Substituting modal coordinates, Eq. (6.27), for physical coordinates in Eq. (7.34), and premultiplying by the transpose of the modal matrix,

$$[\Phi]^T \langle [m][\Phi]\{\ddot{q}\} + (a_0[m] + a_1[k])[\Phi]\{\dot{q}\} + [k][\Phi]\{q\} \rangle = [\Phi]^T \{p\}$$

Or the equation of motion in modal coordinates can now be expressed as

$$[\mathrm{M}]\{\ddot{q}\} + [\mathrm{C}]\{\dot{q}\} + [\mathrm{K}]\{q\} = \{P\} \tag{7.35}$$

where $[M]$, $[C]$, and $[K]$ are diagonal mass, damping, and stiffness matrices, respectively. That is,

$$[M_i] = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_N \end{bmatrix}$$

$$[C_i] = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_N \end{bmatrix}$$

$$[K_i] = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_N \end{bmatrix}$$

Since damping is relatively small, many practical MDOF structural dynamic problems can be solved assuming constant damping for all mode shapes since higher modes usually do not contribute significantly to the displacement response. Thus, we can use the analysis developed for the undamped case to solve for damped frequencies and mode shapes, and then use the uncoupled equations of motion to solve for the response in modal coordinates. We can also divide Eq. (7.30) by the modal mass to obtain the following equation of motion for the *j*th mode,

$$\ddot{q}_j(t) + 2\zeta_j \omega_{nj} \dot{q}_j(t) + \omega_{nj}^2 q_j(t) = P_{oj}(t) \tag{7.36}$$

where

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$$\zeta_j = \frac{c_j}{2m_j\omega_{nj}}$$

$$P_{oj}(t) = \frac{\{\varphi_j\}^T\{p\}}{M_j}$$

The solution to Eq. (7.36) can again be obtained using one of the methods presented in Chaps. 3 and 4 since each of these is equivalent to a SDOF case. However, the results are in modal coordinates, which must then be mapped to physical coordinates using Eq. (7.3). Since we are primarily interested in the maximum response of the system at the various degrees of freedom, we can use all the available solutions, including response spectra. Again, we can use superposition of response quantities (displacement, acceleration, forces, etc.) from different mode shapes to statistically estimate the maximum response using one of the combination methods discussed in Sec. 7.1.2.

7.2.1 Harmonic Forcing Function Response with Damping

Consider a harmonic forcing function vector given by Eq. (7.28). Again, these forcing functions produce bound displacements provided resonance does not occur (i.e., $\omega \neq \omega_{ni}$). The following equations of motion can be solved using the methods described in Chap. 3. Equation (7.36) can be rewritten as

$$\ddot{q}_i(t) + 2\zeta_j\omega_{nj}\dot{q}_j(t) + \omega_{ni}^2 q_i(t) = \frac{[\Phi]^T\{p_{0i}\}\sin\omega t}{[\Phi]^T[m][\Phi]} \tag{7.37}$$

Again, the general solution to these equations is given by Eq. (7.30), which are combinations of particular and complementary solutions. We can use Eq. (3.20) for each of Eq. (7.37), where the arbitrary constants are obtained by evaluating the equations at time $t = 0$ (initial conditions). The complete solution describing the position of the *i*th DOF as a function of time is given by

$$\begin{aligned} q_i(t) = e^{-\zeta_i\omega_{ni}t}\Bigg\{&\left[q_i(0) + \frac{2\zeta_i r_i P_i/K_i}{(1-r_i^2)^2 + (2\zeta_i r_i)^2}\right]\cos\omega_{Di}t \\ &+ \left[\frac{\dot{q}_i(0) + q_i(0)\zeta_i\omega}{\omega_{Di}} + \frac{r_i\omega_{ni}P_i/K_i}{\omega_{Di}}\left[\frac{2\zeta_i^2 - (1-r_i^2)}{(1-r_i^2)^2 + (2\zeta_i r_i)^2}\right]\right]\sin\omega_{Di}t\Bigg\} \\ &+ \frac{P_i/K_i}{(1-r_i^2)^2 + (2\zeta_i r_i)^2}((1-r_i^2)\sin\omega t - 2\zeta_i r_i\cos\omega t) \end{aligned} \tag{7.38}$$

where and are the initial conditions associated with the *i*th modal coordinate, which can be obtained by using the transformation equation to relate modal and physical coordinates,

$$u_i(0) = \{\varphi_i\}\cdot q_i(0) \text{ and } \dot{u}_i(0) = \{\varphi_i\}\cdot\dot{q}_i(0)$$

$$r_i = \frac{\omega}{\omega_{ni}}$$

And,

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$$\omega_{Di} = \omega_{ni}\sqrt{1-\zeta_i^2}$$

All other parameters have been previously defined. The total response is then given by Eq. (7.3). Thus, the maximum response can be obtained directly from Eqs. (7.38) and (7.11) or can be estimated using one of the combination methods discussed in Sec. 7.1.2.

Also, as discussed in Chap. 3, the first two terms in Eq. (7.38) are *transient* because in real structures (even with small damping) they eventually dissipate. The last term corresponds to the *steady-state* response because it persists after the transient vibration dies out and eventually becomes the total solution. The maximum steady-state displacement amplitudes are given by

$$u_i(t) = \sum_{j=1}^{N}\varphi_{ij}\cdot\frac{P_i/K_i}{(1-r_i^2)^2+(2\zeta_i r_i)^2}((1-r_i^2)\sin\omega t - 2\zeta_i r_i\cos\omega t) \tag{7.39}$$

Again, estimates of the maximum steady-state displacement amplitudes can be obtained using response spectra rather than conducting a detailed solution. Recall that the response spectra are graphs of the maximum response, usually given by the maximum dynamic load factor, $\text{DLF}^{\max}$, as a function of frequency (or inverse period). The maximum displacement response vector for each mode, j, is given by Eq. (7.17). We can then estimate the maximum overall response using one of the combination rules presented in Sec. 7.1.2.

Example 3

The building frame introduced in Example 1 ($k_1 = 2k = 40$ kip/in, $k_2 = k = 20$ kip/in, and $m_1 = m_2 = 0.4$ kip $\cdot$ s^2/in) has constant damping of 5% and is subjected to the two harmonic loads $p_1(t) = 4$ kip $\cdot$ sin[(10 rad/s)t] and $p_2(t) = 5$ kip $\cdot$ sin[(10 rad/s)t]. Determine and graph the steady-state response time-history using modal analysis. Modify the MATLAB script developed in Example 1 to perform the computations. Also, assume the initial conditions are zero.

Solution From Example 1, we have the following parameters:

Mass and stiffness matrices:

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \text{kip}\cdot\text{s}^2/\text{in}$$

$$[k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

The natural frequencies and mode shapes can be determined by solving the eigenvalue problem, which results in the following natural frequencies and mode shapes. The natural frequencies are the square root of the eigenvalues: $\omega^{n}_{1} = 5.41$ rad/s and $\omega^{n}_{2} = 13.1$ rad/s. The frequency ratios using Eq. (3.5), r^i:

$$r_1 = \frac{\omega_1}{\omega_{n1}} = \frac{10\ \text{rad/s}}{5.41\ \text{rad/s}} = 1.85$$

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$$r_2 = \frac{\omega_2}{\omega_{n2}} = \frac{10 \text{ rad/s}}{13.1 \text{ rad/s}} = 0.765$$

Mode shapes:

$$[\Phi] = \begin{bmatrix} 0.414 & -2.414 \\ 1 & 1 \end{bmatrix}$$

Modal mass and stiffness matrices:

$$[M] = [\Phi]^T[m][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix}\begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.469 & 0 \\ 0 & 2.73 \end{bmatrix}\frac{\text{kip}\cdot\text{s}^2}{\text{in}}$$

$$[K] = [\Phi]^T[k][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix}\begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix}\begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13.7 & 0 \\ 0 & 466 \end{bmatrix}\frac{\text{kip}}{\text{in}}$$

Modal force vector:

$$\{P\} = [\Phi]^T\{p\} = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix}\begin{Bmatrix} 4\sin(10t) \\ 5\sin(10t) \end{Bmatrix} = \begin{Bmatrix} 6.66 \text{ kip} \\ -4.66 \text{ kip} \end{Bmatrix}\sin(10t) = \{P\}\sin(10t)$$

i. *Determine the damped steady-state response.* The values for the steady-state response in normal coordinates, $q_i(t)$,

$$q_i(t) = \frac{P_i/K_i}{\left(1-r_i^2\right)^2 + \left(2\zeta_i r_i\right)^2}\left(\left(1-r_i^2\right)\sin\omega t - 2\zeta_i r_i\cos\omega t\right)$$

$$q_1(t) = \frac{6.66/13.7}{(1-(1.85)^2)^2 + (2(0.05)1.85)^2}((1-(1.85)^2)\sin(10t) - 2(0.05)(1.85)\cos(10t))$$

$$q_1(t) = -0.20\sin 10t - 0.0153\cos 10t$$

$$q_2(t) = \frac{-4.66/466}{(1-(0.765)^2)^2 + (2(0.05)0.765)^2}((1-(0.765)^2)\sin(10t) - 2(0.05)(0.765)\cos(10t))$$

$$q_2(t) = -0.0233\sin 10t + 0.0043\cos 10t$$

Finally, determine values for the steady-state response in physical coordinates using Eq. (7.3),

$$\{u(t)\} = [\Phi]\{q(t)\} = \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} -0.20\sin 10t - 0.0153\cos 10t \\ -0.0233\sin 10t + 0.0043\cos 10t \end{Bmatrix}$$
$$= \begin{Bmatrix} -0.0264\sin 10t - 0.0167\cos 10t \\ -0.223\sin 10t - 0.011\cos 10t \end{Bmatrix}$$

ii. *Following are graphs of the response, solid line for the first floor and dash line for the top floor.*

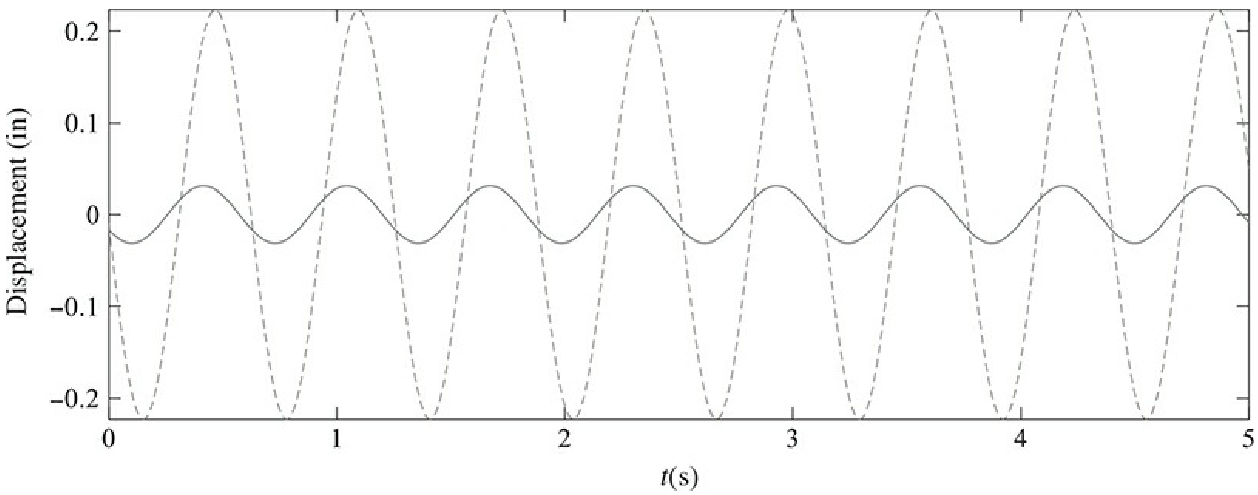

FIGURE E3.1 Steady-state displacement response, dashed line is the top floor and solid line is the first floor.

Note that the results are similar to those of Example 1, which is reasonable considering that only the steady-state response is being considered in both cases and damping is relatively small.

iii. *MATLAB script to perform all the operations:*

```
clear all % Chapter 7, Example 2
m = [1 0; 0 1]*0.4; % mass matrix, kip-sec^2/in
k = [3 -1; -1 1]*20; % stiffness matrix, kip/in
xi = 0.05; % damping ratio
syms t; % define time as a symbolic variable
p = [4; 5]; % amplitudes of forcing functions
%*sin(10*t); % force vector, kip
[phi,lam] = eig(k,m); % compute eigenvalues and eigenvectors
omegasn = diag(sqrt(lam)) % determine and show frequencies from eigenvalues
omegas = [10; 10]; % forcing frequencies, rad/sec
ri = omegas./omegasn % frequency ratios
[N_rows, N_cols] = size(phi); % finds N, the size of the matrices
norm_phi = phi./phi(N_cols,:)% Mode shapes normalized to top displ of 1
M = diag(norm_phi'*m*norm_phi) % display modal mass matrix
K = diag(norm_phi'*k*norm_phi) % display modal stiffness matrix
P = vpa(norm_phi'*p, 4) % display modal force matrix
H = 2*xi*ri;
E = 1-ri.^2;
ust = P./K;
```

```
q = vpa(ust./(E.^2+H.^2).*(E.*sin(10*t)-H.*cos(10*t)), 4)% Normal coords.
u = vpa(norm_phi*q, 4) % Find the physical coordinates
% Graph the results round(3132,2,'significant')
subplot(1,1,1), fplot(t,u(1))
set(gca,'FontSize',12,'FontName','Times New Roman')
ylabel('Displacement(in)','FontAngle','italic')
hold
subplot(1,1,1), fplot(t,u(2), 'LineStyle','--')
set(gca,'FontSize',12,'FontName','Times New Roman')
xlabel('t(sec)','FontAngle', 'italic')
```

The results of this script:

```
omegasn =
    5.4120
   13.0656
ri =
    1.8478
    0.7654
norm_phi =
    0.4142   -2.4142
    1.0000    1.0000
M =
    0.4686
    2.7314
K =
   13.7258
  466.2742
P =
  6.657
-4.657
  q =
 - 0.01529*cos(10.0*t) - 0.1997*sin(10.0*t)
0.004308*cos(10.0*t) - 0.02332*sin(10.0*t)
 u =
 - 0.01673*cos(10.0*t) - 0.02644*sin(10.0*t)
  - 0.01098*cos(10.0*t) - 0.223*sin(10.0*t)
```
▲

7.2.2 General Forcing Function Response with Damping

The formulation for general forcing functions is similar to the undamped case presented in Sec. 7.1.3, except we now include the damping term similar to the previous section. After defining a forcing function, we can use modal analysis and the methods described in Chap. 4 to solve the uncoupled equations of motion [in normal coordinates given by Eq. (7.36)]. Again, the general solution to each of these equations can be expressed as combinations of particular and complementary solutions given by Eq. (4.17):

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$$q_i(t) = e^{-\zeta_i\omega_{ni}t}\left[\frac{\dot{q}_i(0)+q_i(0)\zeta_i\omega}{\omega_{Di}}\sin\omega_{Di}t + q_i(0)\cos\omega_{Di}t\right]$$
$$+\frac{1}{M_i\omega_{Di}}\int_0^t P_i(\tau)e^{-\zeta_i\omega_{ni}(t-\tau)}[\sin\omega_{Di}(t-\tau)]\,d\tau \tag{7.40}$$

where $q_i(0)$ and $\dot{q}_i(0)$ are the initial conditions associated with the ith modal coordinate, which again can be obtained by using the transformation equation to relate modal and physical coordinates,

$$u_i(0) = \{\varphi_i\}\cdot q_i(0) \text{ and } \dot{u}_i(0) = \{\varphi_i\}\cdot\dot{q}_i(0)$$

$$r_i = \frac{\omega}{\omega_{ni}}$$

And,

$$\omega_{Di} = \omega_{ni}\sqrt{1-\zeta_i^2}$$

The last term in Eq. (7.40) is Duhamel's integral and can be evaluated using the procedures described in Chap. 4. All other parameters have been previously defined. The total modal response is then given by Eq. (7.3). The maximum response, which is of most interest, can be obtained directly from Eqs. (7.40) and (7.11) or can be estimated using one of the combination methods discussed in Sec. 7.1.2.

