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# MECHANICS OF SOLIDS



S.S. Bhavikatti



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**S.S. Bhavikatti**

Emeritus Fellow (AICTE)

BVB College of Engineering and Technology, Hubli

(Formerly Principal, RYMEC, Bellary

Professor & Dean

SDMCET, Dharwad and NITK, Surathkal)



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# Preface

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Mechanics of Solids is an important course for all engineering students by which they develop analytical skill. In this course, laws of mechanics are applied to parts of bodies and skill is developed to get solution to engineering problems maintaining continuity of the parts.

The author has clearly explained theories involved and illustrated them by solving a number of engineering problems. Neat diagrams are drawn and solutions are given without skipping any step. SI units and standard notations as suggested by Indian Standard Code are used throughout. The author has made this book to suit the latest syllabus of Gujarat Technical University.

Author hopes, the students and teachers of Gujarat Technical University will receive this book whole-heartedly as most of the earlier books of the author have been received by the students and teachers all over India.

The suggestions and corrections, if any, are most welcome.

The author acknowledges the efforts of M/s. New Age International Publishers in bringing out this book in nice form. He also acknowledges the opportunity given by AICTE for associating him with B.U.B. Engineering College, Hubli.

—Author

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# Introduction to Mechanics of Solids

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The state of rest and the state of motion of the bodies under the action of different forces has engaged the attention of mathematicians and scientists for many centuries. The branch of physical science that deal with the state of rest or the state of motion of bodies is termed as **mechanics**. Starting from the analysis of rigid bodies under gravitational force and application of simple forces the mechanics has grown into the analysis of complex structures like multistorey buildings, aircrafts, space crafts and robotics under complex system of forces like dynamic forces, atmospheric forces and temperature forces.

Archemedes (287–212 BC), Galileo (1564–1642), Sir Issac Newton (1642–1727) and Einstein (1878–1955) have contributed a lot to the development of mechanics. Contributions by Varignon, Euler, and D. Alemberts are also substantial. The mechanics developed by these researchers may be grouped as

- (i) Classical mechanics/Newtonian mechanics
- (ii) Relativistic mechanics
- (iii) Quantum mechanics/Wave mechanics.

Sir Issac Newton, the principal architect of mechanics, consolidated the philosophy and experimental findings developed around the state of rest and state of motion of the bodies and putforth them in the form of three laws of motion as well as the law of gravitation. The mechanics based on these laws is called **Classical mechanics or Newtonian mechanics**.

Albert Einstein proved that Newtonian mechanics fails to explain the behaviour of high speed (speed of light) bodies. He putfourth the theory of **Relativistic mechanics**.

Schrödinger (1887–1961) and Broglie (1892–1965) showed that Newtonian mechanics fails to explain the behaviour of particles when atomic distances are concerned. They putforth the theory of **Quantum mechanics**.

Engineers are keen to use the laws of mechanics to actual field problems. Application of laws of mechanics to field problems is termed as **Engineering mechanics**. For all the problems between atomic distances to high speed distances there are various engineering problems for which Newtonian mechanics has stood the test of time and hence is the mechanics used by engineers.

The various bodies on which engineers are interested to apply laws of mechanics may be classified as

- (i) Solids and
- (ii) Fluids.

The bodies which do not change their shape or size appreciably when the forces are applied are termed as **Solids** while the bodies which change their shape or size appreciably even when small forces are applied are termed as **Fluids**. Stone, steel, concrete etc. are the example of solids while water, gases are the examples of fluids.

In this book application of Newtonian mechanics to solids is dealt with.

### 1.1 BASIC TERMINOLOGIES IN MECHANICS

The following are the terms basic to the study of mechanics, which should be understood clearly.

#### Mass

The quantity of the matter possessed by a body is called mass. The mass of a body will not change unless the body is damaged and part of it is physically separated. If the body is taken out in a space craft, the mass will not change but its weight may change due to the change in gravitational force. The body may even become weightless when gravitational force vanishes but the mass remain the same.

#### Time

The time is the measure of succession of events. The successive event selected is the rotation of earth about its own axis and this is called a day. To have convenient units for various activities, a day is divided into 24 hours, an hour into 60 minutes and a minute into 60 seconds. Clocks are the instruments developed to measure time. To overcome difficulties due to irregularities in the earth's rotation, the unit of time is taken as second which is defined as the duration of 9192631770 period of radiation of the cesium-133 atom.

#### Space

The geometric region in which study of body is involved is called space. A point in the space may be referred with respect to a predetermined point by a set of linear and angular measurements. The reference point is called the *origin* and the set of measurements as *coordinates*. If the coordinates involved are only in mutually perpendicular directions, they are known as *cartesian coordination*. If the coordinates involve angles as well as the distances, it is termed as *Polar Coordinate System*.

#### Length

It is a concept to measure linear distances. The diameter of a cylinder may be 300 mm, the height of a building may be 15 m, the distance between two cities may be 400 km.

Actually metre is the unit of length. However depending upon the sizes involved micro, milli or kilo metre units are used for measurements. A metre is defined as length of the standard bar of platinum-iridium kept at the International Bureau of weights and measures. To overcome the difficulties of accessibility and reproduction now metre is defined as 1690763.73 wavelength of krypton-86 atom.

#### Continuum

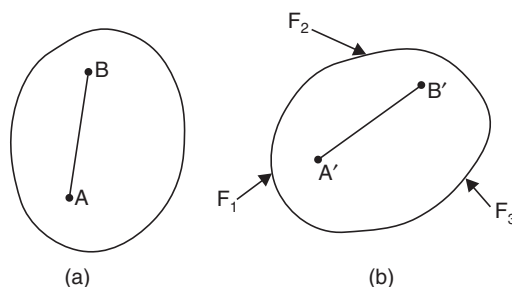
A body consists of several matters. It is a well known fact that each particle can be subdivided into molecules, atoms and electrons. It is not possible to solve any engineering problem by treating a body as conglomeration of such discrete particles. The body is assumed to be a continuous distribution of matter. In other words the body is treated as continuum.

### Rigid Body

A body is said to be rigid, if the relative positions of any two particles do not change under the action of the forces acting on it. In Fig. 1.1 (a), point  $A$  and  $B$  are the original positions in a body. After the application of forces  $F_1, F_2, F_3$ , the body takes the position as shown in Fig. 1.1(b).  $A'$  and  $B'$  are the new positions of  $A$  and  $B$ . If the body is treated as rigid, the relative position of  $A'B'$  and  $AB$  are the same *i.e.*

$$A'B' = AB$$

Many engineering problems can be solved by assuming bodies rigid



**Fig. 1.1**

### Particle

A particle may be defined as an object which has only mass and no size. Theoretically speaking such a body cannot exist. However in dealing with problems involving distances considerably larger compared to the size of the body, the body may be treated as a particle, without sacrificing accuracy.

For example:

- A bomber aeroplane is a particle for a gunner operating from the ground.
- A ship in mid sea is a particle in the study of its relative motion from a control tower.
- In the study of movement of the earth in celestial sphere, earth is treated as a particle.

### Force

Force is an important term used in solid mechanics. Newton's first law states that everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it. This leads to the definition of force as '*force is an external agency which changes or tends to change the state of rest or uniform linear motion of the body*'.

Magnitude of force is defined by Newton's second law. It states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it. Noting that rate of change of velocity is acceleration, and the product of mass and velocity is momentum we can derive expression for the force as given below:

From Newton's second law of motion

$$\begin{aligned} \text{Force} &\propto \text{rate of change of momentum} \\ &\propto \text{rate of change of (mass} \times \text{velocity)} \end{aligned}$$

Since mass do not change,

$$\begin{aligned}
 \text{Force} &\propto \text{mass} \times \text{rate of change of velocity} \\
 &\propto \text{mass} \times \text{acceleration} \\
 F &\propto m \times a \\
 &= k \times m \times a
 \end{aligned}
 \quad \dots(1.1)$$

where  $F$  is the force,  $m$  is the mass and  $a$  is the acceleration and  $k$  is the constant of proportionality.

In all the systems, unit of force is so selected that the constant of the proportionality becomes unity. For example, in S.I. system, unit of force is Newton, which is defined as the force that is required to move one kilogram (kg) mass at an acceleration of  $1 \text{ m/sec}^2$ .

$$\therefore \quad \text{One newton} = 1 \text{ kg mass} \times 1 \text{ m/sec}^2$$

$$\begin{aligned}
 \text{Thus} \quad k &= 1 \\
 F &= m \times a
 \end{aligned}
 \quad \dots(1.2)$$

However in MKS acceleration used is one gravitational acceleration ( $9.81 \text{ m/sec}^2$  on earth surface) and unit of force is defined as kg-wt.

$$\begin{aligned}
 \text{Thus} \quad F \text{ in kg wt} &= m \times g \\
 \text{Thus} \quad 1 \text{ kg-wt} &= 9.81 \text{ newtons}
 \end{aligned}
 \quad \dots(1.3)$$

$$\dots(1.4)$$

It may be noted that in usage kg-wt is often called as kg only.

### Characteristics of a Force

It may be noted that a force is completely specified only when the following four characteristics are specified

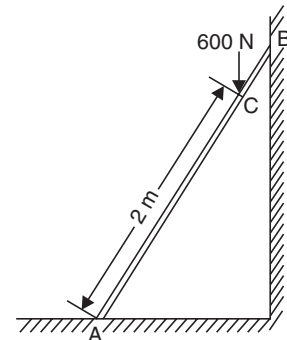
- Magnitude
- Point of application
- Line of action
- Direction.

In Fig. 1.2,  $AB$  is a ladder kept against a wall. At point  $C$ , a person weighing  $600 \text{ N}$  is standing. The force applied by the person on the ladder has the following characters:

- magnitude is  $600 \text{ N}$
- the point of application is  $C$  which is at  $2 \text{ m}$  from  $A$  along the ladder
- the line of action is vertical
- the direction is downward.

It may be noted that in the figure

- magnitude is written near the arrow
- the line of arrow shows the line of application
- the arrow head shows the point of application
- the direction of arrow represents the direction of the force.



**Fig. 1.2**

## 1.2 UNITS

Length (L), mass (M) and time (S) are the fundamental units used in mechanics. The units of all other quantities may be expressed in terms of these basic units. The three commonly used systems are

- Metre, Kilogram, Second (MKS)
- Centimetre, Gram, Second (CGS)
- Foot, Pound, Second (FPS).

The systems are named after the units used to define the fundamental quantities length, mass and time. Using these basic units, the units of other quantities can be found. For example in MKS the units for various quantities are

Quantity	Unit	
Area	$\text{m}^2$	
Volume	$\text{m}^3$	
Velocity	$\text{m/sec}$	
Acceleration	$\text{m/sec}^2$	
Momentum	$\text{kg-m/sec}$	[Since it is = mass $\times$ velocity]
Force	$\text{kg-m/sec}^2$	[Since it is = mass $\times$ acceleration]

### S.I. Units

Presently the whole world is in the process of switching over to SI-system of units. SI units stands for the System International *d'* units or International System of units. As in MKS units in SI also the fundamental units are metre for length, kilogram for mass and second for time. The difference between MKS and SI system arises mainly in selecting the unit of force. In MKS unit of force is kg-wt while in SI units it is newton. As we have already seen one kg-wt is equal to 9.81 newtons.

The prefixes used in SI when quantities are too big or too small are shown in Table 1.1.

**Table 1.1.** Prefixes in SI Units

<i>Multiplying Factors</i>	<i>Prefix</i>	<i>Symbol</i>
$10^{12}$	tera	T
$10^9$	giga	G
$10^6$	mega	M
$10^3$	kilo	k
$10^0$	—	—
$10^{-3}$	milli	m
$10^{-6}$	micro	$\mu$
$10^{-9}$	nano	n
$10^{-12}$	pico	p
$10^{-15}$	femto	f
$10^{-18}$	atto	a



### 1.3 SCALAR AND VECTOR QUANTITIES

Various quantities used in mechanics may be grouped into scalars and vectors. A quantity is said to be scalar, if it is completely defined by its magnitude alone. Examples of scalars are length, area, time and mass.

A quantity is said to be vector if it is completely defined only when its magnitude as well as direction are specified. The example of vectors are displacement, velocity, acceleration, momentum, force etc.

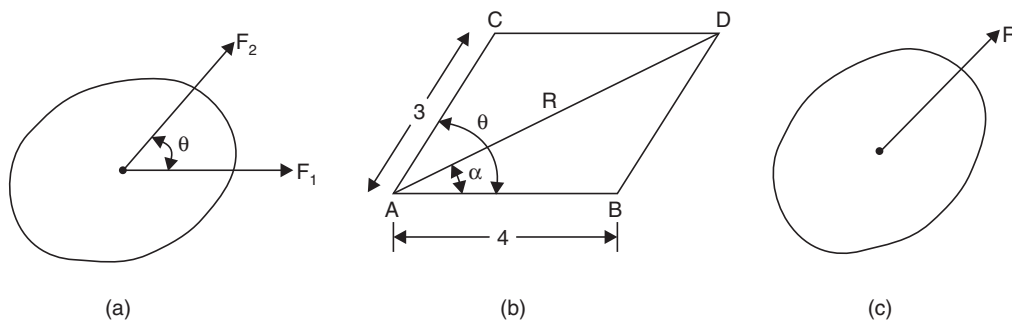
### 1.4 COMPOSITION AND RESOLUTION OF VECTORS

The process of finding a single vector which will have the same effect as a set of vectors acting on a body is known as composition of vectors. The resolution of vectors is exactly the opposite process of composition *i.e.*, it is the process of finding two or more vectors which will have the same effect as that of a vector acting on the body.

#### Parallelogram Law of Vectors

The parallelogram law of vectors enables us to determine the single vector called resultant vector which can replace the two vectors acting at a point with the same effect as that of the two vectors. This law was formulated based on experimental results on a body subjected to two forces. This law can be applied not only to the forces but to any two vectors like velocities, acceleration, momentum etc. Though Stevin employed it in 1586, the credit of presenting it as a law goes to Varignon and Newton (1687). This law states that *if two forces (vectors) acting simultaneously on a body at a point are represented in magnitude and directions by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the forces (vectors).*

In the Fig. 1.3, the force  $F_1 = 4$  units and the force  $F_2 = 3$  unit are acting on a body at a point A. To get the resultant of these forces, according to this law, construct the parallelogram ABCD such that AB is equal to 4 units to the linear scale and AC is equal to 3 units. Then according to this law, the diagonal AD represents the resultant in magnitude and direction. Thus the resultant of the forces  $F_1$  and  $F_2$  is equal to the units corresponding to AD in the direction  $\alpha$  to  $F_1$ .



**Fig. 1.3**

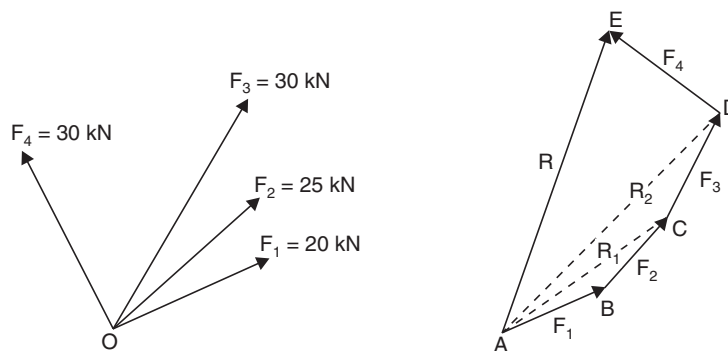
### Triangle Law of Vectors

Referring to Fig. 1.3 (b), it can be observed that the resultant  $AD$  may be obtained by constructing the triangle  $ABD$ . Line  $AB$  is drawn to represent  $F_1$  and  $BD$  to represent  $F_2$ . Then  $AD$  should represent the resultant of  $F_1$  and  $F_2$ . Thus we have derived the triangle law of forces from the fundamental law of parallelogram. The *Triangle Law of Forces (vectors)* may be stated as *if two forces (vectors) acting on a body are represented one after another by the sides of a triangle, their resultant is represented by the closing side of the triangle taken from the first point to the last point.*

### Polygon Law of Forces (Vectors)

If more than two forces (vectors) are acting on a body, two forces (vectors) at a line can be combined by the triangle law, and finally resultant of all forces (vectors) acting on the body may be obtained.

A system of four concurrent forces acting on a body are shown in Fig. 1.4.  $AB$  represents  $F_1$  and  $BC$  represent  $F_2$ . Hence according to triangle law of forces  $AC$  represents the resultant of  $F_1$  and  $F_2$ , say  $R_1$ .



**Fig. 1.4**

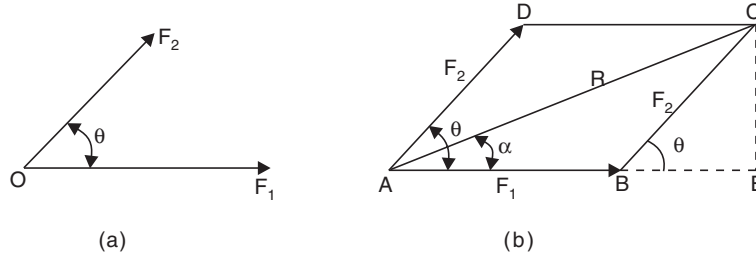
If  $CD$  is drawn to represent  $F_3$ , then from the triangle law of forces  $AD$  represents the resultant of  $R_1$  and  $F_3$ . In other words,  $AD$  represents the resultant of  $F_1$ ,  $F_2$  and  $F_3$ . Let it be called as  $R_2$ .

Similarly the logic can be extended to conclude that  $AE$  represents the resultant of  $F_1$ ,  $F_2$ ,  $F_3$  and  $F_4$ . The resultant  $R$  is represented by the closing line of the polygon  $ABCDE$  in the direction from  $A$  to  $E$ . Thus we have derived the polygon law of the forces (vectors) and it may be stated as *if a number of concurrent forces (vectors) acting simultaneously on a body are represented in magnitude and direction by the sides of a polygon, taken in a order, then the resultant is represented in magnitude and direction by the closing side of the polygon, taken from the first point to the last point.*

### Analytical Method of Composition of Two Vectors

Parallelogram law, triangle law and polygonal law of vectors can be used to find the resultant graphically. This method gives a clear picture of the work being carried out. However the main disadvantage is that it needs drawing aids like pencil, scale, drawing sheets. Hence there is need for analytical method.

Consider the two forces  $F_1$  and  $F_2$  acting on a particle as shown in Fig 1.5(a). Let the angle between the two forces be  $\theta$ . If parallelogram  $ABCD$  is drawn as shown in Fig. 1.5(b) with  $AB$  representing  $F_1$  and  $AD$  representing  $F_2$  to some scale, according to parallelogram law of forces  $AC$  represents the resultant  $R$ . Drop perpendicular  $CE$  to  $AB$ .



**Fig. 1.5**

The resultant  $R$  of  $F_1$  and  $F_2$  is given by

$$R = AC = \sqrt{AE^2 + CE^2} = \sqrt{(AB + BE)^2 + CE^2}$$

But

$$AB = F_1$$

$$BE = BC \cos \theta = F_2 \cos \theta$$

$$CE = BC \sin \theta = F_2 \sin \theta$$

$\therefore$

$$\begin{aligned} R &= \sqrt{(F_1 + F_2 \cos \theta)^2 + (F_2 \sin \theta)^2} \\ &= \sqrt{F_1^2 + 2F_1 F_2 \cos \theta + F_2^2 \cos^2 \theta + F_2^2 \sin^2 \theta} \\ &= \sqrt{F_1^2 + 2F_1 F_2 \cos \theta + F_2^2} \end{aligned}$$

Since,  $\sin^2 \theta + \cos^2 \theta = 1$ .

The inclination of resultant to the direction of  $F_1$  is given by  $\alpha$ , where

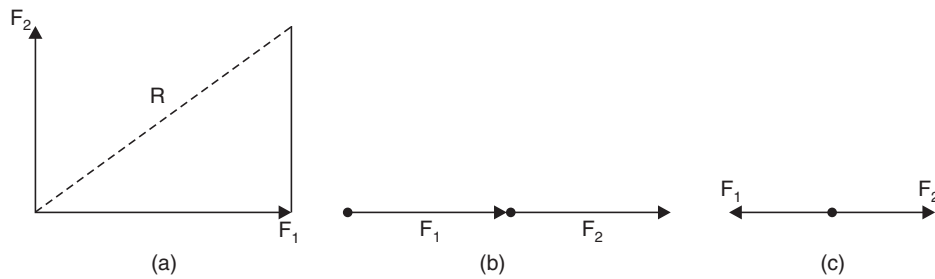
$$\tan \alpha = \frac{CE}{AE} = \frac{CE}{AB + BE} = \frac{F_2 \sin \theta}{F_1 + F_2 \cos \theta}$$

Hence

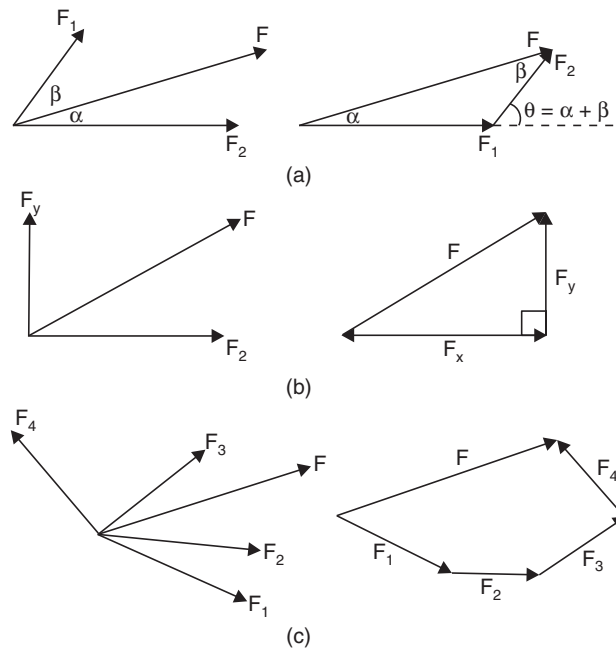
$$\alpha = \tan^{-1} \frac{F_2 \sin \theta}{F_1 + F_2 \cos \theta}$$

Particular cases:

1. When  $\theta = 90^\circ$  [Ref. Fig. 1.6a],  $R = \sqrt{F_1^2 + F_2^2}$
2. When  $\theta = 0^\circ$  [Ref. Fig. 1.6b],  $R = \sqrt{F_1^2 + 2F_1 F_2 + F_2^2} = F_1 + F_2$
3. When  $\theta = 180^\circ$  [Ref. Fig. 1.6c],  $R = \sqrt{F_1^2 - 2F_1 F_2 + F_2^2} = F_1 - F_2$

**Fig. 1.6****Resolution of Vectors**

Since the resolution of vectors is exactly opposite process of composition of vectors, exactly the opposite process of composition can be employed to get the resolved components of a given force.

**Fig. 1.7**

In Fig. 1.7(a), the given force  $F$  is resolved into two components making angles  $\alpha$  and  $\beta$  with  $F$ .

In Fig. 1.7(b) the force  $F$  is resolved into its rectangular components  $F_x$  and  $F_y$ .

In Fig. 1.7(c), the force  $F$  is resolved into its four components  $F_1$ ,  $F_2$ ,  $F_3$  and  $F_4$ .

It may be noted that all component forces act at the same point as the given force. Resolution of forces into its rectangular components is more useful in solving the problems in mechanics. In this case, if the force  $F$  makes angle  $\theta$  with  $x$ -axis, from Fig. 1.7(a), it is clear that

$$F_x = F \cos \theta \quad \text{and} \quad F_y = F \sin \theta.$$

**Example 1.1.** A boat is rowed at a velocity of 20 km/hour across a river. The velocity of stream is 8 km/hour. Determine the resultant velocity of the boat.

**Solution:** Taking downstream direction as  $x$  and direction across the river as  $y$ , it is given that

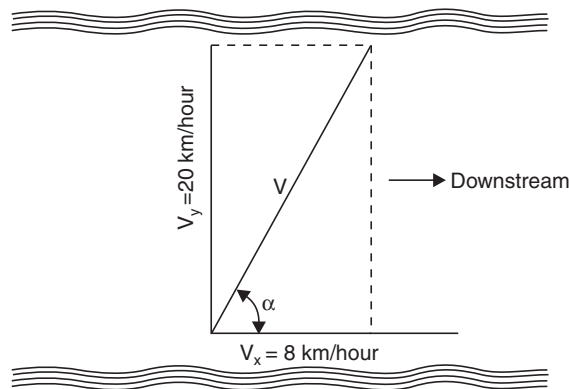
$$V_x = 8 \text{ km/hour}$$

$$V_y = 20 \text{ km/hour}$$

$\therefore$  The resultant velocity

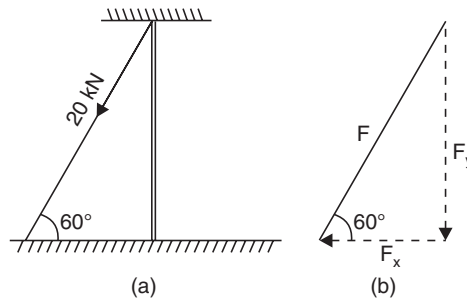
$$V = \sqrt{8^2 + 20^2} = 21.54 \text{ km/hour}$$

$$\alpha = \tan^{-1} \frac{V_y}{V_x} = \tan^{-1} \frac{20}{8} = 68.20^\circ, \text{ as shown in Fig. 1.8}$$



**Fig. 1.8**

**Example 1.2.** The guy wire of the electrical pole shown in Fig. 1.9(a) makes  $60^\circ$  to the horizontal and is carrying a force of 60 kN. Find the horizontal and vertical components of the force.



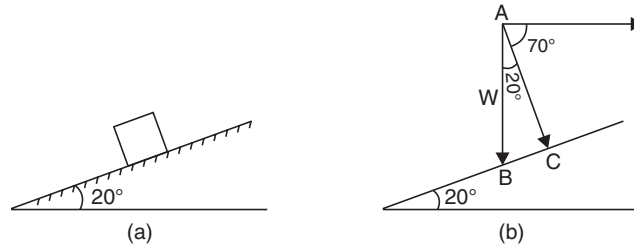
**Fig. 1.9**

**Solution:** Figure 1.9(b) shows the resolution of force  $F = 20 \text{ kN}$  into its components in horizontal and vertical components. From the figure it is clear that

$$F_x = F \cos 60^\circ = 20 \cos 60^\circ = 10 \text{ kN (to the left)}$$

$$F_y = F \sin 60^\circ = 20 \sin 60^\circ = 17.32 \text{ kN (downward)}$$

**Example 1.3.** A block weighing  $W = 10 \text{ kN}$  is resting on an inclined plane as shown in Fig. 1.10(a). Determine its components normal to and parallel to the inclined plane.



**Fig. 1.10**

**Solution:** The plane makes an angle of  $20^\circ$  to the horizontal. Hence the normal to the plane makes an angle of  $70^\circ$  to the horizontal i.e.,  $20^\circ$  to the vertical [Ref. Fig. 1.10(b)]. If  $AB$  represents the given force  $W$  to some scale,  $AC$  represents its component normal to the plane and  $CB$  represents its component parallel to the plane.

Thus from  $\triangle ABC$ ,

$$\begin{aligned}
 \text{Component normal to the plane} &= AC \\
 &= W \cos 20^\circ \\
 &= 10 \cos 20^\circ \\
 &= \mathbf{9.4 \text{ kN}} \text{ as shown in Fig. 1.10(b)} \\
 \text{Component parallel to the plane} &= W \sin 20^\circ = 10 \sin 20^\circ \\
 &= \mathbf{3.42 \text{ kN}}, \text{ down the plane}
 \end{aligned}$$

From the above example, the following points may be noted:

1. Imagine that the arrow drawn represents the given force to some scale.
2. Travel from the tail to head of arrow in the direction of the coordinates selected.
3. Then the direction of travel gives the direction of the component of vector.
4. From the triangle of vector, the magnitudes of components can be calculated.

**Example 1.4.** The resultant of two forces, one of which is double the other is  $260 \text{ N}$ . If the direction of the larger force is reversed and the other remain unaltered, the magnitude of the resultant reduces to  $180 \text{ N}$ . Determine the magnitude of the forces and the angle between the forces.

**Solution:** Let the magnitude of the smaller force be  $F$ . Hence the magnitude of the larger force is  $2F$ .

$$\text{Thus} \quad F_1 = F \quad \text{and} \quad F_2 = 2F$$

Let  $\theta$  be the angle between the two forces.

$\therefore$  From the condition 1, we get

$$R = \sqrt{F_1^2 + 2F_1F_2 \cos \theta + F_2^2} = 260$$

$$\text{i.e.,} \quad F^2 + 2F(2F) \cos \theta + (2F)^2 = 260^2$$

$$5F^2 + 4F^2 \cos \theta = 67600 \quad \dots(i)$$

From condition 2, we get

$$\begin{aligned}\sqrt{F_1^2 + 2F_1F_2 \cos(180 + \theta) + F_2^2} &= 180 \\ F^2 - 2F(2F) \cos \theta + (2F)^2 &= 32400 \quad \dots(ii)\end{aligned}$$

Adding equation (i) and (ii), we get

$$10F^2 = 100000$$

$$\therefore F = 100 \text{ N}$$

Hence  $F_1 = F = 100 \text{ N}$ ;  $F_2 = 2F = 200 \text{ N}$

Substituting the values of  $F_1$  and  $F_2$  in eqn (i), we get,

$$5(100)^2 + 4(100)^2 \cos \theta = 67600$$

$$\therefore \cos \theta = 0.44$$

or

$$\theta = 63.9^\circ$$

**Example 1.5.** Two forces  $F_1$  and  $F_2$  are acting at point A as shown in Fig. 1.11. The angle between the two forces is  $50^\circ$ . It is found that the resultant  $R$  is  $500 \text{ N}$  and makes angles  $20^\circ$  with the force  $F_1$  as shown in the figure. Determine the forces  $F_1$  and  $F_2$ .

**Solution:** Let  $\triangle ABC$  be the triangle of forces drawn to some scale. In this

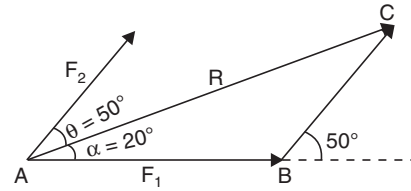


Fig. 1.11

$$\angle BAC = \alpha = 20^\circ$$

$$\angle ABC = 180 - 50 = 130^\circ$$

$$\therefore \angle ACB = 180 - (20 + 130) = 30^\circ$$

Applying sine rule to  $\triangle ABC$ , we get

$$\frac{AB}{\sin 30^\circ} = \frac{BC}{\sin 20^\circ} = \frac{500}{\sin 130^\circ}$$

$$\therefore AB = 326.35 \text{ N}$$

$$\text{and } BC = 223.24 \text{ N.}$$

$$\text{Thus } F_1 = AB = 326.35 \text{ N}$$

$$\text{and } F_2 = BC = 223.24 \text{ N}$$

**Example 1.6.** The resultant of two forces  $F_1 = 400 \text{ N}$  and  $F_2 = 260 \text{ N}$  acting at point A is  $520 \text{ N}$ . Determine the angle between the two forces and the angle between the resultant and force  $F_1$ .

**Solution:** Let  $\triangle ABC$  be the triangle of forces as shown in Fig. 1.12.  $\theta$  be the angle between  $F_1$  and  $F_2$ , and  $\alpha$  be the angle between resultant and  $F_1$

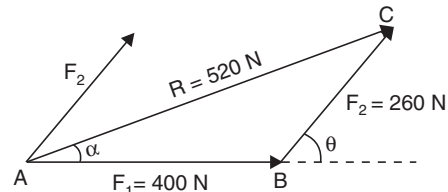


Fig. 1.12

Using the relation

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos \theta},$$

we get,

$$520^2 = 400^2 + 260^2 + 2 \times 400 \times 260 \times \cos \theta$$

$$\therefore \cos \theta = 0.20577$$

$$\therefore \theta = 78.13^\circ$$

Noting that

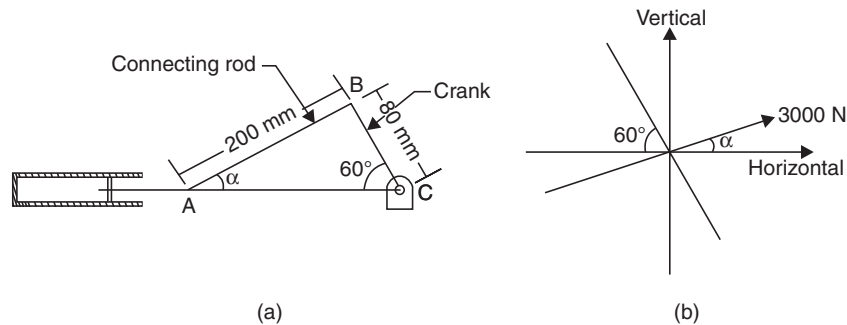
$$R \sin \alpha = F_2 \sin \theta$$

$$\text{we get} \quad \sin \alpha = \frac{260 \sin 78.13^\circ}{520} = 0.489$$

$$\therefore \alpha = 29.29^\circ$$

**Example 1.7.** Fig. 1.13 shows a particular position of 200 mm connecting rod AB and 80 mm long crank BC. At this position, the connecting rod of the engine experience a force of 3000 N on the crank pin at B. Find its

- horizontal and vertical component
- component along BC and normal to it.



**Fig. 1.13**

**Solution:** The force of 3000 N acts along line AB. Let AB make angle  $\alpha$  with horizontal. Then, obviously  $200 \sin \alpha = 80 \sin 60^\circ$

$$\therefore \alpha = 20.268^\circ$$

Referring to Fig. 1.13(b), we get

$$\text{Horizontal component} = 3000 \cos 20.268^\circ = \mathbf{2814.2 \text{ N}}$$

$$\text{Vertical component} = 3000 \sin 20.268^\circ = \mathbf{1039.2 \text{ N}}$$

Components along and normal to crank:

The force makes angle  $\alpha + 60^\circ = 20.268 + 60 = 80.268^\circ$  with crank.

$$\therefore \text{Component along crank} = 3000 \cos 80.268^\circ = \mathbf{507.1 \text{ N}}$$

$$\text{Component normal to crank} = 3000 \sin 80.268^\circ = \mathbf{2956.8 \text{ N}}$$

#### IMPORTANT FORMULAE

- Resultant of two vectors can be obtained by solving the triangle of forces.
- If  $V_1$  and  $V_2$  are the two vectors at angle ' $\theta$ ' between them, then the resultant is

$$R = \sqrt{V_1^2 + V_2^2 + 2V_1V_2 \cos \theta}$$



and acts at ' $\alpha$ ' to  $V_1$  vector, where

$$\tan \alpha = \frac{V_2 \sin \alpha}{V_1 + V_2 \cos \alpha}$$

Vectors may be forces, velocities, momentum etc.

3. If a force makes angle  $\theta$  with  $x$ -axis, then its components are

$$F_x = F \cos \theta$$

$$F_y = F \sin \theta.$$

4. If a body weighing  $W$  rests on an inclined plane, its components normal to and parallel to the plane are

$$F_n = W \cos \theta, \text{ a thrust on the plane.}$$

$$F_t = W \sin \theta, \text{ down the plane.}$$

### THEORY QUESTIONS

1. Explain the following terms:
 

(i) Space	(ii) Continuum
(iii) Particle	(iv) Rigid body.
2. Explain the term 'Force' and list its characteristics.
3. Distinguish between
 

(i) MKS and SI units	(ii) Scalars and vectors.
----------------------	---------------------------
4. State and explain parallelogram law of vectors.
5. State parallelogram law of vector and derive triangle and polygonal law of vectors.

### PROBLEMS FOR EXERCISE

1. The resultant of two forces one of which is 3 times the other is 300 N. When the direction of smaller force is reversed, the resultant is 200 N. Determine the two forces and the angle between them. [Ans.  $F_1 = 80.6 \text{ N}$ ,  $F_2 = 241.8 \text{ N}$ ,  $\theta = 50.13^\circ$ ]
2. A rocket is released from a fighter plane at an angle upward  $20^\circ$  to the vertical with an acceleration of  $8 \text{ m/sec}^2$ . The gravitational acceleration is  $9.1 \text{ m/sec}^2$  downward. Determine the instantaneous acceleration of the rocket when it was fired. [Ans.  $9.849 \text{ m/sec}^2$ ,  $\theta = 49.75^\circ$  to vertical]

# Fundamentals of Statics

In this chapter principles of statics is explained and their applications to concurrent and non-concurrent force system in plane is illustrated by solving several engineering problems.

## 2.1 PRINCIPLES OF STATICS

The statics is based on the following principles of mechanics:

1. Newton's laws of mechanics
2. Law of transmissibility
3. Parallelogram law of forces
4. Principles of physical independence
5. Principles of superposition.

### 2.1.1 Newton's Laws of Mechanics

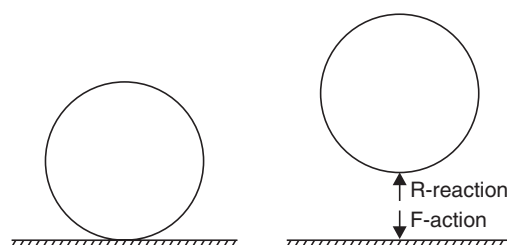
As already discussed in first chapter, Newton's first law gave definition of the force and second law gave basis for quantifying the force. There are two more Newton's laws:

- a. Newton's Third Law
- b. Newton's Law of Gravitation

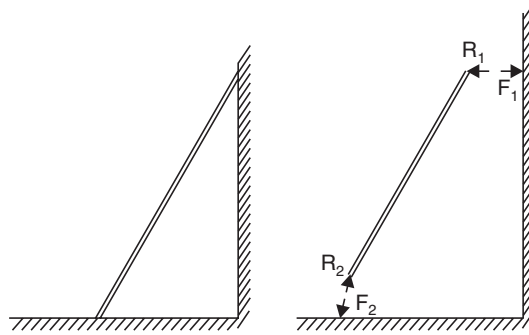
These laws are explained in this article.

#### (a) Newton's Third Law

It states that for every action there is an equal and opposite reaction. Consider the two bodies in contact with each other. Let one body apply a force  $F$  on another. According to this law the second body develops a reactive force  $R$  which is equal in magnitude to force  $F$  and acts in the line same as  $F$  but in the opposite direction. Figure 2.1 shows the action of a ball on the floor and the reaction of floor to this action. In Fig. 2.2 the action of a ladder on the wall and the floor and the reactions from the wall and the floor are shown.



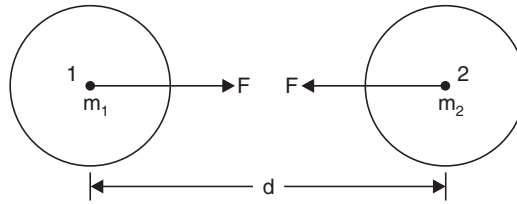
**Fig. 2.1**

**Fig. 2.2****(b) Newton's Law of Gravitation**

It states that everybody attracts the other body. The force of attraction between any two bodies is directly proportional to their masses and inversely proportional to the square of the distance between them. Thus the force of attraction between the bodies of mass  $m_1$  and mass  $m_2$  at distance ' $d$ ' between them as shown in Fig. 2.3 is

$$F = G \frac{m_1 m_2}{d^2} \quad \dots(2.1)$$

where  $G$  is the constant of proportionality and is known as constant of gravitation.

**Fig. 2.3**

From eqn. 2.1,

$$G = \frac{Fd^2}{m_1 m_2}$$

$$\text{Hence unit of } G = \frac{N \times m^2}{kg \times kg} = Nm^2/kg^2$$

It has been proved by experiments that the value of  $G = 6.673 \times 10^{-11} Nm^2/kg^2$ . Thus if two bodies one of mass 10 kg and the other of 5 kg are at a distance of 1 m, they exert a force

$$F = \frac{6.673 \times 10^{-11} \times 10 \times 5}{1^2} = 33.365 \times 10^{-10} N$$

on each other.

Similarly 1 kg-mass on earth surface experiences a force of

$$F = \frac{6.673 \times 10^{-11} \times 1 \times 5.96504 \times 10^{24}}{(6371 \times 10^3)^2} = 9.80665 \text{ N}$$

Since, mass of earth =  $5.96504 \times 10^{24}$  kg

and radius of earth =  $6371 \times 10^3$  m.

This force of attraction is always directed towards the centre of earth.

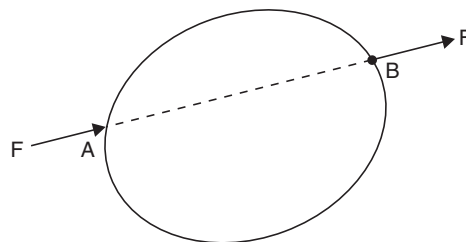
In common usage the force exerted by a earth on a body is known as weight of the body. Thus weight of 1 kg-mass on/near earth surface is 9.80665 N, which is approximated as 9.81 N for all practical problems. Compared to this force the force exerted by two bodies on each other is negligible. Thus in statics:

- a. Weight of a body =  $mg$
- b. Its direction is towards the centre of the earth, in other words, vertically downward.
- c. The force of attraction between the other two objects on the earth is negligible.

### 2.1.2 Law of Transmissibility

According to this law *the state of rest or motion of the rigid body is unaltered, if a force acting on the body is replaced by another force of the same magnitude and direction but acting anywhere on the body along the line of action of the replaced force.*

Let  $F$  be the force acting on a rigid body at point A as shown in Fig. 2.4. According to this law, this force has the same effect on the state of body as the force  $F$  applied at point B, where AB is in the line of force  $F$ .



**Fig. 2.4**

In using law of transmissibility it should be carefully noted that it is applicable only if the body can be treated as rigid. Hence if we are interested in the study of internal forces developed in a body, the deformation of body is to be considered and hence this law cannot be applied in such studies.

### 2.1.3 Parallelogram Law of Forces

This has been already explained in chapter 1 along with the derived laws *i.e.*, triangle and polygonal law.

### 2.1.4 Principles of Physical Independence of Forces

It states that *the action of a force on a body is not affected by the action of any other force on the body.*

### 2.1.5 Principles of Superposition of Forces

It states that *the net effect of a system of forces on a body is same as the combined* of individual forces acting on the body. Since a system of forces in equilibrium do not have any effect on a rigid body this principle is stated in the following form also: 'The effect of a given system of forces on a rigid body is not changed by adding or subtracting another system of forces in equilibrium.'

## 2.2 SYSTEM OF FORCES

When several forces of different magnitude and direction act upon a body, they constitute a *system of forces*. If all the forces in a system lie in a single plane, it is called a *coplanar force system*. If the line of action of all the forces in a system pass through a single point it is called a *concurrent force system*. In a system of *parallel forces* all the forces are parallel to each other. If the line of action of all forces lie along a single line then it is called a *collinear force system*. Various system of forces, their characteristics and examples are given in Table 2.1 below.

**Table 2.1** System of Forces

Force System	Characteristic	Examples
Collinear forces	Line of action of all the forces act along the same line.	Forces on a rope in a tug of war
Coplanar parallel forces	All forces are parallel to each other and lie in a single plane.	System of forces acting on a beam subjected to vertical loads (including reactions)
Coplanar like parallel forces	All forces are parallel to each other, lie in a single plane and are acting in the same direction.	Weight of a stationary train on a rail when the track is straight
Coplanar concurrent forces	Line of action of all forces pass through a single point and forces lie in the same plane.	Forces on a rod resting against a wall
Coplanar non-concurrent forces	All forces do not meet at a point, but lie in a single plane.	Forces on a ladder resting against a wall when a person stands on a rung which is not at its centre of gravity
Non-coplanar parallel forces	All the forces are parallel to each other, but not in the same plane.	The weight of benches in a class room
Non-coplanar concurrent forces	All forces do not lie in the same plane, but their lines of action pass through a single point.	A tripod carrying a camera
Non-coplanar non-concurrent forces	All forces do not lie in the same plane and their lines of action do not pass through a single point.	Forces acting on a moving bus

## 2.3 MOMENT OF A FORCE

Moment of a force about a point is the measure of its rotational effect. *Moment* is defined as the product of the magnitude of the force and the perpendicular distance of the point from the line of

action of the force. The point about which the moment is considered is called *moment centre* and the perpendicular distance of the point from the line of action of the force is called *moment arm*. Referring to Fig. 2.5, if  $d_1$  is the perpendicular distance of point 1 from the line of action of force  $F$ , the moment of  $F$  about point 1 is given by

$$M_1 = F d_1 \quad \dots(2.2)$$

Similarly, moment about point 2 is given by

$$M_2 = F d_2 \quad \dots(2.3)$$

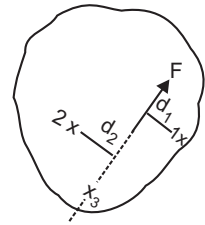
If the moment centre 3 lies on the line of action of the force  $F$ , the moment arm is zero and hence,

$$M_3 = F \times 0 = 0 \quad \dots(2.4)$$

Thus, it may be noted that if a point lie on the line of action of a force, the moment of the force about that point is zero.

The moment of a force has got direction also. In Fig. 2.5 it may be noted that  $M_1$  is *clockwise* and  $M_2$  is *anticlockwise*. To find the direction of the moment, imagine that the line of action of the force is connected to the point by a rigid rod pinned at the point and is free to move around the point. The direction of the rotation indicates the direction of the moment.

If the force is taken in newton unit and the distance in millimetre, the unit of moment will be N-mm. Commonly used units of moment in engineering are kN-m, N-m, kN-mm and N-mm.



**Fig. 2.5**

## 2.4 VARIGNON'S THEOREM

French mathematician Varignon (1654–1722) gave the following theorem which is also known as principle of moments:

*The algebraic sum of the moments of a system of coplanar forces about a moment centre in their plane is equal to the moment of their resultant force about the same moment centre.*

*Proof:* Referring to Fig. 2.6 let  $R$  be the resultant of forces  $F_1$  and  $F_2$  and  $B$  the moment centre. Let  $d$ ,  $d_1$  and  $d_2$  be the moment arms of the forces,  $R$ ,  $F_1$  and  $F_2$ , respectively from the moment centre  $B$ . Then in this case, we have to prove that:

$$Rd = F_1 d_1 + F_2 d_2$$

Join  $AB$  and consider it as  $y$  axis and draw  $x$  axis at right angles to it at  $A$  [Fig. 2.6(b)]. Denoting by  $\theta$  the angle that  $R$  makes with  $x$  axis and noting that the same angle is formed by perpendicular to  $R$  at  $B$  with  $AB$ , we can write:

$$\begin{aligned} Rd &= R \times AB \cos\theta \\ &= AB \times (R \cos\theta) \\ &= AB \times R_x \end{aligned} \quad \dots(a)$$

where  $R_x$  denotes the component of  $R$  in  $x$  direction.

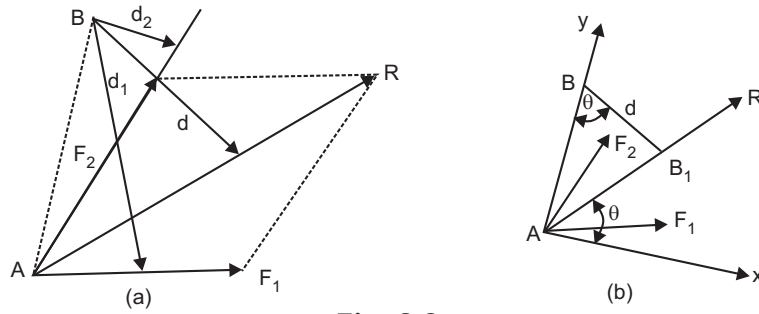


Fig. 2.6

Similarly, if  $F_{1x}$  and  $F_{2x}$  are the components of  $F_1$  and  $F_2$ , in  $x$  direction, respectively, then

$$F_1 d_1 = AB F_{1x} \quad \dots(b)$$

and

$$F_2 d_2 = AB F_{2x} \quad \dots(c)$$

From Eqns. (b) and (c)

$$\begin{aligned} F_1 d_1 + F_2 d_2 &= AB (F_{1x} + F_{2x}) \\ &= AB \times R_x \end{aligned} \quad \dots(d)$$

From equation (a) and (d), we get

$$Rd = F_1 d_1 + F_2 d_2$$

If a system of forces consists of more than two forces, the above result can be extended as given below:

Let  $F_1, F_2, F_3$  and  $F_4$  be four concurrent forces and  $R$  be their resultant. Let  $d_1, d_2, d_3, d_4$  and  $a$  be the distances of line of action of forces  $F_1, F_2, F_3, F_4$  and  $R$ , respectively from the moment centre  $O$ , [Ref. Fig 2.7].

If  $R_1$  is the resultant of  $F_1$  and  $F_2$  and its distance from  $O$  is  $a_1$ , then applying Varignon's theorem:

$$R_1 a_1 = F_1 d_1 + F_2 d_2$$

If  $R_2$  is the resultant of  $R_1$  and  $F_3$  (and hence of  $F_1, F_2$  and  $F_3$ ) and its distance from  $O$  is  $a_2$ , then applying Varignon's theorem:

$$\begin{aligned} R_2 a_2 &= R_1 a_1 + F_3 d_3 \\ &= F_1 d_1 + F_2 d_2 + F_3 d_3 \end{aligned}$$

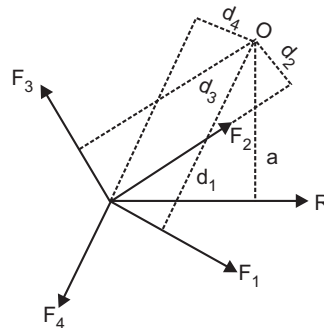


Fig. 2.7

Now considering  $R_2$  and  $F_4$ , we can write:

$$Ra = R_2 a_2 + F_4 d_4$$

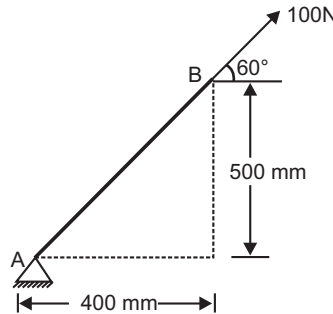
Since  $R$  is the resultant of  $R_2$  and  $F_4$  (i.e.  $F_1, F_2, F_3$  and  $F_4$ ).

$$\therefore Ra = F_1 d_1 + F_2 d_2 + F_3 d_3 + F_4 d_4 \quad \dots(2.5)$$

Thus, the moment of the resultant of a number of forces about a moment centre is equal to the sum of the moments of its component forces about the same moment centre.

**Example 2.1.** Find the moment of 100 N force acting at B about point A as shown in Fig. 2.8.

**Solution:** 100 N force may be resolved into its horizontal components as  $100 \cos 60^\circ$  and vertical component  $100 \sin 60^\circ$ . From Varignon's theorem, moment of 100 N force about the point A is equal to sum of the moments of its components about A.

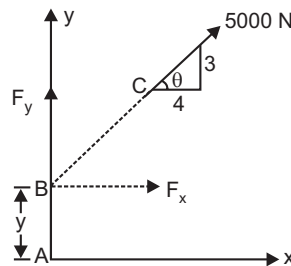


**Fig. 2.8**

Taking clockwise moment as positive,

$$\begin{aligned} M_A &= 100 \cos 60^\circ \times 500 - 100 \sin 60^\circ \times 400 \\ &= 25,000 - 34,641.02 \\ &= -9641.02 \text{ N-mm} \\ &= \mathbf{9641.016 \text{ N-mm Anticlockwise.}} \end{aligned}$$

**Example 2.2.** What will be the y intercept of the 5000 N force if its moment about A is 8000 N-m in Fig. 2.9.



**Fig. 2.9**



**Solution:** 5000 N force is shifted to a point  $B$  along its line of action (law of transmissibility) and it is resolved into its  $x$  and  $y$  components ( $F_x$  and  $F_y$  as shown in Fig. 2.9).

$$F_x = 5000 \cos \theta = 5000 \times \frac{4}{5} = 4000 \text{ N}$$

and

$$F_y = 5000 \sin \theta = 5000 \times \frac{3}{5} = 3000 \text{ N.}$$

By Varignon's theorem, moment of 5000 N force about  $A$  is equal to moment of its component forces about the same point.

$$8000 = 4000 \times y + 3000 \times 0$$

$\therefore$

$$y = 2 \text{ m.}$$

## 2.5 COUPLE

Two parallel forces equal in magnitude and opposite in direction and separated by a definite distance are said to form a couple. The sum of the forces forming a couple is zero, since they are equal and opposite, which means the translatory effect of the couple is zero.

An interesting property can be observed if we consider rotational effect of a couple about any point. Let the magnitude of the forces forming the couple be  $F$  and the perpendicular distance between the two forces be  $d$ . Consider the moment of the two forces constituting a couple about point 1 as shown in Fig. 2.10(a). Let the moment be  $M_1$  then,

$$\begin{aligned} M_1 &= Fd_1 + Fd_2 \\ &= F(d_1 + d_2) = Fd \end{aligned}$$

Now, consider the moment of the forces about point 2 which is outside the two forces as shown in Fig. 2.10(b). Let  $M_2$  be the moment.

Then,

$$\begin{aligned} M_2 &= Fd_3 - Fd_4 \\ &= F(d_3 - d_4) = Fd \end{aligned}$$

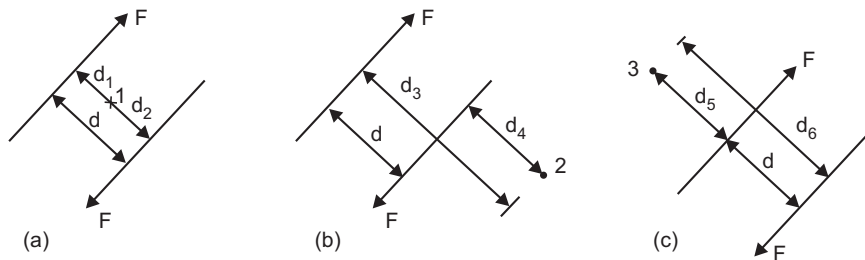
Similarly it can be seen that

$$M_3 = Fd$$

Thus at any point

$$M = Fd$$

...(2.6)



**Fig. 2.10**

Thus, moment of a couple about any point is the same. Now we can list the following characteristics of a couple:

- A couple consists of a pair of equal and opposite parallel forces which are separated by a definite distance;

- The translatory effect of a couple on the body is zero;
- The rotational effect (moment) of a couple about any point is a constant and it is equal to the product of the magnitude of the forces and the perpendicular distance between the two forces.

Since the only effect of a couple is a moment and this moment is the same about any point, the effect of a couple is unchanged if:

- The couple is rotated through any angle;
- The couple is shifted to any other position;
- The couple is replaced by another pair of forces whose rotational effect is the same.

## 2.6 TRANSFER OF A FORCE TO PARALLEL POSITION

It will be advantageous to resolve a force acting at a point on a body into a force acting at some other suitable point on the body and a couple. In Fig. 2.11(a)  $F$  is a force acting on a body at  $A$ .

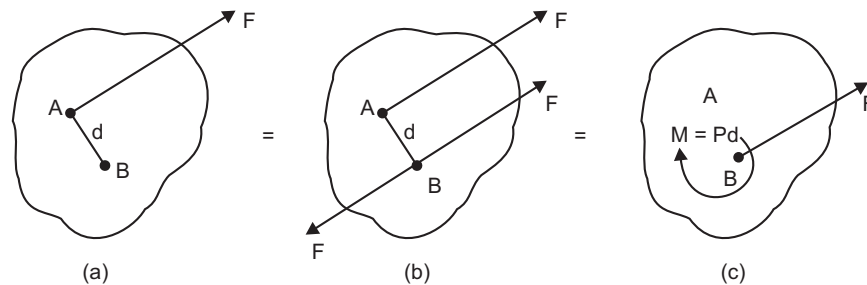


Fig. 2.11

Now it can be shown that  $F$  at  $A$  may be resolved into force  $F$  at  $B$  and a couple of magnitude  $M = F \times d$ , where  $d$  is the perpendicular distance of  $B$  from the line of action of  $F$  through  $A$ .

By applying equal and opposite forces  $F$  at  $B$  the system of forces is not disturbed. Hence the system of forces in Fig. 2.11(b) is the same as the system given in Fig. 2.11(a). Now the original force  $F$  at  $A$  and the opposite force  $F$  at  $B$  form a couple of magnitude  $Fd$ . The system in Fig. 2.11(b) can be replaced by the system shown in Fig. 2.11(c). Thus, the given force  $F$  at  $A$  is replaced by a force  $F$  at  $B$  and a moment  $Fd$ .

## 2.7 COMPOSITION OF CONCURRENT COPLANAR FORCES

### General Approach

In chapter 1, composition of concurrent forces by graphical method and the analytical method of composition of two force system has been discussed. In this article composition of concurrent coplanar forces is explained by a general analytic method.

Analytical method consists in finding the components of given forces in two mutually perpendicular directions and then combining them to get the resultant. Finding the component of a force is called *resolution of forces* and is exactly the opposite to the process of composition of forces. Finding the components of forces in two mutually perpendicular directions is preferable. The following points associated with the analytical method of finding rectangular components may be noted:

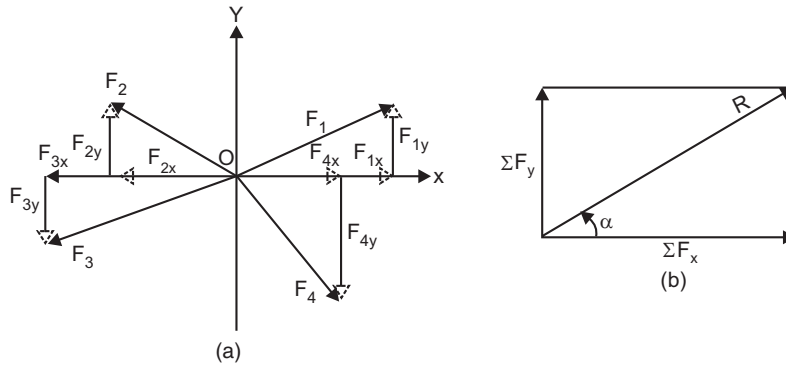
- Imagine that the arrow drawn to show force represents it to some scale
- Travel from tail to head of the arrow in the directions of coordinates

(iii) The direction of the travel gives the direction of component forces

(iv) From the triangle law of forces, the magnitude of the components can be calculated.

After finding the components of all the forces in the system in the two mutually perpendicular directions, the component in each direction are algebraically added to get the two components. These two components, which are mutually perpendicular, are combined to get the resultant.

Let  $F_1, F_2, F_3$  and  $F_4$  shown in Fig. 2.12(a) be the system of four forces the resultant of which is required.



**Fig. 2.12**

The procedure to get the resultant is given below:

**Step 1:** Find the components of all the forces in X and Y directions. Thus,  $F_{1x}, F_{2x}, F_{3x}, F_{4x}, F_{1y}, F_{2y}, F_{3y}$ , and  $F_{4y}$ , are obtained.

**Step 2:** Find the algebraic sum of the component forces in X and Y directions.

$$\Sigma F_x = F_{1x} + F_{2x} + F_{3x} + F_{4x}$$

$$\Sigma F_y = F_{1y} + F_{2y} + F_{3y} + F_{4y}$$

(Note: In the above case  $F_{2x}, F_{3x}, F_{3y}$  and  $F_{4y}$  have negative values.)

**Step 3:** Now the system of forces is equal to two mutually perpendicular forces, namely,  $\Sigma F_x$  and  $\Sigma F_y$  as shown in Fig. 2.12(b). Since these two forces are perpendicular, the parallelogram of forces becomes a rectangle. Hence the resultant  $R$  is given by:

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} \quad \dots(2.7)$$

and its inclination to x axis is given by:

$$\alpha = \tan^{-1} \left( \frac{\Sigma F_y}{\Sigma F_x} \right) \quad \dots(2.8)$$

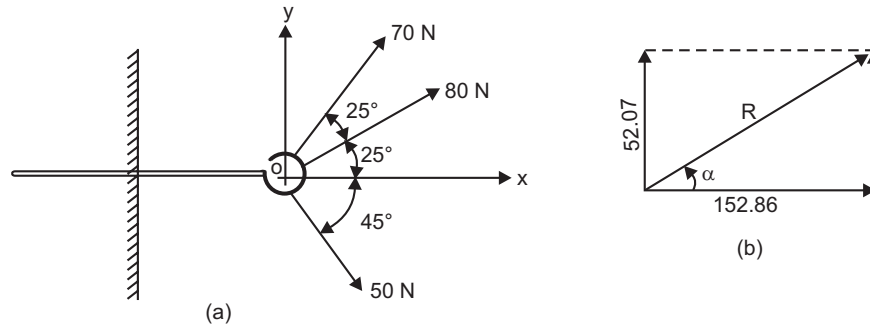
$$\text{Note:} \quad R \cos \alpha = \Sigma F_x = R_x \quad \dots(2.9)$$

$$\text{and} \quad R \sin \alpha = \Sigma F_y = R_y \quad \dots(2.10)$$

i.e.,  $\Sigma F_x$  and  $\Sigma F_y$  are the x and y components of the resultant.

The procedure of finding the component of forces and then finding the resultant is illustrated with examples 2.3 to 2.9.

**Example 2.3.** Determine the resultant of the three forces acting on a hook as shown in Fig. 2.13(a).



**Fig. 2.13**

**Solution:**

Force	$x$ component	$y$ component
70 N	45.00	53.62
80 N	72.50	33.81
50 N	35.36	-35.36

$$R_x = \Sigma F_x = 152.86, \quad R_y = \Sigma F_y = 52.07$$

$$R = \sqrt{152.86^2 + 52.07^2}$$

i.e.,

$$R = 161.48 \text{ N.}$$

$$\alpha = \tan^{-1} \frac{52.07}{152.86}$$

i.e.,

$$\alpha = 18.81^\circ \text{ as shown in Fig. 2.13(b).}$$

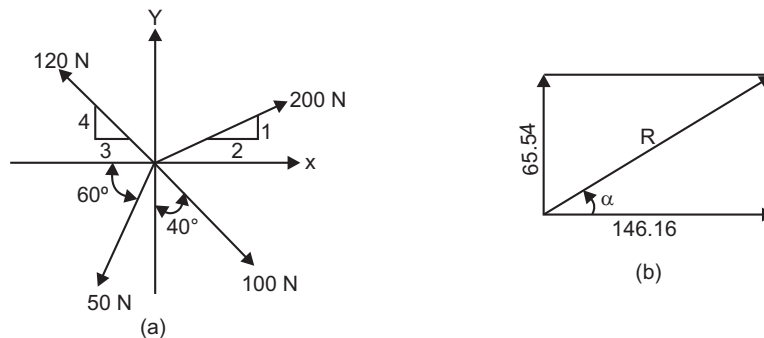
**Example 2.4.** A system of four forces acting on a body is as shown in Fig. 2.14(a). Determine the resultant.

**Solution:** If  $\theta_1$  is the inclination of the 200 N force to  $x$  axis,

$$\tan \theta_1 = \frac{1}{2}$$

$$\sin \theta_1 = \frac{1}{\sqrt{5}}$$

$$\cos \theta_1 = \frac{2}{\sqrt{5}}$$



**Fig. 2.14**

Similarly for the force 120 N,

$$\tan \theta_2 = \frac{4}{3}, \sin \theta_2 = \frac{4}{5}, \cos \theta_2 = \frac{3}{5}$$

Now,

$$\begin{aligned} R_x = \Sigma F_x &= 200 \times \frac{2}{\sqrt{5}} - 120 \times \frac{3}{5} - 50 \cos 60^\circ + 100 \sin 40^\circ \\ &= 146.16 \text{ N.} \end{aligned}$$

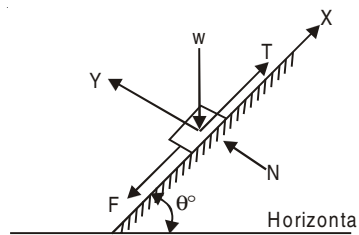
$$\begin{aligned} R_y = \Sigma F_y &= 200 \times \frac{1}{\sqrt{5}} + 120 \times \frac{4}{5} - 50 \times \sin 60^\circ - 100 \cos 40^\circ \\ &= 65.54 \text{ N} \end{aligned}$$

$$R = \sqrt{146.16^2 + 65.54^2}$$

$$R = 160.18 \text{ N.}$$

$$\alpha = \tan^{-1} \frac{65.54}{146.16} = 24.15^\circ \text{ as shown in Fig. 2.14(b).}$$

**Example 2.5.** A system of forces acting on a body resting on an inclined plane is as shown in Fig. 2.15. Determine the resultant force if  $\theta = 60^\circ$  and if  $W = 1000 \text{ N}$ ;  $N = 500 \text{ N}$ ;  $F = 100 \text{ N}$ ; and  $T = 1200 \text{ N}$ .



**Fig. 2.15**

**Solution:** In this problem, note that selecting  $X$  and  $Y$  axes parallel to the plane and perpendicular to the plane is convenient.

$$\begin{aligned} R_x = \Sigma F_x &= T - F - W \sin \theta \\ &= 1200 - 100 - 1000 \sin 60^\circ = 233.97 \text{ N} \end{aligned}$$

$$R_y = \Sigma F_y = N - W \cos 60^\circ = 500 - 1000 \cos 60^\circ = 0.$$

$\therefore$  Resultant is force of 233.97 N directed up the plane.

**Example 2.6.** Two forces acting on a body are 500 N and 1000 N as shown in Fig. 2.16(a). Determine the third force  $F$  such that the resultant of all the three forces is 1000 N directed at  $45^\circ$  to  $x$  axis.

**Solution:** Let the third force  $F$  make an angle  $\theta$  with  $x$  axis.

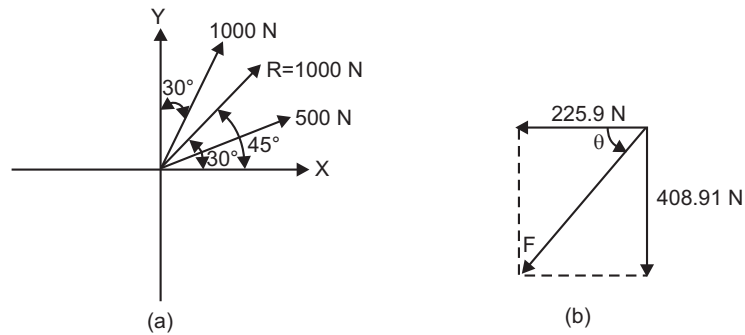


Fig. 2.16

Then,

$$R \cos \alpha = \Sigma F_x$$

i.e.,

$$1000 \cos 45^\circ = 500 \cos 30^\circ + 1000 \sin 30^\circ + F \cos \theta$$

$\therefore$

$$F \cos \theta = -255.9$$

and

$$R \sin \alpha = \Sigma F_y$$

$$1000 \sin 45^\circ = 500 \sin 30^\circ + 1000 \cos 30^\circ + F \sin \theta$$

$\therefore$

$$F \sin \alpha = -408.91 \text{ N}$$

$\therefore$

$$F = \sqrt{225.9^2 + 408.91^2}$$

i.e.,

$$F = 467.2 \text{ N.}$$

$$\theta = \tan^{-1} \left( \frac{408.91}{225.9} \right) = 61.08^\circ \text{ as shown in Fig. 2.16.}$$

**Example 2.7.** Three forces acting at a point are shown in Fig. 2.17. The direction of the 300 N forces may vary, but the angle between them is always  $40^\circ$ . Determine the value of  $\theta$  for which the resultant of the three forces is directed parallel to b-b.

**Solution:** Let the  $x$  and  $y$  axes be as shown in Fig. 2.17. If the resultant is directed along the  $x$  axis, its component in  $y$  direction is zero.

$$i.e., \quad 0 = \Sigma F_y = 300 \sin \theta + 300 \sin (40 + \theta) - 500 \sin 30^\circ$$

$$\therefore \quad \sin \theta + \sin (40 + \theta) = \frac{500 \sin 30^\circ}{300}$$

$$= 0.8333$$

$$\therefore \quad \sin \theta + \sin (40 + \theta) = 0.8333$$

$$2 \sin \left( \frac{40 + \theta + \theta}{2} \right) \times \cos \left( \frac{40 + \theta - \theta}{2} \right) = 0.8333$$

$$2 \sin (20 + \theta) \times \cos (20) = 0.8333$$

$\therefore$

$$\theta = 6.35^\circ$$

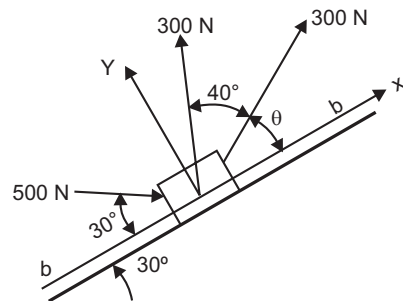


Fig. 2.17

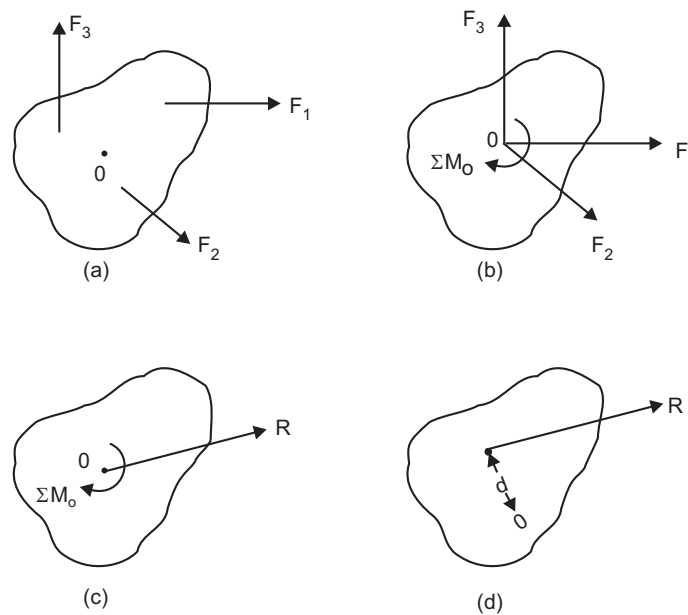
## 2.8 EQUILIBRIANT OF A FORCE SYSTEM

We have seen that the resultant of a system of forces is a single force which will have the same effect as the system of forces. According to Newton's second law of motion, the body starts moving with acceleration in the direction of the resultant force. If we apply a force equal and opposite to the resultant, the body should come to the equilibrium state. Such a force is called equilibrant. Thus an equilibrant of a system of forces may be defined as *the force which brings the body to the state of equilibrium and obviously, this force is equal in magnitude, but opposite in the direction to the resultant*.

## 2.9 COMPOSITION OF COPLANAR NON-CONCURRENT FORCE SYSTEM

Let  $F_1$ ,  $F_2$  and  $F_3$  [Fig. 2.18(a)] constitute a system of forces acting on a body. Each force can be replaced by a force of the same magnitude and acting in the same direction at point  $O$  and a moment about  $O$ . Thus, the given system in Fig. 2.18(a) is equal to the system shown in Fig. 2.18(b) where  $\Sigma M_O$  is the algebraic sum of the moments of the given forces about  $O$ .

At  $O$ , the concurrent force  $F_1$ ,  $F_2$  and  $F_3$  can be combined as usual to get the resultant force  $R$ . Now the resultant of the given system is equal to force  $R$  at  $O$  and a moment  $\Sigma M_O$  as shown in Fig. 2.18(c).



**Fig. 2.18**

The force  $R$  and moment  $\Sigma M_O$  shown in Fig. 2.18(c) can be replaced by a single force  $R$  acting at a distance  $d$  from  $O$  such that the moment produced by this force  $R$  is equal to  $\Sigma M_O$  [Ref. 2.18(d)].

Thus, we get a single force  $R$  acting at a distance  $d$  from the point  $O$  which gives the same effect as the constituent forces of the systems. Thus, the resultant of the given forces may be reduced to a single force.

Mathematically,

$$\left. \begin{aligned} R &= \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} \\ \tan \alpha &= \frac{\Sigma F_y}{\Sigma F_x} \end{aligned} \right\} \quad \dots(2.11)$$

$$d = \frac{\Sigma M_o}{R}$$

where,  $\Sigma F_x$  – algebraic sum of the components of all forces in  $x$  direction  
 $\Sigma F_y$  – algebraic sum of the components of all forces in  $y$  direction  
 $\alpha$  – inclination of the resultant  $R$  to  $x$  direction  
 $\Sigma M_o$  – algebraic sum of the moments of all the forces about point  $O$   
 $d$  – is distance of the resultant  $R$  from the point  $O$ .

**Note:**  $R$  is marked at distance  $d$  such that it produces the same direction of moment about point  $O$  as  $\Sigma M_o$ .

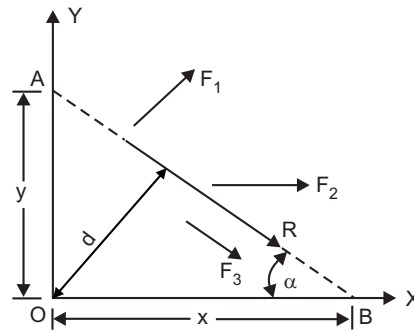
Sometimes the values of  $\Sigma F_x$  and  $\Sigma F_y$  may come out to be zero, but  $\Sigma M_o$  may exist. This means that the resultant of the system gets reduced to a pure couple.

## 2.10 $x$ AND $y$ INTERCEPTS OF RESULTANT

In some situations we may be interested in finding only the distance of  $R$  along  $x$  or  $y$  axis, that is  $x$  and  $y$  intercepts.

Let  $d$  be the distance of the resultant from  $O$  and  $\alpha$  be its inclination to  $x$  axis (Fig. 2.19). Then the intercepts are given by:

$$x = \frac{d}{\sin \alpha} \quad y = \frac{d}{\cos \alpha} \quad \dots(2.12)$$



**Fig. 2.19**

Another method of finding the intercepts is as follows:

Let  $R_x = \Sigma F_x$  and  $R_y = \Sigma F_y$  be the components of the resultant  $R$  in  $x$  and  $y$  directions. Considering the moment of  $R$  about  $O$  as the sum of moments of its components about  $B$  (Varignon's theorem) we get (ref. Fig. 2.20).



$$\begin{aligned}
 Rd &= \Sigma M_O \\
 R_x \times 0 + R_y x &= \Sigma M_O \\
 \therefore x &= \frac{\Sigma M_O}{R_y} = \frac{\Sigma M_o}{\Sigma F_y} \quad \dots(2.13)
 \end{aligned}$$

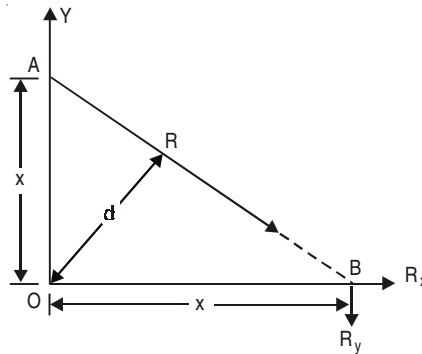


Fig. 2.20

Similarly, resolving the resultant into its components at A, it can be shown that:

$$y = \frac{\Sigma M_O}{R_x} = \frac{\Sigma M_o}{\Sigma F_x} \quad \dots(2.14)$$

**Example 2.8.** A system of loads acting on a beam is shown in Fig. 2.21(a). Determine the resultant of the loads.

**Solution:** Taking horizontal direction towards left as  $x$  axis and the vertical downward direction as  $y$  axis.

$$\begin{aligned}
 \Sigma F_x &= 20 \cos 60^\circ = 10 \text{ kN} \\
 \Sigma F_y &= 20 + 30 + 20 \sin 60^\circ = 67.3205
 \end{aligned}$$

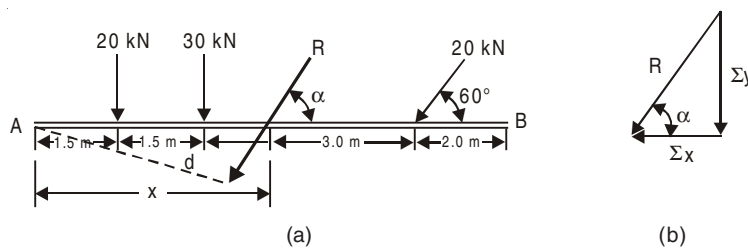


Fig. 2.21

$$\begin{aligned}
 \therefore R &= \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{10^2 + (67.3205)^2} \\
 \text{i.e., } R &= 68.0592 \text{ kN.} \\
 \tan \alpha &= \frac{\Sigma F_y}{\Sigma F_x} = 6.7321 \\
 \therefore \alpha &= 81.55^\circ.
 \end{aligned}$$

Now taking moment about A,

$$\begin{aligned}\Sigma M_A &= 20 \times 1.5 + 30 \times 3.0 + 20 \sin 60^\circ \times 6 \\ &= 223.9231 \text{ kN-m}\end{aligned}$$

$\therefore$  The distance of the resultant from point O is given by:

$$d = \frac{\Sigma M_A}{R} = \frac{223.9231}{68.0592} = 3.290 \text{ m}$$

$$\therefore x = \frac{d}{\sin \alpha} = \frac{3.290}{\sin 81.55^\circ}$$

$$x = 3.326 \text{ m.}$$

The value of  $x$  intercept may be obtained using Eqn. 2.13 also. Thus,

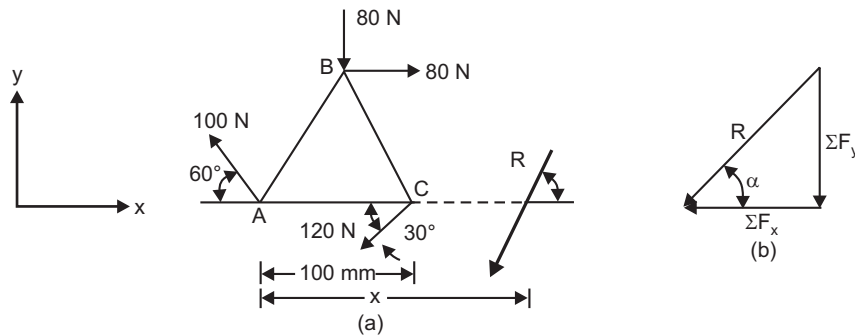
$$x = \frac{\Sigma M_0}{R_y} = \frac{\Sigma M_0}{\Sigma F_y} = \frac{223.9231}{67.3205} = 3.326 \text{ m.}$$

**Example 2.9.** Find the resultant of the force system shown in Fig. 2.22(a) acting on a lamina of equilateral triangular shape.

**Solution:**

$$\Sigma F_x = 80 - 100 \cos 60^\circ - 120 \cos 30^\circ = -73.92 \text{ N}$$

(Negative sign shows that  $R_x$  acts from right to left)



**Fig. 2.22**

$$R_x = 73.92 \text{ N}$$

$$\Sigma F_y = 80 + 120 \sin 30^\circ - 100 \sin 60^\circ$$

$$R_y = 53.40 \text{ N}$$

$$R = \sqrt{73.92^2 + 53.40^2}$$

$$R = 91.19 \text{ N}$$

$$\tan \alpha = \frac{\Sigma F_y}{\Sigma F_x} = \frac{53.40}{73.92}$$

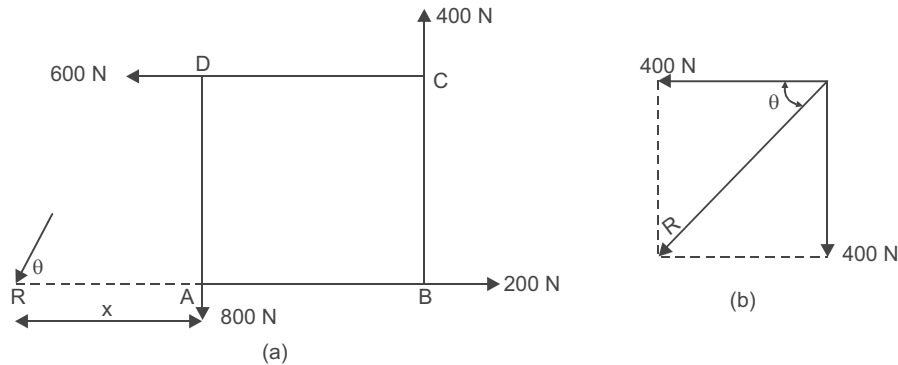
$$\alpha = 35.84^\circ$$

Let  $x$  be the distance from A at which the resultant cuts AC. Then taking A as moment centre,

$$53.40x = 80 \times 100 \sin 60^\circ + 80 \times 50 + 120 \sin 30^\circ \times 100$$

$$x = 317.008 \text{ mm to the right of A as shown in Fig. 2.22(a).}$$

**Example 2.10.** Four forces having magnitudes of 200 N, 400 N, 600 N and 800 N respectively, are acting along the four sides (1 m each) of a square ABCD taken in order, as shown in Fig. 2.23. Determine the magnitude and direction of the resultant force.



**Fig. 2.23**

**Solution:**

$$\Sigma F_x = 200 - 600 = -400 \text{ N} = \overleftarrow{400 \text{ N}}$$

$$\Sigma F_y = 400 - 800 = -400 \text{ N} = 400 \downarrow \text{ N}$$

$\therefore$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{400^2 + 400^2} = 400\sqrt{2} \\ = 565.68 \text{ N}$$

$$\theta = \tan^{-1} \frac{400}{400} = 45^\circ, \text{ as shown in Fig. 2.23(b).}$$

$$\Sigma M_A = 400 \times 1 + 600 \times 1 = 1000 \text{ N-m.}$$

Let  $x$  be the distance from A along  $x$  axis, where resultant cuts  $AB$ . Then

$$x = \frac{\Sigma M_A}{\Sigma F_y} = \frac{1000}{400} = 2.5 \text{ m as shown in the Fig. 2.23(a).}$$

**Example 2.11.** Forces  $2, \sqrt{3}, 5, \sqrt{3}$  and 2 kN respectively act at one of the angular points of a regular hexagon towards five other angular points. Determine the magnitude and direction of the resultant force.

**Solution:** Let the system of forces be as shown in Fig. 2.24 shown below:

Let  $O$  be the centre of the encircling circle  $A, B, C, D, E$  and  $F$ . In regular hexagon each side is equal to the radius  $AO$ . Hence  $OAB$  is equilateral triangle.

$$\therefore \angle OAB = 60^\circ$$

$$\text{In } \triangle ABC, \quad BA = BC$$

$$\therefore \angle CAB = \angle BCA$$

$$\text{But } \angle CAB + \angle BCA = \text{interior angle at } B \\ = 180^\circ - 120^\circ = 60^\circ$$

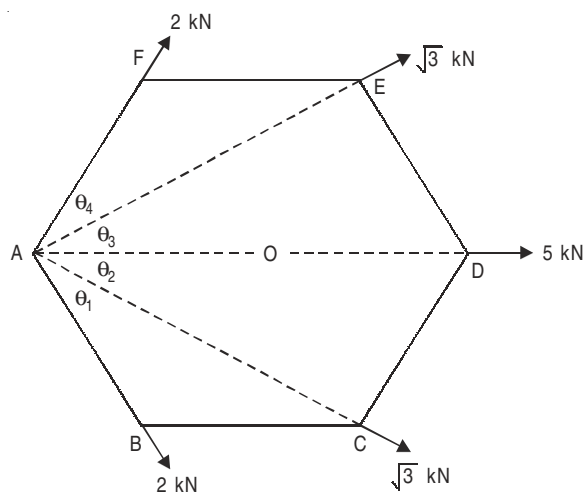


Fig. 2.24

$$\begin{aligned} \therefore \quad \angle CAB = \angle BCA &= \frac{60}{2} = 30^\circ \\ \theta_1 &= 30^\circ \\ \therefore \quad \theta_2 &= 60^\circ - 30^\circ = 30^\circ \\ \text{Similarly} \quad \theta_3 &= 30^\circ \\ \text{and} \quad \theta_4 &= 30^\circ \\ \therefore \quad \Sigma F_x &= 2 \cos 60^\circ + \sqrt{3} \cos 30^\circ + 5 + \sqrt{3} \cos 30^\circ + 2 \cos 60^\circ \\ &= 10 \text{ kN} \\ \Sigma F_y &= -2 \sin 60^\circ - \sqrt{3} \sin 30^\circ + 0 + \sqrt{3} \sin 30^\circ + 2 \sin 60^\circ \\ &= 0 \\ \therefore \quad R &= \sqrt{10^2 + 0^2} = 10 \text{ kN} \\ \theta &= 0^\circ \text{ i.e., the resultant is in the direction } x. \end{aligned}$$

**Example 2.12.** Find the resultant of a set of coplanar forces acting on a lamina as shown in Fig. 2.25(a). Each square has side of 10 mm.