Also, as discussed in Chap. 3, the first two terms in Eq. (7.40) are *transient* and eventually dissipate in real structures (even with little damping). The last term, Duhamel's integral, corresponds to the *steady-state* response because it persists after the transient vibration dies out and eventually becomes the total solution. For such cases, the maximum steady-state displacement amplitudes can be obtained using response spectra rather than conducting a detailed solution. The maximum displacement response vector for each mode, j, is given by Eq. (7.17) with the maximum dynamic load factor, DLF^{max}; and the maximum overall response can be estimated using one of the combination rules presented in Sec. 7.1.2.

Example 4

The building frame and loading introduced in Example 2 ($k^1 = 2k = 40$ kip/in, $k^2 = k = 20$ kip/in, and $m^1 = m^2 = 0.4$ kip · s^2/in) has constant damping of 5% and starts from zero initial displacement and velocity. Estimate the maximum displacement response of each level using the SRSS combination rule. Modify the MATLAB script developed in Example 2 to perform the computations.

Solution From Example 2, we have the following parameters:

Mass and stiffness matrices:

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$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \frac{\text{kip}\cdot\text{s}^2}{\text{in}}$$

$$[k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

Natural frequencies:

$$\omega_{n1} = 5.41 \text{ rad/s}$$

$$\omega_{n2} = 13.1 \text{ rad/s}$$

Periods:

$$T_1 = \frac{2\pi}{\omega_{n1}} = \frac{2\pi}{5.41 \text{ rad/s}} = 1.16 \text{ s}$$

$$T_2 = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{13.1 \text{ rad/s}} = 0.48 \text{ s}$$

Ratio of the pulse duration, t_d = 0.5 s to periods, T_i:

$$tdT_1 = \frac{t_d}{T_1} = \frac{0.5 \text{ s}}{1.16 \text{ s}} = 0.43$$

$$tdT_2 = \frac{t_d}{T_2} = \frac{0.5 \text{ s}}{0.48 \text{ s}} = 1.04$$

Mode shapes:

$$[\Phi] = \begin{bmatrix} 0.414 & -2.414 \\ 1 & 1 \end{bmatrix}$$

Modal mass and stiffness matrices:

$$[M] = [\Phi]^T[m][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.469 & 0 \\ 0 & 2.73 \end{bmatrix} \frac{\text{kip}\cdot\text{s}^2}{\text{in}}$$

$$[K] = [\Phi]^T[k][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13.7 & 0 \\ 0 & 466 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

Modal force vector:

$$\{P\} = [\Phi]^T\{p\} = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{Bmatrix} 10 \text{ kip} \\ 10 \text{ kip} \end{Bmatrix} = \begin{Bmatrix} 14.14 \\ -14.14 \end{Bmatrix} \text{kip}$$

i. *Determine the maximum values for the steady-state response in normal coordinates, , using the dynamic load factor (from Duhamel's integral or Chap. 4, Example 12, but with a triangular load and 5% damping).* This is implemented in the MATLAB script.

$$q_{\max i} = P_i / K_i \cdot \mathrm{DLF}_{\max i}$$

Maximum dynamic load factors:

$$\mathrm{DLF}_{\max 1} = 1.036 \text{ vs. } \mathrm{DLF}_{\max 1} = 1.121 \text{ in Example 2}$$

$$\mathrm{DLF}_{\max 2} = 1.456 \text{ vs. } \mathrm{DLF}_{\max 2} = 1.575 \text{ in Example 2}$$

Notice that there is a reduction in the dynamic normal displacement because of damping, which is what we observed in Chap. 4, Fig. E12.1, for the half-cycle sine pulse force. So,

$$q_{\max 1} = P_1 / K_1 \cdot \mathrm{DLF}_{\max 1} = \ 14.14/13.7(1.036) = 1.068 \text{ in}$$

$$q_{\max 2} = P_2 / K_2 \cdot \mathrm{DLF}_{\max 2} = -14.14/466(1.456) = -0.044 \text{ in}$$

ii. *Determine the maximum values for the steady-state response in physical coordinates, using Eq. (7.17).*

$$\{u_{\max}\}_j = \{\varphi_{ij}\} \cdot q_{\max j}$$

$$\{u_{\max}\}_1 = \{\varphi_{i1}\} \cdot q_{\max 1} = \begin{Bmatrix} 0.414 \\ 1 \end{Bmatrix} \cdot (1.068) = \begin{Bmatrix} 0.442 \\ 1.068 \end{Bmatrix} \text{ in}$$

$$\{u_{\max}\}_2 = \{\varphi_{i2}\} \cdot q_{\max 2} = \begin{Bmatrix} -2.414 \\ 1 \end{Bmatrix} \cdot (-0.0442) = \begin{Bmatrix} 0.107 \\ -0.0442 \end{Bmatrix} \text{ in}$$

iii. *Estimate the maximum displacement response of each level using the SRSS combination rule, Eq. (7.24).* For the *i*th level, we have

$$u_{\mathrm{SRSS}\max i} \approx \sqrt{\sum_{j=1}^{N} (u_{\max i})_j^2} = \sqrt{(u_{\max i})_1^2 + (u_{\max i})_2^2}$$

Level 1:

$$u_{\mathrm{SRSS}\max 1} \approx \sqrt{(u_{\max 1})_1^2 + (u_{\max 1})_2^2} = \sqrt{(0.442)^2 + (0.107)^2} = 0.455 \text{ in}$$

Level 2:

$$u_{\mathrm{SRSS}\max 2} \approx \sqrt{(u_{\max 2})_1^2 + (u_{\max 2})_2^2} = \sqrt{(1.068)^2 + (-0.0442)^2} = 1.069 \text{ in}$$

iv. *Use MATLAB script to perform all the operations.* The Duhamel's integral algorithm includes damping; see Chap. 4, Example 12:

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```
clear all % Chapter 7, Example 4
m = [1 0; 0 1]*0.4; % mass matrix, kip-sec^2/in
k = [3 -1; -1 1]*20; % stiffness matrix, kip/in
xi = 0.05; % damping ratio
p = [10; 10];% force vector, kip
[phi,lam] = eig(k,m); % compute eigenvalues and eigenvectors
omegasn = diag(sqrt(lam)) % determine and show frequencies from eigenvalues
periods = 2*pi./omegasn % determine the periods from omegas, sec
td = [0.5; 0.5]; % pulse duration, sec
tdT = td./periods % td/Tn
[N_rows, N_cols] = size(m); % finds N, the size of the matrices
norm_phi = phi./phi(N_cols,:)% Mode shapes normalized to top displ of 1
M = diag(norm_phi'*m*norm_phi) % display modal mass matrix
K = diag(norm_phi'*k*norm_phi) % display modal stiffness matrix
P = norm_phi'*p % display modal force matrix
par_fac = P./K % participation factor or q2= P./M./omegasn.^2
%%%% Maximum displacements at each level for each mode shape
n = 500;
tT = linspace(0,5,n);
for j = 1:N_cols;
% loop over td/Tn and t/Tn to find DLFmax
    for i = 1:n
        if tT(i) <= tdT(j)
            p(i) = 1-tT(i)/tdT(j);
        else
            p(i) = 0;
        end
    end

    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=exp(-xi*2*pi*tT).*sin(2*pi*sqrt(1-xi^2)*tT);
    uust=(2*pi/sqrt(1-xi^2))*conv(p,h);
    DLFmax=max(abs(uust)) % Select the max values from each response
    ui_max(:,j) = norm_phi(:,j)*DLFmax*par_fac(j);
end
ui_max % display displs.; each column corresponds to a mode shape
% Use the SRSS rule to get max displacements at each level
u_maxsrss=sqrt(sum((ui_max).^2,2)) %.^ operation squares each element
```

The results of this script:

```
omegasn =
    5.4120
   13.0656
periods =
    1.1610
    0.4809
tdT =
    0.4307
    1.0397
norm_phi =
    0.4142   -2.4142
    1.0000    1.0000
M =
    0.4686
    2.7314
K =
   13.7258
  466.2742
P =
   14.1421
  -14.1421
par_fac =
    1.0303
   -0.0303
DLFmax =
    1.0364
    1.4561
ui_max =
    0.4423    0.1066
    1.0678   -0.0442
u_maxsrss =
    0.4550
    1.0688
```

▲

7.2.3 Modal Analysis Method Summary

The following is a brief step-by-step procedure to estimate the maximum response of a building structure subjected to general forcing functions at various levels:

1. Determine the mass matrix, $[m]$; this can be done using given floor weights.
2. Determine the stiffness matrix, $[k]$; this can be done using column material and geometric properties.
3. With the stiffness and mass matrices, solve the associated eigenvalue problem. The eigenvalues, ω_n^2, give the natural frequencies, ω^n, and periods, T_n, and the eigenvectors give the modal matrix, $[\Phi]$, which is usually normalized such that the top floor DOF has

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a value of 1.

4. After selecting an appropriate damping ratio, we can determine the dynamic load factor, $\mathrm{DLF}_{\max j}$ for each of the natural periods using one of the algorithms discussed in Chap. 4.
5. Obtain the equivalent static displacement, $\frac{P_i}{K_i}$.
6. Determine the physical displacements associated with each mode shape, j,

$$\{u\}_j = \{\ _{ij}\} \cdot \frac{P_i}{K_i} \cdot \mathrm{DLF}_{\max j}$$

7. The resultant maximum displacement at each node is obtained using one of the combination rules in Sec. 7.1.2, such as the SRSS rule, $u_{\max}{}^{j} = (\Sigma u^{j2})^{½}$
8. The matrix of forces at each node is [Eq. (7.13)]: $[\,f\,] = [k][u]$
9. The resultant maximum lateral force at each node is obtained using one of the combination rules in Sec. 7.1.2, such as SRSS, $f_{\max}{}^{i} = (\Sigma f^{i2})^{½}$
10. The column vector of total base shear forces for shear buildings is [Eq. (7.20)]$\{V^b\} = [\,f\,]^T\{1\}$
11. The maximum base shear force is obtained using one of the combination rules in Sec. 7.1.2, such as the SRSS rule, $V^b{}_{\max} = (\Sigma V^{i2})^{½}$
12. The overturning moment is obtained using static equilibrium, Eq. (7.22).
13. The maximum overturning moment is obtained using one of the combination rules in Sec. 7.1.2, such as the SRSS rule, $M^{OT}{}_{\max} = (\Sigma M^{OTi2})^{½}$

Example 5

Consider the three-story building frame from Chap. 6, Example 7, subjected to an air blast loading modeled as shown. Estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments using the SRSS combination rule. The damping ratio is assumed to be 5% in the first and third modes and Rayleigh damping is assumed for the second mode. Modify the MATLAB script developed in Example 4 to perform the computations. Also, recall that beams are rigid, and each story has a stiffness k = 326.3 kip/in.

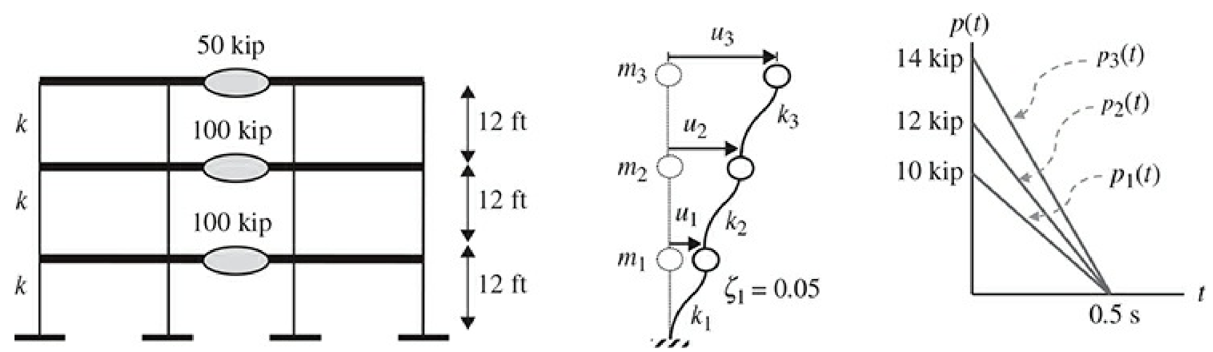

FIGURE E5.1 Building frame schematic (left) and idealized MDOF structural model (right).

Solution From Chap. 6, Example 7, we have the following parameters:

Mass and stiffness matrices:

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$$[m]=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}=\begin{bmatrix} 100\text{ kip} & 0 & 0 \\ 0 & 100\text{ kip} & 0 \\ 0 & 0 & 50\text{ kip} \end{bmatrix}\cdot\frac{1}{386.4\text{ in/s}^2}$$

$$[k]=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\cdot\frac{326.3\text{ kip}}{\text{in}}$$

Natural frequencies:

$$\omega_{n1}=18.4\text{ rad/s},\ \omega_{n2}=50.2\text{ rad/s, and }\omega_{n3}=68.6\text{ rad/s}$$

Recall that with the eigenvalues we can first determine a_0 and a_1 using the given, ζ and ζ_3; from Eq. (6.47),

$$a_0=\frac{2\omega_{n1}\omega_{n3}(\zeta_3\omega_{n1}-\zeta_1\omega_{n3})}{\omega_{n1}^2-\omega_{n3}^2}=\frac{2(18.38)(68.6)(0.05(18.38)-0.05(68.6))}{(18.38)^2-(68.6)^2}=1.45/\text{s}$$

$$a_1=\frac{2(\zeta_1\omega_{n1}-\zeta_3\omega_{n3})}{\omega_{n1}^2-\omega_{n3}^2}=\frac{2(0.05(18.38)-0.05(68.6))}{(18.38)^2-(68.6)^2}=0.0011\text{ s}$$

The damping ratio for the second mode is then estimated using Eq. (6.46),

$$\zeta_2=\frac{a_0}{2\omega_{n2}}+\frac{a_1\omega_{n2}}{2}=\frac{1.45}{2(50.22)}+\frac{(0.0011)(50.22)}{2}=0.042\text{ or }4.2\%$$

Periods:

$$T_1=\frac{2\pi}{\omega_{n1}}=\frac{2\pi}{18.4\text{ rad/s}}=0.342\text{ s}$$

$$T_2=\frac{2\pi}{\omega_{n2}}=\frac{2\pi}{50.2\text{ rad/s}}=0.125\text{ s}$$

$$T_3=\frac{2\pi}{\omega_{n3}}=\frac{2\pi}{68.6\text{ rad/s}}=0.092\text{ s}$$

Ratio of the pulse duration, $t_d=0.5$ s to periods, T_i:

$$tdT_1=\frac{t_d}{T_1}=\frac{0.5\text{ s}}{0.342\text{ s}}=1.463$$

$$tdT_2=\frac{t_d}{T_2}=\frac{0.5\text{ s}}{0.125\text{ s}}=4$$

$$tdT_3=\frac{t_d}{T_3}=\frac{0.5\text{ s}}{0.092\text{ s}}=5.459$$

Mode shapes:

$$[\Phi]=\begin{bmatrix} 0.5 & -1 & 0.5 \\ 0.866 & 0 & -0.866 \\ 1 & 1 & 1 \end{bmatrix}$$

Modal mass and stiffness matrices:

$$[M]=[\Phi]^T[m][\Phi]=\begin{bmatrix} 0.3882 & 0 & 0 \\ 0 & 0.3882 & 0 \\ 0 & 0 & 0.3882 \end{bmatrix}\frac{\text{kip}\cdot\text{s}^2}{\text{in}}$$

$$[K]=[\Phi]^T[k][\Phi]=\begin{bmatrix} 131.1 & 0 & 0 \\ 0 & 978.9 & 0 \\ 0 & 0 & 1827 \end{bmatrix}\frac{\text{kip}}{\text{in}}$$

Modal force vector:

$$\{P\}=[\Phi]^T\{p\}=\begin{bmatrix} 0.5 & 0.866 & 1 \\ -1 & 0 & 1 \\ 0.5 & -0.866 & 1 \end{bmatrix}\begin{Bmatrix} 10 \\ 12 \\ 14 \end{Bmatrix}=\begin{Bmatrix} 29.4 \\ 4 \\ 8.61 \end{Bmatrix}\text{kip}$$

Equivalent static displacements:

$$\frac{P_1}{K_1}=\frac{29.4}{131.1}=0.224\text{ in},\ \frac{P_2}{K_2}=\frac{4}{978.9}=0.0041\text{ in, and }\frac{P_3}{K_3}=\frac{8.61}{1827}=0.0047\text{ in}$$

i. *Determine maximum values for the steady-state response in normal coordinates, , using the dynamic load factor (from Duhamel's integral or Chap. 4, Example 12, but with a triangular load and variable damping).* Just as in the previous example, this is implemented in the MATLAB script.

$$q_{\max i}=P_i/K_i\cdot\text{DLF}_{\max i}$$

Maximum dynamic load factors:

$$\text{DLF}_{\max 1}=1.562,\ \text{DLF}_{\max 2}=1.763,\text{ and DLF}_{\max 3}=1.771$$

Maximum steady-state response in normal coordinates:

$$q_{\max 1}=P_1/K_1\cdot\text{DLF}_{\max 1}=\ 0.224(1.562)=0.350\text{ in}$$

$$q_{\max 2}=P_2/K_2\cdot\text{DLF}_{\max 2}=0.0041(1.763)=0.0072\text{ in}$$

$$q_{\max 3}=P_3/K_3\cdot\text{DLF}_{\max 3}=0.047(1.771)=0.083\text{ in}$$

These are the displacements at the top of the building, which are distributed to the lower floors using the mode shapes. Now, we determine the maximum values for the steady-state response in physical coordinates, , using Eq. (7.18).

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$$\{u_{\max}\}_j = \{\varphi_{ij}\} \cdot q_{\max j}$$

$$\{u_{\max}\}_1 = \{\varphi_{i1}\} \cdot q_{\max 1} = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \cdot (0.35) = \begin{Bmatrix} 0.175 \\ 0.303 \\ 0.350 \end{Bmatrix} \text{ in}$$

$$\{u_{\max}\}_2 = \{\varphi_{i2}\} \cdot q_{\max 2} = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \cdot (0.0072) = \begin{Bmatrix} -0.0072 \\ 0 \\ 0.0072 \end{Bmatrix} \text{ in}$$

$$\{u_{\max}\}_3 = \{\varphi_{i3}\} \cdot q_{\max 3} = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \cdot (0.083) = \begin{Bmatrix} 0.0042 \\ -0.0072 \\ 0.0083 \end{Bmatrix} \text{ in}$$

Finally, we can estimate the maximum displacement response of each level using the SRSS combination rule, Eq. (7.24). For the *i*th level, we have:

$$u_{\text{SRSSmax}\, i} \approx \sqrt{\sum_{j=1}^{N} (u_{\max i})_j^2} = \sqrt{(u_{\max i})_1^2 + (u_{\max i})_2^2 + (u_{\max i})_3^2}$$

Level 1: $u_{\text{SRSSmax 1}} \approx \sqrt{(0.175)^2 + (-0.0072)^2 + (0.0042)^2} = 0.175 \text{ in}$

Level 2: $u_{\text{SRSSmax 2}} \approx \sqrt{(0.303)^2 + (0)^2 + (-0.0072)^2} = 0.303 \text{ in}$

Level 3: $u_{\text{SRSSmax 3}} \approx \sqrt{(0.35)^2 + (0.0072)^2 + (0.0083)^2} = 0.350 \text{ in}$

ii. *Estimate the maximum lateral forces at each level using the SRSS* combination rule. First find the entire set of forces for all levels and modes using Eq. (7.13), the matrix operation [f] = [k][u]:

$$[f] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot 326.3 \frac{\text{kip}}{\text{in}} \cdot \begin{bmatrix} 0.175 & -0.0072 & 0.0042 \\ 0.303 & 0 & -0.0072 \\ 0.350 & 0.0072 & 0.0083 \end{bmatrix} \text{in}$$

$$= \begin{bmatrix} 15.3 & -4.7 & 5.1 \\ 26.5 & 0 & -8.8 \\ 15.3 & 2.4 & 5.1 \end{bmatrix} \text{kip}$$

Graphically,

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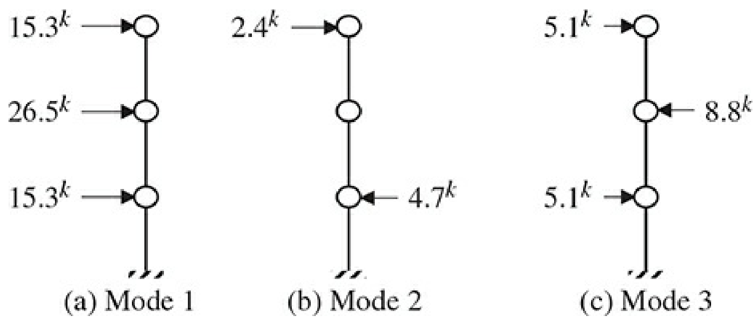

FIGURE E5.2 Resulting lateral forces at each node for a given mode of vibration.

The resultant maximum lateral force at each node is obtained from the SRSS rule [Eq. (7.24)] of each row vector: $f_i^{\max} = (\Sigma f_i^2)^{½}$,

$$\text{Level 1:}\ f_{\text{SRSSmax 1}} \approx \sqrt{(15.3)^2 + (-4.7)^2 + (5.1)^2} = 16.8 \text{ kip}$$

$$\text{Level 2:}\ f_{\text{SRSSmax 2}} \approx \sqrt{(26.5)^2 + (0)^2 + (-8.8)^2} = 27.9 \text{ kip}$$

$$\text{Level 3:}\ f_{\text{SRSSmax 3}} \approx \sqrt{(15.3)^2 + (2.4)^2 + (5.1)^2} = 16.3 \text{ kip}$$

iii. *Estimate the maximum shear forces at each level and the base using the SRSS combination rule.* Story shear forces based on static equilibrium at each level, for each mode shape:

$$\{V_1\} = \begin{Bmatrix} 15.3 + 26.5 + 15.3 \\ 15.3 + 26.5 \\ 15.3 \end{Bmatrix} \text{kip} = \begin{Bmatrix} 57.1 \\ 41.8 \\ 15.3 \end{Bmatrix} \text{kip}$$

$$\{V_2\} = \begin{Bmatrix} -4.7 + 2.4 \\ 2.4 \\ 2.4 \end{Bmatrix} \text{kip} = \begin{Bmatrix} -2.3 \\ 2.4 \\ 2.4 \end{Bmatrix} \text{kip}$$

$$\{V_3\} = \begin{Bmatrix} 5.1 - 8.8 + 5.1 \\ 5.1 - 8.8 \\ 5.1 \end{Bmatrix} \text{kip} = \begin{Bmatrix} 1.4 \\ -3.7 \\ 5.1 \end{Bmatrix} \text{kip}$$

The maximum story shear forces obtained using the SRSS rule at each level:

$$\{V_{\max}\} \approx \begin{Bmatrix} \sqrt{(57.1 \text{ kip})^2 + (-2.3 \text{ kip})^2 + (1.4 \text{ kip})^2} \\ \sqrt{(41.8 \text{ kip})^2 + (2.4 \text{ kip})^2 + (-3.7 \text{ kip})^2} \\ \sqrt{(15.3 \text{ kip})^2 + (2.4 \text{ kip})^2 + (5.1 \text{ kip})^2} \end{Bmatrix} = \begin{Bmatrix} 57.2 \\ 42.0 \\ 16.3 \end{Bmatrix} \text{kip}$$

Column vector of total base shear forces is [Eq. (7.20)]: $\{V_b\} = [\,f\,]^{\mathrm{T}}\{1\}$

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$$[V_b] = \begin{bmatrix} 15.3 & 26.5 & 15.3 \\ -4.7 & 0 & 2.4 \\ 5.1 & -8.8 & 5.1 \end{bmatrix} \text{kip} \cdot \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 57.1 \\ -2.3 \\ 1.4 \end{Bmatrix} \text{kip}$$

Maximum base shear force obtained from SRSS as: $V_b{}^{\max} = (\Sigma V_i^2)^{1/2}$,

$$V_{b\,\text{SRSSmax}} \approx \sqrt{(57.1)^2 + (-2.3)^2 + (1.4)^2} = 57.2 \text{ kip}$$

iv. *Estimate the maximum overturning moment at each level and the base using the SRSS combination rule.* Story overturning moments using static equilibrium at each level, for each mode shape:

$$\{M_1\} = \begin{Bmatrix} 15.3(36) + 26.5(24) + 15.3(12) \\ 15.3(24) + 26.5(12) \\ 15.3(12) \end{Bmatrix} \text{kip} \cdot \text{ft} = \begin{Bmatrix} 1{,}370 \\ 685 \\ 184 \end{Bmatrix} \text{kip} \cdot \text{ft}$$

$$\{M_2\} = \begin{Bmatrix} 2.4(36) - 4.7(12) \\ 2.4(24) \\ 2.4(12) \end{Bmatrix} \text{kip} \cdot \text{ft} = \begin{Bmatrix} 30 \\ 58 \\ 29 \end{Bmatrix} \text{kip} \cdot \text{ft}$$

$$\{M_3\} = \begin{Bmatrix} 5.1(36) - 8.8(24) + 5.1(12) \\ 5.1(24) - 8.8(12) \\ 5.1(12) \end{Bmatrix} \text{kip} \cdot \text{ft} = \begin{Bmatrix} 34 \\ 17 \\ 61 \end{Bmatrix} \text{kip} \cdot \text{ft}$$

The maximum story overturning moments can be obtained using the SRSS rule at each level for all the mode shape:

$$\{M_{\max}\} = \begin{Bmatrix} \sqrt{(1{,}370)^2 + (30)^2 + (34)^2} \\ \sqrt{(685)^2 + (58)^2 + (17)^2} \\ \sqrt{(184)^2 + (29)^2 + (61)^2} \end{Bmatrix} = \begin{Bmatrix} 1{,}371 \\ 688 \\ 202 \end{Bmatrix} \text{kip} \cdot \text{ft}$$

Column vector of overturning moment is [Eq. (7.22)]: $\{M_{OT}\} = [\,f\,]^T\{\text{heights}\}$

$$[M_{OT}] = \begin{bmatrix} 15.3 & 26.5 & 15.3 \\ -4.7 & 0 & 2.4 \\ 5.1 & -8.8 & 5.1 \end{bmatrix} \text{kip} \cdot \begin{Bmatrix} 12 \\ 24 \\ 36 \end{Bmatrix} \text{ft} = \begin{Bmatrix} 1{,}371 \\ 30.0 \\ 33.6 \end{Bmatrix} \text{kip} \cdot \text{ft}$$

Maximum overturning moment obtained from SRSS as: $M_{OT\max} = (\Sigma M_i^2)^{1/2}$,

$$M_{OT\text{SRSSmax}} \approx \sqrt{(1{,}371)^2 + (30)^2 + (33.6)^2} = 1{,}372 \text{ kip} \cdot \text{ft}$$

Figure E5.3 summarizes the results of the maximum response:

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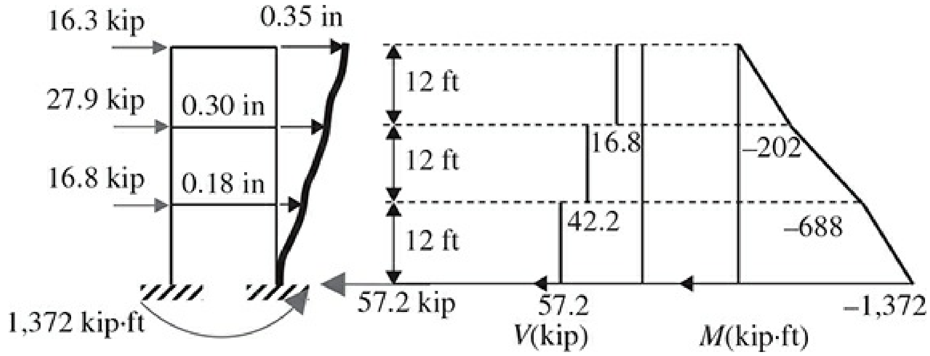

FIGURE E5.3 Graphical summary of all results.

v. *Use MATLAB script to perform all the operations.* The Duhamel's integral algorithm includes damping; see Chap. 4, Example 12:

```
clear all % Chapter 7, Example 5
m = [100 0 0; 0 100 0; 0 0 50]/386.4 % mass matrix in kip-sec2/in
k = [2 -1 0; -1 2 -1; 0 -1 1]*326.3 % stiffness matrix in kip/in
xi = [0.05; 0.042; 0.05]; % damping ratio, for mode 2 see example cals
p = [10; 12; 14];% force vector, kip
[phi,lam] = eig(k,m); % compute eigenvalues and eigenvectors
omegasn = diag(sqrt(lam)) % determine and show frequencies from eigenvalues
periods = 2*pi./omegasn % determine the periods from omegas, sec
td = [0.5; 0.5; 0.5]; % pulse duration, sec
tdT = td./periods % td/Tn
[N_rows, N_cols] = size(m); % finds N, the size of the matrices
norm_phi = phi./phi(N_cols,:)% Mode shapes normalized to top displ of 1
M = diag(norm_phi'*m*norm_phi) % display modal mass matrix
K = diag(norm_phi'*k*norm_phi) % display modal stiffness matrix
P = norm_phi'*p % display modal force matrix
par_fac = P./K % participation factor or q2= P./M./omegasn.^2
%%%% Maximum displacements at each level for each mode shape
n = 500;
tT = linspace(0,5,n);
for j = 1:N_cols
% loop over td/Tn and t/Tn to find DLFmax
    for i = 1:n
        if tT(i) <= tdT(j)
            p(i) = 1-tT(i)/tdT(j);
```

```
        else
            p(i) = 0;
        end
    end
    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=exp(-xi(j)*2*pi*tT).*sin(2*pi*sqrt(1-xi(j)^2)*tT);
    uust=(2*pi/sqrt(1-xi(j)^2))*conv(p,h);
    DLFmax=max(abs(uust)) % Select the max values from each response
    ui_max(:,j) = norm_phi(:,j)*DLFmax*par_fac(j);
end
ui_max % display displs.; each column corresponds to a mode shape
% Use the SRSS rule to get max displacements at each level
u_maxsrss=sqrt(sum((ui_max).^2,2)) %.^ operation squares each element
% Floor forces at each level for each mode shape
f=k*ui_max
% Use SRSS method to get total max forces at each level
f_maxsrss=sqrt(sum((f).^2,2))
% Base shear for each mode shape and max base shear using SRSS rule
V=(sum(f,1))
V_maxsrss=sqrt(sum((V).^2,2))
% Overturning moment for each mode shape and max moment using SRSS
heights=[12;24;36];
OTM = f'*heights
OTM_maxsrss=sqrt(sum((OTM).^2,1))
```


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The results of this script:

```
m =
    0.2588         0         0
         0    0.2588         0
         0         0    0.1294
k =
  652.6000 -326.3000         0
 -326.3000  652.6000 -326.3000
         0 -326.3000  326.3000
omegasn =
   18.3803
   50.2160
   68.5963
periods =
    0.3418
    0.1251
    0.0916
tdT =
    1.4627
    3.9961
    5.4587
norm_phi =
    0.5000   -1.0000    0.5000
    0.8660    0.0000   -0.8660
    1.0000    1.0000    1.0000
M =
    0.3882
    0.3882
    0.3882
```


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```
K =   1.0e+03 *
    0.1311
    0.9789
    1.8267
P =
   29.3923
    4.0000
    8.6077
par_fac =
    0.2241
    0.0041
    0.0047
DLFmax =
    1.5620
    1.7627
    1.7710
ui_max =
    0.1750   -0.0072    0.0042
    0.3032    0.0000   -0.0072
    0.3501    0.0072    0.0083
u_maxsrss =
    0.1752
    0.3033
    0.3502
f =
   15.3039   -4.7005    5.0815
   26.5071         0   -8.8015
   15.3039    2.3503    5.0815
f_maxsrss =
   16.7966
   27.9301
   16.2958
V =
   57.1148   -2.3503    1.3616
V_maxsrss =
   57.1793
OTM =   1.0e+03 *
    1.3708
    0.0282
    0.0327
OTM_maxsrss =
   1.3714e+03  ▲
```

7.3 Support Excitation Vibration Response of MDOF Systems

As discussed in Sec. 3.3, structural vibration can also be induced by support motion. For structures such as buildings and bridges, the most common causes of support excitations are foundation accelerations, $\ddot{u}_g(t)$, caused by earthquakes, traffic, or other vibrating equipment nearby. The formulation is similar for cases where the vibrations are emitted from a structure due to vibrating equipment mounted on it. The dynamic response of a system to support excitation is

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similar to that of systems subjected to time-varying forces, such as those discussed in previous sections.

As in SDOF analysis, ground excitation loading in the formulation of the equations of motion for MDOF systems is determined using the total displacement response of the system at level, j, which is a combination of the relative displacement, $u_j(t)$, and the ground level lateral displacement, $u_g(t)$, caused by base excitation,

$$u_j^t(t) = u_j(t) + u_g(t) \tag{7.41}$$

Therefore, like the SDOF case, the response of a MDOF system can be characterized with either the relative, $u_i(t)$, or total, $u_i^t(t)$, displacements. We can write the equations of motion for the MDOF system by applying equilibrium to the FBDs of the oscillators using D'Alembert's principle; the process is demonstrated using the two-story case shown in Fig. 7.2. Recall that the stiffness and damping forces are proportional to the relative displacement, whereas the inertial force is proportional to the total displacement.

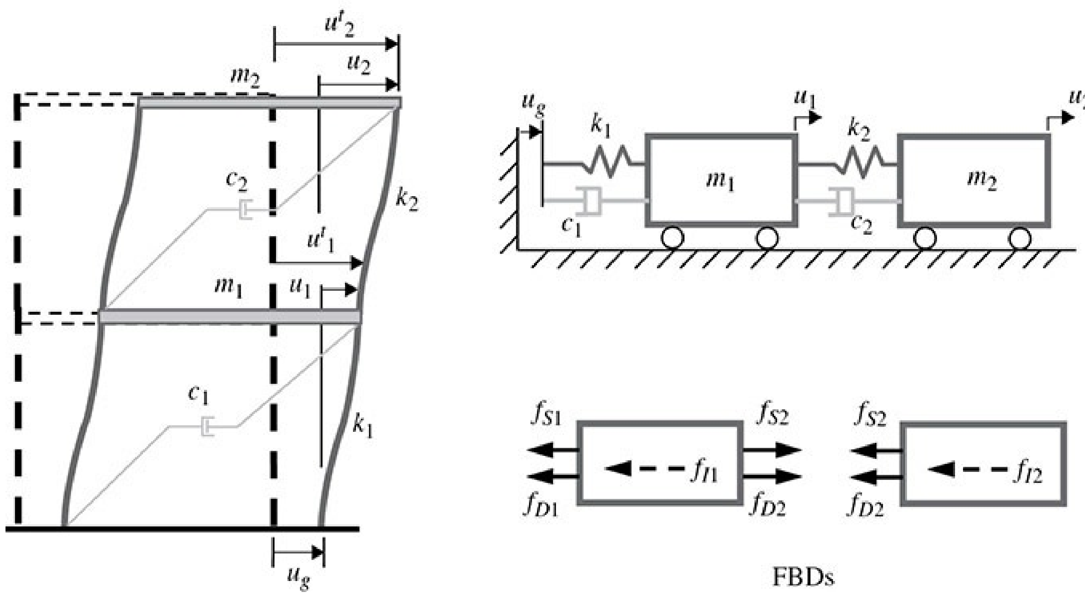

FIGURE 7.2 Idealized MDOF system and free-body diagrams for two-story frame.

Horizontal equilibrium of the FBDs shown in Fig. 7.2 yields the equations of motion,

$$-f_{I1}(t) - f_{D1}(t) - f_{S1}(t) + f_{D2}(t) + f_{S2}(t) = 0$$

$$-f_{I2}(t) - f_{D2}(t) - f_{S2}(t) = 0$$

where

$f_{I1}(t) = m_1\ddot{u}_1^t(t) = m_1[\ddot{u}_1(t) + \ddot{u}_g(t)]$ is the inertial force for FBD 1.

$f_{D1}(t) = c_1\dot{u}_1(t)$ is the damping force for FBD 1.

$f_{S1}(t) = k_1u_1(t)$ is the stiffness force for FBD 1.

$f_{I2}(t) = m_2\ddot{u}_2^t(t) = m_2[\ddot{u}_2(t) + \ddot{u}_g(t)]$ is the inertial force for FBD 2.

$f_{D2}(t) = c_2[\dot{u}_2(t) - \dot{u}_1(t)]$ is the damping force for FBD 2.

$f_{S2}(t) = k_2[u_2(t) - u_1(t)]$ is the stiffness force for FBD 2.

So, the equations of motion in terms of relative displacements and ground motion can be written as

$$
\begin{aligned}
m_1\ddot{u}_1(t) + c_1\dot{u}_1(t) + k_1u_1(t) - c_2\left[\dot{u}_2(t) - \dot{u}_1(t)\right] - k_2\left[u_2(t) - u_1(t)\right] &= -m_1\ddot{u}_g(t) \\
m_2\ddot{u}_2(t) + c_2\left[\dot{u}_2(t) - \dot{u}_1(t)\right] + k_2\left[u_2(t) - u_1(t)\right] &= -m_2\ddot{u}_g(t)
\end{aligned}
\tag{7.42}
$$

Note that the right-hand side is the effective support excitation loading that opposes the sense of ground acceleration and is comparable to time-varying forces discussed in Sec. 7.2. We can rewrite Eq. (7.42) as

$$
\begin{aligned}
m_1\ddot{u}_1(t) + \left[c_1 + c_2\right]\dot{u}_1(t) + \left[k_1 + k_2\right]u_1(t) - c_2\dot{u}_2(t) - k_2u_2(t) &= -m_1\ddot{u}_g(t) \\
m_2\ddot{u}_2(t) - c_2\dot{u}_1(t) + c_2\dot{u}_2(t) - k_2u_1(t) + k_2u_2(t) &= -m_2\ddot{u}_g(t)
\end{aligned}
\tag{7.43}
$$

Alternatively, we can rewrite the equations of motion in terms of the total displacement, in which case the right-hand side is the effective loading that depends on displacement and velocity of the support. The resulting response is the total displacement of the mass from a fixed reference point, rather than displacement relative to the moving base; see Eq. (3.24) for the SDOF case formulation in terms of total displacement.

Note that Eq. (7.43) are coupled and can be rewritten in matrix form as

$$
\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix}\ddot{u}_g(t)
$$

In general, the equations of motion are written as

$$
[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = -[m]\{1\}\ddot{u}_g(t) \tag{7.44}
$$

where

$[m]$ is the mass matrix, which is usually diagonal.

$[c]$ is the damping matrix.

$[k]$ is the stiffness matrix.

$\{1\}$ is a column vector of ones.

$\{u\}$ is the physical relative displacement vector.

$\{\dot{u}\}$ is the corresponding velocity vector.

$\{\ddot{u}\}$ is the corresponding acceleration vector.

Again, an Nth degree-of-freedom (N-DOF) system results in N dependent equations. These

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can be solved directly, or as shown in Chap. 6 can be decoupled using the modal coordinates, q^j. The process requires an undamped free vibration solution for Eq. (7.44), which yields the natural frequencies and the corresponding vibration mode shapes; see Chap. 6. This modal analysis maps the physical coordinates to the modal coordinates, in the process decoupling the differential equations of motion.

Substituting the transformation relationship, $\{u(t)\} = [\Phi]\{q(t)\}$, [Eq. (6.27)], into the equations of motion [Eq. (7.44)], and premultiplying by $[\Phi]^T$, we get

$$[\Phi]^T [m][\Phi]\{\ddot{q}(t)\} + [\Phi]^T [c][\Phi]\{\dot{q}(t)\} + [\Phi]^T [k][\Phi]\{q(t)\} = -[\Phi]^T [m]\{1\}\ddot{u}_g \tag{7.45}$$

Substituting modal mass, damping [Eq. (6.41)], and stiffness relations into [Eq. (7.45)] we get

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = -[\Phi]^T [m]\{1\}\ddot{u}_g \tag{7.46}$$

Alternatively, these equations of motion can be written as N uncoupled equations as

$$\ddot{q}_j + 2\zeta\omega_{nj}\dot{q}_j + \omega_n^2 q_j = -\Gamma_j \ddot{u}_g \tag{7.47}$$

Assuming damping is constant for all mode shapes. For elastic analysis of MDOF systems, where damping effects are significant, there are several approaches used to establish the damping matrix, including a relationship based on the stiffness and mass matrices; see Eq. (6.41). Also, we can now define the *modal participation factor* for support excitation of each mode shape, j,

$$\Gamma_j = \frac{L_j}{M_j} = \frac{\sum_{i=1}^{N} m_i \varphi_{ij}}{\sum_{i=1}^{N} m_i \varphi_{ij}^2} \text{ or } \{\Gamma\} = \frac{[\Phi]^T [m]\{1\}}{[\Phi]^T [m][\Phi]} \tag{7.48}$$

where L^j is the generalized force associated with the jth mode shape.

For this case, a modal participation factor can be viewed as providing a measure of the degree to which the jth mode contributes to (or participates in) the total dynamic response.

7.3.1 Displacements, Nodal Forces, Base Shears, Overturning Moments, and Modal Masses

Equation (7.47) represents N-SDOF systems, each subjected to a ground motion equal to Γ_j times the original ground acceleration, $\ddot{u}_g$; each of which can be solved for $\{q(t)\}$ using the SDOF procedures described in Chaps. 3 and 4. This modal coordinates result must then be mapped to physical coordinates to obtain the total result, $\{u(t)\} = [\Phi]\{q(t)\}$ [Eq. (7.3)]. Once $\{u(t)\}$ is known at any time, t, the nodal force response can be obtained using one of two methods as discussed in Sec. 1.7: (a) directly determining the internal element forces using structural analysis, or (b) first obtaining equivalent static nodal forces that can then be used to conduct a static structural analysis of the system. Again, we focus on the second approach as discussed in Sec. 7.1.1 because it is relatively straight forward, and no additional dynamic analysis is required. That is, we use Eq. (7.12) to determine external forces, $f_{ij}(t)$, at each node i that produce displacements,

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$u_{ij}(t)$, in the stiffness components of the structure for each mode *j*, or $[f(t)]=[k][u(t)]$, [Eq. (7.13)], which can then be used to conduct a static structural analysis to determine the internal element forces and stresses.

This superposition approach produces the entire time-history of the structural response (displacements, forces, etc.), which is only feasible for relatively simple cases of ground displacement excitations that are well defined, such as harmonic. However, for design of various elements of a structure, we are primarily interested in absolute maximum values of the response. In such cases, we can use all the solutions presented in previous chapters, including response spectra. The approach entails using superposition of modal maximum responses, which, in general, do not occur simultaneously; and thus, displacement, acceleration, and lateral story forces in shear buildings resulting from different mode shapes must be combined to obtain the total response. Also, response spectra analysis only provides the values of the modal maxima and not the time at which each value occurs. Thus, we estimate the response using one of the statistically approximate methods described in Sec. 7.1.2.