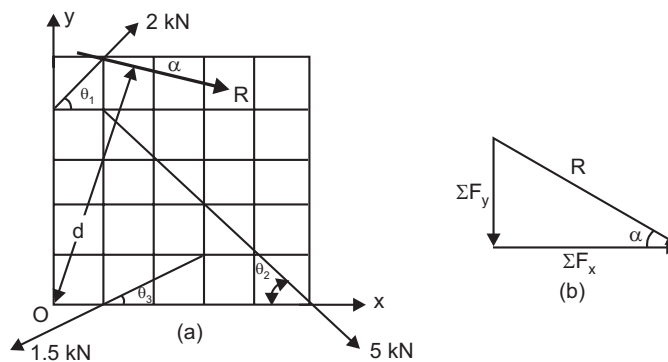


Fig. 2.25

**Solution:** If  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are the slopes of the forces 2 kN, 5 kN and 1.5 kN forces with respect to  $x$  axis, then

$$\tan \theta_1 = \frac{10}{10} = 1 \quad \therefore \theta_1 = 45^\circ$$

$$\tan \theta_2 = \frac{30}{40} \quad \therefore \theta_2 = 36.87^\circ$$

$$\tan \theta_3 = \frac{10}{20} \quad \therefore \theta_3 = 26.565^\circ$$

$$R_x = \Sigma F_x = 2 \cos 45^\circ + 5 \cos 36.87^\circ - 1.5 \cos 26.565^\circ = 4.072 \text{ kN}$$

$$R_y = \Sigma F_y = 2 \sin 45^\circ - 5 \sin 36.87^\circ - 1.5 \sin 26.565^\circ = 2.26 \text{ kN}$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = 4.66 \text{ kN.}$$

$$\tan \alpha = \frac{2.26}{4.072}$$

$\therefore$

$$\alpha = 28.99^\circ.$$

Distance  $d$  of the resultant from  $O$  is given by

$$Rd = \Sigma M_O$$

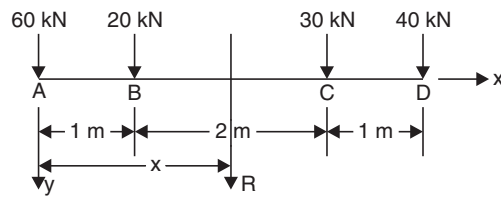
$$4.66d = 2 \times \cos 45^\circ \times 30 + 5 \times \sin 36.87^\circ \times 50 + 1.5 \times \sin 26.565^\circ \times 10$$

$$= 199.13$$

$$d = 42.77 \text{ mm as shown in Fig. 2.25(a).}$$

**Note:** To find moment of forces about  $O$ , 2 kN force is resolved at it's intersection with  $y$  axis and 5 kN and 1.5 kN forces are resolved at their intersection with  $x$  axis, and then Varignon theorem is used.

**Example 2.13** Determine the resultant of four parallel forces acting on the axle of a vehicle as shown in Fig. 2.26.



**Fig. 2.26**

**Solution:** Let  $x$  and  $y$  axes be selected as shown in the Figure

$$R_x = \Sigma F_x = 0$$

$$R_y = \Sigma F_y = 60 + 20 + 30 + 40 = 150 \text{ kN}$$

$$\therefore R = \sqrt{0^2 + 150^2} = 150 \text{ kN}$$

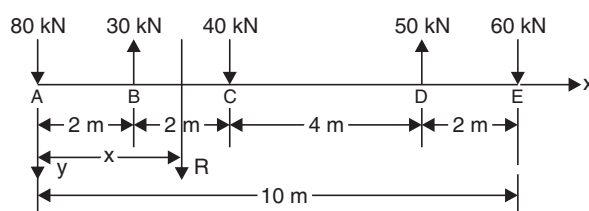
Taking clockwise moment as +ve,

$$\begin{aligned}\Sigma M_A &= 60 \times 0 + 20 \times 1 + 30 \times 3 + 40 \times 4 \\ &= 270 \text{ kN-m}\end{aligned}$$

$\therefore$  Distance of resultant from A

$$x = \frac{270}{150} = 1.8 \text{ m as shown in the figure.}$$

**Example 2.14.** Determine the resultant of system of parallel forces acting on a beam as shown in Fig. 2.27.



**Fig. 2.27**

**Solution:** Selecting  $x$  and  $y$  axes as shown in Fig. 2.27

$$R_x = \Sigma F_x = 0$$

$$R_y = \Sigma F_y = 80 - 30 + 40 - 50 + 60 = 100 \text{ kN}$$

$$\therefore R = \sqrt{0^2 + 100^2} = 100 \text{ kN, in } y\text{-direction}$$

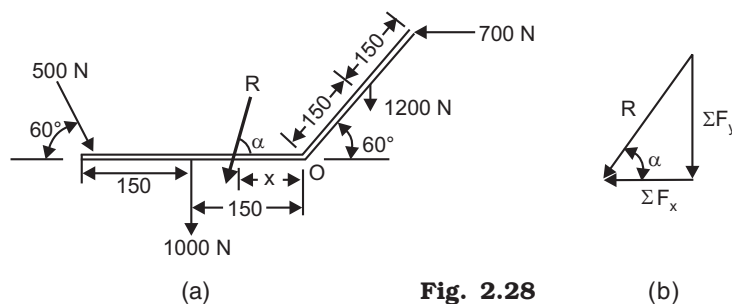
Taking clockwise moment as positive,

$$\Sigma M_A = 80 \times 0 - 30 \times 2 + 40 \times 4 - 50 \times 8 + 60 \times 10 = 300 \text{ kN-m}$$

$\therefore$  The distance of resultant from A is,

$$x = \frac{\Sigma M_A}{R_y} = \frac{300}{100} = 3 \text{ m as shown in Fig. 2.27.}$$

**Example 2.15.** The system of forces acting on a bell crank is shown in Fig. 2.28(a). Determine the magnitude, direction and the point of application of the resultant.



**Fig. 2.28**

**Solution:**  $R_x = \Sigma F_x = 500 \cos 60^\circ - 700 = -450 \text{ N} = 450 \text{ N}$  (from right to left)

$$R_y = \Sigma F_y = -1200 - 1000 - 500 \sin 60^\circ = -2633.01 \text{ N} \\ = 2633.01 \text{ N (downward)}$$

$$R = \sqrt{450^2 + (2633.01)^2}$$

$$R = 2671.19 \text{ N.}$$

$$\tan \alpha = \frac{2633.01}{450}$$

$$\alpha = 80.30^\circ, \text{ as shown in Fig. 2.28(b).}$$

Let the point of application of the resultant be at a distance  $x$  from the point  $O$  along the horizontal arm. Then,

$$x \times 2633.01 = 500 \sin 60^\circ \times 300 + 1000 \times 150 \\ - 1200 \times 150 \cos 60^\circ + 700 \times 300 \sin 60^\circ$$

$$x = 141.195 \text{ mm, as shown in Fig. 2.28(a).}$$

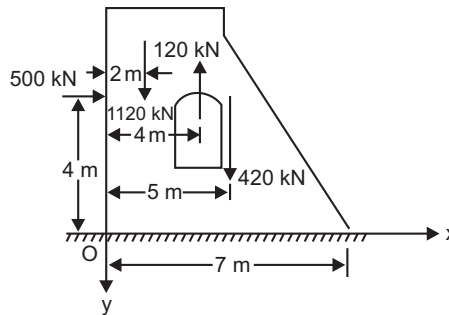
**Example 2.16.** Various forces to be considered for the stability analysis of a dam are shown in the Fig. 2.29. The dam is safe if the resultant force passes through middle third of the base. Verify whether the dam is safe.

**Solution:**

$$R_x = \Sigma F_x = 500 \text{ kN}$$

$$R_y = \Sigma F_y = +1120 - 120 + 420 = 1420 \text{ kN} \\ = 1420 \text{ kN (downward)}$$

Let  $x$  be the distance from  $O$  where the resultant cuts the base.



**Fig. 2.29**

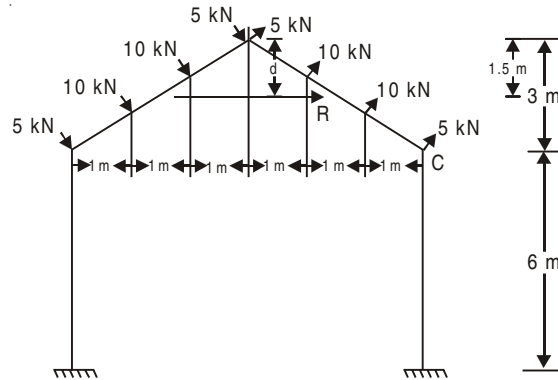
Then

$$x \Sigma F_y = \Sigma M_O \\ x \times 1420 = 500 \times 4 + 1120 \times 2 - 120 \times 4 + 420 \times 5 \\ \therefore x = 4.126 \text{ m}$$

The resultant passes through the middle third of the base i.e., between  $\frac{7}{3}$  m, and  $2 \times \frac{7}{3}$  m.

Hence, the dam is safe.

**Example 2.17.** A building frame is subjected to wind loads as shown in Fig. 2.30. Determine the resultant of the loads.



**Fig. 2.30**

**Solution:** The roof is inclined at  $45^\circ$  to horizontal and loads are at  $90^\circ$  to the roof. Hence, the loads are also inclined at  $45^\circ$  to vertical/horizontal.

Now,

$$\begin{aligned} R_x = \Sigma F_x &= (5 + 10 + 10 + 5 + 5 + 10 + 10 + 5) \cos 45^\circ \\ &= 60 \times \frac{1}{\sqrt{2}} = 42.426 \text{ kN} \end{aligned}$$

$$\begin{aligned} \Sigma F_y &= -(5 + 10 + 10 + 5) \sin 45^\circ + (5 + 10 + 10 + 5) \sin 45^\circ \\ &= 0 \end{aligned}$$

$\therefore$

$$R = \Sigma F_x = 42.426 \text{ kN}$$

and its direction is horizontal.

Let  $R$  be at a distance  $d$  from the ridge  $A$ .

Then,  $Rd = \Sigma M_A$

$$60 \times \frac{1}{\sqrt{2}} \times d = 5 \times \frac{3}{\sqrt{2}} + 10 \times \frac{2}{\sqrt{2}} + 10 \times \frac{1}{\sqrt{2}} + 10 \times \frac{1}{\sqrt{2}} + 10 \times \frac{2}{\sqrt{2}} + 5 \times \frac{3}{\sqrt{2}}$$

$$d = 1.5 \text{ m}$$

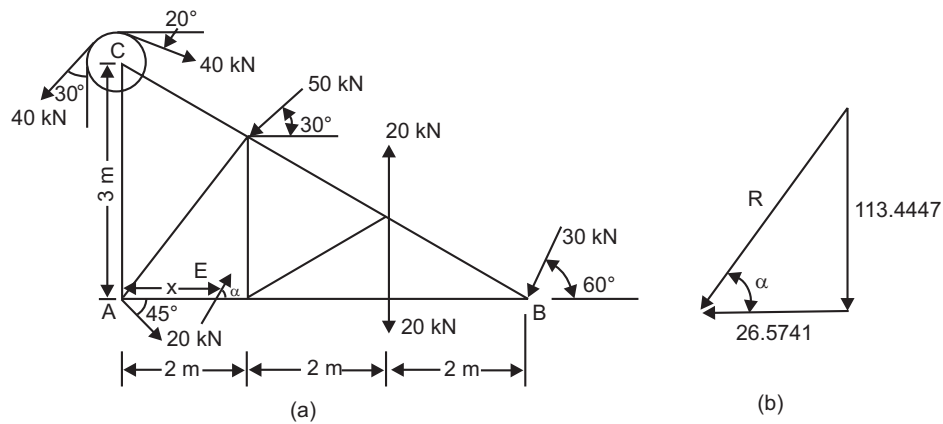
$\therefore$  Resultant is a horizontal force of magnitude 42.426 kN at 1.5 m below  $A$ .

**Example 2.18.** Determine the magnitude, direction and line of action of the equilibrant of the given set of coplanar forces acting on a planar structure shown in Fig. 2.31.

**Solution:** The two 40 kN forces acting on the smooth pulley may be replaced by a pair of 40 kN forces acting at centre of pulley  $C$  and parallel to the given forces, since the sum of moments of the two given forces about  $C$  is zero.

$$\begin{aligned} \text{Now, } R_x = \Sigma F_x &= 20 \cos 45^\circ - 30 \cos 60^\circ - 50 \cos 30^\circ + 40 \cos 20^\circ - 40 \sin 30^\circ \\ \Sigma F_x &= -26.5714 \text{ kN} \\ &= 26.5714 \text{ kN (from right to left)} \end{aligned}$$



**Fig. 2.31**

$$\begin{aligned}\Sigma F_y &= -20 \sin 45^\circ - 20 + 20 - 30 \sin 60^\circ - 50 \sin 30^\circ \\ &\quad - 40 \sin 20^\circ - 40 \cos 30^\circ \\ &= -113.4447 \text{ kN} = 113.4447 \text{ kN downward.}\end{aligned}$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = 116.515 \text{ kN}$$

$$\tan \alpha = \frac{113.4447}{26.5714}$$

$$\alpha = 76.82^\circ \text{ as shown in Fig. 2.31(b)}$$

Let the resultant intersect  $AB$  at a distance  $x$  from  $A$ . Then,

$$\begin{aligned}x \Sigma F_y &= \Sigma M_A \\ x \times 113.4447 &= 20 \times 4 - 20 \times 4 + (30 \sin 60^\circ) \times 6 + (50 \sin 30^\circ) \\ &\quad \times 2 - (50 \cos 30^\circ) \times 2 + (40 \cos 20^\circ) \times 3 \\ &\quad - (40 \sin 30^\circ) \times 3 \\ x &= 1.516 \text{ m}\end{aligned}$$

The equilibrant is equal and opposite to the resultant. Hence, it is as shown in Fig. 2.31(a) in which  $E = 116.515 \text{ kN}$ ,  $\alpha = 76.82^\circ$  and  $x = 1.516 \text{ m}$ .

## 2.11 TYPES OF FORCES ON A BODY

Before taking up equilibrium conditions of a body, it is necessary to identify the various forces acting on it. The various forces acting on a body may be grouped into:

- (a) Applied Forces
- (b) Non-applied Forces

**(a) Applied Forces**

These are the forces applied externally to a body. Each of the forces is having contact with the body. Depending upon type of their contact with the body, the applied forces may be classified as:

- (i) Point Force
- (ii) Distributed Forces

**(i) Point Force:** It is the one which has got contact with the body at a point. Practically there is no force which will have contact with the body at a single point. However, when the contact area is small compared to the other dimensions in the problem, for simplicity of calculation the force may be considered as a point load. If a person stands on a ladder, his weight be taken as an applied point load [Fig. 1.2]. Characteristics of such forces are already explained in Art. 1.1.

**(ii) Distributed Forces:** Distributed forces may act over a line, a surface or a volume. Correspondingly they are known as linear, surface and body forces.

**Linear Force:** A linear force is one that acts along a line on the body. It is usually represented with abscissa representing the position on the body and ordinate representing the magnitude of the load. An example of this force is a beam subjected to uniformly distributed/uniformly varying or general load. The force ' $dF$ ' at any small length ' $ds$ ' is given by

$$dF = w \, ds$$

**Surface Force:** A force acting on the surface of a body is known as surface force. The hydrostatic pressure acting on a Dam is an example of surface force. The force  $dF$  acting on any area  $dA$  is given by

$$dF = p \, dA$$

where ' $p$ ' is the intensity of force per unit area.

**Body Force:** A body force is the force exerted from each and every particle of the mass of the body. Example of this type of force is the weight of a block acting on the body under consideration. The body force  $dF$  of volume  $dV$  is given by

$$dF = \gamma \, dV$$

where  $\gamma$  is the force per unit volume.

**(b) Non-applied Forces**

There are two types of non-applied forces: (a) Self weight and (b) Reactions.

**Self weight:** Everybody subjected to gravitational acceleration and hence has got a selfweight.

$$W = mg$$

where  $m$  is mass of the body and  $g$  is gravitational acceleration ( $9.81 \, \text{m/sec}^2$  near the earth surface)

Selfweight always acts in vertically downward direction. When analysing equilibrium conditions of a body, selfweight is treated as acting through the centre of gravity of the body. If selfweight is very small, it may be neglected.

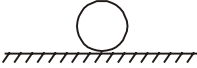

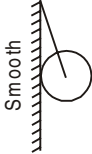
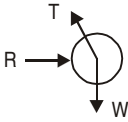
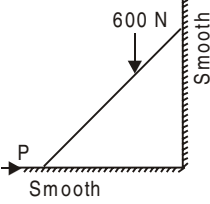
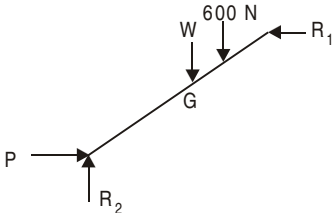
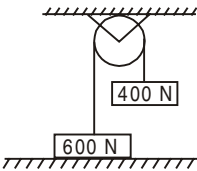
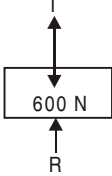
**Reactions:** These are self-adjusting forces developed by the other bodies which come in contact with the body under consideration. According to Newton's third law of motion, the reactions are equal and opposite to the actions. The reactions adjust themselves to bring the body to equilibrium.

If the surface of contact is smooth, the direction of the reaction is normal to the surface of contact. If the surface of contact is not smooth, apart from normal reaction, there will be frictional reaction also. Hence the resultant reaction will not be normal to the surface of contact.

## 2.12 FREE BODY DIAGRAM

In many problems, it is essential to isolate the body under consideration from the other bodies in contact and draw all the forces acting on the body. For this, first the body is drawn and then applied forces, selfweight and the reactions at the points of contact with other bodies are drawn. Such a diagram of the body in which the body under consideration is freed from all the contact surfaces and shows all the forces acting on it (including reactions at contact surfaces), is called a **Free Body Diagram (FBD)**. Free Body Diagrams (FBD) are shown for few typical cases in Table 2.2.

**Table 2.2** Free Body Diagrams (FBD) for a Few Typical Cases

Reacting Bodies	FBD required for	FBD
	Ball	
	Ball	
	Ladder	
	Block weighing 600 N	

## 2.13 EQUILIBRIUM OF BODIES

A body is said to be in equilibrium when it is at rest or has uniform motion. According to Newton's law of motion, it means the resultant of all the forces acting on a body in equilibrium is zero. The resultant of coplanar system of forces acting on a body is zero when—

- (a) The algebraic sum of the component of forces along each of the two mutually perpendicular directions is zero (translatory motion is zero).
- (b) The algebraic sum of moment of all the forces about any point in the plane is zero (rotational moment is zero).

The above conditions for coplanar concurrent and non-concurrent forces is discussed and illustrated in this article.

## 2.14 EQUILIBRIUM OF CONCURRENT FORCE SYSTEMS

If the moment equilibrium condition is considered about the concurrent point of application of the forces, the equation results into zero equal to zero. Hence this is not at all a useful equation. Thus only the conditions to ensure translatory motion is zero gives useful equations. In planar problems the resultant  $R$  of a system of forces is zero only when the following conditions are satisfied.

$$\Sigma F_x = 0 \quad \dots(2.15a)$$

$$\Sigma F_y = 0 \quad \dots(2.15b)$$

It may be observed that only one of the above two conditions is not sufficient. For example,  $\Sigma F_x = 0$  means that  $R \cos \alpha = 0$ . This will ensure that the resultant  $R$  cannot exist in any direction except in  $y$ -direction ( $\alpha = 90^\circ$ ). Hence the condition  $\Sigma F_y = 0$  also should be satisfied to ensure the resultant  $R$  does not exist, that is, the equilibrium condition exists. After drawing free body diagrams for each of the body under the action of concurrent force system equations 2.15a and 2.15b may be written and the problems may be solved.

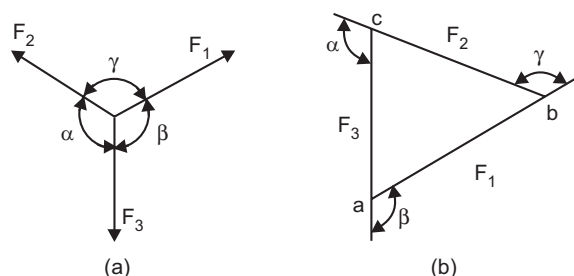
If a body is in equilibrium under the action of only three concurrent forces, Lami's theorem also may be used.

**Lami's theorem** states : If a body is in equilibrium under the action of three forces, each force is proportional to the sine of the angle between the other two forces.

Thus, for the system of forces shown in Fig. 2.32(a).

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma} \quad \dots(2.15c)$$

*Proof:* Draw the three forces  $F_1$ ,  $F_2$  and  $F_3$  one after the other in direction and magnitude starting from point  $a$ . Since the body is in equilibrium (resultant is zero), the last point must coincide with  $a$ . Thus, it results in triangle of forces  $abc$  as shown in Fig. 2.32(b). Now, the external angles at  $a$ ,  $b$  and  $c$  are equal to  $\beta$ ,  $\gamma$  and  $\alpha$ .



**Fig. 2.32**

Since  $ab$  is parallel to  $F_1$ ,  
 $bc$  parallel to  $F_2$  and  
 $ca$  parallel to  $F_3$ .

In the triangle of forces  $abc$

$$ab = F_1,$$

$$bc = F_2 \text{ and } ca = F_3.$$

Applying sine rule for the triangle  $abc$ ,

$$\frac{ab}{\sin(180-\alpha)} = \frac{bc}{\sin(180-\beta)} = \frac{ca}{\sin(180-\gamma)}$$

$$\text{i.e.,} \quad \frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}$$

**Note:** While determining the direction of the reaction on a body note that if the body is in equilibrium under the action of only three coplanar forces, those three forces must be concurrent.

**Example 2.19.** A sphere of weight 100 N is tied to a smooth wall by a string as shown in Fig. 2.33(a). Find the tension  $T$  in the string and reaction  $R$  of the wall.

**Solution:** Free body diagram of the sphere is as shown in Fig. 2.33(b). Figure 2.33(c) shows all the forces moving away from the centre of the ball. Applying Lami's theorem to the system of forces.

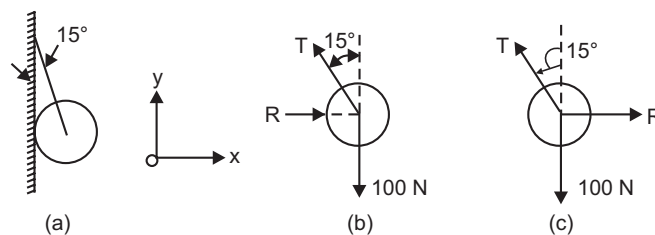
$$\frac{T}{\sin 90^\circ} = \frac{R}{\sin(180-15)} = \frac{100}{\sin(90+15)}$$

$$\therefore \quad T = 103.53 \text{ N.}$$

$$R = 26.79 \text{ N.}$$

The above problem may be solved using equations of equilibrium also.  
 Taking horizontal direction as  $x$  axis and vertical direction as  $y$  axis,

$$\Sigma F_y = 0 \text{ gives}$$



**Fig. 2.33**

$$T \cos 15^\circ - 100 = 0$$

$$T = 103.53 \text{ N.}$$

$$\Sigma F_x = 0 \text{ gives}$$

$$R - T \sin 15^\circ = 0$$

$$R = 26.79 \text{ N.}$$

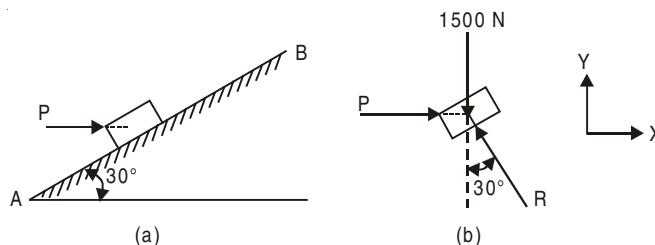
**Notes:**

1. The string can have only tension in it (it can pull a body), but cannot have compression in it (cannot push a body).
2. The wall reaction is a push, but cannot be a pull on the body.
3. If the magnitude of reaction comes out to be negative, then assumed direction of reaction is wrong. It is acting exactly in the opposite to the assumed direction. However, the magnitude will be the same. Hence no further analysis is required. This advantage is not there in using Lami's equation. Hence, it is advisable for beginners to use equations of equilibrium, instead of Lami's theorem even if the body is in equilibrium under the action of only three forces.

**Example 2.20.** Determine the horizontal force  $P$  to be applied to a block of weight  $1500\text{ N}$  to hold it in position on a smooth inclined plane  $AB$  which makes an angle of  $30^\circ$  with the horizontal [Fig. 2.34(a)].

**Solution:** The body is in equilibrium under the action of applied force  $P$ , self-weight  $1500\text{ N}$  and normal reaction  $R$  from the plane. Since  $R$ , which is normal to the plane, makes  $30^\circ$  with the vertical (or  $60^\circ$  with the horizontal),

$$\begin{aligned}\Sigma F_y &= 0, \text{ gives} \\ R \cos 30^\circ - 1500 &= 0 \\ \mathbf{R} &= \mathbf{1732.06\text{ N.}} \\ \Sigma F_x &= 0, \text{ gives} \\ P - R \sin 30^\circ &= 0 \\ P &= R \sin 30^\circ \\ \mathbf{P} &= \mathbf{866.03\text{ N.}}\end{aligned}$$



**Fig. 2.34**

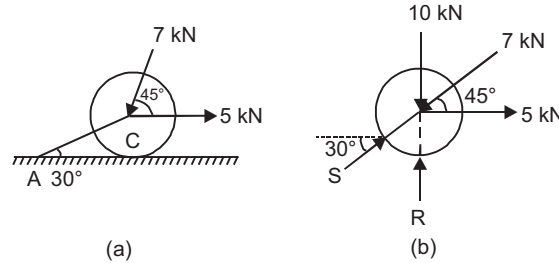
**Note:** Since the body is in equilibrium under the action of only three forces the above problem can be solved using Lami's theorem as given below:

$$\frac{R}{\sin 90^\circ} = \frac{P}{\sin(180 - 30)} = \frac{1500}{\sin(90 + 30)}$$

$$\mathbf{R = 1732.06 \text{ and } P = 866.03.}$$

**Example 2.21.** A roller of weight  $10\text{ kN}$  rests on a smooth horizontal floor and is connected to the floor by the bar  $AC$  as shown in Fig. 2.35 (a). Determine the force in the bar  $AC$  and reaction from floor, if the roller is subjected to a horizontal force of  $5\text{ kN}$  and an inclined force of  $7\text{ kN}$  as shown in the figure.

**Solution:** A bar can develop a tensile force or a compressive force. Let the force developed be a compressive force  $S$  (push on the cylinder). Free Body Diagram of the cylinder is as shown in Fig. 2.35(b).



**Fig. 2.35**

Since there are more than three forces in the system, Lami's equations cannot be applied. Consider the components in horizontal and vertical directions.

$$\Sigma H = 0$$

$$S \cos 30^\circ + 5 - 7 \cos 45^\circ = 0$$

$$S = \frac{7 \cos 45^\circ - 5}{\cos 30^\circ} = -0.058 \text{ kN}$$

Since the value of  $S$  is negative the force exerted by the bar is not a push, but it is pull (tensile force in bar) of magnitude 0.058 kN.

$$\Sigma V = 0$$

$$R - 10 - 7 \sin 45^\circ + S \sin 30^\circ = 0$$

$$R = 10 + 7 \sin 45^\circ - S \sin 30^\circ$$

$$= 10 + 7 \sin 45^\circ - (-0.058) \sin 30^\circ$$

$$R = 14.979 \text{ kN.}$$

**Example 2.22.** A cord  $ACB$  5 m long is attached at points  $A$  and  $B$  to two vertical walls 3 m apart as shown in Fig. 2.36(a). A pulley  $C$  of negligible radius carries a suspended load of 200 N and is free to roll without friction along the cord. Determine the position of equilibrium, as defined by the distance  $X$ , that the pulley will assume and also the tensile force in the cord.

**Solution:** The pulley  $C$  is in equilibrium under the action of tensile forces in  $CA$  and  $CB$  and vertical downward load 200 N. The tensile forces in segment  $CA$  and  $CB$  are the same since the pulley is frictionless. Now consider the equilibrium of pulley  $C$  shown in Fig. 2.36(b).

$$\Sigma H = 0$$

$$T \cos \theta_1 - T \cos \theta_2 = 0$$

$\therefore$

$$\theta_1 = \theta_2, \text{ say, equal to } \theta$$

Now, let  $BC$  be extended to  $D$ .

$$\triangle CFD = \triangle CFA$$

$\therefore$

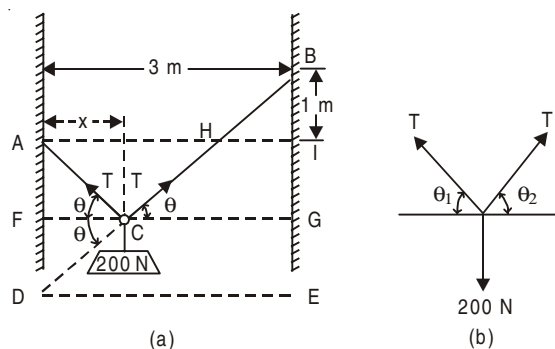
$$CD = AC$$

$$BD = BC + CD = BC + AC = \text{length of chord} = 5 \text{ m}$$

$$DE = 3 \text{ m}$$

$$BE = 4 \text{ m}$$

$\therefore$



**Fig. 2.36**

As  $\triangle BHI$  is similar to  $\triangle BDE$

$$HI = \frac{BI}{BE} \times DE = \frac{1}{4} \times 3 = 0.75$$

$\therefore$

$$AH = 3 - 0.75 = 2.25$$

$$x = 1.125 \text{ m.} \quad \text{Since } AH = 2x.$$

At C,

$$\Sigma V = 0$$

$$2 \times T \sin \theta = 200$$

$$2 \times T \times \frac{4}{5} = 200$$

$\therefore$

$$T = 125 \text{ N.}$$

**Example 2.23.** A roller of radius  $r = 300 \text{ mm}$  and weight  $2000 \text{ N}$  is to be pulled over a curb of height  $150 \text{ mm}$  [Fig. 2.37(a)] by a horizontal force  $P$  applied to the end of a string wound tightly around the circumference of the roller. Find the magnitude of  $P$  required to start the roller move over the curb. What is the least pull  $P$  through the centre of the wheel to just turn the roller over the curb?

**Solution:** When the roller is about to turn over the curb, the contact with the floor is lost and hence there is no reaction from the floor. The reaction  $R$  from the curb must pass through the intersection of  $P$  and the line of action of self weight, since the body is in equilibrium under the action of only three forces (all the three forces must be concurrent).

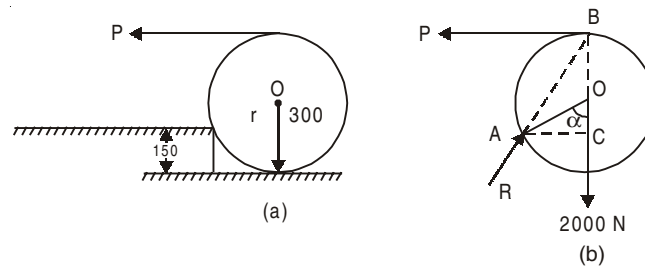
Referring to Fig. 2.37(b),

$$\cos \alpha = \frac{OC}{AO} = \frac{(300 - 150)}{300} = \frac{1}{2}$$

$\therefore$

$$\alpha = 60^\circ$$



**Fig. 2.37**

Now in  $\Delta AOB$ ,  $\angle OAB = \angle OBA$  since  $OA = OB$   
 but  $\angle OAB + \angle OBA = \alpha$

$$2\angle OBA = 60^\circ$$

$$\angle OBA = 30^\circ$$

i.e., the reaction makes  $30^\circ$  with the vertical

$$\Sigma V = 0, \text{ gives}$$

$$R \cos 30^\circ - 2000 = 0$$

$$R = 2309.40 \text{ N}$$

$$\Sigma H = 0, \text{ gives}$$

$$P - R \sin 30^\circ = 0$$

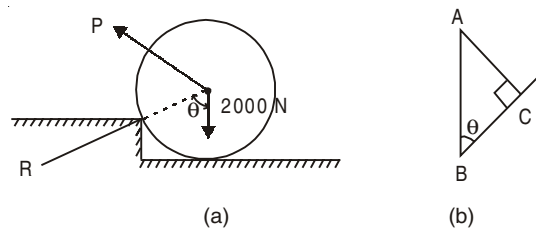
$$P = 2309.40 \times \sin 30^\circ$$

$$P = 1154.70 \text{ N.}$$

*Least force through the centre of wheel:*

Now the reaction from the curb must pass through the centre of the wheel since the other two forces pass through that point. Its inclination to vertical is  $\theta = 60^\circ$ .

If the triangle of forces  $ABC$  is constructed [Fig. 2.38(b)], representing selfweight by  $AB$ , reaction  $R$  by  $BC$  and pull  $P$  by  $AC$ , it may be observed that  $AC$  to be least, it should be perpendicular to  $BC$ . In other words,  $P$  makes  $90^\circ$  with the line of action of  $R$ .

**Fig. 2.38**

From triangle of forces  $ABC$ , we get

$$P = AC = AB \sin \theta = 2000 \sin 60^\circ$$

$$P = 1732.05 \text{ N.}$$

### 2.15 EQUILIBRIUM OF CONNECTED BODIES

When two or more bodies are in contact with one another, the system of forces appears as though it is a non-concurrent forces system. However, when each body is considered separately, in many situations it turns out to be a set of concurrent force system. In such instances, first, the body subjected to only two unknown forces is to be analysed followed by the analysis of other connected body/bodies. Such examples are illustrated below.

**Example 2.24.** A system of connected flexible cables shown in Fig. 2.39(a) is supporting two vertical forces 200 N and 250 N at points B and D. Determine the forces in various segments of the cable.

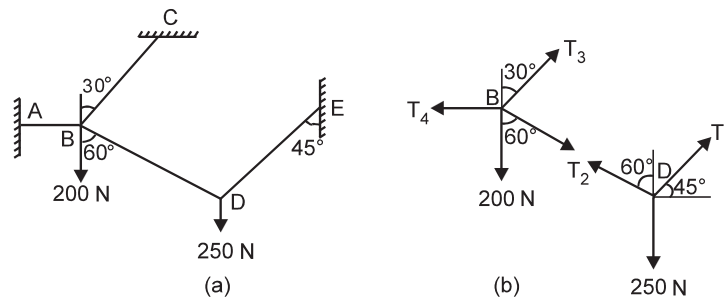


Fig. 2.39

**Solution:** Free body diagrams of points B and D are shown in Fig. 2.39(b). Let the forces in the members be as shown in the figure.

Applying Lami's theorem to the system of forces at point D,

$$\frac{T_1}{\sin 120^\circ} = \frac{T_2}{\sin 135^\circ} = \frac{250}{\sin 105^\circ}$$

$\therefore$

$$T_1 = 224.14 \text{ N.}$$

$$T_2 = 183.01 \text{ N.}$$

Consider the system of forces acting at B.

$$\Sigma V = 0$$

$$T_3 \cos 30^\circ - 200 - T_2 \cos 60^\circ = 0$$

$$T_3 = \frac{200 + 183.01 \cos 60^\circ}{\cos 30^\circ}$$

$$T_3 = 336.60 \text{ N.}$$

$$\Sigma H = 0$$

$$T_4 - T_2 \sin 60^\circ - T_3 \sin 30^\circ = 0$$

$$T_4 = 183.01 \times \sin 60^\circ + 336.60 \sin 30^\circ$$

$$T_4 = 326.79 \text{ N.}$$

**Example 2.25.** A rope AB, 4.5 m long is connected at two points A and B at the same level 4 m apart. A load of 1500 N is suspended from a point C on the rope 1.5 m from A as shown in Fig. 2.40(a). What load connected at a point D on the rope, 1 m from B will be necessary to keep the position CD level ?

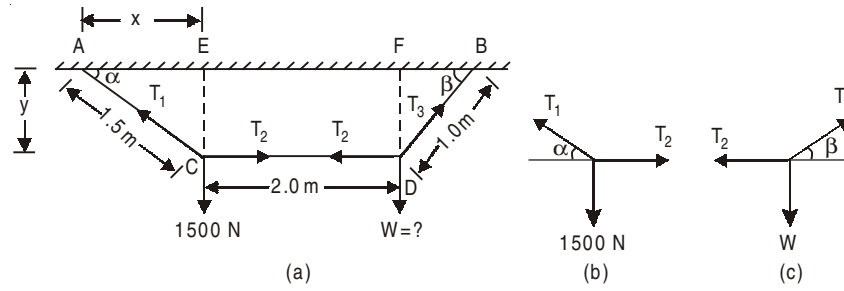


Fig. 2.40

**Solution:** Drop perpendiculars  $CE$  and  $DF$  on  $AB$ .

Let  $CE = y$  and

$$AE = x$$

$$\text{From } \triangle AEC, \quad x^2 + y^2 = 1.5^2 = 2.25 \quad \dots(i)$$

$$\text{Now,} \quad AB = 4 \text{ m}$$

$$\text{and} \quad AC + CD + BD = 4.5 \text{ m}$$

$$\text{i.e.,} \quad CD = 4.5 - 1.5 - 1.0 = 2.0 \text{ m}$$

$$\therefore EF = 2.0 \text{ m}$$

$$\begin{aligned} \therefore BF &= AB - (AE + EF) \\ &= 4 - (x + 2.0) = 2 - x \end{aligned} \quad \dots(ii)$$

$$\begin{aligned} \text{From } \triangle BFD, \quad BF^2 + DF^2 &= 1^2 \\ (2 - x)^2 + y^2 &= 1 \end{aligned} \quad \dots(iii)$$

From (1) and (3)

$$x^2 - (2 - x)^2 = 1.25$$

$$\text{i.e.,} \quad x^2 - 4 + 4x - x^2 = 1.25$$

$$x = 1.3125 \text{ m}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{1.3125}{1.5}\right) = 28.955^\circ$$

$$\beta = \cos^{-1}\left(\frac{2 - 1.3125}{1}\right) = 46.567^\circ$$

Applying Lami's theorem to the system of forces acting at point  $C$  [Fig. 2.40(b)], we get

$$\frac{T_1}{\sin 90^\circ} = \frac{T_2}{\sin 118.955^\circ} = \frac{1500}{\sin (180 - 28.955)^\circ}$$

$$T_1 = 3098.39 \text{ N}$$

$$T_2 = 2711.09 \text{ N}$$

Applying Lami's theorem to the system of forces at  $B$  [Fig. 2.40(c)], we get

$$\frac{T_3}{\sin 90^\circ} = \frac{W}{\sin(180 - 46.567)^\circ} = \frac{T_2}{\sin 136.567^\circ}$$

$\therefore$

$$T_3 = 3993.28 \text{ N}$$

and

$$W = 2863.53 \text{ N.}$$

**Example 2.26.** A wire rope is fixed at two points  $A$  and  $D$  as shown in Fig. 2.41(a). Two weights 20 kN and 30 kN are attached to it at  $B$  and  $C$ , respectively. The weights rest with portions  $AB$  and  $BC$  inclined at angles  $30^\circ$  and  $50^\circ$  respectively, to the vertical as shown in figure. Find the tension in the wire in segments  $AB$ ,  $BC$  and  $CD$  and also the inclination of the segments  $CD$  to vertical.

**Solution:** Applying Lami's theorem for the system of forces at  $B$  [Fig. 2.41(b)], we get

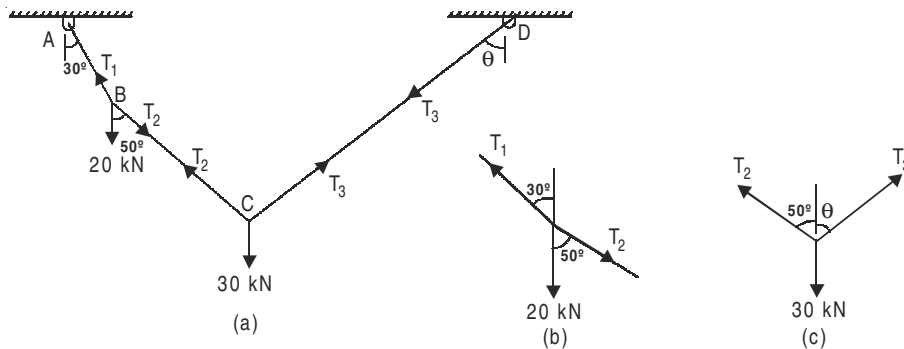
$$\frac{T_1}{\sin 50^\circ} = \frac{T_2}{\sin 150^\circ} = \frac{20}{\sin 160^\circ}$$

$$T_1 = 44.79 \text{ kN.}$$

$$T_2 = 29.24 \text{ kN.}$$

Writing equations of equilibrium for the system of forces at  $C$  [Fig. 2.41(c)], we get

$$T_3 \sin \theta = T_2 \sin 50^\circ = 22.4 \text{ kN} \quad \dots(i)$$



**Fig. 2.41**

and

$$\begin{aligned} T_3 \cos \theta &= 30 - T_2 \cos 50^\circ \\ &= 11.20 \end{aligned} \quad \dots(ii)$$

From (1) and (2),  $\tan \theta = 1.998687$

$\therefore$

$$\theta = 63.422^\circ.$$

and

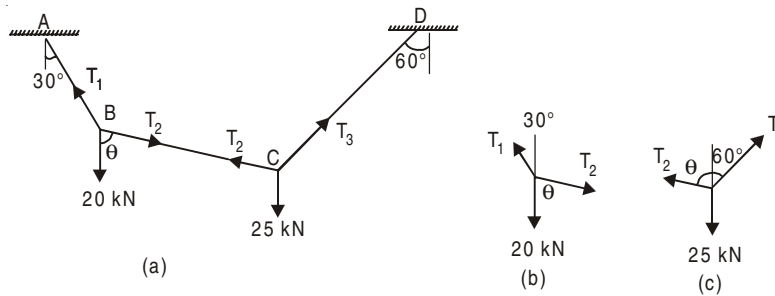
$$T_3 = 25.045 \text{ kN.}$$

**Example 2.27.** A wire is fixed at two points  $A$  and  $D$  as shown in Fig. 2.42(a). Two weights 20 kN and 25 kN are supported at  $B$  and  $C$ , respectively. When equilibrium is reached it is found that inclination of  $AB$  is  $30^\circ$  and that of  $CD$  is  $60^\circ$  to the vertical. Determine the tension in the segments  $AB$ ,  $BC$  and  $CD$  of the rope and also the inclination of  $BC$  to the vertical.

**Solution:** Writing equations of equilibrium for the system of forces at joints  $B$  and  $C$  [Figs. 2.42(b) and (c)], we have

$$T_2 \sin \theta = T_1 \sin 30^\circ \quad \dots(i)$$

$$T_2 \cos \theta = T_1 \cos 30^\circ - 20 \quad \dots(ii)$$



**Fig. 2.42**

$$T_2 \sin \theta = T_3 \sin 60^\circ \quad \dots(iii)$$

$$T_2 \cos \theta = 25 - T_3 \cos 60^\circ \quad \dots(iv)$$

From (1) and (3)

$$T_1 \sin 30^\circ = T_3 \sin 60^\circ$$

$\therefore$

$$T_1 = \sqrt{3}T_3 \quad \dots(v)$$

From (2) and (4),

$$T_1 \cos 30^\circ - 20 = 25 - T_3 \cos 60^\circ$$

$$\sqrt{3}T_3 \frac{\sqrt{3}}{2} + T_3 \frac{1}{2} = 45$$

$$T_3 = 22.5 \text{ kN.}$$

From (5),

$$T_1 = 38.97 \text{ kN.}$$

From (1),

$$T_2 \sin \theta = 19.48$$

From (2),

$$T_2 \cos \theta = 13.75$$

$\therefore$

$$\tan \theta = 1.4167$$

$$\theta = 54.78^\circ$$

$$T_2 = 23.84 \text{ kN.}$$

**Example 2.28.** A 600 N cylinder is supported by the frame  $BCD$  as shown in Fig. 2.43(a). The frame is hinged at  $D$ . Determine the reactions at  $A$ ,  $B$ ,  $C$  and  $D$ .

**Solution:** Free body diagram of sphere and frame are shown in Figs. 2.43(b) and 2.43(c), respectively.

Considering the sphere,

$$\Sigma V = 0, \text{ gives}$$

$$R_B = 600 \text{ N.} \quad \dots(i)$$

$$\Sigma H = 0, \text{ gives}$$

$$R_A = R_C \quad \dots(ii)$$

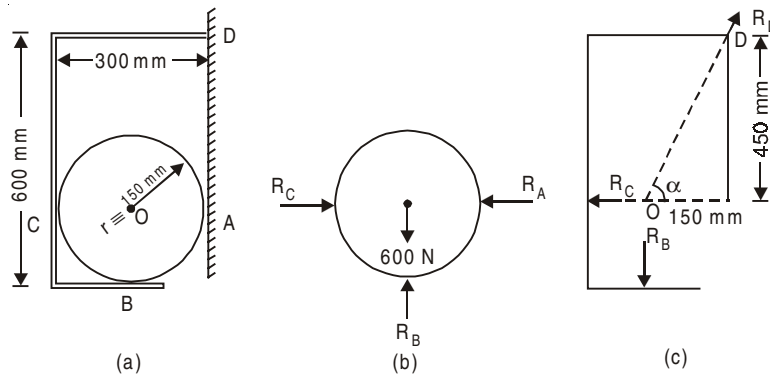


Fig. 2.43

As the frame is in equilibrium under the action of three forces only, they must be concurrent forces. In other words, reaction at  $D$  has line of action along  $OD$ . Hence, its inclination to horizontal is given by:

$$\tan \alpha = \frac{450}{150} = 3$$

$$\alpha = 71.5650^\circ$$

$$\Sigma V = 0, \text{ gives}$$

$$R_D \sin \alpha = R_B = 600$$

$$\therefore R_D = 632.456 \text{ N.}$$

$$\Sigma H = 0, \text{ gives}$$

$$R_C = R_D \cos \alpha$$

$$\therefore R_C = 200 \text{ N.}$$

$$\text{From (2), } \therefore R_A = 200 \text{ N.}$$

**Example 2.29.** Two smooth spheres each of radius 100 mm and weight 100 N, rest in a horizontal channel having vertical walls, the distance between which is 360 mm. Find the reactions at the points of contacts  $A$ ,  $B$ ,  $C$  and  $D$  shown in Fig. 2.44(a).

**Solution:** Let  $O_1$  and  $O_2$  be the centres of the first and second spheres. Drop perpendicular  $O_1P$  to the horizontal line through  $O_2$ . Figures 2.44(b) and 2.44(c) show free body diagram of the sphere 1 and 2, respectively. Since the surface of contact are smooth, reaction of  $B$  is in the radial direction, i.e., in the direction  $O_1O_2$ . Let it make angle  $\alpha$  with the horizontal. Then,

$$\cos \alpha = \frac{O_2P}{O_1O_2} = \frac{360 - O_1A - O_2D}{O_1B + BO_2} = \frac{360 - 100 - 100}{100 + 100} = 0.8$$

$$\therefore \sin \alpha = 0.6.$$

Consider sphere No. 1.

$$\Sigma V = 0, \text{ gives}$$

$$R_B \times 0.6 = 100$$

$$R_B = 166.67 \text{ N.}$$

$$\Sigma H = 0, \text{ gives}$$

$$R_A = R_B \times 0.8$$

$$\therefore R_A = 133.33 \text{ N.}$$

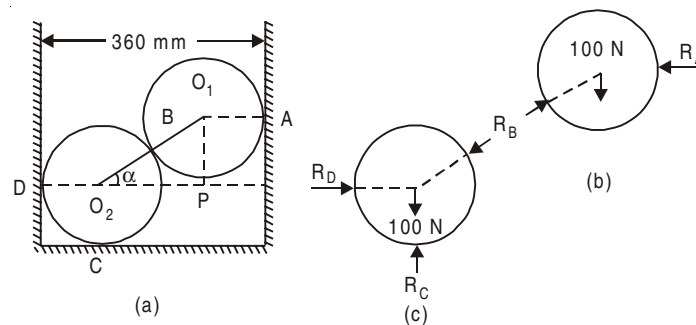


Fig. 2.44

Consider sphere No. 2

$$\Sigma V = 0, \text{ gives}$$

$$R_c = 100 + R_B \times 0.6$$

$\therefore$

$$R_c = 200 \text{ N.}$$

$$\Sigma H = 0$$

$$R_D = R_B \times 0.8$$

$\therefore$

$$R_D = 133.33 \text{ N.}$$

**Example 2.30.** Two cylinders, A of weight 4000 N and B of weight 2000 N rest on smooth inclines as shown in Fig. 2.45(a). They are connected by a bar of negligible weight hinged to each cylinder at its geometric centre by smooth pins. Find the force P to be applied as shown in the figure such that it will hold the system in the given position.

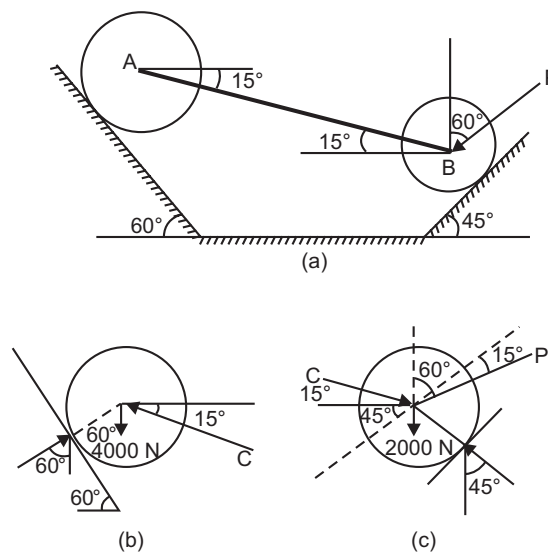


Fig. 2.45

**Solution:** Figures 2.45(b) and 2.45(c) show the free body diagram of the two cylinders. Applying Lami's theorem to the system of forces on cylinder A, we get

$$\frac{C}{\sin 60^\circ} = \frac{4000}{\sin(60 + 90 - 15)}$$

$$C = 4898.98 \text{ N}$$

Consider cylinder B.

Summation of the forces parallel to the inclined plane ( $45^\circ$  to horizontal) = 0, gives:

$$P \cos 15^\circ + 2000 \cos 45^\circ - C \cos 60^\circ = 0$$

$$\therefore P = \frac{4898.98 \cos 60^\circ - 2000 \cos 45^\circ}{\cos 15^\circ}$$

$$P = 1071.8 \text{ N.}$$

## 2.16 EQUILIBRIUM OF NON-CONCURRENT FORCE SYSTEMS

To satisfy the condition that there should not be translatory motion of the body subjected to coplanar forces, in this case also the following two conditions are to be satisfied:

$$\Sigma F_x = 0; \quad \Sigma F_y = 0$$

To satisfy the condition that there should not be rotational motion the following condition should be satisfied.

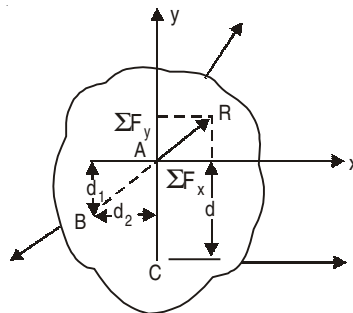
$$\Sigma M_A = 0, \quad \text{where } A \text{ is any point in the plane.}$$

Hence, the necessary and sufficient conditions for equilibrium of coplanar force system are:

$$\left. \begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_A &= 0 \end{aligned} \right\} \quad \dots(2.16)$$

and

Referring to Fig. 2.46, A, B and C are three points in the x-y plane which are not collinear. Any system of forces acting on a body may be replaced by a single force and a couple. Let R be such resultant force.



**Fig. 2.46**



If the condition  $\Sigma M_A = 0$  is satisfied, we can conclude that the resultant  $R$  should be a force passing through  $A$ . If  $\Sigma M_B = 0$  is satisfied, the line of action of the resultant passes through  $B$ , *i.e.*,  $AB$  is the line of action. Now, if  $\Sigma M_C = 0$  is also satisfied, the resultant should be zero since the line of action of the resultant (line  $AB$ ) cannot pass through  $C$  also. Thus if  $A, B, C$  are not collinear points, the following conditions are necessary and sufficient conditions of equilibrium.

$$\text{and} \quad \left. \begin{array}{l} \Sigma M_A = 0 \\ \Sigma M_B = 0 \\ \Sigma M_C = 0 \end{array} \right\} \quad \dots(2.17)$$

The equilibrium Equations 2.16 are not independent of equilibrium Equations 2.15. Two of them are common to the two sets. Referring to Fig. 2.46,  $R$  can be resolved into its components perpendicular to and parallel to  $AC$ .

$$\begin{aligned} \text{Then,} \quad & \Sigma M_C = (\Sigma F_x)d \\ \therefore \quad & \text{if } \Sigma M_C = 0, (\Sigma F_x)d = 0 \quad \text{or } \Sigma F_x = 0 \\ \text{Now,} \quad & \Sigma M_B = 0 \\ \text{i.e.,} \quad & (\Sigma F_x)d_1 + (\Sigma F_y)d_2 = 0 \\ \text{But} \quad & \Sigma F_x = 0 \\ \therefore \quad & (\Sigma F_y)d_2 = 0 \\ \text{Since } ABC \text{ are not collinear,} \quad & d_2 \neq 0 \\ \therefore \quad & \Sigma F_y = 0 \end{aligned}$$

Thus,  $\Sigma M_C = 0$  is identical to  $\Sigma F_x = 0$  and  $\Sigma M_B = 0$  is identical to  $\Sigma F_y = 0$ . Hence any one of the following sets may be used as equations of equilibrium:

$$\left. \begin{array}{l} (1) \quad \Sigma F_x = 0; \Sigma F_y = 0; \Sigma M_A = 0 \\ (2) \quad \text{If line } AB \text{ is not in } y \text{ direction,} \\ \quad \Sigma F_y = 0; \Sigma M_A = 0; \Sigma M_B = 0 \\ (3) \quad \text{If line } AB \text{ is not in } x \text{ direction} \\ \quad \Sigma F_x = 0; \Sigma M_A = 0; \Sigma M_B = 0 \\ (4) \quad \text{If } A, B \text{ and } C \text{ are non-collinear} \\ \quad \Sigma M_A = 0; \Sigma M_B = 0; \Sigma M_C = 0 \end{array} \right\} \quad \dots(2.18)$$

It can be proved that *if a system is in equilibrium under the action of three forces, those three forces must be concurrent.*

Let  $P_1, P_2$  and  $P_3$  be the forces acting on a body as shown in Fig. 2.47 and let  $P_1$  and  $P_2$  intersect at  $A$ . Then applying moment equilibrium condition about  $A$ , we get

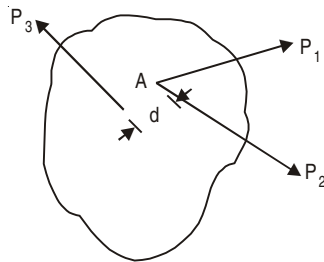
$$\Sigma M_A = 0 \quad \text{i.e. } P_3 d = 0$$

where  $d$  is distance of line of action of  $P_3$  from  $A$ .

Since  $P_3$  is not zero,

$$d = 0, \quad \text{i.e. } P_3 \text{ also must pass through } A.$$

Hence, the proposition is proved.

**Fig. 2.47**

Three problems are solved in this article. The problem of finding the reactions at supports of beams and trusses also fall under this categories. But these problems are solved in the chapters analysis of beams and trusses.

**Example 2.31.** The 12 m boom AB weighs 1 kN, the distance of the centre of gravity G being 6 m from A. For the position shown, determine the tension  $T$  in the cable and the reaction at B [Ref. Fig. 2.48(a)].

**Solution:** The free body diagram of the boom is shown in Fig. 2.48(b).

$$\Sigma M_A = 0, \text{ gives}$$

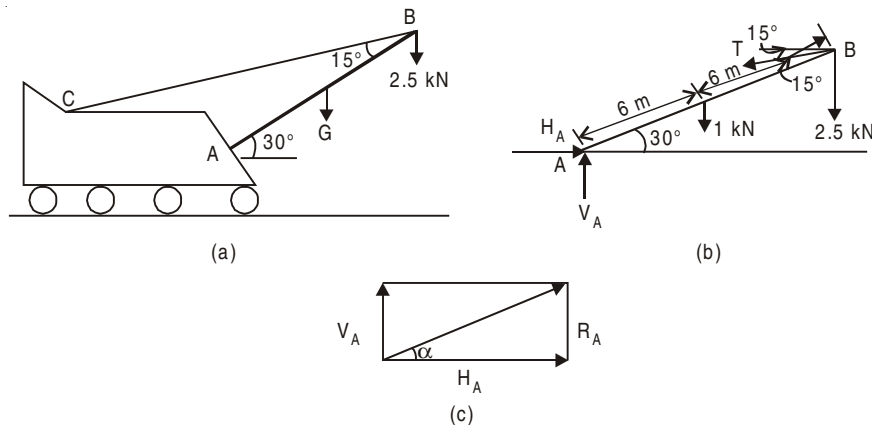
$$T \sin 15^\circ \times 12 - 2.5 \times 12 \cos 30^\circ - 1 \times 6 \cos 30^\circ = 0$$

$$T = 10.0382 \text{ kN.}$$

$$\Sigma H = 0, \text{ gives}$$

$$H_A - T \cos 15^\circ = 0$$

$$H_A = 9.6962 \text{ kN}$$

**Fig. 2.48**

$$\Sigma V = 0, \text{ gives}$$

$$V_A = 1 + 2.5 + T \sin 15^\circ = 6.0981 \text{ kN}$$

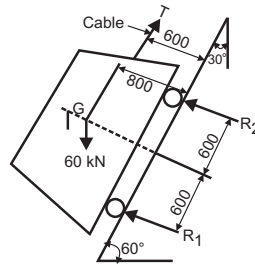
$$R_A = \sqrt{V_A^2 + H_A^2}$$

$$R_A = 11.4544 \text{ kN.}$$

$$\alpha = \tan^{-1} \frac{6.0981}{9.6962}$$

$$= 32.17^\circ \text{ as shown in Fig. 2.48(c).}$$

**Example 2.32.** A cable car used for carrying materials in a hydroelectric project is at rest on a track formed at an angle of  $30^\circ$  with the vertical. The gross weight of the car and its load is 60 kN and its centroid is at a point 800 mm from the track half way between the axles. The car is held by a cable as shown in Fig. 2.49. The axles of the car are at a distance 1.2 m. Find the tension in the cables and reaction at each of the axles neglecting friction of the track.



**Fig. 2.49**

**Solution:** Let  $T$  be the tension in the cable and the reaction at the pair of wheels be  $R_1$  and  $R_2$  as shown in Fig. 2.49.

Now,  $\Sigma$  of forces parallel to the track = 0, gives

$$T - 60 \sin 60^\circ = 0$$

$$T = 51.9615 \text{ kN.}$$

Taking moment equilibrium condition about upper axle point on track, we get

$$R_1 \times 1200 + T \times 600 - 60 \sin 60^\circ \times 800 - 60 \cos 60^\circ \times 600 = 0$$

$$R_1 = 23.6603 \text{ kN.}$$

$\Sigma$  of forces normal to the plane = 0, gives

$$R_1 + R_2 - 60 \cos 60^\circ = 0$$

$$R_2 = 30 - 23.6603$$

$$R_2 = 6.3397 \text{ kN.}$$

**Example 2.33.** A hollow right circular cylinder of radius 800 mm is open at both ends and rests on a smooth horizontal plane as shown in Fig. 2.50(a). Inside the cylinder there are two spheres having weights 1 kN and 3 kN and radii 400 mm and 600 mm, respectively. The lower sphere also rests on the horizontal plane. Neglecting friction find the minimum weight  $W$  of the cylinder for which it will not tip over.

**Solution:** Join the centres of spheres,  $O_1$  and  $O_2$  and drop  $O_1D$  perpendicular to horizontal through  $O_2$ .

Now,

$$O_1O_2 = 400 + 600 = 1000 \text{ mm}$$

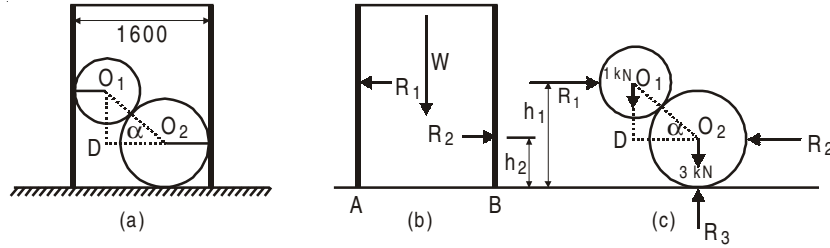
$$O_2D = 1600 - 400 - 600 = 600 \text{ mm}$$

If  $\alpha$  is the inclination of  $O_2O_1$  to horizontal,

$$\cos \alpha = \frac{600}{1000} = 0.6$$

$\therefore$

$$\sin \alpha = 0.8$$



**Fig. 2.50**

Free body diagrams of cylinder and spheres are shown in Fig. 2.50(b) and (c). Considering the equilibrium of the spheres,

$$\Sigma M_{O_2} = 0, \text{ gives}$$

$$R_1 \times O_1O_2 \sin \alpha - 1 \times O_2D = 0$$

$$R_1 \times 1000 \times 0.8 = 1 \times 600$$

$$R_1 = 0.75 \text{ kN}$$

$$\Sigma H = 0, \text{ gives}$$

$$R_2 = R_1 = 0.75 \text{ kN}$$

$$\Sigma V = 0, \text{ gives}$$

$$R_3 = 1 + 3 = 4 \text{ kN}$$

Now consider the equilibrium of cylinder. When it is about to tip over A, there is no reaction from ground at B. The reaction will be only at A.  $\Sigma M_A = 0$  gives,

$$R_1 h_1 - R_2 h_2 - W \times 800 = 0$$

$$R_1(h_1 - h_2) - W \times 800 = 0$$

$$\text{Since } R_1 = R_2$$

$$0.75 \times O_1D = W \times 800$$

$$0.75 \times 1000 \times 0.8 = W \times 800$$

$\therefore$

$$W = 0.75 \text{ kN.}$$

#### IMPORTANT FORMULAE

1. Newton's third law,  $F = R$ .
2. Newton's gravitational law,  $F = G \frac{m_1 m_2}{d^2}$ .
3. Weight of a body,  $W = mg$ .
4. Moment of a force about a point =  $F \times d$ .

5. Varignon's theorem

$$Ra = P_1d_1 + P_2d_2 + P_3d_3 + \dots$$

6. Magnitude of a couple

$$M = Pd.$$

7.  $P$  at any point  $A$  may be resolved into force  $P$  at  $B$  plus a moment  $P \times d$ , where ' $d$ ' is the perpendicular distance of  $B$  from the line of action of  $P$  through  $A$ .

8.  $R_x = R \cos \alpha = \Sigma F_x$

$$R_y = R \sin \alpha = \Sigma F_y$$

$$\tan \alpha = \frac{\Sigma F_y}{\Sigma F_x}$$

and 
$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}.$$

9.  $x$  and  $y$  intercepts of the resultant are given by

$$x = \frac{d}{\sin \alpha} = \frac{\Sigma M_o}{\Sigma F_y}$$

and

$$y = \frac{d}{\cos \alpha} = \frac{\Sigma M_o}{\Sigma F_x}.$$

10. Lami's theorem:

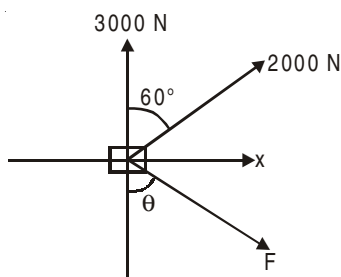
$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}.$$

### THEORY QUESTIONS

1. State and explain Newton's laws of mechanics.
2. Explain the following laws of mechanics:
  - (i) Law of transmissibility of forces
  - (ii) Parallelogram law of forces.
3. Write short notes on
  - (i) Principles of physical independence
  - (ii) Principles of superposition.
4. Explain with examples the following system of forces
  - (i) Coplanar concurrent forces
  - (ii) Coplanar non-concurrent forces
  - (iii) Non-coplanar non-concurrent forces.
5. State and explain Varignon's theorem of moments.
6. Define a couple-moment and list its characteristics.

**PROBLEMS FOR EXERCISE**

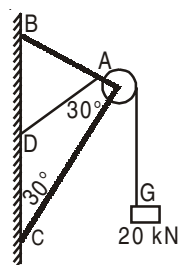
1. A body is subjected to the three forces as shown in Fig. 2.51. If possible determine the direction of the force  $F$  so that the resultant is in  $x$ -direction, when
- (a)  $F = 5000$  N
- (b)  $F = 3000$  N



**Fig. 2.51**

[Ans. (a)  $36.87^\circ$ ; (b) Not possible]

2. The frictionless pulley  $A$  shown in Fig. 2.52 is supported by two bars  $AB$  and  $AC$  which are hinged at  $B$  and  $C$  to a vertical wall. The flexible cable  $DG$  hinged at  $D$ , goes over the pulley and supports a load of 20 kN at  $G$ . The angles between the various members are shown in the figure. Determine the forces in the bars  $AB$  and  $AC$ . Neglect the size of the pulley.

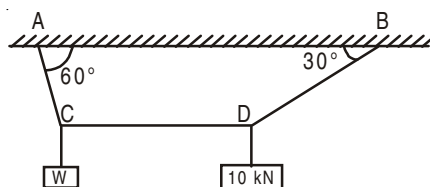


**Fig. 2.52**

[Hint: Force in  $AD = 20$  kN, tensile]

[Ans.  $F_{AB} = 0$ ;  $F_{AC} = 34.641$  kN]

3. A cord supported at  $A$  and  $B$  carries a load of 10 kN at  $D$  and a load of  $W$  at  $C$  as shown in Fig. 2.53. Find the value of  $W$  so that  $CD$  remains horizontal. [Ans.  $W = 30$  kN]



**Fig. 2.53**

4. Three bars, hinged at  $A$  and  $D$  and pinned at  $B$  and  $C$  as shown in Fig. 2.54 form a four-linked mechanism. Determine the value of  $P$  that will prevent movement of bars.

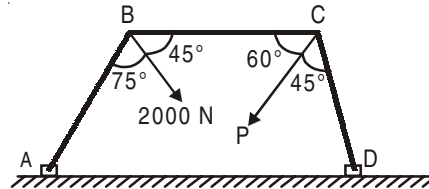


Fig. 2.54

[Ans.  $P = 3047.2$  N]

5. Two identical rollers, each of weights  $100$  N are supported by an inclined plane and a vertical wall as shown in Fig. 2.55. Assuming smooth surfaces, find the reactions induced at the points of supports  $A$ ,  $B$ ,  $C$  and  $D$ .

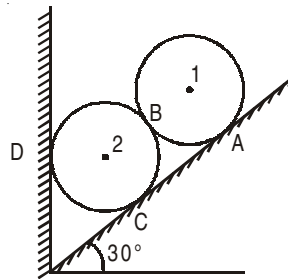


Fig. 2.55

[Ans.  $R_A = 86.67$  N;  $R_B = 50.00$  N;  $R_C = 144.34$  N;  $R_D = 115.47$  N]

6. Two cylinders of diameters  $100$  mm and  $50$  mm, weighing  $200$  N and  $50$  N, respectively are placed in a trough as shown in Fig. 2.56. Neglecting friction, find the reactions at contact surfaces 1, 2, 3 and 4.

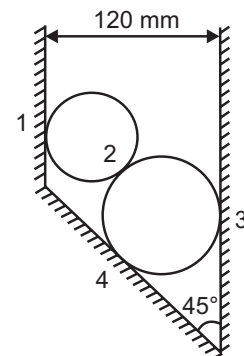
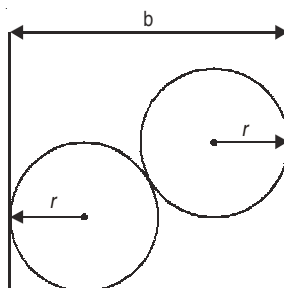
[Ans.  $R_1 = 37.5$  N;  $R_2 = 62.5$  N;  $R_3 = 287.5$  N;  $R_4 = 353.5$  N]

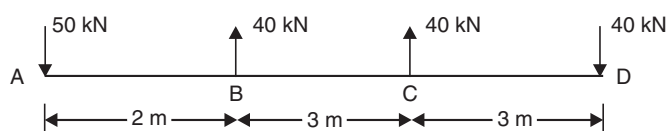
Fig. 2.56

7. Two smooth spheres each of weight  $W$  and radius ' $r$ ' are in equilibrium in a horizontal channel of width ' $b$ ' ( $b < 4r$ ) and vertical sides as shown in Fig. 2.57. Find the three reactions from the sides of the channel which are all smooth. Also find the force exerted by each spheres on the other.

**Fig. 2.57**

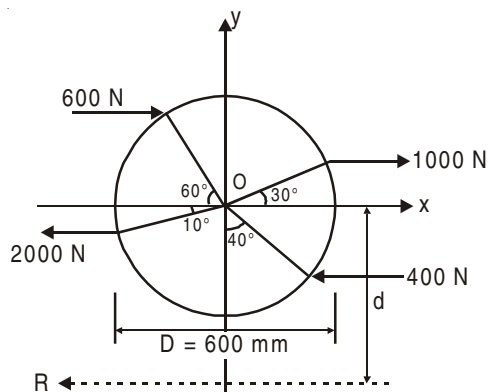
[Ans.  $R_A = R_D = W \cot \alpha$ ,  $R_C = 2W$  and  $R_B = \frac{W}{\sin \alpha}$   
 where  $\cos \alpha = \frac{b-2r}{2r}$ ]

8. Determine the resultant of parallel force system shown in Fig. 2.58.

**Fig. 2.58**

[Ans.  $R = 10$  kN, downward at  $x = 4$  m from A]

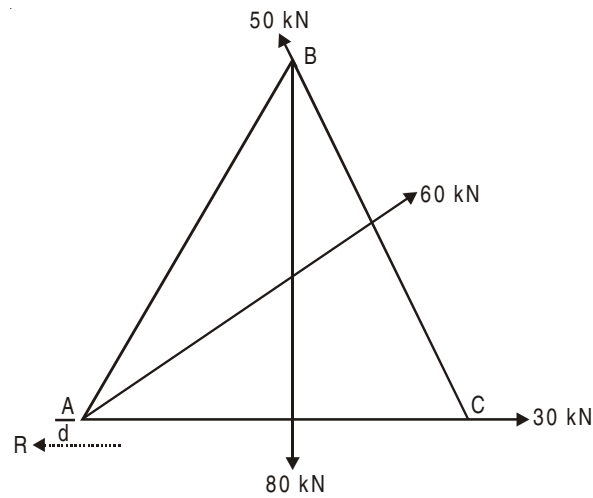
9. Determine the resultant of the parallel coplanar force system shown in Fig. 2.59.

**Fig. 2.59**

[Ans.  $R = 800$  N (towards left);  $d = 627.50$  mm]

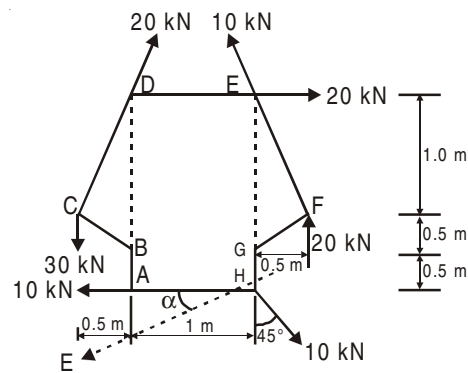
10. An equilateral triangular plate of sides 200 mm is acted upon by four forces as shown in Fig. 2.60. Determine the magnitude and direction of the resultant of this system of forces and its position.  
 [Ans.  $R = 57.3541$  kN;  $\alpha = 6.70^\circ$ ;  $d = 11.51$  mm]



**Fig. 2.60**

11. Determine the magnitude, direction and the line of application of the equilibrant of the set of forces shown in Fig. 2.61, which will keep the plane body ABCDEFGH in equilibrium.

[Ans.  $E = 23.6517$  kN;  $\alpha = 24.37^\circ$ ;  $x$  from A = 1.041 m]

**Fig. 2.61**

12. Determine the resultant of the four forces acting on a body as shown in Fig. 2.62.

[Ans.  $R = 200$ ;  $\alpha = 60^\circ$ ; at a distance  $y = 8.7679$  m below  $O$ ]

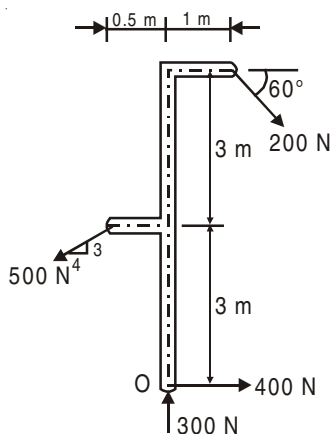


Fig. 2.62

13. A bracket is subjected to the system of forces and couples as shown in Fig. 2.63. Find the resultant of the system and the point of intersection of its line of action with (a) line  $AB$ , (b) line  $BC$  and (c) line  $CD$ .

[Ans.  $R = 485.4$  N;  $\alpha = 34.50^\circ$ ;  $Y_{BA} = 112.5$  mm;  $X_{BC} = 163.6$  mm;  $Y_{CD} = 93.75$  mm]

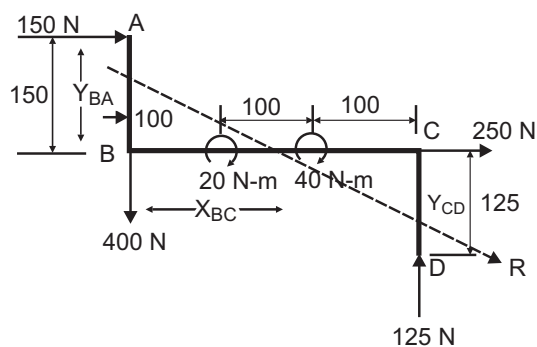


Fig. 2.63

14. Determine the resultant of the three forces acting on the dam section shown in Fig. 2.64 and locate its intersection with the base  $AB$ . For a safe design this intersection should occur within the middle third. Is it a safe design?

[Ans. Resultant intersects  $AB$  3.333 m from  $A$ . It is a safe design]

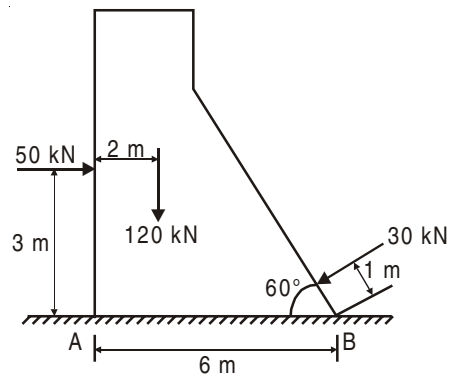


Fig. 2.64

15. A 1000 N cylinder is supported by a horizontal rod  $AB$  and a smooth uniform rod  $CD$  which weighs 500 N as shown in Fig. 2.65. Assuming the pins at  $A$ ,  $B$ ,  $C$  and  $D$ , to be frictionless and weight of  $AB$  negligible, find the reactions at  $C$  and  $D$ .

[Ans.  $V_C = 937.1$  N (upward);  $H_C = 577.3$  N (towards left)  $R_D = 562.9$  N (upward)]

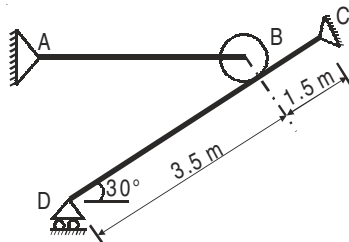


Fig. 2.65

16. The frame shown in Fig. 2.66 is supported by a hinge at  $E$  and by a roller at  $D$ . Determine the horizontal and vertical components of the reactions at hinge  $C$  as it acts upon member  $BD$ .

[Ans.  $H_C = 140$  N (towards right);  $V_C = 35$  N (upward)]

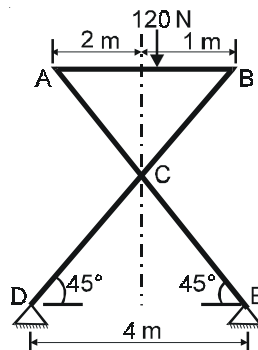


Fig. 2.66

# Trusses

A truss is a structure made up of slender members pin-connected at ends and is capable of taking loads at joints. They are used as roof trusses to support sloping roofs and as bridge trusses to support deck. In many machines steel trusses are used. Transmission towers are also the examples of trusses. In the case of wooden trusses, the ends are connected by making suitable joints or by nailing and bolting whereas in steel trusses ends are connected by bolting or welding. The trusses are also known as '*pinjointed frames*'.

A truss in which all the members lie in a single plane is called as a *plane truss*. In such trusses loads act in the plane of the truss only. Roof trusses and bridge trusses can be considered as plane trusses. If all the members of a truss do not lie in a single plane, then it is called a space truss. Tripod and transmission towers are the examples of space trusses. In this chapter, the analysis of only plane trusses is considered.

## 3.1 PERFECT, DEFICIENT AND REDUNDANT TRUSSES

A pinjointed truss which has got just sufficient number of members to resist the loads without undergoing appreciable deformation in shape is called a *perfect truss*. Triangular truss is the simplest perfect truss and it has three joints and three members (Fig. 3.1). Perfect trusses with four and five joints are shown in Figs. 3.2 and 3.3 respectively.

It may be observed that to increase one joint in a perfect truss, two more members are required. Hence the following expression may be written down as the relationship between number of joints  $j$ , and the number of members  $m$ , in a perfect truss.

$$m = 2j - 3 \quad \dots(3.1)$$

However, the above equation gives only a necessary, but not a sufficient condition of a perfect truss. For example, the two trusses shown in Fig. 3.4(a) and (b) have the same number of members and joints. The truss shown in Fig. 3.4(a) is perfect whereas the one shown in Fig. 3.4(b) is not capable of retaining its shape if loaded at the joint marked 6. Therefore, the only necessary and sufficient condition of a perfect truss is that it should retain its shape when load is applied at any joint in any direction.

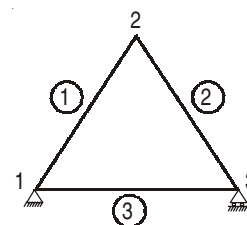


Fig. 3.1

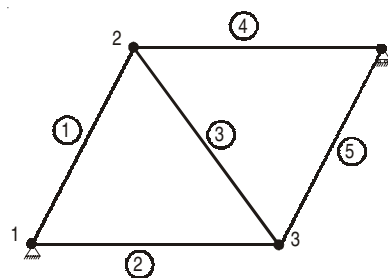


Fig. 3.2

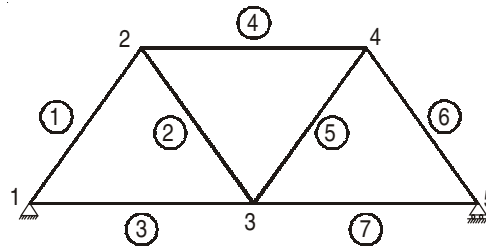


Fig. 3.3

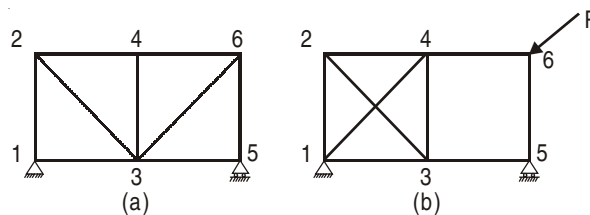


Fig. 3.4

A truss is said to be *deficient* if the number of members in it are less than that required for a perfect truss. Such trusses cannot retain their shape when loaded. A deficient truss is shown in Fig. 3.5.

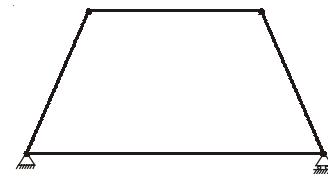


Fig. 3.5

A truss is said to be *redundant* if the number of members in it are more than that required in a perfect truss. Such trusses cannot be analysed by making use of the equations of equilibrium alone. Thus, a redundant truss is statically indeterminate. Each extra member adds one degree of indeterminacy. For the analysis of such members the consistency of deformations is to be considered. The truss shown in the Fig. 3.6 is a typical redundant truss. In this truss one diagonal member in each panel is extra. Hence it is a two-degree redundant truss.

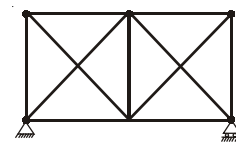


Fig. 3.6

In this chapter, only the analysis of perfect frames is considered.

### 3.2 ASSUMPTIONS

In the theory that is going to be developed in this chapter, the following assumptions are made:

- (1) The ends of the members are pin-connected (hinged);
- (2) The loads act only at the joints;
- (3) Self-weights of the members are negligible;
- (4) Cross-section of the members is uniform.

If at all the cross-section varies, the centre of gravity of the section is assumed to be located along the same longitudinal line.

In reality the members are connected by bolting, riveting or by welding. No special care is taken to ensure perfect pin-connections. However, experiments have shown that assuming pin-connected ends is quite satisfactory since the members used are slender.

In most of the frames the loads act at the joints. Even if a load is not acting at a joint, it can be replaced by its reaction at the joint and a local bending effect on the member. The frame may be analysed for the joint loads and the local bending effect on the member superposed in the design of that member.

In most of the trusses, the self-weight is really small compared to the loads they carry. Hence self-weight of the members may be neglected.

It is the duty of construction engineer to see that the centroid of all cross-sections lie along a single axis so that the member is held in equilibrium by the two forces acting at its ends.

Because of the assumption of pin-connected ends, it is more appropriate to call the theory that is going to be developed in this chapter as *analysis of pin-connected plane trusses*. Analysis of rigid frames is not covered in this book.

### 3.3 NATURE OF FORCES IN MEMBERS

The members of a truss are subjected to either tensile or compressive forces. A typical truss  $ABCDE$  loaded at joint  $E$  is shown in Fig. 3.7(a). The member  $BC$  is subjected to compressive force  $C$  as shown in Fig. 3.7(b). Effect of this force on the joint  $B$  (or  $C$ ) is equal and opposite to the force  $C$  as shown in Fig. 3.7(b).

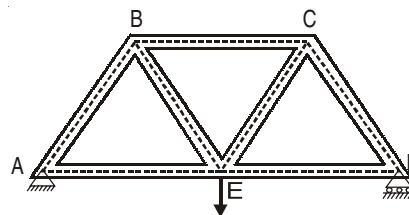


Fig. 3.7(a)

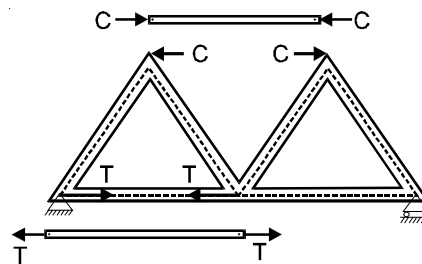


Fig. 3.7(b)

The member  $AE$  is subjected to tensile force  $T$ . Its effect on the joints  $A$  and  $E$  are as shown in Fig. 3.7(b). In the analysis of frame we mark the forces on the joints, instead of the forces in the members as shown in Fig. 3.7(c). It may be noted that compressive force in a member is represented in a figure by two arrows going away from each other and a tensile force by two arrows coming towards each other. This is quite logical considering the fact that the markings on the members represent the internal reactive forces developed which are opposite in direction to the applied forces.

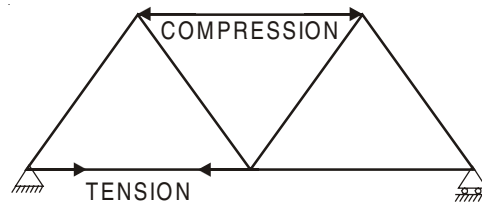


Fig. 3.7(c)

### 3.4 METHODS OF ANALYSIS

The following three methods are available for the analysis of pin-connected frames:

- (1) Method of joints
- (2) Method of section
- (3) Graphical method.

The first two are analytical methods and they are dealt in this chapter.

### 3.5 METHOD OF JOINTS

At each joint the forces in the members meeting and the loads acting, if any, constitute a system of concurrent forces. Hence, two independent equations of equilibrium can be formed at each joint. First, a joint is selected where there are only two unknown forces. Many times such a joint can be identified only after finding the reaction at the support by considering the equilibrium of the entire frame. Then making use of the two equations of equilibrium at that joint the two unknown forces are found. Then, the next joint is selected for analysis where there are now only two unknown forces. Thus, the analysis proceeds from joints to joint to find the forces in all the members.

It may be noted that if there are  $j$  number of joints,  $2j$  number of the equations can be formed. There will be three reactions in a general determinate truss. The force in each member is unknown. Hence, if there are  $m$  number of members, the total number of unknowns will be  $m+3$ . A problem can be analysed if there are as many equations as there are unknowns. Hence, a frame analysis problem is determinate if:

$$2j = m + 3 \quad \dots(3.2)$$

This equation is the same as Equation 3.1 which was derived on the consideration of a perfect frame. Hence, a perfect frame is determinate. If  $m > 2j - 3$ , then the number of unknowns is more than the number of equations. Hence, a redundant frame is indeterminate. If  $m < 2j - 3$ , then the number of equations is more than the number of unknowns. Since a set of solutions can satisfy such equations, it shows instability of the structure. Hence, a deficient frame is not stable.

The method of joints is illustrated with the examples 3.1 to 3.7.

#### EXAMPLES

**Example 3.1.** Find the forces in all the members of the truss shown in Fig. 3.8(a). Tabulate the results.

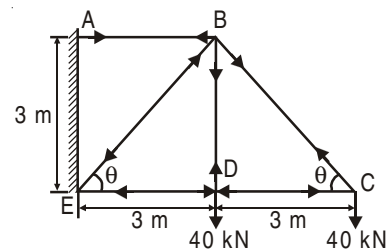


Fig. 3.8(a)

**Solution:** Step 1: Determine the inclinations of all inclined members. In this case,

$$\tan \theta = \frac{3}{3} = 1$$

$$\therefore \theta = 45^\circ$$

Step 2: Look for a joint at which there are only two unknowns. If such a joint is not available, determine the reactions at the supports, and then at the supports these unknowns may reduce to only two.

Now at joints  $C$ , there are only two unknowns, i.e., forces in members  $CB$  and  $CD$ , say  $F_{CB}$  and  $F_{CD}$ .

*Note:* Usually in cantilever type frames, we find such joints without the need to find reactions.

Step 3: Now there are two equations of equilibrium for the forces meeting at the joint and two unknown forces. Hence, the unknown forces can be determined.

At joint  $C$  [Ref. Fig. 3.8(b)]  $\sum V = 0$  condition shows that the force  $F_{CB}$  should act away from the joint  $C$  so that its vertical component balances the vertical downward load at  $C$ .