The solution to Eq. (7.47) can be given in terms of relative displacement or total acceleration using the definition of DLF given by Eq. (4.37), repeated here for convenience:

$$\mathrm{DLF}=\frac{u_o}{u_{st}}=\frac{\omega_n^2 u_o}{\ddot{u}_{go}}=\frac{\ddot{u}_o^t}{\ddot{u}_{go}}$$

The relative modal displacement and acceleration of the *j*th vibration mode are

$$q_{oj}=\Gamma_j\cdot\frac{\ddot{u}_{go}}{\omega_{nj}^2}\cdot \mathrm{DLF}_j \tag{7.49}$$

$$\ddot{q}_{oj}^t=\Gamma_j\cdot \mathrm{DLF}_j\cdot \ddot{u}_{go} \tag{7.50}$$

Or in the case of response spectrum analysis based on values of maximum (spectrum) displacement, D^j, (the ordinates of the response spectrum corresponding to the periods T^{nj} associated with the *j*th mode of vibration and an appropriate damping ratio),

$$q_{oj}=\Gamma_j\cdot D_j \tag{7.51}$$

where q_{oj} represents the maximum displacements at the top of the building (relative modal displacement) as shown in Fig. 7.3.

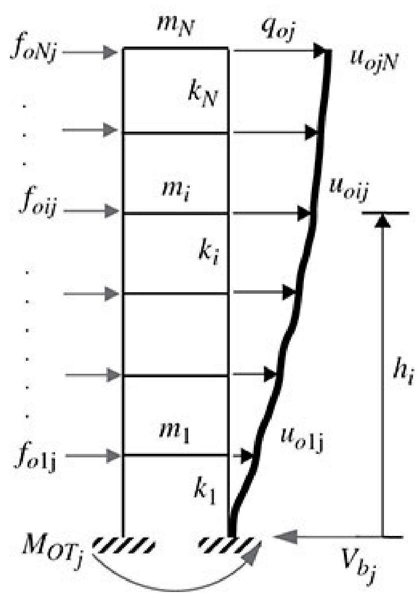

FIGURE 7.3 Maximum dynamic displacements and associated equivalent static forces.

We can now obtain the maximum physical displacement and nodal force responses for each mode, *j*, in terms of the maximum relative modal displacement. The maximum modal displacements, q_{oj}, can be distributed to the other floors using the mode shapes, as shown in Fig. 7.3. That is, the maximum displacement value of the *i*th floor induced when only the *j*th mode shape is excited is

$$\{u_{oi}\}_j = \{\varphi_{ij}\} \cdot q_{oj} \tag{7.52}$$

The displacement subscript *ij* can be interpreted as the contribution to the total response at level *i* due to mode of vibration *j*. Notice that we use the subscript *o* instead of max to distinguish this from the excitation caused by a time-dependent force. In matrix form, each matrix column represents the floor displacements associated with each mode shape; that is,

$$[u_o] = [\Phi][q_o] \tag{7.53}$$

where $[q_o]$ is a diagonal matrix of maximum modal coordinates.

The equivalent static forces associated with these floor displacements can be obtained using the product of the associated story stiffness and maximum story displacement [Eq. (7.12)], that is, the stiffness force at level *i* due to vibration mode *j* is

$$\{f_{oi}\}_j = [k]\{u_{oi}\}_j = [k]\{\varphi_{ij}\} \cdot q_{oj} \tag{7.54}$$

As with displacements, this can be represented in matrix form as

$$[f] = [k][u_o] = [k][\Phi][q_o] \tag{7.55}$$

From structural analysis we can determine the force and moment at the base of the building shown in Fig. 7.3, or we can use Eqs. (7.19) to (7.22) to determine the *base shear*, V_{bj},

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overturning moment, M_{OTj}.

For most cases of support excitation, the major portion of the response is contained in the first few modes, particularly systems with large numbers of degrees of freedom. For such cases, we only have to superimpose the response from a subset of the total N mode shapes. In most practical cases, when N is large, only the first few modes are included in the analysis; this requires a rational approach to obtain reasonable accuracy. Earlier we discussed modal participation or modal contribution factors; in this section we discuss the effective mass or weight commonly used in seismic resistant design. Once an acceptable number of modes has been established, we can estimate the maximum total response using one of the combination rules presented in Sec. 7.1.2.

To determine the modal mass, we first write the lateral nodal forces at level i due to vibration mode j in terms of the acceleration,

$$\{f_{oi}\}_j = [m]\{\varphi_{ij}\}\cdot \ddot{q}^{t}_{oj} \tag{7.56}$$

Substituting Eq. (7.50) into Eq. (7.56)

$$\{f_{oi}\}_j = [m]\{\varphi_{ij}\}\cdot \Gamma_j \cdot \mathrm{DLF}_j \cdot \ddot{u}_{go} \tag{7.57}$$

Or in the case of response spectrum analysis based on values of maximum (spectrum) acceleration, A^j, (the ordinates of the response spectrum corresponding to the periods, T^{nj}, associated with the jth mode of vibration and an appropriate damping ratio), we can replace $\mathrm{DLF}_j \cdot \ddot{u}_{go}$ with A^j. That is, Eq. (7.56) represents Newton's second law ($F = mA$), where the effective mass is given by $[m]\{\ _{ij}\}\cdot\Gamma_j$. Also, $[m]\{\ _{ij}\}$ is the generalized force associated with the jth mode shape, L^j. So, the *effective mass* for mode j, M^e_j, is given as

$$M^e_j = L_j\Gamma_j = \frac{L_j^2}{M_j} = \frac{\left(\sum_{i=1}^{N} m_i\varphi_{ij}\right)^2}{\sum_{i=1}^{N} m_i\varphi_{ij}^2} \tag{7.58}$$

Substituting into Eq. (7.57) the definition of the mass in terms of the floor weights, $m^i = w^i/g$ (weight associated with the ith floor divided by the acceleration due to gravity),

$$\{f_{oi}\}_j = \frac{\left(\sum_{i=1}^{N} w_i\varphi_{ij}\right)^2}{\sum_{i=1}^{N} w_i\varphi_{ij}^2}\,\frac{\mathrm{DLF}_j\cdot\ddot{u}_{go}}{g} = W^e_j\cdot\mathrm{DLF}_j\cdot\frac{\ddot{u}_{go}}{g} \tag{7.59}$$

where W^e_j is the *effective weight* given as

$$W^e_j = \frac{\left(\sum_{i=1}^{N} w_i\varphi_{ij}\right)^2}{\sum_{i=1}^{N} w_i\varphi_{ij}^2} \tag{7.60}$$

Combining the contribution of all the modes of vibration, we get the *total weight* of the

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building as

$$W_{\text{total}} = \sum_{j=1}^{N} W_j^e \tag{7.61}$$

This implies that the effective weight of the jth mode shape is the fraction of the total weight that participates in the jth mode of vibration. Effective weight is used in some seismic specifications to establish the acceptable number of vibration modes to be included in a modal analysis. For example, ASCE7 requires an effective weight of at least 90% of the actual building weight.

Example 6

Consider the two-story building frame from Example 1 with $k^1 = 2k$ and $k^2 = k$, where $k = 20$ kip/in, and $m^1 = m^2 = 0.4$ kip · s^2/in and shown in Fig. E6.1. Assuming 12-ft story heights, determine (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, (d) maximum overturning moment, and (e) effective masses. Assume beams are rigid, damping of 5%, and that the frame is subjected to a ground acceleration caused by a large vehicle passing by, which can be modeled as the triangular pulse shown (Chap. 4, Fig. E9.2 gives the shock spectrum). Use the SRSS combination rule to estimate the maximum response. Also, write a MATLAB script to perform the computations.

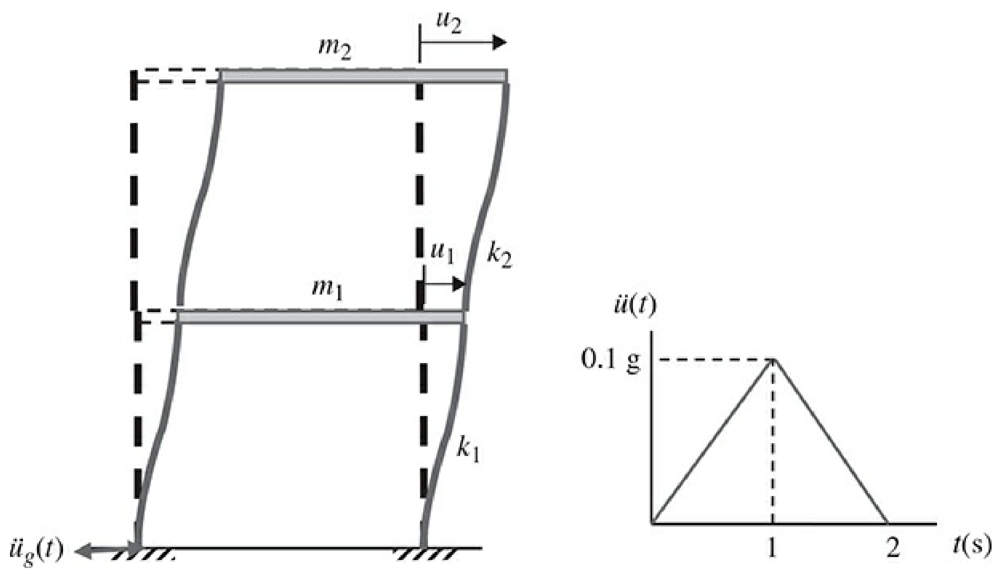

FIGURE E6.1 Schematic of two-story building frame with pulse loading.

Solution From Example 2, we have the following parameters:

Mass and stiffness matrices:

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$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

$$[k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

Natural frequencies:

$$\omega_{n1} = 5.41 \text{ rad/s}$$

$$\omega_{n2} = 13.1 \text{ rad/s}$$

Mode shapes:

$$[\Phi] = \begin{matrix} 0.414 & -2.414 \\ 1 & 1 \end{matrix}$$

Modal mass and stiffness matrices:

$$[M] = [\Phi]^T [m][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.469 & 0 \\ 0 & 2.73 \end{bmatrix} \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

$$[K] = [\Phi]^T [k][\Phi] = \begin{bmatrix} 0.414 & 1 \\ -2.41 & 1 \end{bmatrix} \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} 0.414 & -2.41 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13.7 & 0 \\ 0 & 466 \end{bmatrix} \frac{\text{kip}}{\text{in}}$$

i. *Determine the participation factors using Eq. (7.48).*

$$\Gamma_i = \frac{L_j}{M_j} = \frac{\sum_{i=1}^{N} m_i \ {}_{ij}}{\sum_{i=1}^{N} m_i \ {}_{ij}^{2}}$$

$$\Gamma_1 = \frac{L_1}{M_1} = \frac{\sum_{i=1}^{2} m_i \ {}_{i1}}{\sum_{i=1}^{2} m_i \ {}_{i1}^{2}} = \frac{m_1 \ {}_{11} + m_2 \ {}_{21}}{m_1 \ {}_{11}^{2} + m_2 \ {}_{21}^{2}} = \frac{0.4(0.414) + 0.4(1)}{0.4(0.414)^2 + 0.4(1)^2} = \frac{0.566}{0.469} = 1.207$$

$$\Gamma_2 = \frac{L_2}{M_2} = \frac{m_1 \ {}_{12} + m_2 \ {}_{22}}{m_1 \ {}_{12}^{2} + m_2 \ {}_{22}^{2}} = \frac{0.4(-2.414) + 0.4(1)}{0.4(-2.414)^2 + 0.4(1)^2} = \frac{-0.566}{2.731} = -0.207$$

ii. *Determine peak (maximum) floor displacements.* First, determine the maximum displacement at the top of the building for each vibration mode, *j*, using Eq. (7.49), $q_{oj} = \Gamma_j \cdot \text{DLF}_j \cdot \frac{\ddot{u}_{go}}{\omega_{nj}^2}$, where $\ddot{u}_{go} = 0.1\,g$ and DLF_j is obtained using the algorithm developed in Chap. 4, Example 9, with ratio of the pulse duration, $t^d = 2$ seconds to periods, T^i; then use Eq. (7.52) to determine the floor displacements for each mode; finally determine the peak floor displacements using the SRSS:

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Periods:

$$T_1 = \frac{2\pi}{\omega_{n1}} = \frac{2\pi}{5.41\ \text{rad/s}} = 1.16\ \text{s}$$

$$T_2 = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{13.1\ \text{rad/s}} = 0.48\ \text{s}$$

Ratios of the pulse duration, t_d = 2 s, to periods, T_i:

$$tdT_1 = \frac{t_d}{T_1} = \frac{2\ \text{s}}{1.16\ \text{s}} = 1.72$$

$$tdT_2 = \frac{t_d}{T_2} = \frac{2\ \text{s}}{0.48\ \text{s}} = 4.16$$

Maximum dynamic load factors:

$$\text{DLF}_1 = 1.130 \quad \text{and} \quad \text{DLF}_2 = 1.008$$

Maximum steady-state response in normal coordinates:

$$q_{o1} = \Gamma_1 \cdot \text{DLF}_1 \cdot \frac{\ddot{u}_{go}}{\omega_{n1}^2} = 1.207(1.130)\frac{(0.1)386.4\ \text{in/s}^2}{(5.41\ \text{rad/s})^2} = 1.80\ \text{in}$$

$$q_{o2} = \Gamma_2 \cdot \text{DLF}_2 \cdot \frac{\ddot{u}_{go}}{\omega_{n2}^2} = -0.207(1.008)\frac{(0.1)386.4\ \text{in/s}^2}{(13.1\ \text{rad/s})^2} = -0.047\ \text{in}$$

These are the displacements at the top of the building, which must now be distributed to the lower floors using the mode shapes. We determine the maximum values for the steady-state response in physical coordinates, , using Eq. (7.52).

$$\{u_{oi}\}_j = \{\varphi_{ij}\} \cdot q_{oj}$$

$$\{u_o\}_1 = \{\varphi_{i1}\} \cdot q_{o1} = \begin{Bmatrix} 0.414 \\ 1 \end{Bmatrix} \cdot (1.80\ \text{in}) = \begin{Bmatrix} 0.745 \\ 1.80 \end{Bmatrix}\ \text{in}$$

$$\{u_o\}_2 = \{\varphi_{i2}\} \cdot q_{o2} = \begin{Bmatrix} -2.414 \\ 1 \end{Bmatrix} \cdot (-0.047\ \text{in}) = \begin{Bmatrix} 0.114 \\ -0.047 \end{Bmatrix}\ \text{in}$$

Finally, estimate the maximum displacement response of each level using the SRSS combination rule, Eq. (7.24). For the *i*th level, we have:

$$u_{\text{SRSS}oi} \approx \sqrt{\sum_{j=1}^{N} (u_{oi})_j^2} = \sqrt{(u_{oi})_1^2 + (u_{oi})_2^2}$$

Level 1: $u_{\text{SRSSo1}} \approx \sqrt{(0.745)^2 + (0.114)^2} = 0.754$ in

Level 2: $u_{\text{SRSSo2}} \approx \sqrt{(1.80)^2 + (-0.047)^2} = 1.80$ in

iii. *Estimate the maximum lateral forces at each level using the SRSS combination rule.* First find the entire set of forces for all levels and modes using Eq. (7.13), the matrix operation $[f_o] = [k][u_o]$:

$$[f_o] = [k][u_o] = \begin{bmatrix} 60 & -20 \\ -20 & 20 \end{bmatrix} \frac{\text{kip}}{\text{in}} \cdot \begin{bmatrix} 0.745 & 0.114 \\ 1.80 & -0.047 \end{bmatrix} \text{in} = \begin{bmatrix} 8.73 & 7.79 \\ 21.1 & -3.22 \end{bmatrix} \text{kip}$$

Graphically,

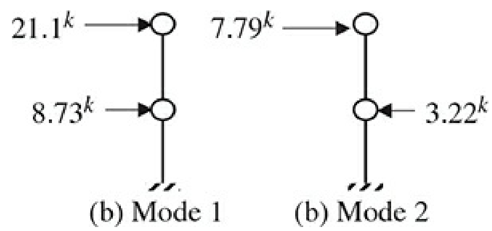

FIGURE E6.2 Resulting lateral forces at each node for a given mode of vibration.

The resultant maximum lateral force at each node is obtained from the SRSS rule [Eq. (7.24)] of each row vector: $f_{io} = (\Sigma f_i^2)^{1/2}$,

$$\text{Level 1: } f_{\text{SRSS}o1} \approx \sqrt{(8.73)^2 + (-3.22)^2} = 11.7 \text{ kip}$$

$$\text{Level 2: } f_{\text{SRSS}o2} \approx \sqrt{(21.1)^2 + (7.79)^2} = 21.3 \text{ kip}$$

iv. *Estimate the maximum base shear force using the SRSS combination rule.* Column vector of total base shear forces [Eq. (7.20)], $\{V_b\} = [f_o]^T\{1\}$, is

$$[V_b] = \begin{bmatrix} 8.73 & 21.1 \\ 7.79 & -3.22 \end{bmatrix} \text{kip} \cdot \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 29.8 \\ 4.56 \end{Bmatrix} \text{kip}$$

Maximum base shear force obtained from SRSS as $V_{bo} = (\Sigma V_i^2)^{1/2}$,

$$V_{b\text{SRSS}o} \approx \sqrt{(29.8)^2 + (4.56)^2} = 30.2 \text{ kip}$$

v. *Estimate the maximum overturning moment at the base using the SRSS combination rule.* Column vector of overturning moment is [Eq. (7.22)] $\{M_{OT}\} = [f_o]^T\{\text{heights}\}$

$$[M_{OT}] = \begin{bmatrix} 8.73 & 21.1 \\ 7.79 & -3.22 \end{bmatrix} \text{kip} \cdot \begin{Bmatrix} 12 \\ 24 \end{Bmatrix} \text{ft} = \begin{Bmatrix} 611 \\ 16.0 \end{Bmatrix} \text{kip} \cdot \text{ft}$$

Maximum overturning moment obtained from SRSS as $M_{OTo} = (\Sigma M_i^2)^{1/2}$,

$$M_{OT\text{SRSS}o} \approx \sqrt{(611)^2 + (16)^2} = 611 \text{ kip} \cdot \text{ft}$$

vi. *Determine the effective masses and the percent contribution of each mode to the total modal mass.* Use Eq. (7.58):

$$M_1^e = \frac{L_1^2}{M_1} = \frac{\left(\sum_{i=1}^{2} m_i \varphi_{i1}\right)^2}{\sum_{i=1}^{2} m_i \varphi_{i1}^2} = \frac{(m_1\varphi_{11} + m_2\varphi_{21})^2}{m_1\varphi_{11}^2 + m_2\varphi_{21}^2} = \frac{(0.4(0.414) + 0.4(1))^2}{0.4(0.414)^2 + 0.4(1)^2} = \frac{0.320}{0.469}$$
$$= 0.683 \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

$$M_2^e = \frac{L_2^2}{M_2} = \frac{\left(\sum_{i=1}^{2} m_i \varphi_{i1}\right)^2}{\sum_{i=1}^{2} m_i \varphi_{i1}^2} = \frac{(m_1\varphi_{11} + m_2\varphi_{21})^2}{m_1\varphi_{11}^2 + m_2\varphi_{21}^2} = \frac{(0.4(-2.414) + 0.4(1))^2}{0.4(-2.414)^2 + 0.4(1)^2}$$
$$= \frac{0.320}{2.731} = 0.117 \frac{\text{kip} \cdot \text{s}^2}{\text{in}}$$

Given that the total mass of the system is $0.8 \text{ kip} \cdot \text{s}^2/\text{in}$, the percent contribution of each mode is

$$\% M_1^e = \frac{M_1^e}{\sum_{i=1}^{2} m_i} 100 = \frac{0.683 \text{ kip} \cdot \text{s}^2/\text{in}}{0.8 \text{ kip} \cdot \text{s}^2/\text{in}} 100 = 85.4\%$$

$$\% M_2^e = \frac{M_2^e}{\sum_{i=1}^{2} m_i} 100 = \frac{0.117 \text{ kip} \cdot \text{s}^2/\text{in}}{0.8 \text{ kip} \cdot \text{s}^2/\text{in}} 100 = 14.6\%$$

That is, the first mode contributes approximately 85% of the response, not quite the 90% required by ASCE7, so we would include both modes in a modal analysis.

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vii. *Use MATLAB script to perform all the operations.* The Duhamel's integral algorithm includes damping; see Chap. 4, Example 9:

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```
clear all % Chapter 7, Example 6
m = [1 0; 0 1]*0.4; % mass matrix, kip-sec^2/in
k = [3 -1; -1 1]*20; % stiffness matrix, kip/in
xi = [0.05; 0.05]; % damping ratio
g = 386.4; % acceleration due to gravity in in/s^2
pA = 0.1*g; % peak ground acceleration
[phi,lam] = eig(k,m); % compute eigenvalues and eigenvectors
omegasn = diag(sqrt(lam)) % determine and show frequencies from eigenvalues
periods = 2*pi./omegasn % determine the periods from omegas, sec
td = [2; 2]; % pulse duration, sec
tdT = td./periods % td/Tn
[N_rows, N_cols] = size(m); % finds N, the size of the matrices
norm_phi = phi./phi(N_cols,:)% Mode shapes normalized to top displ of 1
M = diag(norm_phi'*m*norm_phi) % display modal mass matrix
K = diag(norm_phi'*k*norm_phi) % display modal stiffness matrix
% determine the participation factors
LT = sum(norm_phi'*m,2); % sum(A,2) returns a vector of sums of each row
MT= sum(norm_phi'*m*norm_phi,2);
par_fac = LT./MT % ./ operation divides each element of the two vectors
%%%% Maximum displacements at each level for each mode shape
n = 500;
tT = linspace(0,5,n);
for j = 1:N_cols
% loop over td/Tn and t/Tn to find DLFmax
    for i = 1:n
        if tT(i) <= tdT(j)/2
            p(i) = 2*tT(i)/tdT(j);
        elseif tT(i) <= tdT(j)
            p(i) = 2-2*tT(i)/tdT(j);
        else
            p(i) = 0;
        end
    end
    % integrate pulse function to get response
    dt=tT(2)-tT(1);
    p=p*dt;
    h=exp(-xi(j)*2*pi*tT).*sin(2*pi*sqrt(1-xi(j)^2)*tT);
    uust=(2*pi/sqrt(1-xi(j)^2))*conv(p,h);
    DLFmax=max(abs(uust)) % Select the max values from each response
    ui_max(:,j) = norm_phi(:,j)*DLFmax*par_fac(j)*pA/(omegasn(j))^2;
end
ui_max % display displs.; each column corresponds to a mode shape
% Use the SRSS rule to get max displacements at each level
u_maxsrss=sqrt(sum((ui_max).^2,2)) %.^ operation squares each element
% Floor forces at each level for each mode shape and max by SRSS method
f=k*ui_max
f_maxsrss=sqrt(sum((f).^2,2))
% Base shear for each mode shape and max base shear using SRSS rule
V=(sum(f,1))
V_maxsrss=sqrt(sum((V).^2,2))
% Overturning moment for each mode shape and max moment using SRSS
heights=[12;24];
OTM = f'*heights
OTM_maxsrss=sqrt(sum((OTM).^2,1))
```


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```
Eff_mass = LT.^2./MT % .^ operation squares each element of LT vector
Eff_masspercent = Eff_mass/sum(diag(m))*100
```


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The results of this script:

```
omegasn =
    5.4120
   13.0656
periods =
    1.1610
    0.4809
tdT =
    1.7227
    4.1589
norm_phi =
    0.4142   -2.4142
    1.0000    1.0000
M =
    0.4686
    2.7314
K =
   13.7258
  466.2742
par_fac =
    1.2071
   -0.2071
DLFmax =
    1.1301
    1.0077
ui_max =
    0.7454    0.1140
    1.7996   -0.0472
u_maxsrss =
    0.7541
    1.8002
f =
    8.7332    7.7873
   21.0838   -3.2256
f_maxsrss =
   11.7009
   21.3291
V =
   29.8170    4.5617
V_maxsrss =
   30.1639
OTM =
  610.8095
   16.0330
OTM_maxsrss =
  611.0199
Eff_mass =
    0.6828
    0.1172
Eff_masspercent =
   85.3553
   14.6447
```

▲

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7.3.2 Modal Analysis Method Summary

The following is a brief step-by-step procedure for estimating the maximum response of a structure subjected to a ground acceleration characterized by a response spectrum:

1. Determine the mass matrix, [m]; this can be done using the given floor weights.
2. Determine the stiffness matrix, [k]; this can be done using column material and geometric properties.
3. With the stiffness and mass matrices, solve the associated eigenvalue problem. The eigenvalues, ω_n^2, give the natural frequencies, ω_n, and periods, T_n, and the eigenvectors give the modal matrix, [Φ], which is usually normalized such that the top floor DOF has a value of 1.
4. After selecting an appropriate damping ratio, we can determine the dynamic load factor, DLF_{oj}, or spectral displacements/accelerations for each of the natural periods using one of the algorithms discussed in Chap. 4.
5. Obtain the modal participation factors [Eq. (7.48)], $\{\Gamma\} = [\Phi]^T[m]\{1\}/[\Phi]^T[m][\Phi]$; (the contribution of each mode shape to the total response, which is different for this case and the applied story force case).
6. Determine the physical displacements associated with each mode shape, j, [Eq. (7.52)], $\{u_{oi}\}_j = \{\varphi_{ij}\} \cdot q_{oj}$..
7. The resultant maximum displacement at each node is obtained using one of the combination rules in Sec. 7.1.2, such as the SRSS rule, $u_o^j = (\Sigma u^{j2})^{½}$.
8. The matrix of forces at each node is [Eq. (7.13)]: $[f] = [k][u]$.
9. The resultant maximum lateral force at each node is obtained using one of the combination rules in Sec. 7.1.2, such as SRSS, $f_o^i = (\Sigma f^{i2})^{½}$.
10. The column vector of total base shear forces for shear buildings is [Eq. (7.20)]: $\{V^b\} = [f]^T\{1\}$.
11. The maximum base shear force is obtained using one of the combination rules in Sec. 7.1.2, such as the SRSS rule, $V^b{}_o = (\Sigma V^{i2})^{½}$.
12. The overturning moment is obtained using static equilibrium or Eq. (7.22).
13. The maximum overturning moment is obtained using one of the combination rules in Sec. 7.1.2, such as the SRSS rule, $M^{OT}{}_o = (\Sigma M^{OTi2})^{½}$.
14. Determine the effective masses and the percent contribution of each mode to the total modal mass.

Example 7

Consider the three-story building frame from Chap. 6, Example 7, subjected to a ground acceleration due to the 2021 Haiti earthquake presented in Chap. 4, Example 16. Estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments using SRSS combination rule. For this case, assume a uniform damping ratio of 2%. Determine the effective masses and modify the MATLAB script developed in Example 5 to perform the computations. Also, recall that beams are rigid, and each story has a stiffness k = 326.3 kip/in.

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Solution From Chap. 6, Example 7, we have the following parameters:

Mass and stiffness matrices:

$$[m]=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}=\begin{bmatrix} 100\text{ kip} & 0 & 0 \\ 0 & 100\text{ kip} & 0 \\ 0 & 0 & 50\text{ kip} \end{bmatrix}/386.4\,\frac{\text{in}}{\text{s}^2}$$

$$[k]=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\cdot 326.3\text{ kip/in}$$

Natural frequencies:

$\omega_{n1} = 18.4$ rad/s, $\omega_{n2} = 50.2$ rad/s, and $\omega_{n3} = 68.6$ rad/s

Periods:

$$T_1=\frac{2\pi}{\omega_{n1}}=\frac{2\pi}{18.4\text{ rad/s}}=0.342\text{ s}$$

$$T_2=\frac{2\pi}{\omega_{n2}}=\frac{2\pi}{50.2\text{ rad/s}}=0.125\text{ s}$$

$$T_3=\frac{2\pi}{\omega_{n3}}=\frac{2\pi}{68.6\text{ rad/s}}=0.092\text{ s}$$

The damping ratio for all floors, $\zeta = 2\%$

Mode shapes:

$$[\Phi]=\begin{bmatrix} 0.5 & -1 & 0.5 \\ 0.866 & 0 & -0.866 \\ 1 & 1 & 1 \end{bmatrix}$$

Modal mass and stiffness matrices:

$$[M]=[\Phi]^T[m][\Phi]=\begin{bmatrix} 0.3882 & 0 & 0 \\ 0 & 0.3882 & 0 \\ 0 & 0 & 0.3882 \end{bmatrix}\frac{\text{kip}\cdot\text{s}^2}{\text{in}}$$

$$[K]=[\Phi]^T[k][\Phi]=\begin{bmatrix} 131.1 & 0 & 0 \\ 0 & 978.9 & 0 \\ 0 & 0 & 1{,}827 \end{bmatrix}\frac{\text{kip}}{\text{in}}$$

i. *Obtain the modal participation factors, {Γ} = [Φ]^T[m]{1}/[Φ]^T[m][Φ].* First, determine the modal force, [L] = [Φ]^T[m]{1},

$$[L]=\frac{1}{386.4\ \text{in/s}^2}\begin{bmatrix}0.5 & 0.866 & 1\\ -1 & 0 & 1\\ 0.5 & -0.866 & 1\end{bmatrix}\begin{bmatrix}100 & 0 & 0\\ 0 & 100 & 0\\ 0 & 0 & 50\end{bmatrix}\begin{Bmatrix}1\\1\\1\end{Bmatrix}=\begin{Bmatrix}0.483\\-0.129\\0.0347\end{Bmatrix}\frac{\text{lb}\cdot\text{s}^2}{\text{in}}$$