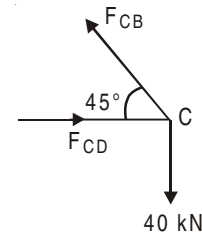
$$F_{CB} \sin 45^\circ = 40$$

$$\therefore F_{CB} = 40\sqrt{2} \text{ kN}$$

Now  $\sum H = 0$  indicates that  $F_{CD}$  should act towards  $C$ .

$$F_{CD} - F_{CB} \cos 45^\circ = 0$$

$$F_{CD} = F_{CB} \cos 45^\circ = 40\sqrt{2} \times \frac{1}{\sqrt{2}} = 40 \text{ kN}$$



**Fig. 3.8(b)**

**Note:** If the assumed direction of unknown force is opposite, the value will be negative. Then reverse the direction and proceed.

Step 4: On the diagram of the truss, mark arrows on the members near the joint analysed to indicate the forces on the joint. At the other end, mark the arrows in the reverse direction.

In the present case, near the joint  $C$ , the arrows are marked on the members  $CB$  and  $CD$  to indicate forces  $F_{CB}$  and  $F_{CD}$  directions as found in the analysis of joint  $C$ . Then reversed directions are marked in the members  $CB$  and  $CD$  near joints  $B$  and  $D$ , respectively.

Step 5: Look for the next joint where there are only two unknown forces and analyse that joint.

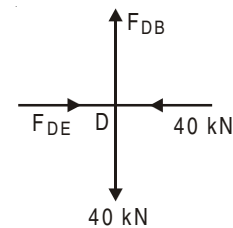
In this case, there are only two unknown forces at the joint  $D$  as shown in Fig. 3.8(c).

$$\sum V = 0$$

$$F_{DB} = 40 \text{ kN}$$

$$\sum H = 0$$

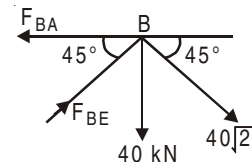
$$F_{DE} = 40 \text{ kN}$$



**Fig. 3.8(c)**

Step 6: Repeat steps 4 and 5 till forces in all the members are found.

In the present case, after marking the forces in the members  $DB$  and  $DE$ , we find that analysis of joint  $B$  can be taken up. Referring to Fig. 3.8(d).



**Fig. 3.8(d)**



$\Sigma V = 0$ , gives

$$F_{BE} \sin 45^\circ - 40 - 40\sqrt{2} \times \sin 45^\circ = 0$$

$$\therefore F_{BE} = 80\sqrt{2} \text{ kN}$$

$\Sigma H = 0$

$$F_{BA} - F_{BE} \cos 45^\circ - 40\sqrt{2} \times \cos 45^\circ = 0$$

$$F_{BA} = 80\sqrt{2} \times \frac{1}{\sqrt{2}} + 40\sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$\therefore F_{BA} = 120 \text{ kN}$$

The directions of these forces are marked on the diagram. Now the analysis is complete since the forces in all the members are determined.

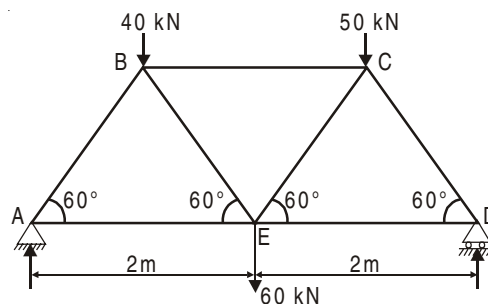
*Step 7:* Determine the nature of forces in each member and tabulate the results. Note that if the arrow marks on a member are towards each other, then the member is in tension and if the arrow marks are away from each other, the member is in compression [Ref. Fig. 3.8(e)]. In this case,



**Fig. 3.8(e)**

Member	Magnitude of Force in kN	Nature
AB	120	Tension
BC	$40\sqrt{2}$	Tension
CD	40	Compression
DE	40	Compression
BE	$80\sqrt{2}$	Compression
BD	40	Tension

**Example 3.2.** Determine the forces in all the members of the truss shown in Fig. 3.9(a) and indicate the magnitude and nature of forces on the diagram of the truss. All inclined members are at  $60^\circ$  to horizontal and length of each member is 2 m.



**Fig. 3.9(a)**

**Solution:** Now, we cannot find a joint with only two unknown forces without finding reactions.

Consider the equilibrium of the entire frame.

$$\Sigma M_A = 0, \text{ gives}$$

$$R_D \times 4 - 40 \times 1 - 60 \times 2 - 50 \times 3 = 0$$

$$\therefore R_D = 77.5 \text{ kN}$$

$$\sum H = 0, \text{ gives}$$

$$\therefore H_A = 0$$

$\therefore$  Reaction at A is vertical only

$$\sum V = 0, \text{ gives}$$

$$R_A + 77.5 = 40 + 60 + 50$$

$$\therefore R_A = 72.5 \text{ kN}$$

Joint A:  $\sum V = 0, \text{ gives}$

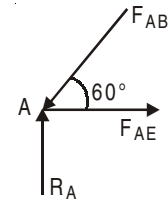
$$F_{AB} \sin 60^\circ = R_A = 72.5$$

$$F_{AB} = 83.7158 \text{ kN (Comp.)}$$

$$\sum H = 0, \text{ gives}$$

$$F_{AE} - 83.7158 \cos 60^\circ = 0$$

$$F_{AE} = 41.8579 \text{ kN (Tension)}$$



**Fig. 3.9(b)**

Joint D:  $\sum V = 0, \text{ gives}$

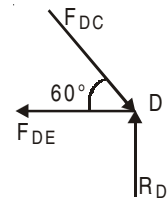
$$F_{DC} \sin 60^\circ = R_D = 77.5$$

$$\therefore F_{DC} = 89.4893 \text{ kN (Comp.)}$$

$$\sum H = 0, \text{ gives}$$

$$F_{DE} - 89.4893 \cos 60^\circ = 0$$

$$\therefore F_{DE} = 44.7446 \text{ kN (Tension)}$$



**Fig. 3.9(c)**

Joint B:  $\sum V = 0, \text{ gives}$

$$F_{BE} \sin 60^\circ - F_{AB} \sin 60^\circ + 40 = 0$$

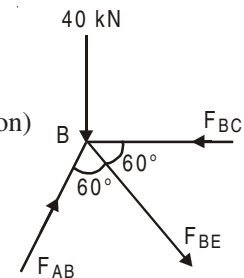
$$\therefore F_{BE} = \frac{72.5 - 40}{\sin 60^\circ} = 37.5278 \text{ (Tension)}$$

$$\sum H = 0, \text{ gives}$$

$$F_{BC} - F_{AB} \cos 60^\circ - F_{BE} \cos 60^\circ = 0$$

$$F_{BC} = (83.7158 + 37.5274) \times 0.5$$

$$F_{BC} = 60.6218 \text{ kN (Comp.)}$$

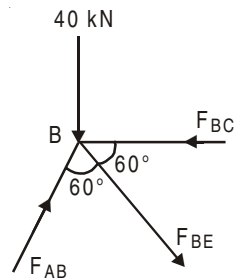


**Fig. 3.9(d)**

Joint C:  $\Sigma V = 0$ , gives

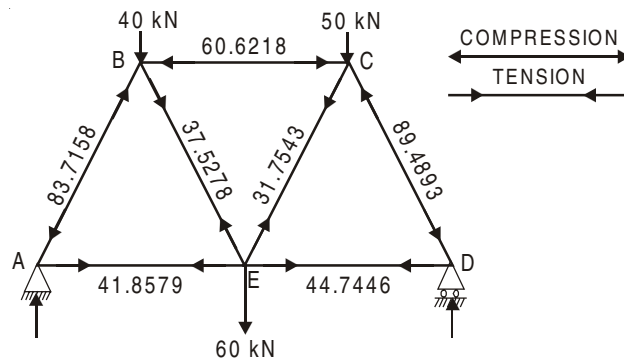
$$F_{CE} \sin 60^\circ + 50 - F_{DC} \sin 60^\circ = 0$$

$$F_{CE} = \frac{77.5 - 50}{\sin 60^\circ} = \mathbf{31.7543 \text{ kN (Tension)}}$$



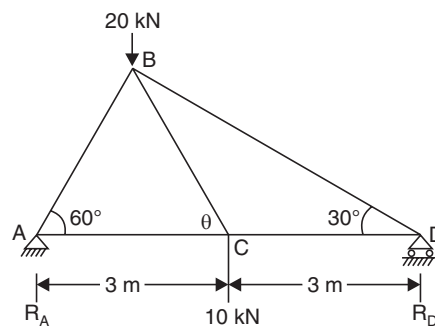
**Fig. 3.9(e)**

Now the forces in all the members are known. If joint E is analysed it will give the check for the analysis. The results are shown on the diagram of the truss in Fig. 3.9(f).



**Fig. 3.9(f)**

**Example 3.3.** Determine the forces in all the members of the truss shown in Fig. 3.10 (a).



**Fig. 3.10(a)**

**Solution:**  $AB = 6 \sin 60^\circ = 3 \text{ m}; \quad BD = 6 \cos 30^\circ = 23.09 \text{ m}$

In  $\triangle ABC$ ,  $AC = AB = 3 \text{ m}$

$\therefore \angle ABC = \angle BCA = \text{say } \theta$

$\therefore 2\theta = 180 - 60 = 120$

or  $\theta = 60^\circ$

Taking moment about A, we get

$$R_D \times 6 - 10 \times 3 - 20 \times 3 \cos 60^\circ = 0$$

$$R_D = 10 \text{ kN}$$

$$\therefore R_A = 20 + 10 - 10 = 20 \text{ kN}$$

[Note:  $H_4 = 0$ ]

Joint A: Referring to Fig. 3.10(b)

$$\Sigma V = 0, \text{ gives}$$

$$20 - F_{AB} \sin 60^\circ = 0$$

$$\therefore F_{AB} = 23.09 \text{ kN [Comp.]}$$

$$\Sigma H = 0, \text{ gives}$$

$$F_{AC} - F_{AB} \cos 60^\circ = 0 \quad \therefore F_{AC} = 23.09 \cos 60^\circ = 11.55 \text{ kN [Tensile]}$$

Joint D:

$$\Sigma V = 0, \text{ gives}$$

$$F_{DB} \sin 30^\circ = 10 \quad \text{or} \quad F_{DB} = 20 \text{ kN [Comp.]}$$

$$\Sigma H = 0, \text{ gives}$$

$$-F_{DC} + F_{DB} \cos 30^\circ = 0$$

$$\text{i.e.,} \quad F_{DC} = 20 \cos 30^\circ = 17.32 \text{ kN [Tensile]}$$

Joint C:

$$\Sigma V = 0, \text{ gives}$$

$$F_{CB} \sin 60^\circ = 10$$

$$\therefore F_{CB} = 11.55 \text{ kN}$$

$$\text{check} \quad \Sigma H = 0, \text{ gives}$$

$$F_{CB} \cos 60^\circ = F_{CD} - F_{CA} = 17.32 - 11.55$$

$$\therefore F_{CB} = 11.55, \text{ Checked}$$

These forces are marked in Fig. 3.10(e).

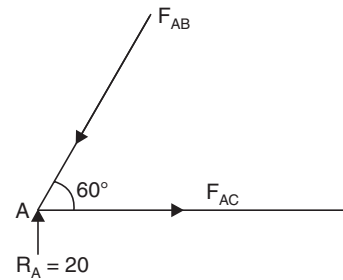


Fig. 3.10(b)

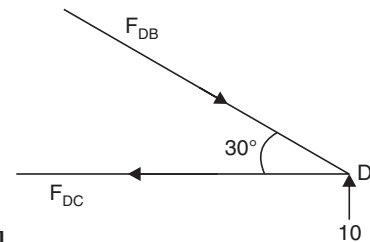


Fig. 3.10(c)

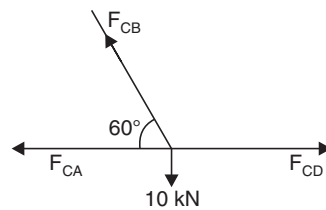


Fig. 3.10(d)

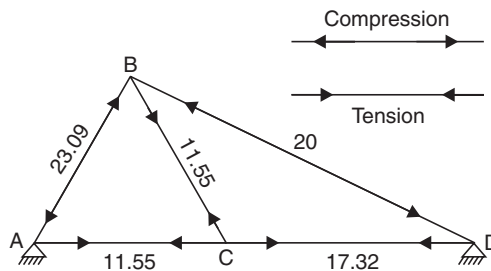
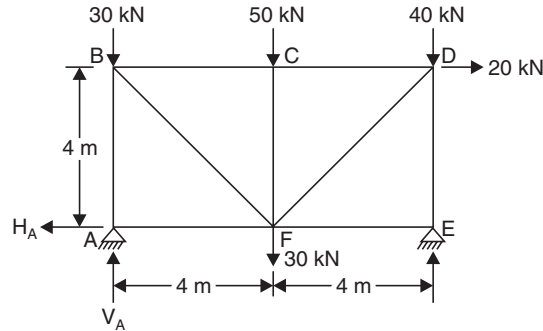


Fig. 3.10(e)

**Example 3.4.** Determine the forces in the members of truss shown in Fig. 3.11(a).



**Fig. 3.11(a)**

**Solution:**  $\Sigma$  Forces in horizontal direction = 0, gives,

$$H_A = 20 \text{ kN}$$

$$\Sigma M_A = 0 \rightarrow$$

$$-R_E \times 8 + 50 \times 4 + 40 \times 8 + 20 \times 4 + 30 \times 4 = 0$$

$$\therefore R_E = 90 \text{ kN}$$

$$\Sigma V_A = 0, \text{ gives}$$

$$V_A + R_E = 30 + 50 + 40 + 30$$

$$\therefore V_A = 150 - R_E = 150 - 90 = 60 \text{ kN}$$

Joint A:  $\Sigma V = 0 \rightarrow F_{AB} = 60 \text{ kN (Comp.)}$

$$\Sigma H = 0 \rightarrow F_{AF} = 20 \text{ kN (Tensile)}$$

Joint E:

$$\Sigma V = 0, \text{ gives, } F_{ED} = 90 \text{ kN (Comp.)}$$

$$\Sigma H = 0, \text{ gives, } F_{EF} = 0$$

Joint B: Noting that inclined member is at  $45^\circ$ ,

$$\Sigma V = 0; \text{ gives,}$$

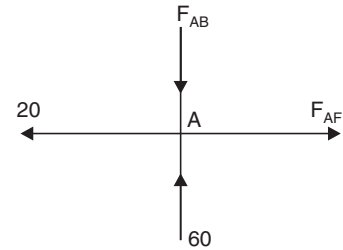
$$-F_{BF} \sin 45^\circ - 30 + 60 = 0$$

or 
$$F_{BF} = \frac{30}{\sin 45^\circ} = 42.43 \text{ kN [Tensile]}$$

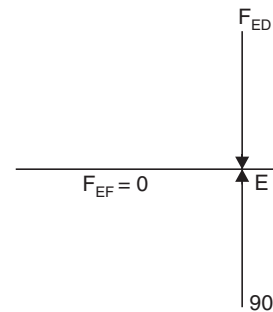
$$\Sigma H = 0, \text{ gives}$$

$$-F_{BC} + F_{BF} \cos 45^\circ = 0$$

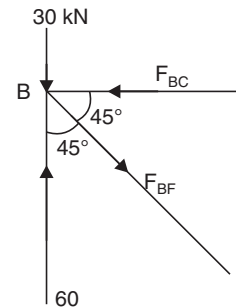
or 
$$F_{BC} = 42.43 \times \cos 45^\circ = 30 \text{ kN [Comp.]}$$



**Fig. 3.11(b)**



**Fig. 3.11(c)**



**Fig. 3.11(d)**

Joint C:  $\Sigma V = 0$ , gives  $F_{CF} = 50 \text{ kN}$  [Comp.]  
 $\Sigma H = 0$ , gives  $30 - F_{CD} = 0$   
 or  $F_{CD} = 30 \text{ kN}$  [Comp.]

Joint D: Noting that  $DF$  is at  $45^\circ$  as shown in Fig. 3.11(f)

$$\Sigma V = 0 \rightarrow$$

$$-F_{DF} \cos 45^\circ + 90 - 40 = 0$$

or 
$$F_{DF} = \frac{50}{\cos 45^\circ} = 70.71 \text{ kN} \text{ [Tensile]}$$

Check  $\Sigma H = 0$ , gives

$$-F_{DF} \cos 45^\circ + 30 + 20 = 0 \text{ or } F_{DF} = 70.71 \text{ kN checked.}$$

Final result is shown in Fig. 3.11(g)

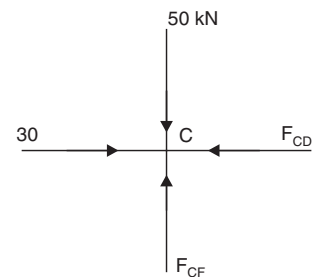


Fig. 3.11(e)

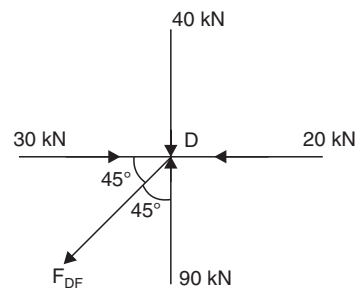


Fig. 3.11(f)

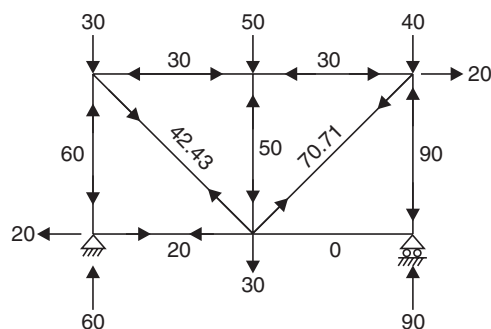


Fig. 3.11(g)

**Example 3.5.** Analyse the truss shown in Fig. 3.12(a).

**Solution:** All inclined members have the same inclination to horizontal. Now, length of an inclined member is

$$= \sqrt{3^2 + 4^2} = 5 \text{ m}$$

$$\therefore \sin \theta = \frac{4}{5}$$

$$\cos \theta = \frac{3}{5}$$

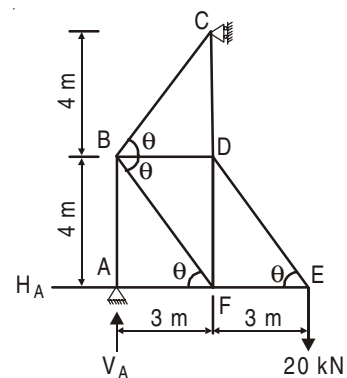
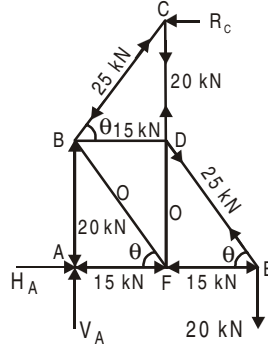


Fig. 3.12(a)

As soon as a joint is analysed the forces on the joint are marked on members [Fig. 3.12(b)]



**Fig. 3.12(b)**

Joint E:  $\sum V = 0$ , gives

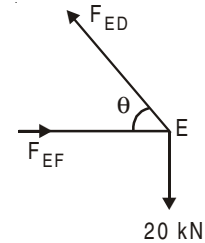
$$F_{ED} \times \frac{4}{5} - 20 = 0$$

$\therefore F_{ED} = 25 \text{ kN (Tension)}$

$\sum H = 0$ , gives

$$F_{EF} - F_{ED} \cos \theta = 0$$

$\therefore F_{EF} = 25 \times \frac{3}{5} = 15 \text{ kN (Comp.)}$



**Fig. 3.12(c)**

At this stage as no other joint is having only two unknowns, no further progress is possible. Let us find the reactions at the supports considering the whole structure. Let the reaction be as shown in Fig. 3.12(b).

$\sum M_A = 0$ , gives

$$R_C \times 8 - 20 \times 6 = 0$$

$$R_C = 15 \text{ kN}$$

$\sum V = 0$ , gives

$$V_A = 20 \text{ kN}$$

$\sum H = 0$ , gives

$$H_A = R_C = 15 \text{ kN}$$

Joint A:  $\sum V = 0$ , gives

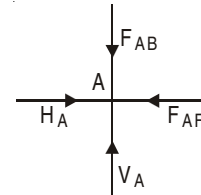
$$F_{AB} - V_A = 0$$

$$F_{AB} = 20 \text{ kN (Comp.)}$$

$\sum H = 0$ , gives

$$F_{AF} - H_A = 0$$

$$F_{AF} = 15 \text{ kN (Comp.)}$$



**Fig. 3.12(d)**

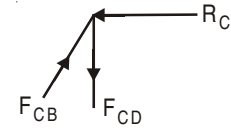
Joint C:  $\sum H = 0$ , gives

$$F_{CB} \times \frac{3}{5} - R_C = 0$$

$$F_{CB} = 15 \times \frac{5}{3} = 25 \text{ kN (Comp.)}$$

$\sum V = 0$ , gives

$$\begin{aligned} F_{CD} &= F_{CB} \sin \theta \\ &= 25 \times \frac{4}{5} = 20 \text{ kN (Tension)} \end{aligned}$$



**Fig. 3.12(e)**

Joint B:  $\sum V = 0$ , gives

$$F_{BF} \times \frac{4}{5} - F_{BC} \times \frac{4}{5} + F_{AB} = 0$$

$$F_{BF} \times \frac{4}{5} = 25 \times \frac{4}{5} - 20 = 0$$

$\therefore$

$$F_{BF} = 0$$

$\sum H = 0$ , gives

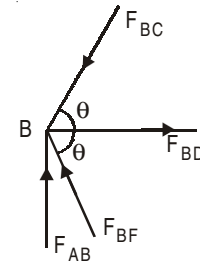
$$F_{BD} - 25 \times \frac{3}{5} = 0$$

$$F_{BD} = 15 \text{ kN (Tension)}$$

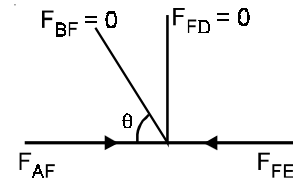
Joint F:

$$\sum V = 0$$

$$F_{FD} = 0 \text{ (since } F_{BF} = 0 \text{)}$$

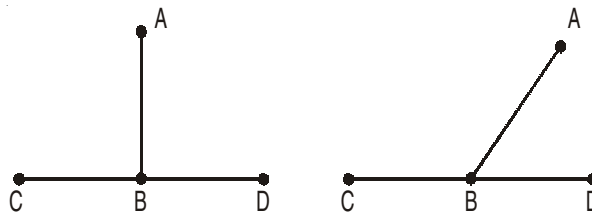


**Fig. 3.12(f)**



**Fig. 3.12(g)**

**Note:** When three members are meeting at an unloaded joint and out of them two are collinear, then the force in third member will be zero. Such situations are illustrated in Fig. 3.12(h) and (i).



**Fig. 3.12(h) and (i)**



**Example 3.6.** Find the forces in all the members of the truss shown in Fig. 3.13(a).

**Solution:**

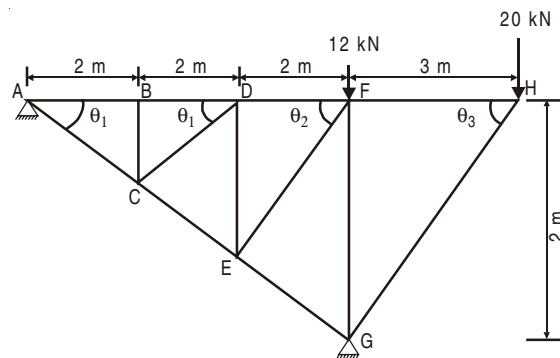
$$\tan \theta_1 = \frac{4}{6} = 33.69^\circ$$

$$\theta_2 = \tan^{-1} \left( \frac{8}{3} \times \frac{1}{2} \right) = 53.13^\circ$$

$$\theta_3 = \tan^{-1} \left( \frac{4}{3} \right) = 53.13^\circ$$

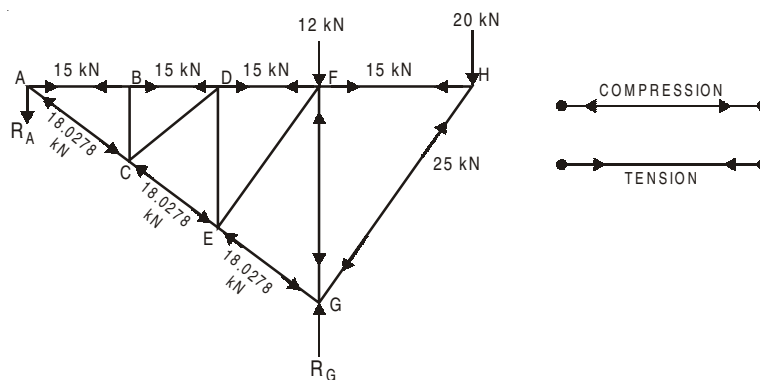
$$\theta_2 = \theta_3 = \theta$$

$$\sin \theta = \frac{4}{5} \text{ and } \cos \theta = \frac{3}{5}$$



**Fig. 3.13(a)**

Joint-by-joint analysis is carried out as given below and the joint forces are marked in Fig. 3.13(b). Then nature of the force in the members is determined.



**Fig. 3.13(b)**

Joint  $H$ :

$$\sum V = 0, \text{ gives}$$

$$F_{HG} \sin \theta_3 = 20$$

$$\therefore F_{HG} = 20 \times \frac{5}{4} = \mathbf{25 \text{ kN (Comp.)}}$$

$$\Sigma H = 0, \text{ gives}$$

$$F_{HF} - F_{HG} \cos \theta_2 = 0$$

$$F_{HF} = 25 \times \frac{3}{5} = \mathbf{15 \text{ kN (Tension)}}$$

$$\text{Now } \Sigma M_G = 0, \text{ gives}$$

$$R_A \times 6 - 20 \times 3 = 0$$

$$R_A = 10 \text{ kN (Downward)}$$

$$\Sigma V = 0, \text{ gives}$$

$$R_G - 10 - 12 - 20 = 0$$

$$R_G = 42 \text{ kN}$$

$$\text{Joint A: } \Sigma V = 0, \text{ gives}$$

$$F_{AC} \sin \theta_1 - 10 = 0$$

$$F_{AC} = \mathbf{18.0278 \text{ kN (Comp.)}}$$

$$\Sigma H = 0, \text{ gives}$$

$$F_{AB} - F_{AC} \cos \theta_1 = 0$$

$$F_{AB} = \mathbf{15 \text{ kN (Tension)}}$$

$$\text{Joint B: } \Sigma V = 0, \text{ gives}$$

$$F_{BC} = 0$$

$$\Sigma H = 0, \text{ gives}$$

$$F_{BD} = F_{BA} = \mathbf{15 \text{ kN (Tension)}}$$

$$\text{Joint C: } \Sigma \text{ Forces normal to AC} = 0, \text{ gives}$$

$$F_{CD} = \mathbf{0} \text{ since } F_{BC} = 0$$

$$\Sigma \text{ Forces parallel to CE} = 0$$

$$F_{CE} = F_{CA} = \mathbf{18.0278 \text{ (Comp.)}}$$

$$\text{Joint D: } \Sigma V = 0, \text{ gives}$$

$$F_{DE} = \mathbf{0}$$

$$\Sigma H = 0, \text{ gives}$$

$$F_{DF} = F_{DB} = \mathbf{15 \text{ kN (Tension)}}$$

$$\text{Joint E: } \Sigma \text{ Forces normal to CG} = 0, \text{ gives}$$

$$F_{EF} = \mathbf{0} \text{ and}$$

$$\Sigma \text{ Forces in the direction of CG} = 0, \text{ gives}$$

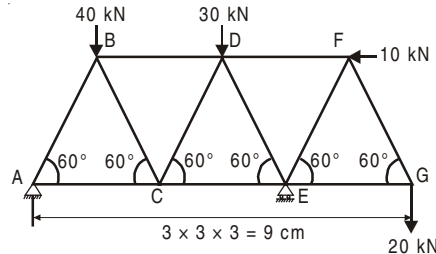
$$F_{EG} = F_{CE} = \mathbf{18.0278 \text{ kN (Comp.)}}$$

$$\text{Joint F: } \Sigma V = 0, \text{ gives}$$

$$F_{FG} - 12 = 0$$

$$F_{FG} = \mathbf{12 \text{ kN (Tension)}}$$

**Example 3.7.** Analyse the truss shown in Fig. 3.14(a). All the members are of 3 m length.



**Fig. 3.14(a)**

**Solution:** Since all members are 3 m long, all triangles are equilateral and hence all inclined members are at  $60^\circ$  to horizontal. Joint-by-joint analysis is carried out and the forces are represented in Fig. 3.14(b). Then nature of the force is determined.

Joint G:  $\sum V = 0$ , gives

$$F_{GF} \sin 60^\circ = 20$$

$$F_{GF} = 23.0940 \text{ kN (Tension)}$$

$\sum H = 0$ , gives

$$F_{GE} - F_{GF} \cos 60^\circ = 0$$

$$F_{GE} = 11.5470 \text{ kN (Comp.)}$$

Joint F:  $\sum V = 0$ , gives

$$F_{FG} \sin 60^\circ - F_{GF} \sin 60^\circ = 0$$

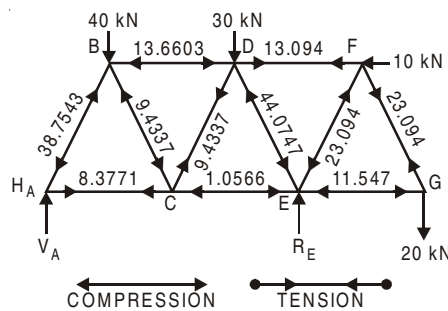
$$F_{FG} = F_{GF} = 23.0940 \text{ kN (Comp.)}$$

$\sum H = 0$ , gives

$$F_{FD} + 10 - F_{GF} \cos 60^\circ - F_{FE} \cos 60^\circ = 0$$

$$F_{FD} = 13.0940 \text{ kN (Tension)}$$

Now, without finding reaction we cannot proceed. Hence, consider equilibrium of the entire truss.



**Fig. 3.14(b)**

$$\sum M_A = 0$$

$$R_E \times 6 + 10 \times 3 \times \sin 60^\circ - 40 \times 1.5 - 30 \times 4.5 - 20 \times 9 = 0$$

$\therefore$

$$R_E = 58.1699 \text{ kN}$$

$$\sum V = 0, \text{ gives}$$

$$V_A = 40 + 30 + 20 - R_E = 31.8301 \text{ kN}$$

$$\sum H = 0, \text{ gives}$$

$$H_A = 10 \text{ kN}$$

$$\text{Joint A: } \sum V = 0, \text{ gives}$$

$$F_{AB} \sin 60^\circ - 31.8301 = 0$$

$$F_{AB} = \mathbf{36.7543 \text{ kN (Comp.)}}$$

$$\sum H = 0, \text{ gives}$$

$$F_{AC} - F_{AB} \cos 60^\circ + 10 = 0$$

$$F_{AC} = 8.3771 \text{ kN (Tension)}$$

$$\text{Joint B: } \sum V = 0, \text{ gives}$$

$$F_{BC} \sin 60^\circ + F_{AB} \sin 60^\circ - 40 = 0$$

$$F_{BC} = \mathbf{9.4337 \text{ kN (Comp.)}}$$

$$\sum H = 0, \text{ gives}$$

$$F_{BD} + F_{BC} \cos 60^\circ - F_{BA} \cos 60^\circ = 0$$

$$F_{BD} = \mathbf{13.6603 \text{ kN (Comp.)}}$$

$$\text{Joint C: } \sum V = 0, \text{ gives}$$

$$F_{CD} \sin 60^\circ - F_{BC} \sin 60^\circ = 0$$

$$F_{CD} = F_{BC} = \mathbf{9.4337 \text{ kN (Tension)}}$$

$$\sum H = 0, \text{ gives}$$

$$F_{CE} + F_{AC} - F_{CD} \cos 60^\circ - F_{BC} \cos 60^\circ = 0$$

$$F_{CE} = 2 \times 9.4337 \times \frac{1}{2} - 8.3771 = \mathbf{1.0566 \text{ kN (Comp.)}}$$

$$\text{Joint D: } \sum V = 0, \text{ gives}$$

$$F_{DE} \sin 60^\circ - F_{CD} \sin 60^\circ - 30 = 0$$

$$F_{DE} = \mathbf{44.0747 \text{ kN (Comp.)}}$$

### 3.6 METHOD OF SECTION

In the method of section, after determining the reactions, a section line is drawn passing through not more than three members in which forces are not known such that the frame is cut into two separate parts. Each part should be in equilibrium under the action of loads, reactions and forces in the members that are cut by the section line. Equilibrium of any one of these two parts is considered and the unknown forces in the members cut by the section line are determined. The system of forces acting on either part of truss constitutes a non-concurrent force system. Since there are only three independent equation of equilibrium, there should be only three unknown forces. Hence, in this method it is an essential condition that the section line should pass through not more than three members in which forces are not known and it should separate the frame into two parts.

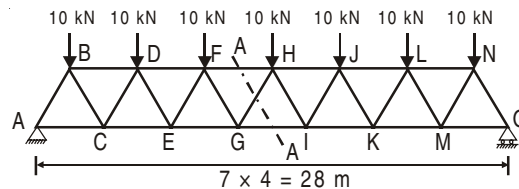
Thus, the method of section is the application of nonconcurrent force system analysis whereas the method of joints, described in previous article was the application of analysis of concurrent force system.

Under the following two conditions the method of section is preferred over the method of joints:

- (1) In a large truss in which forces in only few members are required;
- (2) In the situation where the method of joints fails to start/proceed with analysis.

The method of section is illustrated with the examples 3.8 to 3.11. Examples 3.8 and 3.9 are the cases in which method of section is advantageous since forces in only few members are required. Examples 3.10 and 3.11 are the cases in which method of joints fails to start/proceed to get the solution. In practice the frames may be analysed partly by method of section and partly by method of joints as illustrated in example 3.11.

**Example 3.8.** Determine the forces in the members  $FH$ ,  $HG$  and  $GI$  in the truss shown in Fig. 3.15(a). Each load is 10 kN and all triangles are equilateral with sides 4 m.

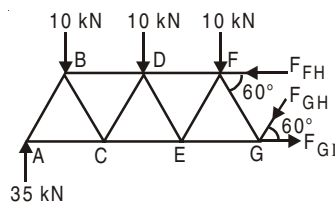


**Fig. 3.15(a)**

**Solution:** Due to symmetry,

$$R_A = R_O = \frac{1}{2} \times 10 \times 7 = 35 \text{ kN}$$

Take section (A)–(A), which cuts the members  $FH$ ,  $GH$  and  $GI$  and separates the truss into two parts. Consider the equilibrium of left hand side part as shown in Fig. 3.15(b) (Prefer part in which number of forces are less).



**Fig. 3.15(b)**

$$\Sigma M_G = 0, \text{ gives}$$

$$F_{FH} \times 4 \sin 60^\circ - 35 \times 12 + 10 \times 10 + 10 \times 6 + 10 \times 2 = 0$$

$$F_{FH} = 69.2820 \text{ kN (Comp.)}$$

$$\Sigma V = 0, \text{ gives}$$

$$F_{GH} \sin 60^\circ + 10 + 10 + 10 - 35 = 0$$

$$F_{GH} = 5.7735 \text{ kN (Comp.)}$$

$$\Sigma H = 0, \text{ gives}$$

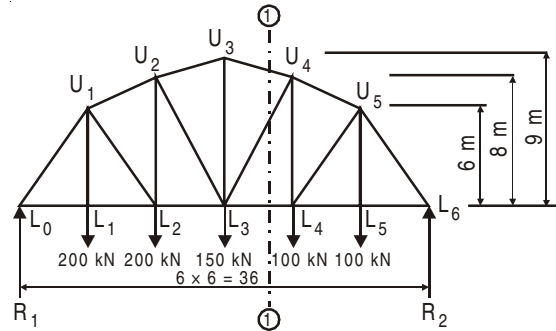
$$F_{GI} - F_{FH} - F_{GH} \cos 60^\circ = 0$$

$$F_{GI} = 69.2820 + 5.7735 \cos 60^\circ$$

$$= 72.1688 \text{ kN (Tension)}$$

**Example 3.9.** Find the magnitude and nature of the forces in the members  $U_3U_4$ ,  $L_3L_4$  and  $U_4L_3$  of the loaded truss shown in Fig. 3.16(a).

**Solution:** To determine reactions, consider equilibrium equations.



**Fig. 3.16(a)**

Now,  $\sum M_{L0} = 0$ , gives

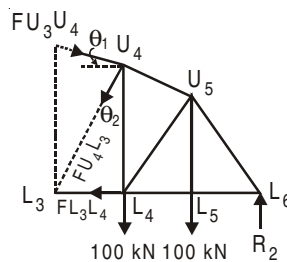
$$R_2 \times 36 - 200 \times 6 - 200 \times 12 - 150 \times 18 - 100 \times 24 - 100 \times 30 = 0$$

$$R_2 = 325 \text{ kN}$$

$\sum V = 0$ , gives

$$R_1 = 200 + 200 + 150 + 100 + 100 - 325 = 425 \text{ kN}$$

Take the section (I)-(I) and consider the right hand side part.



**Fig. 3.16(b)**

Now,  $U_3U_4 = \sqrt{1^2 + 6^2} = 6.0828$

$$\sin \theta_1 = \frac{1}{6.0828} = 0.1644$$

$$\cos \theta_1 = \frac{6}{6.0828} = 0.9864$$

$$L_3U_4 = \sqrt{6^2 + 8^2} = 10$$

$$\sin \theta_2 = 0.6 \quad \cos \theta_2 = 0.8$$

$$\Sigma M_{U_4} = 0, \text{ gives}$$

$$F_{L_3L_4} \times 8 - 325 \times 12 + 100 \times 6 = 0$$

$$F_{L_3L_4} = 412.5 \text{ kN (Tension)}$$

$$\Sigma M_{L_3} = 0, \text{ gives}$$

$$F_{U_4U_3} \times \cos \theta_1 \times 9 + 100 \times 6 + 100 \times 12 - 325 \times 18 = 0$$

$$F_{U_4U_3} = 456.2072 \text{ kN (Comp.)}$$

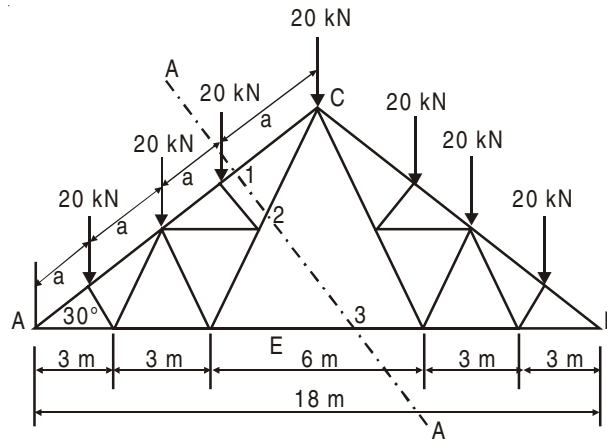
$$\Sigma H = 0$$

$$F_{U_4U_3} \sin \theta_2 - F_{U_4U_3} \cos \theta_1 + F_{L_4L_3} = 0$$

$$F_{U_4U_3} = \frac{456.2072 \times 0.9864 - 412.5}{0.6}$$

$$= 62.5 \text{ kN (Tension)}$$

**Example 3.10.** Find the forces in the members (1), (2) and (3) of French truss shown in Fig. 3.17(a).



**Fig. 3.17(a)**

**Solution:** Due to symmetry

$$R_A = R_B = \frac{1}{2} \times 20 \times 7 = 70 \text{ kN}$$

Now

$$AC = 4 \times a = \frac{9}{\cos 30^\circ}$$

$\therefore$

$$a = 2.5981 \text{ m.}$$

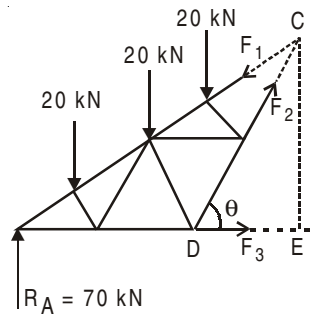


Fig. 3.17(b)

Take Section (A)–(A) and consider the equilibrium of left hand side part of the French Truss shown in Fig. 3.17(b).

Drop perpendicular  $CE$  on  $AB$ .

Now  $CE = 9 \tan 30^\circ$  and  $DE = 3$  m

$$\therefore \tan \theta = \frac{9 \tan 30^\circ}{3} = 3 \times \frac{1}{\sqrt{3}} = \sqrt{3}$$

$$\therefore \theta = 60^\circ$$

$\sum M_A = 0$ , gives

$$F_2 \sin 60^\circ \times 6 - 20 \times 2.5981 \cos 30^\circ - 20 \times 2 \times 2.5981 \cos 30^\circ - 20 \times 3 \times 2.5981 \cos 30^\circ = 0$$

$$F_2 = 20 \times 2.5981 \frac{(1 + 2 + 3)}{6} \quad \text{since } \sin 60^\circ = \cos 30^\circ$$

$$F_2 = 51.9615 \text{ kN (Tension)}$$

$\sum V = 0$ , gives

$$F_1 \sin 30^\circ - 70 + 20 + 20 + 20 - 51.9615 \sin 60^\circ = 0$$

$$F_1 = 110 \text{ kN (Comp.)}$$

$\sum H = 0$ , gives

$$F_3 + F_2 \cos 60^\circ - F_1 \cos 30^\circ = 0$$

$$F_3 = 69.2820 \text{ (Tension)}$$

**Note:** In this problem, the method of joints cannot give complete solution.

**Example 3.11.** Find the forces in all the members of the truss shown in Fig. 3.18(a).

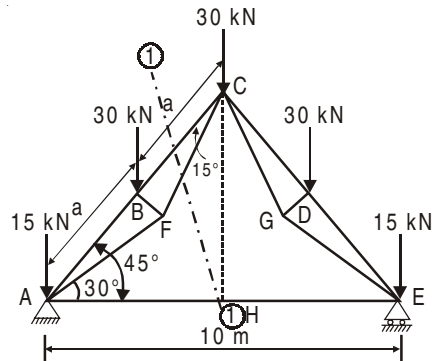


Fig. 3.18(a)



**Solution:** Due to symmetry, the reactions are equal.

$$R_A = R_E = \frac{1}{2} \times \text{Total load}$$

$$= \frac{1}{2} (15 + 30 + 30 + 30 + 15) = 60 \text{ kN}$$

Drop perpendicular  $CH$  on  $AF$ .

In  $\triangle ACH$ ,  $\angle ACH = 45^\circ$

$\therefore FC$  is inclined at  $30^\circ$  to vertical *i.e.*,  $60^\circ$  to horizontal and  $CH = 5 \text{ m}$

It is not possible to find a joint where there are only two unknowns. Hence, consider section (1)–(1).

For left hand side part of the frame:

$$\sum M_C = 0, \text{ gives}$$

$$F_{AE} \times 5 - 60 \times 5 + 15 \times 5 + 30 \times 2.5 = 0$$

$$\therefore F_{AE} = 30 \text{ kN (Tension)}$$

Assuming the directions for  $F_{FC}$  and  $F_{BC}$  as shown in Fig. 3.18(b)

$$\sum V = 0, \text{ gives}$$

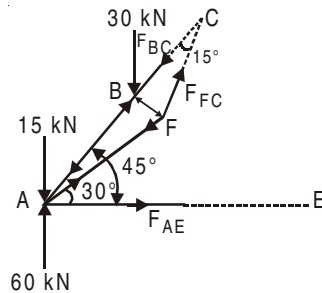
$$F_{FC} \sin 60^\circ - F_{BC} \sin 45^\circ + 60 - 15 - 30 = 0 \quad \dots(i)$$

$$\sum H = 0, \text{ gives}$$

$$F_{FC} \cos 60^\circ - F_{BC} \cos 45^\circ + F_{AE} = 0$$

$$F_{FC} \times \frac{1}{2} - F_{BC} \frac{1}{\sqrt{2}} + 30 = 0$$

$$\therefore F_{FC} = F_{BC} \sqrt{2} - 60 \quad \dots(ii)$$



**Fig. 3.18(b)**

Substituting this value of  $F_{FC}$  in (1), we get

$$(F_{BC} \sqrt{2} - 60) \sin 60^\circ - F_{BC} \frac{1}{\sqrt{2}} + 15 = 0$$

$$F_{BC} \left( \sqrt{2} \sin 60^\circ - \frac{1}{\sqrt{2}} \right) = 60 \sin 60^\circ - 15$$

$$F_{BC} = 71.40 \text{ kN (Comp.)}$$

$$\therefore F_{FC} = 71.4042 \times \sqrt{2} - 60$$

$$= \mathbf{40.98 \text{ kN (Tension)}}$$

Assumed directions of  $F_{BC}$  and  $F_{FC}$  are correct.

Therefore,  $F_{BC}$  is in compression and  $F_{FC}$  is in tension.

Now we can proceed with method of joints to find the forces in other members. Since it is a symmetric truss, analysis of half the truss is sufficient. Other values may be written down by making use of symmetry.

Joint B:

$\sum$  forces normal to AC = 0, gives

$$F_{BF} - 30 \cos 45^\circ = 0$$

$$F_{BF} = 21.21 \text{ kN (Comp.)}$$

$\sum$  forces parallel to AC = 0, gives

$$F_{AB} - F_{BC} - 30 \sin 45^\circ = 0$$

$$F_{AB} = 71.40 + 21.21$$

$$= \mathbf{92.61 \text{ kN (Comp)}}$$

Joint A:

$\sum V = 0$ , gives

$$F_{AF} \sin 30^\circ - F_{AB} \sin 45^\circ - 15 + 60 = 0$$

$$F_{AF} = \mathbf{40.98 \text{ kN (Tension)}}$$

The results are tabulated below:

Members	Force in kN
AB and ED	- 92.61
BC and DC	- 71.40
BF and DG	- 21.21
AF and EG	+ 40.98
FC and GC	+ 40.98
AE	+ 30.00

+ means tension and - means compression

### IMPORTANT FORMULA

1. In a perfect truss

$$m = 2j - 3$$

where  $m$  = number of members

$j$  = number of joints.

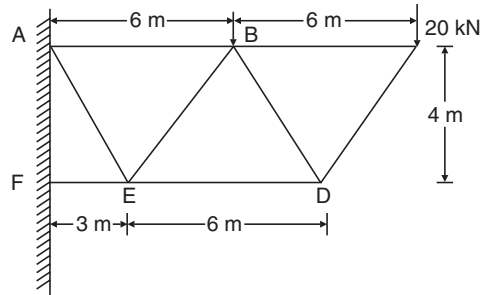
### THEORY QUESTIONS

1. Bring out the differences among perfect, deficient and redundant trusses.
2. State the assumptions made in the analysis of pin jointed trusses.
3. How method of joint differs from the method of section in the analysis of pin jointed trusses?

### PROBLEMS FOR EXERCISE

1 to 17: Determine the forces in all the members of the trusses shown in Fig. 3.19 to 3.35. Indicate the nature of forces using the convention tension as +ve and compression as -ve.

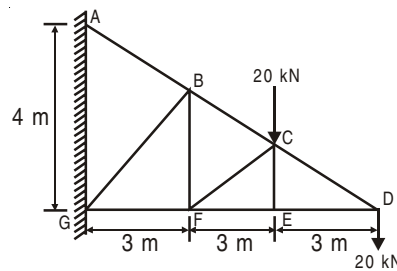
1.



**Fig. 3.19 (Prob. 1)**

[Ans.  $F_{AB} = +67.5$  kN;  $F_{BC} = +15$  kN;  $F_{CD} = -25$  kN;  $F_{DE} = -30$  kN;  
 $F_{EF} = -105$  kN;  $F_{AE} = +62.5$  kN;  $F_{BE} = -62.5$  kN;  $F_{BD} = +25$  kN]

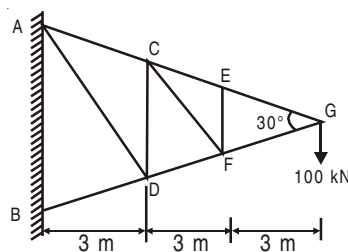
2.



**Fig. 3.20 (Prob. 2)**

[Ans.  $F_{AB} = +82.0738$  kN;  $F_{BC} = +73.866$  kN;  $F_{CD} = 49.2443$  kN;  $F_{DE} = -45$  kN;  
 $F_{EF} = -45$  kN;  $F_{FG} = -67.5$  kN;  $F_{BG} = -10.0$  kN;  $F_{FC} = +24.622$  kN;  $F_{CE} = 0$ ;  $F_{BF} = 10$  kN]

3.



**Fig. 3.21 (Prob. 3)**

[Ans.  $F_{AC} = F_{CE} = F_{EG} = +193.1852$  kN;  $F_{BD} = F_{DE} = F_{FG} = -193.1852$  kN; all others are zero members]

4.

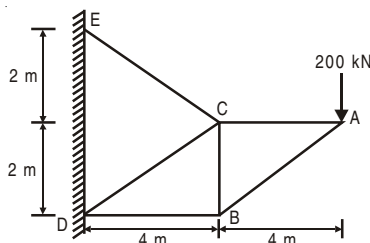


Fig. 3.22 (Prob. 4)

[Ans.  $F_{EC} = +447.2136$  kN;  $F_{CA} = +400$  kN;  $F_{AB} = -447.2136$  kN;  $F_{BD} = -400$  kN;  $F_{CD} = 0$ ;  $F_{CB} = -200$  kN]

5. [Ans.  $F_{DB} = F_{BA} = +5.7735$  kN;  $F_{BC} = F_{DE} = -5.7738$  kN;  $F_{AC} = -2.8868$  kN;  $F_{CE} = -14.4338$  kN;  $F_{DC} = +17.3205$  kN;  $F_{DF} = +20.0$  kN]

6. [Ans.  $F_{AB} = -30$  kN;  $F_{AC} = -160$  kN;  $F_{BC} = +50$  kN;  $F_{BD} = -200$  kN;  $F_{CD} = -50$  kN;  $F_{CE} = -120$  kN;  $F_{DF} = -266.67$  kN;  $F_{DE} = +83.33$  kN]

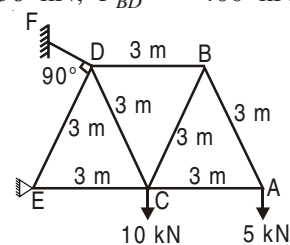


Fig. 3.23 (Prob. 5)

7. [Ans.  $F_{AB} = -200$  kN;  $F_{AD} = -100$  kN;  $F_{BC} = F_{CE} = 0$ ;  $F_{BD} = 100\sqrt{2}$  kN;  $F_{BF} = -100\sqrt{2}$  kN;  $F_{DE} = -100$  kN;  $F_{DG} = 0$ ;  $F_{EF} = +100$  kN;  $F_{EH} = -100\sqrt{2}$  kN;  $F_{EG} = +100\sqrt{2}$  kN;  $F_{GH} = +100$  kN]

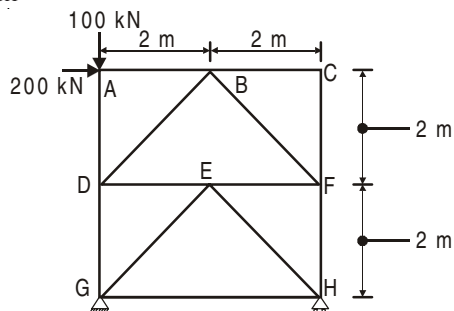


Fig. 3.25 (Prob. 7)

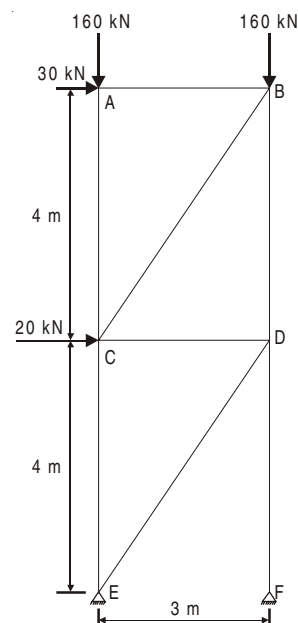


Fig. 3.24 (Prob. 6)

8. [Ans.  $F_{BD} = -2\sqrt{2}$  kN;  $F_{BA} = +3$  kN;  $F_{AC} = +3\sqrt{2}$  kN;  $F_{AD} = -3$  kN;  $F_{DC} = -2$  kN;  $F_{DF} = -5$  kN;  $F_{CF} = -\sqrt{2}$  kN;  $F_{CE} = +6$  kN;  $F_{FE} = +1$  kN;  $F_{FH} = -4$  kN;  $F_{EH} = -\sqrt{2}$  kN;  $F_{EG} = +5$  kN;  $F_{GH} = +1$  kN]
9. [Ans.  $F_{AC} = -100\sqrt{5}$  kN;  $F_{AB} = +200$  kN;  $F_{BD} = +200$  kN;  $F_{BC} = -100$  kN;  $F_{CD} = +50\sqrt{5}$  kN;  $F_{CE} = -150\sqrt{5}$  kN;  $F_{DE} = +35.0$  kN;  $F_{DF} = 300\sqrt{2}$  kN;  $F_{EF} = -300$  kN]
10. [Ans.  $F_{AB} = +5\sqrt{2}$  kN;  $F_{AC} = -5$  kN;  $F_{BC} = -5$  kN;  $F_{BD} = +5$  kN;  $F_{CD} = +15\sqrt{2}$  kN;  $F_{CE} = -20$  kN;  $F_{DE} = -15$  kN;  $F_{DF} = +20\sqrt{2}$  kN;  $F_{EH} = -15$  kN;  $F_{EF} = -20$  kN;  $F_{FG} = +30\sqrt{2}$  kN;  $F_{FH} = +10\sqrt{2}$  kN]

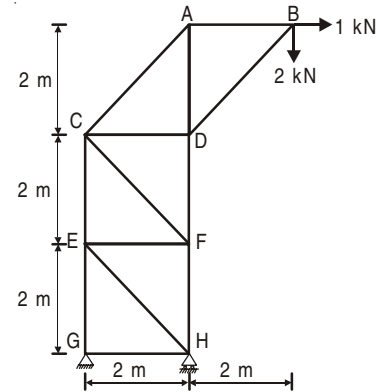


Fig. 3.26 (Prob. 8)

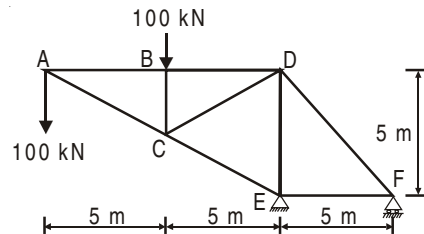


Fig. 3.27 (Prob. 9)

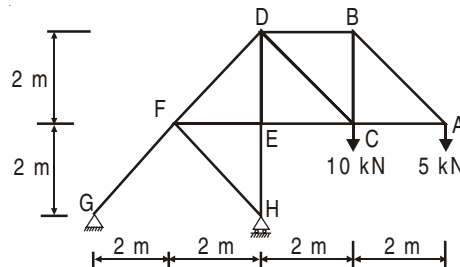


Fig. 3.28 (Prob. 10)

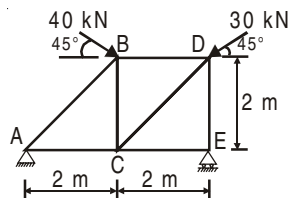
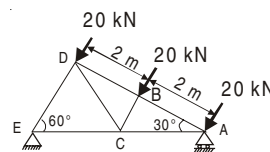
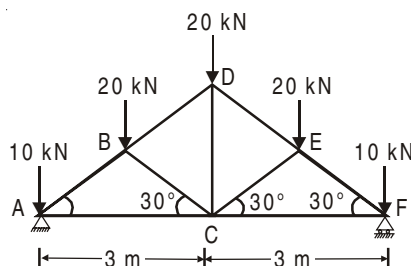


Fig. 3.29 (Prob. 11)

11. [Ans.  $F_{AB} = -15$  kN;  $F_{AC} = +12\sqrt{2}$  kN;  $F_{BD} = -27.5\sqrt{2}$  kN;  $F_{BC} = -12.5\sqrt{2}$  kN;  $F_{CE} = 0$ ;  $F_{CD} = +25$  kN;  $F_{ED} = -27.5\sqrt{2}$  kN]
12. [Ans.  $F_{AB} = -17.32$  kN;  $F_{AC} = +5$  kN;  $F_{BC} = -20$  kN;  $F_{BD} = -17.32$  kN;  $F_{CD} = +20$  kN;  $F_{CE} = -15$  kN;  $F_{DE} = -30$  kN]
13. [Ans.  $F_{AB} = 60$  kN;  $F_{AC} = +51.96$  kN;  $F_{BC} = -20$  kN;  $F_{BD} = -40$  kN;  $F_{CD} = +40$  kN; symmetry]
- 
- Fig. 3.30 (Prob. 12)**

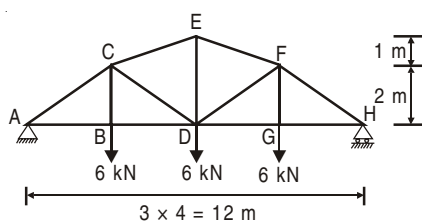


**Fig. 3.30 (Prob. 12)**



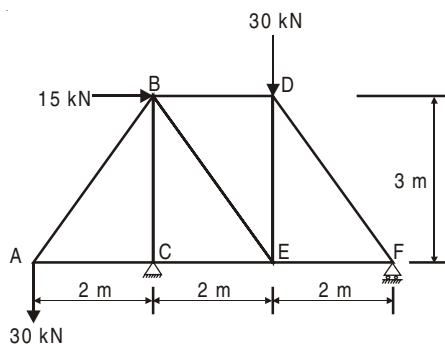
**Fig. 3.31 (Prob. 13)**

- 14. [Ans.  $F_{AC} = -4.5\sqrt{13}$  kN;  $F_{AB} = +13.5$  kN;  $F_{BC} = +6$  kN;  $F_{BD} = +13.5$  kN;  $F_{CD} = -0.5\sqrt{13}$  kN;  $F_{CE} = -4\sqrt{10}$  kN;  $F_{DE} = 8$  kN]**



**Fig. 3.32 (Prob. 14)**

- 15.** [Ans.  $F_{AB} = +10\sqrt{13}$  kN;  $F_{AC} = -20$  kN;  $F_{CB} = -48.75$  kN;  $F_{CE} = -20$  kN;  $F_{CD} = -7.5$  kN;  $F_{BE} = +6.25\sqrt{13}$  kN;  $F_{DE} = 18.75$  kN;  $F_{DF} = -3.75\sqrt{13}$  kN;  $F_{FE} = -7.5$  kN]



**Fig. 3.33 (Prob. 15)**

16. [Ans.  $F_{AB} = 16.91$  kN;  $F_{AF} = +31.55$  kN;  $F_{BF} = +23.91$  kN;  $F_{BD} = -23.91$  kN;  $F_{BC} = +40$  kN;  $F_{CD} = -40$  kN;  $F_{DE} = -63.1$  kN;  $F_{DF} = +23.91$  kN;  $F_{EF} = +31.55$  kN]

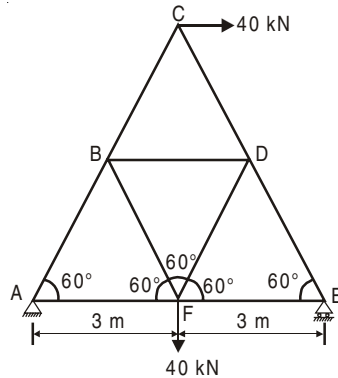


Fig. 3.34 (Prob. 16)

17. [Ans.  $F_{AC} = -67.48$  kN;  $F_{AB} = +53.99$  kN;  $F_{BC} = +10$  kN;  $F_{CD} = -8.33$  kN;  $F_{CE} = -59.15$  kN;  $F_{EF} = -24.5$  kN;  $F_{ED} = +52.81$  kN;  $F_{FD} = +47.21$  kN;  $F_{FG} = -34.64$  kN;  $F_{DG} = +47.32$  kN]

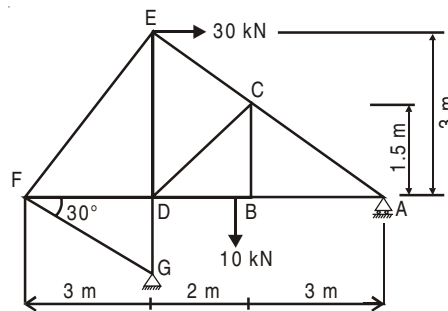


Fig. 3.35 (Prob. 17)

18. Find the force in the member  $FG$  of the triangular Howe truss shown in Fig. 3.36.

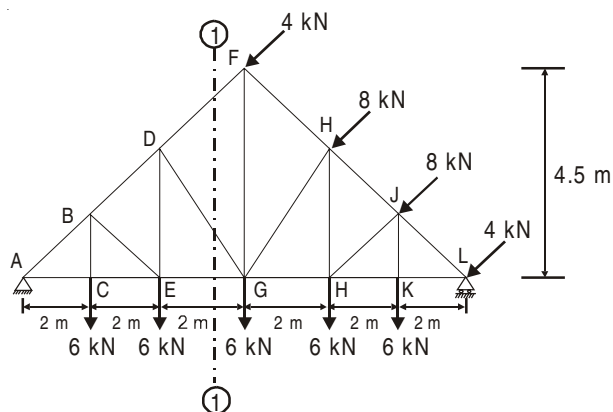


Fig. 3.36

(Hint: Take section (1)–(1) and find force in  $FD$ . Then analyse joint  $F$ ) [Ans. +28 kN]

19. Determine the forces in the members  $AB$ ,  $AC$ ,  $DF$  and  $CE$  of the scissors truss shown in Fig. 3.37.

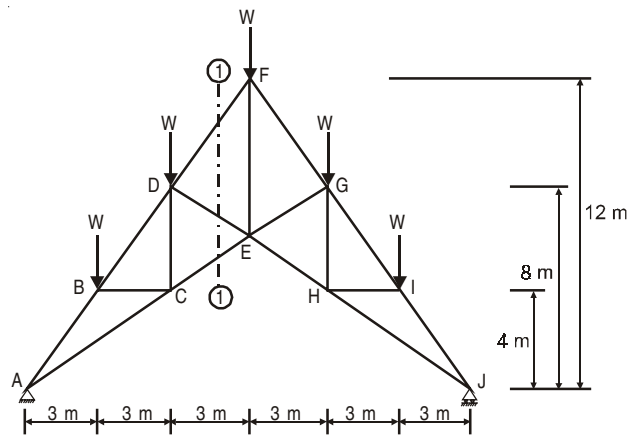


Fig. 3.37

(Hint: Find reaction  $R_A$  and analyse joint A. Take section (1)–(1) and find force in  $DF$  and  $CE$ )

[Ans.  $F_{AB} = -6.25 W$ ;  $F_{AC} = 4.51 W$ ;  $F_{DF} = -3.75 W$ ;  $F_{CE} = +2.75 W$ ]

20. Find the force in member  $KL$  of the French truss shown in Fig. 3.38.

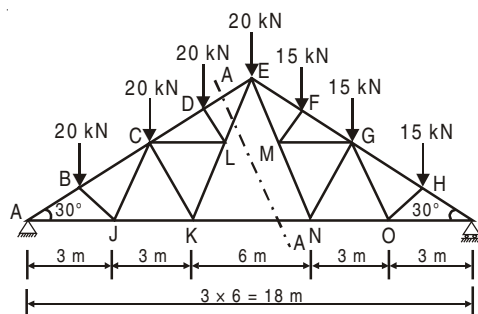


Fig. 3.38

(Hint: Take section (A)–(A) and find  $F_{LE}$  and  $F_{DE}$ . From joint  $D$  find  $F_{DL}$ . Then analyse joint  $L$  to get  $F_{KL}$ )

[Ans. +41.96 kN]



# 4

## Distributed Forces, Centre of Gravity and Moment of Inertia

The term 'distributed force' has been explained in Art 2.11 and this applied force has been classified into linear surface and the body force. The number of such forces acting on a body is infinite. However, these forces can be replaced by their resultant which acts through a point, known as the centre of gravity of the body. In this chapter the method of finding areas of given figures and volumes is explained. Then the terms centroid and centre of gravity and second moment of area (moment of inertia of areas) are explained and method of finding them is illustrated with examples. Theorem of Pappus-Guldinus is introduced which is very useful for finding surface areas and volumes of solids then the method of finding centre of gravity and moment of inertia (mass moment of inertia) of solids is illustrated.

### 4.1 DETERMINATION OF AREAS AND VOLUMES

In the school education methods of finding areas and volumes of simple cases are taught by many methods. Here we will see the general approach which is common to all cases *i.e.* by the method of integration. In this method the expression for an elemental area will be written then suitable integrations are carried out so as to take care of entire surface/volume. This method is illustrated with standard cases below, first for finding the areas and latter for finding the volumes:

#### A: Area of Standard Figures

##### (i) Area of a rectangle

Let the size of rectangle be  $b \times d$  as shown in Fig. 4.1.  $dA$  is an elemental area of side  $dx \times dy$ .

$$\begin{aligned} \text{Area of rectangle, } A &= \oint dA = \int_{-b/2}^{b/2} \int_{-d/2}^{d/2} dx dy \\ &= [x]_{-b/2}^{b/2} [y]_{-d/2}^{d/2} \\ &= bd. \end{aligned}$$

If we take element as shown in Fig. 4.2,

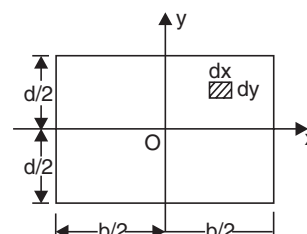


Fig. 4.1

$$\begin{aligned}
 A &= \int_{-d/2}^{d/2} dA = \int_{-d/2}^{d/2} b \cdot dy \\
 &= b [y]_{-d/2}^{d/2} \\
 &= bd
 \end{aligned}$$

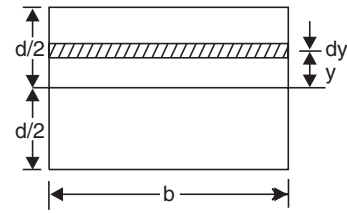


Fig. 4.2

- (ii) Area of a triangle of base width 'b' height 'h': Referring to Fig. 4.3, let the element be selected as shown by hatched lines

Then  $dA = b' dy = b \frac{y}{h} dy$

$$\begin{aligned}
 A &= \int_0^h dA = \int_0^h b \frac{y}{h} dy \\
 &= \frac{b}{h} \left[ \frac{y^2}{2} \right]_0^h = \frac{bh}{2}
 \end{aligned}$$

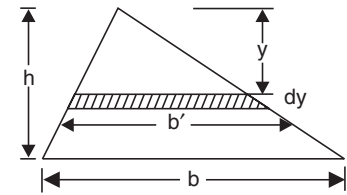


Fig. 4.3

- (iii) Area of a circle

Consider the elemental area  $dA = r d\theta dr$  as shown in Fig. 4.4. Now,

$$dA = r d\theta dr$$

$r$  varies from 0 to  $R$  and  $\theta$  varies from 0 to  $2\pi$

$$\begin{aligned}
 \therefore A &= \int_0^{2\pi} \int_0^R r d\theta dr \\
 &= \int_0^{2\pi} \left[ \frac{r^2}{2} \right]_0^R d\theta \\
 &= \int_0^{2\pi} \frac{R^2}{2} d\theta \\
 &= \frac{R^2}{2} [\theta]_0^{2\pi} \\
 &= \frac{R^2}{2} \cdot 2\pi = \pi R^2
 \end{aligned}$$

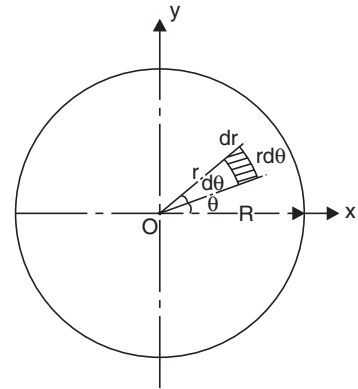


Fig. 4.4

In the above derivation, if we take variation of  $\theta$  from 0 to  $\pi$ , we get the area of semicircle as  $\frac{\pi R^2}{2}$  and if the limit is from 0 to  $\pi/2$  the area of quarter of a circle is obtained as  $\frac{\pi R^2}{4}$ .

## (iv) Area of a sector of a circle

Area of a sector of a circle with included angle  $2\alpha$  shown in Fig. 4.5 is to be determined. The elemental area is as shown in the figure

$$dA = r d\theta \cdot dr$$

$\theta$  varies from  $-\alpha$  to  $\alpha$  and  $r$  varies from 0 to  $R$

$$\begin{aligned} \therefore A &= \oint dA = \int_{-\alpha}^{\alpha} \int_0^R r \, d\theta \, dr \\ &= \int_{-\alpha}^{\alpha} \left[ \frac{r^2}{2} \right]_0^R d\theta = \int_{-\alpha}^{\alpha} \frac{R^2}{2} d\theta \\ &= \left[ \frac{R^2}{2} \theta \right]_{-\alpha}^{\alpha} = \frac{R^2}{2} (2\alpha) = R^2 \alpha. \end{aligned}$$

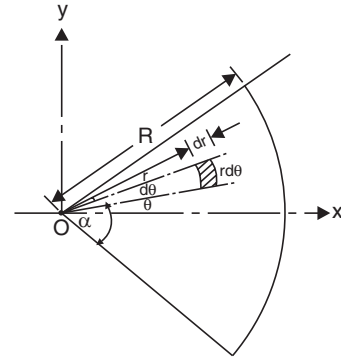


Fig. 4.5

## (v) Area of a parabolic spandrel

Two types of parabolic curves are possible

(a)  $y = kx^2$

(b)  $y^2 = kx$

Case a: This curve is shown in Fig. 4.6.

The area of the element

$$\begin{aligned} dA &= y \, dx \\ &= kx^2 \, dx \\ \therefore A &= \int_0^a dA = \int_0^a kx^2 \, dx \\ &= k \left[ \frac{x^3}{3} \right]_0^a = \frac{ka^3}{3} \end{aligned}$$

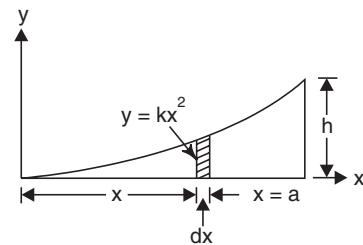


Fig. 4.6

We know, when  $x = a$ ,  $y = h$

$$\text{i.e., } h = ka^2 \text{ or } k = \frac{h}{a^2}$$

$$\therefore A = \frac{ka^3}{3} = \frac{h}{a^2} \frac{a^3}{3} = \frac{1}{3} ha = \frac{1}{3} \text{rd the area of rectangle of size } a \times h$$

Case b: In this case  $y^2 = kx$

Referring to Fig. 4.7

$$\begin{aligned} dA &= y \, dx = \sqrt{kx} \, dx \\ A &= \int_0^a y \, dx = \int_0^a \sqrt{kx} \, dx \end{aligned}$$

$$= \sqrt{k} \left[ x^{3/2} \frac{2}{3} \right]_0^a = \sqrt{k} \frac{2}{3} a^{3/2}$$

We know that, when  $x = a$ ,  $y = h$

$$\therefore h^2 = ka \quad \text{or} \quad k = \frac{h^2}{a}$$

$$\text{Hence } A = \frac{h}{\sqrt{a}} \cdot \frac{2}{3} \cdot a^{3/2}$$

$$\text{i.e., } A = \frac{2}{3} ha = \frac{2}{3} \text{rd the area of rectangle of size } a \times h.$$

(vi) Surface area of a cone

Consider the cone shown in Fig. 4.8. Now,

$$y = \frac{x}{h} R$$

Surface area of the element,

$$dA = 2\pi y \, dl = 2\pi \frac{x}{h} R \, dl$$

$$= 2\pi \frac{x}{h} R \frac{dx}{\sin \alpha}$$

$$\therefore A = \frac{2\pi R}{h \sin \alpha} \left[ \frac{x^2}{2} \right]_0^h$$

$$= \frac{\pi R h}{\sin \alpha} = \pi R l$$

(vii) Surface area of a sphere

Consider the sphere of radius  $R$  shown in Fig. 4.9. The element considered is the parallel circle at distance  $y$  from the diametral axis of sphere.

$$dS = 2\pi x \, R d\theta$$

$$= 2\pi R \cos \theta \, R d\theta, \text{ since } x = R \cos \theta$$

$$\therefore S = 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta$$

$$= 2\pi R^2 [\sin \theta]_{-\pi/2}^{\pi/2}$$

$$= 4\pi R^2$$

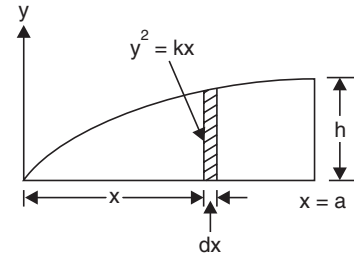


Fig. 4.7

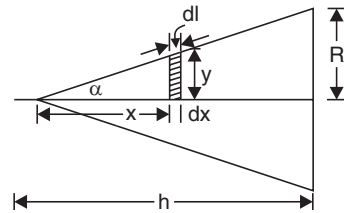


Fig. 4.8

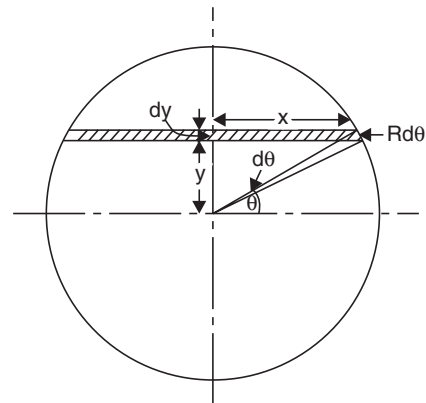


Fig. 4.9

**B: Volume of Standard Solids**

(i) Volume of a parallelopiped

Let the size of the parallelopiped be  $a \times b \times c$ . The volume of the element is

$$dV = dx \, dy \, dz$$

$$\begin{aligned} V &= \int_0^a \int_0^b \int_0^c dx \, dy \, dz \\ &= [x]_0^a [y]_0^b [z]_0^c = abc \end{aligned}$$

(ii) Volume of a cone

Referring to Fig. 4.8

$$dV = \pi y^2 \cdot dx = \pi \frac{x^2}{h^2} R^2 dx, \quad \text{since } y = \frac{x}{h} R$$

$$\begin{aligned} V &= \frac{\pi}{h^2} R^2 \int_0^h x^2 \, dx = \frac{\pi}{h^2} R^2 \left[ \frac{x^3}{3} \right]_0^h \\ &= \frac{\pi}{h^2} R^2 \frac{h^3}{3} = \frac{\pi R^2 h}{3} \end{aligned}$$

(iii) Volume of a sphere

Referring to Fig. 4.9

$$dV = \pi x^2 \cdot dy$$

$$\text{But } x^2 + y^2 = R^2$$

$$\text{i.e., } x^2 = R^2 - y^2$$

$$\therefore dV = \pi (R^2 - y^2) dy$$

$$V = \int_{-R}^R \pi (R^2 - y^2) dy$$

$$= \pi \left[ R^2 y - \frac{y^3}{3} \right]_{-R}^R$$

$$= \pi \left[ R^2 \cdot R - \frac{R^3}{3} - \left\{ -R^3 - \frac{(-R)^3}{3} \right\} \right]$$

$$= \pi R^3 \left[ 1 - \frac{1}{3} + 1 - \frac{1}{3} \right] = \frac{4}{3} \pi R^3$$

The surface areas and volumes of solids of revolutions like cone, spheres may be easily found using theorems of Pappus and Guldinus. This will be taken up latter in this chapter, since it needs the term centroid of generating lines.

## 4.2 CENTRE OF GRAVITY AND CENTROIDS

Consider the suspended body shown in Fig. 4.10(a). The self weight of various parts of this body are acting vertically downward. The only upward force is the force  $T$  in the string. To satisfy the equilibrium condition the resultant weight of the body,  $W$  must act along the line of string 1–1. Now, if the position is changed and the body is suspended again (Fig. 4.10(b)), it will reach equilibrium condition in a particular position. Let the line of action of the resultant weight be 2–2 intersecting 1–1 at  $G$ . It is obvious that if the body is suspended in any other position, the line of action of resultant weight  $W$  passes through  $G$ . This point is called the centre of gravity of the body. Thus *centre of gravity can be defined as the point through which the resultant of force of gravity of the body acts.*

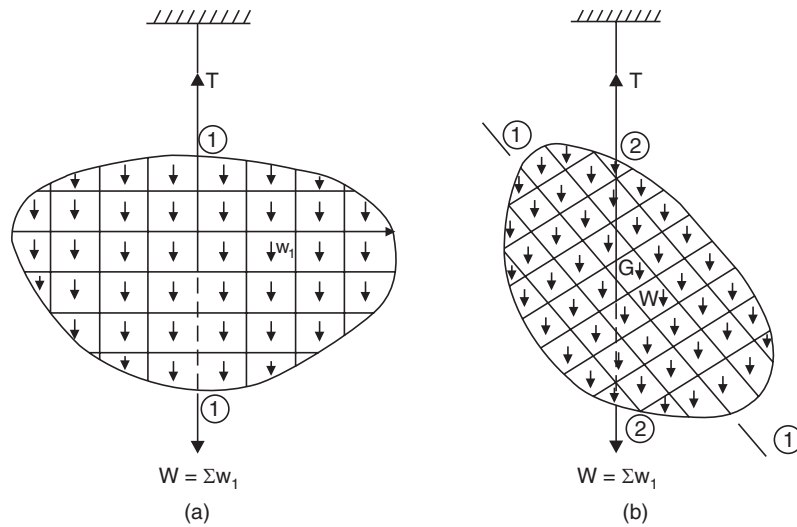


Fig. 4.10

The above method of locating centre of gravity is the practical method. If one desires to locate centre of gravity of a body analytically, it is to be noted that the resultant of weight of various portions of the body is to be determined. For this Varignon's theorem, which states the moment of resultant force is equal to the sum of moments of component forces, can be used.

Referring to Fig. 4.11, let  $W_i$  be the weight of an element in the given body.  $W$  be the total weight of the body. Let the coordinates of the element be  $x_i, y_i, z_i$  and that of centroid  $G$  be  $x_c, y_c, z_c$ . Since  $W$  is the resultant of  $W_i$  forces,

$$W = W_1 + W_2 + W_3 + \dots \\ = \Sigma W_i$$

$$\text{and} \quad Wx_c = W_1x_1 + W_2x_2 + W_3x_3 + \dots$$

$$\therefore Wx_c = \Sigma W_ix_i = \oint xdw$$

$$\text{Similarly,} \quad Wy_c = \Sigma W_iy_i = \oint ydw$$

$$\text{and} \quad Wz_c = \Sigma W_iz_i = \oint zdw$$

...(4.1)

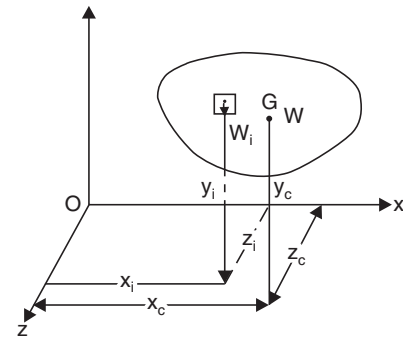


Fig. 4.11

If  $M$  is the mass of the body and  $m_i$  that of the element, then

$$M = \frac{W}{g} \quad \text{and} \quad m_i = \frac{W_i}{g}, \quad \text{hence we get}$$

$$\left. \begin{aligned} Mx_c &= \sum m_i x_i = \oint x_i dm \\ My_c &= \sum m_i y_i = \oint y_i dm \\ Mz_c &= \sum m_i z_i = \oint z_i dm \end{aligned} \right\} \quad \dots(4.2)$$

and

If the body is made up of uniform material of unit weight  $\gamma$  then we know  $W_i = V_i \gamma$ , where  $V$  represents volume, then equation 4.1 reduces to

$$\left. \begin{aligned} Vx_c &= \sum V_i x_i = \oint x dV \\ Vy_c &= \sum V_i y_i = \oint y dV \\ Vz_c &= \sum V_i z_i = \oint z dV \end{aligned} \right\} \quad \dots(4.3)$$

If the body is a flat plate of uniform thickness, in  $x$ - $y$  plane,  $W_i = \gamma A_i t$  (Ref. Fig. 4.12). Hence equation 4.1 reduces to

$$\left. \begin{aligned} Ax_c &= \sum A_i x_i = \oint x dA \\ Ay_c &= \sum A_i y_i = \oint y dA \end{aligned} \right\} \quad \dots(4.4)$$

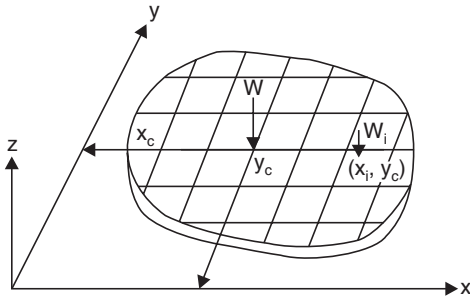


Fig. 4.12

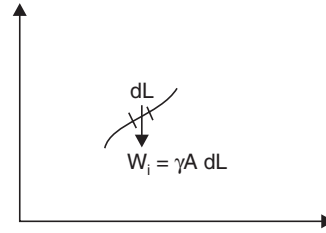


Fig. 4.13

If the body is a wire of uniform cross-section in plane  $x$ ,  $y$  (Ref. Fig. 4.13) the equation 4.1 reduces to

$$\left. \begin{aligned} Lx_c &= \sum L_i x_i = \oint x dL \\ Ly_c &= \sum L_i y_i = \oint y dL \end{aligned} \right\} \quad \dots(4.5)$$

The term centre of gravity is used only when the gravitational forces (weights) are considered. This term is applicable to solids. Equations 4.2 in which only masses are used the point obtained is termed as *centre of mass*. The central points obtained for volumes, surfaces and line segments (obtained by eqn. 4.3, 4.4 and 4.5) are termed as *centroids*.

### 4.3 CENTROID OF A LINE

Centroid of a line can be determined using equation 4.5. Method of finding the centroid of a line for some standard cases is illustrated below:

#### (i) Centroid of a straight line

Selecting the  $x$ -coordinate along the line (Fig. 4.14)

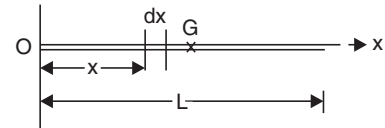


Fig. 4.14

$$Lx_c = \int_0^L x \, dx = \left[ \frac{x^2}{2} \right]_0^L = \frac{L^2}{2}$$

$$\therefore x_c = \frac{L}{2}$$

Thus the centroid lies at midpoint of a straight line, whatever be the orientation of line (Ref. Fig. 4.15).

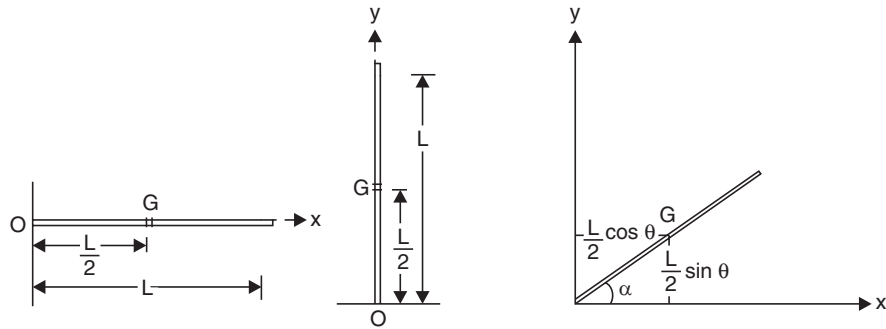


Fig. 4.15

(ii) *Centroid of an arc of a circle*

Referring to Fig. 4.16,

$$L = \text{Length of arc} = R 2\alpha$$

$$dL = R d\theta$$

Hence from eqn. 4.5

$$x_c L = \int_{-\alpha}^{\alpha} x dL$$

i.e.,

$$x_c R 2\alpha = \int_{-\alpha}^{\alpha} R \cos \theta \cdot R d\theta$$

$$= R^2 \left[ \sin \theta \right]_{-\alpha}^{\alpha}$$

$$\therefore x_c = \frac{R^2 \times 2 \sin \alpha}{2R\alpha} = \frac{R \sin \alpha}{\alpha}$$

and  $y_c L = \int_{-\alpha}^{\alpha} y dL = \int_{-\alpha}^{\alpha} R \sin \theta \cdot R d\theta$

$$= R^2 \left[ -\cos \theta \right]_{-\alpha}^{\alpha}$$

$$= 0$$

$$\therefore y_c = 0$$

From equation (i) and (ii) we can get the centroid of semicircle shown in Fig. 4.17 by putting  $\alpha = \pi/2$  and for quarter of a circle shown in Fig. 4.18 by putting  $\alpha$  varying from zero to  $\pi/2$ .

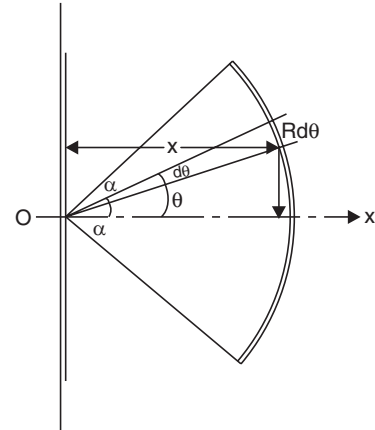


Fig. 4.16



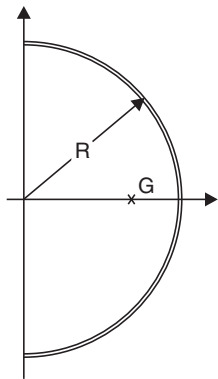


Fig. 4.17

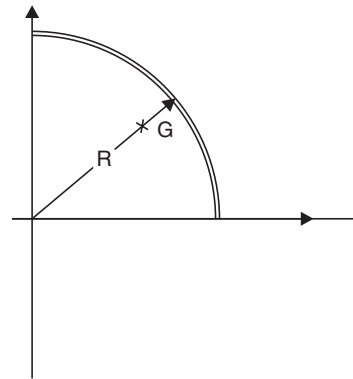


Fig. 4.18

For semicircle  $x_c = \frac{2R}{\pi}$

$$y_c = 0$$

For quarter of a circle,

$$x_c = \frac{2R}{\pi}$$

$$y_c = \frac{2R}{\pi}$$

(iii) **Centroid of composite line segments**

The results obtained for standard cases may be used for various segments and then the equations 4.5 in the form

$$x_c L = \sum L_i x_i$$

$$y_c L = \sum L_i y_i$$

may be used to get centroid  $x_c$  and  $y_c$ . If the line segments is in space the expression  $z_c L = \sum L_i z_i$  may also be used. The method is illustrated with few examples below:

**Example 4.1.** Determine the centroid of the wire shown in Fig. 4.19.

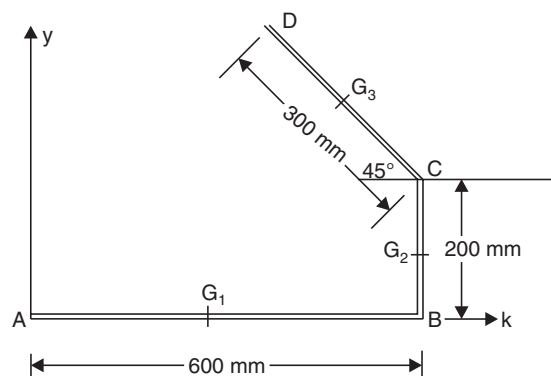


Fig. 4.19

**Solution:** The wire is divided into three segments  $AB$ ,  $BC$  and  $CD$ . Taking  $A$  as origin the coordinates of the centroids of  $AB$ ,  $BC$  and  $CD$  are

$$G_1(300, 0); G_2(600, 100) \text{ and } G_3(600 - 150 \cos 45^\circ, 200 + 150 \sin 45^\circ)$$

$$\text{i.e., } G_3(493.93, 306.07)$$

$$L_1 = 600 \text{ mm}, L_2 = 200 \text{ mm}, L_3 = 300 \text{ mm}$$

$$\therefore \text{ Total length } L = 600 + 200 + 300 = 1100 \text{ mm}$$

$\therefore$  From the eqn.  $Lx_c = \sum L_i x_i$ , we get

$$\begin{aligned} 1100 x_c &= L_1 x_1 + L_2 x_2 + L_3 x_3 \\ &= 600 \times 300 + 200 \times 600 + 300 \times 493.93 \end{aligned}$$

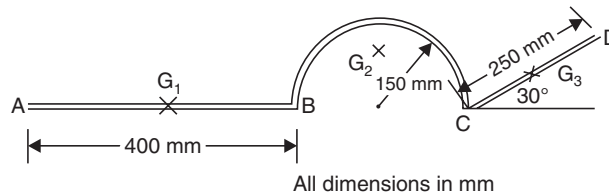
$$\therefore x_c = 407.44 \text{ mm}$$

$$\text{Now, } Ly_c = \sum L_i y_i$$

$$1100 y_c = 600 \times 0 + 200 \times 100 + 300 \times 306.07$$

$$\therefore y_c = 101.66 \text{ mm}$$

**Example 4.2.** Locate the centroid of the uniform wire bent as shown in Fig. 4.20.



**Fig. 4.20**

**Solution:** The composite figure is divided into three simple figures and taking  $A$  as origin coordinates of their centroids noted down as shown below:

$AB$ —a straight line

$$L_1 = 400 \text{ mm}, \quad G_1(200, 0)$$

$BC$ —a semicircle

$$L_2 = 150 \pi = 471.24, \quad G_2 \left( 475, \frac{2 \times 150}{\pi} \right)$$

$$\text{i.e. } G_2(475, 95.49)$$

$CD$ —a straight line

$$L_3 = 250; x_3 = 400 + 300 + \frac{250}{2} \cos 30^\circ = 808.25 \text{ mm}$$

$$y_3 = 125 \sin 30^\circ = 62.5 \text{ mm}$$

$$\therefore \text{ Total length } L = L_1 + L_2 + L_3 = 1121.24 \text{ mm}$$

$$\therefore Lx_c = \sum L_i x_i \quad \text{gives}$$

$$1121.24 x_c = 400 \times 200 + 471.24 \times 475 + 250 \times 808.25$$

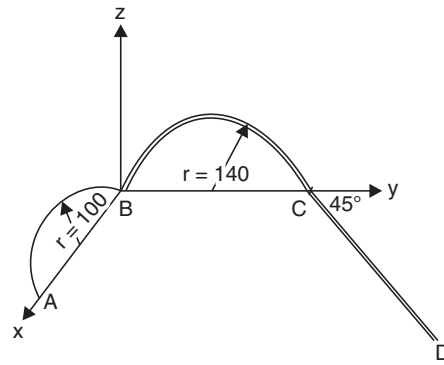
$$x_c = 451.20 \text{ mm}$$

$$Ly_c = \sum L_i y_i \quad \text{gives}$$

$$1121.24 y_c = 400 \times 0 + 471.24 \times 95.49 + 250 \times 62.5$$

$$y_c = 54.07 \text{ mm}$$

**Example 4.3.** Locate the centroid of uniform wire shown in Fig. 4.21. Note: portion AB is in x-z plane, BC in y-z plane and CD in x-y plane. AB and BC are semi circular in shape.



**Fig. 4.21**

**Solution:** The length and the centroid of portions AB, BC and CD are as shown in table below:

**Table 4.1**

Portion	$L_i$	$x_i$	$y_i$	$z_i$
AB	$100\pi$	100	0	$\frac{2 \times 100}{\pi}$
BC	$140\pi$	0	140	$\frac{2 \times 140}{\pi}$
CD	300	$300 \sin 45^\circ$	$280 + 300 \cos 45^\circ$ $= 492.13$	0

$$\therefore L = 100\pi + 140\pi + 300 = 1053.98 \text{ mm}$$

From eqn.  $Lx_c = \sum L_i x_i$ , we get

$$1053.98 x_c = 100\pi \times 100 + 140\pi \times 0 + 300 \times 300 \sin 45^\circ$$

$$x_c = 90.19 \text{ mm}$$

$$\text{Similarly, } 1053.98 y_c = 100\pi \times 0 + 140\pi \times 140 + 300 \times 492.13$$

$$y_c = 198.50 \text{ mm}$$

$$\text{and } 1053.98 z_c = 100\pi \times \frac{200}{\pi} + 140\pi \times \frac{2 \times 140}{\pi} + 300 \times 0$$

$$z_c = 56.17 \text{ mm}$$

#### 4.4 FIRST MOMENT OF AREA AND CENTROID

From equation 4.1, we have

$$x_c = \frac{\sum W_i x_i}{W}, \quad y_c = \frac{\sum W_i y_i}{W} \quad \text{and} \quad z_c = \frac{\sum W_i z_i}{W}$$

From the above equation we can make the statement that distance of centre of gravity of a body from an axis is obtained by dividing moment of the gravitational forces acting on the body, about the axis, by the total weight of the body. Similarly from equation 4.4, we have,

$$x_c = \frac{\Sigma A_i x_i}{A}, \quad y_c = \frac{\Sigma A_i y_i}{A}$$

By terming  $\Sigma A_i x_i$  as the moment of area about the axis, we can say centroid of plane area from any axis is equal to moment of area about the axis divided by the total area. The moment of area  $\Sigma A_i x_i$  is termed as first moment of area also just to differentiate this from the term  $\Sigma A_i x_i^2$ , which will be dealt latter. It may be noted that since the moment of area about an axis divided by total area gives the distance of the centroid from that axis, the moment of area is zero about any centroidal axis.

### Difference between Centre of Gravity and Centroid

From the above discussion we can draw the following differences between centre of gravity and centroid:

- (1) The term centre of gravity applies to bodies with weight, and centroid applies to lines, plane areas and volumes.
- (2) Centre of gravity of a body is a point through which the resultant gravitational force (weight) acts for any orientation of the body whereas centroid is a point in a line plane area volume such that the moment of area about any axis through that point is zero.

### Use of Axis of Symmetry

*Centroid of an area lies on the axis of symmetry if it exists.* This is useful theorem to locate the centroid of an area.

This theorem can be proved as follows:

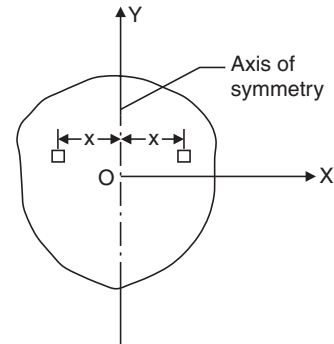
Consider the area shown in Fig. 4.22. In this figure y-y is the axis of symmetry. From eqn. 4.4, the distance of centroid from this axis is given by:

$$\frac{\Sigma A_i x_i}{A}$$

Consider the two elemental areas shown in Fig. 4.22, which are equal in size and are equidistant from the axis, but on either side. Now the sum of moments of these areas cancel each other since the areas and distances are the same, but signs of distances are opposite. Similarly, we can go on considering an area on one side of symmetric axis and corresponding image area on the other side, and prove that total moments of area ( $\Sigma A_i x_i$ ) about the symmetric axis is zero. Hence the distance of centroid from the symmetric axis is zero, i.e., centroid always lies on symmetric axis.

Making use of the symmetry we can conclude that:

- (1) Centroid of a circle is its centre (Fig. 4.23);
- (2) Centroid of a rectangle of sides  $b$  and  $d$  is at distance  $\frac{b}{2}$  and  $\frac{d}{2}$  from the corner as shown in Fig. 4.24.



**Fig. 4.22**

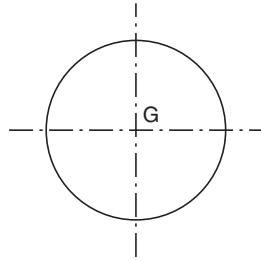


Fig. 4.23

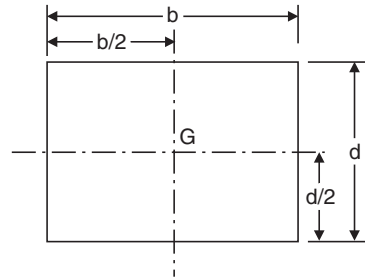


Fig. 4.24

### Determination of Centroid of Simple Figures From First Principle

For simple figures like triangle and semicircle, we can write general expression for the elemental area and its distance from an axis. Then equation 4.4 reduces to:

$$\bar{y} = \frac{\int y dA}{A}$$

$$\bar{x} = \frac{\int x dA}{A}$$

The location of the centroid using the above equations may be considered as finding centroid from first principle. Now, let us find centroid of some standard figures from first principle.

#### Centroid of a Triangle

Consider the triangle  $ABC$  of base width  $b$  and height  $h$  as shown in Fig. 4.25. Let us locate the distance of centroid from the base. Let  $b_1$  be the width of elemental strip of thickness  $dy$  at a distance  $y$  from the base. Since  $\triangle AEF$  and  $\triangle ABC$  are similar triangles, we can write:

$$\frac{b_1}{b} = \frac{h-y}{h}$$

$$b_1 = \left( \frac{h-y}{h} \right) b = \left( 1 - \frac{y}{h} \right) b$$

$$\begin{aligned} \therefore \text{Area of the element} &= dA = b_1 dy \\ &= \left( 1 - \frac{y}{h} \right) b dy \end{aligned}$$

$$\text{Area of the triangle} \quad A = \frac{1}{2} bh$$

$\therefore$  From eqn. 4.4

$$\bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\int y dA}{A}$$

$$\text{Now,} \quad \int y dA = \int_0^h y \left( 1 - \frac{y}{h} \right) b dy$$

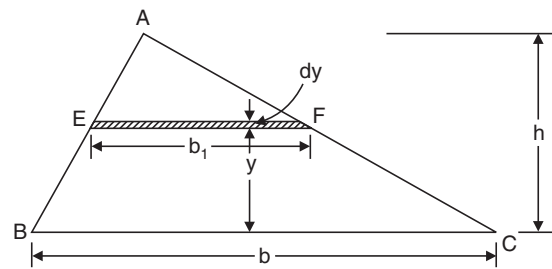


Fig. 4.25

$$\begin{aligned}
 &= \int_0^h \left( y - \frac{y^2}{h} \right) b \, dy \\
 &= b \left[ \frac{y^2}{2} - \frac{y^3}{3h} \right]_0^h \\
 &= \frac{bh^2}{6} \\
 \therefore \bar{y} &= \frac{\int y dA}{A} = \frac{bh^2}{6} \times \frac{1}{\frac{1}{2}bh} \\
 \therefore \bar{y} &= \frac{h}{3}
 \end{aligned}$$

Thus the centroid of a triangle is at a distance  $\frac{h}{3}$  from the base (or  $\frac{2h}{3}$  from the apex) of the triangle, where  $h$  is the height of the triangle.

### Centroid of a Semicircle

Consider the semicircle of radius  $R$  as shown in Fig. 4.26. Due to symmetry centroid must lie on  $y$  axis. Let its distance from diametral axis be  $\bar{y}$ . To find  $\bar{y}$ , consider an element at a distance  $r$  from the centre  $O$  of the semicircle, radial width being  $dr$  and bound by radii at  $\theta$  and  $\theta + d\theta$ .

Area of element =  $r \, d\theta \, dr$ .

Its moment about diametral axis  $x$  is given by:

$$r d\theta \times dr \times r \sin \theta = r^2 \sin \theta \, dr \, d\theta$$

$\therefore$  Total moment of area about diametral axis,

$$\begin{aligned}
 \int_0^\pi \int_0^R r^2 \sin \theta \, dr \, d\theta &= \int_0^\pi \left[ \frac{r^3}{3} \right]_0^R \sin \theta \, d\theta \\
 &= \frac{R^3}{3} \left[ -\cos \theta \right]_0^\pi \\
 &= \frac{R^3}{3} [1 + 1] = \frac{2R^3}{3}
 \end{aligned}$$

Area of semicircle  $A = \frac{1}{2} \pi R^2$

$$\begin{aligned}
 \therefore \bar{y} &= \frac{\text{Moment of area}}{\text{Total area}} = \frac{\frac{2R^3}{3}}{\frac{1}{2} \pi R^2} \\
 &= \frac{4R}{3\pi}
 \end{aligned}$$

Thus, the centroid of the circle is at a distance  $\frac{4R}{3\pi}$  from the diametral axis.

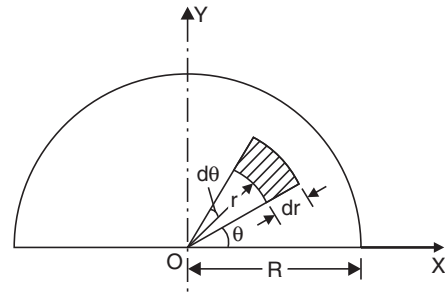


Fig. 4.26

### Centroid of Sector of a Circle

Consider the sector of a circle of angle  $2\alpha$  as shown in Fig. 4.27. Due to symmetry, centroid lies on  $x$  axis. To find its distance from the centre  $O$ , consider the elemental area shown.

Area of the element  $= r d\theta \times dr$

Its moment about  $y$  axis

$$= r d\theta \times dr \times r \cos \theta$$

$$= r^2 \cos \theta dr d\theta$$

$\therefore$  Total moment of area about  $y$  axis

$$= \int_{-\alpha}^{\alpha} \int_0^R r^2 \cos \theta dr d\theta$$

$$= \left[ \frac{r^3}{3} \right]_0^R \left[ \sin \theta \right]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3} 2 \sin \alpha$$

Total area of the sector

$$= \int_{-\alpha}^{\alpha} \int_0^R r dr d\theta$$

$$= \int_{-\alpha}^{\alpha} \left[ \frac{r^2}{2} \right]_0^R d\theta$$

$$= \frac{R^2}{2} \left[ \theta \right]_{-\alpha}^{\alpha}$$

$$= R^2 \alpha$$

$\therefore$  The distance of centroid from centre  $O$

$$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$$

$$= \frac{\frac{2R^3}{3} \sin \alpha}{R^2 \alpha} = \frac{2R}{3\alpha} \sin \alpha$$

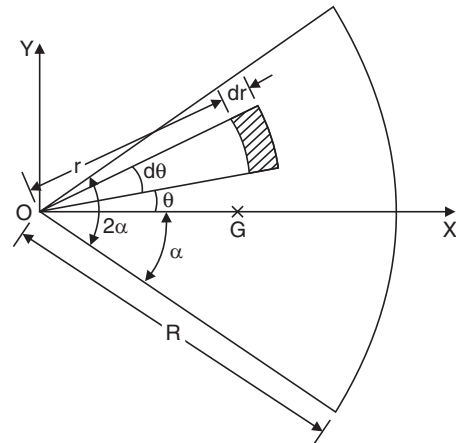


Fig. 4.27

### Centroid of Parabolic Spandrel

Consider the parabolic spandrel shown in Fig. 4.28. Height of the element at a distance  $x$  from  $O$  is  $y = kx^2$

$$\text{Width of element} = dx$$

$$\therefore \text{Area of the element} = kx^2 dx$$

$$\therefore \text{Total area of spandrel} = \int_0^a kx^2 dx = \left[ \frac{kx^3}{3} \right]_0^a$$

$$= \frac{ka^3}{3}$$

Moment of area about y axis

$$= \int_0^a kx^2 dx \times x$$

$$= \int_0^a kx^3 dx$$

$$= \left[ \frac{kx^4}{4} \right]_0^a$$

$$= \frac{ka^4}{4}$$

$$\text{Moment of area about } x \text{ axis} = \int_0^a dA \cdot \frac{y}{2}$$

$$= \int_0^a kx^2 dx \cdot \frac{kx^2}{2} = \int_0^a \frac{k^2 x^4}{2} dx$$

$$= \frac{k^2 a^5}{10}$$

$$\therefore \bar{x} = \frac{ka^4}{4} \div \frac{ka^3}{3} = \frac{3a}{4}$$

$$\bar{y} = \frac{k^2 a^5}{10} \div \frac{ka^3}{3} = \frac{3}{10} ka^2$$

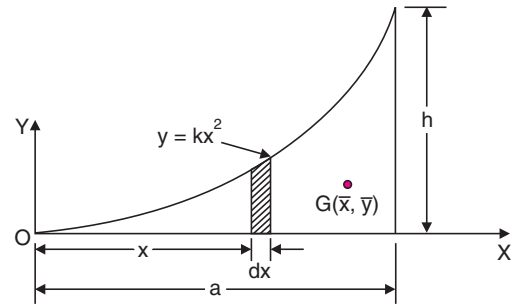
From the Fig. 4.28, at  $x = a$ ,  $y = h$

$$\therefore h = ka^2 \text{ or } k = \frac{h}{a^2}$$

$$\therefore \bar{y} = \frac{3}{10} \times \frac{h}{a^2} a^2 = \frac{3h}{10}$$

Thus, centroid of spandrel is  $\left( \frac{3a}{4}, \frac{3h}{10} \right)$

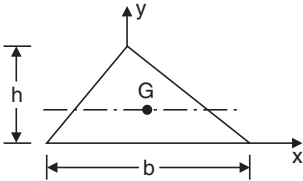
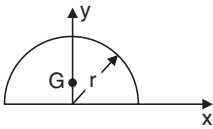
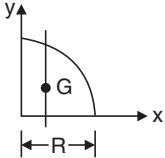
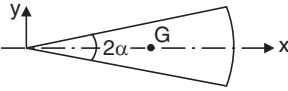
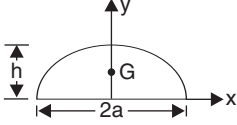
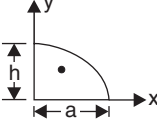
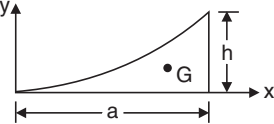
Centroids of some common figures are shown in Table 4.2.



**Fig. 4.28**



**Table 4.2** Centroid of Some Common Figures

Shape	Figure	$\bar{x}$	$\bar{y}$	Area
Triangle		—	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{4}$
Sector of a circle		$\frac{2R}{3\alpha} \sin \alpha$	0	$\alpha R^2$
Parabola		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Semi parabola		$\frac{3}{8}a$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

**Centroid of Composite Sections**

So far, the discussion was confined to locating the centroid of simple figures like rectangle, triangle, circle, semicircle, etc. In engineering practice, use of sections which are built up of many simple sections is very common. Such sections may be called as built-up sections or composite sections. To locate the centroid of composite sections, one need not go for the first principle (method of integration). The given composite section can be split into suitable simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae listed in Table 4.2. Assuming the area of the simple figure as concentrated at its centroid, its moment about an axis can be

found by multiplying the area with distance of its centroid from the reference axis. After determining moment of each area about reference axis, the distance of centroid from the axis is obtained by dividing total moment of area by total area of the composite section.

**Example 4.4.** Locate the centroid of the T-section shown in the Fig. 4.29.

**Solution:** Selecting the axis as shown in Fig. 4.29, we can say due to symmetry centroid lies on y axis, i.e.  $\bar{x} = 0$ . Now the given T-section may be divided into two rectangles  $A_1$  and  $A_2$  each of size  $100 \times 20$  and  $20 \times 100$ . The centroid of  $A_1$  and  $A_2$  are  $g_1(0, 10)$  and  $g_2(0, 70)$  respectively.

$\therefore$  The distance of centroid from top is given by:

$$\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100}$$

$$= 40 \text{ mm}$$

Hence, **centroid of T-section is on the symmetric axis at a distance 40 mm from the top.**

**Example 4.5.** Find the centroid of the unequal angle  $200 \times 150 \times 12 \text{ mm}$ , shown in Fig. 4.30.

**Solution:** The given composite figure can be divided into two rectangles:

$$A_1 = 150 \times 12 = 1800 \text{ mm}^2$$

$$A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2$$

$$\text{Total area } A = A_1 + A_2 = 4056 \text{ mm}^2$$

Selecting the reference axis  $x$  and  $y$  as shown in Fig. 4.30. The centroid of  $A_1$  is  $g_1(75, 6)$  and that of  $A_2$  is:

$$g_2 \left[ 6, 12 + \frac{1}{2} (200 - 12) \right]$$

$$\text{i.e., } g_2(6, 106)$$

$$\therefore \bar{x} = \frac{\text{Moment about y axis}}{\text{Total area}}$$

$$= \frac{A_1 x_1 + A_2 x_2}{A}$$

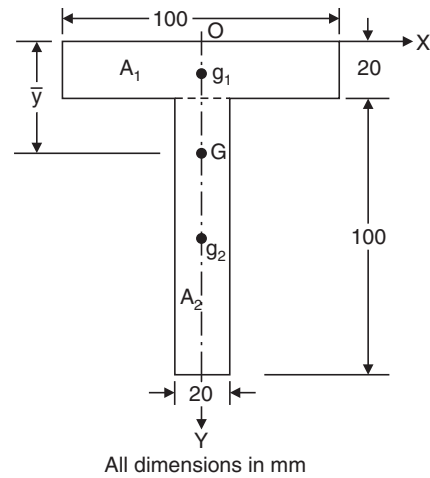
$$= \frac{1800 \times 75 + 2256 \times 6}{4056} = 36.62 \text{ mm}$$

$$\bar{y} = \frac{\text{Moment about x axis}}{\text{Total area}}$$

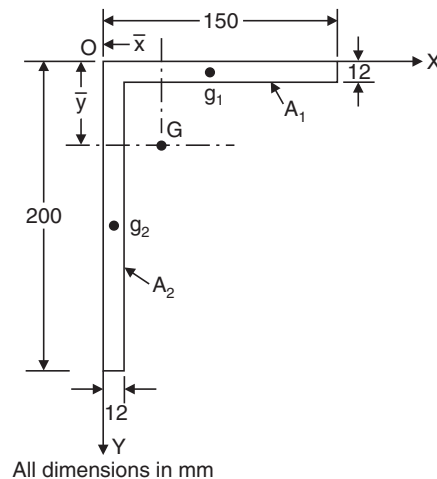
$$= \frac{A_1 y_1 + A_2 y_2}{A}$$

$$= \frac{1800 \times 6 + 2256 \times 106}{4056} = 61.62 \text{ mm}$$

Thus, the centroid is at  $\bar{x} = 36.62 \text{ mm}$  and  $\bar{y} = 61.62 \text{ mm}$  as shown in the figure.

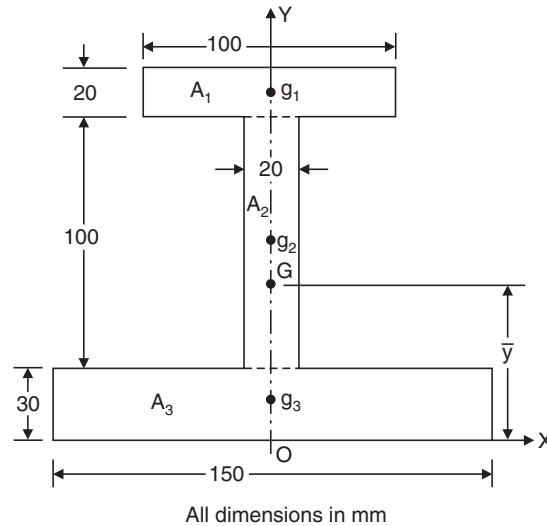


**Fig. 4.29**



**Fig. 4.30**

**Example 4.6.** Locate the centroid of the I-section shown in Fig. 4.31.



**Fig. 4.31**

**Solution:** Selecting the coordinate system as shown in Fig. 4.31, due to symmetry centroid must lie on y axis,

i.e.,  $\bar{x} = 0$

Now, the composite section may be split into three rectangles

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2$$

Centroid of  $A_1$  from the origin is:

$$y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}$$

Similarly

$$A_2 = 100 \times 20 = 2000 \text{ mm}^2$$

$$y_2 = 30 + \frac{100}{2} = 80 \text{ mm}$$

$$A_3 = 150 \times 30 = 4500 \text{ mm}^2,$$

and

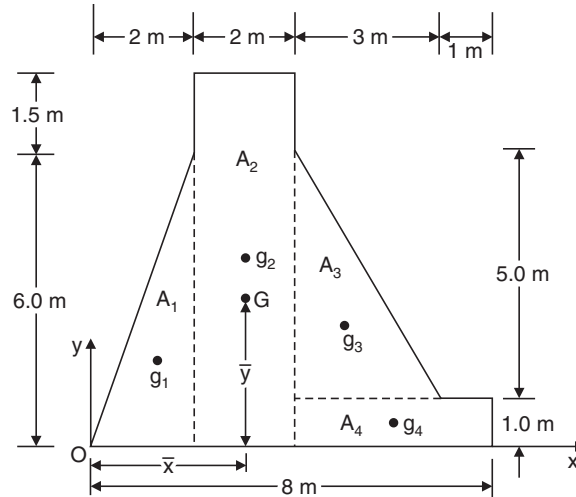
$$y_3 = \frac{30}{2} = 15 \text{ mm}$$

$\therefore$

$$\begin{aligned} \bar{y} &= \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A} \\ &= \frac{2000 \times 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500} \\ &= 59.71 \text{ mm} \end{aligned}$$

Thus, the centroid is on the symmetric axis at a distance 59.71 mm from the bottom as shown in Fig. 4.31.

**Example 4.7.** Determine the centroid of the section of the concrete dam shown in Fig. 4.32.



**Fig. 4.32**

**Solution:** Let the axis be selected as shown in Fig. 4.32. Note that it is convenient to take axis in such a way that the centroids of all simple figures are having positive coordinates. If coordinate of any simple figure comes out to be negative, one should be careful in assigning the sign of moment of area of that figure.

The composite figure can be conveniently divided into two triangles and two rectangles, as shown in Fig. 4.32.

Now,

$$A_1 = \frac{1}{2} \times 2 \times 6 = 6 \text{ m}^2$$

$$A_2 = 2 \times 7.5 = 15 \text{ m}^2$$

$$A_3 = \frac{1}{2} \times 3 \times 5 = 7.5 \text{ m}^2$$

$$A_4 = 1 \times 4 = 4 \text{ m}^2$$

$$A = \text{total area} = 32.5 \text{ m}^2$$

Centroids of simple figures are:

$$x_1 = \frac{2}{3} \times 2 = \frac{4}{3} \text{ m}$$

$$y_1 = \frac{1}{3} \times 6 = 2 \text{ m}$$

$$x_2 = 2 + 1 = 3 \text{ m}$$

$$y_2 = \frac{7.5}{2} = 3.75 \text{ m}$$

$$x_3 = 2 + 2 + \frac{1}{3} \times 3 = 5 \text{ m}$$

$$y_3 = 1 + \frac{1}{3} \times 5 = \frac{8}{3} \text{ m}$$

$$x_4 = 4 + \frac{4}{2} = 6 \text{ m}$$

$$y_4 = 0.5 \text{ m}$$

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4}{A}$$

$$= \frac{6 \times \frac{4}{3} + 15 \times 3 + 7.5 \times 5 + 4 \times 6}{32.5}$$

$$= 3.523 \text{ m}$$

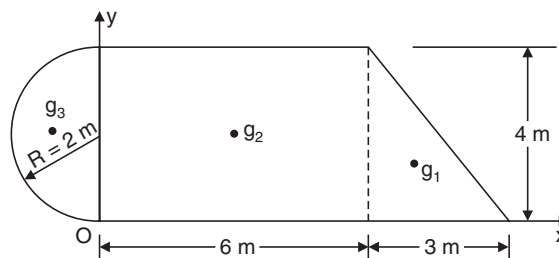
$$\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4}{A}$$

$$= \frac{6 \times 2 + 15 \times 3.75 + 7.5 \times \frac{8}{3} + 4 \times 0.5}{32.5}$$

$$= 2.777 \text{ m}$$

The centroid is at  $\bar{x} = 3.523 \text{ m}$   
and  $\bar{y} = 2.777 \text{ m}$

**Example 4.8.** Determine the centroid of the area shown in Fig. 4.33 with respect to the axis shown.



**Fig. 4.33**

**Solution:** The composite section is divided into three simple figures, a triangle, a rectangle and a semicircle

Now, area of triangle  $A_1 = \frac{1}{2} \times 3 \times 4 = 6 \text{ m}^2$

Area of rectangle  $A_2 = 6 \times 4 = 24 \text{ m}^2$

Area of semicircle  $A_3 = \frac{1}{2} \times \pi \times 2^2 = 6.2832 \text{ m}^2$

$\therefore$  Total area  $A = 36.2832 \text{ m}^2$

The coordinates of centroids of these three simple figures are:

$$x_1 = 6 + \frac{1}{3} \times 3 = 7 \text{ m}$$

$$y_1 = \frac{4}{3} \text{ m}$$

$$x_2 = 3 \text{ m}$$

$$y_2 = 2 \text{ m}$$

$$x_3 = \frac{-4R}{3\pi} = -\frac{4 \times 2}{3\pi} = -0.8488 \text{ m}$$

$$y_3 = 2 \text{ m} \quad (\text{Note carefully the sign of } x_3).$$

$$\begin{aligned} \bar{x} &= \frac{A_1x_1 + A_2x_2 + A_3x_3}{A} \\ &= \frac{6 \times 7 + 24 \times 3 + 6.2832 \times (-0.8488)}{36.2832} \end{aligned}$$

i.e.,

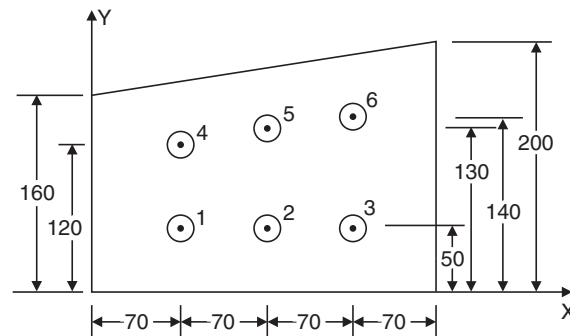
$$\bar{x} = 2.995 \text{ m}$$

$$\begin{aligned} \bar{y} &= \frac{A_1y_1 + A_2y_2 + A_3y_3}{A} \\ &= \frac{\frac{6 \times 4}{3} + 24 \times 2 + 6.2832 \times 2}{36.2832} \end{aligned}$$

i.e.,

$$\bar{y} = 1.890 \text{ m}$$

**Example 4.9.** In a gusset plate, there are six rivet holes of 21.5 mm diameter as shown in Fig. 4.34. Find the position of the centroid of the gusset plate.



**Fig. 4.34**

**Solution:** The composite area is equal to a rectangle of size 160 × 280 mm *plus* a triangle of size 280 mm base width and 40 mm height and *minus* areas of six holes. In this case also the Eqn. 4.4 can be used for locating centroid by treating area of holes as negative. The area of simple figures and their centroids are as shown in Table 4.3.

Table 4.3

Figure	Area in mm <sup>2</sup>	$x_i$ in mm	$y_i$ in mm
Rectangle	$160 \times 280 = 44,800$	140	80
Triangle	$\frac{1}{2} \times 280 \times 40 = 5600$	$\frac{560}{3}$	$160 + \frac{40}{3} = 173.33$
1st hole	$\frac{-\pi \times 21.5^2}{4} = -363.05$	70	50
2nd hole	-363.05	140	50
3rd hole	-363.05	210	50
4th hole	-363.05	70	120
5th hole	-363.05	140	130
6th hole	-363.05	210	140

$$\therefore A = \Sigma A_i = 48221.70$$

$$\therefore \Sigma A_i x_i = 44800 \times 140 + 5600 \times \frac{560}{3} - 363.05 (70 + 140 + 210 + 70 + 140 + 210)$$

$$= 7012371.3 \text{ mm}^3$$

$$\bar{x} = \frac{\Sigma A_i x_i}{A} = 145.42 \text{ mm}$$

$$\Sigma A_i y_i = 44800 \times 80 + 5600 \times 173.33 - 363.05 (50 \times 3 + 120 + 130 + 140)$$

$$= 4358601 \text{ mm}^3$$

$$\bar{y} = \frac{\Sigma A_i y_i}{A} = \frac{4358601}{48221.70}$$

$$= 90.39 \text{ mm}$$

Thus, the coordinates of centroid of composite figure is given by:

$$\bar{x} = 145.42 \text{ mm}$$

$$\bar{y} = 90.39 \text{ mm}$$

**Example 4.10.** Determine the coordinates  $x_c$  and  $y_c$  of the centre of a 100 mm diameter circular hole cut in a thin plate so that this point will be the centroid of the remaining shaded area shown in Fig. 4.35 (All dimensions are in mm).

**Solution:** If  $x_c$  and  $y_c$  are the coordinates of the centre of the circle, centroid also must have the coordinates  $x_c$  and  $y_c$  as per the condition laid down in the problem. The shaded area may be considered as a rectangle of size 200 mm  $\times$  150 mm minus a triangle of sides 100 mm  $\times$  75 mm and a circle of diameter 100 mm.

$\therefore$  Total area

$$= 200 \times 150 - \frac{1}{2} \times 100 \times 75 - \left(\frac{\pi}{4}\right) 100^2$$

$$= 18396 \text{ mm}^2$$

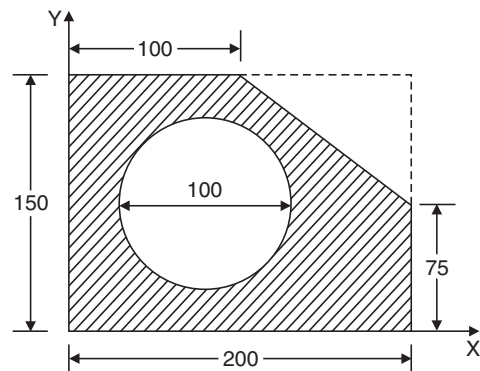


Fig. 4.35

$$x_c = \bar{x} = \frac{200 \times 150 \times 100 - \frac{1}{2} \times 100 \times 75 \times \left[ 200 - \left( \frac{100}{3} \right) \right] - \frac{\pi}{4} \times 100^2 \times x_c}{18396}$$

$$\therefore x_c(18396) = 200 \times 150 \times 100 - \frac{1}{2} \times 100 \times 75 \times 166.67 - \frac{\pi}{4} \times 100^2 x_c$$

$$x_c = \frac{2375000}{26250} = 90.48 \text{ mm}$$

Similarly,

$$18396 y_c = 200 \times 150 \times 75 - \frac{1}{2} \times 100 \times 75 \times (150 - 25) - \frac{\pi}{4} \times 100^2 y_c$$

$$\therefore y_c = \frac{1781250.0}{26250} = 67.86 \text{ mm}$$

Centre of the circle should be located at (90.48, 67.86) so that this point will be the centroid of the remaining shaded area as shown in Fig. 4.35.

**Note:** The centroid of the given figure will coincide with the centroid of the figure without circular hole. Hence, the centroid of the given figure may be obtained by determining the centroid of the figure without the circular hole also.

**Example 4.11.** Determine the coordinates of the centroid of the plane area shown in Fig. 4.36 with reference to the axis shown. Take  $x = 40 \text{ mm}$ .

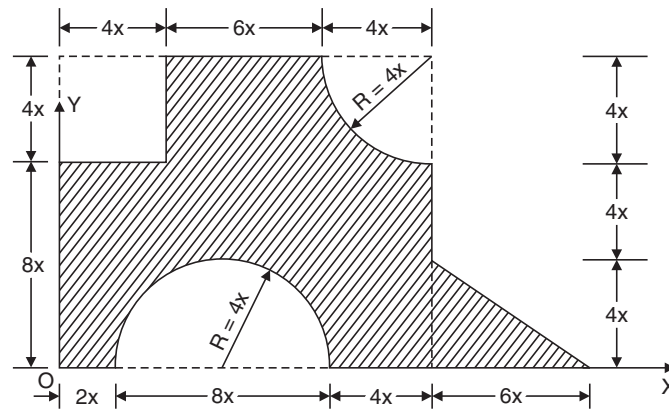


Fig. 4.36

**Solution:** The composite figure is divided into the following simple figures:

- (1) A rectangle  $A_1 = (14x) \times (12x) = 168x^2$   
 $x_1 = 7x$ ;  $y_1 = 6x$



(2) A triangle  $A_2 = \frac{1}{2} (6x) \times (4x) = 12x^2$   
 $x_2 = 14x + 2x = 16x$   
 $y_2 = \frac{4x}{3}$

(3) A rectangle to be subtracted  
 $A_3 = (-4x) \times (4x) = -16x^2$   
 $x_3 = 2x; y_3 = 8x + 2x = 10x$

(4) A semicircle to be subtracted  
 $A_4 = -\frac{1}{2} \pi (4x)^2 = -8\pi x^2$   
 $x_4 = 6x$   
 $y_4 = \frac{4R}{3\pi} = 4 \times \frac{4(x)}{3\pi} = \frac{16x}{3\pi}$

(5) A quarter of a circle to be subtracted  
 $A_5 = -\frac{1}{4} \times \pi (4x)^2 = -4\pi x^2$   
 $x_5 = 14x - \frac{4R}{3\pi} = 14x - (4) \left( \frac{4x}{3\pi} \right) = 12.3023x$   
 $y_5 = 12x - 4 \times \left( \frac{4x}{3\pi} \right) = 10.3023x$

Total area  $A = 168x^2 + 12x^2 - 16x^2 - 8\pi x^2 - 4\pi x^2$   
 $= 126.3009x^2$

$$\bar{x} = \frac{\Sigma A_i x_i}{A}$$

$$\Sigma A_i x_i = 168x^2 \times 7x + 12x^2 \times 16x - 16x^2 \times 2x - 8\pi x^2 \times 6x - 4\pi x^2 \times 12.3023x$$

$$= 1030.6083x^3$$

$$\therefore \bar{x} = \frac{1030.6083x^3}{126.3009x^2}$$

$$= 8.1599x = 8.1599 \times 40 \quad (\text{since } x = 40 \text{ mm})$$

$$= \mathbf{326.40 \text{ mm}}$$

$$\bar{y} = \frac{\Sigma A_i y_i}{A}$$

$$\begin{aligned}
 \Sigma A_i y_i &= 168x^2 \times 6x + 12x^2 \times \frac{4x}{3} - 16x^2 \times 10x \\
 &\quad - 8\pi x^2 \times \frac{16x}{3\pi} - 4\pi x^2 \times 10.3023x \\
 &= 691.8708x^3 \\
 \therefore \bar{y} &= \frac{691.8708x^3}{126.3009x^2} \\
 &= 5.4780x \\
 &= \mathbf{219.12 \text{ mm}} \quad (\text{since } x = 40 \text{ mm})
 \end{aligned}$$

Centroid is at (326.40, 219.12).

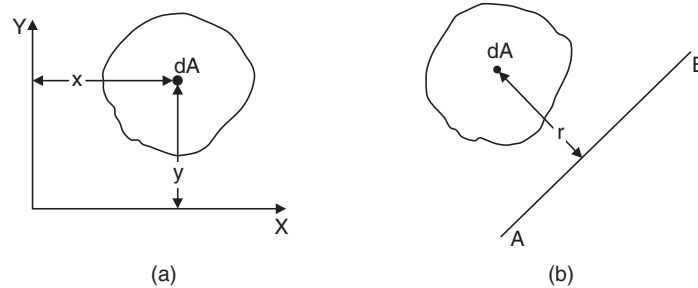
#### 4.5 SECOND MOMENTS OF PLANE AREA

Consider the area shown in Fig. 4.37(a).  $dA$  is an elemental area with coordinates as  $x$  and  $y$ . The term  $\Sigma y_i^2 dA_i$  is called *moment of inertia* of the area about  $x$  axis and is denoted as  $I_{xx}$ . Similarly, the moment of inertia about  $y$  axis is

$$I_{yy} = \Sigma x_i^2 dA_i$$

In general, if  $r$  is the distance of elemental area  $dA$  from the axis  $AB$  [Fig. 4.37(b)], the sum of the terms  $\Sigma r^2 dA$  to cover the entire area is called moment of inertia of the area about the axis  $AB$ . If  $r$  and  $dA$  can be expressed in general term, for any element, then the sum becomes an integral. Thus,

$$I_{AB} = \Sigma r_i^2 dA_i = \int r^2 dA \quad \dots(4.6)$$



**Fig. 4.37**

The term  $r dA$  may be called as moment of area, similar to moment of a force, and hence  $r^2 dA$  may be called as moment of area or the second moment of area. Thus, the moment of inertia of area is nothing but second moment of area. In fact, the term '*second moment of area*' appears to correctly signify the meaning of the expression  $\Sigma r^2 dA$ . The term '*moment of inertia*' is rather a misnomer. However, the term moment of inertia has come to stay for long time and hence it will be used in this book also.

Though moment of inertia of plane area is a purely mathematical term, it is one of the important properties of areas. The strength of members subject to bending depends on the moment of inertia of its cross-sectional area. Students will find this property of area very useful when they study subjects like strength of materials, structural design and machine design.

The moment of inertia is a fourth dimensional term since it is a term obtained by multiplying area by the square of the distance. Hence, in SI units, if metre (m) is the unit for linear measurements

used then  $m^4$  is the unit of moment of inertia. If millimetre (mm) is the unit used for linear measurements, then  $mm^4$  is the unit of moment of inertia. In MKS system  $m^4$  or  $cm^4$  and in FPS system  $ft^4$  or  $in^4$  are commonly used as units for moment of inertia.

### Polar Moment of Inertia

Moment of inertia about an axis perpendicular to the plane of an area is known as *polar moment of inertia*. It may be denoted as  $J$  or  $I_{zz}$ . Thus, the moment of inertia about an axis perpendicular to the plane of the area at  $O$  in Fig. 4.38 is called polar moment of inertia at point  $O$ , and is given by

$$I_{zz} = \Sigma r^2 dA \quad \dots(4.7)$$

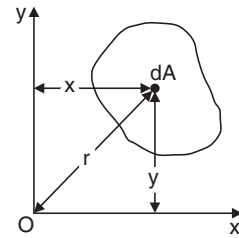


Fig. 4.38

### Radius of Gyration

Radius of gyration is a mathematical term defined by the relation

$$k = \sqrt{\frac{I}{A}} \quad \dots(4.8)$$

where  $k$  = radius of gyration,

$I$  = moment of inertia,

and  $A$  = the cross-sectional area

Suffixes with moment of inertia  $I$  also accompany the term radius of gyration  $k$ . Thus, we can have,

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$k_{yy} = \sqrt{\frac{I_{yy}}{A}}$$

$$k_{AB} = \sqrt{\frac{I_{AB}}{A}}$$

and so on.

The relation between radius of gyration and moment of inertia can be put in the form:

$$I = Ak^2 \quad \dots(4.9)$$

From the above relation a geometric meaning can be assigned to the term 'radius of gyration.' We can consider  $k$  as the distance at which the complete area is squeezed and kept as a strip of negligible width (Fig. 4.39) such that there is no change in the moment of inertia.

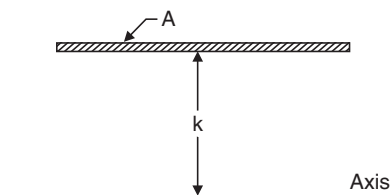


Fig. 4.39

### Theorems of Moments of Inertia

There are two theorems of moment of inertia:

- (1) Perpendicular axis theorem, and
- (2) Parallel axis theorem.

These are explained and proved below.

#### Perpendicular Axis Theorem

The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point  $O$  is equal to the sum of moments of inertia about any two mutually perpendicular axis through the same point  $O$  and lying in the plane of the area.

Referring to Fig. 4.40, if  $z$ - $z$  is the axis normal to the plane of paper passing through point  $O$ , as per this theorem,

$$I_{zz} = I_{xx} + I_{yy} \quad \dots (4.10)$$

The above theorem can be easily proved. Let us consider an elemental area  $dA$  at a distance  $r$  from  $O$ . Let the coordinates of  $dA$  be  $x$  and  $y$ . Then from definition:

$$\begin{aligned} I_{zz} &= \Sigma r^2 dA \\ &= \Sigma (x^2 + y^2) dA \\ &= \Sigma x^2 dA + \Sigma y^2 dA \end{aligned}$$

$$I_{zz} = I_{xx} + I_{yy}$$

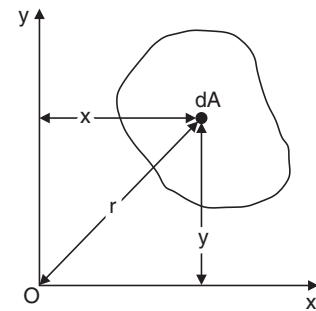


Fig. 4.40

#### Parallel Axis Theorem

Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axis. Referring to Fig. 4.41 the above theorem means:

$$I_{AB} = I_{GG} + A y_c^2 \quad \dots (4.11)$$

where

$I_{AB}$  = moment of inertia about axis  $AB$

$I_{GG}$  = moment of inertia about centroidal axis  $GG$  parallel to  $AB$ .

$A$  = the area of the plane figure given and

$y_c$  = the distance between the axis  $AB$  and the parallel centroidal axis  $GG$ .

*Proof:* Consider an elemental parallel strip  $dA$  at a distance  $y$  from the centroidal axis (Fig. 4.41).

Then,

$$\begin{aligned} I_{AB} &= \Sigma (y + y_c)^2 dA \\ &= \Sigma (y^2 + 2y y_c + y_c^2) dA \\ &= \Sigma y^2 dA + \Sigma 2y y_c dA + \Sigma y_c^2 dA \end{aligned}$$

Now,

$$\begin{aligned} \Sigma y^2 dA &= \text{Moment of inertia about the axis } GG \\ &= I_{GG} \end{aligned}$$

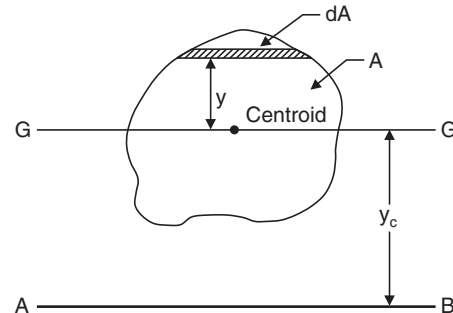


Fig. 4.41

$$\begin{aligned}\Sigma 2yy_c dA &= 2y_c \Sigma y dA \\ &= 2y_c A \frac{\Sigma y dA}{A}\end{aligned}$$

In the above term  $2y_c A$  is constant and  $\frac{\Sigma y dA}{A}$  is the distance of centroid from the reference axis  $GG$ . Since  $GG$  is passing through the centroid itself  $\frac{y dA}{A}$  is zero and hence the term  $\Sigma 2yy_c dA$  is zero.

Now, the third term,

$$\begin{aligned}\Sigma y_c^2 dA &= y_c^2 \Sigma dA \\ &= Ay_c^2 \\ \therefore I_{AB} &= I_{GG} + Ay_c^2\end{aligned}$$

**Note:** The above equation cannot be applied to any two parallel axis. One of the axis ( $GG$ ) must be centroidal axis only.

#### 4.6 MOMENT OF INERTIA FROM FIRST PRINCIPLES

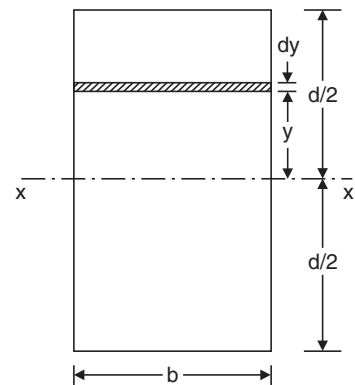
For simple figures, moment of inertia can be obtained by writing the general expression for an element and then carrying out integration so as to cover the entire area. This procedure is illustrated with the following three cases:

- (1) Moment of inertia of a rectangle about the centroidal axis
- (2) Moment of inertia of a triangle about the base
- (3) Moment of inertia of a circle about a diametral axis

(1) *Moment of Inertia of a Rectangle about the Centroidal Axis:* Consider a rectangle of width  $b$  and depth  $d$  (Fig. 4.42). Moment of inertia about the centroidal axis  $x-x$  parallel to the short side is required.

Consider an elemental strip of width  $dy$  at a distance  $y$  from the axis. Moment of inertia of the elemental strip about the centroidal axis  $xx$  is:

$$\begin{aligned}&= y^2 dA \\ &= y^2 b dy \\ \therefore I_{xx} &= \int_{-d/2}^{d/2} y^2 b dy \\ &= b \left[ \frac{y^3}{3} \right]_{-d/2}^{d/2} \\ &= b \left[ \frac{d^3}{24} + \frac{d^3}{24} \right] \\ I_{xx} &= \frac{bd^3}{12}\end{aligned}$$



**Fig. 4.42**

(2) *Moment of Inertia of a Triangle about its Base:* Moment of inertia of a triangle with base width  $b$  and height  $h$  is to be determined about the base  $AB$  (Fig. 4.43).

Consider an elemental strip at a distance  $y$  from the base  $AB$ . Let  $dy$  be the thickness of the strip and  $dA$  its area. Width of this strip is given by:

$$b_1 = \frac{(h-y)}{h} \times b$$

Moment of inertia of this strip about  $AB$

$$\begin{aligned} &= y^2 dA \\ &= y^2 b_1 dy \\ &= y^2 \frac{(h-y)}{h} \times b \times dy \end{aligned}$$

$\therefore$  Moment of inertia of the triangle about  $AB$ ,

$$\begin{aligned} I_{AB} &= \int_0^h \frac{y^2 (h-y)b dy}{h} \\ &= \int_0^h \left( y^2 - \frac{y^3}{h} \right) b dy \\ &= b \left[ \frac{y^3}{3} - \frac{y^4}{4h} \right]_0^h \\ &= b \left[ \frac{h^3}{3} - \frac{h^4}{4h} \right] \\ I_{AB} &= \frac{bh^3}{12} \end{aligned}$$

(3) *Moment of Inertia of a Circle about its Diametral Axis:* Moment of inertia of a circle of radius  $R$  is required about its diametral axis as shown in Fig. 4.44

Consider an element of sides  $r d\theta$  and  $dr$  as shown in the figure. Its moment of inertia about the diametral axis  $x-x$ :

$$\begin{aligned} &= y^2 dA \\ &= (r \sin \theta)^2 r d\theta dr \\ &= r^3 \sin^2 \theta d\theta dr \end{aligned}$$

$\therefore$  Moment of inertia of the circle about  $x-x$  is given by

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2 \theta d\theta dr$$

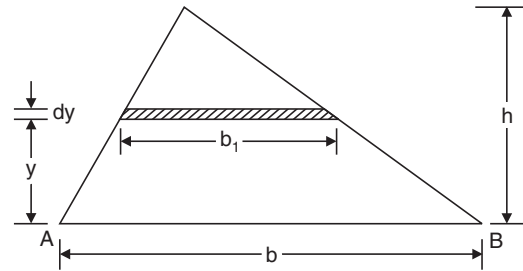


Fig. 4.43

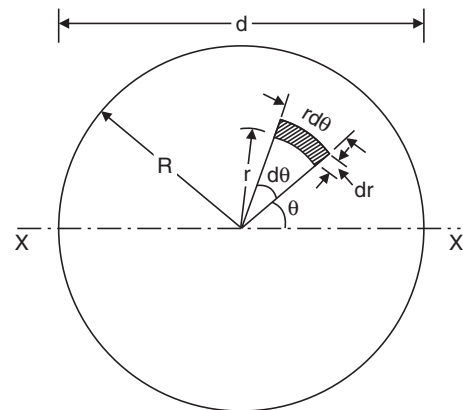


Fig. 4.44

$$\begin{aligned}
 &= \int_0^R \int_0^{2\pi} \frac{(1 - \cos 2\theta)}{2} d\theta dr \\
 &= \int_0^R \frac{r^3}{2} \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} dr \\
 &= \left[ \frac{r^4}{8} \right]_0^R [2\pi - 0 + 0 - 0] = \frac{2\pi}{8} R^4
 \end{aligned}$$

$$I_{xx} = \frac{\pi R^4}{4}$$

If  $d$  is the diameter of the circle, then

$$R = \frac{d}{2}$$

$$\therefore I_{xx} = \frac{\pi}{4} \left( \frac{d}{2} \right)^4$$

$$I_{xx} = \frac{\pi d^4}{64}$$

### Moment of Inertia of Standard Sections

*Rectangle:* Referring to Fig. 4.45.

(a)  $I_{xx} = \frac{bd^3}{12}$  as derived from first principle.

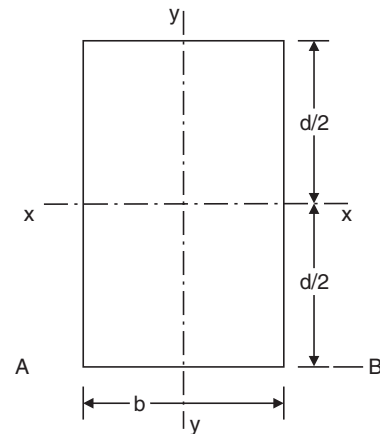
(b)  $I_{yy} = \frac{db^3}{12}$  can be derived on the same lines.

(c) About the base  $AB$ , from parallel axis theorem,

$$\begin{aligned}
 I_{AB} &= I_{xx} + Ay_c^2 \\
 &= \frac{bd^3}{12} + bd \left( \frac{d}{2} \right)^2, \quad \text{since } y_c = \frac{d}{2}
 \end{aligned}$$

$$= \frac{bd^3}{12} + \frac{bd^3}{4}$$

$$= \frac{bd^3}{3}$$



**Fig. 4.45**

**Hollow Rectangular Section:** Referring to Fig. 4.46, Moment of inertia  $I_{xx}$  = Moment of inertia of larger rectangle – Moment of inertia of hollow portion. That is,

$$\begin{aligned} &= \frac{BD^3}{12} - \frac{bd^3}{12} \\ &= \frac{1}{12} (BD^3 - bd^3) \end{aligned}$$

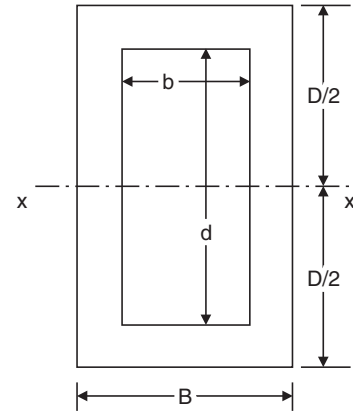


Fig. 4.46

**Triangle**—Referring to Fig. 4.47.

(a) *About the base:*

As found from first principle

$$I_{AB} = \frac{bh^3}{12}$$

(b) *About centroidal axis, x-x parallel to base:*

From parallel axis theorem,

$$I_{AB} = I_{xx} + Ay_c^2$$

Now,  $y_c$ , the distance between the non-centroidal axis AB and centroidal axis x-x, is equal to  $\frac{h}{3}$ .

$$\therefore \frac{bh^3}{12} = I_{xx} + \frac{1}{2}bh\left(\frac{h}{3}\right)^2$$

$$= I_{xx} + \frac{bh^3}{18}$$

$$\therefore I_{xx} = \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$= \frac{bh^3}{36}$$

Moment of Inertia of a Circle about any diametral axis

$$= \frac{\pi d^4}{64} \quad (\text{as found from first principle})$$

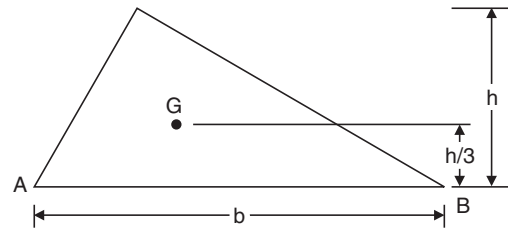


Fig. 4.47

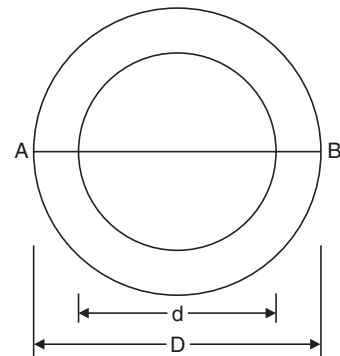


**Moment of Inertia of a Hollow Circle:** Referring to Fig. 4.48.

$I_{AB}$  = Moment of inertia of solid circle of diameter  $D$  about  $AB$

– Moment of inertia of circle of diameter  $d$  about  $AB$ . That is,

$$\begin{aligned} &= \frac{\pi D^4}{64} - \frac{\pi d^4}{64} \\ &= \frac{\pi}{64} (D^4 - d^4) \end{aligned}$$

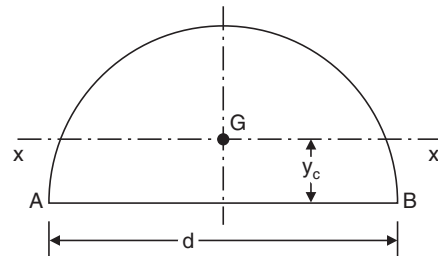


**Fig. 4.48**

**Moment of Inertia of a Semicircle: (a) About Diametral Axis:**

If the limit of integration is put as 0 to  $\pi$  instead of 0 to  $2\pi$  in the derivation for the moment of inertia of a circle about diametral axis the moment of inertia of a semicircle is obtained. It can be observed that the moment of inertia of a semicircle (Fig. 4.49) about the diametral axis  $AB$ :

$$= \frac{1}{2} \times \frac{\pi d^4}{64} = \frac{\pi d^4}{128}$$



**Fig. 4.49**

(b) About Centroidal Axis  $x$ - $x$ :

Now, the distance of centroidal axis  $y_c$  from the diametral axis is given by:

$$y_c = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

and, 
$$\text{Area } A = \frac{1}{2} \times \frac{\pi d^2}{4} = \frac{\pi d^2}{8}$$

From parallel axis theorem,

$$\begin{aligned} I_{AB} &= I_{xx} + Ay_c^2 \\ \frac{\pi d^4}{128} &= I_{xx} + \frac{\pi d^2}{8} \times \left( \frac{2d}{3\pi} \right)^2 \\ I_{xx} &= \frac{\pi d^4}{128} - \frac{d^4}{18\pi} \\ &= 0.0068598 d^4 \end{aligned}$$

**Moment of Inertia of a Quarter of a Circle: (a) About the Base:** If the limit of integration is put as 0 to  $\frac{\pi}{2}$  instead of 0 to  $2\pi$  in the derivation for moment of inertia of a circle the moment of inertia of a quarter of a circle is obtained. It can be observed that moment of inertia of the quarter of a circle about the base  $AB$ .

$$= \frac{1}{4} \times \frac{\pi d^4}{64\pi} = \frac{\pi d^4}{256}$$

(b) About Centroidal Axis  $x$ - $x$ :

Now, the distance of centroidal axis  $y_c$  from the base is given by:

$$y_c = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

and the area  $A = \frac{1}{4} \times \frac{\pi d^2}{4} = \frac{\pi d^2}{16}$

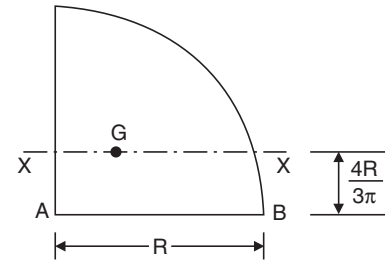
From parallel axis theorem,

$$I_{AB} = I_{xx} + Ay_c^2$$

$$\frac{\pi d^4}{256} = I_{xx} + \frac{\pi d^2}{16} \left( \frac{2d}{3\pi} \right)^2$$

$$I_{xx} = \frac{\pi d^4}{256} - \frac{d^4}{36\pi}$$

$$= 0.00343 d^4$$



**Fig. 4.50**

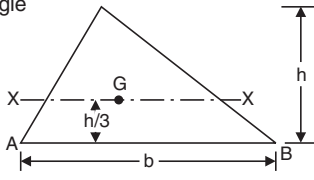
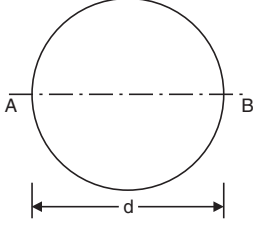
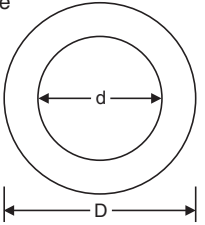
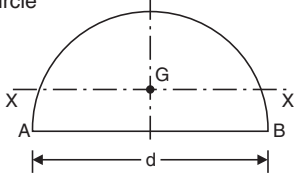
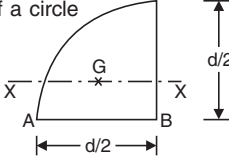
The moment of inertia of common standard sections are presented in Table 4.4.

**Table 4.4** Moment of Inertia of Standard Sections

Shape	Axis	Moment of Inertia
<p>Rectangle</p>	<p>(a) Centroidal axis <math>x</math>-<math>x</math></p> <p>(b) Centroidal axis <math>y</math>-<math>y</math></p> <p>(c) <math>A - B</math></p>	$I_{xx} = \frac{bd^3}{12}$ $I_{yy} = \frac{db^3}{12}$ $I_{AB} = \frac{bd^3}{3}$
<p>Hollow Rectangle</p>	Centroidal axis $x$ - $x$	$I_{xx} = \frac{BD^3 - bd^3}{12}$

(Contd.)

Table 4.4 (Contd.)

Shape	Axis	Moment of Inertia
<p>Triangle</p> 	<p>(a) Centroidal axis <math>x-x</math></p> <p>(b) Base <math>AB</math></p>	$I_{xx} = \frac{bh^3}{36}$ $I_{AB} = \frac{bh^3}{12}$
<p>Circle</p> 	Diametral axis	$I = \frac{\pi d^4}{64}$
<p>Hollow circle</p> 	Diametral axis	$I = \frac{\pi}{64} (D^4 - d^4)$
<p>Semicircle</p> 	<p>(a) <math>A - B</math></p> <p>(b) Centroidal axis</p>	$I_{AB} = \frac{\pi d^4}{128}$ $I_{xx} = 0.0068598 d^4$
<p>Quarter of a circle</p> 	<p>(a) <math>A - B</math></p> <p>(b) Centroidal axis <math>x-x</math></p>	$I_{AB} = \frac{\pi d^4}{256}$ $I_{xx} = 0.00343 d^4$

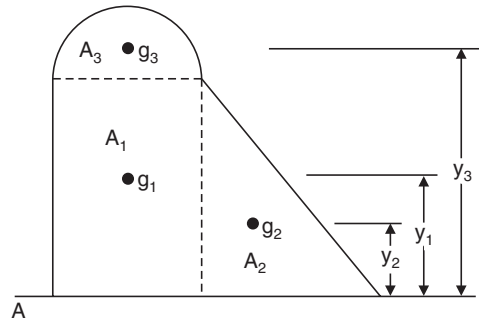
#### 4.7 MOMENT OF INERTIA OF COMPOSITE SECTIONS

Beams and columns having composite sections are commonly used in structures. Moment of inertia of these sections about an axis can be found by the following steps:

- (1) Divide the given figure into a number of simple figures.
- (2) Locate the centroid of each simple figure by inspection or using standard expressions.
- (3) Find the moment of inertia of each simple figure about its centroidal axis. Add the term  $Ay^2$  where  $A$  is the area of the simple figure and  $y$  is the distance of the centroid of the simple figure from the reference axis. This gives moment of inertia of the simple figure about the reference axis.
- (4) Sum up moments of inertia of all simple figures to get the moment of inertia of the composite section.

The procedure given above is illustrated below. Referring to the Fig. 4.51, it is required to find out the moment of inertia of the section about axis  $A-B$ .

(1) The section in the figure is divided into a rectangle, a triangle and a semicircle. The areas of the simple figures  $A_1$ ,  $A_2$  and  $A_3$  are calculated.



**Fig. 4.51**

(2) The centroids of the rectangle ( $g_1$ ), triangle ( $g_2$ ) and semicircle ( $g_3$ ) are located. The distances  $y_1$ ,  $y_2$  and  $y_3$  are found from the axis  $AB$ .

(3) The moment of inertia of the rectangle about its centroid ( $I_{g_1}$ ) is calculated using standard expression. To this, the term  $A_1 y_1^2$  is added to get the moment of inertia about the axis  $AB$  as:

$$I_1 = I_{g_1} + A_1 y_1^2$$

Similarly, the moment of inertia of the triangle ( $I_2 = I_{g_2} + A_2 y_2^2$ ) and of semicircle ( $I_3 = I_{g_3} + A_3 y_3^2$ ) about axis  $AB$  are calculated.

(4) Moment of inertia of the composite section about  $AB$  is given by:

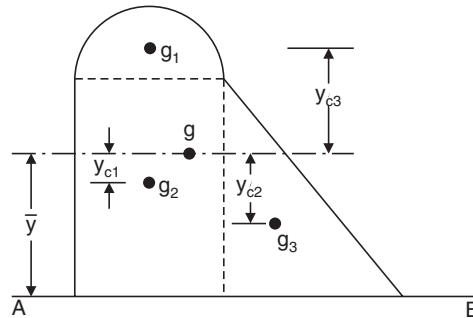
$$\begin{aligned} I_{AB} &= I_1 + I_2 + I_3 \\ &= I_{g_1} + A_1 y_1^2 + I_{g_2} + A_2 y_2^2 + I_{g_3} + A_3 y_3^2 \end{aligned} \quad \dots(4.12)$$

In most engineering problems, moment of inertia about the centroidal axis is required. In such cases, first locate the centroidal axis as discussed in 4.4 and then find the moment of inertia about this axis.

Referring to Fig. 4.52, first the moment of area about any reference axis, say  $AB$  is taken and is divided by the total area of section to locate centroidal axis  $x-x$ . Then the distances of centroid of

individual figures  $y_{c1}$ ,  $y_{c2}$  and  $y_{c3}$  from the axis  $x-x$  are determined. The moment of inertia of the composite section about the centroidal axis  $x-x$  is calculated using the expression:

$$I_{xx} = I_{g1} + A_1^2 y_{c1} + I_{g2} + A_2^2 y_{c2} + I_{g3} + A_3^2 y_{c3} \quad \dots(4.13)$$



**Fig. 4.52**

Sometimes the moment of inertia is found about a convenient axis and then using parallel axis theorem, the moment of inertia about centroidal axis is found.

In the above example, the moment of inertia  $I_{AB}$  is found and  $y_c$ , the distance of CG from axis  $AB$  is calculated. Then from parallel axis theorem,

$$I_{AB} = I_{xx} + Ay_c^2$$

$$I_{xx} = I_{AB} - Ay_c^2$$

where  $A$  is the area of composite section.

**Example 4.12.** Determine the moment of inertia of the section shown in Fig. 4.53 about an axis passing through the centroid and parallel to the top most fibre of the section. Also determine moment of inertia about the axis of symmetry. Hence find radii of gyration.

**Solution:** The given composite section can be divided into two rectangles as follows:

$$\text{Area } A_1 = 150 \times 10 = 1500 \text{ mm}^2$$

$$\text{Area } A_2 = 140 \times 10 = 1400 \text{ mm}^2$$

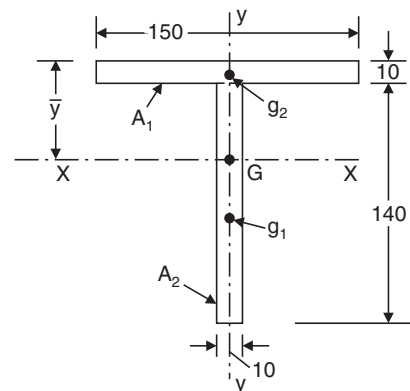
$$\text{Total Area } A = A_1 + A_2 = 2900 \text{ mm}^2$$

Due to symmetry, centroid lies on the symmetric axis  $y-y$ .

The distance of the centroid from the top most fibre is given by:

$$\begin{aligned} y_c &= \frac{\text{Sum of moment of the areas about the top most fibre}}{\text{Total area}} \\ &= \frac{1500 \times 5 + 1400(10 + 70)}{2900} \\ &= 41.21 \text{ mm} \end{aligned}$$

Referring to the centroidal axis  $x-x$  and  $y-y$ , the centroid of  $A_1$  is  $g_1(0.0, 36.21)$  and that of  $A_2$  is  $g_2(0.0, 38.79)$ .



**Fig. 4.53**

Moment of inertia of the section about  $x$ - $x$  axis

$I_{xx}$  = moment of inertia of  $A_1$  about  $x$ - $x$  axis + moment of inertia of  $A_2$  about  $x$ - $x$  axis.

$$\therefore I_{xx} = \frac{150 \times 10^3}{12} + 1500 (36.21)^2 + \frac{10 \times 140^3}{12} + 1400 (38.79)^2$$

i.e.,  $I_{xx} = 63\,72442.5 \text{ mm}^4$

Similarly,

$$I_{yy} = \frac{10 \times 150^3}{12} + \frac{140 \times 10^3}{12} = 2824166.7 \text{ mm}^4$$

Hence, the moment of inertia of the section about an axis passing through the centroid and parallel to the top most fibre is  $6372442.5 \text{ mm}^4$  and moment of inertia of the section about the axis of symmetry is  $2824166.66 \text{ mm}^4$ .

The radius of gyration is given by:

$$k = \sqrt{\frac{I}{A}}$$

$$\therefore k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$= \sqrt{\frac{6372442.5}{2900}}$$

$$k_{xx} = 46.88 \text{ mm}$$

Similarly,  $k_{yy} = \sqrt{\frac{2824166.66}{2900}}$

$$k_{yy} = 31.21 \text{ mm}$$

**Example 4.13.** Determine the moment of inertia of the L-section shown in the Fig. 4.54 about its centroidal axis parallel to the legs. Also find out the polar moment of inertia.

**Solution:** The given section is divided into two rectangles  $A_1$  and  $A_2$ .

Area  $A_1 = 125 \times 10 = 1250 \text{ mm}^2$

Area  $A_2 = 75 \times 10 = 750 \text{ mm}^2$

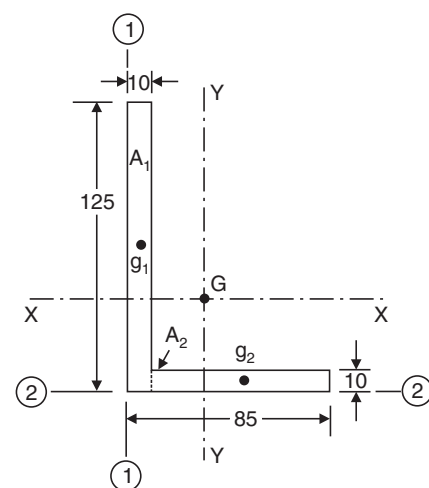
Total Area =  $2000 \text{ mm}^2$

First, the centroid of the given section is to be located.

Two reference axis (1)–(1) and (2)–(2) are chosen as shown in Fig. 4.54.

The distance of centroid from the axis (1)–(1)

$$= \frac{\text{sum of moment of areas } A_1 \text{ and } A_2 \text{ about (1) – (1)}}{\text{Total area}}$$



**Fig. 4.54**

$$\text{i.e., } \bar{x} = \frac{120 \times 5 + 750 \left(10 + \frac{75}{2}\right)}{2000}$$

$$= 20.94 \text{ mm}$$

Similarly,

the distance of the centroid from the axis (2)–(2)

$$= \bar{y} = \frac{1250 \times \frac{125}{2} + 750 \times 5}{2000} = 40.94 \text{ mm}$$

With respect to the centroidal axis  $x$ - $x$  and  $y$ - $y$ , the centroid of  $A_1$  is  $g_1$  (15.94, 21.56) and that of  $A_2$  is  $g_2$  (26.56, 35.94).

$\therefore I_{xx}$  = Moment of inertia of  $A_1$  about  $x$ - $x$  axis + Moment of inertia of  $A_2$  about  $x$ - $x$  axis

$$\therefore I_{xx} = \frac{10 \times 125^3}{12} + 1250 \times 21.56^2 + \frac{75 \times 10^3}{12} + 750 \times 39.94^2$$

$$\text{i.e., } I_{xx} = 3411298.9 \text{ mm}^4$$

Similarly,

$$I_{yy} = \frac{125 \times 10^3}{12} + 1250 \times 15.94^2 + \frac{10 \times 75^3}{12} + 750 \times 26.56^2$$

$$\text{i.e., } I_{yy} = 1208658.9 \text{ mm}^4$$

$$\begin{aligned} \text{Polar moment of inertia} &= I_{xx} + I_{yy} \\ &= 3411298.9 + 12,08658.9 \end{aligned}$$

$$I_{zz} = 4619957.8 \text{ mm}^4$$

**Example 14.** Determine the moment of inertia of the symmetric I-section shown in Fig. 4.55 about its centroidal axis  $x$ - $x$  and  $y$ - $y$ .

Also, determine moment of inertia of the section about a centroidal axis perpendicular to  $x$ - $x$  axis and  $y$ - $y$  axis.

**Solution:** The section is divided into three rectangles  $A_1$ ,  $A_2$  and  $A_3$ .

$$\text{Area } A_1 = 200 \times 9 = 1800 \text{ mm}^2$$

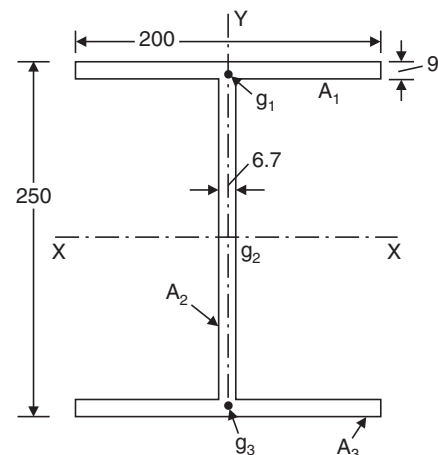
$$\text{Area } A_2 = (250 - 9 \times 2) \times 6.7 = 1554.4 \text{ mm}^2$$

$$\text{Area } A_3 = 200 \times 9 = 1800 \text{ mm}^2$$

$$\text{Total Area } A = 5154.4 \text{ mm}^2$$

The section is symmetrical about both  $x$ - $x$  and  $y$ - $y$  axis. Therefore, its centroid will coincide with the centroid of rectangle  $A_2$ .

With respect to the centroidal axis  $x$ - $x$  and  $y$ - $y$ , the centroid of rectangle  $A_1$  is  $g_1$  (0.0, 120.5), that of  $A_2$  is  $g_2$  (0.0, 0.0) and that of  $A_3$  is  $g_3$  (0.0, 120.5).



**Fig. 4.55**

$I_{xx}$  = Moment of inertia of  $A_1$  + Moment of inertia of  $A_2$   
+ Moment of inertia of  $A_3$  about  $x$ - $x$  axis

$$I_{xx} = \frac{200 \times 9^3}{12} + 1800 \times 120.5^2 + \frac{6.7 \times 232^3}{12} + 0$$

$$+ \frac{200 \times 9^3}{12} + 1800(120.5)^2$$

$$I_{xx} = 5,92,69,202 \text{ mm}^4$$

Similarly,

$$I_{yy} = \frac{9 \times 200^3}{12} + \frac{232 \times 6.7^3}{12} + \frac{9 \times 200^3}{12}$$

$$I_{yy} = 1,20,05,815 \text{ mm}^4$$

Moment of inertia of the section about a centroidal axis perpendicular to  $x$ - $x$  and  $y$ - $y$  axis is nothing but polar moment of inertia, and is given by:

$$I_{xx} = I_{xx} + I_{yy}$$

$$= 59269202 + 12005815$$

$$I_{yy} = 7,12,75,017 \text{ mm}^4$$

**Example 4.15.** Compute the second moment of area of the channel section shown in Fig. 4.56 about centroidal axis  $x$ - $x$  and  $y$ - $y$ .

**Solution:** The section is divided into three rectangles  $A_1$ ,  $A_2$  and  $A_3$ .

Area  $A_1 = 100 \times 13.5 = 1350 \text{ mm}^2$

Area  $A_2 = (400 - 27) \times 8.1 = 3021.3 \text{ mm}^2$

Area  $A_3 = 100 \times 13.5 = 1350.00 \text{ mm}^2$

Total Area  $A = 5721.3 \text{ mm}^2$

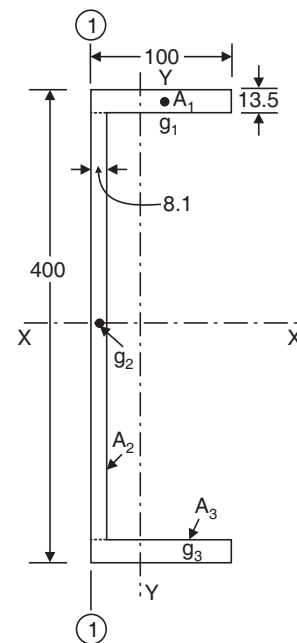
The given section is symmetric about horizontal axis passing through the centroid  $g_2$  of the rectangle  $A_2$ . A reference axis (1)–(1) is chosen as shown in Fig. 4.56.

The distance of the centroid of the section from (1)–(1)

$$= \frac{1350 \times 50 + 3021.3 \times \frac{8.1}{2} + 1350 \times 50}{5721.3}$$

$$= 25.73 \text{ mm}$$

With reference to the centroidal axis  $x$ - $x$  and  $y$ - $y$ , the centroid of the rectangle  $A_1$  is  $g_1(24.27, 193.25)$  that of  $A_2$  is  $g_2(21.68, 0.0)$  and that of  $A_3$  is  $g_3(24.27, 193.25)$ .



**Fig. 4.56**

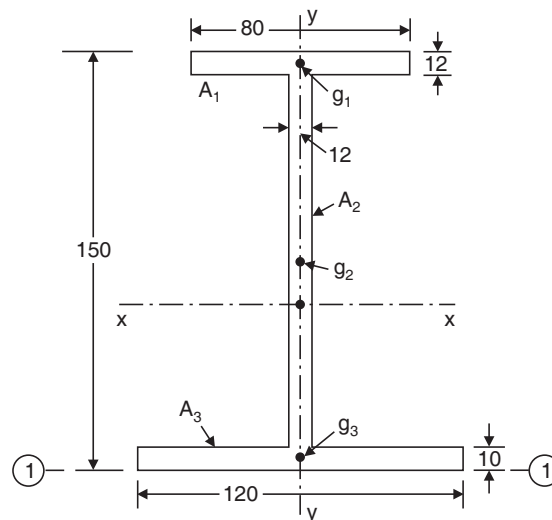


$$\begin{aligned}
 \therefore I_{xx} &= \text{Moment of inertia of } A_1, A_2 \text{ and } A_3 \text{ about } x-x \\
 &= \frac{100 \times 13.5^3}{12} + 1350 \times 193.25^2 \\
 &\quad + \frac{8.1 \times 373^3}{12} + \frac{100 \times 13.5^3}{12} + 1350 \times 193.25^2 \\
 I_{xx} &= 1.359 \times 10^8 \text{ mm}^4
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 I_{yy} &= \frac{13.5 \times 100^3}{12} + 1350 \times 24.27^2 + \frac{273 \times 8.1^3}{12} + 3021.3 \\
 &\quad \times 21.68^2 + \frac{13.5 \times 100^3}{12} + 1350 \times 24.27^2 \\
 I_{yy} &= 52,72557.6 \text{ mm}^4
 \end{aligned}$$

**Example 4.16.** Determine the polar moment of inertia of the I-section shown in the Fig. 4.57. Also determine the radii of gyration with respect to x-x axis and y-y axis.



**Fig. 4.57**

**Solution:** The section is divided into three rectangles as shown in Fig. 4.57

Area	$A_1 = 80 \times 12 = 960 \text{ mm}^2$
Area	$A_2 = (150 - 22) \times 12 = 1536 \text{ mm}^2$
Area	$A_3 = 120 \times 10 = 1200 \text{ mm}^2$
Total area	$A = 3696 \text{ mm}^2$

Due to symmetry, centroid lies on axis y-y. The bottom fibre (1)–(1) is chosen as reference axis to locate the centroid.

The distance of the centroid from (1)–(1)

$$= \frac{\text{Sum of moments of the areas of the rectangles about (1)–(1)}}{\text{Total area of section}}$$

$$= \frac{960 \times (150 - 6) + 1536 \times \left(\frac{128}{2} + 10\right) + 1200 \times 5}{3696}$$

$$= 69.78 \text{ mm}$$

With reference to the centroidal axis  $x$ - $x$  and  $y$ - $y$ , the centroid of the rectangles  $A_1$  is  $g_1$  (0.0, 74.22), that of  $A_2$  is  $g_2$  (0.0, 4.22) and that of  $A_3$  is  $g_3$  (0.0, 64.78).

$$I_{xx} = \frac{80 \times 12^3}{12} + 960 \times 74.22^2 + \frac{12 \times 128^3}{12} + 1536 \times 4.22^2 + \frac{120 \times 10^3}{12} + 1200 \times 64.78^2$$

$$I_{xx} = 1,24,70,028 \text{ mm}^4$$

$$I_{yy} = \frac{12 \times 80^3}{12} + \frac{128 \times 12^3}{12} + \frac{10 \times 120^3}{12}$$

$$= 19,70,432 \text{ mm}^4$$

**Polar moment of inertia**  $= I_{xx} + I_{yy}$

$$= 1,24,70,027 + 19,70,432$$

$$= \mathbf{1,44,40459 \text{ mm}^4}$$

$$\therefore k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{1,24,70,027}{3696}}$$

$$= \mathbf{58.09 \text{ mm}}$$

$$k_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{19,70,432}{3696}}$$

$$= \mathbf{23.09 \text{ mm.}}$$

**Example 4.17.** Determine the moment of inertia of the built-up section shown in Fig. 4.58 about its centroidal axis  $x$ - $x$  and  $y$ - $y$ .

**Solution:** The given composite section may be divided into simple rectangles and triangles as shown in the Fig. 4.58

Area	$A_1 = 100 \times 30 = 3000 \text{ mm}^2$
Area	$A_2 = 100 \times 25 = 2500 \text{ mm}^2$
Area	$A_3 = 200 \times 20 = 4000 \text{ mm}^2$
Area	$A_4 = \frac{1}{2} \times 87.5 \times 20 = 875 \text{ mm}^2$
Area	$A_5 = \frac{1}{2} \times 87.5 \times 20 = 875 \text{ mm}^2$
Total area	$A = 11250 \text{ mm}^2$

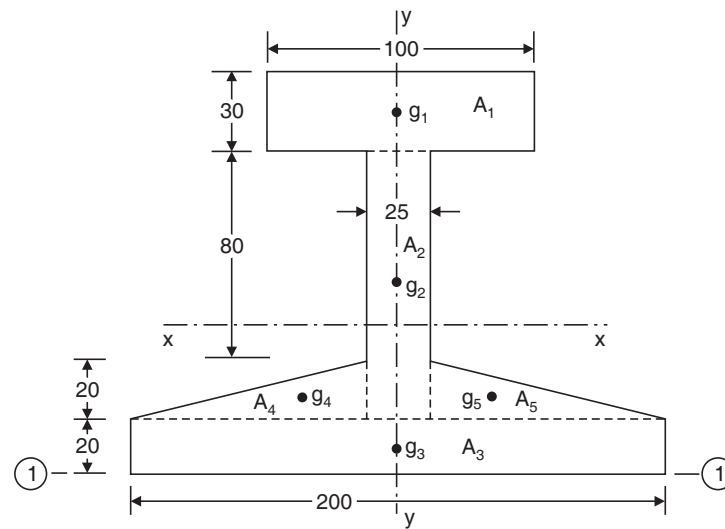


Fig. 4.58

Due to symmetry, centroid lies on the axis  $y$ - $y$ .

A reference axis (1)–(1) is chosen as shown in the figure.