Now get the participation factors, {γ } = [Φ]^T[m]{1}/[M],

$$\{\Gamma\}=[M]^{-1}\{L\}=\begin{bmatrix}2.576 & 0 & 0\\ 0 & 2.576 & 0\\ 0 & 0 & 2.576\end{bmatrix}\frac{\text{in}}{\text{lb}\cdot\text{s}^2}\cdot\begin{Bmatrix}0.483\\-0.129\\0.0347\end{Bmatrix}\frac{\text{lb}\cdot\text{s}^2}{\text{in}}=\begin{Bmatrix}1.244\\-0.333\\0.0893\end{Bmatrix}$$

ii. *Determine the effective masses and the percent contribution of each mode to the total mass.*

$$M_j^e=\Gamma_j L_j$$

$$M_1^e=\Gamma_1 L_1=(1.244)(0.483\ \text{lb}\cdot\text{s}^2/\text{in})=0.601\ \text{lb}\cdot\text{s}^2/\text{in}$$

$$M_2^e=\Gamma_2 L_2=(-0.333)(-0.129\ \text{lb}\cdot\text{s}^2/\text{in})=0.043\ \text{lb}\cdot\text{s}^2/\text{in}$$

$$M_3^e=\Gamma_3 L_3=(0.0893)(0.0347\ \text{lb}\cdot\text{s}^2/\text{in})=0.003\ \text{lb}\cdot\text{s}^2/\text{in}$$

Given that the total mass of the system is , the percent contribution of each mode is

$$\%M_1^e=\frac{M_1^e}{\sum_{i=1}^2 m_i}100=\frac{0.60\ \text{kip}\cdot\text{s}^2/\text{in}}{0.647\ \text{kip}\cdot\text{s}^2/\text{in}}100=92.8\%$$

$$\%M_2^e=\frac{M_2^e}{\sum_{i=1}^2 m_i}100=\frac{0.043\ \text{kip}\cdot\text{s}^2/\text{in}}{0.647\ \text{kip}\cdot\text{s}^2/\text{in}}100=6.7\%$$

$$\%M_3^e=\frac{M_3^e}{\sum_{i=1}^2 m_i}100=\frac{0.003\ \text{kip}\cdot\text{s}^2/\text{in}}{0.647\ \text{kip}\cdot\text{s}^2/\text{in}}100=0.5\%$$

The first mode contributes over 90% of the response; thus, the first mode is sufficient to satisfy the 90% required by ASCE7!

iii. *Determine peak (maximum) floor displacements.* First, determine the maximum displacement at the top of the building using Eq. (7.51), $q_{oj}=\Gamma_j\cdot D_j$; where D_j is obtained from the response spectra in Chap. 4, Fig. E16.4, using periods, T_j; then use Eq. (7.52) to determine the floor displacements for each mode. Finally, determine the peak floor displacements using SRSS:
Spectral displacements for each period (obtained from the MATLAB code in Chap. 4, Example 16):

$$D_1 = 0.11 \text{ in}, D_2 = 0.0056 \text{ in, and } D_3 = 0.0024 \text{ in}$$

Maximum steady-state response in normal coordinates:

$$q_{o1} = \Gamma_1 \cdot D_1 = (1.244)(0.11 \text{ in}) = 0.137 \text{ in}$$

$$q_{o2} = \Gamma_2 \cdot D_2 = (-0.333)(0.0056 \text{ in}) = -0.0019 \text{ in}$$

$$q_{o3} = \Gamma_3 \cdot D_3 = (0.0893)(0.0024 \text{ in}) = 0.0002 \text{ in}$$

These are the displacements at the top of the building, which must now be distributed to the lower floors using the mode shapes, Eq. (7.52), to get physical displacements, u_{oi}.

$$\{u_{oi}\}_j = \{\varphi_{ij}\} \cdot q_{oj}$$

$$\{u_o\}_1 = \{\varphi_{i1}\} \cdot q_{o1} = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \cdot (0.137) = \begin{Bmatrix} 0.068 \\ 0.118 \\ 0.137 \end{Bmatrix} \text{ in}$$

$$\{u_o\}_2 = \{\varphi_{i2}\} \cdot q_{o2} = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \cdot (-0.0019) = \begin{Bmatrix} 0.0019 \\ 0 \\ -0.0019 \end{Bmatrix} \text{ in}$$

$$\{u_o\}_3 = \{\varphi_{i3}\} \cdot q_{o3} = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \cdot (0.0002) = \begin{Bmatrix} 0.0001 \\ -0.0002 \\ 0.0002 \end{Bmatrix} \text{ in}$$

Finally, estimate the maximum displacement response of each level using the SRSS combination rule, Eq. (7.24). For the *i*th level, we have

$$u_{\text{SRSS}oi} \approx \sqrt{\sum_{j=1}^{N} (u_{oi})_j^2} = \sqrt{(u_{oi})_1^2 + (u_{oi})_2^2 + (u_{oi})_3^2}$$

$$\text{Level 1: } u_{\text{SRSS}o1} \approx \sqrt{(0.068)^2 + (0.0019)^2 + (0.0001)^2} = 0.068 \text{ in}$$

$$\text{Level 2: } u_{\text{SRSS}o2} \approx \sqrt{(0.118)^2 + (0)^2 + (-0.0002)^2} = 0.1185 \text{ in}$$

$$\text{Level 3: } u_{\text{SRSS}o3} \approx \sqrt{(0.137)^2 + (-0.0019)^2 + (0.0002)^2} = 0.137 \text{ in}$$

iv. *Estimate the maximum lateral forces at each level using the SRSS combination rule.* First find the entire set of forces for all levels and modes using Eq. (7.13), the matrix operation [*f*] = [*k*][*u*]:

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$$[f]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\cdot 326.3\frac{\text{kip}}{\text{in}}\cdot\begin{bmatrix} 0.068 & 0.0019 & 0.0001 \\ 0.118 & 0 & -0.0002 \\ 0.137 & -0.0019 & 0.0002 \end{bmatrix}\text{in}$$

$$=\begin{bmatrix} 5.98 & 1.22 & 0.13 \\ 10.4 & 0 & -0.23 \\ 5.98 & -0.61 & 0.13 \end{bmatrix}\text{kip}$$

The forces are shown graphically in Fig. E7.1.

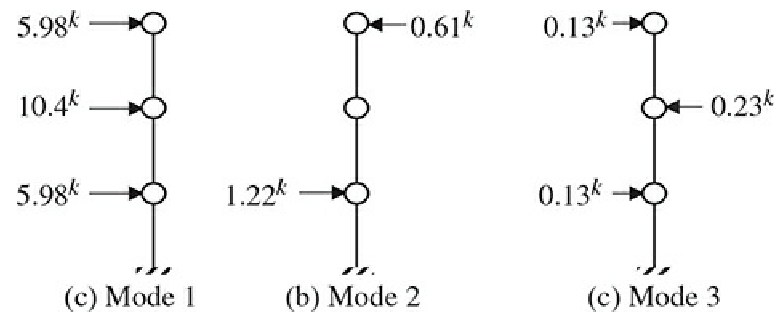

FIGURE E7.1 Resulting lateral forces at each node for a given mode of vibration.

The resultant maximum lateral force at each node is obtained from the SRSS rule [Eq. (7.24)] of each row vector: $f_i^{\,o} = (\Sigma f_i^2)^{½}$,

$$\text{Level 1: } f_{\text{SRSS}o1} \approx \sqrt{(5.98)^2+(-1.22)^2+(0.13)^2} = 6.1\text{ kip}$$

$$\text{Level 2: } f_{\text{SRSS}o2} \approx \sqrt{(10.4)^2+(0)^2+(-0.23)^2} = 10.4\text{ kip}$$

$$\text{Level 3: } f_{\text{SRSS}o3} \approx \sqrt{(5.98)^2+(0.61)^2+(0.13)^2} = 6.0\text{ kip}$$

v. *Estimate the maximum shear forces at the base using the SRSS combination rule.* Column vector of total base shear forces is [Eq. (7.20)] $\{V_b\} = [f]^{\mathrm{T}}\{1\}$

$$[V_b]=\begin{bmatrix} 5.98 & 10.4 & 5.98 \\ -1.22 & 0 & 0.61 \\ 0.13 & -0.23 & 0.13 \end{bmatrix}\text{kip}\cdot\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}=\begin{Bmatrix} 22.3 \\ 0.61 \\ 0.035 \end{Bmatrix}\text{kip}$$

Maximum base shear force obtained from SRSS as $V_b^{\,o} = (\Sigma V_i^2)^{½}$,

$$V_{b\text{SRSS}o} \approx \sqrt{(22.3)^2+(0.61)^2+(0.035)^2} = 22.3\text{ kip}$$

vi. *Estimate the maximum overturning moment at the base using the SRSS combination rule.* Column vector of overturning moment is [Eq. (7.22)] $\{M^{OT}\} = [f]^{\mathrm{T}}\{\text{heights}\}$

$$[M_{OT}]=\begin{matrix} 5.98 & 10.4 & 5.98 \\ -1.22 & 0 & 0.61 \\ 0.13 & -0.23 & 0.13 \end{matrix}\ \text{kip}\cdot\ \begin{matrix} 12 \\ 24 \\ 36 \end{matrix}\ \text{ft}=\ \begin{matrix} 536 \\ -7.35 \\ 0.85 \end{matrix}\ \text{kip}\cdot\text{ft}$$

Maximum overturning moment obtained from SRSS as $M_{OT}^{\,o} = (\Sigma M_i^2)^{½}$,

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$$M_{OTSRSSo} \approx \sqrt{(536)^2 + (-7.35)^2 + (0.85)^2} = 536 \text{ kip} \cdot \text{ft}$$

Clearly the first mode contains most of the contribution to the response (displacement, story forces, base shear, and overturning moment) as noted by the 93% modal mass. Thus, there would be no need to carry the second and third mode analyses.

The following figure summarizes the results of the maximum response for the first mode:

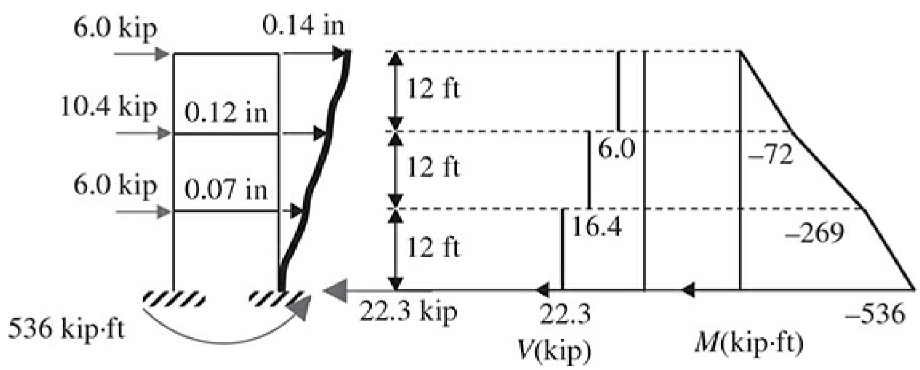


FIGURE E7.2 Graphical summary of all results.

vii. *We can write a MATLAB script to perform all the operation as follows:*

```
clear all % clears any previously defined variables
m = [100 0 0; 0 100 0; 0 0 50]/386.4 % mass matrix in kip-sec2/in
k = [2 -1 0; -1 2 -1; 0 -1 1]*326.3 % stiffness matrix in kip/in
[phi,lam] = eig(k,m); % compute eigenvalues and eigenvectors
omegasn = diag(sqrt(lam)) % determine frequencies from eigenvalues
periods = 2*pi./omegasn % determine the periods from omegas, sec
[N_rows, N_cols] = size(m); % finds N, the size of the matrices
norm_phi = phi./phi(N_cols,:)% Mode shapes normalized to top displ of 1
M = diag(norm_phi'*m*norm_phi) % display modal mass matrix
K = diag(norm_phi'*k*norm_phi) % display modal stiffness matrix
% determine the participation factors
LT = sum(norm_phi'*m,2); % sum(A,2) returns a vector of sums of each row
MT= sum(norm_phi'*m*norm_phi,2);
par_fac = LT./MT % ./ operation divides each element of the two vectors
Eff_mass = par_fac.*LT % effective modal mass and percent of total mass
Eff_masspercent = Eff_mass/sum(diag(m))*100
%%%% Maximum displacements at each level for each mode shape
Di_max=[0.110;0.00563; 0.00242];% spectral displacements, in
qi_max=par_fac.*Di_max % top floor displacements, in
for j = 1:N_cols
    ui_max(:,j) = norm_phi(:,j)*qi_max(j);
end
ui_max % display displs.; each column corresponds to a mode shape
% Use the SRSS rule to get max displacements at each level
u_maxsrss=sqrt(sum((ui_max).^2,2)) %.^ operation squares each element
% Floor forces at each level for each mode shape and max by SRSS method
f=k*ui_max
f_maxsrss=sqrt(sum((f).^2,2))
% Base shear for each mode shape and max base shear using SRSS rule
V=(sum(f,1))
V_maxsrss=sqrt(sum((V).^2,2))
% Overturning moment for each mode shape and max moment using SRSS
heights=[12;24;36];
OTM = f'*heights
OTM_maxsrss=sqrt(sum((OTM).^2,1))
```

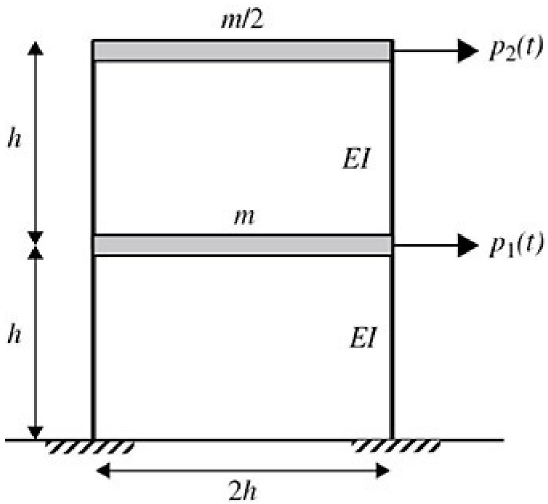
The results of the script are

```
m =
    0.2588         0         0
         0    0.2588         0
         0         0    0.1294
k =
  652.6000 -326.3000         0
 -326.3000  652.6000 -326.3000
         0 -326.3000  326.3000
omegasn =
   18.3803
   50.2160
   68.5963
periods =
    0.3418
    0.1251
    0.0916
```

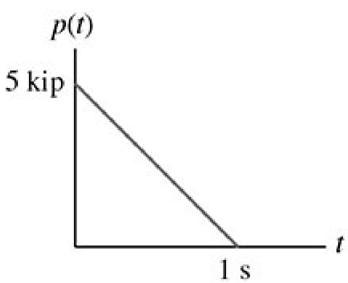
```
norm_phi =
     0.5000    -1.0000     0.5000
     0.8660     0.0000    -0.8660
     1.0000     1.0000     1.0000
M =
     0.3882
     0.3882
     0.3882
K = 1.0e+03 *
     0.1311
     0.9789
     1.8267
par_fac =
     1.2440
    -0.3333
     0.0893
Eff_mass =
     0.6008
     0.0431
     0.0031
Eff_masspercent =
    92.8547
     6.6667
     0.4786
qi_max =
     0.1368
    -0.0019
     0.0002
ui_max =
     0.0684     0.0019     0.0001
     0.1185    -0.0000    -0.0002
     0.1368    -0.0019     0.0002
u_maxsrss =
     0.0684
     0.1185
     0.1369
f =
     5.9822     1.2247     0.1316
    10.3614          0    -0.2280
     5.9822    -0.6124     0.1316
f_maxsrss =
     6.1077
    10.3639
     6.0149
V =
    22.3257     0.6124     0.0353
V_maxsrss =
    22.3342
OTM =
   535.8180
    -7.3483
     0.8463
OTM_maxsrss =
   535.8691
```

7.4 Problems

7.1 Given the following two-story building frame with k = 40 kip/in and m = 0.4 kip · s^2/in, determine (a) mass and stiffness matrices, (b) natural frequencies and modal matrix, (c) modal mass and stiffness matrices and modal force vector, and (d) steady-state displacement response. Also, graph the steady-state displacement response time-history. The two loads are harmonic and are given as $p^1(t)$ = 4 kip · sin[(26 rad/s)t] and $p^2(t)$ = 5 kip · sin[(26 rad/s)t]. Assume the initial conditions are zero.