The distance of the centroidal axis from (1)–(1)

$$\bar{y} = \frac{\text{sum of moment of areas about (1)–(1)}}{\text{Total area}}$$

$$= \frac{3000 \times 135 + 2500 \times 70 + 4000 \times 10 + 875 \left( \frac{1}{3} \times 20 + 20 \right) \times 2}{11250}$$

$$= 59.26 \text{ mm}$$

With reference to the centroidal axis  $x$ - $x$  and  $y$ - $y$ , the centroid of the rectangle  $A_1$  is  $g_1$  (0.0, 75.74), that of  $A_2$  is  $g_2$  (0.0, 10.74), that of  $A_3$  is  $g_3$  (0.0, 49.26), the centroid of triangle  $A_4$  is  $g_4$  (41.66, 32.59) and that of  $A_5$  is  $g_5$  (41.66, 32.59).

$$I_{xx} = \frac{100 \times 30^3}{12} + 3000 \times 75.74^2 + \frac{25 \times 100^3}{12} + 2500 \times 10.74^2 + \frac{200 \times 20^3}{12} + 4000$$

$$\times 49.26^2 + \frac{87.5 \times 20^3}{36} + 875 \times 32.59^2 + \frac{87.5 \times 20^3}{36} + 875 \times 32.59^2$$

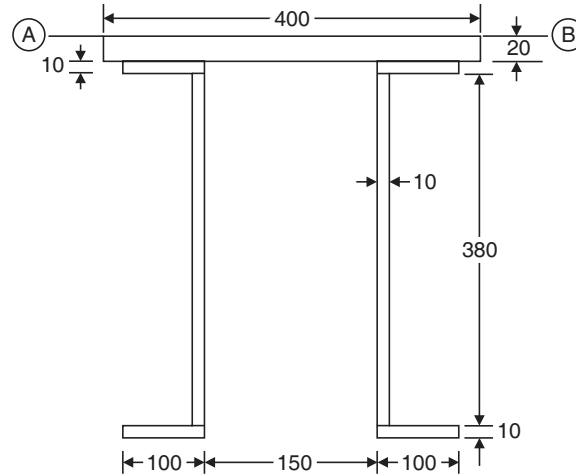
$$I_{xx} = 3,15,43,447 \text{ mm}^4$$

$$I_{yy} = \frac{30 \times 100^3}{12} + \frac{100 \times 25^3}{12} + \frac{20 \times 200^3}{12} + \frac{20 \times 87.5^3}{36} + 875 \times 41.66^2$$

$$+ \frac{20 \times 87.5^3}{36} + 875 \times 41.66^2$$

$$I_{yy} = 1,97,45,122 \text{ mm}^4.$$

**Example 4.18.** Determine the moment of inertia of the built-up section shown in the Fig. 4.59 about an axis  $AB$  passing through the top most fibre of the section as shown.



**Fig. 4.59**

**Solution:** In this problem, it is required to find out the moment of inertia of the section about an axis  $AB$ . So there is no need to find out the position of the centroid.

The given section is split up into simple rectangles as shown in Fig. 4.59.

Now,

Moment of inertia about  $AB$  = Sum of moments of inertia of the rectangle about  $AB$

$$\begin{aligned}
 &= \frac{400 \times 20^3}{12} + 400 \times 20 \times 10^2 + \left[ \frac{100 \times 10^3}{12} + 100 \times 10 \times (20 + 5)^2 \right] \times 2 \\
 &\quad + \left[ \frac{100 \times 380^3}{12} + 10 \times 380 \times (30 + 190)^2 \right] \times 2 \\
 &\quad + \left[ \frac{100 \times 10^3}{12} + 100 \times 10 \times (20 + 10 + 380 + 5)^2 \right] \times 2
 \end{aligned}$$

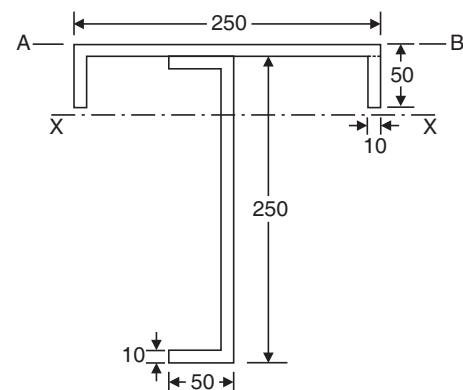
$$I_{AB} = 8.06093 \times 10^8 \text{ mm}^4.$$

**Example 4.19.** Calculate the moment of inertia of the built-up section shown in Fig. 4.60 about a centroidal axis parallel to  $AB$ . All members are 10 mm thick.

**Solution:** The built-up section is divided into six simple rectangles as shown in the figure.

The distance of centroidal axis from  $AB$

$$\begin{aligned}
 &= \frac{\text{Sum of the moment of areas about } AB}{\text{Total area}} \\
 &= \frac{\sum A_i y_i}{A}
 \end{aligned}$$



**Fig. 4.60**

Now,

$$\begin{aligned}\Sigma A_i y_i &= 250 \times 10 \times 5 + 2 \times 40 \times 10 \times (10 + 20) + 40 \times 10 \times (10 + 5) \\ &\quad + 40 \times 10 \times 255 + 250 \times 10 \times (10 + 125) \\ &= 4,82,000 \text{ mm}^3 \\ A &= 2 \times 250 \times 10 + 40 \times 10 \times 4 \\ &= 6600 \text{ mm}^2 \\ \therefore \bar{y} &= \frac{\Sigma A_i y_i}{A} = \frac{482000}{6600} \\ &= 73.03 \text{ mm}\end{aligned}$$

Now,

Moment of inertia about the } = { \begin{matrix} \text{Sum of the moment of inertia} \\ \text{centroidal axis} \end{matrix} \text{ of the individual rectangles}

$$\begin{aligned}&= \frac{250 \times 10^3}{12} + 250 \times 10 \times (73.03 - 5)^2 \\ &\quad + \left[ \frac{10 \times 40^3}{12} + 40 \times 10 (73.03 - 30)^2 \right] \times 2 \\ &\quad + \frac{40 \times 10^3}{12} + 40 \times 10 (73.03 - 15)^2 + \frac{10 \times 250^3}{12} + 250 \\ &\quad \times 10 (73.03 - 135)^2 + \frac{40 \times 10^3}{12} + 40 \times 10 (73.03 - 255)^2\end{aligned}$$

$$I_{xx} = 5,03,99,395 \text{ mm}^4.$$

**Example 4.20.** A built-up section of structural steel consists of a flange plate 400 mm × 20 mm, a web plate 600 mm × 15 mm and two angles 150 mm × 150 mm × 10 mm assembled to form a section as shown in Fig. 4.61. Determine the moment of inertia of the section about the horizontal centroidal axis.

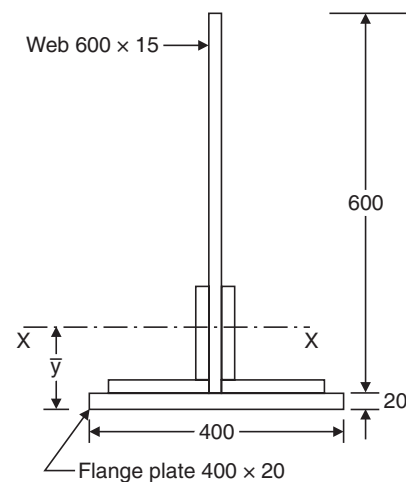
**Solution:** Each angle is divided into two rectangles as shown in Fig. 4.61.

The distance of the centroidal axis from the bottom fibres of section

$$\begin{aligned}&= \frac{\text{Sum of the moment of the areas about bottom fibres}}{\text{Total area of the section}} \\ &= \frac{\Sigma A_i y_i}{A}\end{aligned}$$

Now,

$$\begin{aligned}\Sigma A_i y_i &= 600 \times 15 \times \left( \frac{600}{2} + 20 \right) + 140 \times 10 \\ &\quad \times (70 + 30) \times 2 + 150 \times 10 \times (5 + 20) \\ &\quad \times 2 + 400 \times 20 \times 10 \\ &= 33,15,000 \text{ mm}^3\end{aligned}$$



**Fig. 4.61**

$$A = 600 \times 15 + 140 \times 10 \times 2 + 150 \times 10 \times 2 + 400 \times 20$$

$$= 22,800 \text{ mm}^2$$

$$\therefore \bar{y} = \frac{\sum A_i y_i}{A} = \frac{3315000}{22800}$$

$$= 145.39 \text{ mm}$$

Moment of inertia of the section about centroidal axis  $\left\{ \begin{array}{l} \text{Sum of the moments of inertia of the} \\ \text{all simple figures about centroidal axis} \end{array} \right\} =$

$$= \frac{15 \times 600^3}{12} + 600 \times 15(145.39 - 320)^2$$

$$+ \left[ \frac{10 \times 140^3}{12} + 1400(145.39 - 100)^2 \right] \times 2$$

$$+ \left[ \frac{150 \times 10^3}{12} + 1500 \times (145.39 - 15)^2 \right] \times 2$$

$$+ \frac{400 \times 20^3}{12} + 400 \times 20 \times (145.39 - 10)^2$$

$$I_{xx} = 7.45156 \times 10^8 \text{ mm}^4.$$

**Example 4.21.** Compute the moment of inertia of the  $100 \text{ mm} \times 150 \text{ mm}$  rectangle shown in Fig. 4.62 about  $x$ -axis to which it is inclined at an angle

$$\theta = \sin^{-1} \left( \frac{4}{5} \right).$$

**Solution:** The rectangle is divided into four triangles as shown in the figure. [The lines  $AE$  and  $FC$  are parallel to  $x$ -axis].

$$\text{Now } \theta = \sin^{-1} \left( \frac{4}{5} \right) = 53.13^\circ$$

From the geometry of the Fig. 4.62,

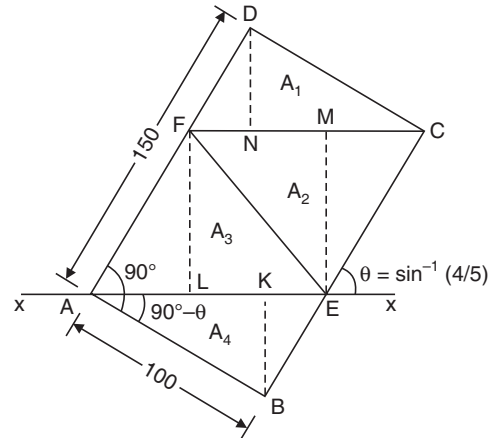
$$BK = AB \sin (90^\circ - \theta)$$

$$= 100 \sin (90^\circ - 53.13^\circ)$$

$$= 60 \text{ mm}$$

$$ND = BK = 60 \text{ mm}$$

$$\therefore FD = \frac{60}{\sin \theta} = \frac{60}{\sin 53.13} = 75 \text{ mm}$$



**Fig. 4.62**

$$\therefore AF = 150 - FD = 75 \text{ mm}$$

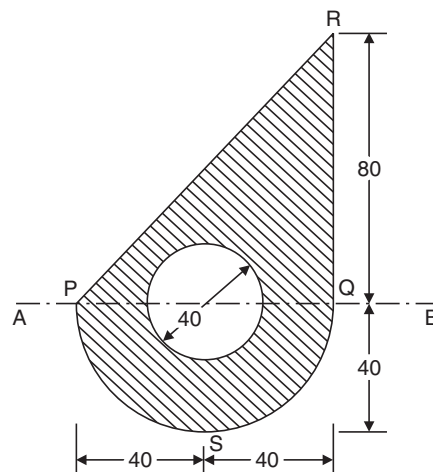
$$\text{Hence } FL = ME = 75 \sin \theta = 60 \text{ mm}$$

$$AE = FC = \frac{AB}{\cos(90^\circ - \theta)} = \frac{100}{0.8} = 125 \text{ mm}$$

Moment of inertia of the section about  $x-x$  axis  $\left. \vphantom{\begin{matrix} \text{Sum of the moments of inertia of individual triangular areas about } x-x \text{ axis} \end{matrix}} \right\} = \left\{ \begin{matrix} \text{Sum of the moments of inertia of individual triangular areas about } x-x \text{ axis} \end{matrix} \right.$

$$\begin{aligned} &= I_{DFC} + I_{FCE} + I_{FEA} + I_{AEB} \\ &= \frac{125 \times 60^3}{36} + \frac{1}{2} \times 125 \times 60 \times \left( 60 + \frac{1}{3} \times 60 \right)^2 \\ &\quad + \frac{125 \times 60^3}{36} + \frac{1}{2} \times 125 \times 60 \times \left( \frac{2}{3} \times 60 \right)^2 + \frac{125 \times 60^3}{36} + \frac{1}{2} \times 125 \\ &\quad \times 60 \times \left( \frac{1}{3} \times 60 \right)^2 + \frac{125 \times 60^3}{36} + \frac{1}{2} \times 125 \times 60 \times \left( \frac{1}{3} \times 60 \right)^2 \\ &I_{xx} = 3,60,00,000 \text{ mm}^4. \end{aligned}$$

**Example 4.22.** Find moment of inertia of the shaded area shown in the Fig. 4.63 about the axis AB.



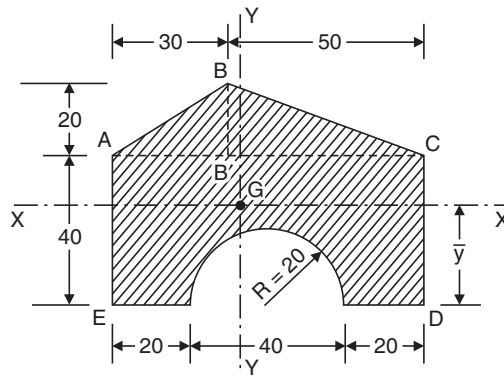
**Fig. 4.63**

**Solution:** The section is divided into a triangle  $PQR$ , a semicircle  $PSQ$  having base on axis  $AB$  and a circle having its centre on axis  $AB$ .

Now,

$$\begin{aligned} \text{Moment of inertia of the section about axis } AB &= \begin{cases} \text{Moment of inertia of triangle } PQR \text{ about } AB \\ + \text{Moment of inertia of semicircle } PSQ \text{ about } AB \\ - \text{moment of inertia of circle about } AB \end{cases} \\ &= \frac{80 \times 80^3}{12} + \frac{\pi}{128} \times 80^4 - \frac{\pi}{64} \times 40^4 \\ I_{AB} &= 42,92,979 \text{ mm}^4. \end{aligned}$$

**Example 4.23.** Find the second moment of the shaded portion shown in the Fig. 4.64 about its centroidal axis.



**Fig. 4.64**

**Solution:** The section is divided into three simple figures viz., a triangle  $ABC$ , a rectangle  $ACDE$  and a semicircle.

Total Area = Area of triangle  $ABC$  + Area of rectangle  $ACDE$  – Area of semicircle

$$\begin{aligned} A &= \frac{1}{2} \times 80 \times 20 + 40 \times 80 - \frac{1}{2} \times \pi \times 20^2 \\ &= 3371.68 \end{aligned}$$

$$\begin{aligned} A \bar{y} &= \frac{1}{2} \times 80 \times 20 \left( \frac{1}{3} \times 20 + 40 \right) + 40 \times 80 \times 20 - \frac{1}{2} \times \pi \times 20^2 \times \frac{4 \times 20}{3\pi} \\ &= 95991.77 \end{aligned}$$

$$\therefore \bar{y} = \frac{95991.77}{3371.6} = 28.47 \text{ mm}$$

$$\begin{aligned} A \bar{x} &= \frac{1}{2} \times 30 \times 20 \times \frac{2}{3} \times 30 + \frac{1}{2} \times 50 \times 20 \times \left( \frac{1}{3} \times 50 \times 30 \right) \\ &\quad + 40 \times 80 \times 40 - \frac{1}{2} \times \pi \times 20^2 \times 40 \end{aligned}$$



$$= 132203.6$$

$$\therefore \bar{x} = \frac{A\bar{x}}{A} = \frac{132203.6}{3371.68} = 37.21 \text{ mm}$$

$$\left. \begin{array}{l} \text{Moment of inertia about} \\ \text{centroidal } x\text{-}x \text{ axis} \end{array} \right\} = \left\{ \begin{array}{l} \text{Moment of inertia of triangle } ABC \text{ about} \\ x\text{-}x \text{ axis} + \text{Moment of inertia of rectangle} \\ \text{about } x\text{-}x \text{ axis} - \text{moment of semicircle} \\ \text{about } x\text{-}x \text{ axis} \end{array} \right.$$

$$\begin{aligned} \therefore I_{xx} &= \frac{80 \times 20^3}{36} + \frac{1}{2} \times 80 \times 20 \left( 60 - \frac{2}{3} \times 20 - 28.47 \right)^2 \\ &\quad + \frac{80 \times 40^3}{12} + 80 \times 40 \times (28.47 - 20)^2 \\ &\quad - \left[ 0.0068598 \times 20^4 + \frac{1}{2} \pi \times 20^2 \left( 28.47 - \frac{4 \times 20}{3\pi} \right)^2 \right] \end{aligned}$$

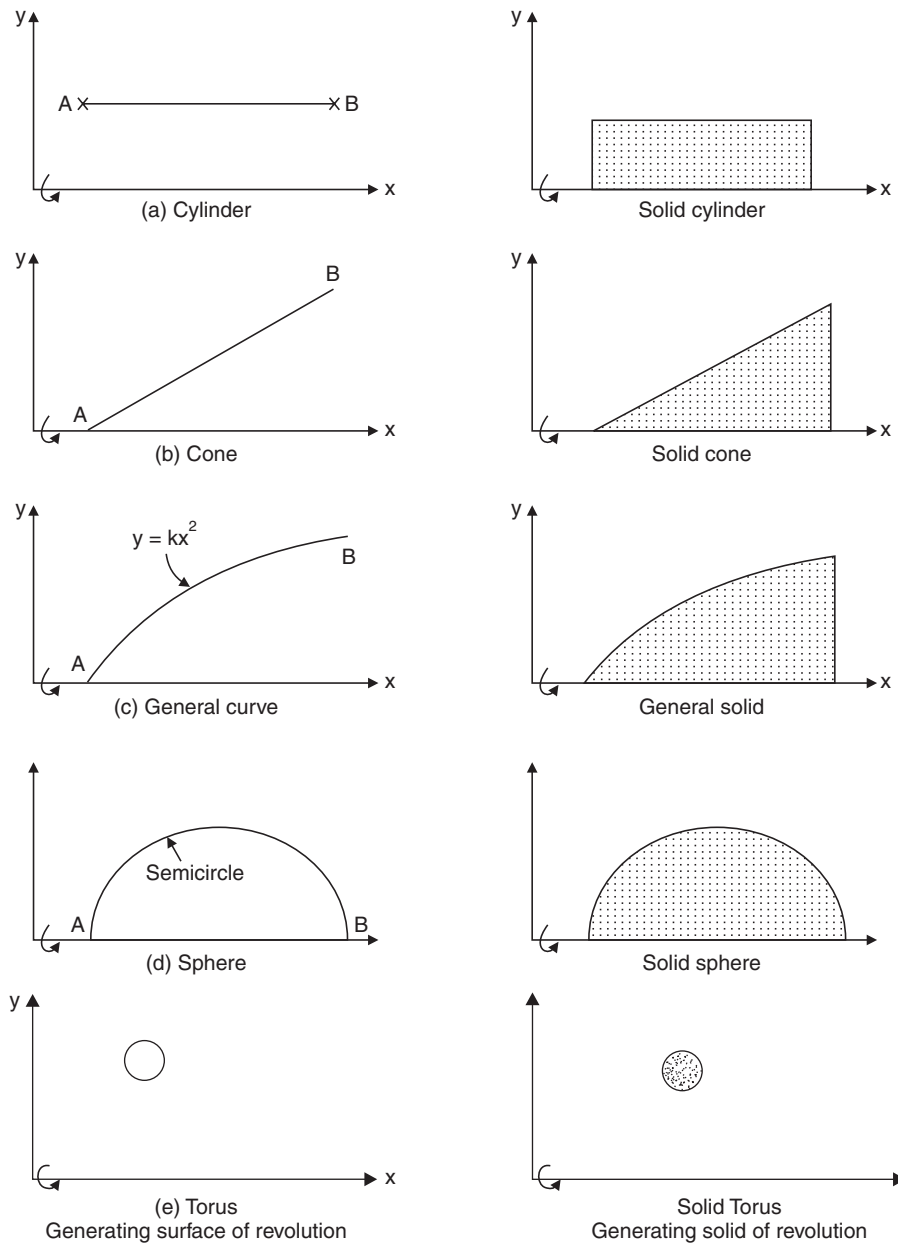
$$I_{xx} = 6,86,944 \text{ mm}^4.$$

Similarly,

$$\begin{aligned} I_{yy} &= \frac{20 \times 30^3}{36} + \frac{1}{2} \times 20 \times 30 \left( 39.21 - \frac{2}{3} \times 30 \right)^2 + \frac{20 \times 50^3}{36} \\ &\quad + \frac{1}{2} \times 20 \times 50 \times \left[ 39.21 - \left( 30 + \frac{1}{3} \times 50 \right) \right]^2 + \frac{40 \times 80^3}{12} \\ &\quad + 40 \times 80(39.21 - 40)^2 - \frac{1}{2} \times \frac{\pi}{64} \times 40^4 - \frac{1}{2} \times \frac{\pi}{4} \\ &\quad \times 40^2 (40 - 39.21)^2 \\ &= 1868392 \text{ mm}^4. \end{aligned}$$

#### 4.8 THEOREMS OF PAPPUS-GULDINUS

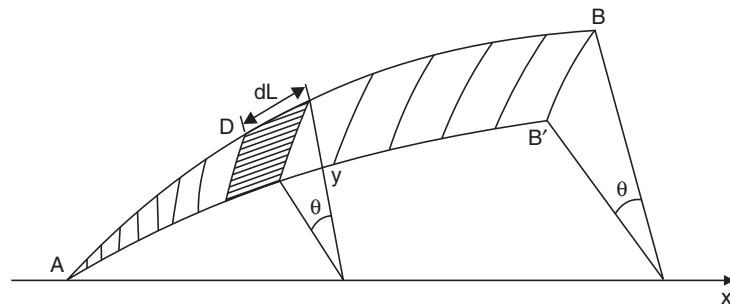
There are two important theorems, first proposed by Greek scientist (about 340 AD) and then restated by Swiss mathematician Paul Guldinus (1640) for determining the surface area and volumes generated by rotating a curve and a plane area about a non-intersecting axis, some of which are shown in Fig. 4.65. These theorems are known as Pappus-Guldinus theorems.

**Fig. 4.65****Theorem I**

*The area of surface generated by revolving a plane curve about a non-intersecting axis in the plane of the curve is equal to the length of the generating curve times the distance travelled by the centroid of the curve in the rotation.*

*Proof:* Figure 4.66 shows the isometric view of the plane curve rotated about  $x$ -axis by angle  $\theta$ . We are interested in finding the surface area generated by rotating the curve  $AB$ . Let  $dL$  be the elemental length on the curve at  $D$ . Its coordinate be  $y$ . Then the elemental surface area generated by this element at  $D$

$$\begin{aligned} dA &= dL(y \theta) \\ \therefore A &= \int dL(y \theta) \\ &= \theta \int y dL \\ &= \theta Ly_c \\ &= L(y_c \theta) \end{aligned}$$



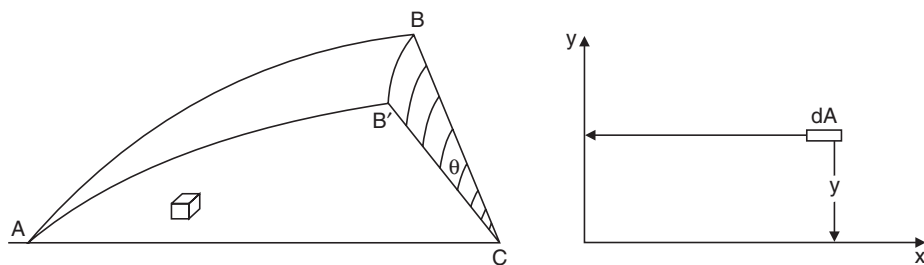
**Fig. 4.66**

Thus we get area of the surface generated as length of the generating curve times the distance travelled by the centroid.

### Theorem II

*The volume of the solid generated by revolving a plane area about a non-intersecting axis in the plane is equal to the area of the generating plane times the distance travelled by the centroid of the plane area during the rotation.*

*Proof:* Consider the plane area  $ABC$ , which is rotated through an angle  $\theta$  about  $x$ -axis as shown in Fig. 4.67.



**Fig. 4.67**

Let  $dA$  be the elemental area of distance  $y$  from  $x$ -axis. Then the volume generated by this area during rotation is given by

$$\begin{aligned} dV &= dA/y\theta \\ \therefore V &= \int dA/y\theta \\ &= \theta \int y dA \\ &= \theta A y_c \\ &= A(y_c \theta) \end{aligned}$$

Thus the volume of the solid generated is area times the distance travelled by its centroid during the rotation. Using Pappus-Guldinus theorems surface area and volumes of cones and spheres can be calculated as shown below:

(i) *Surface area of a cone*: Referring to Fig. 4.68(a),  
Length of the line generating cone =  $L$

$$\text{Distance of centroid of the line from the axis of rotation} = y = \frac{R}{2}$$

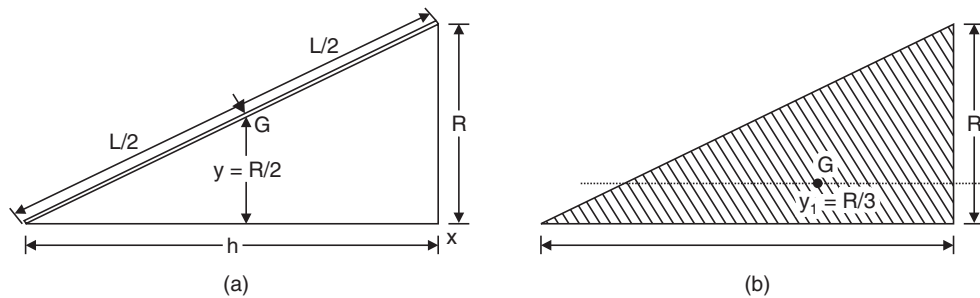
$$\text{In one revolution centroid moves by distance} = 2\pi y = \pi R$$

$$\therefore \text{Surface area} = L \times (\pi R) = \pi RL$$

(ii) *Volume of a cone*: Referring to Fig. 4.68(b),

$$\text{Area generating solid cone} = \frac{1}{2} hR$$

$$\text{Centroid } G \text{ is at a distance } y = \frac{R}{3}$$



**Fig. 4.68**

$$\therefore \text{The distance moved by the centroid in one revolution} = 2\pi y = 2\pi \frac{R}{3}$$

$$\begin{aligned} \therefore \text{Volume of solid cone} &= \frac{1}{2} hR \times \frac{2\pi R}{3} \\ &= \frac{\pi R^2 h}{3} \end{aligned}$$

(iii) *Surface area of sphere*: Sphere of radius  $R$  is obtained by rotating a semi circular arc of radius  $R$  about its diametral axis. Referring to Fig. 4.69(a),

Length of the arc  $= \pi R$

Centroid of the arc is at  $y = \frac{2R}{\pi}$  from the diametral axis (i.e. axis of rotation)

$\therefore$  Distance travelled by centroid of the arc in one revolution

$$= 2\pi y = 2\pi \frac{2R}{\pi} = 4R$$

$\therefore$  Surface area of sphere  $= \pi R \times 4R$   
 $= 4\pi R^2$

(iv) *Volume of sphere*: Solid sphere of radius  $R$  is obtained by rotating a semicircular area about its diametral axis. Referring to Fig. 4.69(b).

$$\text{Area of semicircle} = \frac{\pi R^2}{2}$$

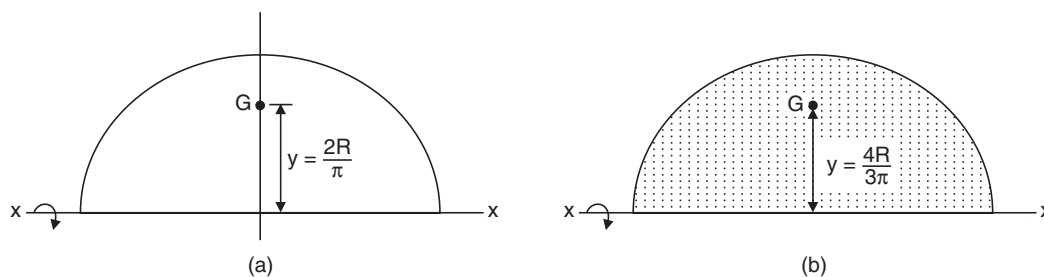
Distance of centroid of semicircular area from its centroidal axis

$$= y = \frac{4R}{3\pi}$$

$\therefore$  The distance travelled by the centroid in one revolution

$$= 2\pi y = 2\pi \frac{4R}{3\pi} = \frac{8R}{3}$$

$\therefore$  Volume of sphere  $= \frac{\pi R^2}{2} \times \frac{8R}{3}$   
 $= \frac{4\pi R^3}{3}$



**Fig. 4.69**

#### 4.9 CENTRE OF GRAVITY OF SOLIDS

Centre of gravity of solids may be found using eqn. (4.1) which will be same as those found from eqns. (4.2) and (4.3) if the mass is uniform. Hence centre of gravity of solids, centre of gravity of mass or centroid of volumes is the same for all solids with uniform mass. For standard solids, the

centre of gravity may be found from first principle and the results obtained for standard solids may be used to find centre of gravity of composite solids. The procedure is illustrated with examples 4.24 to 4.27.

**Example 4.24.** Locate the centre of gravity of the right circular cone of base radius  $r$  and height  $h$  shown in Fig. 4.70.

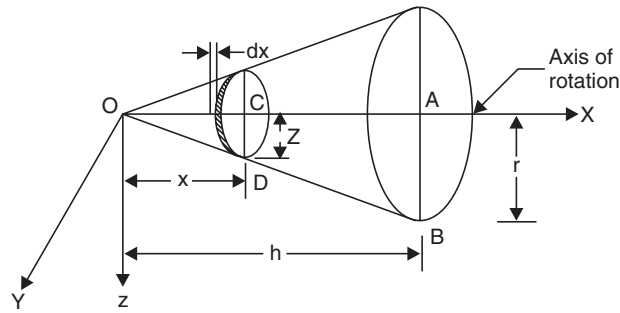


Fig. 4.70

**Solution:** Taking origin at the vertex of the cone and selecting the axis as shown in Fig. 4.70, it can be observed that due to symmetry the coordinates of centre of gravity  $\bar{y}$  and  $\bar{z}$  are equal to zero, i.e. the centre of gravity lies on the axis of rotation of the cone. To find its distance  $\bar{x}$  from the vertex, consider an elemental plate at a distance  $x$ . Let the thickness of the elemental plate be  $dx$ . From the similar triangles  $OAB$  and  $OCD$ , the radius of elemental plate  $z$  is given by

$$z = \frac{x}{h} r$$

$\therefore$  Volume of the elemental plate  $dv$

$$dv = \pi z^2 dx = \pi x^2 \frac{r^2}{h^2} dx$$

If  $\gamma$  is the unit weight of the material of the cone, then weight of the elemental plate is given by:

$$dW = \gamma \pi x^2 \frac{r^2}{h^2} dx \quad \dots(i)$$

$$\begin{aligned} W &= \int_0^h \gamma \frac{\pi r^2}{h^2} x^2 dx \\ &= \gamma \frac{\pi r^2}{h^2} \left[ \frac{x^3}{3} \right]_0^h \\ &= \gamma \pi \frac{r^2 h}{3} \quad \dots(ii) \end{aligned}$$

$$\left[ \text{Note: } \frac{\pi r^2 h}{3} \text{ is volume of cone} \right]$$

Now, substituting the value of  $dW$  in (i), above, we get:

$$\int x \cdot dW = \int_0^h \gamma \frac{\pi r^2}{h^2} x^2 \cdot x \cdot dx$$

$$\begin{aligned}
 &= \gamma \frac{\pi r^2}{h^2} \left[ \frac{x^4}{4} \right]_0^h \\
 &= \gamma \frac{\pi r^2 h^2}{4} \quad \dots(iii)
 \end{aligned}$$

From eqn. 4.1,

$$\begin{aligned}
 W\bar{x} &= \int x dW \\
 \text{i.e., } \frac{\pi r^2 h}{3} \bar{x} &= \frac{\gamma \pi r^2 h^2}{4} \\
 \therefore \bar{x} &= \frac{3}{4} h
 \end{aligned}$$

Thus, in a right circular cone, centre of gravity lies at a distance  $\frac{3}{4} h$  from vertex along the axis of rotation i.e., at a distance  $\frac{h}{4}$  from the base.

**Example 4.25.** Determine the centre of gravity of a solid hemisphere of radius  $r$  from its diametral axis.

**Solution:** Due to symmetry, centre of gravity lies on the axis of rotation. To find its distance  $\bar{x}$  from the base along the axis of rotation, consider an elemental plate at a distance  $x$  as shown in Fig. 4.71.

$$\begin{aligned}
 \text{Now, } x^2 + z^2 &= r^2 \\
 z^2 &= r^2 - x^2
 \end{aligned}$$

Volume of elemental plate

$$dv = \pi z^2 dx = \pi(r^2 - x^2)dx$$

$\therefore$  Weight of elemental plate

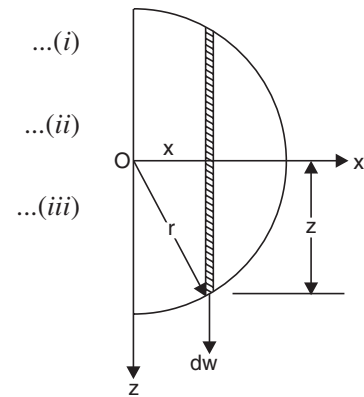
$$dW = \gamma dv = \gamma \pi(r^2 - x^2)dx$$

$\therefore$  Weight of hemisphere

$$\begin{aligned}
 W &= \int dW = \int_0^r \gamma \pi(r^2 - x^2)dx \\
 &= \gamma \pi \left[ r^2 x - \frac{x^3}{3} \right]_0^r \\
 &= \frac{2\gamma \pi r^3}{3} \quad \dots(iv)
 \end{aligned}$$

Moment of weight about  $z$  axis

$$= \int_0^r x dW$$



**Fig. 4.71**

$$\begin{aligned}
 &= \int_0^r x \pi(r^2 - x^2) dx \\
 &= \pi \left[ r^2 \frac{x^2}{2} - \frac{x^4}{4} \right]_0^r \\
 &= \frac{\pi r^4}{4} \quad \dots(v)
 \end{aligned}$$

$\therefore \bar{x}$ , the distance of centre of gravity from base is given by:

$$W\bar{x} = \int_0^r x dw$$

i.e., From (iv) and (v) above, we get

$$\frac{2\gamma\pi r^3}{3} \bar{x} = \frac{\gamma\pi r^4}{4} \quad \bar{x} = \frac{3}{8} r$$

Thus, the centre of gravity of a solid hemisphere of radius  $r$  is at a distance  $\frac{3}{8}r$  from its diametral axis.

**Example 4.26.** Determine the maximum height  $h$  of the cylindrical portion of the body with hemispherical base shown in Fig. 4.72 so that it is in stable equilibrium on its base.

**Solution:** The body will be stable on its base as long as its centre of gravity is in hemispherical base. The limiting case is when it is on the plane  $x-x$  shown in the figure.

Centroid lies on the axis of rotation.

Mass of cylindrical portion

$$m_1 = \pi r^2 h \rho, \text{ where } \rho \text{ is unit mass of material.}$$

Its centre of gravity  $g_1$  is at a height

$$z_1 = \frac{h}{2} \text{ from } x \text{ axis.}$$

Mass of hemispherical portion

$$m_2 = \rho \frac{2\pi r^3}{3}$$

and its CG is at a distance

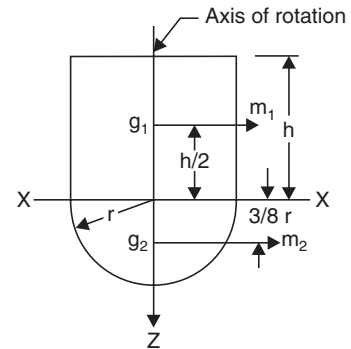
$$z_2 = \frac{3r}{8} \text{ from } x-x \text{ plane.}$$

Since centroid is to be on  $x-x$  plane  $\bar{z} = 0$

i.e.,  $\Sigma m_i z_i = 0$

$$\therefore \frac{m_1 h}{2} - m_2 \frac{3}{8} r = 0$$

$$\pi r^2 h \rho \frac{h}{2} = \rho \frac{2\pi r^3}{3} \frac{3}{8} r$$



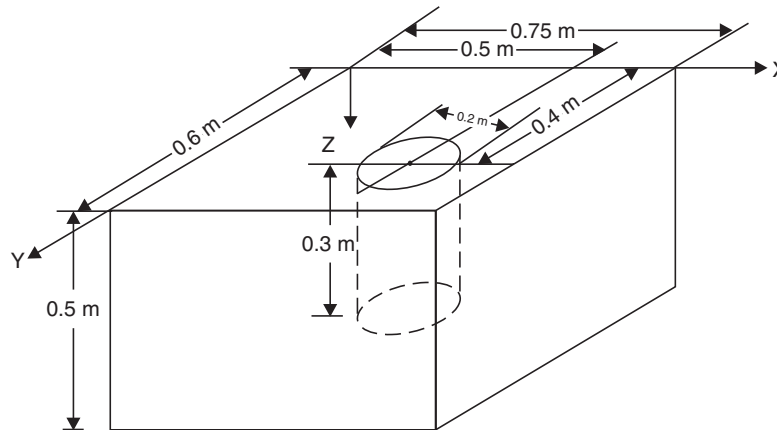
**Fig. 4.72**



$$\therefore h^2 = \frac{1}{2} r^2$$

or 
$$h = \frac{r}{\sqrt{2}} = 0.707 r$$

**Example 4.27.** A concrete block of size  $0.60 \text{ m} \times 0.75 \text{ m} \times 0.5 \text{ m}$  is cast with a hole of diameter  $0.2 \text{ m}$  and depth  $0.3 \text{ m}$  as shown in Fig. 4.73. The hole is completely filled with steel balls weighing  $2500 \text{ N}$ . Locate the centre of gravity of the body. Take the weight of concrete  $= 25000 \text{ N/m}^3$ .



**Fig. 4.73**

**Solution:** Weight of solid concrete block:

$$W_1 = 0.6 \times 0.75 \times 0.5 \times 25000 = 5625 \text{ N}$$

Weight of concrete ( $W_2$ ) removed for making hole:

$$W_2 = \frac{\pi}{4} \times 0.2^2 \times 0.3 \times 25000 = 235.62 \text{ N}$$

Taking origin as shown in the figure, the centre of gravity of solid block is  $(0.375, 0.3, 0.25)$  and that of hollow portion is  $(0.5, 0.4, 0.15)$ . The following table may be prepared now:

**Table**

Simple Body	$W_i$	$x_i$	$W_i x_i$	$y_i$	$W_i y_i$	$z_i$	$W_i z_i$
1. Solid block	5625	0.375	2109.38	0.3	1687.5	0.25	1406.25
2. Hole in concrete block	-235.62	0.5	-117.81	0.4	-94.25	0.15	-35.34
3. Steel balls	2500	0.5	1250.0	0.4	1000.0	0.15	375.0

$$\Sigma W_i = 7889.38 \quad \Sigma W_i x_i = 3241.57 \quad \Sigma W_i y_i = 2593.25 \quad \Sigma W_i z_i = 1745.91$$

$$\therefore \bar{x} = \frac{\Sigma W_i x_i}{W} = \frac{\Sigma W_i x_i}{\Sigma W_i} \quad \bar{x} = \frac{3241.57}{7889.38} = 0.411 \text{ m}$$

Similarly, 
$$\bar{y} = \frac{2593.25}{7889.38} = 0.329 \text{ m}$$

$$\bar{z} = \frac{1745.91}{7889.38} = 0.221 \text{ m}$$

**IMPORTANT FORMULAE**

1. Area of sector of a circle =  $R^2 \alpha$
2. Area of parabolic spandrel
  - (i) if  $y = kx^2$ ,  $A = \frac{1}{3} ha = \frac{1}{3} \times$  the area of rectangle of size  $a \times h$
  - (ii) if  $y^2 = kx$ ,  $A = \frac{2}{3} ha = \frac{2}{3} \times$  the area of rectangle of size  $a \times h$ .
3. Surface area of the cone =  $\pi Rl$
4. Surface area of the sphere =  $4\pi R^2$
5. Volume of a cone =  $\frac{\pi R^2 h}{3}$
6. Volume of a sphere =  $\frac{4}{3} \pi R^3$
7. Centroid of a arc of a circle is at  $x_c = \frac{R \sin \alpha}{\alpha}$  from the centre of circle on the symmetric axis.
8. Centroid of a composite figure is given by
 
$$x_c = \frac{\sum A_i x_i}{A}, \quad y_c = \frac{\sum A_i y_i}{A}.$$
9. Centroid of simple figure from the reference axis
 
$$\bar{y} = \frac{\int y dA}{A}.$$
10. For centroid of standard figures refer Table 4.2.
11.  $I_{yy} = \sum x_i^2 dA_i$  and  $I_{xx} = \sum y_i^2 dA_i$ ,  $I_{zz} = \sum r_i^2 dA_i = \int r^2 dA$ .
12. Radius of gyration  $k = \sqrt{\frac{I}{A}}$  i.e.  $I = Ak^2$ .
13.  $I_{zz} = I_{xx} + I_{yy}$ .
14.  $I_{AB} = I_{GG} + Ay_c^2$ .
15. Moment of inertia of standard sections are as shown in Table 4.4.
16. Pappus-Guldinus Theorems:
  - (i) The area of surface generated by revolving a plane curve about a non-intersecting axis in the plane of the curve is equal to the length of the generating curve times the distance travelled by the centroid of the curve in the rotation.
  - (ii) The volume of the solid generated by a plane area about a non-intersecting axis in the plane is equal to the area of the generating plane times the distance travelled by the centroid of the plane area during the rotation.

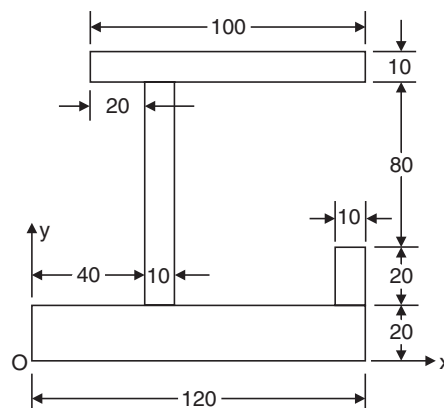
17. In a right circular cone, the centre of gravity lies at a distance  $\frac{3}{4} \times \text{height}$  from the vertex along the axis of rotation.
18. The centre of gravity of a solid hemisphere of radius  $r$  is at a distance  $\frac{3}{8} \times r$  from its diametral axis.

### THEORY QUESTIONS

1. Determine the centroid of an arc of radius  $R$  from first principle.
2. Distinguish between centroid and centre of gravity.
3. Determine the centroid of a triangle of base width ' $b$ ' and height ' $h$ ' by the method of integration.
4. Locate the centroid of a semicircle from its diametral axis using the method of integration.
5. Explain the terms moment of inertia and radius of gyration of a plane figure.
6. State and prove
  - (a) Perpendicular axis theorem
  - (b) Parallel axis theorem of moment of inertia.
7. Determine the moment of inertia of the areas specified below by first principle:
  - (i) Triangle of base width  $b$  and height  $h$  about its centroid axis parallel to the base.
  - (ii) A semicircle about its centroidal axis parallel to the diametral axis.
8. State and explain theorems of Pappus-Guldinus.
9. Locate the centre of gravity of the right circular cone of base radius  $R$  and height  $h$ .
10. Determine the centre of gravity of a solid hemisphere of radius  $R$  from its diametral axis.

### PROBLEMS FOR EXERCISE

1. Determine the centroid of the built-up section in Fig. 4.74. Express the coordinates of centroid with respect to  $x$  and  $y$  axes shown. [Ans.  $\bar{x} = 48.91$  mm;  $\bar{y} = 61.30$  mm]



**Fig. 4.74**

2. Determine the centroid of the reinforced concrete retaining wall section shown in Fig. 4.75.

[Ans.  $\bar{x} = 1.848$  m;  $\bar{y} = 1.825$  m]

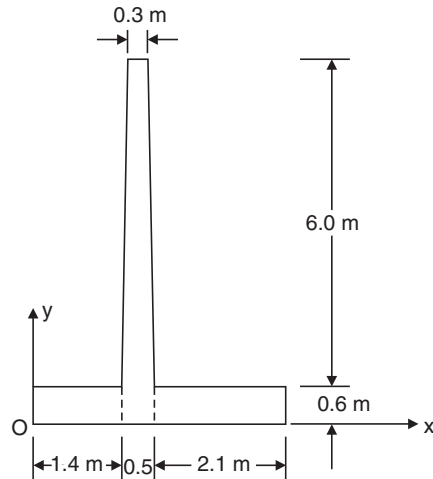


Fig. 4.75

3. Find the coordinates of the centroid of the shaded area with respect to the axes shown in Fig. 4.76.

[Ans.  $x = 43.98$  mm;  $y = 70.15$  mm]

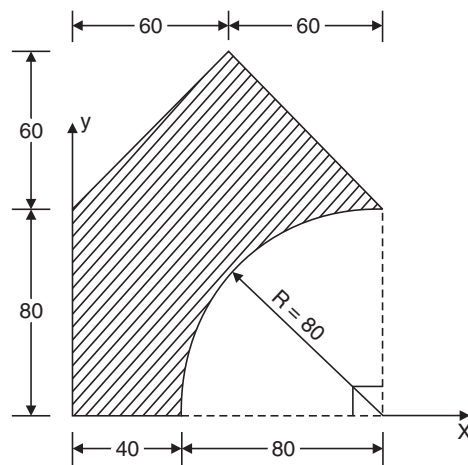


Fig. 4.76

4. A circular plate of uniform thickness and of diameter 500 mm as shown in Fig. 4.77 has two circular holes of 40 mm diameter each. Where should a 80 mm diameter hole be drilled so that the centre of gravity of the plate will be at the geometric centre.

[Ans.  $x = 50$  mm;  $y = 37.5$  mm]

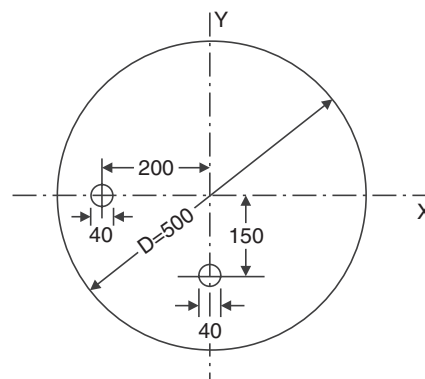


Fig. 4.77

5. With respect to the coordinate axes  $x$  and  $y$  locate the centroid of the shaded area shown in Fig. 4.78. [Ans.  $\bar{x} = 97.47$  mm;  $\bar{y} = 70.77$  mm]

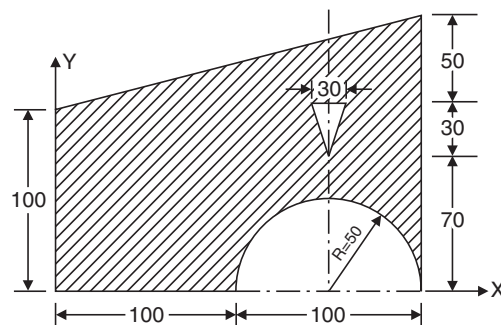


Fig. 4.78

6. Locate the centroid of the plane area shown in Fig. 4.79. [Ans.  $\bar{x} = 104.10$  mm;  $\bar{y} = 44.30$  mm]

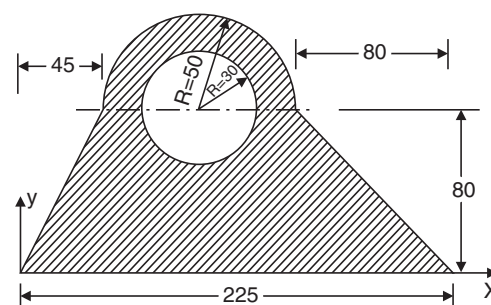


Fig. 4.79

7. Determine the coordinates of the centroid of shaded as shown in Fig. 4.80 with respect to the corner point  $O$ . Take  $x = 40$  mm. [Ans.  $\bar{x} = 260.07$  mm;  $\bar{y} = 113.95$  mm]

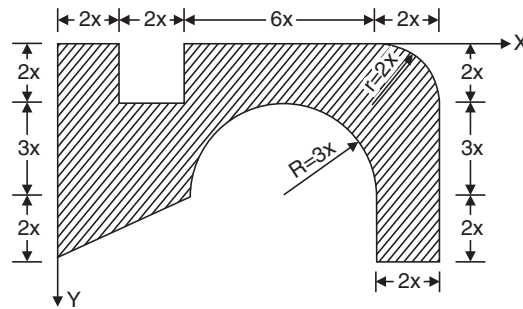


Fig. 4.80

8.  $ABCD$  is a square section of sides 100 mm. Determine the ratio of moment of inertia of the section about centroidal axis parallel to a side to that about diagonal  $AC$ . [Ans. 1]
9. The cross-section of a rectangular hollow beam is as shown in Fig. 4.81. Determine the polar moment of inertia of the section about centroidal axes.

[Ans.  $I_{xx} = 1,05,38,667 \text{ mm}^4$ ;  $I_{yy} = 49,06,667 \text{ mm}^4$ ;  $I_{zz} = 1,54,45,334 \text{ mm}^4$ ]

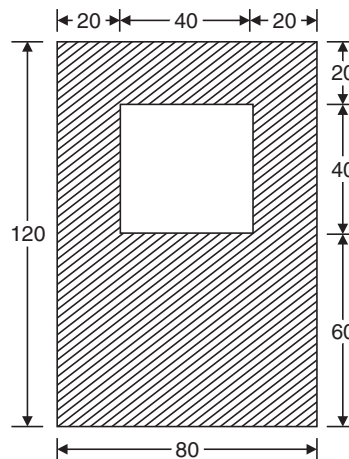


Fig. 4.81

10. The cross-section of a prestressed concrete beam is shown in Fig. 4.82. Calculate the moment of inertia of this section about the centroidal axes parallel to and perpendicular to top edge. Also determine the radii of gyration. [Ans.  $I_{xx} = 1.15668 \times 10^{10} \text{ mm}^4$ ;  $k_{xx} = 231.95 \text{ mm}$ ;  $I_{yy} = 8.75729 \times 10^9 \text{ mm}^4$ ;  $k_{yy} = 201.82 \text{ mm}$ ]

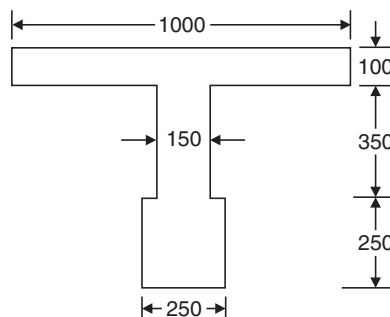


Fig. 4.82

11. The strength of a 400 mm deep and 200 mm wide *I*-beam of uniform thickness 10 mm, is increased by welding a 250 mm wide and 20 mm thick plate to its upper flanges as shown in Fig. 4.83. Determine the moment of inertia and the radii of gyration of the composite section with respect to centroidal axes parallel to and perpendicular to the bottom edge *AB*.

[Ans.  $I_{xx} = 3.32393 \times 10^8 \text{ mm}^4$ ;  $k_{xx} = 161.15 \text{ mm}$ ;  
 $I_{yy} = 3,94,06,667 \text{ mm}^4$ ;  $k_{yy} = 55.49 \text{ mm}$ ]

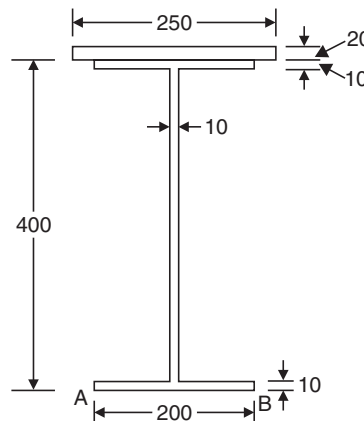


Fig. 4.83

12. The cross-section of a gantry girder is as shown in Fig. 4.84. It is made up of an *I*-section of depth 450 mm, flange width 200 mm and a channel of size 400 mm  $\times$  150 mm. Thickness of all members is 10 mm. Find the moment of inertia of the section about the horizontal centroid axis.

[Ans.  $I_{xx} = 4.2198 \times 10^8 \text{ mm}^4$ ]

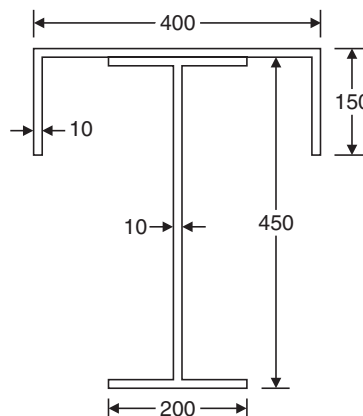
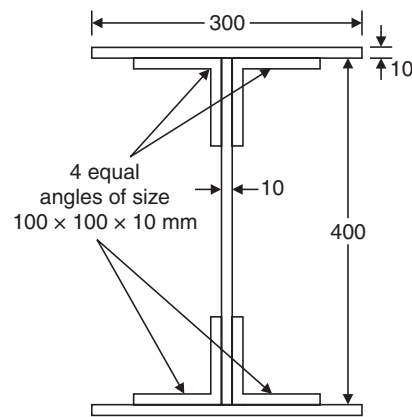


Fig. 4.84

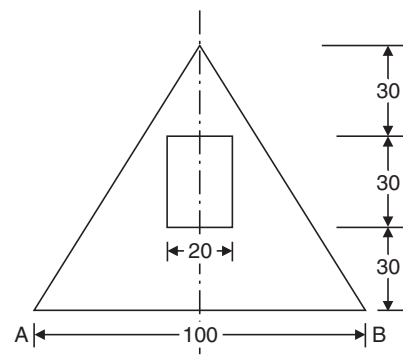
13. A plate girder is made up of a web plate of size 400 mm  $\times$  10 mm, four angles of size 100 mm  $\times$  100 mm  $\times$  10 mm and cover plates of size 300 mm  $\times$  10 mm as shown in Fig. 4.85. Determine the moment of inertia about horizontal and vertical centroidal axes.

[Ans.  $I_{xx} = 5.35786 \times 10^8 \text{ mm}^4$ ;  $I_{yy} = 6,08,50,667 \text{ mm}^4$ ]

**Fig. 4.85**

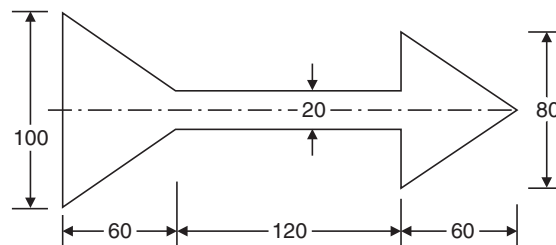
14. Determine the moment of inertia and radii of gyration of the area shown in Fig. 4.86 about the base  $A-B$  and the centroidal axis parallel to  $AB$ .

[Ans.  $I_{AB} = 48,15,000 \text{ mm}^4$ ;  $I_{xx} = 18,24,231 \text{ mm}^4$   
 $k_{AB} = 35.14 \text{ mm}$ ;  $k_{xx} = 21.62 \text{ mm}$ ]

**Fig. 4.86**

15. Determine the moment of inertia of the section shown in Fig. 4.87 about the vertical centroidal axis.

[Ans.  $I_{yy} = 5,03,82,857 \text{ mm}^4$ ]

**Fig. 4.87**



16. A semi-circular cut is made in rectangular wooden beam as shown in Fig. 4.88. Determine the polar moment of inertia of the section about the centroidal axis.

[Ans.  $I_{xx} = 3,35,81,456 \text{ mm}^4$ ;  $I_{yy} = 1,00,45,631 \text{ mm}^4$ ;  $I_{zz} = 2,20,98,980 \text{ mm}^4$ ]

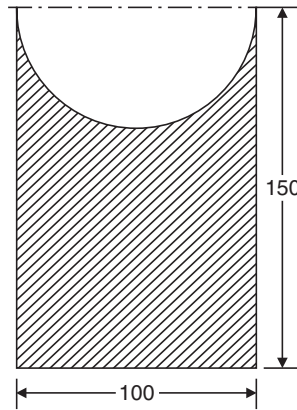


Fig. 4.88

17. Determine the moment of inertia of the section shown in the Fig. 4.89 about the horizontal centroidal axis. Also find the moment of inertia of the section about the symmetrical axis. Hence find the polar moment of inertia.

[Ans.  $I_{xx} = 54,09,046 \text{ mm}^4$ ;  $I_{yy} = 14,55,310 \text{ mm}^4$ ;  $I_{zz} = 68,64,356 \text{ mm}^4$ ]

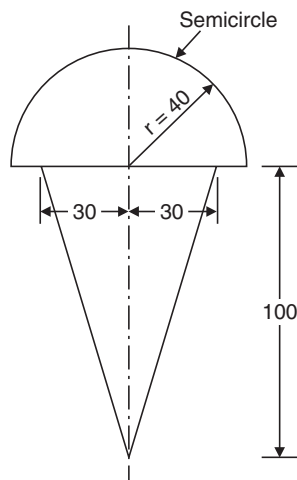


Fig. 4.89

18. The cross-section of a machine part is as shown in Fig. 4.90. Determine its moment of inertia and radius of gyration about the horizontal centroidal axis.

[Ans.  $I_{xx} = 5249090.85 \text{ mm}^4$ ;  $k_{xx} = 27.05 \text{ mm}$ ]

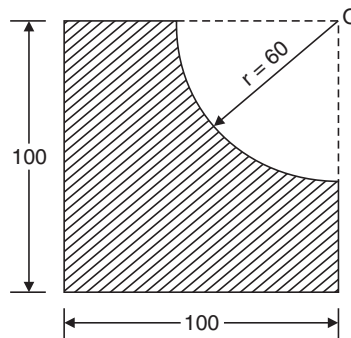


Fig. 4.90

19. The cross-section of a plain concrete culvert is as shown in Fig. 4.91. Determine the moment of inertia about the horizontal centroidal axes. [Ans.  $I_{xx} = 5.45865 \times 10^{10} \text{ mm}^4$ ]

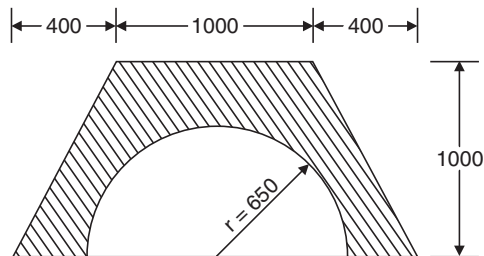


Fig. 4.91

20. Determine the centroid of the built-up section shown in Fig. 4.92 and find the moment of inertia and radius of gyration about the horizontal centroidal axis.

[Ans.  $I_{xx} = 1267942 \text{ mm}^4$ ;  $k_{xx} = 18.55 \text{ mm}$ ]

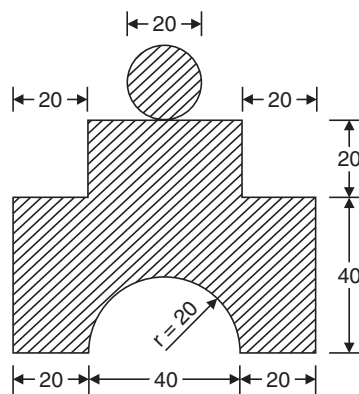
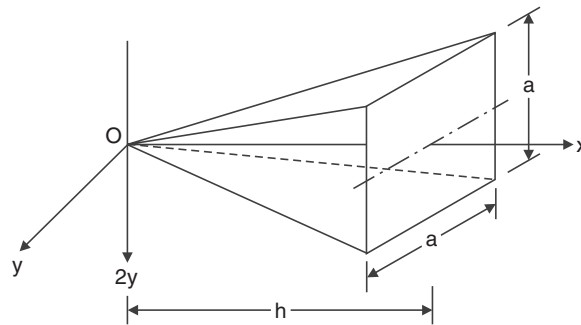


Fig. 4.92

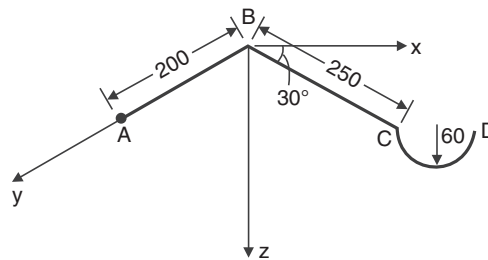
21. Determine the centre of gravity of the pyramid shown in Fig. 4.93.

[Ans.  $x = \frac{3}{4} h$ ]

**Fig. 4.93**

22. A steel ball of diameter 150 mm rests centrally over a concrete cube of size 150 mm. Determine the centre of gravity of the system, taking weight of concrete =  $25000 \text{ N/m}^3$  and that of steel  $80000 \text{ N/m}^3$ .  
 [Ans. 168.94 mm from base]
23. Locate the centre of gravity of the wire shown in Fig. 4.94. Portion  $BC$  is in  $x$ - $y$  plane and semicircle  $CD$  is parallel to  $x$ - $z$  plane.

[Ans.  $\bar{x} = 124.02 \text{ mm}$  ;  $\bar{y} = 110.41 \text{ mm}$  ;  $\bar{z} = 11.28 \text{ mm}$ ]

**Fig. 4.94**

# 5

## Friction

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When a body moves or tends to move over another body, a force opposing the motion develops at the contact surfaces. The force which opposes the movement or the tendency of movement is called the **frictional force** or simply **friction**. So far, in earlier chapters, we had ignored this force and considered contacting surfaces are smooth. Actually in almost all cases the contacting surfaces are not smooth. There are minutely projecting particles which develop frictional force to oppose the tendency to movement of one surface over the other surface. In this chapter, the additional terminology used in connection with frictional forces are explained and laws of dry friction (wet friction excluded) are presented. Applications of these laws to many engineering problems are illustrated.

### 5.1 COEFFICIENT OF FRICTION

Whenever a resultant force acts in the direction of contacting surfaces frictional force develops to oppose that force. The frictional force, like any other reaction, has a remarkable property of adjusting itself in magnitude to the tangential force. However, there is a limit beyond which the magnitude of the frictional force will not develop. If the applied tangential force is more than this maximum frictional force, there will be movement of one body over the other body with an acceleration as per Newton's second law of mass times acceleration equal to the resultant force. This maximum value of frictional force, which comes into play when the motion is impending is known as **Limiting Friction**. It may be noted that when the applied tangential force is less than the limiting friction, the body remains at rest and such friction is called **Static Friction**, which will have any value between zero and limiting friction. If the value of applied tangential force exceeds the limiting friction, the body starts moving over another body and the frictional resistance experienced while moving is known as **Dynamic Friction**. The magnitude of dynamic friction is found to be less than limiting friction. Dynamic friction may be further classified into two groups:

- (i) *Sliding Friction*: It is the friction experienced by a body when it slides over the other body.
- (ii) *Rolling Friction*: It is the friction experienced by a body when it rolls over another body.

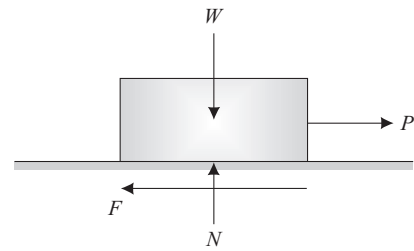
It has been experimentally proved that, between two contacting surfaces, the magnitude of limiting friction bears a constant ratio to the normal reaction between the two and this ratio is called '*Coefficient*

of Friction'. Referring to Fig. 5.1 a body weighing  $W$  is being pulled by a force  $P$  and the motion is impending. Let  $N$  be normal reaction and  $F$  the limiting frictional force. Then

$$\text{Coefficient of Friction} = \frac{F}{N}$$

Coefficient of friction is denoted by  $\mu$ . Then

$$\mu = \frac{F}{N}$$



**Fig. 5.1**

...(5.1)

## 5.2 LAWS OF FRICTION

The principles discussed in a previous article are mainly due to the experimental studies by Coulomb (1781) and by Mozin (1831). These principles constitute the laws of dry friction and may be called as *Coulomb's laws of dry friction*. These laws are listed below:

- (i) The frictional force always acts in a direction opposite to that in which the body tends to move.
- (ii) Till the limiting value is reached, the magnitude of frictional force is exactly equal to the tangential force which tends to move the body.
- (iii) The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two contacting surfaces.
- (iv) The force of friction depends upon the roughness/smoothness of the surfaces.
- (v) The force of friction is independent of the area of contact between the two surfaces.
- (vi) After the body starts moving, the dynamic friction comes into play, the magnitude of which is less than that of limiting friction and it bears a constant ratio to the normal force. This ratio is called coefficient of dynamic friction.

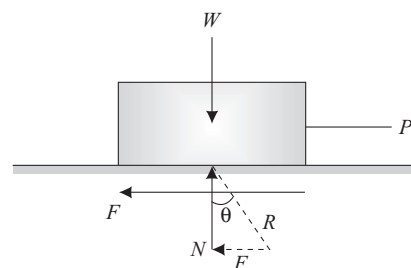
## 5.3 ANGLE OF FRICTION, ANGLE OF REPOSE AND CONE OF FRICTION

### Angle of Friction

Consider the block shown in Fig. 5.2 subject to pull  $P$ . Let  $F$  be the frictional force developed and  $N$  the normal reaction. Thus, at contact surface, the reactions are  $F$  and  $N$ . They can be combined to get the resultant reaction  $R$  which acts at angle  $\theta$  to normal reaction. This angle is given by

$$\tan \theta = \frac{F}{N}$$

As frictional force increases the angle  $\theta$  increases and it can reach maximum value  $\alpha$  when limiting value of friction is reached. Thus, when motion is impending



**Fig. 5.2**

$$\tan \alpha = \frac{F}{N} = \mu \quad \dots(5.2)$$

and this value of  $\alpha$  is called *angle of limiting friction*. Hence, the angle of limiting friction can be defined as the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.

### Angle of Repose

It is very well-known that when grains (food grains, sand, cement, soil etc.) are heaped, there exists a limit for the inclination of the heap. Beyond that the grains start rolling down. The limiting angle up to which the grains repose (sleep) is called *angle of repose*.

Now consider the block of weight  $W$  shown in Fig. 5.3 which is resting on an inclined plane that makes angle  $\theta$  with the horizontal. When  $\theta$  is a small, block rests on the plane. If  $\theta$  is increased gradually a stage is reached at which the block starts sliding. The angle made by the plane with the horizontal is called angle of friction for the contacting surfaces. Thus, the maximum inclination of the plane on which the body, free from external forces, can repose is called *angle of repose*.

Consider the equilibrium of the block shown in Fig. 5.3. Since the surface of contact is not smooth, not only normal reaction but frictional force also develops. As the body tends to slide down, the frictional resistance will be up the plane.

$\Sigma$  Forces normal to plane = 0, gives

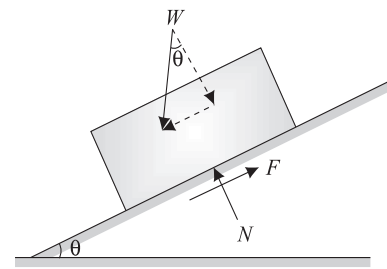
$$N = W \cos \theta \quad \dots(i)$$

$\Sigma$  Forces parallel to plane = 0, gives

$$F = W \sin \theta \quad \dots(ii)$$

Dividing eqn. (ii) by eqn. (i) we get,

$$\frac{F}{N} = \tan \theta$$



**Fig. 5.3**

If  $\phi$  is the value of  $\theta$  when motion is impending, frictional force will be limiting friction and hence

$$\tan \phi = \frac{F}{N} \quad \dots(5.3)$$

i.e.,

$$\tan \phi = \mu = \tan \alpha$$

or

$$\phi = \alpha.$$

Thus, the value of angle of repose is the same as the value of limiting angle of friction.

### Cone of Friction

When a body is having impending motion in the direction of  $P$ , the frictional force will be the limiting friction and the resultant reaction  $R$  will make limiting frictional angle  $\alpha$  with the normal as shown in

Fig. 5.4. If the body is having impending motion in some other direction, the resultant reaction makes limiting frictional angle  $\alpha$  with the normal. Thus, if the direction of force  $P$  is gradually changed through  $360^\circ$ , the resultant  $R$  generates a right circular cone with semicentral angle equal to  $\alpha$ .

If the resultant reaction lies on the surface of this inverted right circular cone whose semicentral angle is limiting frictional angle  $\alpha$ , the motion of the body is impending. If the resultant is within this cone the body is stationary. This inverted cone with semicentral angle, equal to limiting frictional angle  $\alpha$ , is called *cone of friction*.

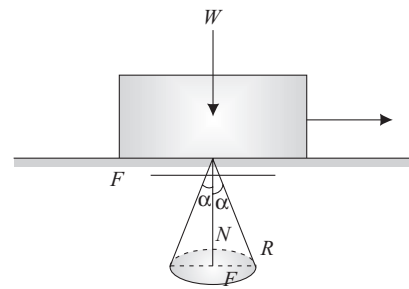


Fig. 5.4

#### 5.4 PROBLEMS ON BLOCKS RESTING ON HORIZONTAL AND INCLINED PLANES

Analysis of such problems, when motion is impending is illustrated in this article by solving typical problems.

**Example 5.1.** Block A weighing 1000 N rests over block B which weighs 2000 N as shown in Fig. 5.5(a). Block A is tied to a wall with a horizontal string. If the coefficient of friction between A and B is  $1/4$  and that between B and the floor is  $1/3$ , what value of force  $P$  is required to create impending motion if (a)  $P$  is horizontal, (b)  $P$  acts  $30^\circ$  upwards to horizontal?

**Solution:** (a) **When  $P$  is horizontal:** The free body diagrams of the two blocks are shown in Fig. 5.5(b). Note the frictional forces are to be marked in the opposite directions of impending relative motion. In this problem, block B is having impending motion to the right. Hence on it  $F_1$  and  $F_2$  are towards right. The relative motion of block A w.r.t. B is to the left. Hence the direction of  $F_1$  in this is towards the right. Another way of thinking for the direction of  $F_1$  in case of block A can be 'actions and reactions are equal and opposite'. Hence on block B if  $F_1$  is towards left, on A it should be towards right.

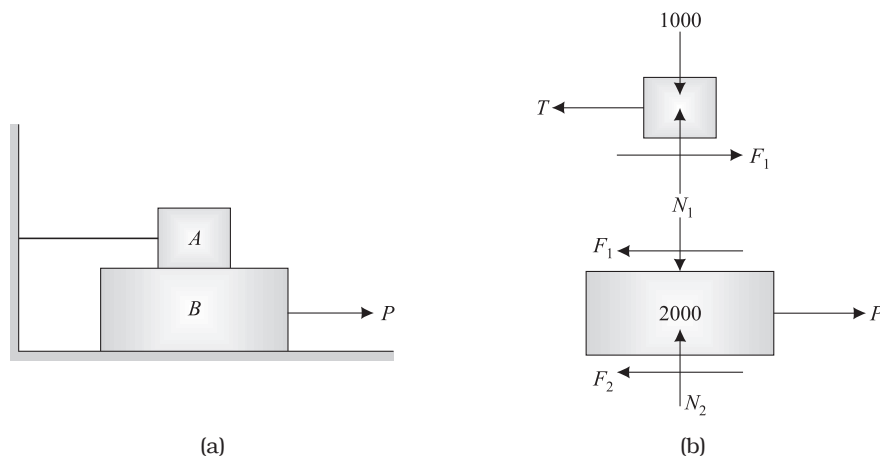


Fig. 5.5

Now consider the equilibrium of block A.

$$\begin{aligned}\sum F_V &= 0 \rightarrow \\ N_1 - 1000 &= 0 \quad \text{or} \quad N_1 = 1000 \text{ newton.}\end{aligned}$$

Since  $F_1$  is limiting friction,

$$\frac{F_1}{N_1} = \mu_1 = \frac{1}{4}$$

$$\therefore F_1 = \frac{1}{4} \times 1000 = 250 \text{ newton.}$$

$$\sum F_H = 0 \rightarrow$$

$$F_1 - T = 0 \quad \text{or} \quad T = F_1, \quad \text{i.e.} \quad T = 250 \text{ newton.}$$

Consider the equilibrium of block B.

$$\sum F_V = 0 \rightarrow$$

$$N_2 - N_1 - 2000 = 0.$$

$$\therefore N_2 = N_1 + 2000 = 1000 + 2000 = 3000 \text{ newton.}$$

Since  $F_2$  is limiting friction,

$$F_2 = \mu_2 N_2 = \frac{1}{3} \times 3000 = 1000 \text{ newton.}$$

$$\sum F_H = 0 \rightarrow$$

$$P - F_1 - F_2 = 0$$

$$\therefore P = F_1 + F_2 = 250 + 1000 = \mathbf{1250 \text{ newton.}}$$

**(b) When  $P$  is inclined:** Free body diagrams for this case are shown in Fig. 5.5(c).

Considering equilibrium of block A, we get

$$\sum F_V = 0 \rightarrow N_1 = 1000 \text{ newton.}$$

$$\therefore F_1 = \frac{1}{4} \times 1000 = 250 \text{ newton.}$$

$$\sum F_H = 0 \rightarrow T = F_1 = 250 \text{ newton.}$$

Consider the equilibrium of block B.

$$\sum F_V = 0 \rightarrow$$



$$N_2 - 2000 - N_1 + P \sin 30 = 0$$

or  $N_2 + 0.5P = 3000$ , since  $N_1 = 1000$  newton.

From law of friction

$$F_2 = \mu_2 N_2 = \frac{1}{3} \times (3000 - 0.5P)$$

$$= 1000 - \frac{0.5}{3}P.$$

$$\sum F_H = 0 \rightarrow$$

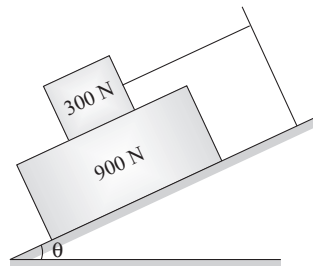
$$P \cos 30 - F_1 - F_2 = 0$$

$$\therefore P \cos 30 - 250 - \left(1000 - \frac{0.5}{3}P\right) = 0$$

$$\therefore P \left( \cos 30 + \frac{0.5}{3} \right) = 1250$$

$$\therefore P = 1210.4 \text{ newton}$$

**Example 5.2.** What should be the value of  $\theta$  in Fig. 5.6(a) which will make the motion of 900 N block down the plane to impend? The coefficient of friction for all contact surfaces is  $1/3$ .



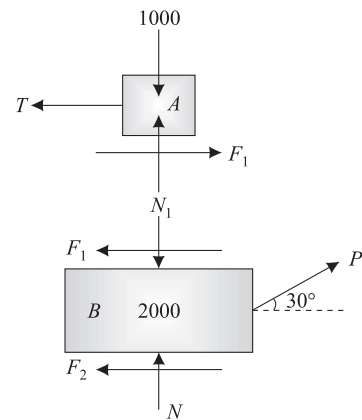
**Fig. 5.6(a)**

**Solution:** 900 N block is on the verge of moving downward. Hence frictional forces  $F_1$  and  $F_2$  [Ref. Fig. 5.6(b)] act up the plane on 900 N block. Free body diagrams of the blocks are as shown in Fig. 5.6(b).

Consider the equilibrium of 300 N block.

$$\sum \text{Forces normal to plane} = 0 \rightarrow$$

$$N_1 - 300 \cos \theta = 0 \quad \text{or} \quad N_1 = 300 \cos \theta \quad \dots(i)$$



**Fig. 5.5(c)**

From law of friction,

$$F_1 = \frac{1}{3} N_1 = 100 \cos \theta$$

For 900 N block:

$\Sigma$  Forces normal to plane = 0  $\rightarrow$

$$N_2 - N_1 - 900 \cos \theta = 0$$

or

$$\begin{aligned} N_2 &= N_1 + 900 \cos \theta \\ &= 300 \cos \theta + 900 \cos \theta \\ &= 1200 \cos \theta. \end{aligned}$$

From law of friction,

$$F_2 = \mu_2 N_2 = \frac{1}{3} \times 1200 \cos \theta = 400 \cos \theta.$$

$\Sigma$  Forces parallel to the plane = 0  $\rightarrow$

$$F_1 + F_2 - 900 \sin \theta = 0$$

$$100 \cos \theta + 400 \cos \theta = 900 \sin \theta$$

$$\therefore \tan \theta = \frac{500}{900}$$

$$\therefore \theta = 29.05^\circ$$

**Example 5.3.** A block weighing 500 N just starts moving down a rough inclined plane when it is subjected to 200 N force acting up the inclined plane and it is at the point of moving up the plane when pulled up by a force of 300 N parallel to the plane. Find the inclination of the plane and the coefficient of friction between the inclined plane and the block.

**Solution:** Free body diagram of the block when its motion is impending down the plane is shown in Fig. 5.7(a) and that when it is moving up the plane is shown in Fig. 5.7(b).

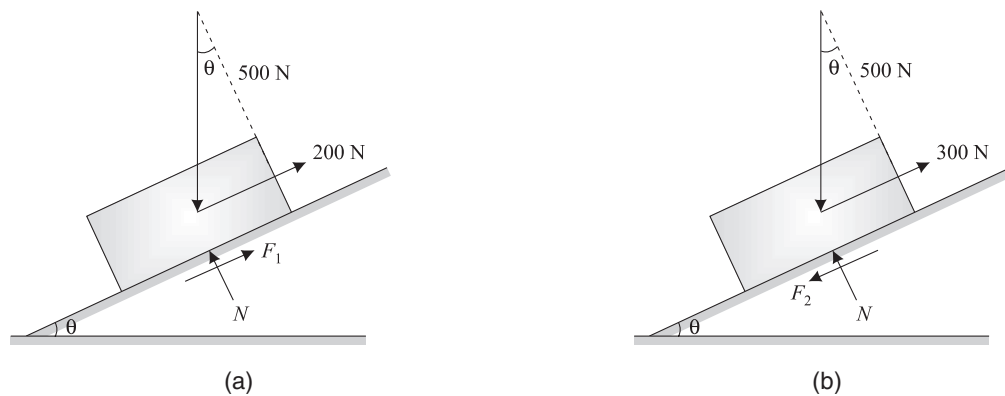


Fig. 5.7

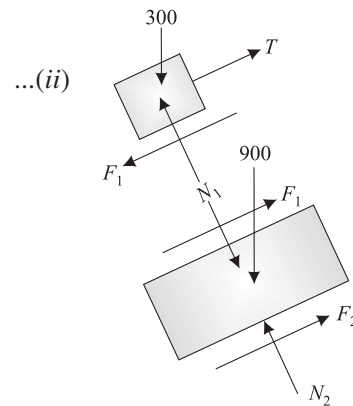


Fig. 5.6(b)

**When block starts moving down the plane [Ref. Fig. 5.7(a)]**

Frictional forces oppose the direction of the movement. Hence  $F_1$  is up the plane and  $F_2$  down the plane. Since it is limiting case

$$\frac{F}{N} = \mu.$$

$\Sigma$  forces perpendicular to the plane = 0  $\rightarrow$

$$N - 500 \cos \theta = 0 \quad \text{or} \quad N = 500 \cos \theta \quad \dots(i)$$

From law of friction,

$$F_1 = \mu N = 500 \mu \cos \theta \quad \dots(ii)$$

$\Sigma$  Forces parallel to the plane = 0  $\rightarrow$

$$F_1 + 200 - 500 \sin \theta = 0$$

Substituting the value of  $F_1$  from eqn. (ii), we get

$$500 \sin \theta - 500 \mu \cos \theta = 200 \quad \dots(iii)$$

**When the block starts moving up the plane [Fig. 5.7(b)]**

$\Sigma$  Forces perpendicular to the plane = 0  $\rightarrow$

$$N - 500 \cos \theta = 0 \quad \text{i.e.,} \quad N = 500 \cos \theta \quad \dots(iv)$$

From law of friction,

$$F_2 = \mu N = 500 \mu \cos \theta \quad \dots(v)$$

$\Sigma$  Forces parallel to the plane = 0  $\rightarrow$

$$300 - 500 \sin \theta - F_2 = 0$$

$$\text{i.e.,} \quad 500 \sin \theta + 500 \mu \cos \theta = 300 \quad \dots(vi)$$

Adding eqns. (iii) and (vi), we get

$$1000 \sin \theta = 500$$

$$\text{i.e.,} \quad \sin \theta = 0.5$$

Hence

$$\theta = 30^\circ$$

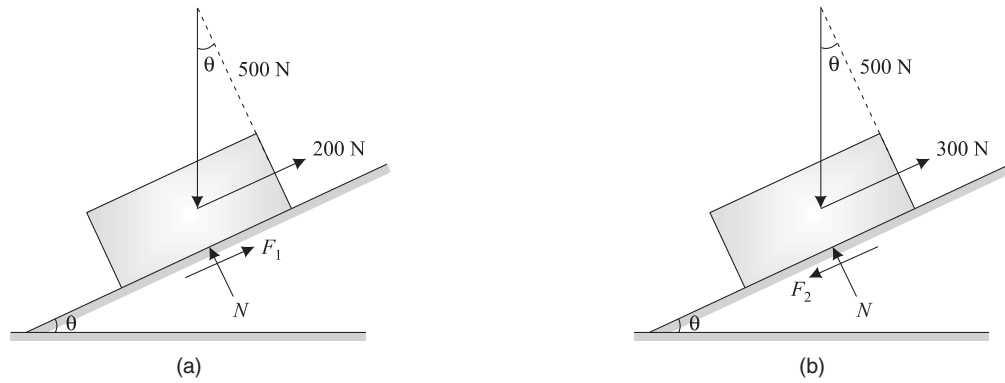
Substituting it in eqn. (vi), we get

$$500 \sin 30 + 500 \mu \cos 30 = 300$$

$$500 \mu \cos 30 = 300 - 250 = 50$$

$$\therefore \mu = \frac{50}{500 \cos 30} = 0.115$$

**Example 5.4.** Block A weighing 1000 N and block B weighing 500 N are connected by flexible wire. The coefficient of friction between block A and the plane is 0.5 while that for block B and the plane is 0.2. Determine what value of inclination of the plane the system will have impending motion down the plane? [Ref. Fig. 5.8].

**Fig. 5.8**

**Solution:** Let  $\theta$  be the inclination of the plane for which motion is impending. Free body diagrams of blocks A and B are as shown in Fig. 5.8(b). Considering equilibrium of block A,

$$\Sigma \text{ Forces normal to plane} = 0 \rightarrow$$

$$N_1 - 1000 \cos \theta = 0 \quad \text{or} \quad N_1 = 1000 \cos \theta \quad \dots(i)$$

$\therefore$  From law of friction

$$F_1 = \mu_1 N_1 = 0.5 \times 1000 \cos \theta = 500 \cos \theta \quad \dots(ii)$$

$$\Sigma \text{ Forces parallel to plane} = 0 \rightarrow$$

$$F_1 - T - 1000 \sin \theta = 0$$

or

$$T = 500 \cos \theta - 1000 \sin \theta \quad \dots(iii)$$

Consider the equilibrium of block B,

$$\Sigma \text{ Forces normal to plane} = 0 \rightarrow$$

$$N_2 - 500 \cos \theta = 0 \quad \text{or} \quad N_2 = 500 \cos \theta \quad \dots(iv)$$

From law of friction,

$$F_2 = \mu_2 N_2 = 0.2 \times 500 \cos \theta = 100 \cos \theta \quad \dots(v)$$

$$\Sigma \text{ Forces parallel to plane} = 0 \rightarrow$$

$$F_2 + T - 500 \sin \theta = 0$$

Using the values of  $F_2$  and  $T$  from eqn. (v) and eqn. (iii),

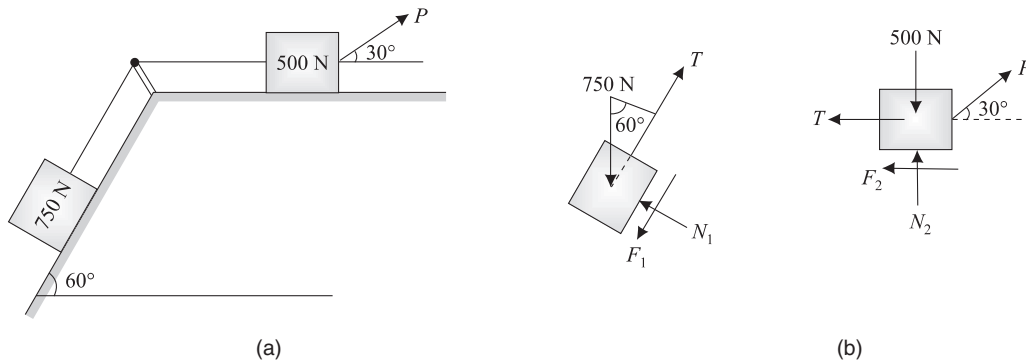
$$100 \cos \theta + 500 \cos \theta - 1000 \sin \theta - 500 \sin \theta = 0$$

$$600 \cos \theta = 1500 \sin \theta$$

$$\therefore \tan \theta = \frac{600}{1500}$$

$$\therefore \theta = 21.8^\circ$$

**Example 5.5.** What is the value of  $P$  in the system shown in Fig. 5.9(a) to cause the motion to impend? Assume the pulley is smooth and coefficient of friction between the other contact surfaces is 0.2.



**Fig. 5.9**

**Solution:** Free body diagrams of the blocks are as shown in Fig. 5.9(b). Consider the equilibrium of 750 N block.

$\Sigma$  Forces normal to the plane = 0  $\rightarrow$

$$N_1 - 750 \cos 60 = 0 \quad \therefore N_1 = 375 \text{ newton} \quad \dots(i)$$

Since the motion is impending, from law of friction,

$$F_1 = \mu N_1 = 0.2 \times 375 = 75 \text{ newton} \quad \dots(ii)$$

$\Sigma$  Forces parallel to the plane = 0  $\rightarrow$

$$T - F_1 - 750 \sin 60 = 0$$

$$\therefore T = 75 + 750 \sin 60 = 724.5 \text{ newton.} \quad \dots(iii)$$

Consider the equilibrium of 500 N block.

$$\Sigma F_V = 0 \rightarrow$$

$$N_2 - 500 + P \sin 30 = 0$$

$$\text{i.e.,} \quad N_2 + 0.5P = 500 \quad \dots(iv)$$

From law of friction,

$$F_2 = \mu N_2 = 0.2 (500 - 0.5P) = 100 - 0.1P \quad \dots(v)$$

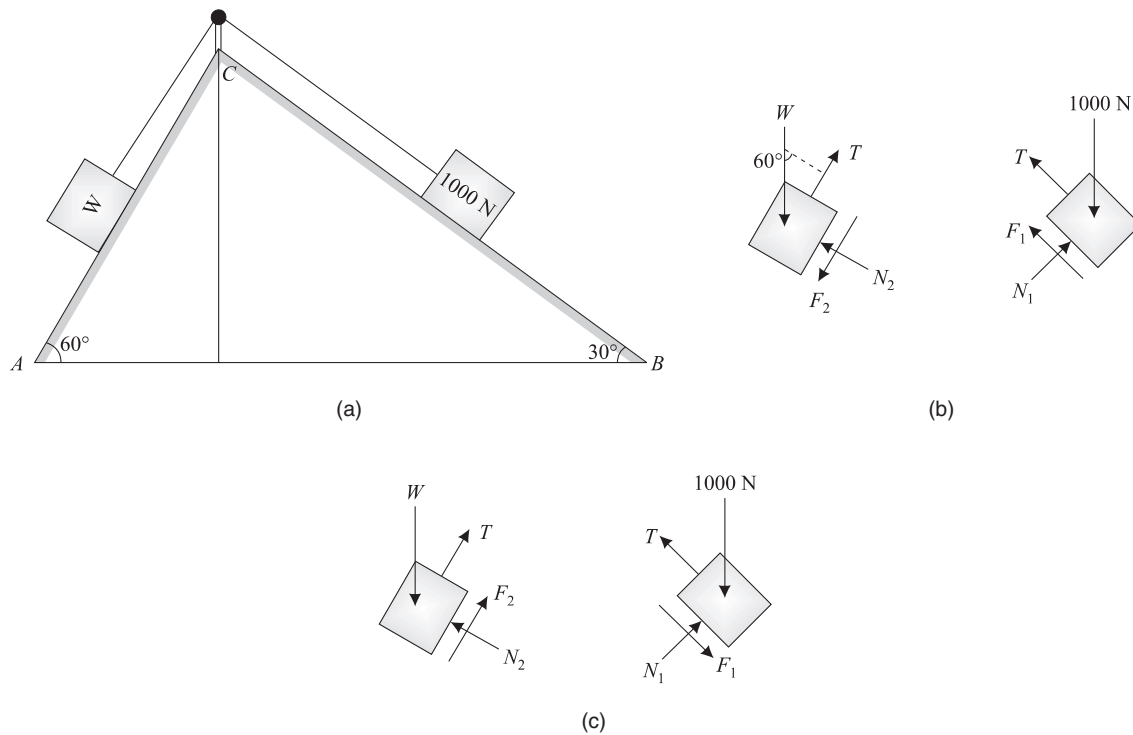
$$\Sigma F_H = 0 \rightarrow$$

$$P \cos 30 - T - F_2 = 0$$

$$\text{i.e.,} \quad P \cos 30 - 724.5 - 100 + 0.1P = 0$$

$$\therefore P = 853.5 \text{ N}$$

**Example 5.6.** Two identical planes AC and BC, inclined at  $60^\circ$  and  $30^\circ$  to the horizontal meet at C as shown in Fig. 5.10. A load of 1000 N rests on the inclined plane BC and is tied by a rope passing over a pulley to a block weighing  $W$  newtons and resting on the plane AC. If the coefficient of friction between the load and the plane BC is 0.28 and that between the block and the plane AC is 0.20, find the least and greatest values of  $W$  for the equilibrium of the system.



**Fig. 5.10**

**Solution:**

**(a) Least value of  $W$ :**

In this case motion of 1000 N block is impending down the plane and block  $W$  has impending motion up the plane. Hence free body diagrams for the blocks are as shown in Fig. 5.10(b). Considering the equilibrium of 1000 N block,

$$\Sigma \text{ Forces normal to the plane} = 0 \rightarrow$$

$$N_1 - 1000 \cos 30 = 0 \quad \therefore N_1 = 866.0 \text{ newton} \quad \dots(i)$$

From the law of friction

$$F_1 = \mu_1 N_1 = 0.28 \times 866.0 = 242.5 \text{ newton} \quad \dots(ii)$$

$$\Sigma \text{ Forces parallel to the plane} = 0 \rightarrow$$

$$T - 1000 \sin 30 + F_1 = 0$$

$$\therefore T = 500 - 242.5 = 257.5 \text{ newton} \quad \dots(iii)$$

Now consider the equilibrium of block weighing  $W$ .

$\Sigma$  Forces normal to the plane = 0  $\rightarrow$

$$N_2 - W \cos 60 = 0 \quad \text{i.e.,} \quad N_2 = 0.5 W \quad \dots(iv)$$

From law of friction

$$F_2 = \mu_2 N_2 = 0.2 \times 0.5 W = 0.1 W \quad \dots(v)$$

$\Sigma$  Forces parallel to the plane = 0  $\rightarrow$

$$T - F_2 - W \sin 60 = 0$$

Substituting the values of  $T$  and  $F_2$  from eqns. (iii) and (v), we get

$$257.5 - 0.1 W - W \sin 60 = 0$$

$$\therefore W = \frac{257.5}{0.1 + \sin 60} = 266.6 \text{ N.}$$

**(b) For the greatest value of  $W$ :**

In such case 1000 N block is on the verge of moving up the plane and  $W$  is on the verge of moving down the plane. For this case free body diagrams of the blocks are as shown in Fig. 5.10(c).

Considering the block of 1000 N,

$\Sigma$  Forces normal to plane = 0  $\rightarrow$

$$N_1 - 1000 \cos 30 = 0 \quad \therefore N_1 = 866.0 \text{ newton} \quad \dots(vi)$$

From law of friction,

$$F_1 = \mu_1 N_1 = 0.28 \times 866.0 = 242.5 \text{ N} \quad \dots(vii)$$

$\Sigma$  Forces parallel to the plane = 0  $\rightarrow$

$$T - 1000 \sin 30 - F_1 = 0$$

$$\therefore T = 500 + 242.5 = 742.5 \text{ newton} \quad \dots(viii)$$

Considering the equilibrium of block weighing  $W$ ,

$\Sigma$  Forces normal to plane = 0  $\rightarrow$

$$N_2 - W \cos 60 = 0 \quad \text{or} \quad N_2 = 0.5 W \quad \dots(ix)$$

$$\therefore F_2 = \mu_2 N_2 = 0.2 \times 0.5 W = 0.1 W \quad \dots(x)$$

$\Sigma$  Forces parallel to plane = 0  $\rightarrow$

$$T - W \sin 60 + F_2 = 0 \quad \dots(xi)$$

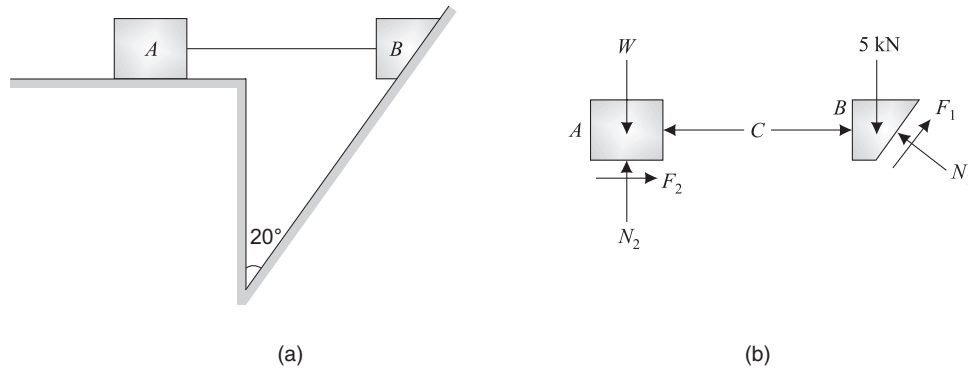
Substituting the values of  $T$  and  $F_2$  from eqns. (viii) and (x), we get,

$$742.5 - W \sin 60 + 0.1 W = 0$$

$$\text{or} \quad W = \frac{742.5}{\sin 60 - 0.1} = 969.3 \text{ newton}$$

The system of blocks are, in equilibrium for  $W = 266.6 \text{ N}$  to  $969.3 \text{ N}$ .

**Example 5.7.** Two blocks connected by a horizontal link  $AB$  are supported on two rough planes as shown in Fig. 5.11(a). The coefficient of friction on the horizontal plane is 0.4. The limiting angle of friction for block  $B$  on the inclined plane is  $20^\circ$ . What is the smallest weight  $W$  of the block  $A$  for which equilibrium of the system can exist if weight of block  $B$  is  $5 \text{ kN}$ ?



**Fig. 5.11**

**Solution:** Free body diagrams for blocks  $A$  and  $B$  are as shown in Fig. 5.11(b). Consider block  $B$ .

$$\mu = \tan 20^\circ, \text{ given.}$$

$$\therefore F_1 = N_1 \tan 20^\circ$$

$$\sum F_V = 0 \rightarrow$$

$$N_1 \sin 30 + F_1 \sin 60 - 5 = 0$$

$$0.5N_1 + N_1 \tan 20 \sin 60 = 5$$

$$N_1 = 6.133 \text{ kN}$$

$$\text{Hence, } F_1 = 6.13 \tan 20 = 2.232 \text{ kN}$$

$$\sum F_H = 0 \rightarrow$$

$$C + F_1 \cos 60 - N_1 \cos 30 = 0$$

$$C + 2.232 \cos 60 - 6.133 \cos 30 = 0$$

$$\therefore C = 4.196 \text{ kN}$$

Now consider the equilibrium of block  $A$ .

$$\sum F_H = 0 \rightarrow$$



$$F_2 - C = 0 \quad \text{or} \quad F_2 = C = 4.196 \text{ kN}$$

From law of friction,

$$F_2 = \mu N_2$$

$$\text{i.e.,} \quad 4.196 = 0.4 N_2$$

$$\therefore \quad N_2 = 10.49 \text{ kN}$$

$$\text{Then,} \quad \sum F_V = 0 \rightarrow$$

$$N_2 - W = 0$$

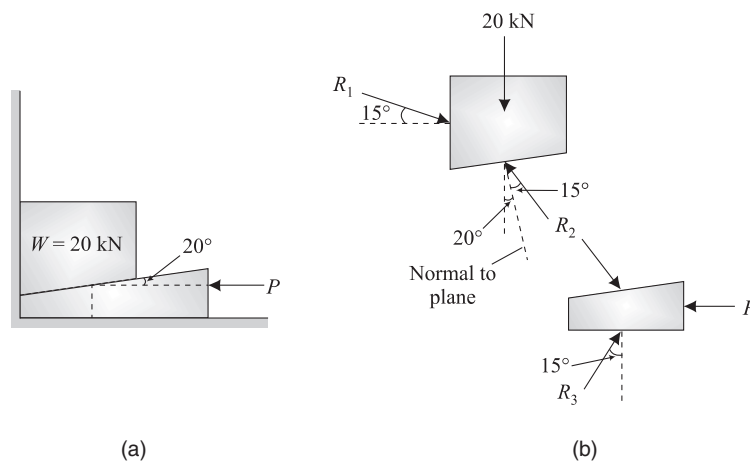
$$\text{or} \quad W = N_2 = 10.49 \text{ kN}$$

### 5.5 APPLICATION TO WEDGE PROBLEMS

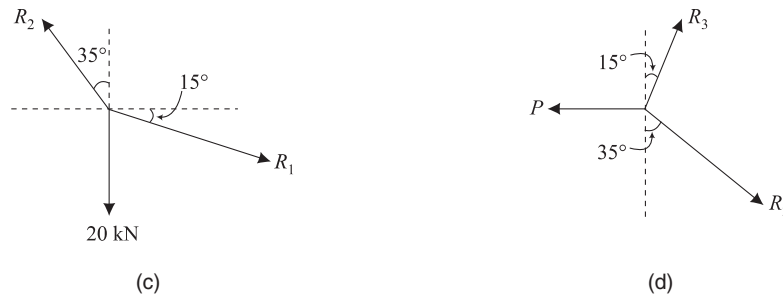
Wedges are small pieces of hard materials with two of their opposite surfaces not parallel to each other. They are used to slightly lift heavy blocks, machinery, precast beams etc. for making final alignment or to make place for inserting lifting devices. In any problem weight of wedge is very small compared to the weight lifted. Hence in all problems self weight of wedge is neglected. It is found that in the analysis instead of treating normal reaction and frictional force independently, it is advantageous to treat their resultant.

If  $F$  is limiting friction, then resultant  $R$  makes limiting angle  $\alpha$  with the normal. Its direction should be marked correctly. Note that the tangential component of the resultant reaction  $R$  is the frictional force and it will always oppose impending motion. Application to wedge problems is illustrated below by solving problems.

**Example 5.8.** Determine the force  $P$  required to start the movement of the wedge as shown in Fig. 5.12(a). The angle of friction for all surfaces of contact is  $15^\circ$ .



**Fig. 5.12 (Contd.)**

**Fig. 5.12**

**Solution:** As wedge is driven, it moves towards left and the block moves upwards. When motion is impending limiting friction develops. Hence resultant force makes limiting angle of  $15^\circ$  with normal. The care is taken to mark  $15^\circ$  inclination such that the tangential component of the resultant opposes the impending motion.

The free body diagrams of the block and wedge are shown in Fig. 5.12(b). The forces on block and wedge are redrawn in Figs. 5.12(c) and (d) so that Lami's theorem can be applied conveniently. Applying Lami's theorem to the system of forces on block

$$\frac{R_1}{\sin (180-15-20)} = \frac{R_2}{\sin (90-15)} = \frac{20}{\sin (15+20+90+15)}$$

$$\text{i.e.,} \quad \frac{R_1}{\sin 145} = \frac{R_2}{\sin 75} = \frac{20}{\sin 140}$$

$$\therefore R_1 = 17.85 \text{ kN}$$

$$\text{and} \quad R_2 = 30.05 \text{ kN}$$

Applying Lami's theorem to system of forces on the wedge, we get

$$\frac{P}{\sin 130} = \frac{R_2}{\sin 105}$$

$$\therefore P = 23.84 \text{ kN}$$

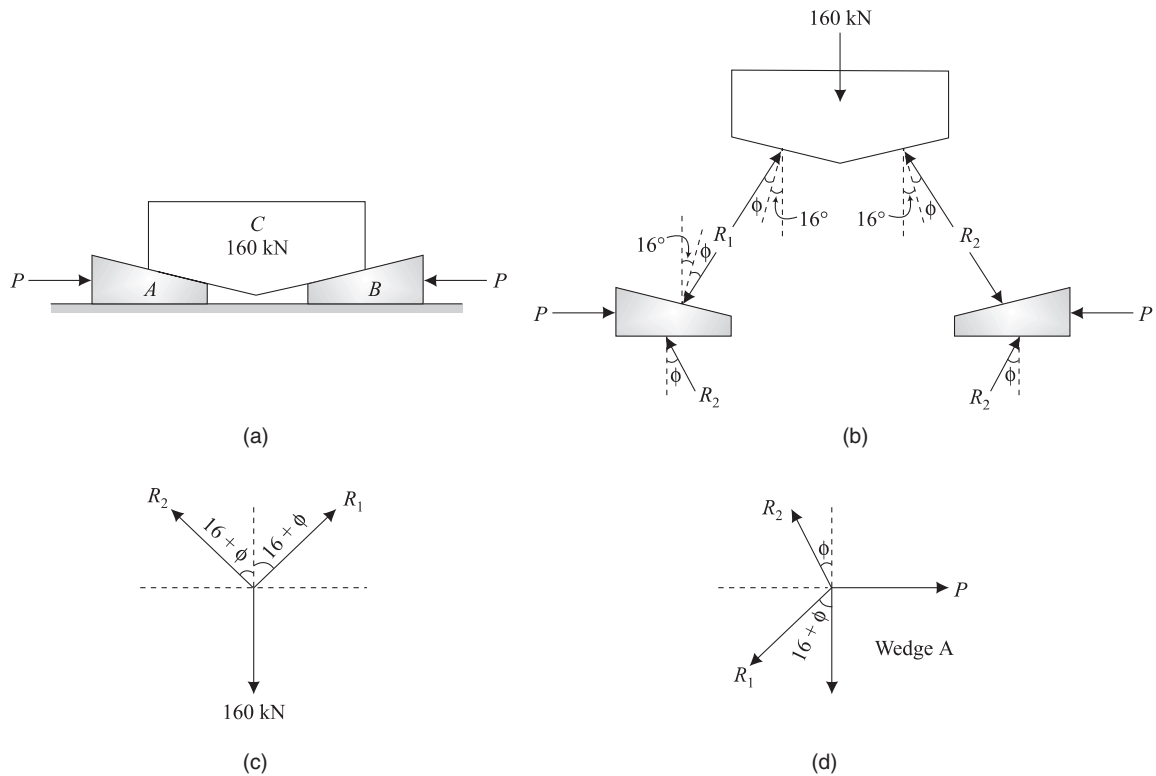
**Example 5.9.** A block weighing 160 kN is to be raised by means of the wedges A and B as shown in Fig. 5.13(a). Find the value of force P for impending motion of block C upwards, if coefficient of friction is 0.25 for all contact surfaces. The self weight of wedges may be neglected.

**Solution:** Let  $\phi$  be the angle of limiting friction.

$$\therefore \phi = \tan^{-1} (0.25) = 14.036^\circ$$

The free body diagrams of wedges A, B and block C are shown in Fig. 5.13(b). The problem being symmetric, the reactions  $R_1$  and  $R_2$  on wedges A and B are equal. The system of forces on block C and

on wedge A are shown in the form convenient for applying Lami's theorem [Ref. Figs. 5.13(c) and (d)].



**Fig. 5.13**

Consider the equilibrium of block C.

$$\frac{R_1}{\sin (180-16-\phi)} = \frac{160}{\sin 2(\phi+16)}$$

$$\text{i.e.,} \quad \frac{R_1}{\sin 149.96} = \frac{160}{\sin 60.072}, \quad \text{since } \phi = 14.036^\circ.$$

$$\therefore \quad R_1 = 92.42 \text{ kN}$$

Consider the equilibrium of the wedge A. Applying Lami's theorem, we get

$$\frac{P}{\sin (180-\phi-\phi-16)} = \frac{R_1}{\sin (90+\phi)}$$

$$\therefore \quad P = \frac{92.42 \sin 135.928}{\sin 104.036}, \quad \text{since } \phi = 14.036.$$

$$\text{i.e.,} \quad P = 66.26 \text{ kN}$$

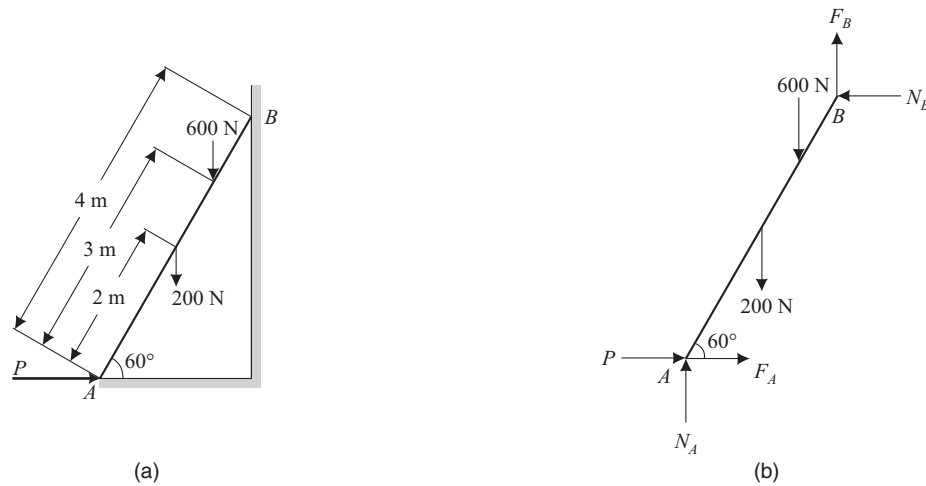
### 5.6 APPLICATION TO LADDER PROBLEMS

A ladder resting against a wall is a typical case of friction problems in non-concurrent system of forces. Hence we have three equations of equilibrium available. From law of friction we have the

equation  $\frac{F}{N} = \mu$ . Using equilibrium equations and friction law the problems can be solved. The

procedure is illustrated with the examples below:

**Example 5.10.** A ladder of length 4 m, weighing 200 N is placed against a vertical wall as shown in Fig. 5.14(a). The coefficient of friction between the wall and the ladder is 0.2 and that between floor and the ladder is 0.3. The ladder, in addition to its own weight, has to support a man weighing 600 N at a distance of 3 m from A. Calculate the minimum horizontal force to be applied at A to prevent slipping.



**Fig. 5.14**

**Solution:** The free body diagram of the ladder is as shown in Fig. 5.14(b).

$$\sum M_A = 0 \rightarrow$$

$$200 \times 2 \cos 60 + 600 \times 3 \cos 60 - F_B \times 4 \cos 60 - N_B \times 4 \sin 60 = 0$$

Dividing throughout by 4 and rearranging the terms, we get

$$0.866 N_B + 0.5 F_B = 275$$

From law of friction,

$$F_B = \mu N_B = 0.2 N_B$$

$$\therefore 0.866 N_B + 0.5 \times 0.2 N_B = 275$$

$$\text{or } N_B = 284.7 \text{ newton.}$$

$$\therefore F_B = 56.94 \text{ newton.}$$

$$\sum F_V = 0 \rightarrow$$

$$N_A - 200 - 600 + F_B = 0$$

$$N_A = 743.06 \text{ newton, since } F_B = 56.94$$

$\therefore$

$$F_A = \mu_A N_A$$

$$= 0.3 \times 743.06 = 222.9 \text{ newton}$$

$$\sum F_H = 0 \rightarrow$$

$$P + F_A - N_B = 0$$

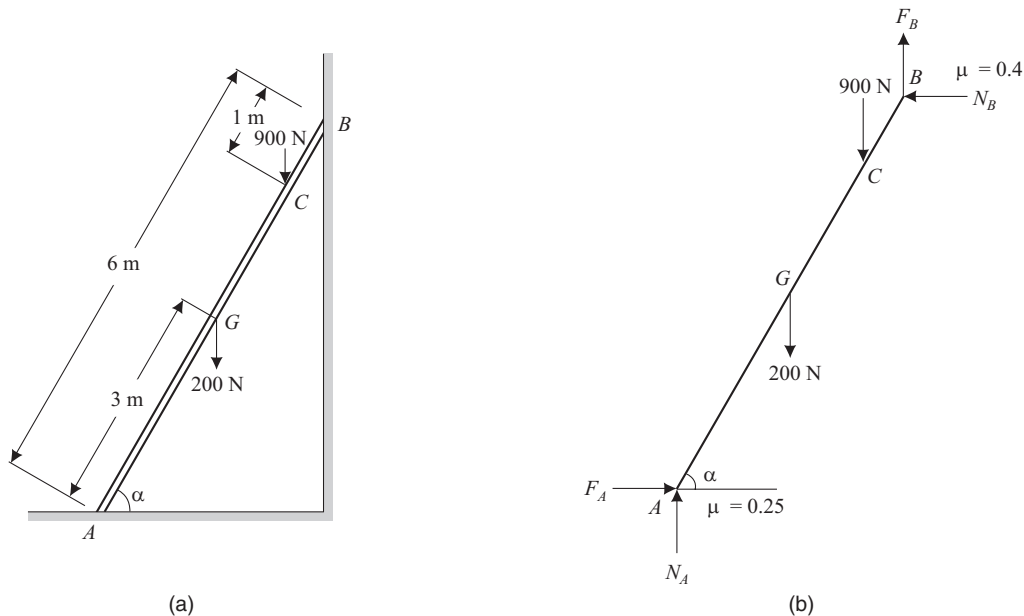
$\therefore$

$$P = N_B - F_A = 284.7 - 222.9$$

i.e.,

$$P = 61.8 \text{ newton}$$

**Example 5.11.** The ladder shown in Fig. 5.15(a) is 6 m long and is supported by a horizontal floor and a vertical wall. The coefficient of friction between the floor and the ladder is 0.25 and between the wall and the ladder is 0.4. The weight of the ladder is 200 N and may be considered as a concentrated load at G. The ladder supports a vertical load of 900 N at C which is at a distance of 1 m from B. Determine the least value of  $\alpha$  at which the ladder may be placed without slipping. Determine the reaction at that stage.



**Fig. 5.15**

**Solution:** Figure 5.15(b) shows the free body diagram of the ladder.

From law of friction,

$$F_A = 0.25 N_A \quad \dots(i)$$

$$F_B = 0.40 N_B \quad \dots(ii)$$

$$\Sigma F_V = 0 \rightarrow$$

$$N_A - 200 - 900 + F_B = 0$$

$$\text{i.e.,} \quad N_A + 0.4 N_B = 900 + 200 = 1100 \quad \dots(iii)$$

$$\Sigma F_H = 0 \rightarrow$$

$$F_A - N_B = 0$$

$$\text{i.e.,} \quad F_A = N_B$$

$$\text{i.e.,} \quad 0.25 N_A = N_B \quad \dots(iv)$$

From eqns. (iii) and (iv), we get

$$N_A (1 + 0.4 \times 0.25) = 1100$$

or

$$N_A = \mathbf{1000 \text{ newton}}$$

$\therefore$

$$F_A = 0.25 \times N_A = 0.25 \times 1000 = \mathbf{250 \text{ N}}$$

From eqn. (iv)

$$N_B = 0.25 N_A = \mathbf{250 \text{ N}}$$

$\therefore$

$$F_B = 0.4 \times N_B = 0.4 \times 250 = \mathbf{100 \text{ N}}$$

$$\Sigma M_A = 0 \rightarrow$$

$$200 \times 3 \cos \alpha + 900 \times 5 \cos \alpha - F_B \times 6 \cos \alpha - N_B \times 6 \sin \alpha = 0$$

$\therefore$  Substituting the values of  $F_B$  and  $N_B$ , we get

$$200 \times 3 \cos \alpha + 900 \times 5 \cos \alpha - 100 \times 6 \cos \alpha - 250 \times 6 \sin \alpha = 0.$$

or

$$4500 \cos \alpha = 1500 \sin \alpha$$

or

$$\tan \alpha = 3$$

$\therefore$

$$\alpha = \mathbf{71.57^\circ}$$

**Example 5.12.** A ladder of length  $L$  rests against a wall, the angle of inclination being  $45^\circ$ . If the coefficient of friction between the ladder and the ground and that between ground and the wall is 0.5 each, what will be the maximum distance on ladder to which a man whose weight is 1.5 times the weight of ladder may ascend before the ladder begins to slip?

**Solution:** Figure 5.16(a) shows the ladder when it is about to slip when the man weighing 1.5  $W$  is at a distance, ' $aL$ ' from the end A. Its free body diagram is shown in Fig. 5.16(b).

Since ladder is on the verge of slipping, from law of friction,

$$F_A = \mu N_B = 0.5 N_B \quad \dots(1)$$

and

$$F_B = \mu N_A = 0.5 N_A \quad \dots(2)$$

$$\Sigma H = 0 \rightarrow$$

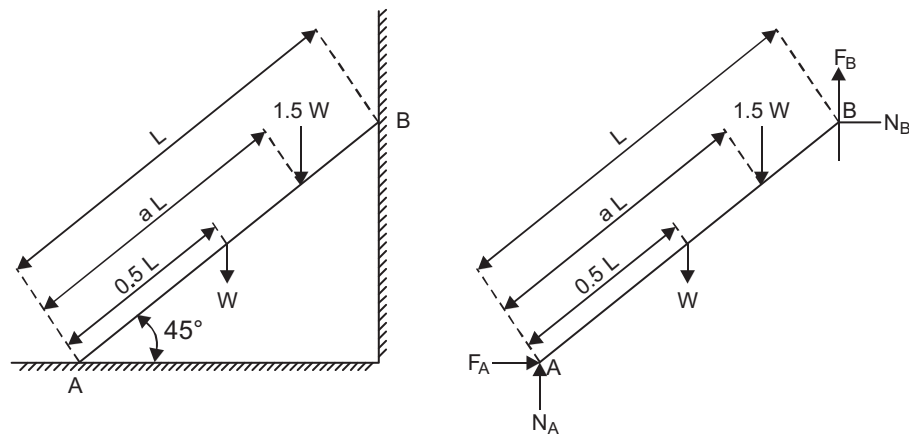


Fig. 5.16

$$F_A - N_B = 0 \quad \text{or} \quad N_B = F_A = 0.5 N_A \quad \dots(3)$$

$$\therefore F_B = 0.5 N_B = 0.25 N_A \quad \dots(4)$$

$$\Sigma V = 0 \rightarrow$$

$$N_A + F_B = W + 1.5 W$$

$$\text{i.e.,} \quad N_A + 0.25 N_A = 2.5 W$$

$$\text{or} \quad N_A = \frac{2.5}{1.5} W = 1.667 W \quad \dots(5)$$

$$\Sigma M_A = 0 \rightarrow$$

$$-F_B L \cos 45^\circ - N_B L \sin 45^\circ + 1.5 W aL \cos 45^\circ + W 0.5 L \cos 45^\circ = 0$$

Since  $\sin 45^\circ = \cos 45^\circ$ , we get

$$F_B + N_B = 1.5 aW + 0.5 W$$

$$0.25 N_A + 0.5 N_A = 1.5 aW + 0.5 W$$

$$\text{i.e.,} \quad 0.75 \times 1.667 W = 1.5 aW + 0.5 W$$

$$\text{i.e.,} \quad 1.25 = 1.5 a + 0.5$$

$$\text{or} \quad a = \frac{0.75}{1.5} = 0.5$$

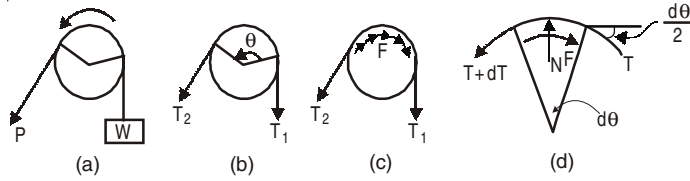
Thus in this case the man can ascend up to '0.5 L' of ladder.

## 5.7 BELT FRICTION

The transmission of power by means of belt or rope drives is possible because of friction which exists between the wheels and the belt. Similarly, band brakes stops the rotating discs because of friction between the belt and the disc. All along the contact surface the frictional resistance develops. Hence, the tension in the rope is more on the side it is pulled and is less on the other side. Accordingly, the two sides of the rope may be called as tight side and slack side.

### Relationship between Tight Side and Slack Side Forces in a Rope

Figure 5.17 (a) shows a load  $W$  being pulled by a force  $P$  over a fixed drum. Let the force on slack side be  $T_1$  and on tight side by  $T_2$  [Fig. 5.17(b)].  $T_2$  is more than  $T_1$  because frictional force develops between drum and the rope [Fig. 5.17(c)]. Let  $\theta$  be the angle of contact between rope and the drum. Now, consider an elemental length of rope as shown in Fig. 5.17(d). Let  $T$  be the force on slack side and  $T + dT$  on tight side. There will be normal reaction  $N$  on the rope in the radial direction and frictional force  $F = \mu N$  in the tangential direction. Then,



**Fig. 5.17**

$\Sigma$  Forces in radial direction = 0, gives

$$N - T \sin \frac{d\theta}{2} - (T + dT) \sin \frac{d\theta}{2} = 0$$

Since  $d\theta$  is small,  $\sin \frac{d\theta}{2} = \frac{d\theta}{2}$

$$\therefore N - T \frac{d\theta}{2} - (T + dT) \frac{d\theta}{2} = 0 \quad \dots(1)$$

i.e., 
$$N = \left( T + \frac{dT}{2} \right) d\theta$$

From the law of friction,

$$F = \mu N = \mu \left( T + \frac{dT}{2} \right) d\theta \quad \dots(2)$$

where  $\mu$  is coefficient of friction.

Now,  $\Sigma$  forces in tangential direction = 0, gives

$$(T + dT) \cos \frac{d\theta}{2} = F + T \cos \frac{d\theta}{2}$$

Since  $\frac{d\theta}{2}$  is small,  $\cos \frac{d\theta}{2} = 1$

$$\therefore T + dT = F + T$$

or 
$$dT = F \quad \dots(3)$$

From (2) and (3), 
$$dT = \mu \left( T + \frac{dT}{2} \right) d\theta$$



Neglecting small quantity of higher order, we get

$$dT = \mu T d\theta$$

$$\frac{dT}{T} = \mu d\theta.$$

Integrating both sides over 0 to  $\theta$ .

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^\theta \mu d\theta$$

$$\therefore [\log T]_{T_1}^{T_2} = \mu [\theta]_0^\theta$$

$$\log \frac{T_2}{T_1} = \mu \theta; \text{ since } \log T_2 - \log T_1 = \log \frac{T_2}{T_1}$$

$$\frac{T_2}{T_1} = e^{\mu \theta}$$

$$\text{i.e., } T_2 = T_1 e^{\mu \theta} \quad \dots(5.4)$$

**Note:**  $\theta$  should be in radians.

**Example 5.13.** A rope making  $1\frac{1}{4}$  turns around a stationary horizontal drum is used to support a weight  $W$  (Fig. 5.18). If the coefficient of friction is 0.3 what range of weight can be supported by exerting a 600 N force at the other end of the rope?

**Solution:** Angle of contact =  $1.25 \times 2\pi = 2.5\pi$

(1) Let the impending motion of the weight be downward.

Then,

$$T_1 = 600 \text{ N}; T_2 = W$$

$$\frac{W}{600} = e^{\mu 2.5\pi} = e^{0.3 \times 2.5\pi} = e^{0.75\pi}$$

$$W = 6330.43 \text{ N}$$

(2) Let the impending motion of weight be upwards. Then

$$T_1 = W; T_2 = 600 \text{ N}$$

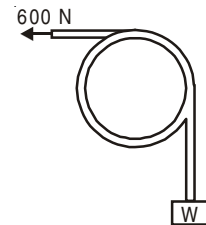
$$T_2 = T_1 e^{\mu \theta}$$

$$600 = W e^{0.75\pi}$$

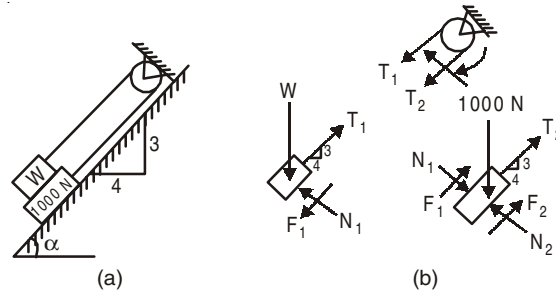
$$W = 56.87 \text{ N}$$

**Thus, a 600 N force can support a range of loads between 56.87 N to 6330.43 N weight on the other side of drum.**

**Example 5.14.** In Fig. 5.19 (a) The coefficient of friction is 0.20 between the rope and the fixed drum, and between other surface of contact  $\mu = 0.3$ . Determine the minimum weight  $W$  to prevent downward motion of the 1000 N body.



**Fig. 5.18**

**Fig. 5.19**

**Solution:** Since 1000 N weight is on the verge of sliding downwards the rope connecting it is the tight side and the rope connecting  $W$  is the slack side. Free body diagrams for  $W$  and 1000 N body are shown in Fig. 5.19(b).

Now,  $\cos \alpha = 0.8$   
 $\sin \alpha = 0.6$

Consider the equilibrium of weight  $W$ ,

$\Sigma$  Forces perpendicular to the plane = 0, gives

$$N_1 = W \cos \alpha$$

$$N_1 = 0.8 W \quad \dots(1)$$

$\therefore$

$$F_1 = \mu N_1 = 0.3 \times 0.8 W$$

$$F_1 = 0.24 W \quad \dots(2)$$

$\Sigma$  Forces parallel to the plane = 0, gives

$$T_1 = F_1 + W \sin \alpha = 0.24 W + 0.6 W$$

$$= 0.84 W \quad \dots(3)$$

Angle of contact of rope with the pulley =  $180^\circ = \pi$  radians

Applying friction equation, we get

$$T_2 = T_1 e^{\mu \theta} = T_1 e^{0.3\pi}$$

$$T_2 = 2.566 T_1$$

Substituting the value of  $T_1$  from (3)

$$T_2 = 2.156 W \quad \dots(4)$$

Now, consider 1000 N body,

$\Sigma$  forces perpendicular to the plane = 0, gives

$$N_2 - N_1 - 1000 \cos \alpha = 0$$

Substituting the value of  $N_1$  from (1),

$$N_2 = 0.8 W + 1000 \times 0.8 = 0.8 W + 800$$

$\therefore$

$$F_2 = 0.3 N_2 = 0.24 W + 240 \quad \dots(5)$$

$\Sigma$  forces parallel to the plane = 0, gives

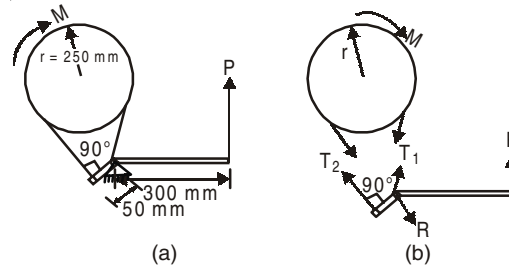
$$F_1 + F_2 - 1000 \sin \alpha + T_2 = 0$$

Substituting the values from (2), (4) and (5),

$$0.24 W + 0.24 W + 240 - 1000 \times 0.6 + 2.156 W = 0$$

$$W = 136.57 \text{ N.}$$

**Example 5.15.** A torque of 300 N-m acts on the brake drum shown in Fig. 5.20(a). If the brake band is in contact with the brake drum through  $250^\circ$  and the coefficient of friction is 0.3 determine the force  $P$  applied at the end of the brake lever for the position shown in the figure.



**Fig. 5.20**

**Solution:** Figure 5.20 (b) shows free body diagrams of brake drum and the lever arm.

Now

$$T_2 = T_1 e^{\mu\theta}$$

$$\theta = \frac{250\pi}{180} \text{ radians and } \mu = 0.3$$

$\therefore$

$$\mu \theta = 1.309$$

$\therefore$

$$T_2 = T_1 e^{1.309} = 3.7025 T_1$$

Now,

$$(T_2 - T_1) r = M$$

$$(3.7025 - 1) T_1 \times 250 = 300 \times 10^3$$

$\therefore$

$$T_1 = 444.04 \text{ N}$$

$\therefore$

$$T_2 = 1644.058 \text{ N}$$

Consider the lever arm. Taking moment about the hinge, we get

$$T_2 \times 50 = P \times 300$$

$$P = 274.0 \text{ N.}$$

**Example 5.16.** Two parallel shafts 3 m apart are to be connected by a belt running over the pulleys of diameter 500 mm and 100 mm respectively. Determine the length of belt required and the angle of contacts between belt and each pulley if it is crossed. What power can be transferred if the larger pulley rotates at 220 revolutions per minute. Given: Maximum permissible tension in the belt = 1 kN and coefficient of friction between the belt and the pulley is 0.25.

**Solution:** The crossed belt drive system is shown in Fig. 5.21.

From the geometry of the system,

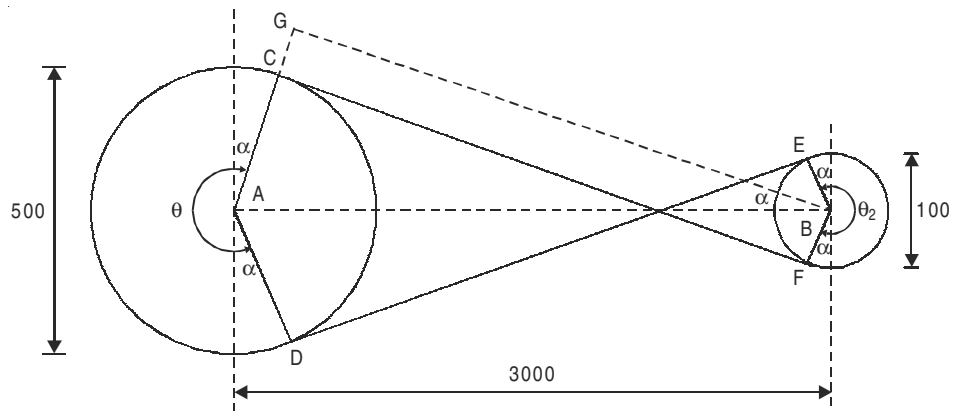
$$\sin \alpha = \frac{250 + 50}{3000} = 0.1$$

$\therefore$

$$\alpha = 5.739^\circ$$

$\therefore$

$$\begin{aligned} \theta_2 = \theta_1 &= 180 + 2\alpha = 191.478^\circ \\ &= 3.342 \text{ radians.} \end{aligned}$$

**Fig. 5.21**

$$\begin{aligned}
 \text{Length of belt} &= \text{Arc length } DC + \text{Arc length } FE + 2BG \\
 &= 250 \theta_1 + 50 \theta_2 + 2 \times 3000 \cos \alpha \\
 &= 250 \times 3.342 + 50 \times 3.342 + 2 \times 3000 \cos 5.734^\circ \\
 &= \mathbf{6972.5 \text{ mm.}}
 \end{aligned}$$

$$\text{Max. tension in the belt} = 1 \text{ kN} = 1000 \text{ N.}$$

From rope friction formula

$$\frac{T_2}{T_1} = e^{\mu\theta} = e^{0.25 \times 3.342} = e^{0.6684} = 2.306$$

In this case

$$T_2 = \text{maximum tension} = 1000 \text{ N}$$

$\therefore$

$$T_1 = \frac{1000}{2.306} = 433.66 \text{ N}$$

$$\text{Velocity of the belt} = r \theta$$

$$= 250 \times 220 \times 2\pi \text{ mm/min.}$$

$$= \frac{250 \times 220 \times 2\pi}{60} \text{ mm/sec.}$$

$$= 5799.6 \text{ mm/sec.}$$

$\therefore$

$$\begin{aligned}
 \text{Power transmitted} &= (T_2 - T_1) \times \text{Velocity} \\
 &= (1000 - 433.66) \times 5799.6 \\
 &= 3261884 \text{ N-mm/sec.} \\
 &= \mathbf{3.261884 \text{ kW.}}
 \end{aligned}$$

**Example 5.17.** In the above example, if the belt is parallel connected find angle of contact, length of belt and minimum that can be transmitted.

**Solution:** The parallel drive system is as shown in Fig. 5.22.

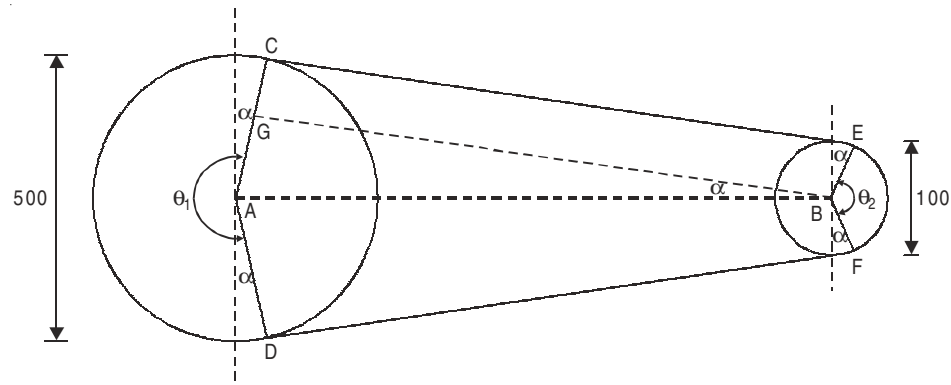


Fig. 5.22

From the geometry of the system

$$\sin \alpha = \frac{250 - 50}{3000}$$

$$\therefore \alpha = 3.822^\circ$$

$$\begin{aligned} \therefore \theta_1 &= \pi + 2\alpha = 180 + 2 \times 3.822 \\ &= 187.644^\circ \\ &= \mathbf{3.275 \text{ radians.}} \end{aligned}$$

$$\begin{aligned} \theta_2 &= \pi - 2\alpha = 180 - 2 \times 3.822 \\ &= 172.356^\circ \\ &= \mathbf{3.008 \text{ radians.}} \end{aligned}$$

Length of belt required

$$\begin{aligned} &= \text{Arc length } CD + \text{Arc length } EF + 2BG \\ &= 250 \theta_1 + 50 \theta_2 + \sqrt{(3000)^2 - (250 - 50)^2} \\ &= 250 \times 3.275 + 50 \times 3.008 + 2 \times 2993.325 \\ &= \mathbf{6955.8 \text{ mm.}} \end{aligned}$$

As frictional force is less on smaller pulley, slippage takes place on smaller pulley. Hence

$$\frac{T_2}{T_1} = e^{\mu \theta_2} = e^{0.25 \times 3.008} = 2.1212$$

$$\therefore T_1 = \frac{T_2}{2.1212} = \frac{1000}{2.1212} = 471.4 \text{ Newton}$$

$$\begin{aligned}\text{Velocity of belt} &= r \times \frac{2\pi \times 220}{60} = 250 \times \frac{2\pi \times 220}{60} \\ &= 5759.6 \text{ mm/sec.}\end{aligned}$$

$$\begin{aligned}\therefore \text{Maximum power that can be transferred without slippage} &= (T_2 - T_1) \times \text{Velocity} \\ &= (1000 - 471.4) \times 5759.6 = 3044517 \text{ N mm/sec} \\ &= \mathbf{3.044517 \text{ kW.}}\end{aligned}$$

#### IMPORTANT FORMULAE

1.  $\mu = \frac{\text{Limiting } F}{N} = \tan \alpha$
2.  $T_2 = T_1 e^{\mu\theta}$

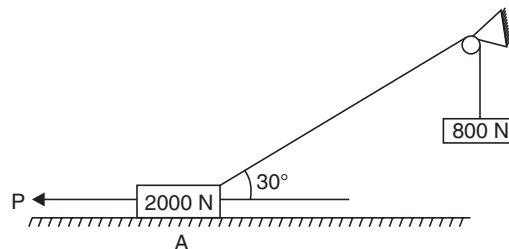
#### THEORY QUESTIONS

1. State the laws of dry friction.
2. Explain the terms: coefficient of friction, angle of friction and cone of friction.
3. Derive the expression for the relationship between tight side and slack side forces in a belt friction problem.

#### PROBLEMS FOR EXERCISE

1. A pull of 180 N applied upward at  $30^\circ$  to a rough horizontal plane was required to just move a body resting on the plane while a push of 220 N applied along the same line of action was required to just move the same body downwards. Determine the weight of the body and the coefficient of friction. [Ans.  $W = 990 \text{ N}$ ;  $\mu = 0.1732$ ]
2. The block A shown in Fig. 5.23 weighs 2000 N. The cord attached to it passes over a frictionless pulley and supports a weight equal to 800 N. The value of coefficient friction between A and the horizontal plane is 0.35. Determine the horizontal force  $P$ : (i) If the motion is impending towards the left. (ii) if the motion is impending towards the right.

[Ans. (i) 1252.82 N (ii) 132.82 N]



**Fig. 5.23**

3. A 3000 N block is placed on an inclined plane as shown in Fig. 5.24. Find the maximum value of  $W$  for equilibrium if tipping does not occur. Assume coefficient of friction as 0.2. [Ans. 2636.15]

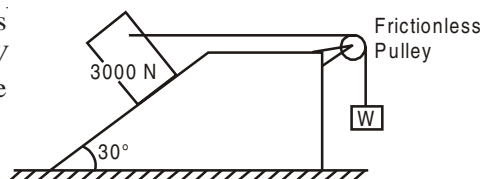


Fig. 5.24

4. Find whether block  $A$  is moving up or down the plane in Fig. 5.25 for the data given below. Weight of block  $A = 300$  N. Weight of block  $B = 600$  N. Coefficient of limiting friction between plane  $AB$  and block  $A$  is 0.2. Coefficient of limiting friction between plane  $BC$  and block  $B$  is 0.25. Assume pulley as smooth.

[Ans. The block  $A$  is stationary since  $F$  developed  $< F_{min}$ ]

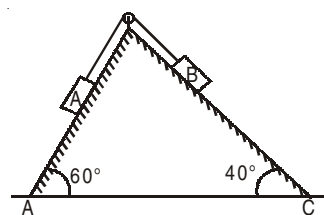


Fig. 5.25

5. Two identical blocks  $A$  and  $B$  are connected by a rod and they rest against vertical and horizontal planes respectively as shown in Fig. 5.26. If sliding impends when  $\theta = 45^\circ$ , determine the coefficient of friction, assuming it to be same for both floor and wall. [Ans. 0.414]

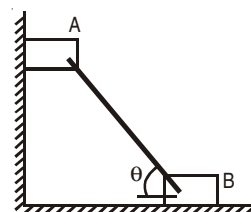


Fig. 5.26

6. Determine the force  $P$  required to start the wedge as shown in Fig. 5.27. The angle of friction for all surfaces of contact is  $15^\circ$ . [Ans. 26.6784 kN]

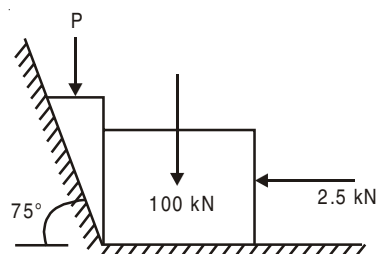


Fig. 5.27

7. Two blocks  $A$  and  $B$  weighing 3 kN and 15 kN, respectively, are held in position against an inclined plane by applying a horizontal force  $P$  as shown in Fig. 5.28. Find the least value of  $P$  which will induce motion of the block  $A$  upwards. Angle of friction for all contact surfaces is  $12^\circ$ .

[Ans. 14.025 kN]

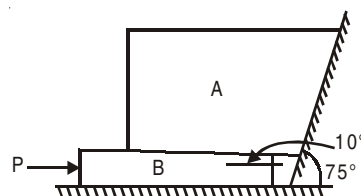


Fig. 5.28

8. In Fig. 5.29,  $C$  is a stone block weighing 6 kN. It is being raised slightly by means of two wooden wedges  $A$  and  $B$  with a force  $P$  on wedge  $B$ . The angle between the contacting surfaces of the wedge is  $5^\circ$ . If coefficient of friction is 0.3 for all contacting surfaces, compute the value of  $P$  required to impend upward motion of the block  $C$ . Neglect weight of the wedges. [Ans. 2.344 kN]

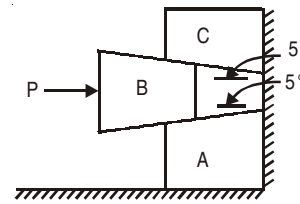


Fig. 5.29

9. Find the horizontal force  $P$  required to push the block  $A$  of weight 150 N which carries block  $B$  of weight 1280 N as shown in Fig. 5.30. Take angle of limiting friction between floor and block  $A$  as  $14^\circ$  and that between vertical wall and block  $B$  as  $13^\circ$  and coefficient of limiting friction between the blocks as 0.3. [Ans.  $P = 1294.2$  N]

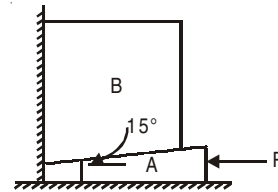


Fig. 5.30

10. The level of precast beam weighing 20,000 N is to be adjusted by driving a wedge as shown in Fig. 5.31. If coefficient of friction between the wedge and pier is 0.35 and that between beam and the wedge is 0.25, determine the minimum force  $P$  required on the wedge to make adjustment of the beam. Angle of the wedge is  $15^\circ$ . (Hint: Vertical component of reaction on wedge at contact with beam =  $1/2$  vertical load on beam = 10,000 kN)

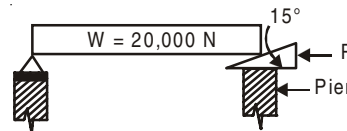


Fig. 5.31

11. A ladder 5 m long rests on a horizontal ground and leans against a smooth vertical wall at an angle of  $70^\circ$  with the horizontal. The weight of the ladder is 300 N. The ladder is on the verge of sliding when a man weighing 750 N stands on a rung 1.5 m high. Calculate the coefficient of friction between the ladder and the floor.

[Ans.  $\mu = 0.1837$ ]

12. A 4 m ladder weighing 200 N is placed against a vertical wall as shown in Fig. 5.32 as a man weighing 800 N, reaches a point 2.7 m from  $A$ , the ladder is about to slip. Assuming that the coefficient of friction between the ladder and the wall is 0.2, determine the coefficient of friction between the ladder and the floor. [Ans. 0.3548]

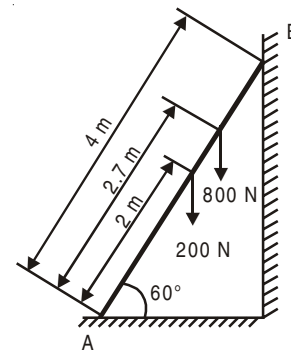


Fig. 5.32

13. A uniform ladder of length ' $L$ ' rests against a vertical wall making an angle of  $60^\circ$  with the horizontal. Coefficient of friction between wall and ladder and ground and ladder are 0.3 and 0.25 respectively. A man weighing 650 N ascends the ladder. How high he will be able to go before the ladder slips? Find the magnitude of weight to be put at the bottom of the ladder so as to make it just sufficient to permit the man to go to the top. Take ladder's weight = 900 N.

(Hint. Find  $P$  as found in Example 11. Then  $W = \frac{P}{\mu}$ )

[Ans. (i) To a length of  $0.4345 L$  i.e., to a height of  $0.435 L \sin 60^\circ = 0.367 L$  metres.

(ii)  $W = 777.68$  N]



14. Determine the maximum weight that can be lowered by a person who can exert a 300 N pull on rope if the rope is wrapped  $2\frac{1}{2}$  turns round a horizontal spur as shown in Fig. 5.33. Coefficient of friction between spur and the rope is 0.3.

[Ans. 33395.33 N]

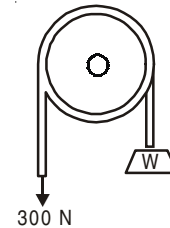


Fig. 5.33

15. Determine the minimum value of  $W$  required to cause motion of blocks  $A$  and  $B$  towards right (Ref. Fig. 5.34). Each block weighs 3000 N and coefficient of friction between blocks and the planes is 0.2. Coefficient of friction between the drum and rope is 0.1. Angle of wrap over the drum is  $90^\circ$ .

[Ans. 3065.18 N]

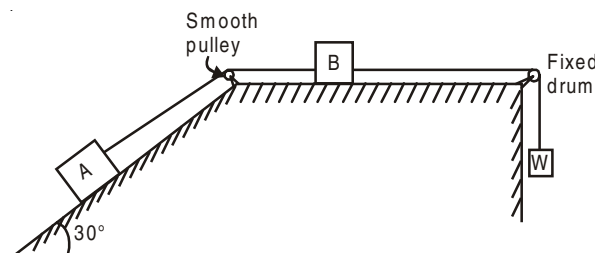


Fig. 5.34

16. Block  $A$  shown in Fig. 5.35 weighs 2000 N. The cord attached to  $A$  passes over a fixed drum and supports a weight equal to 800 N. The value of coefficient of friction between  $A$  and the horizontal plane is 0.25 and between the rope and the fixed drum is 0.1. Solve for  $P$ : (1) if motion is impending towards the left, (2) if the motion is impending towards the right.

[Ans. (1) 1230.94 N; (2) 143.0 N]

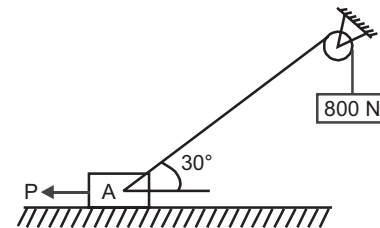


Fig. 5.35

17. The dimension of a brake drum is as shown in Fig. 5.36. Determine the torque  $M$  exerted on the drum if the load  $P = 50$  N. Assume coefficient of kinetic friction between rope and drum to be 0.15.

[Ans. 747.685 N-m]

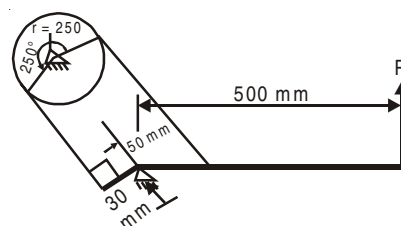


Fig. 5.36

18. A belt drive is required to transmit 12 kW power. The velocity of the belt is 9.6 m/sec. If coefficient of friction is 0.25 and the angle of contact is  $150^\circ$ , determine the maximum tension in the belt.
- [Ans.  $T_1 = 2.602$  kN]
19. Two parallel shafts are having pulleys of diameters 300 mm and 500 mm. The distance between the shafts is 2.5 m. They are connected by crossed belt drive system. The speed of the larger pulley is 500 rpm and coefficient of friction  $\mu = 0.3$ , determine the maximum power that can be transmitted, if maximum permissible tension in the belt is 2.4 kN.
- [Ans.  $P = 20.3$  kW]

# 6

## Simple Machines

---

A **simple machine** is a device with the help of which heavy loads are lifted by applying small effects in a convenient direction. Pulley used to lift water from a well and screw jacks used to lift motor car are some of the common examples of simple machines. In this chapter some of the terms connected with simple machines are explained first followed by the description of the characteristic features of levers, systems of pulleys, wheel and axle, Weston differential pulley block, inclined plane, simple screw jack, differential screw jack and winch crab.

### 6.1 DEFINITIONS

The terms commonly used while dealing with simple machines are defined below:

**Load:** This is the resistance to be overcome by the machine.

**Effort:** This is the force required to overcome the resistance to get the work done by the machine.

**Mechanical Advantage:** This is the ratio of load lifted to effort applied. Thus, if  $W$  is the load and  $P$  is the corresponding effort, then

$$\text{Mechanical Advantage} = \frac{W}{P} \quad \dots(6.1)$$

**Velocity Ratio:** This is the ratio of the distance moved by the effort to the distance moved by the load in the same interval of time. Thus,

$$\text{Velocity Ratio} = \frac{D}{d} \quad \dots(6.2)$$

where,  $D$  – distance moved by effort

$d$  – distance moved by the load.

**Input:** The work done by the effort is known as input to the machine. Since work done by a force is defined as the product of the force and the distance moved in the direction of the force,

$$\text{Input} = P \times D \quad \dots(6.3)$$

If force  $P$  is in newton and distance  $D$  is in metre, the unit of input will be N-m. One N-m work is also known as one Joule (J).

**Output:** It is defined as useful work got out of the machine, *i.e.*, the work done by the load. Thus,

$$\text{Output} = W \times d \quad \dots(6.4)$$

**Efficiency:** This is defined as the ratio of output to the input. Thus, if we use notation  $\eta$  for efficiency,

$$\begin{aligned}\eta &= \frac{\text{Output}}{\text{Input}} = \frac{W \times d}{P \times D} = \frac{W}{P} \times \frac{d}{D} \\ &= \text{Mechanical Advantage (MA)} \times \frac{1}{\text{Velocity Ratio (VR)}} \\ &= \frac{MA}{VR} \quad \dots(6.5)\end{aligned}$$

$$\text{i.e.,} \quad \text{Efficiency} = \frac{\text{Mechanical Advantage}}{\text{Velocity Ratio}}$$

**Ideal Machine:** A machine whose efficiency is 1 (i.e., 100%) is called an ideal machine. In other words, in an ideal machine, the output is equal to the input. From eqn. (6.5), in an ideal machine,

$$\text{Velocity Ratio} = \text{Mechanical Advantage}$$

**Ideal Effort:** Ideal effort is the effort required to lift the given load by the machine assuming the machine to be ideal.

For ideal machine,

$$VR = MA$$

If  $P_i$  is the ideal effort, then

$$VR = \frac{W}{P_i}$$

$$\therefore P_i = \frac{W}{VR} \quad \dots(6.6)$$

**Ideal Load:** Ideal load is the load that can be lifted using the given effort by the machine, assuming it to be ideal.

For the ideal machine,

$$VR = MA$$

If  $W_i$  is the ideal load, then

$$VR = \frac{W_i}{P}$$

$$\therefore W_i = VR \times P \quad \dots(6.7)$$

## 6.2 PRACTICAL MACHINES

In practice, it is difficult to get an ideal machine. Friction exists between all surfaces of contacts of movable parts. Some of the work done by the effort is utilised to overcome frictional resistance. Hence, the useful work done in lifting the load is reduced, resulting in reduction of efficiency.

Let

$P$  = actual effort required

$P_i$  = ideal effort required

$W$  = actual load to be lifted

$W_i$  = ideal load to be lifted

Then,

$P - P_i$  is called *effort lost in friction* and  $W - W_i$  is called *frictional resistance*.

Now, 
$$\eta = \frac{MA}{VR} = \frac{W}{P} \times \frac{1}{VR}$$

From eqn. (6.6), 
$$P_i = \frac{W}{VR}$$

$$\therefore \eta = \frac{P_i}{P}$$

Similarly from eqn. (6.7),  $W_i = VR \times P$

$$\therefore \eta = \frac{W}{W_i}$$

Thus, 
$$\eta = \frac{P_i}{P} = \frac{W}{W_i} \quad \dots(6.8)$$

**Example 6.1.** In a lifting machine, an effort of 500 N is to be moved by a distance of 20 m to raise a load of 10,000 N by a distance of 0.8 m. Determine the velocity ratio, mechanical advantage and efficiency of the machine. Determine also ideal effort, effort lost in friction, ideal load and frictional resistance.

**Solution:** Load,  $W = 10,000$  N

Effort  $P = 500$  N

Distance moved by the effort  $D = 20$  m

Distance moved by the load  $d = 0.8$  m

**Mechanical advantage,** 
$$MA = \frac{W}{P} = \frac{10,000}{500}$$
  

$$= 20$$

**Velocity ratio,** 
$$VR = \frac{D}{d} = \frac{20}{0.8}$$
  

$$= 25$$

**Efficiency,** 
$$\eta = \frac{MA}{VR} = \frac{20}{25}$$
  

$$= 0.8 = 80 \text{ per cent}$$

**Ideal effort,** 
$$P_i = \frac{W}{VR} = \frac{10,000}{25} = 400 \text{ N}$$

**Effort lost in friction** 
$$= P - P_i$$
  

$$= 500 - 400$$
  

$$= 100 \text{ N}$$

**Ideal load,** 
$$W_i = P \times VR = 500 \times 25$$
  

$$= 12,500 \text{ N}$$

$$\begin{aligned}
 \text{Frictional resistance} &= W_i - W \\
 &= 12,500 - 10,000 \\
 &= 2500 \text{ N}
 \end{aligned}$$

### 6.3 LAW OF MACHINE

The relationship between the load lifted and the effort required in a machine is called the **law of machine**. This is found by conducting experiments in which efforts required for lifting different loads are determined and then load *versus* effort graph as shown in Fig. 6.1 is plotted. This is generally a straight line which does not pass through the origin.

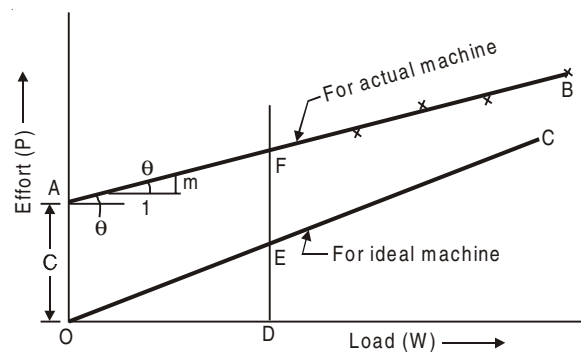


Fig. 6.1

The law of machine can be expressed mathematically in the form:

$$P = mW + C \quad \dots(6.9)$$

where,  $C$  is the intercept  $OA$  and  $m = \tan \theta$ , the slope of  $AB$ . For the ideal machine

$$MA = VR$$

$$\frac{W}{P} = VR$$

This is a straight line relationship passing through the origin and is shown by line  $OC$  in Fig. 6.1.

After plotting the law for the actual machine ( $AB$ ) and the law for ideal machine ( $OC$ ), it is easy to determine efficiency at any given load. The vertical line  $DEF$  corresponding to given load  $OD$  is drawn. Then,

$$DE = P_i, \text{ effort required in ideal machine}$$

$$DF = P, \text{ effort required in actual machine}$$

$$\begin{aligned}
 \therefore \text{Friction loss} &= P - P_i \\
 &= DF - DE \\
 &= EF
 \end{aligned}$$

$$\begin{aligned}
 \text{Efficiency, } \eta &= \frac{P_i}{P} \\
 &= \frac{DE}{DF}
 \end{aligned}$$

### 6.4 VARIATION OF MECHANICAL ADVANTAGE

Mechanical Advantage (MA) is given by:

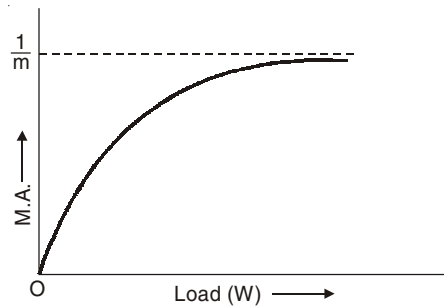
$$MA = \frac{W}{P}$$

From the law of machine,

$$P = mW + C$$

$$\begin{aligned} \therefore MA &= \frac{W}{mW + C} \\ &= \frac{1}{m + \frac{C}{W}} \end{aligned} \quad \dots(6.10)$$

As the load increases,  $\frac{C}{W}$  which is in denominator, decreases and hence mechanical advantage increases. In limiting case when  $W$  tends to infinity,  $\frac{C}{W} = 0$  and hence *maximum mechanical advantage equals*  $\frac{1}{m}$ . The variation of mechanical advantage with respect to load is as shown in Fig. 6.2.



**Fig. 6.2**

### 6.5 VARIATION OF EFFICIENCY

From eqn. (6.5), the efficiency of the machine is given by  $\eta = \frac{MA}{VR}$ . Using the eqn. (6.10),

$$\eta = \frac{1}{VR} \times \frac{1}{m + \frac{C}{W}} \quad \dots(6.11)$$

Since the velocity ratio (VR) is constant for a machine, variation of efficiency with load is similar to the variation of mechanical advantage with the load.

The *maximum efficiency* is approached as the load approaches infinity ( $W \rightarrow \infty$ ) and its value is equal to  $\frac{1}{VR} \times \frac{1}{m}$ . The variation of the efficiency with load is shown in Fig. 6.3.

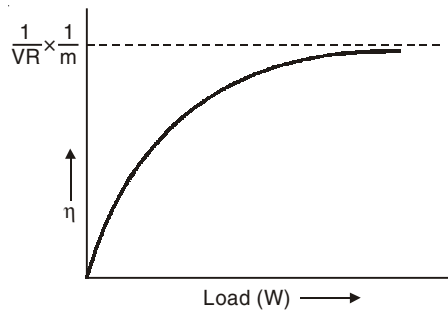


Fig. 6.3

**Example 6.2.** In a simple machine, whose velocity ratio is 30, a load of 2400 N is lifted by an effort of 150 N and a load of 3000 N is lifted by an effort of 180 N. Find the law of machine and calculate the load that could be lifted by a force of 200 N. Calculate also:

- (1) The amount of effort wasted in overcoming the friction,
- (2) Mechanical advantage, and
- (3) The efficiency.

**Solution:** Let the law of machine be

$$P = mW + C$$

In the first case,  $P = 150$  N, and  $W = 2400$  N

In the second case,  $P = 180$  N and  $W = 3000$  N

$$\therefore 150 = 2400 m + C \quad \dots(1)$$

$$180 = 3000 m + C \quad \dots(2)$$

Subtracting eqn. (1) from eqn. (2), we get

$$30 = 600 m$$

$$m = 0.05$$

Substituting this value in eqn. (1), we get

$$150 = 120 + C$$

$$\therefore C = 30$$

Hence, the law of machine is

$$P = 0.05 W + 30 \quad \dots(3)$$

When a force of 200 N is applied:

From the law of machine (3),

$$200 = 0.05 W + 30$$

$$\therefore W = 3400 \text{ N}$$

Ideal effort is given by:

$$P_i = \frac{W}{VR} = \frac{3400}{30} = 113.33 \text{ N}$$

∴ Effort wasted in overcoming the friction

$$= P - P_i = 200 - 113.33$$

$$= 86.67 \text{ N}$$

$$\text{Mechanical advantage} = \frac{W}{P} = \frac{3400}{200}$$

$$= 17$$

$$\text{Efficiency} = \frac{MA}{VR} = \frac{17}{30} = .5667$$

$$= 56.67\%$$

**Example 6.3.** In a lifting machine an effort of 150 N raised a load of 7700 N. What is the mechanical advantage? Find the velocity ratio if the efficiency at this load is 60%. If by the same machine, a load of 13,200 N is raised by an effort of 250 N, what is the efficiency? Calculate the maximum mechanical advantage and the maximum efficiency.

**Solution:** Effort,  $P = 150 \text{ N}$

Load,  $W = 7700 \text{ N}$

∴ Mechanical advantage:

$$MA = \frac{W}{P} = \frac{7700}{150} = 51.33$$

If the efficiency is 60%,

$$\eta = 0.6$$

$$\eta = \frac{MA}{VR}$$

$$\therefore 0.6 = \frac{51.33}{VR}$$

$$\text{or } VR = \frac{51.33}{0.6} \quad \text{i.e., } \mathbf{VR = 85.55}$$

When an effort of 250 N raised a load of 13,200 N,

$$MA = \frac{W}{P} = \frac{13,200}{250} = 52.8$$

$$\therefore \eta = \frac{MA}{VR} = \frac{52.8}{85.55} = 0.6172$$

$$\text{i.e., } \eta = 61.72\%$$

Let the law of machine be

$$P = mW + C$$

$$\text{In the first case, } 150 = 7700 m + C \quad \dots(1)$$

$$\text{In the second case, } 250 = 13,200 m + C \quad \dots(2)$$

Subtracting eqn. (1) from eqn. (2), we get

$$100 = 5500 m$$



$$\therefore m = 0.01818$$

$\therefore$  Maximum mechanical advantage

$$= \frac{1}{m} = \frac{1}{0.01818}$$

$$= 55$$

Maximum efficiency

$$= \frac{1}{m} \times \frac{1}{VR} = \frac{1}{0.01818} \times \frac{1}{85.55}$$

$$= 0.6429$$

$$= 64.29\%$$

**Example 6.4.** The efforts required for lifting various loads by a lifting machine are tabulated below:

Load lifted in N	100	200	300	400	500	600
Effort required in N	16.0	22.5	28.0	34.0	40.5	46.5

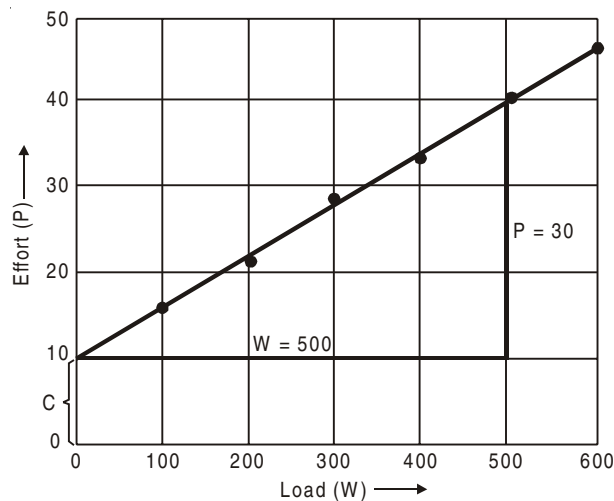
Determine the law of machine. If the velocity ratio is 25, calculate efficiency at each load and plot efficiency versus load curve. From this curve, determine the maximum efficiency.

**Solution:** Figure 6.4 shows the graph of effort versus load. From this figure,  $C = 10$  N and slope

$$m = \frac{30}{500} = 0.06$$

$\therefore$  The law of machine is

$$P = 0.06 W + 10$$

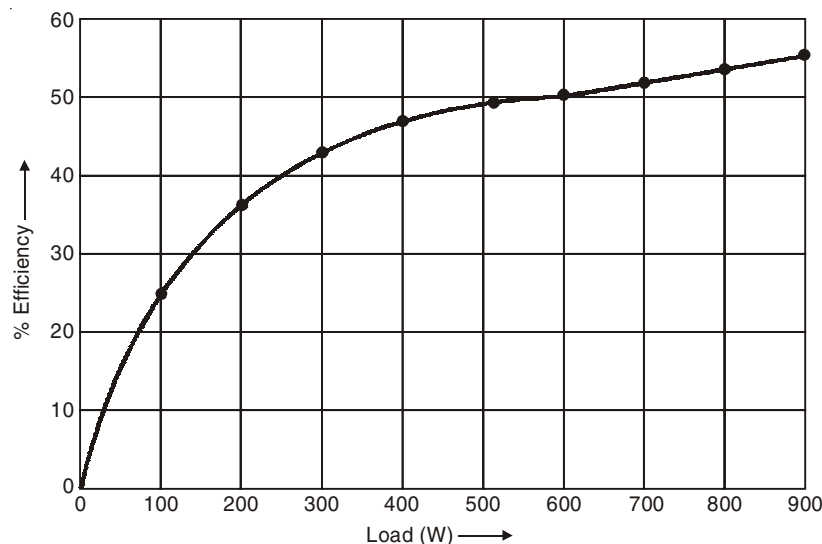


**Fig. 6.4**

$$\eta = \frac{MA}{VR} = \frac{W}{P} \times \frac{1}{VR} = \frac{W}{P} \times \frac{1}{25} = \frac{W}{25P}$$

Table below shows the calculation of efficiency for various loads:

<i>Load in N</i>	100	200	300	400	500	600
<i>Effort in N</i>	16.0	22.5	28.0	34.0	40.5	46.5
<i>Efficiency in %</i>	25	36.56	42.86	47.06	49.38	51.61



**Fig. 6.5**

From the graph (Fig. 6.5) maximum efficiency is seen as 57%. Actually if it is plotted for infinitely large load, **maximum efficiency** will be equal to

$$\frac{1}{m} \times \frac{1}{VR} = \frac{1}{0.06} \times \frac{1}{25} = 0.6667 = \mathbf{66.67\%}$$

## 6.6 REVERSIBILITY OF A MACHINE

If the removal of effort while lifting results in lowering of the load, the machine is said to be **reversible**. The machine is said to be **self-locking** if the load is not lowered on removal of the effort.

For example, while lifting water from the well, the pot falls back if the effort to pull it up is removed whereas the screw jack used to lift the motor car will hold the car at the same position even if the application of the effort is stopped. Hence, the former is a reversible and later is a self-locking type simple lifting machine.

A simple lifting machine will be reversible or self-locking solely based on its efficiency. It can be shown that a lifting machine is reversible if its efficiency is greater than 50 per cent and self-locking if its efficiency is less than 50 per cent.

Let

- $W$  – load being lifted
- $P$  – effort required
- $VR$  – Velocity ratio
- $D$  – distance moved by the effort

$d$  – distance moved by the load

Then,

$$\text{Input} = P \times D$$

$$\text{Output} = W \times d$$

$$\therefore \text{Work lost in friction} = PD - Wd$$

When effort is removed, the load can start moving down if it can overcome the frictional resistance  $= PD - Wd$ . Hence the condition for the reversibility is:

$$Wd > (PD - Wd)$$

$$\therefore 2Wd > PD$$

$$\left(\frac{W}{P}\right)\left(\frac{d}{D}\right) > \frac{1}{2}$$

$$MA \times \frac{1}{VR} > \frac{1}{2}$$

$$\text{i.e.,} \quad \eta > \frac{1}{2} \quad \text{or} \quad 50\%.$$

Hence, a machine is reversible if its efficiency is greater than 50%.

**Example 6.5.** In a lifting machine in which velocity ratio is 30, a load of 5000 N is lifted with an effort of 360 N. Determine whether it is self-locking or reversible machine. How much is the frictional resistance?

**Solution:**  $VR = 30$

$$W = 5000 \text{ N}$$

$$P = 360 \text{ N}$$

$$MA = \frac{W}{P} = \frac{5000}{360} = 13.889$$

$$\begin{aligned} \text{efficiency, } \eta &= \frac{MA}{VR} = \frac{13.889}{30} \\ &= 0.4630 \\ &= 46.30\% \end{aligned}$$

Since the efficiency is less than 50%, **it is self-locking machine.**

$$\text{Ideal load, } W_i = P \times VR$$

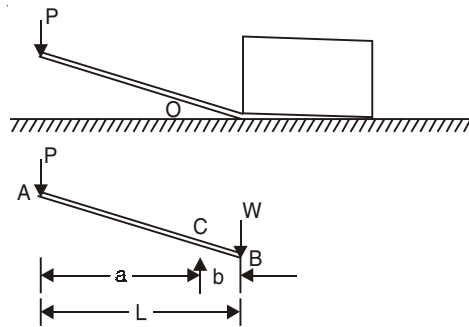
$$= 360 \times 30$$

$$= 10,800 \text{ N}$$

$$\begin{aligned} \therefore \text{Frictional resistance} &= W_i - W = 10,800 - 5000 \\ &= 5,800 \text{ N} \end{aligned}$$

## 6.7 LEVER ARM

Probably lever arm is the first machine to be invented by man. It is a strong rod used to push or lift slightly a heavy body. Figure 6.6 shows a lever arm  $AB$  used for sliding a block. Let  $W$  be the weight to be slightly lifted for pushing. A rigid body is inserted very close to the body (say at  $C$ ) between the rod (lever) and floor. The rigid body acts as a hinge about which the lever is to be rotated. If 'a' is the effort arm and 'b' is the load arm, obviously.

**Fig. 6.6**

$$Pa = Wb$$

or

$$P = W \frac{b}{a}$$

$$\therefore \text{Mechanical advantage} = \frac{W}{P} = \frac{a}{b} \quad \dots(6.12)$$

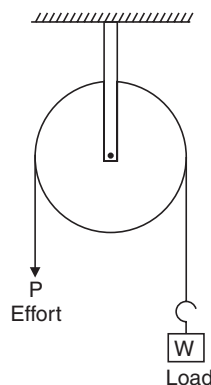
## 6.8 PULLEYS

A systematic arrangement of one or more pulleys may provide a simple and convenient lifting machine. In its simplest form, it consists only one pulley over which a rope or chain passes as shown in the Fig. 6.7. In this case, velocity ratio is equal to one since distance moved by effort is equal to the distance moved by the load. It just performs changing the direction of the applied force.

Depending on the arrangement, pulleys are classified as:

- (1) First order pulley system
- (2) Second order pulley system
- (3) Third order pulley system.

At times, it may be difficult or may be detour to find velocity ratio directly. In such cases ideal conditions may be assumed (neglecting friction) and mechanical advantage may be found first. Then applying  $VR = MA$  for ideal machine, the velocity ratio is found. This method of determining velocity ratio is used for various pulley systems considered here.

**Fig. 6.7**

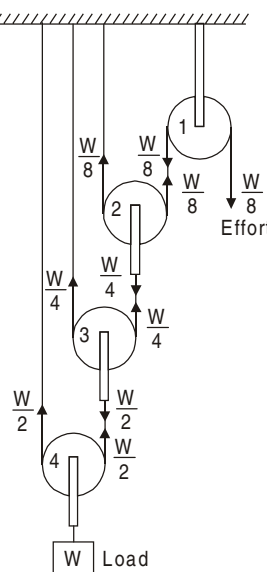
### First Order Pulley System

A first order pulley system is shown in the Fig. 6.8. Pulley No. 1 is fixed in position to a support at top. A rope passes over this pulley and one end of this rope is tied to the support at the top, making a loop, in which pulley No. 2 is suspended and effort is applied at the other end. One end of another rope is tied to pulley No. 2 and the other end to the top support which makes a loop in which pulley No. 3 is suspended. Similarly, a number of pulleys can be arranged as shown, when an effort is applied to lift the load except first pulley all other pulleys move vertically. Therefore, first pulley is termed as fixed pulley and the others as movable pulleys.

Let an effort  $P$  be applied to lift a load  $W$ . In an ideal pulley system (friction = 0), the rope which passes over pulley No. 4 is subjected to a tension  $\frac{W}{2}$ . Then tension in rope which passes over pulley No. 3 is  $\frac{W}{4}$  and tension in the rope which passes over pulley No. 2 is  $\frac{W}{8}$ . Hence, an effort equal to  $\frac{W}{8}$  is required to lift a load  $W$ .

$$\therefore P = \frac{W}{8} \quad \text{and hence } MA = \frac{W}{P} = \frac{W}{\frac{W}{8}} = 8 = 2^3$$

$$\text{But in an ideal machine, } VR = MA \quad \therefore VR = 2^3$$



**Fig. 6.8**

It is to be noted that in the system considered, there are three movable pulleys and the velocity ratio is  $2^3$ . If there are only two movable pulleys, then velocity ratio would be 4 (i.e.,  $2^2$ ). In general, in the first order pulley system, velocity ratio (VR) is given by  $2^n$ , where,  $n$  is the number of movable pulleys present in the system. Thus, in first order pulley system

$$VR = 2^n \quad \dots(6.13)$$

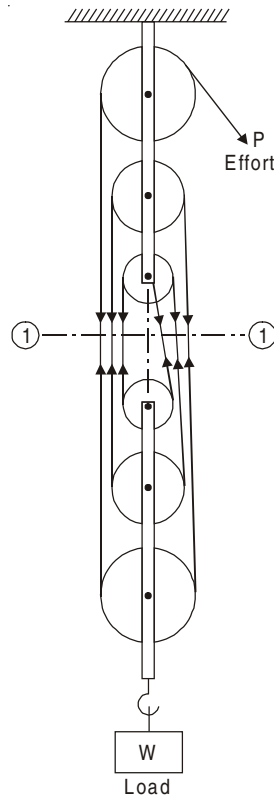
**Fig. 6.9****Second Order Pulley System**

Figure 6.9 shows a second order pulley system. This system consists of a top pulley block and a bottom pulley block. In a pulley block pulleys may be arranged side by side or may be one below the other as shown in Fig. 6.9. The top pulley block is fixed in position to the top support whereas bottom pulley block can move vertically along with the load which is attached to it. One end of the rope is attached to the hook provided at the bottom of the top pulley block and the effort is applied at the other end. A single rope goes round all the pulleys. Let an effort  $P$  be applied to lift a load  $W$ .

Neglecting frictional losses, the tension in the rope all along the length is  $P$ . Take the section along (1)–(1) and consider the equilibrium of the bottom pulley block.

The load  $W$  is lifted using six ropes having equal tension  $P$

$$\therefore W = 6P, \quad \therefore P = \frac{W}{6}, \quad \therefore MA = \frac{W}{P} = \frac{W}{\frac{W}{6}} = 6.$$

But  $VR = MA$  in ideal condition

$$\therefore VR = 6.$$

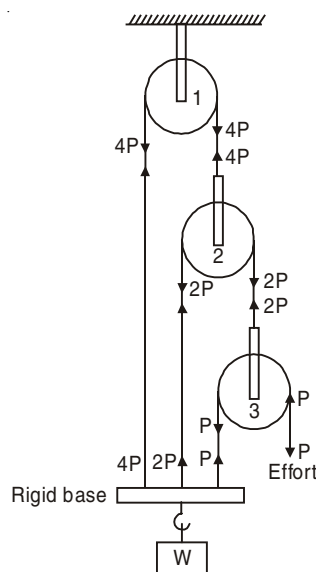
In general, in the second order pulley system velocity ratio is equal to twice the number of movable pulleys in the system.

That is,  $VR = 2n$  ... (6.14)

where,  $n$  is total number of movable pulley in the system.

### Third Order Pulley System

The arrangement of the pulleys in the third order system is shown in the Fig. 6.10. In this system a pulley (No. 1) is fixed to the top support, over which a rope passes. One end of the rope is attached to a rigid base at the bottom. The other end is attached to a second pulley. Over this pulley another rope passes, whose one end is attached to the same rigid base and the other end to a third pulley as shown. Likewise a series of pulleys can be arranged. The load to be lifted will be attached to the rigid base.



**Fig. 6.10**

Referring to the Fig. 6.10, let the effort required be  $P$  to lift a load  $W$ . Then neglecting friction,

Tension in the rope which passes over pulley No. 3 =  $P$

Tension in the rope which passes over pulley No. 2 =  $2P$

Tension in the rope which passes over pulley No. 1 =  $4P$

$\therefore$  A total force of  $7P$  is acting on the base.

$\therefore$  Lifting force produced on the base =  $7P$

Considering the equilibrium of rigid base,

$$7P = W \quad \therefore \quad MA = \frac{W}{P} = 7$$

But in an ideal machine,  $VR = MA$ , and hence  $VR = 7$ .

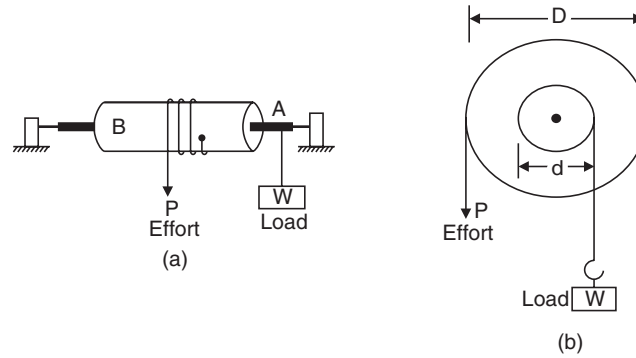
It can be easily seen that, if there are only two pulleys,  $VR = 3$  and if there is only one pulley,  $VR = 1$ . Therefore, in general, for the third order pulley system:

$$VR = 2^n - 1 \quad \dots(6.15)$$

where,  $n$  = number of pulleys.