7.2 For the two-story building frame given in Prob. 7.1, estimate the maximum displacement response of each level using the SRSS combination rule. Assume the initial conditions are zero and the two loads, $p_1(t) = p_2(t) = p(t)$, are given by the following triangular pulse.



7.3 Given the following three-story building with rigid beams and flexible steel (E = 29,000 ksi) columns and total moment of inertias for each floor shown, estimate the maximum displacement response of each level using the SRSS combination rule. Assume the initial conditions are zero and loading at each floor is given by the triangular pulses shown.

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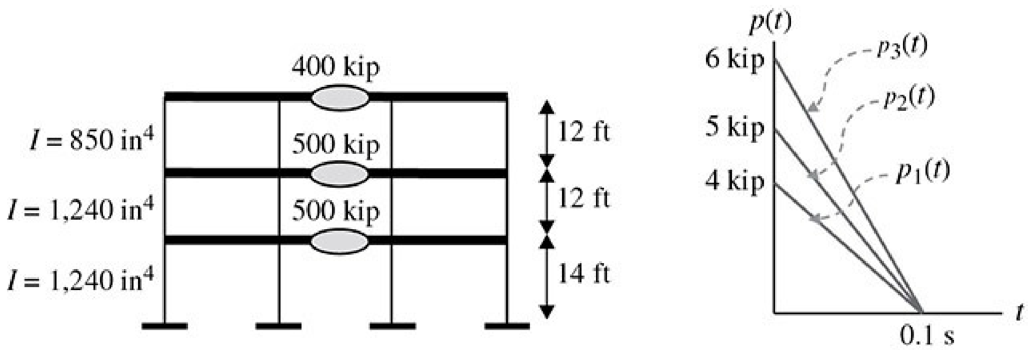
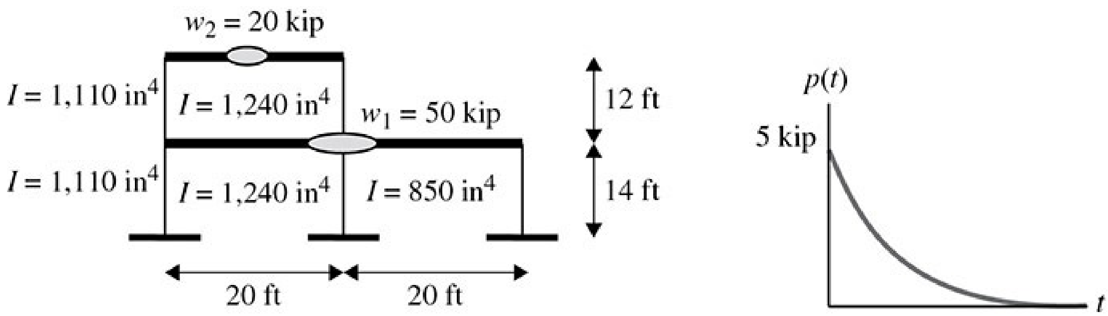
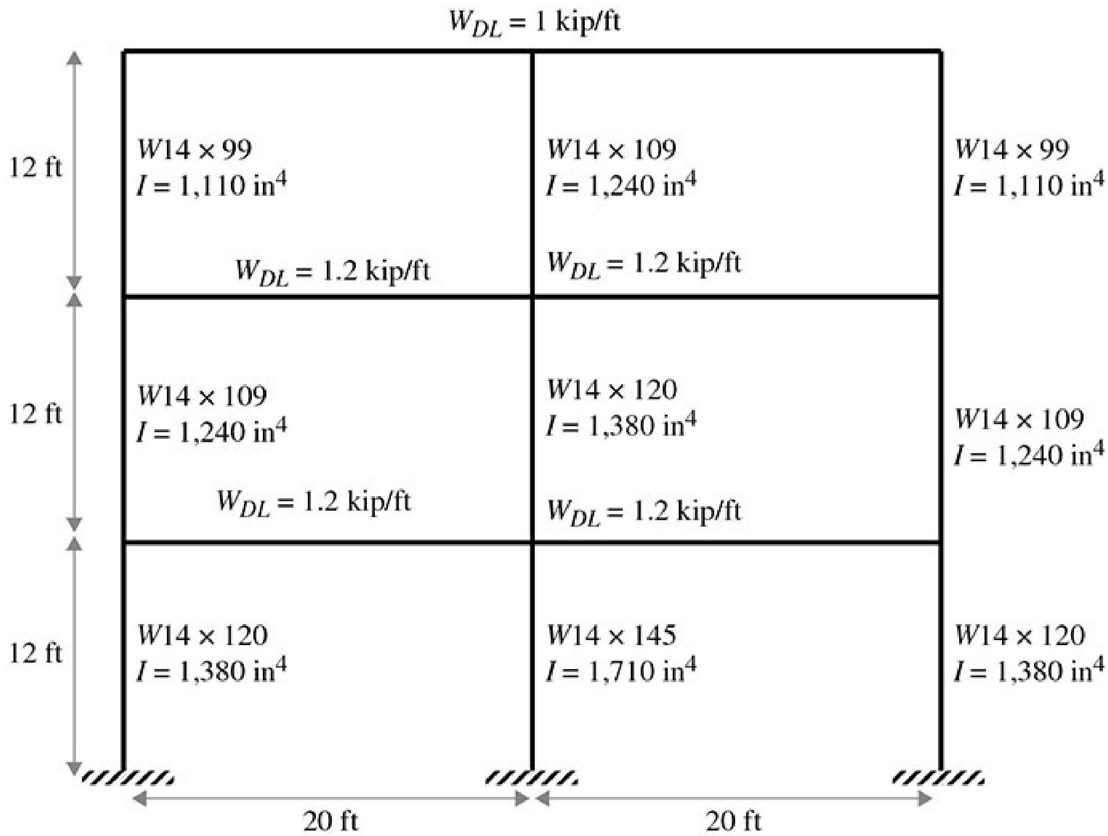


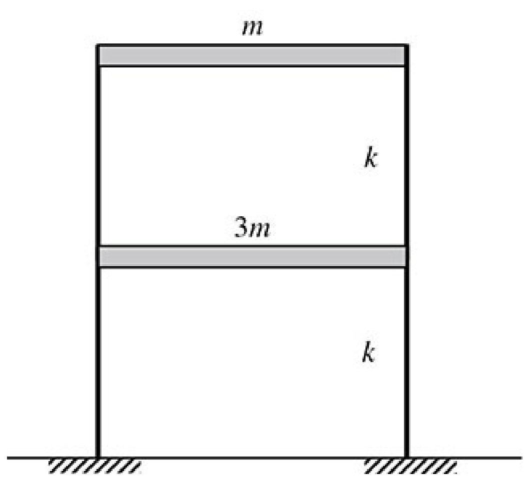
7.4 Given the following two-story building with rigid beams and flexible steel (E = 29,000 ksi) columns and moment of inertias for each column shown, estimate the maximum displacement response of each level using the SRSS combination rule. Assume the initial conditions are zero and the loading on each floor is given by the following exponential pulse, $p^1(t) = p^2(t) = p(t) = 5\text{ kip} \cdot e^{-10t}$.



7.5 Given the following three-story building with rigid beams and flexible steel (E = 29,000 ksi) columns, estimate the maximum displacement response of each level using the SRSS combination rule. Assume the initial conditions are zero and loading at each floor given as $p^1(t) = 5\text{ kip} \cdot \sin[(15\text{ rad/s})t]$, $p^2(t) = 10\text{ kip} \cdot \sin[(15\text{ rad/s})t]$, and $p^3(t) = 15\text{ kip} \cdot \sin[(15\text{ rad/s})t]$.



7.6 Given the following two-story building frame with k = 40 kip/in and m = 0.4 kip · s^2/in, determine (a) mass and stiffness matrices, (b) natural frequencies and mode shape matrix, (c) modal mass and stiffness matrices and modal force vector, and (d) steady-state displacement response. Also, graph the steady-state displacement response time-history. Assume the initial conditions are zero and the two loads are harmonic and are given as $p^1(t)$ = 4 kip · sin[(26 rad/s)t] and $p^2(t)$ = 5 kip · sin[(26 rad/s)t]. Also, the system has a uniform damping ratio of 5%.



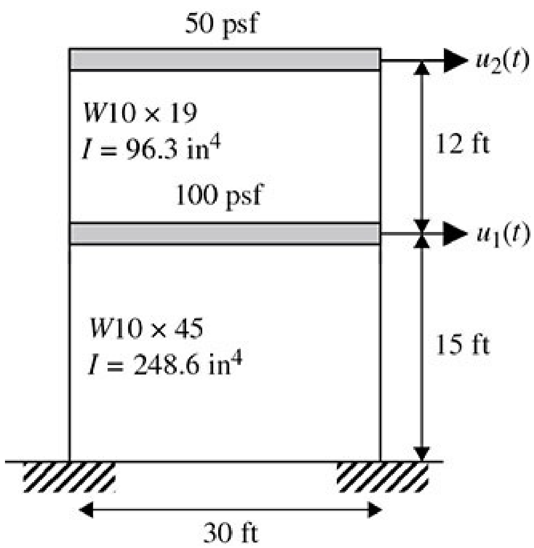
7.7 Consider the frame in Prob. 7.1 and assume a uniform damping ratio of 5%. Estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments using the SRSS combination rule.

7.8 Consider the frame in Prob. 7.3 and assume a uniform damping ratio of 5%. Estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments using the SRSS combination rule.

7.9 Consider the frame in Prob. 7.4 and assume a damping ratio of 3% for the top floor and 5% for the first floor. Estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments using the SRSS combination rule.

7.10 Consider the frame in Prob. 7.5 and assume a damping ratio of 5% in the first and third modes and Rayleigh damping for the second mode. Estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments using the SRSS combination rule.

7.11 The following two-story building is supported with steel frames spaced at 15 ft on center and has the floor load shown plus wall load of 20 psf. Formulate the equation of motion using D'Alembert's principle and determine the natural frequencies, periods, normalized modal matrix, and participation factors. Also, estimate (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, and (d) maximum floor overturning moments using the SRSS combination rule for the two harmonic displacement excitations given as, $u_1(t) = 0.5 \text{ in} \cdot \sin[(16 \text{ rad/s})t]$ and $u_2(t) = 0.9 \text{ in} \cdot \sin[(16 \text{ rad/s})t]$. Assume a damping ratio of 5%.



7.12 Given the following mode shapes and frequencies, compute the participation factors for a *three*-story building with floor weights of 120 kip for the first floor, and 80 kip for the second floor and roof.

$$[\Phi] = \begin{bmatrix} 1.68 & -1.208 & -0.714 \\ 1.22 & 0.704 & 1.697 \\ 0.572 & 1.385 & -0.984 \end{bmatrix} \text{ and } \{\omega\} = \begin{Bmatrix} 8.77 \\ 25.18 \\ 48.13 \end{Bmatrix} \text{ rad/s}$$

7.13 Use a MDOF analysis for the following building frame to determine the story displacements, story forces, base shear, and overturning moment due to a harmonic base displacement having an amplitude, u_0 = 0.5 in and an excitation frequency, ω = 10 rad/s.

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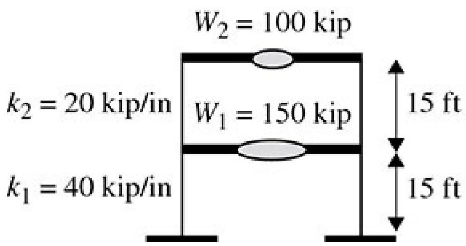
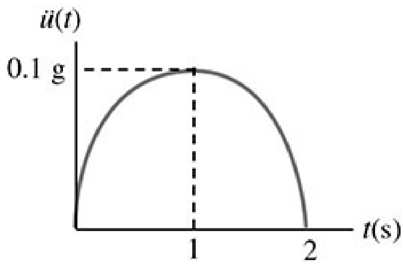
Assume rigid beams and damping of 5%.



7.14 Consider the two-story building frame given in Prob. 7.13 and determine (a) peak displacements, (b) maximum equivalent static floor forces, (c) maximum base shear, (d) maximum overturning moment, and (e) effective masses. Assume beams are rigid, damping of 2%, and the frame is subjected to a ground acceleration caused by a large vehicle passing by, which can be modeled as a half-cycle pulse shown (Chap. 4, Fig. E12.1 gives the shock spectrum). Use the SRSS combination rule to estimate the maximum response.



7.15 Given a five-story building frame subjected to a ground acceleration due to the 2021 Haiti earthquake presented in Chap. 4, Example 16, use a modal analysis to determine (a) peak displacements, (b) maximum base shear, and (c) maximum floor overturning moments. Each story has a stiffness $k^1 = k^2 = k^3 = k^4 = k^5 = k = 326.3$ kip/in, the weight of each of the first four floors is $w^1 = w^2 = w^3 = w^4 = 100$ kip. The fifth floor or roof has a weight $w^5 = 50$ kip. Assume rigid beams and damping of 2%.

7.16 Solve Prob. 7.15 but use the fewest number of modes such that they contribute 90% of the effective mass.

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Index

A

B

C

D

@seismicisolation

@seismicisolation

E

@seismicisolation

F

@seismicisolation

G

@seismicisolation

H

@seismicisolation

I

@seismicisolation

@seismicisolation

M

N

O

P

R

S

@seismicisolation

@seismicisolation

@seismicisolation

@seismicisolation

U

@seismicisolation

V

@seismicisolation