### 6.9 WHEEL AND AXLE

This machine consists of an axle  $A$  having diameter  $d$  and a wheel  $B$  having diameter  $D$  ( $D > d$ ) co-axially fitted as shown in Fig. 6.11. The whole assembly is mounted on ball bearing so that wheel and axle can be rotated.



**Fig. 6.11**

One end of a rope is tied to the pin provided on the wheel and the rope is wound around the wheel. The other end of the rope provides the means for the application of the effort. One end of another rope is tied to the pin provided on the axle and wound around the axle in the opposite direction to that of rope wound to the wheel. To the other end of this rope the load is attached. If the whole assembly is rotated, one rope gets wound up and the other gets unwound.

Suppose the assembly is moved by one complete revolution, then the distance moved by the effort  $= \pi D$  and distance moved by the load  $= \pi d$

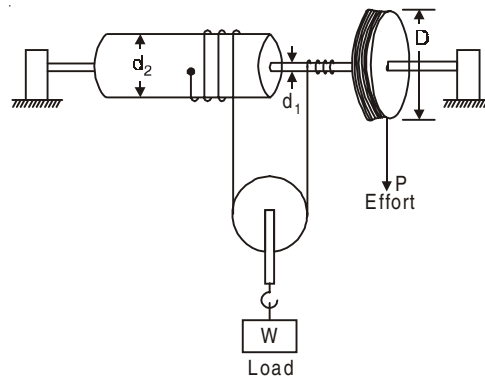
$$\therefore \quad VR = \frac{\text{distance moved by effort}}{\text{distance moved by load}}$$

$$= \frac{\pi D}{\pi d},$$

$$\text{or} \quad VR = \frac{D}{d} \quad \dots(6.16)$$

### 6.10 WHEEL AND DIFFERENTIAL AXLE

An improvement over wheel and axle machine is made by using one more wheel of bigger diameter and it is called wheel and differential axle.



**Fig. 6.12**



This consists of a differential axle of diameter  $d_1$  and  $d_2$  ( $d_1 < d_2$ ) and a wheel of diameter  $D$ , fixed uniaxially as shown in Fig. 6.12. One end of the rope is tied to the pin provided on the axle portion having diameter  $d_1$  and a part of the rope is wound around it. The other end of the rope is wound around axle on the portion having diameter  $d_2$  in the opposite direction. This pattern of winding forms a loop and a simple pulley is installed in this loop as shown in the figure. The load is attached to this pulley. The second rope is wound to the wheel in such a direction that if it is unwound, the rope around the bigger diameter axle gets wound up and the rope around smaller diameter axle gets unwound. One end of this second rope provides means for application of the effort.

Suppose the whole system makes one complete revolution due to the applied effort, then

Total distance moved by the effort at the differential axle =  $\pi D$

Length of winding of the rope =  $\pi d_2$

Length of unwinding of rope =  $\pi d_1$

$\therefore$  Net wound length =  $\pi d_2 - \pi d_1$   
 $= \pi(d_2 - d_1)$

But, the rope is continuous and the load is to be lifted by the pulley block in the loop.

$\therefore$  Total height over which pulley is lifted =  $\frac{\pi(d_2 - d_1)}{2}$

and hence the distance moved by the load =  $\frac{\pi(d_2 - d_1)}{2}$

$\therefore$  
$$VR = \frac{\pi D}{\frac{\pi(d_2 - d_1)}{2}}$$

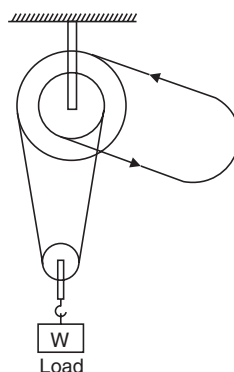
$$VR = \frac{2D}{d_2 - d_1}$$

Hence, velocity ratio in wheel and differential axle is given by:

$$VR = \frac{2D}{d_2 - d_1} \quad \dots(6.17)$$

### 6.11 WESTON DIFFERENTIAL PULLEY BLOCK

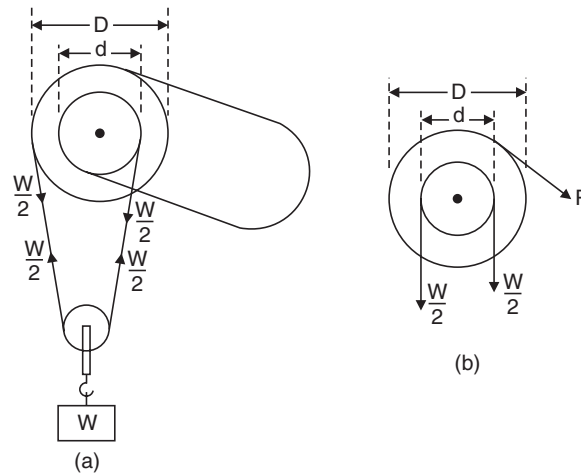
This is a special type of simple pulley system. It is shown in the Fig. 6.13.



**Fig. 6.13**

This system consists of two pulley blocks, one at the top attached to the support and the other at the bottom hanging in the chain loop. The top block consists of two wheels of different diameters, but fixed co-axially. The bottom block is a simple pulley to which the load  $W$  is attached. An endless chain is wound around the pulley system as shown in the figure. All the wheels are made with teeth so as to accommodate the links of the chain. The chain is essentially used to avoid slipping.

To determine velocity ratio of the system, let us consider pulley block as an ideal machine and determine its mechanical advantage first. In ideal machine  $VR = MA$ . Let the diameter of the larger wheel of the top block be  $D$  and the diameter of the smaller wheel of the top block be  $d$  [Fig. 6.14(a)]. Let the effort required to lift the load  $W$  be  $P$ .



**Fig. 6.14**

Then the tension in the chain loop in which pulley is hanging is  $\frac{W}{2}$ .

Now, taking moment about the axis of top block [Fig. 6.14(b)],

$$\begin{aligned} \frac{W}{2} \times \frac{D}{2} &= \frac{W}{2} \times \frac{d}{2} + P \times \frac{D}{2} \\ \therefore \frac{W}{4}(D - d) &= \frac{PD}{2} \\ \frac{W}{P} &= \frac{2D}{(D - d)} \\ \text{i.e., } MA &= \frac{2D}{(D - d)} \end{aligned}$$

In an ideal machine,

$$VR = MA = \frac{2D}{(D - d)} \quad \dots(6.18)$$

### 6.12 INCLINED PLANE

Inclined plane is a very simple lifting device. The lift is essentially accomplished with the horizontal displacement.

An inclined plane consists of a plane surface at a definite angle over which the load is to be lifted.

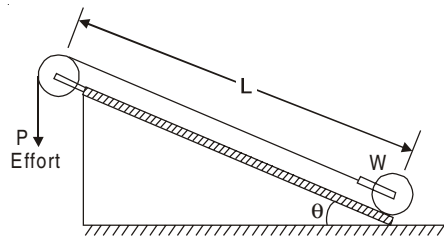


Fig. 6.15

Figure 6.15 shows a typical inclined plane. Here the load is a roller which is to be lifted to a higher elevation. One end of a rope is tied to the roller and the rope is passed over a pulley attached at the top of the inclined plane. At the other end of the rope the effort is applied.

Let the angle of inclination of the plane be  $\theta$  and the length of the inclined plane be  $L$ . Then, if the roller is made to roll from bottom to top, applying an effort  $P$ , the load is lifted through a height of  $L \sin \theta$ . In this process the effort  $P$  moves through a distance  $L$  vertically downwards.

$$\begin{aligned} \therefore \quad VR &= \frac{\text{distance moved by the effort}}{\text{distance through which load is lifted}} \\ &= \frac{L}{L \sin \theta} = \frac{1}{\sin \theta} \\ \therefore \quad VR &= \frac{1}{\sin \theta} \quad \dots(6.19) \end{aligned}$$

**Example 6.6.** In a first order system of pulleys there are three movable pulleys. What is the effort required to raise a load of 6000 N? Assume efficiency of the system to be 80%.

If the same load is to be raised using 520 N, find the number of movable pulleys that are necessary.

Assume a reduction of efficiency of 5% for each additional pulley used in the system.

**Solution:**  $VR = 2^n$ , where  $n$  is the number of movable pulleys.

$$VR = 2^3 = 8$$

$$\begin{aligned} \text{Now,} \quad MA &= \eta \times VR \\ &= 0.8 \times 8 \\ &= 6.4 \end{aligned}$$

$$\text{i.e.,} \quad \frac{W}{P} = 6.4$$

$$\therefore \quad P = \frac{W}{6.4} = \frac{6000}{6.4}$$

*i.e.*,  $P = 937.5 \text{ N}$

In the second case,

$$\text{Effort} = 520 \text{ N}$$

$$\text{Efficiency} \quad \eta = 0.80 - n_1 \times 0.05$$

where  $n_1$  = number of additional pulleys required and equal to  $(n - 3)$ .

$$MA = \eta \times VR$$

*i.e.*,  $\frac{W}{P} = \eta \times VR$

$$\begin{aligned} \therefore W &= P \times \eta \times 2^n \\ &= P(0.8 - n_1 \times 0.05) \times 2^n \\ &= P[0.8 - (n - 3) \times 0.05] 2^n \end{aligned}$$

By going for a trial and error solution, starting with one additional pulley *i.e.*, totally with four pulleys,

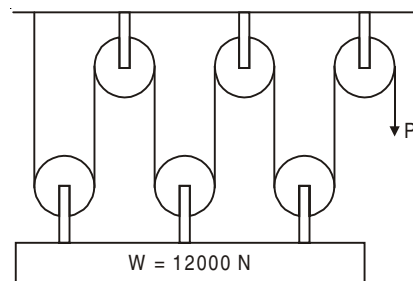
$$W = 520 [0.8 - (4 - 3) \times 0.05] 2^4 = 6240 \text{ N}$$

*i.e.*, if four pulleys are used, a load of 6240 N can be raised with the help of 520 N effort.

$\therefore$  **Number of movable pulleys required = 4**

**Example 6.7.** What force is required to raise the load  $W$  shown in Fig. 6.16 ? Assume efficiency of the system to be 85%.

**Solution:** The pulley system shown in the Fig. 6.16 is a variation of the second order pulley system.



**Fig. 6.16**

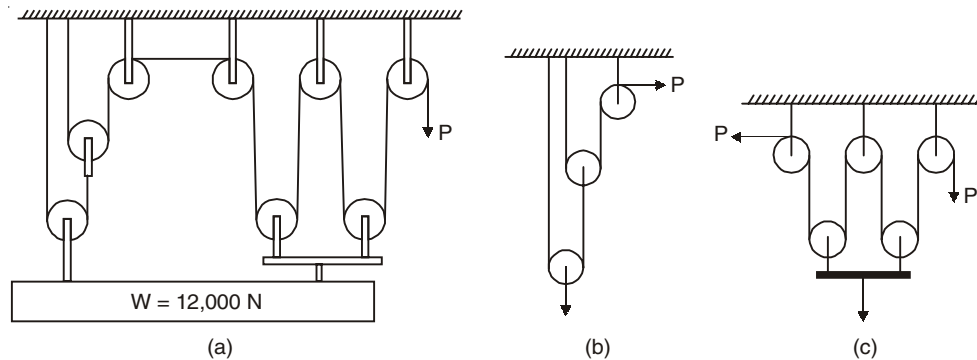
$$VR = 2 \times \text{number of movable pulleys} = 6$$

$$\begin{aligned} MA &= \eta \times VR \\ &= 0.85 \times 6 = 5.1 \end{aligned}$$

*i.e.*,  $\frac{W}{P} = 5.1$

$$\therefore P = \frac{W}{5.1} = \frac{12,000}{5.1} = 2352.94 \text{ N}$$

**Example 6.8.** Find the pull required to lift the load  $W$  shown in Fig. 6.17(a) assuming the efficiency of the system to be 78%.



**Fig. 6.17**

**Solution:** The pulley system shown in Fig. 6.17(a) is a combination of a first order system and a second order system as shown in Figs. 6.17(b) and (c).

Let load  $W$  be lifted by a distance  $x$ . Consider the first order system portion [Fig. 6.17(b)].

Here there are two movable pulleys. Hence

$$VR = 2^2 = 4$$

In this portion  $P$  moves by  $4x$ .

Now, consider the second order pulley system portion [Fig. 6.17(c)]. Here there are two movable pulleys. Hence  $VR = 2 \times 2 = 4$ .

$\therefore$  Distance moved by the effort in this system =  $4x$ .

Hence, the total distance moved by the effort in the given system =  $4x + 4x = 8x$

$$\therefore VR = \frac{8x}{x} = 8$$

$$\text{Now, } MA = \eta \times VR = 0.78 \times 8 \\ = 6.24$$

$$\text{i.e., } \frac{W}{P} = 6.24$$

$$\therefore P = \frac{12,000}{6.24} = 1923.08 \text{ N}$$

**Example 6.9.** A lifting machine consists of pulleys arranged in the third order system. There are three pulleys in the system. A load of 1000 N is lifted by an effort of 180 N. Find the efficiency of the machine and the effort lost in friction.

**Solution:** For the third order system of pulleys,

$$VR = 2^n - 1$$

where,  $n$  is the number of pulleys in the system.

$$VR = 2^3 - 1 = 7$$

$$\begin{aligned}
 \text{Now,} \quad MA &= \eta \times VR \\
 \text{i.e.,} \quad \frac{W}{P} &= \eta \times VR \\
 \therefore \quad \eta &= \frac{W}{P} \times \frac{1}{VR} \\
 &= \frac{1000}{180} \times \frac{1}{7} = 0.7937 \\
 \text{i.e.,} \quad \eta &= 79.37\% \\
 \text{Now,} \\
 \text{Ideal effort,} \quad P_i &= \frac{W}{VR} \\
 &= \frac{1000}{7} \\
 &= 142.86 \text{ N}
 \end{aligned}$$

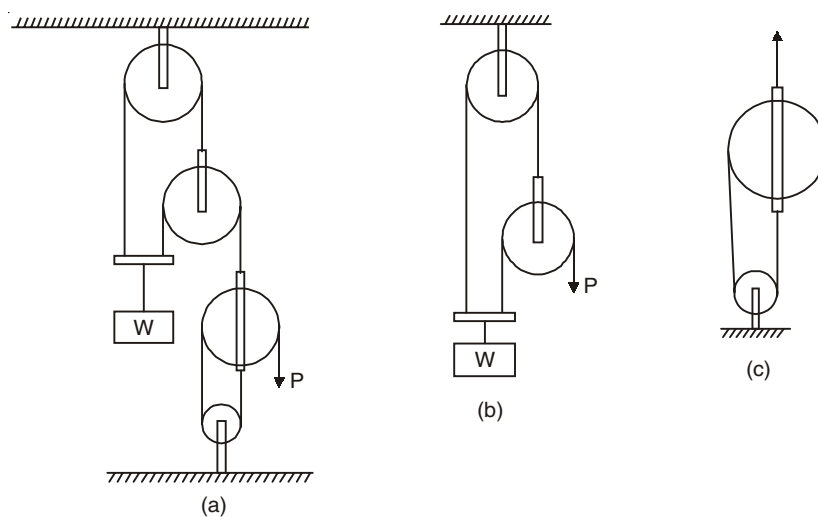
$$\begin{aligned}
 \therefore \text{ Effort lost in friction} &= P - P_i \\
 &= 180 - 142.86 \\
 &= 37.14 \text{ N}
 \end{aligned}$$

**Example 6.10.** What force  $P$  is required to raise a load of 2500 N in the system of pulleys shown in Fig. 6.18(a). Assume efficiency of the system to be equal to 70%.

**Solution:** Figure 6.18(a) can be split into two simple systems as shown in Figs. 6.18(b) and 6.18(c).

What is shown in Fig. 6.18(b) is a third order pulley system having two pulleys.

$$\begin{aligned}
 \therefore \quad VR &= 2^n - 1 \\
 &= 2^2 - 1 = 3
 \end{aligned}$$



**Fig. 6.18**

Figure 6.18(c) is also a third order system, having two pulleys.

$$\therefore VR = 2^2 - 1 = 3$$

$$\therefore VR \text{ of the whole system} = 3 + 3 = 6$$

$$\text{Now, } MA = \eta \times VR$$

$$\text{i.e., } \frac{W}{P} = \eta \times VR$$

$$\therefore P = \frac{W}{\eta \times VR}$$

$$= \frac{2500}{0.7 \times 6}$$

$$\text{i.e., } P = 595.24 \text{ N}$$

**Example 6.11.** In a wheel and axle, diameter of the wheel is 500 mm and that of the axle is 200 mm. The thickness of the cord on the wheel is 6 mm and that of the axle is 20 mm. Find the velocity ratio of the machine. If the efficiency when lifting a load of 1200 N with a velocity of 10 metres per minute is 70%, find the effort necessary.

$$\begin{aligned} \text{Solution: Effective wheel diameter} &= \frac{6}{2} + 500 + \frac{6}{2} \\ &= 506 \text{ mm} \end{aligned}$$

$$\begin{aligned} \text{Effective axle diameter} &= \frac{20}{2} + 200 + \frac{20}{2} \\ &= 220 \text{ mm.} \end{aligned}$$

For a wheel and axle, the velocity ratio is given by  $\frac{D}{d}$

$$\therefore VR = \frac{506}{220} = 2.30$$

$$\begin{aligned} \text{Mechanical advantage} &= \text{Efficiency} \times \text{velocity ratio} \\ &= 0.7 \times 2.30 = 1.61 \end{aligned}$$

$$MA = \frac{W}{P}$$

$$\therefore P = \frac{1200}{1.61}$$

$$\text{i.e., } P = 745.34 \text{ N}$$

**Example 6.12.** A load of 20 kN is to be lifted by a differential wheel and axle. It consists of differential axle of 250 mm and 300 mm diameter and the wheel diameter is 800 mm. Find the effort required if the efficiency of the machine is 55%.

**Solution:** Differential axle diameters,

$$d_1 = 250 \text{ mm and}$$

$$d_2 = 300 \text{ mm}$$

Wheel diameter,  $D = 800 \text{ mm}$

Load,  $W = 20 \text{ kN}$

Efficiency,  $\eta = 55\%$

Velocity ratio,  $VR = \frac{2D}{d_2 - d_1} = \frac{2 \times 800}{300 - 250} = 32$

Mechanical advantage

$$\begin{aligned} MA &= \text{Efficiency} \times \text{velocity ratio} \\ &= 0.55 \times 32 = 17.6 \end{aligned}$$

$$MA = \frac{W}{P}$$

$$\therefore P = \frac{20,000}{17.6}$$

*i.e.*,  $P = 1136.4 \text{ N}$

**Example 6.13.** A Weston differential pulley block of diameter 500 mm and 200 mm is used to lift a load of 5000 N. Find the effort required if the efficiency is 60%.

**Solution:** Diameter of pulley block  $D = 500 \text{ mm}$ , and

$$d = 200 \text{ mm}$$

Load,  $W = 5000 \text{ N}$

Efficiency,  $\eta = 60\%$

$$\begin{aligned} \text{Velocity ratio} &= \frac{2D}{D - d} \\ &= \frac{2 \times 500}{500 - 200} = 3.33 \end{aligned}$$

$$\begin{aligned} \text{Mechanical advantage} &= \text{Efficiency} \times \text{Velocity ratio} \\ &= 0.6 \times 3.33 = 2 \end{aligned}$$

$$\text{Effort required, } P = \frac{W}{MA} = \frac{5000}{2}$$

*i.e.*,  $P = 2500 \text{ N}$

### 6.13 SCREW JACK

This is a device commonly used to lift heavy loads. Screw jack works on the principle same as that of inclined plane. A typical section of the screw jack is shown in the Fig. 6.19.



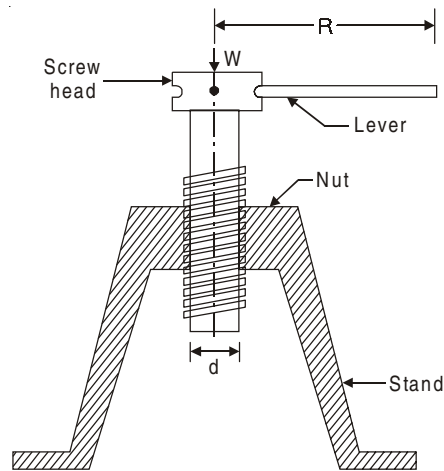


Fig. 6.19

The device consists of a nut and a screw. Monolithically cast nut and stand form the body of the jack. The load is carried by the screw head fitted onto the screw as shown in the figure. The body (consisting of nut) is fixed and the screw is rotated by means of a lever.

The axial distance moved by the nut (or by the screw, relative to each other) when it makes one complete revolution is known as **lead of the screw head**. The distance between consecutive threads is called *pitch* (of a screw thread). If the screw is single threaded, then lead of the screw is equal to the pitch. If the screw is double threaded then lead of the screw is twice the pitch.

Let  $R$  be the length of the lever and  $d$  be the mean diameter of the screw.

Let a load  $W$  be lifted using an effort  $P$ .

If an effort  $P$  is applied at the lever end, it is equivalent to an effort  $P_1$  at the screw [Fig. 6.20(a)] and  $P_1$  is given by the condition:

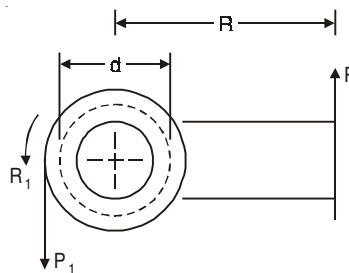
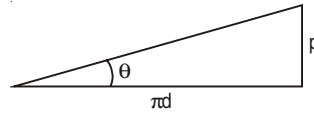


Fig. 6.20(a)

$$P \times R = P_1 \times \frac{d}{2}$$

$$\therefore P_1 = \frac{2PR}{d}$$

Now, consider one complete revolution of the lever. The load  $W$  is lifted up by a distance  $p$  equal to the lead of the screw.

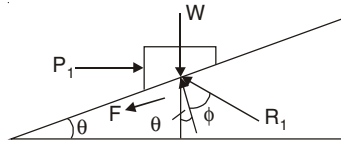
**Fig. 6.20(b)**

This can be compared with that of inclined plane having inclination  $= \tan^{-1} \frac{p}{\pi d}$

where,  $p$  – lead of the screw

$d$  – mean diameter of screw.

Applying an effort  $P$  at the end of the lever is as good as applying an effort  $P_1$  (at the screw) on this inclined plane. [Fig. 6.20(c)].

**Fig. 6.20(c)**

Resolving horizontally *i.e.*, parallel to  $P_1$

$$P_1 = R_1 \sin (\theta + \phi), \text{ where} \quad \dots(1)$$

$R_1$  is resultant reaction and

$\phi$  is limiting angle of friction. Resolving vertically

$$W = R_1 \cos (\theta + \phi) \quad \dots(2)$$

Dividing eqn. (1) by (2)

$$\frac{P_1}{W} = \tan (\theta + \phi)$$

$$\therefore P_1 = W \tan (\theta + \phi)$$

But,

$$P_1 = \frac{2PR}{d}$$

$$\frac{2PR}{d} = W \tan (\theta + \phi)$$

$$\therefore P = \frac{d}{2R} W \tan (\theta + \phi) \quad \dots(6.20)$$

We have  $\tan \phi = \mu$

where  $\mu$  is the coefficient of friction.

Then,

$$P = \frac{d}{2R} W \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$$

$$P = \frac{d}{2R} W \frac{\mu + \tan \theta}{1 - \mu \tan \theta} \quad \dots(6.20(a))$$

where,  $\tan \theta = \frac{p}{\pi d}$

If the load is descending, then the friction will be acting in the reverse direction so that the resultant reaction  $R$  shifts as shown in Fig. 6.20(d).

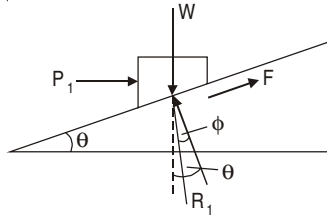


Fig. 6.20(d)

Then eqn. 6.20 changes to

$$P = \frac{d}{2R} W \tan (\theta - \phi) \quad \dots(6.20(b))$$

Torque required,  $= T = PR$

$$= \frac{d}{2} W \tan (\theta + \phi)$$

Hence torque required while ascending

$$T = \frac{d}{2} W \tan (\theta + \phi) \quad \dots(6.21)$$

and torque required while descending

$$T = \frac{d}{2} W \tan (\theta - \phi) \quad \dots(6.21(a))$$

Now,

$$\begin{aligned} VR &= \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} \\ &= \frac{2\pi R}{p} \quad \dots(6.22) \end{aligned}$$

**Example 6.14.** A screw jack raises a load of 40 kN. The screw is square threaded having three threads per 20 mm length and 40 mm in diameter. Calculate the force required at the end of a lever 400 mm long measured from the axis of the screw, if the coefficient of friction between screw and nut is 0.12.

**Solution:** Screw diameter,  $d = 40$  mm

$$\text{Lead of the screw, } p = \frac{20}{3} = 6.667 \text{ mm}$$

$$\text{Load, } W = 40 \text{ kN}$$

$$\text{Lever length, } R = 400 \text{ mm}$$

$$\mu = 0.12$$

We have

$$P = \frac{d}{2R} W \frac{\mu + \tan \theta}{1 - \mu \tan \theta}$$

$$\begin{aligned} \text{and } \tan \theta &= \frac{p}{\pi D} \\ &= \frac{6.667}{\pi \times 40} = 0.05305 \end{aligned}$$

$$\therefore P = \frac{40}{2 \times 400} \times 40,000 \left[ \frac{0.12 + 0.05305}{1 - (0.12 \times 0.05305)} \right]$$

$$\text{i.e., } \quad \mathbf{P = 348.32 \text{ kN}}$$

**Example 6.15.** A screw jack has square threads 50 mm mean diameter and 10 mm pitch. The load on the jack revolves with the screw. The coefficient of friction at the screw thread is 0.05.

(1) Find the tangential force required at the end of 300 mm lever to lift a load of 6000 N.

(2) State whether the jack is self-locking. If not, find the torque which must be applied to keep the load from descending.

**Solution:**  $\tan \theta = \frac{p}{\pi d} = \frac{10}{\pi \times 50} = 0.0637$

$$\therefore \theta = 3.6426^\circ$$

$$\tan \phi = 0.05$$

$$\therefore \phi = 2.8624^\circ$$

$$\begin{aligned} (1) \quad P &= \frac{d}{2R} \times W \tan (\theta + \phi) \\ &= \frac{50}{2 \times 300} \times 6000 \tan (3.6426^\circ + 2.8624^\circ) \end{aligned}$$

$$\mathbf{P = 57.01 \text{ N}}$$

(2) We have

$$\begin{aligned} VR &= \frac{2\pi R}{p} = \frac{2\pi \times 300}{10} \\ &= 188.496 \end{aligned}$$

$$MA = \frac{W}{P} = \frac{6000}{57.01}$$

$$= 105.245$$

$$\begin{aligned} \text{Efficiency} &= \frac{MA}{VR} = \frac{105.245}{188.496} \\ &= 0.5583 \end{aligned}$$

i.e., Efficiency = 55.83% > 50.

**Hence the screw jack is not self-locking.**

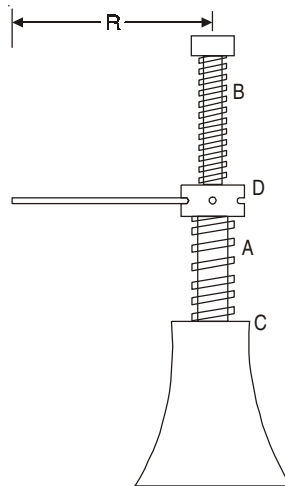
∴ The torque required to keep the load from descending

$$\begin{aligned}
 &= \frac{d}{2} W \tan (\theta - \phi) \\
 &= \frac{50}{2} \times 600 \times \tan (3.6426^\circ - 2.8624^\circ)
 \end{aligned}$$

$$T = 204.3 \text{ N-mm}$$

### 6.14 DIFFERENTIAL SCREW JACK

Differential screw jack is an improvement over simple screw jack. A typical differential screw jack is shown in Fig. 6.21. It consists of two threaded elements *A* and *B*. Both *A* and *B* have threads in the same direction (right-handed). The element *A* is a cylinder which has threads on both its outer and inner surfaces. The threads on the outer surface of the element *A* fits into the nut *C* which also functions as the base of the whole mechanism. The threads on the element *B* fit into the threads cut on the inner surface of *A*. Thus, the element *A* acts as a screw for the nut *C* and also as a nut for the element *B*. With the help of a lever inserted in the holes made on the top of the block *D*, which is attached to the element *B*, block *D* can be rotated. When *D* is rotated, *A* rotates with it. Rotation of *B* is prevented by suitable arrangement.



**Fig. 6.21**

Let  $D$  and  $d$  be the mean diameters of the screws *A* and *B*, respectively.

Let  $p_A$  and  $p_B$  be the pitch of the screws *A* and *B*, respectively and  $p_A$  be greater than  $p_B$ .

If the lever is rotated through one complete revolution, the height through which the element *A* moves up =  $p_A$

In the mean time, the element *B* moves down with respect to *C*.

The distance through which *B* comes down =  $p_B$

∴ Net height through which load is lifted =  $(p_A - p_B)$

Let  $R$  be the radial distance (from the centre line of  $A$  and  $B$ ) at which an effort  $P$  is applied

$$\text{Now, } VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{2\pi R}{p_A - p_B} \quad \dots(6.23)$$

It can be seen from eqn. (6.23) that the velocity ratio in the differential screw jack is increased as compared to that of simple screw jack (eqn. 6.22).

**Example 6.16.** The following are the specifications for a differential screw jack:

- (1) Pitch of smaller screw, 5.0 mm
- (2) Pitch of larger screw, 10.0 mm
- (3) Lever arm length from centre of screw = 500 mm.

The screw jack raises a load of 15 kN with an effort of 185 N. Determine the efficiency of the differential screw jack at this load.

If the above jack can raise a load of 40 kN with an effort of 585 N, determine the law of machine.

**Solution:** Now,  $p_A = 10.0$  mm

$$p_B = 5.0 \text{ mm}$$

Lever arm length,  $R = 500$  mm

$$VR = \frac{2\pi R}{p_A - p_B} = \frac{2\pi \times 500}{10 - 5.0} = 628.32$$

$$MA = \frac{W}{P} = \frac{15,000}{185} = 81.08$$

$$\begin{aligned} \therefore \eta &= \frac{MA}{VR} = \frac{81.08}{628.32} \\ &= 0.129 = \mathbf{12.9 \text{ per cent.}} \end{aligned}$$

To find law of machine:

Let law of machine be  $P = mW + C$

$$\text{From first case: } 185 = m \times 15,000 + C \quad \dots(1)$$

$$\text{From second case: } 585 = m \times 50,000 + C \quad \dots(2)$$

$$(2) \text{ minus } (1) \text{ gives, } 400 = 35,000 m.$$

$$\text{or } m = \frac{4}{350}$$

Substituting this value in eqn. (1), we get

$$185 = 171.43 + C$$

$$\therefore C = 13.57 \text{ N}$$

$$\therefore \text{ Law of machine is } P = \frac{4}{350} W + 13.57$$

## 6.15 WINCH CRABS

Winch crabs are lifting machines in which velocity ratio is increased by a gear system. If only one set of gears is used, the winch crab is called a **single purchase winch crab** and if two sets are used it is called **double purchase winch crab**.

### Single Purchase Winch Crab

Line diagram of a single purchase winch crab is shown in Fig. 6.22. It consists of a load drum of radius  $r$  connected to an axle by gears. The toothed wheel on load drum is called **spur wheel** and the toothed wheel on axle is called **pinion**. Pinion is always smaller in size and it contains less number of teeth as compared to that on the spur wheel. The axle is provided with a handle of arm length  $R$ . Let the number of teeth on pinion and spur wheel be  $T_1$  and  $T_2$ , respectively. Let the effort be applied at the end of the handle. When one revolution is made, the distance moved by the effort is given by:

$$D = 2\pi R$$

When axle makes one revolution, due to gear arrangement load drum moves by  $T_1$  number of teeth, which means that it makes a revolution of  $\frac{T_1}{T_2}$ .

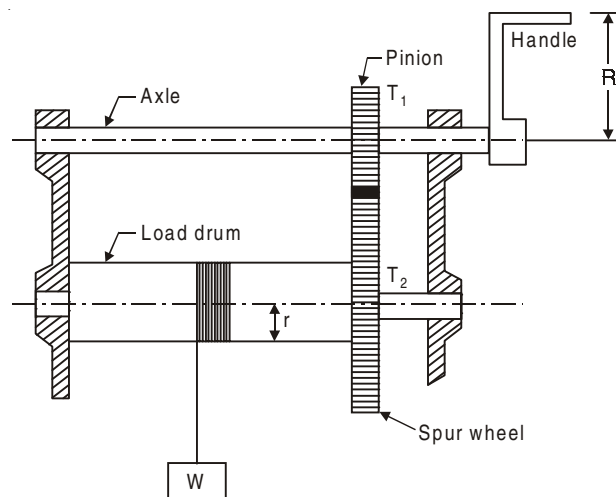
$\therefore$  The distance over which the load moves:

$$d = 2\pi r \times \frac{T_1}{T_2}$$

$\therefore$  Velocity ratio,

$$VR = \frac{D}{d} = \frac{2\pi R}{2\pi r \times \frac{T_1}{T_2}}$$

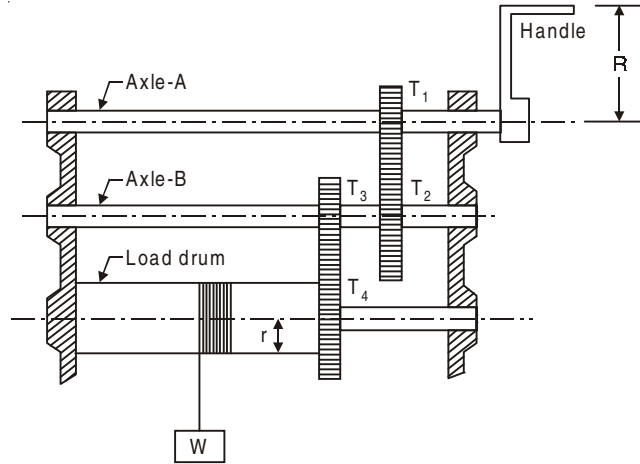
$$i.e., \quad VR = \frac{R}{r} \times \frac{T_2}{T_1} \quad \dots(6.24)$$



**Fig. 6.22**

### Double Purchase Winch Crab

Velocity ratio of a winch crab can be increased by providing another axle with a pair of pinion and gear as shown in Fig. 6.23. Since two pairs of pinion and gear are used it is called a double purchase winch crab. This is used for lifting heavier loads.

**Fig. 6.23**

Let the number of teeth on various wheels be  $T_1$ ,  $T_2$ ,  $T_3$  and  $T_4$  as shown in Fig. 6.23. Let the handle makes one revolution.

Distance moved by effort  $P$  is given by:

$$D = 2\pi R \quad \dots(6.25)$$

When axle  $A$  makes one revolution, axle  $B$  is moved by  $T_1$  teeth, *i.e.*, it makes  $\frac{T_1}{T_2}$  revolutions.

The number of teeth by which spur wheel is moved is  $\frac{T_1}{T_2} \times T_3$  and hence load drum makes

$$\left(\frac{T_1}{T_2}\right) \times \left(\frac{T_3}{T_4}\right) \text{ revolutions.}$$

$\therefore$  The distance moved by the load

$$d = 2\pi r \times \left(\frac{T_1}{T_2}\right) \times \left(\frac{T_3}{T_4}\right)$$

$$VR = \frac{D}{d} = \frac{2\pi R}{2\pi r \times \left(\frac{T_1}{T_2}\right) \times \left(\frac{T_3}{T_4}\right)}$$

$$\text{i.e.,} \quad VR = \frac{R}{r} \times \left(\frac{T_2}{T_1}\right) \times \left(\frac{T_4}{T_3}\right) \quad \dots(6.26)$$

**Example 6.17.** Following are the specifications of a single purchase crab:

Diameter of the load drum = 200 mm

Length of lever arm  $R$  = 1.2 m



Number of teeth on pinion,  $T_1 = 10$

Number of teeth on spur wheel,  $T_2 = 100$ .

Find the velocity ratio of this machine. On this machine efforts of 100 N and 160 N are required to lift the load of 3 kN and 9 kN, respectively. Find the law of the machine and the efficiencies at the above loads.

**Solution:** Radius of the load drum,  $r = \frac{200}{2} = 100$  mm

Length of lever arm,  $R = 1.2$  m = 1200 mm

Velocity ratio of the single purchase crab is given by:

$$\begin{aligned} VR &= \frac{R}{r} \times \frac{T_2}{T_1} \\ &= \frac{1200}{100} \times \frac{100}{10} \end{aligned}$$

$$\mathbf{VR = 120}$$

Let the law of machine be  $P = mW + C$

In first case:  $P = 100$  N;  $W = 3$  kN = 3000 N

$$\therefore 100 = m \times 3000 + C \quad \dots(1)$$

In the second case:  $P = 160$  N; and  $W = 9$  kN = 9000 N

$$\therefore 160 = m \times 9000 + C \quad \dots(2)$$

Subtracting eqn. (1) from eqn. (2), we get

$$60 = 6000 m$$

$$\therefore m = \frac{1}{100} = 0.01$$

Substituting this value of  $m$  in eqn. (1), we get

$$100 = \frac{1}{100} \times 3000 + C$$

$$\therefore C = 70$$

Hence, the law of machine is

$$\mathbf{P = 0.01 W + 70}$$

*Efficiencies:*

In the first case,

$$MA = \frac{W}{P} = \frac{3000}{100} = 30$$

$$\therefore \eta = \frac{MA}{VR} = \frac{30}{120} = 0.25 = \mathbf{25\%}$$

In the second case,

$$MA = \frac{W}{P} = \frac{9000}{160} = 56.25$$

$$\therefore \eta = \frac{MA}{VR} = \frac{56.25}{120}$$

$$\text{i.e., } \eta = 0.4688 = \mathbf{46.88\%}$$

**Example 6.18.** In a double purchase crab, the pinions have 15 and 20 teeth, while the spur wheels have 45 and 40 teeth. The effort handle is 400 mm while the effective diameter of the drum is 150 mm. If the efficiency of the winch is 40%, what load will be lifted by an effort of 250 N applied at the end of the handle?

**Solution:**  $T_1 = 15$ ;  $T_2 = 45$ ;  $T_3 = 20$ ;  $T_4 = 40$

Length of handle,  $R = 400$  mm

$$\text{Radius of the load drum, } r = \frac{150}{2} = 75 \text{ mm}$$

$$\therefore VR = \frac{R}{r} \times \frac{T_2}{T_1} \times \frac{T_4}{T_3}$$

$$= \frac{400}{75} \times \frac{45}{15} \times \frac{40}{20}$$

$$= 32$$

$$\text{Now, } \eta = \frac{MA}{VR}$$

$$\therefore 0.40 = \frac{MA}{32}$$

$$\text{or } MA = 12.8$$

$$\text{i.e., } \frac{W}{P} = 12.8, \text{ but } P = 250 \text{ N}$$

$$\therefore W = 12.8 \times 250$$

$$\text{i.e., } W = 3200 \text{ N}$$

**Applied effort lifts a load of 3200 N**

<b>IMPORTANT FORMULAE</b>
---------------------------

1. Mechanical advantage =  $\frac{\text{Load}}{\text{Effort}}$ .
2. Velocity ratio =  $\frac{\text{Distance moved by effort}}{\text{Distance moved by load}}$ .
3. Input = Effort  $\times$  Distance moved by effort.
4. Output = Load  $\times$  Distance moved by the load.
5. Efficiency =  $\frac{\text{Output}}{\text{Input}} = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}}$ .
6. Law of machine:  
 $P = mW + C$ .
7. A machine is reversible, if  $\eta > 50\%$ .

8. In case of first order pulleys,  
 $VR = 2^n$ , where  $n$  is the number of movable pulleys.
9. In case of second order pulleys,  
 $VR = 2n$ , is the where  $n$  is number of pulleys.
10. In case of third order pulleys  
 $VR = 2^n - 1$ , where  $n$  is number of pulleys.
11. In wheel and axle,

$$VR = \frac{\text{Diameter of wheel}}{\text{Diameter of axle}}.$$

12. In case of wheel and differential axle,

$$VR = \frac{2D}{d_2 - d_1}.$$

13. In Weston differential pulley block,

$$VR = \frac{2D}{D - d}.$$

14. In inclined plane

$$VR = \frac{1}{\sin \theta}$$

15. In screw jack

$$VR = \frac{2\pi R}{p}.$$

16. In double purchase winch crab

$$VR = \frac{R}{r} \times \frac{T_2}{T_1} \times \frac{T_4}{T_3}.$$

17. In single purchase winch crab

$$VR = \frac{R}{r} \times \frac{T_2}{T_1}.$$

### THEORY QUESTIONS

1. Define the following terms:
  - (i) Mechanical advantage
  - (ii) Velocity ratio
  - (iii) Efficiency of a lifting machine.
2. Explain the law of machine. How it is determined ?
3. What is meant by reversibility of machine ? Prove that a machine is reversible if its efficiency is more than 50%.

4. With neat sketches explain the different types of pulley systems and derive the expressions for their efficiency.
5. With neat sketches explain the following simple machines:
  - (i) Wheel and axle
  - (ii) Wheel and differential axle
  - (iii) Weston differential pulley block.

6. Show that in a screw jack effort  $P$  required to lift a load  $W$  is given by  $P = \frac{d}{2R} W \frac{\mu + \tan \theta}{1 - \mu \tan \theta}$

where  $\phi$  is angle of friction and  $\tan \theta = \frac{p}{\pi d}$ ,  $p$  being the pitch of screw.

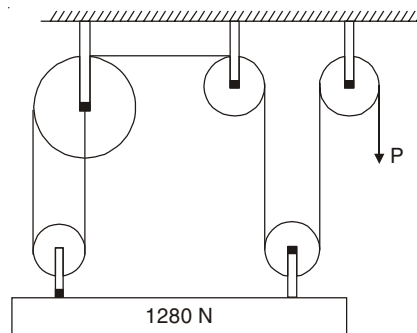
7. Give a neat sketch of
  - (i) Single purchase winch crab
  - (ii) Double purchase winch crab

#### PROBLEMS FOR EXERCISE

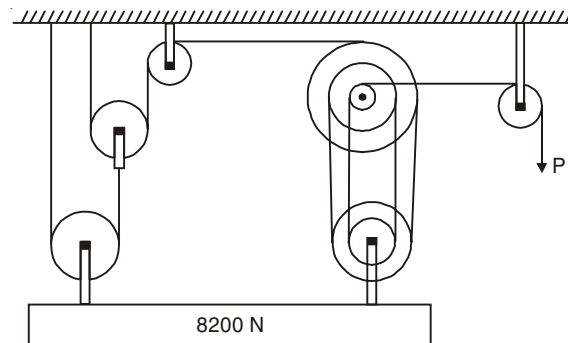
1. In a lifting machine an effort of 1.5 kN is to be moved through a distance of 7.2 m to move a load of 24 kN through a distance of 300 mm. Determine: (1) mechanical advantage, (2) velocity ratio, (3) efficiency, (4) ideal effort, (5) effort lost in friction, (6) ideal load and (7) frictional resistance.  
[Ans. (1) 16; (2) 24; (3) 66.67%; (4) 1.0 kN; (5) 0.5 kN; (6) 36 kN and (7) 12 kN]
2. In a lifting machine an effort of 400 N is required to raise a load of 3000 N and an effort of 640 N, to raise a load of 5000 N. How much load can be lifted with an effort of 760 N? If the velocity ratio is 16, determine the efficiency of the machine when an effort of 760 N is applied. Is it a reversible machine?  
[Ans.  $W = 6000$  N;  $\eta = 49.34$ ; It is not reversible]
3. The following observations were made in an experiment on a lifting machine:
 

Load in N	500	1000	1500	2000	2500	3000
Effort in N	26	47	76	95	105	127

 Draw the load *versus* effort graph and determine the law of machine. If the velocity ratio is 30 determine the efficiency while lifting a load of 1800 N.  
What is the maximum efficiency of the machine ?  
[Ans.  $P = 0.04W + 6$ ;  $\eta = 51.28\%$ ; maximum  $\eta = 83.33\%$ ]
4. A lifting machine having velocity ratio 28 starts raising a load of 6420 N on applying an effort of 450 N to it. If suddenly the effort is removed find whether the load comes down or not?  
[Ans.  $\eta = 50.95\%$ , hence the load comes down]
5. In the first order pulley system having three movable pulleys, how much effort is required to raise a load of 5780 N if the same system raises a load of 1200 N with an effort of 200 N? Assume the efficiency to be constant for the pulley system.  
[Ans.  $P = 963.33$  N]
6. For the arrangement of pulleys shown in Fig. 6.24, find the effort required to raise the given load 7280 N. Assume efficiency of the system as 75%.  
[Ans.  $P = 2436.7$  N]

**Fig. 6.24**

7. For the combination of first order and second order pulley system shown in Fig. 6.25 what will be velocity ratio?

**Fig. 6.25**

Assuming efficiency to be 80%, calculate what effort is required to raise the load of 8200 N.

[Ans.  $VR = 8$ ;  $P = 1281.25 \text{ N}$ ]

8. For a third order pulley system having six movable pulleys, an effort of 720 N is required to raise a load of 30 kN. Calculate the efficiency of the system. [Ans. 66.14%]
9. For a wheel and axle, the following details are available:  
 Diameter of the wheel = 540 mm  
 Diameter of the axle = 270 mm  
 Thickness of the cord on the wheel = 6 mm  
 Thickness of the cord on the axle = 20 mm  
 Calculate the efficiency of the device if an effort of 725 N is required to lift a load of 1000 N.  
 [Ans.  $\eta = 73.26\%$ ]
10. A differential axle and wheel consists of a differential axle of 240 mm and 320 mm diameter; and a wheel of diameter 750 mm. Assuming efficiency to be 57%, find the effort required to raise a load of 24 kN. [Ans.  $P = 2.2456 \text{ kN}$ ]
11. A Weston differential pulley block of diameters 400 mm and 800 mm is used to lift a load of 40,000 N. Find the effort required if efficiency of the system is 60%. [Ans. 26,667 N]

12. The following particulars refer to a screw jack:

- (1) Diameter of the screw rod = 62.5 mm
- (2) Length of the handle = 250 mm
- (3) Pitch of the square threads = 12.5 mm
- (4) Coefficient of friction = 0.05

Find the effort required to lift a load of 5000 N. Find also the efficiency of the jack.

[Ans.  $P = 71.27$  N;  $\eta = 55.83\%$ ]

13. A screw jack carries a load of 10 kN. It has a square threaded screw of pitch 25 mm and mean diameter 60 mm. The coefficient of friction between screw and nut is 0.20. Calculate the torque required to raise the load and the efficiency of the screw. Find also the force required at the end of the handle 500 mm long to lower the load. [Ans.  $T = 102.5802$  kN-mm;  $P = 0.2052$  kN]

14. The following are the specifications of a single purchase crab:

Diameter of the load drum = 220 mm

Length of lever arm,  $R = 1.0$  m

Number of teeth on pinion,  $T_1 = 10$

Number of teeth on spur wheel,  $T_2 = 120$

Determine the velocity ratio of the machine. On this machine, it is found in one trial that an effort of 120 N is required to lift a load 4.2 kN and in another trial an effort of 150 N is required to lift a load of 11.4 kN. Establish the law of machine. Also calculate efficiencies corresponding to 120 N and 150 N efforts.

( Ans.  $VR = 109.091$ ;  $P = \frac{1}{240}W + 102.5$ ;  $\eta_1 = 32.08\%$ ;  $\eta_2 = 69.67\%$  )

15. In a double purchase crab, the pinions have 12 and 18 teeth while the spur wheels have 24 and 36 teeth. The effective diameter of the drum is 142 mm. The effort handle is 450 mm. What load can be lifted by an effort of 220 N applied at the end of handle if efficiency of winch is 42%.

[Ans.  $P = 2342.54$  N].

# Physical and Mechanical Properties of Structural Materials

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The principle materials used in building structures and machines include metals and their alloys, concrete, wood, stones, bricks and other clay products, bituminous materials and plastics. The products finally produced with these materials should be strong enough to resist the forces coming on it without undergoing deformation so as to put it out of use. The product should be durable. The required properties for different usage is different. Hence an engineer must know the required properties for his products. Various properties of materials may be grouped into:

- (a) Physical Properties
- (b) Mechanical Properties.

## 7.1 PHYSICAL PROPERTIES

The various physical properties of the materials are briefly explained below:

1. **Density:** It is defined as mass of a material per unit volume. Kilogram is the unit of mass in SI system. It is found by weighing the material without voids.
2. **Bulk density or unit weight:** It is defined as weight of material per unit volume. Its unit in SI is  $\text{kN/m}^3$ . Volume of the material is found from the natural state of material *i.e.* from volume of voids and solids together.
3. **Specific gravity:** It is the ratio of density of materials to the unit weight of water. It may be noted that for water density and unit weight are one and the same. Specific gravity is a dimensionless quantity.
4. **Porosity:** It is the ratio of the volume of voids to the volume of granular materials.
5. **Hygroscopy:** It is the property of the material to absorb water from the atmosphere. Timber possesses this property.
6. **Water absorption:** It is the property of the material to absorb water and retain it. It is expressed as percentage in weight of absorbed water to weight of dry material. Concrete and bricks possess this property to significant extent.
7. **Permeability:** It is the property of material by virtue of which it allow water to flow through it. Soil possesses this property and materials like bitumen and metals are impervious.
8. **Weather resistance:** It is the ability to sustain alternating changes in wet and dry conditions of weather. We look for materials with good weather resistance without undergoing considerable changes in dimensions and losing mechanical strength.
9. **Frost resistance:** It is the property to resist repeated freezing and thawing of absorbed water without losing mechanical strength.

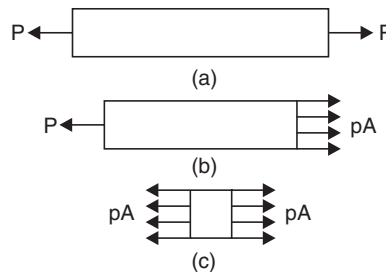
- 10. Thermal conductivity:** It is the ability of the material to transfer heat. Metals have higher thermal conductivity. Moist materials have higher heat conductivity compared to dry porous materials. Rubber is having very low thermal conductivity.
- 11. Thermal resistivity:** It is the reciprocal of thermal conductivity and is defined as time taken for the flow of unit heat.
- 12. Fire resistance:** It is the property by virtue of which a material resists the action of high temperature without undergoing substantial changes in shape and loss of strength. Steel has poor fire resistance. Concrete is better than steel in resisting fire. Bricks are having very good fire resistance.
- 13. Sound absorption:** It is the property of reducing the reflection of sound waves. Porous materials have better sound absorption property. In auditoriums and cinema halls we have to look for materials with good sound absorption properties.
- 14. Chemical resistance:** It is the ability of the material to withstand the action of acids, alkalies, gases etc. Materials are subjected to the action of these chemicals, which are present in air or water or land. Stone and wood have poor resistance to chemical actions while bricks have good chemical resistance.
- 15. Corrosion:** It is the destruction of the material due to slow oxidation. Steel is susceptible to corrosion. The corrosion rate is high in marine environment.
- 16. Soundness:** A material is said to be sound, if it has good resistance to heat, alternate freezing and thawing and the other destructive actions of the atmosphere.
- 17. Durability:** It is the ability to resist the combined effects of atmosphere, rain and other effects and maintain the original strength characters for a long period.

## 7.2 MECHANICAL PROPERTIES

Resistance to various applied forces is referred as mechanical properties. Some of these properties are briefly presented below:

**1. Properties related to axial loading:** For designing structural elements to resist the applied loads, the following strength properties should be known:

- (a) **Tensile strength:** If an element is subjected to pulling forces, the resistance developed per unit area is termed as tensile stress and the maximum tensile stress the material can resist is termed as tensile strength. To find the resisting forces developed a section plane may be taken through the element and the equilibrium of any one part may be considered. Each part is in equilibrium under the action of applied forces and internal resisting forces. Fig. 7.1



**Fig. 7.1**

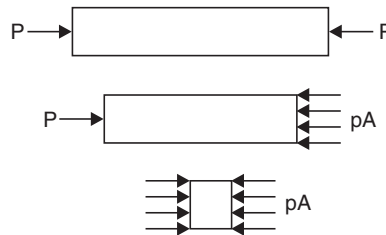


shows a bar under tension and stresses at a cross-section and also an element of bar under tensile stress. In this  $P$  is the tensile force applied,  $p$  is the tensile stress (resistance per unit area) and  $A$  is the cross-sectional area.

$$\therefore P = pA \quad \dots(7.1)$$

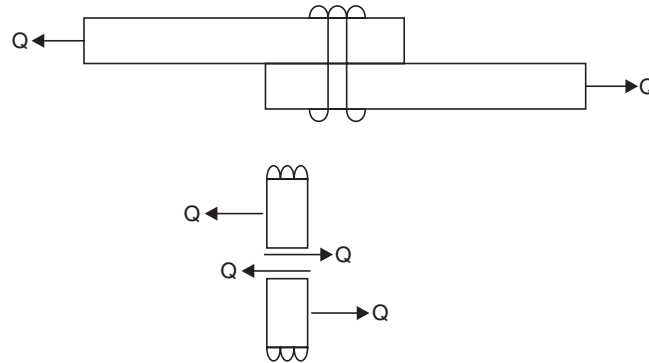
- (b) **Compressive stress:** Instead of pull, if push ' $P$ ' acts on the element, it tries to shorten the bar and the internal resistance developed per unit area is called compressive stress

(Ref. Fig. 7.2). Here also 
$$p = \frac{P}{A}.$$



**Fig. 7.2**

**2. Properties related to shear loading:** If the applied force is trying to shear off a particular section of the element, the resistance developed for unit area in such case along that section is called shearing stress. The maximum stress that material can resist is called shearing strength of the material. Figure 7.3 shows a section of bolt under shearing stress.



**Fig. 7.3**

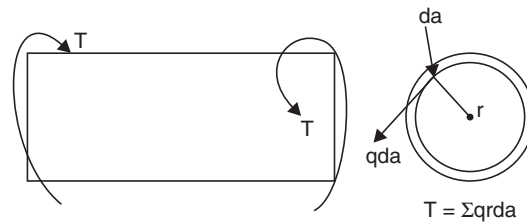
If  $Q$  is the shear force,  $q$  is the shear stress, then the equilibrium condition of any one part considered shows,

$$Q = qA, \text{ where } A \text{ is the cross-sectional area of the bolt.}$$

$$\therefore q = \frac{Q}{A} \quad \dots(7.2)$$

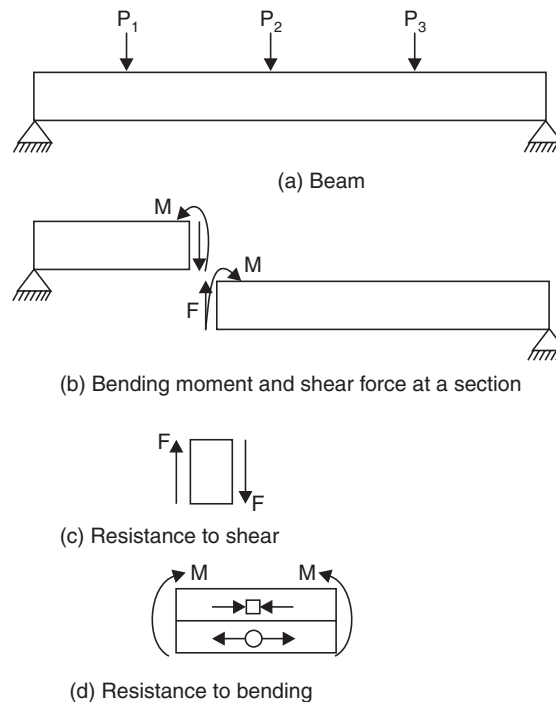
**3. Properties related to torsional moment:** A member is said to be in torsion when it is subjected to a moment about its axis, Fig. 7.5 shows a shaft in torsion. The effect of a torsional member is to twist it and hence a torsional moment is also called as a twisting moment. In engineering

problems many members are subjected to torsion. Shafts transmitting power from engine to rear axle of automobile, from a motor to machine tool and from a turbine to electric motor are the common examples of members in torsion. Ring beams of circular water tanks and beams of grid floors are also the examples of members in torsion. The applied torsion is resisted by the material of the member by developing shear stresses. This shear stress varies linearly from zero at centre of the shaft to the maximum value at the extreme fibre.



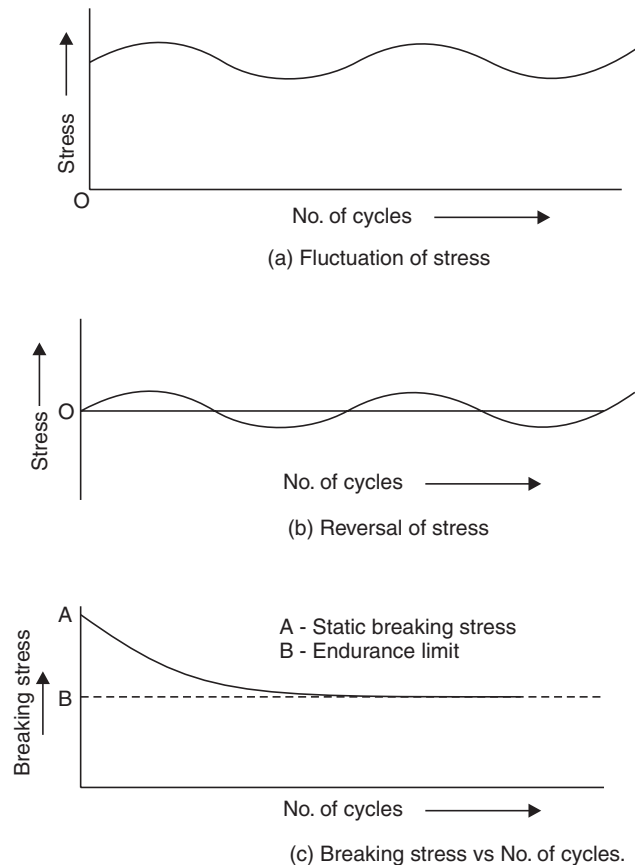
**Fig. 7.4**

**4. Properties related to bending:** When a member is supported at two or more points and subjected to transverse load it bends and develops resistance to the load. The cross-sections of the members are subjected to bending moment and shear force. Finally the load gets transferred to the support by end shear. The shear force introduces shear stresses in the material while bending moment introduces tension in some parts and compression in other parts as shown in Fig. 7.5.



**Fig. 7.5**

**5. Fatigue:** Many structural members are subjected to fluctuation and reversal of stresses as shown in Fig. 7.6 (a) and (b). A bridge deck is a common example of fluctuation of stresses and the shaft is the mechanical component subjected to reversal of stresses. This type of stress when applied repeatedly reduces the strength of material and this effect is called fatigue. This property is shown in Fig. 7.6 (c). The maximum stress at which even a billion reversal of stresses cannot cause failure of the material is called endurance limit.



**Fig. 7.6**

**6. Abrasive resistance:** This is the property by virtue of which a material resists the forces acting at contact surfaces when one material rubs/moves over the other. This is an important property to be studied for the materials used for road surfaces and flooring.

**7. Impact strength:** This property refers to the ability of the material to resist shock loads due to heavy loads falling on the surface.

**8. Hardness:** It is the ability of the material to resist penetration from another material. A number called hardness number is used to measure hardness of various materials. It is based on hardnesses of ten minerals arranged in the increasing hardness. Diamond has hardness number 10. Sometimes hardness is measured by size of indentation of steel balls under standard pressure using hardness testing machine.

**9. Elasticity:** It is the property of the material by virtue of which it regains its original shape and size after the removal of external load. The maximum stress level before which if the load is removed the material regains its shape and size fully is called its elastic limit.

**10. Plasticity:** It is the property of the material to retain its changed shape and size after the loads are removed. It is a required property when a material is to be moulded into different shape.

**11. Creep:** It is the property of the material by virtue of which it undergoes changes in size with time under the action of constant load. Concrete possesses this property.

**12. Toughness:** It is the property of a material whereby it absorbs energy due to straining actions by undergoing plastic deformation.

The other terms proof stress, factor of safety, working stress and load factors are explained in the next chapter after explaining the stress strain curves. There are standard tests specified by codes to quantify various properties of the materials. Knowing the requirements of materials properties for different purposes, the engineer has to select or reject the materials to be used in construction.

### THEORY QUESTIONS

1. Explain the terms:  
Weather resistance, frost resistance, thermal resistance and fire resistance.
2. What do you understand by the terms: sound absorption, soundness, corrosion resistance and durability ?
3. Differentiate between:  
(a) Impact strength and toughness  
(b) Elasticity and plasticity  
(c) Tensile stress and shear stress.

# Simple Stresses and Strains

In this chapter general meaning of stress is explained. Expressions for stresses and strains is derived with the following assumptions:

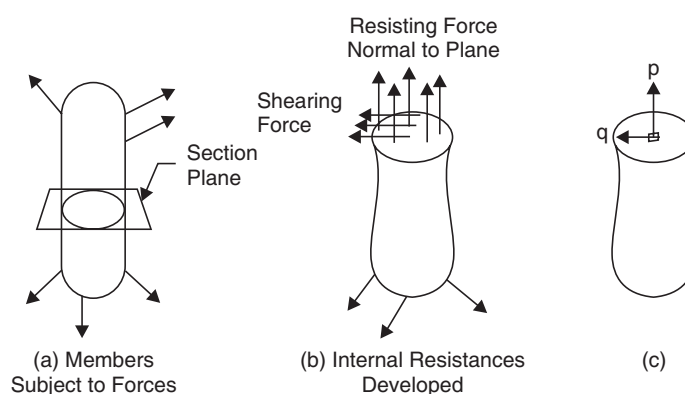
1. For the range of forces applied the material is elastic *i.e.* it can regain its original shape and size, if the applied force is removed.
2. Material is homogeneous *i.e.* every particle of the material possesses identical mechanical properties.
3. Material is isotropic *i.e.* the material possesses identical mechanical property at any point in any direction.

Presenting the typical stress-strain curve for a typical steel, the commonly referred terms like limits of elasticity and proportionality, yield points, ultimate strength and strain hardening are explained.

Linear elastic theory is developed to analyse different types of members subject to axial, shear, thermal and hoop stresses.

## 8.1 MEANING OF STRESS

When a member is subjected to loads it develops resisting forces. To find the resisting forces developed a section plane may be passed through the member and equilibrium of any one part may be considered. Each part is in equilibrium under the action of applied forces and internal resisting forces. The resisting forces may be conveniently split into normal and parallel to the section plane. The resisting force parallel to the plane is called *shearing resistance*. The intensity of resisting force normal to the sectional plane is called *intensity of Normal Stress* (Ref. Fig. 8.1).



**Fig. 8.1**

In practice, intensity of stress is called as “stress” only. Mathematically

$$\begin{aligned}\text{Normal Stress} = p &= \lim_{\Delta A \rightarrow 0} \frac{\Delta R}{\Delta A} \\ &= \frac{dR}{dA}\end{aligned}\quad \dots(8.1)$$

where  $R$  is normal resisting force.

The intensity of resisting force parallel to the sectional plane is called *Shearing Stress* ( $q$ ).

$$\text{Shearing Stress} = q = \lim_{\Delta A \rightarrow 0} \frac{\Delta Q}{\Delta A} = \frac{dQ}{dA}\quad \dots(8.2)$$

where  $Q$  is Shearing Resistance.

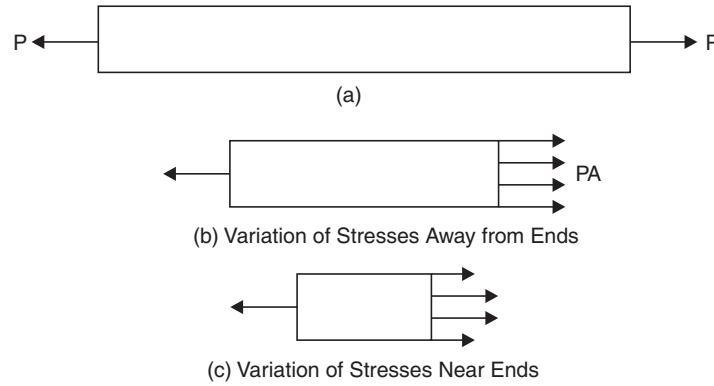
Thus, *stress at any point may be defined as resistance developed per unit area*. From equations (8.1) and (8.2), it follows that

$$dR = p dA$$

$$\text{or} \quad R = \int p dA \quad \dots(8.3a)$$

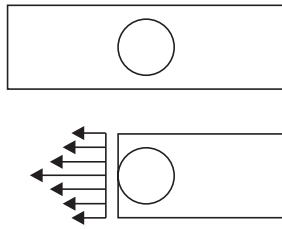
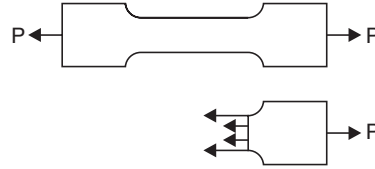
$$\text{and} \quad Q = \int q dA \quad \dots(8.3b)$$

At any cross-section, stress developed may or may not be uniform. In a bar of uniform cross-section subject to axial concentrated loads as shown in Fig. 8.2a, the stress is uniform at a section away from the applied loads (Fig. 8.2b); but there is variation of stress at the section near the applied loads (Fig. 8.2c).



**Fig. 8.2**

Similarly stress near the hole or at fillets will not be uniform as shown in Figs. 8.3 and 8.4. It is very common that at some points in such regions maximum stress will be as high as 2 to 4 times the average stresses.

**Fig. 8.3.** Stresses in a Plate with a Hole**Fig. 8.4**

## 8.2 UNIT OF STRESS

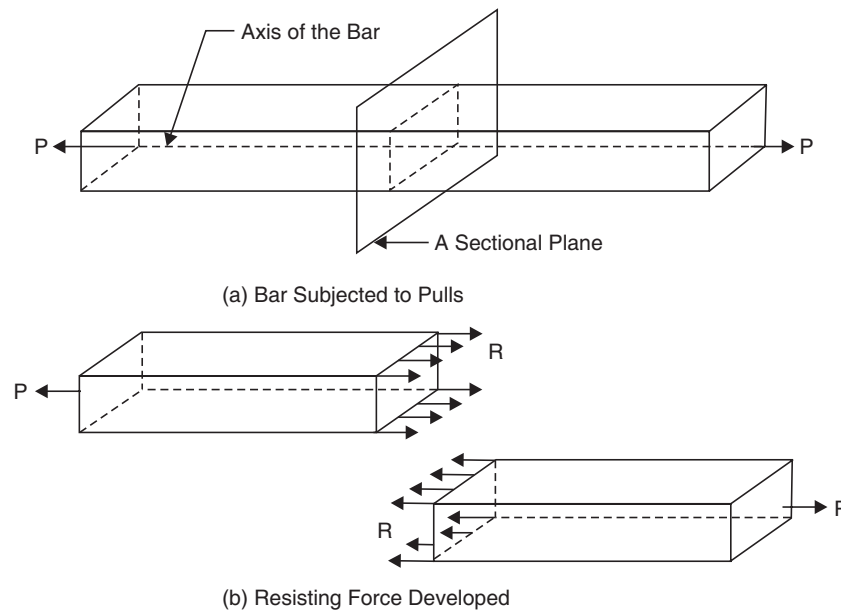
When Newton is taken as unit of force and millimetre as unit of area, unit of stress will be  $\text{N/mm}^2$ . The other derived units used in practice are  $\text{kN/mm}^2$ ,  $\text{N/m}^2$ ,  $\text{kN/m}^2$  or  $\text{MN/m}^2$ . A stress of one  $\text{N/m}^2$  is known as Pascal and is represented by Pa.

Hence,  $1 \text{ MPa} = 1 \text{ MN/m}^2 = 1 \times 10^6 \text{ N/(1000 mm)}^2 = 1 \text{ N/mm}^2$ .

Thus one Mega Pascal is equal to  $1 \text{ N/mm}^2$ . In most of the standard codes published unit of stress has been used as Mega Pascal (MPa or  $\text{N/mm}^2$ ).

## 8.3 AXIAL STRESS

Consider a bar subjected to force  $P$  as shown in Fig. 8.5. To maintain the equilibrium the end forces applied must be the same, say  $P$ .

**Fig. 8.5.** Tensile Stresses

The resisting forces acting on a section are shown in Fig. 8.5b. Now since the stresses are uniform

$$R = \int p dA = p \int dA = pA \quad \dots(8.4)$$

where  $A$  is the cross-sectional area.

Considering the equilibrium of a cut piece of the bar, we get

$$P = R \quad \dots(8.5)$$

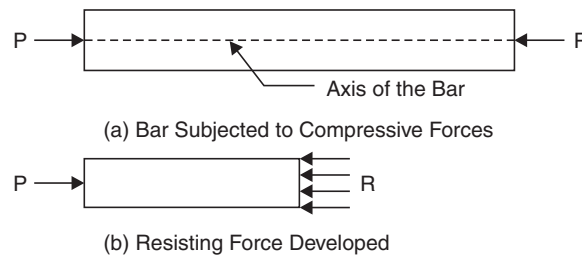
From equations (8.4) and (8.5), we get

$$P = pA$$

$$\text{or} \quad p = \frac{P}{A} \quad \dots(8.6)$$

Thus, in case of axial load 'P' the stress developed is equal to the load per unit area. Under this type of normal stresses the bar is being extended. Such stress which is causing extension of the bar is called tensile stress.

A bar subjected to two equal forces pushing the bar is shown in Fig. 8.6. It causes shortening of the bar. Such forces which are causing shortening, are known as compressive forces and corresponding stresses as compressive stresses.



**Fig. 8.6.** Compressive Stresses

Now  $R = \int p dA = p \int dA$  (as stress is assumed uniform)

For equilibrium of the piece of the bar

$$P = R = pA$$

$$\text{or} \quad p = \frac{P}{A} \text{ as in equation 8.6}$$

Thus, whether it is tensile or compressive, the stress developed in a bar subjected to axial forces, is equal to load per unit area.

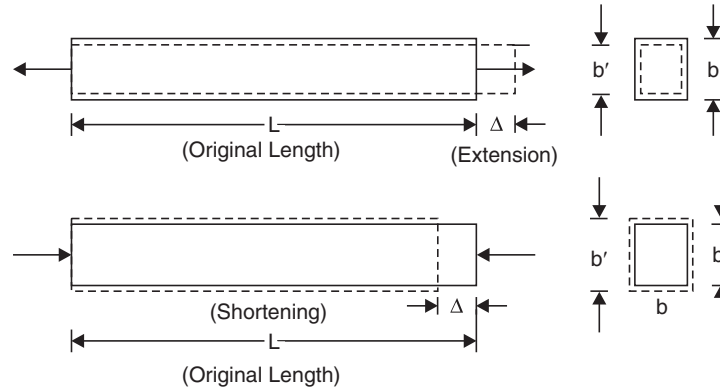
## 8.4 STRAIN

No material is perfectly rigid. Under the action of forces a rubber undergoes changes in shape and size. This phenomenon is very well known to all since in case of rubber, even for small forces deformations are quite large. Actually all materials including steel, cast iron, brass, concrete, etc. undergo similar deformation when loaded. But the deformations are very small and hence we cannot see them with naked eye. There are instruments like extensometer, electric strain gauges which can measure extension of magnitude 1/100th, 1/1000th of a millimetre. There are machines like universal testing machines in which bars of different materials can be subjected to accurately known forces of magnitude as high as 1000 kN. The studies have shown that the bars extend under tensile force and shorten under compressive forces as shown in Fig. 8.7. The change in length per unit length is known as linear strain. Thus,

$$\text{Linear Strain} = \frac{\text{Change in Length}}{\text{Original Length}}$$



$$e = \frac{\Delta}{L} \quad \dots(8.7)$$



**Fig. 8.7**

When changes in longitudinal direction is taking place changes in lateral direction also take place. The nature of these changes in lateral direction are exactly opposite to that of changes in longitudinal direction *i.e.*, if extension is taking place in longitudinal direction, the shortening of lateral dimension takes place and if shortening is taking place in longitudinal direction extension takes place in lateral directions (See Fig. 8.7). The lateral strain may be defined as changes in the lateral dimension per unit lateral dimension. Thus,

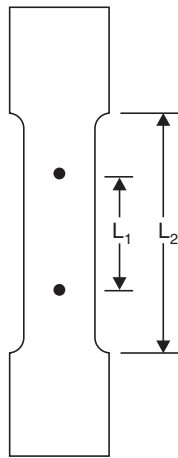
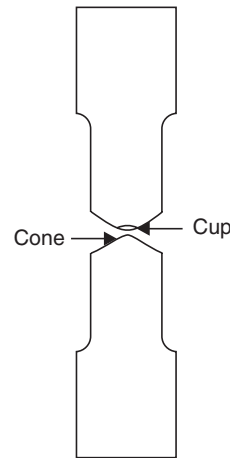
$$\begin{aligned} \text{Lateral Strain} &= \frac{\text{Change in Lateral Dimension}}{\text{Original Lateral Dimension}} \\ &= \frac{b' - b}{b} = \frac{\delta b}{b} \quad \dots(8.8) \end{aligned}$$

## 8.5 STRESS-STRAIN RELATION

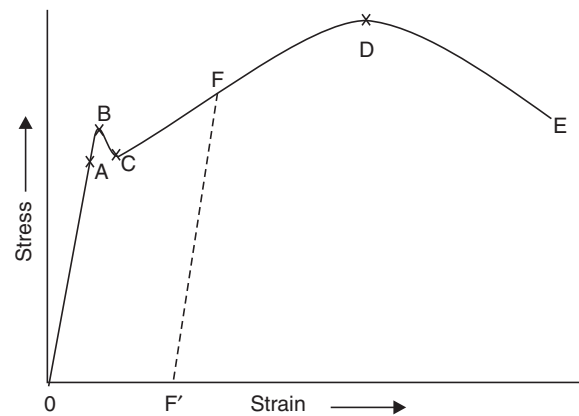
The stress-strain relation of any material is obtained by conducting tension test in the laboratories on standard specimen. Different materials behave differently and their behaviour in tension and in compression differ slightly.

### 8.5.1 Behaviour in Tension

**Mild steel.** Figure 8.8 shows a typical tensile test specimen of mild steel. Its ends are gripped into universal testing machine. Extensometer is fitted to test specimen which measures extension over the length  $L_1$ , shown in Fig. 8.8. The length over which extension is measured is called *gauge length*. The load is applied gradually and at regular interval of loads extension is measured. After certain load, extension increases at faster rate and the capacity of extensometer to measure extension comes to an end and, hence, it is removed before this stage is reached and extension is measured from scale on the universal testing machine. Load is increased gradually till the specimen breaks.

**Fig. 8.8.** Tension Test Specimen**Fig. 8.9.** Tension Test Specimen after Breaking

Load divided by original cross-sectional area is called as nominal stress or simply as stress. Strain is obtained by dividing extensometer readings by gauge length of extensometer ( $L_1$ ) and by dividing scale readings by grip to grip length of the specimen ( $L_2$ ). Figure 8.10 shows stress vs strain diagram for the typical mild steel specimen. The following salient points are observed on stress-strain curve:

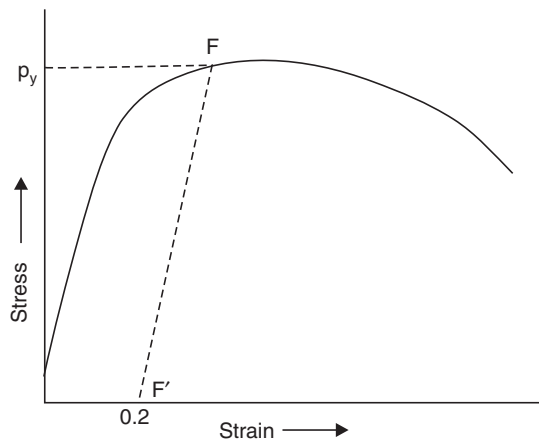
**Fig. 8.10**

- (a) **Limit of Proportionality (A):** It is the limiting value of the stress up to which stress is proportional to strain.
- (b) **Elastic Limit:** This is the limiting value of stress up to which if the material is stressed and then released (unloaded) strain disappears completely and the original length is regained. This point is slightly beyond the limit of proportionality.
- (c) **Upper Yield Point (B):** This is the stress at which, the load starts reducing and the extension increases. This phenomenon is called yielding of material. At this stage strain is about 0.125 per cent and stress is about  $250 \text{ N/mm}^2$ .
- (d) **Lower Yield Point (C):** At this stage the stress remains same but strain increases for some time.
- (e) **Ultimate Stress (D):** This is the maximum stress the material can resist. This stress is about  $370\text{--}400 \text{ N/mm}^2$ . At this stage cross-sectional area at a particular section starts reducing very fast (Fig. 8.9). This is called neck formation. After this stage load resisted and hence the stress developed starts reducing.
- (f) **Breaking Point (E):** The stress at which finally the specimen fails is called breaking point. At this strain is 20 to 25 per cent.

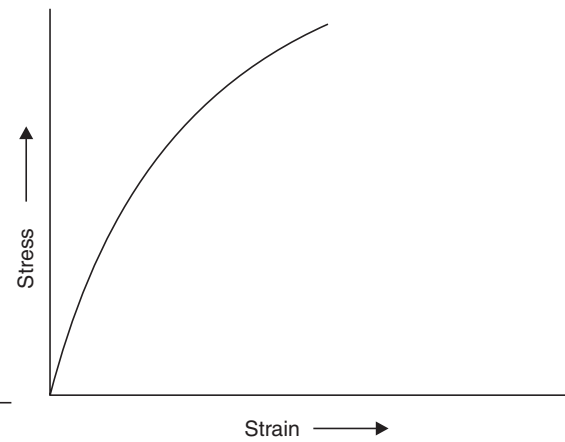
If unloading is made within elastic limit the original length is regained *i.e.*, the stress-strain curve

follows down the loading curve shown in Fig. 8.6. If unloading is made after loading the specimen beyond elastic limit, it follows a straight line parallel to the original straight portion as shown by line  $FF'$  in Fig. 8.10. Thus if it is loaded beyond elastic limit and then unloaded a permanent strain ( $OF$ ) is left in the specimen. This is called *permanent set*.

**Stress-strain relation in aluminium and high strength steel.** In these elastic materials there is no clear cut yield point. The necking takes place at ultimate stress and eventually the breaking point is lower than the ultimate point. The typical stress-strain diagram is shown in Fig. 8.11. The stress  $p$  at which if unloading is made there will be 0.2 per cent permanent set is known as 0.2 per cent proof stress and this point is treated as yield point for all practical purposes.



**Fig. 8.11.** Stress-Strain Relation in Aluminium and High Strength Steel



**Fig. 8.12.** Stress-Strain Relation for Brittle Material

**Stress-strain relation in brittle material.** The typical stress-strain relation in a brittle material like cast iron, is shown in Fig. 8.12.

In these material, there is no appreciable change in rate of strain. There is no yield point and no necking takes place. Ultimate point and breaking point are one and the same. The strain at failure is very small.

**Percentage elongation and percentage reduction in area.** Percentage elongation and percentage reduction in area are the two terms used to measure the ductility of material.

(a) **Percentage Elongation:** It is defined as the ratio of the final extension at rupture to original length expressed, as percentage. Thus,

$$\text{Percentage Elongation} = \frac{L' - L}{L} \times 100 \quad \dots(8.9)$$

where  $L$  – original length,  $L'$  – length at rupture.

The code specify that original length is to be five times the diameter and the portion considered must include neck (whenever it occurs). Usually marking are made on tension rod at every '2.5  $d$ ' distance and after failure the portion in which necking takes place is considered. In case of ductile material percentage elongation is 20 to 25.

(b) **Percentage Reduction in Area:** It is defined as the ratio of maximum changes in the cross-sectional area to original cross-sectional area, expressed as percentage. Thus,

$$\text{Percentage Reduction in Area} = \frac{A - A'}{A} \times 100 \quad \dots(8.10)$$

where  $A$ —original cross-sectional area,  $A'$ —minimum cross-sectional area. In case of ductile material,  $A'$  is calculated after measuring the diameter at the neck. For this, the two broken pieces of the specimen are to be kept joining each other properly. For steel, the percentage reduction in area is 60 to 70.

### 8.5.2 Behaviour of Materials under Compression

As there is chance to buckling (laterally bending) of long specimen, for compression tests short specimens are used. Hence, this test involves measurement of smaller changes in length. It results into lesser accuracy. However precise measurements have shown the following results:

- (a) In case of ductile materials stress-strain curve follows exactly same path as in tensile test up to and even slightly beyond yield point. For larger values the curves diverge. There will not be necking in case of compression tests.
- (b) For most brittle materials ultimate compressive stress in compression is much larger than in tension. It is because of flaws and cracks present in brittle materials which weaken the material in tension but will not affect the strength in compression.

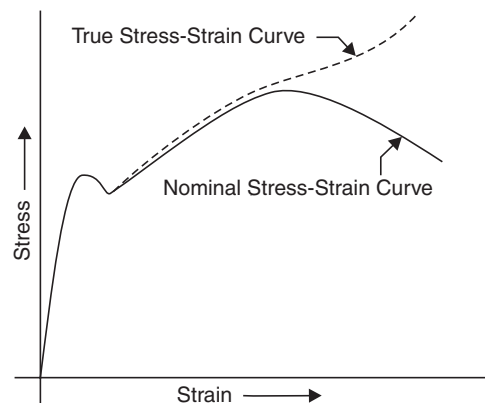
### 8.6 NOMINAL STRESS AND TRUE STRESS

So far our discussion on direct stress is based on the value obtained by dividing the load by original cross-sectional area. That is the reason why the value of stress started dropping after neck is formed in mild steel (or any ductile material) as seen in Fig. 8.10. But actually as material is stressed its cross-sectional area changes. We should divide load by the actual cross-sectional area to get true stress in the material. To distinguish between the two values we introduce the terms nominal stress and true stress and define them as given below:

$$\text{Nominal Stress} = \frac{\text{Load}}{\text{Original Cross-sectional Area}} \quad \dots(8.11a)$$

$$\text{True Stress} = \frac{\text{Load}}{\text{Actual Cross-sectional Area}} \quad \dots(8.11b)$$

So far discussion was based on nominal stress. That is why after neck formation started (after ultimate stress), stress-strain curve started sloping down and the breaking took place at lower stress (nominal). If we consider true stress, it is increasing continuously as strain increases as shown in Fig. 8.13.



**Fig. 8.13.** Nominal Stress-Strain Curve and True Stress-Strain Curve for Mild Steel.

### 8.7 FACTOR OF SAFETY

In practice it is not possible to design a mechanical component or structural component permitting stressing up to ultimate stress for the following reasons:

1. Reliability of material may not be 100 per cent. There may be small spots of flaws.
2. The resulting deformation may obstruct the functional performance of the component.
3. The loads taken by designer are only estimated loads. Occasionally there can be overloading. Unexpected impact and temperature loadings may act in the lifetime of the member.
4. There are certain ideal conditions assumed in the analysis (like boundary conditions). Actually ideal conditions will not be available and, therefore, the calculated stresses will not be 100 per cent real stresses.

Hence, *the maximum stress to which any member is designed is much less than the ultimate stress, and this stress is called Working Stress. The ratio of ultimate stress to working stress is called factor of safety.* Thus

$$\text{Factor of Safety} = \frac{\text{Ultimate Stress}}{\text{Working Stress}} \quad \dots(8.12)$$

In case of elastic materials, since excessive deformation create problems in the performance of the member, working stress is taken as a factor of yield stress or that of a 0.2 proof stress (if yield point do not exist).

Factor of safety for various materials depends up on their reliability. The following values are commonly taken in practice:

1. For steel – 1.85
2. For concrete – 3
3. For timber – 4 to 6

### 8.8 HOOKE'S LAW

Robert Hooke, an English mathematician conducted several experiments and concluded that *stress is proportional to strain up to elastic limit*. This is called Hooke's law. Thus Hooke's law is, up to elastic limit

$$p \propto e \quad \dots(8.13a)$$

where  $p$  is stress and  $e$  is strain

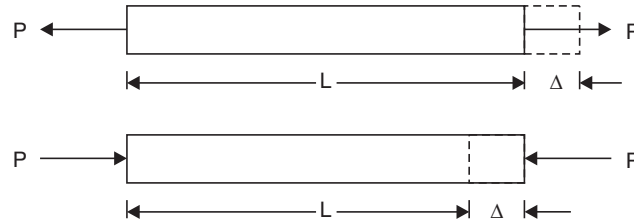
$$\text{Hence, } p = Ee \quad \dots(8.13b)$$

where  $E$  is the constant of proportionality of the material, known as modulus of elasticity or Young's modulus, named after the English scientist Thomas Young (1773–1829).

However, present day sophisticated experiments have shown that for mild steel the Hooke's law holds good up to the proportionality limit which is very close to the elastic limit. For other materials, as seen in art. 1.5, Hooke's law does not hold good. However, in the range of working stresses, assuming Hooke's law to hold good, the relationship does not deviate considerably from actual behaviour. Accepting Hooke's law to hold good, simplifies the analysis and design procedure considerably. Hence Hooke's law is widely accepted. The analysis procedure accepting Hooke's law is known as Linear Analysis and the design procedure is known as the working stress method.

### 8.9 EXTENSION/SHORTENING OF A BAR

Consider the bars shown in Fig. 8.14



**Fig. 8.14**

From equation (8.6), Stress  $p = \frac{P}{A}$

From equation (8.7), Strain,  $e = \frac{\Delta}{L}$

From Hooke's Law we have,

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{p}{e} = \frac{P/A}{\Delta/L} = \frac{PL}{A\Delta}$$

or

$$\Delta = \frac{PL}{AE} \quad \dots(8.14)$$

**Example 8.1.** A circular rod of diameter 16 mm and 500 mm long is subjected to a tensile force 40 kN. The modulus of elasticity for steel may be taken as 200 kN/mm<sup>2</sup>. Find stress, strain and elongation of the bar due to applied load.

**Solution:**

$$\text{Load } P = 40 \text{ kN} = 40 \times 1000 \text{ N}$$

$$E = 200 \text{ kN/mm}^2 = 200 \times 10^3 \text{ N/mm}^2$$

$$L = 500 \text{ mm}$$

$$\text{Diameter of the rod } d = 16 \text{ mm}$$

$$\begin{aligned} \text{Therefore, sectional area } A &= \frac{\pi d^2}{4} = \frac{\pi}{4} \times 16^2 \\ &= 201.06 \text{ mm}^2 \end{aligned}$$

$$\text{Stress } p = \frac{P}{A} = \frac{40 \times 1000}{201.06} = 198.94 \text{ N/mm}^2$$

$$\text{Strain } e = \frac{p}{E} = \frac{198.94}{200 \times 10^3} = 0.0009947$$

$$\text{Elongation } \Delta = \frac{PL}{AE} = \frac{4.0 \times 1000 \times 500}{201.06 \times 200 \times 10^3} = 0.497 \text{ mm}$$

**Example 8.2.** A Surveyor's steel tape 30 m long has a cross-section of 15 mm × 0.75 mm. With this, line AB is measure as 150 m. If the force applied during measurement is 120 N more than the force applied at the time of calibration, what is the actual length of the line?

Take modulus of elasticity for steel as 200 kN/mm<sup>2</sup>.

**Solution:**

$$A = 15 \times 0.75 = 11.25 \text{ mm}^2$$

$$P = 120 \text{ N}, L = 30 \text{ m} = 30 \times 1000 \text{ mm}$$

$$E = 200 \text{ kN/mm}^2 = 200 \times 10^3 \text{ N/mm}^2$$

$$\text{Elongation } \Delta = \frac{PL}{AE} = \frac{120 \times 30 \times 1000}{11.25 \times 200 \times 10^3} = 1.600 \text{ mm}$$

Hence, if measured length is 30 m.

Actual length is 30 m + 1.600 mm = 30.001600 m

$$\therefore \text{Actual length of line AB} = \frac{150}{30} \times 30.001600 = \mathbf{150.008 \text{ m}}$$

**Example 8.3.** A hollow steel tube is to be used to carry an axial compressive load of 160 kN. The yield stress for steel is 250 N/mm<sup>2</sup>. A factor of safety of 1.75 is to be used in the design. The following three class of tubes of external diameter 101.6 mm are available.

Class	Thickness
Light	3.65 mm
Medium	4.05 mm
Heavy	4.85 mm

Which section do you recommend?

**Solution:** Yield stress = 250 N/mm<sup>2</sup>

Factor of safety = 1.75

Therefore, permissible stress

$$p = \frac{250}{1.75} = 142.857 \text{ N/mm}^2$$

$$\text{Load } P = 160 \text{ kN} = 160 \times 10^3 \text{ N}$$

$$\text{but } p = \frac{P}{A}$$

$$\text{i.e. } 142.857 = \frac{160 \times 10^3}{A}$$

$$\therefore A = \frac{160 \times 10^3}{142.857} = 1120 \text{ mm}^2$$

For hollow section of outer diameter 'D' and inner diameter 'd'

$$A = \frac{\pi}{4}(D^2 - d^2) = 1120$$

$$\frac{\pi}{4}(101.6^2 - d^2) = 1120$$

$$d^2 = 8896.53 \quad \therefore \quad d = 94.32 \text{ mm}$$

$$\therefore \quad t = \frac{D - d}{2} = \frac{101.6 - 94.32}{2} = 3.63 \text{ mm}$$

**Hence, use of light section is recommended.**

**Example 8.4.** A specimen of steel 20 mm diameter with a gauge length of 200 mm is tested to destruction. It has an extension of 0.25 mm under a load of 80 kN and the load at elastic limit is 102 kN. The maximum load is 130 kN.

The total extension at fracture is 56 mm and diameter at neck is 15 mm. Find

- (i) The stress at elastic limit.
- (ii) Young's modulus.
- (iii) Percentage elongation.
- (iv) Percentage reduction in area.
- (v) Ultimate tensile stress.

**Solution:** Diameter  $d = 20 \text{ mm}$

$$\text{Area } A = \frac{\pi d^2}{4} = 314.16 \text{ mm}^2$$

$$\begin{aligned} \text{(i) Stress at elastic limit} &= \frac{\text{Load at elastic limit}}{\text{Area}} \\ &= \frac{102 \times 10^3}{314.16} = \mathbf{324.675 \text{ N/mm}^2} \end{aligned}$$

$$\begin{aligned} \text{(ii) Young's modulus } E &= \frac{\text{Stress}}{\text{Strain}} \quad \text{within elastic limit} \\ &= \frac{P/A}{\Delta/L} = \frac{80 \times 10^3 / 314.16}{0.25/200} \\ &= \mathbf{203718 \text{ N/mm}^2} \end{aligned}$$

$$\begin{aligned} \text{(iii) Percentage elongation} &= \frac{\text{Final extension}}{\text{Original length}} \\ &= \frac{56}{200} \times 100 = \mathbf{28} \end{aligned}$$

$$\begin{aligned} \text{(iv) Percentage reduction in area} &= \frac{\text{Initial area} - \text{Final area}}{\text{Initial area}} \times 100 \\ &= \frac{\frac{\pi}{4} \times 20^2 - \frac{\pi}{4} \times 15^2}{\frac{\pi}{4} \times 20^2} \times 100 = \mathbf{43.75} \end{aligned}$$

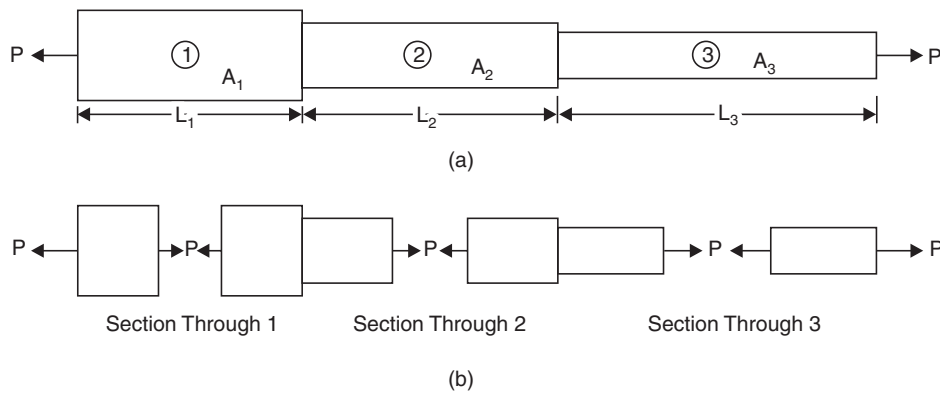


$$\begin{aligned}
 \text{(v) Ultimate Tensile Stress} &= \frac{\text{Ultimate Load}}{\text{Area}} \\
 &= \frac{130 \times 10^3}{314.16} = 413.80 \text{ N/mm}^2.
 \end{aligned}$$

### 8.10 BARS WITH CROSS-SECTIONS VARYING IN STEPS

A typical bar with cross-sections varying in steps and subjected to axial load is as shown in Fig. 8.15(a). Let the length of three portions be  $L_1$ ,  $L_2$  and  $L_3$  and the respective cross-sectional areas of the portion be  $A_1$ ,  $A_2$ ,  $A_3$  and  $E$  be the Young's modulus of the material and  $P$  be the applied axial load.

Figure 8.15(b) shows the forces acting on the cross-sections of the three portions. It is obvious that to maintain equilibrium the load acting on each portion is  $P$  only. Hence stress, strain and extension of each of these portions are as listed below:



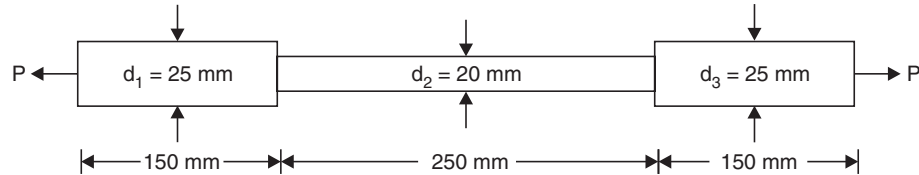
**Fig. 8.15.** Typical Bar with Cross-section Varying in Step

Portion	Stress	Strain	Extension
1	$p_1 = \frac{P}{A_1}$	$e_1 = \frac{p_1}{E} = \frac{P}{A_1 E}$	$\Delta_1 = \frac{PL_1}{A_1 E}$
2	$p_2 = \frac{P}{A_2}$	$e_2 = \frac{p_2}{E} = \frac{P}{A_2 E}$	$\Delta_2 = \frac{PL_2}{A_2 E}$
3	$p_3 = \frac{P}{A_3}$	$e_3 = \frac{p_3}{E} = \frac{P}{A_3 E}$	$\Delta_3 = \frac{PL_3}{A_3 E}$

Hence total change in length of the bar

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = \frac{PL_1}{A_1 E} + \frac{PL_2}{A_2 E} + \frac{PL_3}{A_3 E} \quad \dots(8.15)$$

**Example 8.5.** The bar shown in Fig. 8.16 is tested in universal testing machine. It is observed that at a load of 40 kN the total extension of the bar is 0.280 mm. Determine the Young's modulus of the material.



**Fig. 8.16**

**Solution:** Extension of portion 1,

$$\frac{PL_1}{A_1E} = \frac{40 \times 10^3 \times 150}{\frac{\pi}{4} \times 25^2 E}$$

Extension of portion 2,

$$\frac{PL_2}{A_2E} = \frac{40 \times 10^3 \times 250}{\frac{\pi}{4} \times 20^2 E}$$

Extension of portion 3,

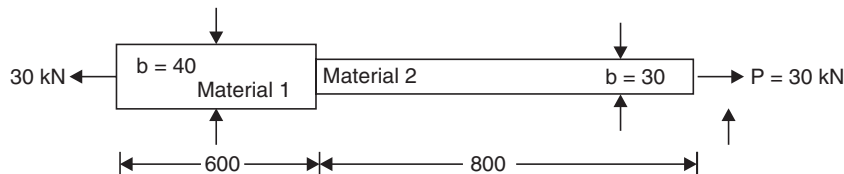
$$\frac{PL_3}{A_3E} = \frac{40 \times 10^3 \times 150}{\frac{\pi}{4} \times 25^2 E}$$

$$\text{Total extension} = \frac{40 \times 10^3}{E} \times \frac{4}{\pi} \left\{ \frac{150}{625} + \frac{250}{400} + \frac{150}{625} \right\}$$

$$0.280 = \frac{40 \times 10^3}{E} \times \frac{4}{\pi} \times \frac{1.112}{E}$$

$$E = 200990 \text{ N/mm}^2$$

**Example 8.6.** The stepped bar shown in Fig. 8.17 is made up of two different materials. The material 1 has Young's modulus  $= 2 \times 10^5 \text{ N/mm}^2$ , while that of material 2 is  $1 \times 10^5 \text{ N/mm}^2$ . Find the extension of the bar under a pull of 30 kN if both the portions are 20 mm in thickness.



**Fig. 8.17**

**Solution:**

$$A_1 = 40 \times 20 = 800 \text{ mm}^2$$

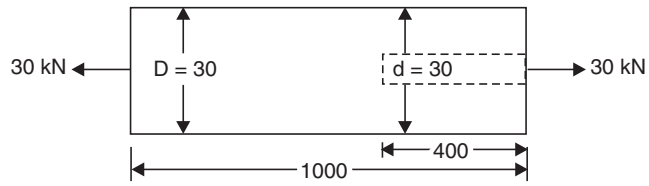
$$A_2 = 30 \times 20 = 600 \text{ mm}^2$$

$$\text{Extension of portion 1, } \frac{PL_1}{A_1E_1} = \frac{30 \times 10^3 \times 600}{800 \times 2 \times 10^5} = 0.1125 \text{ mm.}$$

$$\text{Extension of portion 2, } \frac{PL_2}{A_2E_2} = \frac{30 \times 10^3 \times 800}{600 \times 1 \times 10^5} = 0.4000 \text{ mm.}$$

$$\therefore \text{ Total extension of the bar} = 0.1125 + 0.4000 = 0.5125 \text{ mm.}$$

**Example 8.7.** A bar of length 1000 mm and diameter 30 mm is centrally bored for 400 mm, the bore diameter being 10 mm as shown in Fig. 8.18. Under a load of 30 kN, if the extension of the bar is 0.222 mm, what is the modulus of elasticity of the bar?



**Fig. 8.18**

**Solution:** Now

$$L_1 = 1000 - 400 = 600 \text{ mm}$$

$$L_2 = 400 \text{ mm}$$

$$A_1 = \frac{\pi}{4} \times 30^2 = 225 \pi$$

$$A_2 = \frac{\pi}{4} \times (30^2 - 10^2) = 200 \pi$$

$$\Delta_1 = \frac{PL_1}{A_1E}$$

$$\Delta_2 = \frac{PL_2}{A_2E}$$

$$\therefore \Delta = \Delta_1 + \Delta_2 = \frac{P}{E} \left( \frac{L_1}{A_1} + \frac{L_2}{A_2} \right)$$

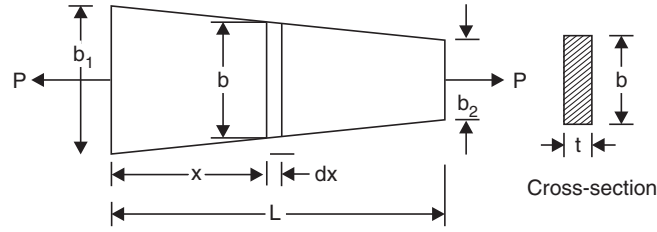
$$\text{i.e., } 0.222 = \frac{30 \times 10^3}{E} \left( \frac{600}{225 \pi} + \frac{400}{200 \pi} \right)$$

$$\therefore E = 200736 \text{ N/mm}^2.$$

### 8.11 BARS WITH CONTINUOUSLY VARYING CROSS-SECTIONS

When the cross-section varies continuously, an elemental length of the bar should be considered and general expression for elongation of the elemental length derived. Then the general expression should be integrated over entire length to get total extension. The procedure is illustrated with Examples 8.8 and 8.9.

**Example 8.8.** A bar of uniform thickness 't' tapers uniformly from a width of  $b_1$  at one end to  $b_2$  at other end in a length 'L' as shown in Fig. 8.18. Find the expression for the change in length of the bar when subjected to an axial force P.

**Fig. 8.19**

**Solution:** Consider an elemental length  $dx$  at a distance  $x$  from larger end. Rate of change of breadth is  $\frac{b_1 - b_2}{L}$ .

Hence, width at section  $x$  is  $b = b_1 - \frac{b_1 - b_2}{L} x = b_1 - kx$

where  $k = \frac{b_1 - b_2}{L}$

$\therefore$  Cross-section area of the element  $= A = t(b_1 - kx)$

Since force acting at all sections is  $P$  only,

$$\text{Extension of element} = \frac{Pdx}{AE} \quad [\text{where length} = dx]$$

$$= \frac{Pdx}{(b_1 - kx)tE}$$

$$\text{Total extension of the bar} = \int_0^L \frac{Pdx}{(b_1 - kx)tE} = \frac{P}{tE} \int_0^L \frac{dx}{(b_1 - kx)}$$

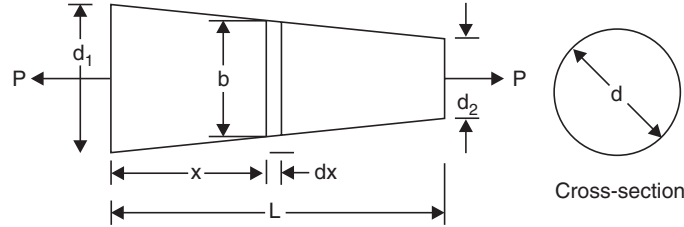
$$= \frac{P}{tE} \left( \frac{1}{-k} \right) \left[ \log(b_1 - kx) \right]_0^L$$

$$= \frac{P}{tEk} \left[ -\log \left( b_1 - \frac{b_1 - b_2}{L} x \right) \right]_0^L$$

$$= \frac{P}{tEk} [-\log b_2 + \log b_1] = \frac{P}{tEk} \log \frac{b_1}{b_2}$$

$$= \frac{PL}{tE(b_1 - b_2)} \log \frac{b_1}{b_2} \quad \dots(8.16)$$

**Example 8.9.** A tapering rod has diameter  $d_1$  at one end and it tapers uniformly to a diameter  $d_2$  at the other end in a length  $L$  as shown in Fig. 8.20. If modulus of elasticity of the material is  $E$ , find its change in length when subjected to an axial force  $P$ .



**Fig. 8.20**

**Solution:** Change in diameter in length  $L$  is  $d_1 - d_2$

$$\therefore \text{Rate of change of diameter, } k = \frac{d_1 - d_2}{L}$$

Consider an elemental length of bar  $dx$  at a distance  $x$  from larger end. The diameter of the bar at this section is

$$d = d_1 - kx.$$

$$\text{Cross-sectional area } A = \frac{\pi d^2}{4} = \frac{\pi}{4} (d_1 - kx)^2$$

$$\therefore \text{Extension of the element} = \frac{P dx}{\frac{\pi}{4} (d_1 - kx)^2 E}$$

$$\begin{aligned} \text{Extension of the entire bar } \Delta &= \int_0^L \frac{P dx}{\frac{\pi}{4} (d_1 - kx)^2 E} \\ &= \frac{4P}{\pi E} \int_0^L \frac{dx}{(d_1 - kx)^2} \\ &= \frac{4P}{\pi E k} \left( \frac{1}{d_1 - kx} \right)_0^L \\ &= \frac{4P}{\frac{\pi E (d_1 - d_2)}{L}} \left( \frac{1}{d_2} - \frac{1}{d_1} \right), \text{ since } d_1 - kL = d_2 \end{aligned}$$

$$\therefore \Delta = \frac{4PL}{\pi E (d_1 - d_2)} \times \frac{(d_1 - d_2)}{d_1 d_2} = \frac{4PL}{\pi E d_1 d_2}. \quad \dots(8.17)$$

**Note:** For bar of uniform diameter extension is  $\frac{PL}{\frac{\pi d^2}{4}E}$  and for tapering rod it is  $\frac{PL}{\frac{\pi}{4}d_1d_2E}$ .

Thus the change is, instead of  $d^2$  for uniform bar,  $d_1 d_2$  term appears if the bar tapers uniformly.

**Example 8.10.** A steel flat of thickness 10 mm tapers uniformly from 60 mm at one end to 40 mm at other end in a length of 600 mm. If the bar is subjected to a load of 80 kN, find its extension. Take  $E = 2 \times 10^5$  MPa. What is the percentage error if average area is used for calculating extension?

**Solution:** Now,  $t = 10$  mm  $b_1 = 60$  mm  $b_2 = 40$  mm  
 $L = 600$  mm  $P = 80$  kN = 80000 N

Now,  $1$  MPa =  $1$  N/mm<sup>2</sup>

Hence  $E = 2 \times 10^5$  N/mm<sup>2</sup>

Extension of the tapering bar of rectangular section

$$\begin{aligned}\Delta &= \frac{PL}{tE(b_1 - b_2)} \log \frac{b_1}{b_2} \\ &= \frac{80000 \times 600}{10 \times 2 \times 10^5 (60 - 40)} \log \frac{60}{40} \\ &= 0.4865 \text{ mm}\end{aligned}$$

If averages cross-section is considered instead of tapering cross-section, extension is given by

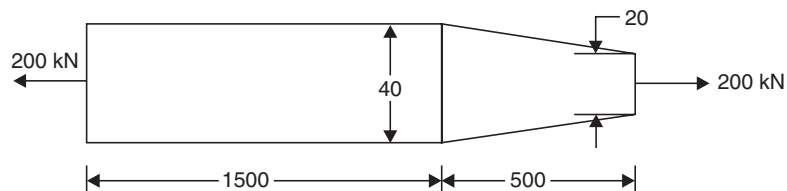
$$\Delta = \frac{PL}{A_{av}E}$$

Now  $A_{av} = \frac{60 \times 10 + 40 \times 10}{2} = 500 \text{ mm}^2$

$$\Delta = \frac{80000 \times 600}{500 \times 2 \times 10^5} = 0.480 \text{ mm}$$

$$\begin{aligned}\therefore \text{Percentage error} &= \frac{0.4865 - 0.48}{0.4865} \times 100 \\ &= 1.348\end{aligned}$$

**Example 8.11.** A 2.0 m long steel bar is having uniform diameter of 40 mm for a length of 1 m and in the next 0.5 m its diameter gradually reduces from 40 mm to 20 mm as shown in Fig. 8.21. Determine the elongation of this bar when subjected to an axial tensile load of 200 kN. Given  $E = 200$  GN/m<sup>2</sup>.



**Fig. 8.21**

**Solution:** Now,

$$P = 200 \times 10^3 \text{ N}$$

$$\begin{aligned} E &= 200 \text{ GN/m}^2 = \frac{200 \times 10^9}{(1000)^2} \text{ N/mm}^2 \\ &= 200 \times 1000 \text{ N/mm}^2 \\ &= 2 \times 10^5 \text{ N/mm}^2 \end{aligned}$$

Extensions of uniform portion and tapering portion are worked out separately and then added to get extension of the given bar.

Extension of uniform portion

$$\Delta_1 = \frac{PL}{AE} = \frac{200 \times 10^3 \times 1500}{\frac{\pi}{4} \times 40^2 \times 2 \times 10^5} = 1.194 \text{ mm.}$$

Extension of tapering portion

$$\begin{aligned} \Delta_2 &= \frac{4 PL}{E \pi d_1 d_2} = \frac{4 \times 200 \times 10^3 \times 500}{2 \times 10^5 \times \pi \times 60 \times 40} \\ &= 0.265 \text{ mm} \end{aligned}$$

**Total extension**

$$= \Delta_1 + \Delta_2 = 1.194 + 0.265 = \mathbf{1.459 \text{ mm}}$$

**Example 8.12.** The extension of a bar uniformly tapering from a diameter of  $d + a$  to  $d - a$  in a length  $L$  is calculated by treating it as a bar of uniform cross-section of average diameter  $d$ . What is the percentage error?

**Solution:** Actual extension under load

$$\begin{aligned} P &= \frac{4 PL}{\pi d_1 d_2 E} = \frac{4 PL}{\pi (d + a)(d - a) E} \\ &= \frac{4 PL}{\pi E (d^2 - a^2)} \end{aligned}$$

If it is treated as a bar of uniform diameter ' $d$ ', erroneous extension calculated

$$= \frac{PL}{\frac{\pi}{4} \times d^2 E} = \frac{4 PL}{\pi d^2 E}$$

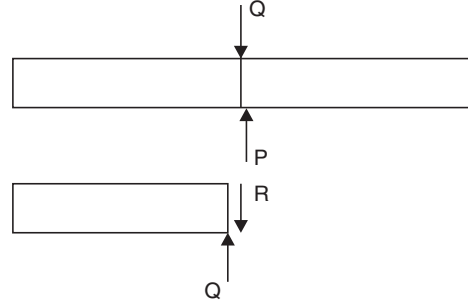
$$\begin{aligned} \therefore \text{Percentage error} &= \frac{\frac{4 PL}{\pi E (d^2 - a^2)} - \frac{4 PL}{\pi E d^2}}{\frac{4 PL}{\pi E (d^2 - a^2)}} \times 100 \\ &= \frac{\frac{1}{(d^2 - a^2)} - \frac{1}{d^2}}{\frac{1}{(d^2 - a^2)}} \times 100 \end{aligned}$$

$$= \left( 1 - \frac{d^2 - a^2}{d^2} \right) \times 100$$

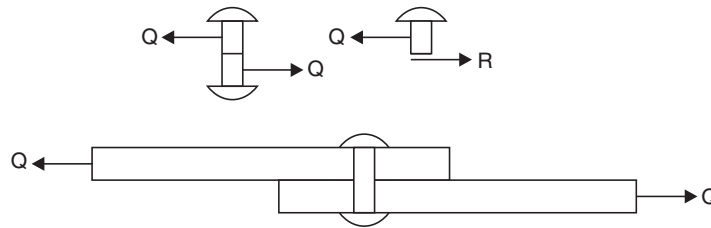
$$= 100 \frac{a^2}{d^2}$$

### 8.12 SHEAR STRESS

Figure 8.22 shows a bar subject to direct shearing force *i.e.*, the force parallel to the cross-section of bar. The section of a rivet/bolt subject to direct shear is shown in Fig. 8.23. Let  $Q$  be the shearing force and  $q$  the shearing stress acting on the section. Then, with usual assumptions that stresses are uniform we get,



**Fig. 8.22.** Direct Shear Force on a Section



**Fig. 8.23.** Rivet in Direct Shear

$$R = \int q \, dA = q \int dA = qA$$

For equilibrium

$$Q = R = qA$$

*i.e.*,

$$q = \frac{Q}{A} \quad \dots(8.18)$$

Thus, the direct stress is equal to shearing force per unit area.

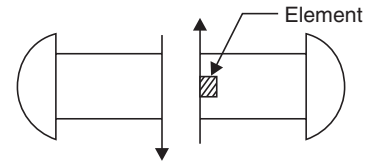
### 8.13 SIMPLE SHEAR

A material is said to be in a state of simple shear if it is subjected to only shearing stress. A shearing force tries to shear off the cross-section of the body. Consider a bolt subjected to pure shear as shown in Fig. 8.24a. A rectangular element at this section is shown in Fig. 8.24b.

Let the intensity of shear stress be  $q_{ab}$  and thickness of element be ' $t$ '. Consider the equilibrium of the element.

Vertical force on  $AB = q_{ab} \times AB \times t$

This can be balanced by vertical downward force on  $CD$ .



(a) Bolt Subjected to Shearing

**Fig. 8.24**



$$q_{ab} \times AB \times t = q_{cd} \times CD \times t$$

$$q_{ab} = q_{cd} = q(\text{say})$$

Since  $q_{ab}$  and  $q_{cd}$  are giving rise to equal and opposite forces  $q \times AB \times t$ , with moment arm =  $AD$ , they form a couple of magnitude equal to

$$q \times AB \times t \times AD \quad \dots(1)$$

This can be balanced by another couple only. *i.e.*  $q_{bc}$  and  $q_{cd}$  should form an equal and opposite couple. Hence their direction will be as shown in Fig. 8.24b.

Let  $q_{bc} = q_{ad} = q'$

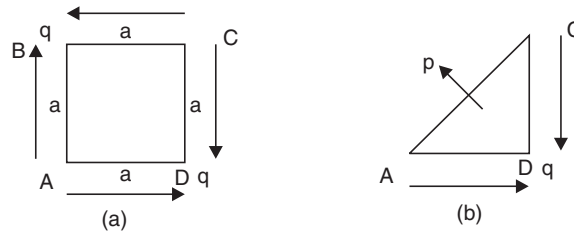
The couple formed by these forces

$$= q' \times AD \times t \times AB \quad \dots(2)$$

Equating (1) and (2) we get,  $q = q'$ .

Thus if a section is subjected to pure shear, the state of stress in any element at that section is as shown in Fig. 8.24c.

State of stress at  $45^\circ$  to shearing direction is of engineering interest. To study it, for simplicity, consider a square element of side ' $a$ ' and thickness ' $t$ ', under simple shear as shown in Fig. 8.25a.



**Fig. 8.25**

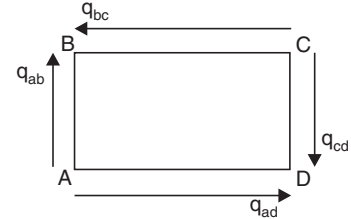
Since it is square element  $AC = a\sqrt{2}$ . Consider section along  $AC$  and let ' $p$ ' be the stress on this section. From equilibrium condition of system of forces in the direction normal to  $AC$ , we get

$$p \times AC \times t = q \times CD \times t \cos 45^\circ + q \times AD \times t \sin 45^\circ$$

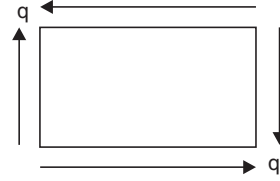
$$\begin{aligned} pa\sqrt{2}t &= qat \frac{1}{\sqrt{2}} + qat \frac{1}{\sqrt{2}} \\ &= qat\sqrt{2} \end{aligned}$$

$$\text{i.e.,} \quad p = q$$

Thus in case of simple shear tensile stress of the same magnitude as shearing stress develops at  $45^\circ$  to shearing plane. By taking stresses on diagonal  $BD$  it can be shown that compressive stress of magnitude  $q$  acts on this plane. Thus simple shear gives rise to tensile and compressive stresses across planes inclined at  $45^\circ$  to the shearing planes, the intensity of direct stresses being of same magnitude as shearing stress.

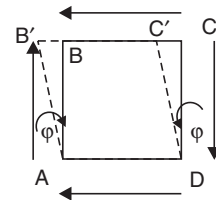


(b) Element under Shear



(c) State of Simple Shear

**Fig. 8.24**



**Fig. 8.26**

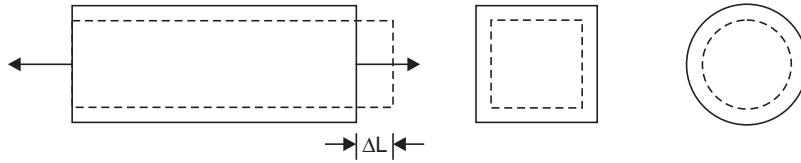
*Shearing strain.* Shearing stress has a tendency to distort the element to position  $AB'C'D$  from the original position  $ABCD$  as shown in Fig. 8.26.

This deformation is expressed in terms of angular displacement and is called shear strain. Thus

$$\begin{aligned}\text{Shear strain} &= \frac{BB'}{AB} = \tan \phi \\ &= \phi, \text{ since angle } \phi \text{ is small}\end{aligned}$$

#### 8.14 POISSON'S RATIO

When a material undergoes changes in length, it undergoes changes of opposite nature in lateral directions. For example, if a bar is subjected to direct tension in its axial direction it elongates and at the same time its sides contract (Fig. 8.27).



**Fig. 8.27.** Changes in Axial and Lateral Directions

If we define the ratio of change in axial direction to original length as linear strain and change in lateral direction to the original lateral dimension as lateral strain, it is found that *within elastic limit there is a constant ratio between lateral strain and linear strain. This constant ratio is called Poisson's ratio.* Thus,

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Linear strain}} \quad \dots(8.19)$$

It is denoted by  $\frac{1}{m}$ , or  $\mu$ . For most of metals its value is between 0.25 to 0.33. Its value for steel is 0.3 and for concrete 0.15.

#### 8.15 VOLUMETRIC STRAIN

When a member is subjected to stresses, it undergoes deformation in all directions. Hence, there will be change in volume. The *ratio of the change in volume to original volume is called volumetric strain.*

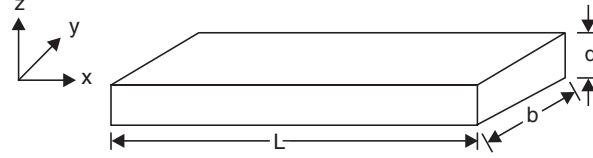
$$\text{Thus} \quad e_v = \frac{\delta V}{V} \quad \dots(8.20)$$

where  $e_v$  = Volumetric strain  
 $\delta V$  = Change in volume  
 $V$  = Original volume

It can be shown that volumetric strain is sum of strains in three mutually perpendicular directions.  
*i.e.,*

$$e_v = e_x + e_y + e_z$$

For example consider a bar of length  $L$ , breadth  $b$  and depth  $d$  as shown in Fig. 8.28.



**Fig. 8.28**

Now,

$$V = Lbd$$

Since volume is function of  $L$ ,  $b$  and  $d$ .

$$\delta V = \delta L \, bd + L \, \delta b \, d + Lb \, \delta d$$

$$\frac{\delta V}{V} = \frac{\delta v}{Lbd}$$

$$e_V = \frac{\delta L}{L} + \frac{\delta b}{b} + \frac{\delta d}{d}$$

$$e_V = e_x + e_y + e_z$$

Now, consider a circular rod of length  $L$  and diameter ' $d$ ' as shown in Fig. 8.29.



**Fig. 8.29**

Volume of the bar  $V = \frac{\pi}{4} d^2 L$

$$\therefore \delta V = \frac{\pi}{4} 2d \delta d \, L + \frac{\pi}{4} d^2 \delta L \quad (\text{since } v \text{ is function of } d \text{ and } L).$$

$$\therefore \frac{\delta V}{\frac{\pi}{4} d^2 L} = 2 \frac{\delta d}{d} + \frac{\delta L}{L}$$

$$e_V = e_x + e_y + e_z; \text{ since } e_y = e_z = \frac{\delta d}{d}$$

In general for any shape volumetric strain may be taken as sum of strains in three mutually perpendicular directions.

### 8.16 ELASTIC CONSTANTS

Modulus of elasticity, modulus of rigidity and bulk modulus are the three elastic constants. Modulus of elasticity (Young's Modulus) ' $E$ ' has been already defined as the ratio of linear stress to linear strain within elastic limit. Rigidity modulus and Bulk modulus are defined in this article.

**Modulus of Rigidity:** It is defined as the ratio of shearing stress to shearing strain within elastic limit and is usually denoted by letter  $G$  or  $N$ . Thus

$$G = \frac{q}{\phi} \quad \dots(8.21)$$

where  $G$  = Modulus of rigidity

$q$  = Shearing stress

and  $\phi$  = Shearing strain

**Bulk Modulus:** When a body is subjected to identical stresses  $p$  in three mutually perpendicular directions, (Fig. 8.30), the body undergoes uniform changes in three directions without undergoing distortion of shape. The ratio of change in volume to original volume has been defined as volumetric strain ( $e_v$ ). Then the bulk modulus,  $K$  is defined as

$$K = \frac{p}{e_v}$$

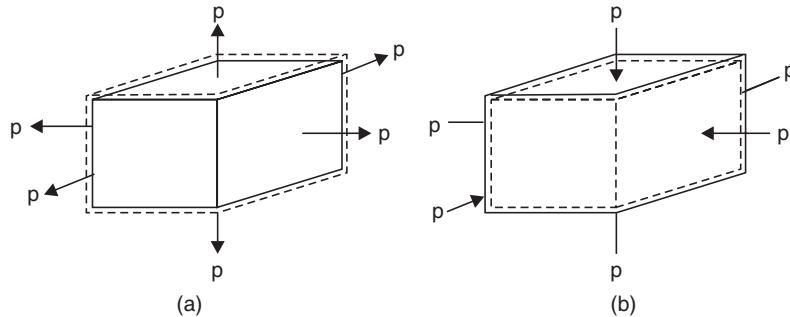
where  $p$  = identical pressure in three mutually perpendicular directions

$e_v = \frac{\Delta_v}{v}$ , Volumetric strain

$\Delta_v$  = Change in volume

$v$  = Original volume

Thus bulk modulus may be defined as the ratio of identical pressure ' $p$ ' acting in three mutually perpendicular directions to corresponding volumetric strain.

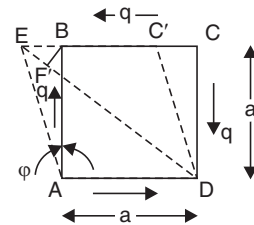


**Fig. 8.30**

Figure 8.30 shows a body subjected to identical compressive pressure ' $p$ ' in three mutually perpendicular directions. Since hydrostatic pressure, the pressure exerted by a liquid on a body within it, has this nature of stress, such a pressure ' $p$ ' is called as hydrostatic pressure.

### 8.17 RELATIONSHIP BETWEEN MODULUS OF ELASTICITY AND MODULUS OF RIGIDITY

Consider a square element  $ABCD$  of sides ' $a$ ' subjected to pure shear ' $q$ ' as shown in Fig. 8.31.  $AEC'D$  shown is the deformed shape due to shear  $q$ . Drop perpendicular  $BF$  to diagonal  $DE$ . Let  $\phi$  be the shear strain and  $G$  modulus of rigidity.



**Fig. 8.31**

$$\begin{aligned} \text{Now, strain in diagonal } BD &= \frac{DE - DF}{DF} \\ &= \frac{EF}{DB} \\ &= \frac{EF}{AB\sqrt{2}} \end{aligned}$$

Since angle of deformation is very small we can assume  $\angle BEF = 45^\circ$ , hence  $EF = BE \cos 45^\circ$

$$\begin{aligned} \therefore \text{Strain in diagonal } BD &= \frac{EF}{BD} = \frac{BE \cos 45^\circ}{AB\sqrt{2}} \\ &= \frac{a \tan \phi \cos 45^\circ}{a\sqrt{2}} \\ &= \frac{1}{2} \tan \phi = \frac{1}{2} \phi \quad (\text{Since } \phi \text{ is very small}) \\ &= \frac{1}{2} \times \frac{q}{G}, \text{ since } \phi = \frac{q}{G} \quad \dots(1) \end{aligned}$$

Now, we know that the above pure shear gives rise to axial tensile stress  $q$  in the diagonal direction of  $DB$  and axial compression  $q$  at right angles to it. These two stresses cause tensile strain along the diagonal  $DB$ .

$$\text{Tensile strain along the diagonal } DB = \frac{q}{E} + \mu \frac{q}{E} = \frac{q}{E} (1 + \mu) \quad \dots(2)$$

From equations (1) and (2), we get

$$\begin{aligned} \frac{1}{2} \times \frac{q}{G} &= \frac{q}{E} (1 + \mu) \\ E &= 2G(1 + \mu) \quad \dots(8.22) \end{aligned}$$

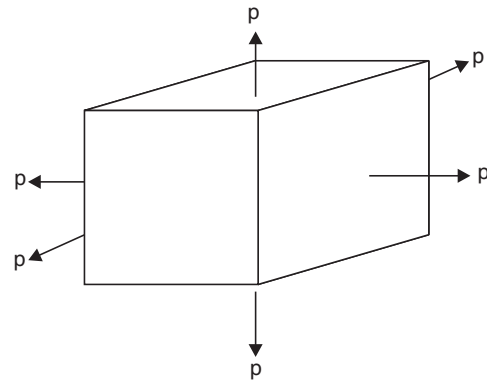
### 8.18 RELATIONSHIP BETWEEN MODULUS OF ELASTICITY AND BULK MODULUS

Consider a cubic element subjected to stresses  $p$  in the three mutually perpendicular direction  $x$ ,  $y$ ,  $z$  as shown in Fig. 8.32.

Now the stress  $p$  in  $x$  direction causes tensile strain  $\frac{p}{E}$  in  $x$  direction while the stress  $p$  in  $y$  and  $z$  direction cause compressive strains  $\mu \frac{p}{E}$  in  $x$  direction.

$$\begin{aligned} \text{Hence, } e_x &= \frac{p}{E} - \mu \frac{p}{E} - \mu \frac{p}{E} \\ &= \frac{p}{E} (1 - 2\mu) \end{aligned}$$

$$\text{Similarly } e_y = \frac{p}{E} (1 - 2\mu)$$



**Fig. 8.32**

$$e_z = \frac{p}{E}(1 - 2\mu) \quad \dots(1)$$

$$\therefore \text{Volumetric strain} \quad e_v = e_x + e_y + e_z = \frac{3p}{E}(1 - 2\mu)$$

From definition, bulk modulus  $K$  is given by

$$K = \frac{p}{e_v} = \frac{p}{\frac{3p(1-2\mu)}{E}}$$

$$\text{or} \quad E = 3K(1 - \mu) \quad \dots(2)$$

*Relationship between EGK:*

$$\text{We know} \quad E = 2G(1 + \mu) \quad \dots(a)$$

$$\text{and} \quad E = 3K(1 - 2\mu) \quad \dots(b)$$

By eliminating  $\mu$  between the above two equations we can get the relationship between  $E$ ,  $G$ ,  $K$ , free from the term  $\mu$ .

$$\text{From equation (a)} \quad \mu = \frac{E}{2G} - 1$$

Substituting it in equation (b), we get

$$\begin{aligned} E &= 3K \left[ 1 - 2 \left( \frac{E}{2G} - 1 \right) \right] \\ &= 3K \left( 1 - \frac{E}{G} + 2 \right) = 3K \left( 3 - \frac{E}{G} \right) \\ &= 9K - \frac{3KE}{G} \end{aligned}$$

$$\therefore \quad E \left( 1 + \frac{3K}{G} \right) = 9K$$

$$\text{or} \quad E \left( \frac{G + 3K}{G} \right) = 9K \quad \dots(c)$$

$$\text{or} \quad E = \frac{9KG}{G + 3K} \quad \dots(8.23a)$$

Equation (c) may be expressed as

$$\begin{aligned} \frac{9}{E} &= \frac{G + 3K}{KG} \\ \text{i.e.,} \quad \frac{9}{E} &= \frac{3}{G} + \frac{1}{K} \quad \dots(8.23b) \end{aligned}$$

**Example 8.13.** A bar of 25 mm diameter is tested in tension. It is observed that when a load of 60 kN is applied, the extension measured over a gauge length of 200 mm is 0.12 mm and contraction in diameter is 0.0045 mm. Find Poisson's ratio and elastic constants  $E$ ,  $G$ ,  $K$ .

**Solution:** Now,

$$P = 60 \text{ kN} = 60000 \text{ N}$$

$$\text{Area } A = \frac{\pi}{4} \times 25^2 = 156.25\pi \text{ mm}^2$$

$$\text{Gauge length } L = 200 \text{ mm}$$

$$\Delta = 0.12 \text{ mm}$$

$$\Delta d = 0.0045 \text{ mm}$$

$$\text{Linear strain} = \frac{\Delta}{L} = \frac{0.12}{200} = 0.0006$$

$$\text{Lateral strain} = \frac{\Delta d}{d} = \frac{0.0045}{25} = 0.00018$$

$$\therefore \text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Linear strain}} = \frac{0.00018}{0.0006}$$

$$\mu = 0.3$$

$$\text{Now, } \Delta = \frac{PL}{AE}$$

$$0.12 = \frac{60000 \times 200}{156.25\pi \times E}$$

$$\text{or } E = 203718.3 \text{ N/mm}^2$$

$$\text{Using the relation } E = 2G(1 + \mu)$$

$$\text{We get } G = \frac{E}{2(1 + \mu)} = \frac{203718.3}{2(1 + 0.3)} = 78353.2 \text{ N/mm}^2$$

$$\text{From the relation, } E = 3K(1 - 2\mu), \text{ we get}$$

$$K = \frac{E}{3(1 - 2\mu)} = \frac{203718.3}{3(1 - 2 \times 0.3)} = 169765.25 \text{ N/mm}^2$$

**Example 8.14.** A circular rod of 25 mm diameter and 500 mm long is subjected to a tensile force of 60 kN. Determine modulus of rigidity, bulk modulus and change in volume if Poisson's ratio = 0.3 and Young's modulus  $E = 2 \times 10^5 \text{ N/mm}^2$ .

**Solution:** From the relationship

$$E = 2G(1 + \mu) = 3k(1 - 2\mu)$$

$$\text{We get, } G = \frac{E}{2(1 + \mu)} = \frac{2 \times 10^5}{2(1 + 0.3)} = 0.7692 \times 10^5 \text{ N/mm}^2$$

$$\text{and } K = \frac{E}{3(1 + 2\mu)} = \frac{2 \times 10^5}{3(1 - 2 \times 0.3)} = 1.667 \times 10^5 \text{ N/mm}^2$$

$$\text{Longitudinal stress} = \frac{P}{A} = \frac{60 \times 10^3}{\frac{\pi}{4} \times 25^2} = 122.23 \text{ N/mm}^2$$

$$\text{Linear strain} = \frac{\text{Stress}}{E} = \frac{122.23}{2 \times 10^5} = 61.115 \times 10^{-5}$$

$$\text{Lateral strain} = e_y = -\mu e_x \quad \text{and} \quad e_z = -\mu e_x$$

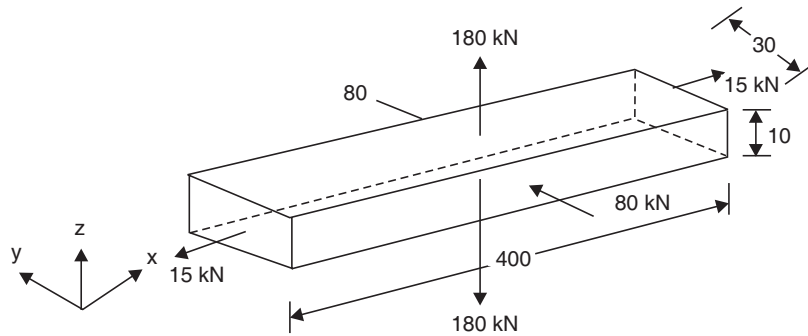
$$\begin{aligned} \text{Volumetric strain } e_v &= e_x + e_y + e_z \\ &= e_x(1 - 2\mu) \\ &= 61.115 \times 10^{-5} (1 - 2 \times 0.3) \\ &= 24.446 \times 10^{-5} \end{aligned}$$

$$\text{but } \frac{\text{Change in volume}}{v} = e_v$$

$$\begin{aligned}
 \therefore \quad \text{Change in volume} &= e_v \times v \\
 &= 24.446 \times 10^{-5} \times \frac{\pi}{4} \times (25^2) \times 500 \\
 &= \mathbf{60 \text{ mm}^3}
 \end{aligned}$$

**Example 8.15.** A 400 mm long bar has rectangular cross-section 10 mm × 30 mm. This bar is subjected to

- (i) 15 kN tensile force on 10 mm × 30 mm faces,
  - (ii) 80 kN compressive force on 10 mm × 400 mm faces, and
  - (iii) 180 kN tensile force on 30 mm × 400 mm faces.
- Find the change in volume if  $E = 2 \times 10^5 \text{ N/mm}^2$  and  $\mu = 0.3$ .



**Fig. 8.33**

**Solution:** The forces on the bar are as shown in Fig. 8.33. Let the  $x, y, z$  be the mutually perpendicular directions as shown in the figure.

$$\begin{aligned}
 \text{Now,} \quad p_x &= \frac{15 \times 10^3}{10 \times 30} = 50 \text{ N/mm}^2 && \text{(Tensile)} \\
 p_y &= \frac{80 \times 10^3}{10 \times 400} = 20 \text{ N/mm}^2 && \text{(Comp.)} \\
 p_z &= \frac{180 \times 10^3}{30 \times 400} = 15 \text{ N/mm}^2 && \text{(Tensile)}
 \end{aligned}$$

Noting that a stress produces a strain of  $p/E$  in its own direction, the nature being same as that of stress and  $\mu \frac{p}{E}$  in lateral direction of opposite nature, and taking tensile stress as +ve, we can write expression for strains  $e_x, e_y, e_z$  as

$$\begin{aligned}
 e_x &= \frac{50}{E} + \mu \frac{20}{E} - \mu \frac{15}{E} \\
 &= \frac{1}{E} (50 + 0.3 \times 20 - 0.3 \times 15) = \frac{1}{E} (51.5) \\
 e_y &= -\mu \frac{50}{E} - \frac{20}{E} - \mu \frac{15}{E} \\
 &= \frac{1}{E} (-0.3 \times 50 - 20 - 0.3 \times 15) = -\frac{39.5}{E}
 \end{aligned}$$



$$e_z = -\mu \frac{50}{E} + \mu \frac{20}{E} + \frac{15}{E}$$

$$= \frac{1}{E} (-0.3 \times 50 + 20 \times 0.3 + 15) = \frac{6}{E}$$

Volumetric strain:

$$e_v = e_x + e_y + e_z$$

$$= \frac{1}{E} (51.5 - 39.5 + 6) = \frac{18}{E}$$

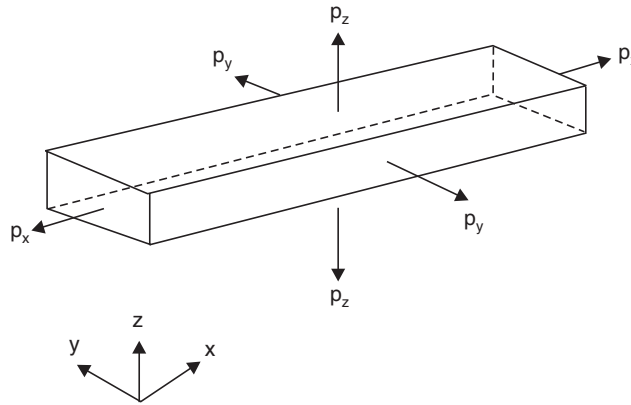
But  $\frac{\text{Change in volume}}{\text{Volume}} = e_v$

$$\therefore \text{Change in volume} = e_v \times \text{Volume}$$

$$= \frac{18}{2 \times 10^5} \times 10 \times 30 \times 400$$

$$= 10.8 \text{ mm}^3$$

**Example 8.16.** A bar of rectangular section shown in Fig. 8.34 is subjected to stresses  $p_x$ ,  $p_y$  and  $p_z$  in  $x$ ,  $y$  and  $z$  directions respectively. Show that if sum of these stresses is zero, there is no change in volume of the bar.



**Fig. 8.34**

**Solution:** A stress  $p$  produces strain of magnitude  $p/E$  in its direction and a strain  $\mu \frac{p}{E}$  of opposite nature at right angles to its direction.

Hence

$$e_x = \frac{p_x}{E} - \mu \frac{p_y}{E} - \mu \frac{p_z}{E}$$

$$e_y = -\mu \frac{p_x}{E} + \frac{p_y}{E} - \mu \frac{p_z}{E}$$

$$e_z = -\mu \frac{p_x}{E} - \mu \frac{p_y}{E} + \frac{p_z}{E}$$

Now,

$$e_v = e_x + e_y + e_z$$

$$= \frac{p_x}{E}(1 - 2\mu) + \frac{p_y}{E}(1 - 2\mu) + \frac{p_z}{E}(1 - 2\mu)$$

i.e., 
$$\frac{dV}{V} = \frac{(p_x + p_y + p_z)}{E}(1 - 2\mu)$$

Hence if  $p_x + p_y + p_z = 0$

$$e_v = \frac{dV}{V} = 0$$

$\therefore dV = 0$

i.e., there is no volumetric change, if  $p_x + p_y + p_z = 0$ .

**Example 8.17.** In a laboratory, tensile test is conducted and Young's modulus of the material is found to be  $2.1 \times 10^5 \text{ N/mm}^2$ . On the same material torsion test is conducted and modulus of rigidity is found to be  $0.78 \times 10^5 \text{ N/mm}^2$ . Determine Poisson's Ratio and bulk modulus of the material.

[Note: This is usual way of finding material properties in the laboratory].

**Solution:**

$$E = 2.1 \times 10^5 \text{ N/mm}^2$$

$$G = 0.78 \times 10^5 \text{ N/mm}^2$$

Using relation

$$E = 2G(1 + \mu)$$

we get

$$2.1 \times 10^5 = 2 \times 0.78 \times 10^5 (1 + \mu)$$

$$1.346 = 1 + \mu$$

or

$$\mu = 0.346$$

From relation

$$E = 3K(1 - 2\mu)$$

we get

$$2.1 \times 10^5 = 3 \times K(1 - 2 \times 0.346)$$

$$K = 2.275 \times 10^5 \text{ N/mm}^2$$

**Example 8.18.** A material has modulus of rigidity equal to  $0.4 \times 10^5 \text{ N/mm}^2$  and bulk modulus equal to  $0.8 \times 10^5 \text{ N/mm}^2$ . Find its Young's Modulus and Poisson's Ratio.

**Solution:**

$$G = 0.4 \times 10^5 \text{ N/mm}^2$$

$$K = 0.8 \times 10^5 \text{ N/mm}^2$$

Using the relation 
$$E = \frac{9GK}{3K + G}$$

$$E = \frac{9 \times 0.4 \times 10^5 \times 0.8 \times 10^5}{3 \times 0.8 \times 10^5 + 0.4 \times 10^5}$$

$$E = 1.0286 \times 10^5 \text{ N}$$

From the relation

$$E = 2G(1 + \mu)$$

we get

$$1.0286 \times 10^5 = 2 \times 0.4 \times 10^5 (1 + \mu)$$

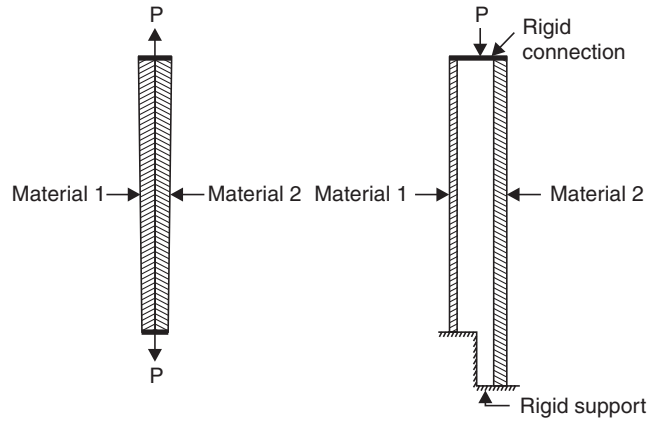
$$1.2857 = 1 + \mu$$

or

$$\mu = 0.2857$$

### 8.19 COMPOSITE/COMPOUND BARS

Bars made up of two or more materials are called composite/compound bars. They may have same length or different lengths as shown in Fig. 8.35. The ends of different materials of the bar are held together under loaded conditions.



**Fig. 8.35**

Consider a member with two materials. Let the load shared by material 1 be  $P_1$  and that by material 2 be  $P_2$ . Then

(i) From equation of equilibrium of the forces, we get

$$P = P_1 + P_2 \quad \dots(8.24a)$$

(ii) Since the ends are held securely, we get

$$\Delta l_1 = \Delta l_2$$

where  $\Delta l_1$  and  $\Delta l_2$  are the extension of the bars of material 1 and 2 respectively

$$\text{i.e.} \quad \frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L_2}{A_2 E_2} \quad \dots(8.24b)$$

Using equations 8.24(a) and (b),  $P_1$  and  $P_2$  can be found uniquely. Then extension of the system can be found using the relation  $\Delta l = \frac{P_1 L_1}{A_1 E_1}$  or  $\Delta l = \frac{P_2 L_2}{A_2 E_2}$  since  $\Delta l = \Delta l_1 = \Delta l_2$ .

The procedure of the analysis of compound bars is illustrated with the examples below:

**Example 8.19.** A compound bar of length 600 mm consists of a strip of aluminium 40 mm wide and 20 mm thick and a strip of steel 60 mm wide  $\times$  15 mm thick rigidly joined at the ends. If elastic modulus of aluminium and steel are  $1 \times 10^5 \text{ N/mm}^2$  and  $2 \times 10^5 \text{ N/mm}^2$ , determine the stresses developed in each material and the extension of the compound bar when axial tensile force of 60 kN acts.

**Solution:** The compound bar is shown in the figure 8.36.

Data available is

$$\begin{aligned} L &= 600 \text{ mm} \\ P &= 60 \text{ kN} = 60 \times 1000 \text{ N} \\ A_a &= 40 \times 20 = 800 \text{ mm}^2 \\ A_s &= 60 \times 15 = 900 \text{ mm}^2 \\ E_a &= 1 \times 10^5 \text{ N/mm}^2, E_s = 2 \times 10^5 \text{ N/mm}^2. \end{aligned}$$

Let the load shared by aluminium strip be  $P_a$  and that shared by steel be  $P_s$ . Then from equilibrium condition

$$P_a + P_s = 60 \times 1000 \quad \dots(1)$$

From compatibility condition, we have

$$\Delta_a = \Delta_s$$

$$\frac{P_a L}{A_a E_a} = \frac{P_s L}{A_s E_s}$$

$$\text{i.e.} \quad \frac{P_a \times 600}{800 \times 1 \times 10^5} = \frac{P_s \times 600}{900 \times 2 \times 10^5}$$

$$P_s = 2.25 P_a \quad \dots(2)$$

Substituting it in eqn. (1), we get

$$P_a + 2.25 P_a = 60 \times 1000$$

$$\text{i.e.} \quad P_a = 18462 \text{ N.}$$

$$\therefore P_s = 2.25 \times 18462 = 41538 \text{ N.}$$

$$\begin{aligned} \therefore \text{Stress in aluminium strip} &= \frac{P_a}{A_a} = \frac{18462}{800} \\ &= 23.08 \text{ N/mm}^2 \end{aligned}$$

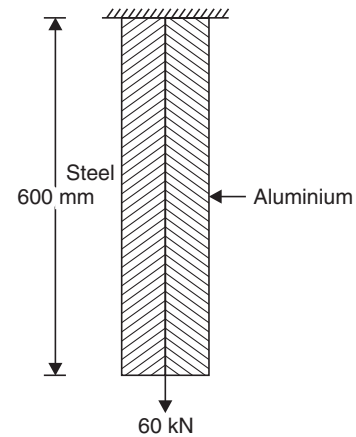
$$\text{Stress in steel strip} = \frac{P_s}{A_s} = \frac{41538}{900} = 46.15 \text{ N/mm}^2$$

$$\text{Extension of the compound bar} = \frac{P_a L}{A_a E_a} = \frac{18462 \times 600}{800 \times 1 \times 10^5}$$

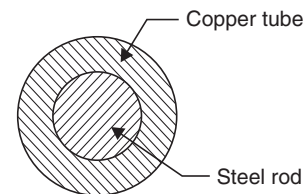
$$\Delta l = 0.138 \text{ mm.}$$

**Example 8.20.** A compound bar consists of a circular rod of steel of 25 mm diameter rigidly fixed into a copper tube of internal diameter 25 mm and external diameter 40 mm as shown in Fig. 8.37. If the compound bar is subjected to a load of 120 kN, find the stresses developed in the two materials.

$$\begin{aligned} \text{Take } E_s &= 2 \times 10^5 \text{ N/mm}^2 \\ \text{and } E_c &= 1.2 \times 10^5 \text{ N/mm}^2. \end{aligned}$$



**Fig. 8.36**



**Fig. 8.37**

**Solution:** Area of steel rod  $A_s = \frac{\pi}{4} \times 25^2 = 490.87 \text{ mm}^2$

Area of copper tube  $A_c = \frac{\pi}{4} (40^2 - 25^2) = 765.76 \text{ mm}^2$

From equation of equilibrium,

$$P_s + P_c = 120 \times 1000 \quad \dots(1)$$

where  $P_s$  is the load shared by steel rod and  $P_c$  is the load shared by the copper tube.

From compatibility condition, we have

$$\Delta_s = \Delta_c$$

$$\frac{P_s L}{A_s E_s} = \frac{P_c L}{A_c E_c}$$

$$\frac{P_s}{490.87 \times 2 \times 10^5} = \frac{P_c}{765.76 \times 1.2 \times 10^5}$$

$$\therefore P_s = 1.068 P_c \quad \dots(2)$$

From eqns. (1) and (2), we get

$$1.068 P_c + P_c = 120 \times 1000$$

$$\therefore P_c = \frac{120 \times 1000}{2.068} = 58027 \text{ N}$$

$$\therefore P_s = 1.068 P_c = 61973 \text{ N}$$

$$\therefore \text{Stress in copper} = \frac{58027}{9765.76} = 75.78 \text{ N/mm}^2$$

$$\text{Stress in steel} = \frac{61973}{490.87} = 126.25 \text{ N/mm}^2$$

**Example 8.21.** A reinforced concrete column of size  $300 \text{ mm} \times 500 \text{ mm}$  has 8 steel bars of 16 mm diameter as shown in Fig. 8.38. If the column is subjected to an axial compressive force of 800 kN, find the stresses developed in steel and concrete. Take  $\frac{E_s}{E_c} = 18$ .

**Solution:** Area of steel  $= 8 \times \frac{\pi}{4} \times 16^2 = 1608.5 \text{ mm}^2$

$$\therefore \text{Area of concrete} = 300 \times 500 - 1608.5 = 148391.5 \text{ mm}^2$$

From equilibrium condition,

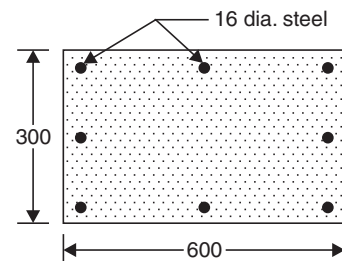
$$P_c + P_s = 800 \times 1000 \quad \dots(1)$$

From compatibility condition,

$$\Delta_s = \Delta_c$$

$$\frac{P_s L}{A_s E_s} = \frac{P_c L}{A_c E_c}$$

$$\frac{P_s}{1608.5 E_s} = \frac{P_c}{148391.5 E_c}$$



**Fig. 8.38**

$$\begin{aligned}
 P_c &= \frac{148391.5}{1608.5} \times \frac{E_c}{E_s} P_s \\
 &= \frac{148391.5}{1608.5} \times \frac{1}{18} P_s \\
 &= 5.125 P_s
 \end{aligned}
 \quad \dots(2)$$

From eqns. (1) and (2), we get

$$P_s + 5.125 P_s = 800 \times 1000$$

$$\therefore P_s = 130612 \text{ N.}$$

$$\text{Hence } P_c = 5.125 P_s = 669388 \text{ N}$$

$$\text{Hence stress in concrete} = \frac{P_c}{A_c} = \frac{669388}{148391.5} = 4.51 \text{ N/mm}^2$$

$$\text{and stress in steel} = \frac{P_s}{A_s} = \frac{130612}{1608.5} = 81.2 \text{ N/mm}^2$$

**Example 8.22.** Three pillars, two of aluminium and one of steel support a rigid platform of 250 kN as shown in Fig. 8.39. If area of each aluminium pillar is  $1200 \text{ mm}^2$  and that of steel pillar is  $1000 \text{ mm}^2$ , find the stresses developed in each pillar.

Take  $E_s = 2 \times 10^5 \text{ N/mm}^2$  and  $E_a = 1 \times 10^6 \text{ N/mm}^2$ .

**Solution:** Let force shared by each aluminium pillar be  $P_a$  and that shared by steel pillar be  $P_s$ .

$\therefore$  The forces in vertical direction = 0  $\rightarrow$

$$P_a + P_s + P_a = 250$$

$$2P_a + P_s = 250 \quad \dots(1)$$

From compatibility condition, we get

$$\Delta_s = \Delta_a$$

$$\frac{P_s L_s}{A_s E_s} = \frac{P_a L_a}{A_a E_a}$$

$$\frac{P_s \times 240}{1000 \times 2 \times 10^5} = \frac{P_a \times 160}{1200 \times 1 \times 10^6}$$

$$\therefore P_s = 1.111 P_a \quad \dots(2)$$

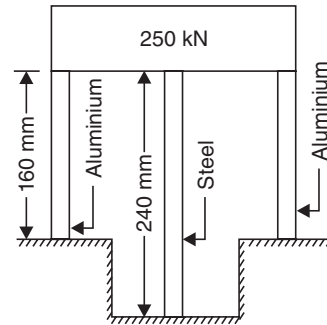
From eqns. (1) and (2), we get

$$P_a (2 + 1.111) = 250$$

$$\therefore P_a = 80.36 \text{ kN}$$

Hence from eqn. (1),

$$P_s = 250 - 2 \times 80.36 = 89.28 \text{ kN}$$



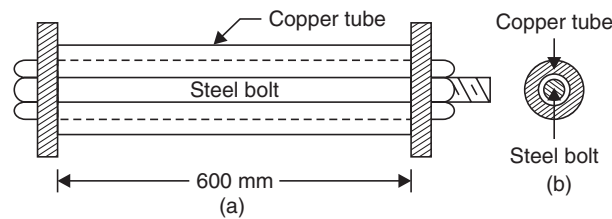
**Fig. 8.39**

∴ Stresses developed are

$$\sigma_s = \frac{P_s}{A_s} = \frac{89.28 \times 1000}{1000} = 89.28 \text{ N/mm}^2$$

$$\sigma_a = \frac{80.36 \times 1000}{1200} = 66.97 \text{ N/mm}^2$$

**Example 8.23.** A steel bolt of 20 mm diameter passes centrally through a copper tube of internal diameter 28 mm and external diameter 40 mm. The length of whole assembly is 600 mm. After tight fitting of the assembly, the nut is over tightened by quarter of a turn. What are the stresses introduced in the bolt and tube, if pitch of nut is 2 mm? Take  $E_s = 2 \times 10^5 \text{ N/mm}^2$  and  $E_c = 1.2 \times 10^5 \text{ N/mm}^2$ .



**Fig. 8.40**

**Solution:** Figure 8.40 shows the assembly. Let the force shared by bolt be  $P_s$  and that by tube be  $P_c$ . Since there is no external force, static equilibrium condition gives

$$P_s + P_c = 0 \quad \text{or} \quad P_s = -P_c$$

i.e., the two forces are equal in magnitude but opposite in nature. Obviously bolt is in tension and tube is in compression.

Let the magnitude of force be  $P$ . Due to quarter turn of the nut, the nut advances by  $\frac{1}{4} \times \text{pitch}$   
 $= \frac{1}{4} \times 2 = 0.5 \text{ mm}.$

[Note. Pitch means advancement of nut in one full turn]

During this process bolt is extended and copper tube is shortened due to force  $P$  developed. Let  $\Delta_s$  be extension of bolt and  $\Delta_c$  shortening of copper tube. Final position of assembly be  $\Delta$ , then

$$\Delta_s + \Delta_c = \Delta$$

$$\text{i.e.} \quad \frac{P_s L_s}{A_s E_s} + \frac{P_c L_c}{A_c E_c} = 0.5$$

$$\frac{P \times 600}{(\pi/4) \times 20^2 \times 2 \times 10^5} + \frac{P \times 600}{(\pi/4) (40^2 - 28^2) \times 1.2 \times 10^5} = 0.5$$

$$\frac{P \times 600}{(\pi/4) \times 10^5} \left[ \frac{1}{20^2 \times 2} + \frac{1}{(40^2 - 28^2) \times 1.2} \right] = 0.5$$

∴

$$P = 28816.8 \text{ N}$$

$$\therefore p_s = \frac{P_s}{A_s} = \frac{28816.8}{(\pi / 4) \times 20^2} = 91.72 \text{ N/mm}^2$$

$$p_c = \frac{P_c}{A_c} = \frac{28816.8}{(\pi / 4) (40^2 - 28^2)} = 44.96 \text{ N/mm}^2$$

## 8.20 THERMAL STRESSES

Every material expands when temperature rises and contracts when temperature falls. It is established experimentally that the change in length  $\Delta$  is directly proportional to the length of the member  $L$  and change in temperature  $t$ . Thus

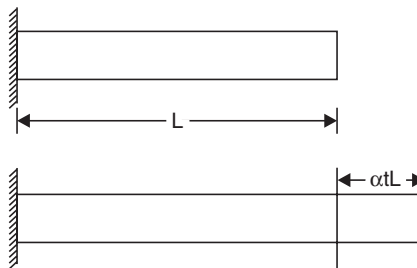
$$\begin{aligned} \Delta &\propto tL \\ &= \alpha tL \end{aligned} \quad \dots(8.25)$$

The constant of proportionality  $\alpha$  is called coefficient of thermal expansion and is defined as change in unit length of material due to unit change in temperature. Table 8.1 shows coefficient of thermal expansion for some of the commonly used engineering materials:

**Table 8.1**

Material	Coefficient of thermal expansion
Steel	$12 \times 10^{-6}/^\circ\text{C}$
Copper	$17.5 \times 10^{-6}/^\circ\text{C}$
Stainless steel	$18 \times 10^{-6}/^\circ\text{C}$
Brass, Bronze	$19 \times 10^{-6}/^\circ\text{C}$
Aluminium	$23 \times 10^{-6}/^\circ\text{C}$

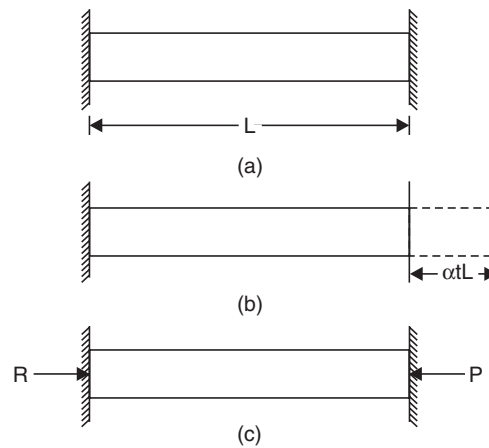
If the expansion of the member is freely permitted, as shown in Fig. 8.41, no temperature stresses are induced in the material.



**Fig. 8.41** Free Expansion Permitted

If the free expansion is prevented fully or partially the stresses are induced in the bar, by the support forces. Referring to Fig. 8.42,



**Fig. 8.42**

If free expansion is permitted the bar would have expanded by

$$\Delta = \alpha tL$$

Since support is not permitting it, the support force  $P$  develops to keep it at the original position. Magnitude of this force is such that contraction is equal to free expansion, *i.e.*

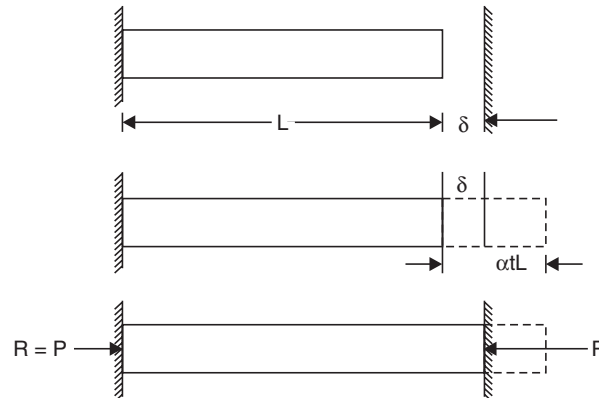
$$\frac{PL}{AE} = \alpha tL$$

or

$$p = E \alpha t. \quad \dots(8.26)$$

which is the temperature stress. It is compressive in nature in this case.

Consider the case shown in Fig. 8.43 in which free expansion is prevented partially.

**Fig. 8.43**

In this case free expansion  $= \alpha tL$

Expansion prevented  $\Delta = \alpha tL - \delta$

The expansion is prevented by developing compressive force  $P$  at supports

$$\therefore \frac{PL}{AE} = \Delta = \alpha tL - \delta. \quad \dots(8.27)$$

**Example 8.24.** A steel rail is 12 m long and is laid at a temperature of  $18^\circ\text{C}$ . The maximum temperature expected is  $40^\circ\text{C}$ .

- (i) Estimate the minimum gap between two rails to be left so that the temperature stresses do not develop.
- (ii) Calculate the temperature stresses developed in the rails, if:
  - (a) No expansion joint is provided.
  - (b) If a 1.5 mm gap is provided for expansion.
- (iii) If the stress developed is  $20 \text{ N/mm}^2$ , what is the gap provided between the rails?  
Take  $E = 2 \times 10^5 \text{ N/mm}^2$  and  $\alpha = 12 \times 10^{-6}/^\circ\text{C}$ .

**Solution:**

- (i) The free expansion of the rails

$$\begin{aligned} &= \alpha tL = 12 \times 10^{-6} \times (40 - 18) \times 12.0 \times 1000 \\ &= 3.168 \text{ mm} \end{aligned}$$

**$\therefore$  Provide a minimum gap of 3.168 mm between the rails, so that temperature stresses do not develop.**

- (ii) (a) If no expansion joint is provided, free expansion prevented is equal to 3.168 mm.

i.e.  $\Delta = 3.168 \text{ mm}$

$$\therefore \frac{PL}{AE} = 3.168$$

$$\therefore p = \frac{P}{A} = \frac{3.168 \times 2 \times 10^5}{12 \times 1000} = 52.8 \text{ N/mm}^2$$

- (b) If a gap of 1.5 mm is provided, free expansion prevented  $\Delta = \alpha tL - \delta = 3.168 - 1.5 = 1.668 \text{ mm}$ .

$$\therefore \text{The compressive force developed is given by } \frac{PL}{AE} = 1.668$$

or 
$$p = \frac{P}{A} = \frac{1.668 \times 2 \times 10^5}{12 \times 1000} = 27.8 \text{ N/mm}^2$$

- (iii) If the stress developed is  $20 \text{ N/mm}^2$ , then  $p = \frac{P}{A} = 20$

If  $\delta$  is the gap,  $\Delta = \alpha tL - \delta$

$$\therefore \frac{PL}{AE} = 3.168 - \delta$$

i.e.  $20 \times \frac{12 \times 1000}{2 \times 10^5} = 3.168 - \delta$

$$\therefore \delta = 3.168 - 1.20 = 1.968 \text{ mm}$$

**Example 8.25.** The temperature of a steel ring is raised through  $150^\circ\text{C}$  in order to fit it on a wooden wheel of 1.2 m diameter. Find the original diameter of the steel ring and also the stresses developed in the ring, when it cools back to normal temperature. Assume  $E_s = 2 \times 10^5 \text{ N/mm}^2$  and  $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$ .

**Solution:** Let  $D$  be the diameter of ring after heating and ' $d$ ' be its diameter before heating

$$\therefore D = 1.2 \text{ m} = 1200 \text{ mm.}$$

Circumference of ring after heating  $= \pi D$

Circumference of ring before heating ( $L$ )  $= \pi d$

$$\therefore \pi D = \pi d + \alpha_s t (\pi d)$$

$$D = d + \alpha_s t d = (1 + \alpha_s t) d$$

$$\text{i.e., } 1200 = [1 + 12 \times 10^{-6} \times 150] d$$

$$\therefore d = 1197.84 \text{ mm.}$$

when it cools expansion prevented

$$\begin{aligned} \Delta &= \pi(D - d) = \alpha_s t \pi d = 12 \times 10^{-6} \times 150 \pi d \\ &= 1800 \times 10^{-6} \pi d \end{aligned}$$

$$\therefore \frac{PL}{AE} = 1800 \times 10^{-6} \pi d$$

$$\left(\frac{P}{A}\right) \times \frac{\pi \times d}{E} = 1800 \times 10^{-6} \pi d$$

$$\text{Stress } p = \frac{P}{A} = 1800 \times 10^{-6} \times 2 \times 10^5 = 360 \text{ N/mm}^2$$

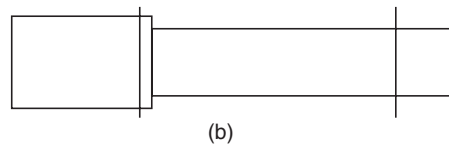
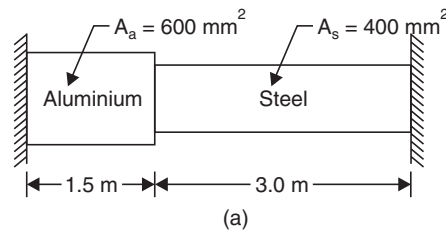
**Example 8.26.** The composite bar shown in Fig. 8.44 is rigidly fixed at the ends A and B. Determine the reaction developed at ends when the temperature is raised by  $18^\circ\text{C}$ . Given

$$E_a = 70 \text{ kN/mm}^2$$

$$E_s = 200 \text{ kN/mm}^2$$

$$\alpha_a = 11 \times 10^{-6}/^\circ\text{C}$$

$$\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$$



**Fig. 8.44**

**Solution:** Free expansion =  $\alpha_a tL_a + \alpha_s tL_s$   
 $= 11 \times 10^{-6} \times 18 \times 1500 + 12 \times 10^{-6} \times 18 \times 3000$   
 $= 0.945 \text{ mm}$

Since this is prevented

$$\Delta = 0.945 \text{ mm.}$$

$$E_a = 70 \text{ kN/mm}^2 = 70000 \text{ N/mm}^2 ;$$

$$E_s = 200 \text{ kN/mm}^2 = 200 \times 1000 \text{ N/mm}^2$$

If  $P$  is the support reaction,

$$\Delta = \frac{PL_a}{A_a E_a} + \frac{PL_s}{A_s E_s}$$

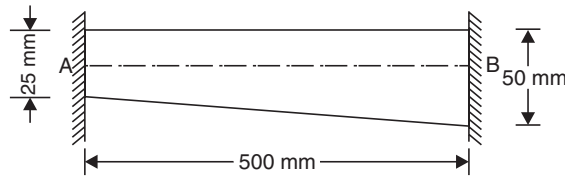
i.e.  $0.945 = P \left[ \frac{1500}{600 \times 70000} + \frac{3000}{400 \times 200 \times 1000} \right]$

$$0.945 = 73.214 \times 10^{-6} P$$

or

$$P = 12907.3 \text{ N}$$

**Example 8.27.** The steel bar AB shown in Fig. 8.45 varies linearly in diameter from 25 mm to 50 mm in a length 500 mm. It is held between two unyielding supports at room temperature. What is the stress induced in the bar, if temperature rises by  $25^\circ\text{C}$ ? Take  $E_s = 2 \times 10^5 \text{ N/mm}^2$  and  $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$ .



**Fig. 8.45**

**Solution:** Free expansion of bar =  $\alpha tL$   
 $= 12 \times 10^{-6} \times 25 \times 500$   
 $= 0.15 \text{ mm.}$

This expansion is prevented  $\therefore \Delta = 0.15 \text{ mm.}$

If  $P$  is the force developed by supports, then it can cause a contraction of

$$\frac{4PL}{\pi d_1 d_2 E} = \frac{4 \times P \times 500}{\pi \times 25 \times 50 \times 2 \times 10^5}$$

Equating it to the contraction prevented, for satisfying the compatibility, we get

$$\frac{4P \times 500}{\pi \times 25 \times 50 \times 2 \times 10^5} = 0.15$$

$$\therefore P = 58905 \text{ N}$$

$$\therefore \text{Corresponding maximum stress} = \frac{P}{A_{\min}} = \frac{58905}{\frac{\pi}{4} \times 25^2}$$

$$= 120 \text{ N/mm}^2.$$

### 8.21 THERMAL STRESSES IN COMPOUND BARS

When temperature rises the two materials of the compound bar experience different free expansion. Since they are prevented from separating, the two bars will have common position. This is possible only by extension of the bar which has less free expansion and contraction of the bar which has more free expansion. Thus one bar develops tensile force and another develops the compressive force. In this article we are interested to find such stresses.

Consider the compound bar shown in Fig. 8.46(a). Let  $\alpha_1, \alpha_2$  be coefficient of thermal expansion and  $E_1, E_2$  be moduli of elasticity of the two materials respectively. If rise in temperature is ' $t$ ',

$$\text{Free expansion of bar 1} = \alpha_1 tL$$

$$\text{Free expansion of bar 2} = \alpha_2 tL$$

Let  $\alpha_1 > \alpha_2$ . Hence the position of the two bars, if the free expansions are permitted are at AA and BB as shown in Fig. 8.46(b)

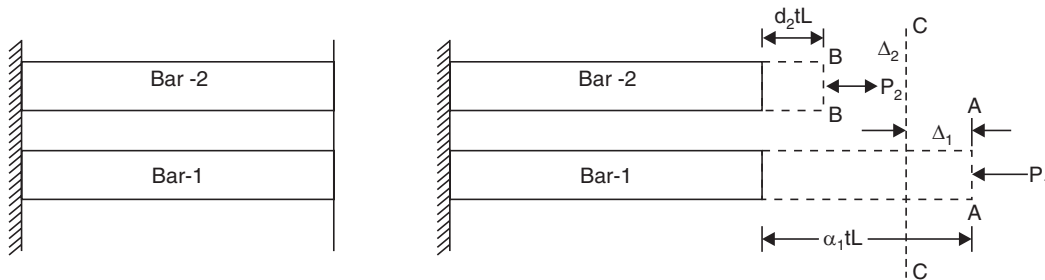


Fig. 8.46

Since the two bars are rigidly connected at the ends, the final position of the end will be somewhere between AA and BB, say at CC. It means Bar-1 will experience compressive force  $P_1$  which contracts it by  $\Delta_1$  and Bar-2 experience tensile force  $P_2$  which will expand it by  $\Delta_2$ .

The equilibrium of horizontal forces gives,

$$P_1 = P_2, \text{ say } P$$

From the Fig. 8.46 (b), it is clear,

$$\alpha_1 tL - \Delta_1 = \alpha_2 tL + \Delta_2$$

$$\therefore \Delta_1 + \Delta_2 = \alpha_1 tL - \alpha_2 tL = (\alpha_1 - \alpha_2) tL.$$

If the cross-sectional areas of the bars are  $A_1$  and  $A_2$ , we get

$$\frac{PL}{A_1 E_1} + \frac{PL}{A_2 E_2} = (\alpha_1 - \alpha_2) tL \quad \dots(8.28)$$

From the above equation force  $P$  can be found and hence the stresses  $P_1$  and  $P_2$  can be determined.

**Example 8.28.** A bar of brass 20 mm is enclosed in a steel tube of 40 mm external diameter and 20 mm internal diameter. The bar and the tubes are initially 1.2 m long and are rigidly fastened at both ends using 20 mm diameter pins. If the temperature is raised by 60°C, find the stresses induced in the bar, tube and pins.

Given:

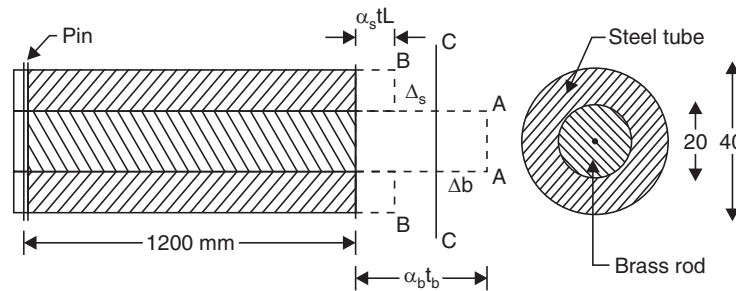
$$E_s = 2 \times 10^5 \text{ N/mm}^2$$

$$E_b = 1 \times 10^5 \text{ N/mm}^2$$

$$\alpha_s = 11.6 \times 10^{-6}/^\circ\text{C}$$

$$\alpha_b = 18.7 \times 10^{-6}/^\circ\text{C}$$

**Solution:**



**Fig. 8.47**

$$t = 60^\circ \quad E_s = 2 \times 10^5 \text{ N/mm}^2 \quad E_b = 1 \times 10^5 \text{ N/mm}^2$$

$$\alpha_s = 11.6 \times 10^{-6}/^\circ\text{C} \quad \alpha_b = 18.7 \times 10^{-6}/^\circ\text{C}$$

$$A_s = \frac{\pi}{4}(40^2 - 20^2) \quad A_b = \frac{\pi}{4} \times 20^2$$

$$= 942.48 \text{ mm}^2 \quad = 314.16 \text{ mm}^2$$

Since free expansion of brass ( $\alpha_b tL$ ) is more than free expansion of steel ( $\alpha_s tL$ ), compressive force  $P_b$  develops in brass and tensile force  $P_s$  develops in steel to keep the final position at CC (Ref: Fig. 8.47).

Horizontal equilibrium condition gives  $P_b = P_s$ , say  $P$ . From the figure, it is clear that

$$\Delta_s + \Delta_b = \alpha_b tL - \alpha_s tL = (\alpha_b - \alpha_s)tL$$

where  $\Delta_s$  and  $\Delta_b$  are the changes in length of steels and brass bars.

$$\therefore \frac{PL}{A_s E_s} + \frac{PL}{A_b E_b} = (18.7 - 11.6) \times 10^{-6} \times 60 \times 1200.$$

$$P \times 1200 \left[ \frac{1}{942.48 \times 2 \times 10^5} + \frac{1}{314.16 \times 1 \times 10^5} \right] = 7.1 \times 10^{-6} \times 60 \times 1200$$

$$\therefore P = 11471.3 \text{ N}$$

$$\therefore \text{Stress in steel} = \frac{P}{A_s} = \frac{11471.3}{942.48} = 12.17 \text{ N/mm}^2$$

and  $\text{Stress in brass} = \frac{P}{A_b} = \frac{11471.3}{314.16} = 36.51 \text{ N/mm}^2$

The pin resist the force  $P$  at the two cross-sections at junction of two bars.

$$\begin{aligned}\therefore \text{Shear stress in pin} &= \frac{P}{2 \times \text{Area of pin}} \\ &= \frac{11471.3}{2 \times \pi/4 \times 20^2} = 18.26 \text{ N/mm}^2\end{aligned}$$

**Example 8.29.** A compound bar is made of a steel plate 50 mm wide and 10 mm thick to which copper plates of size 40 mm wide and 5 mm thick are connected rigidly on each side as shown in Fig. 8.48. The length of the bar at normal temperature is 1 m. If the temperature is raised by  $80^\circ$ , determine the stresses in each metal and the change in length. Given  $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$ ,  $\alpha_c = 17 \times 10^{-6}/^\circ\text{C}$ ,  $E_s = 2 \times 10^5 \text{ N/mm}^2$ ,  $E_c = 1 \times 10^5 \text{ N/mm}^2$ .

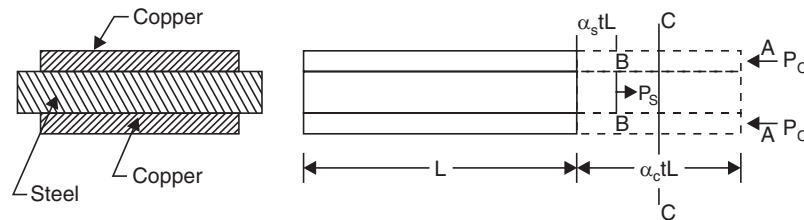


Fig. 8.48

**Solution:** Now,  $L = 1000 \text{ mm}$ ,  $A_s = 50 \times 10 = 500 \text{ mm}^2$ ,  $A_c =$  Free expansion of copper  $\alpha_c t L$  is greater than free expansion of steel  $\alpha_s t L$ . To bring them to the same position, tensile force  $P_s$  acts on steel plate and compressive force  $P_c$  acts on each copper plate.

$$\therefore P_s = 2P_c$$

If  $\Delta_s$  and  $\Delta_c$  are changes in the length of steel and copper plates.

$$\Delta_s + \Delta_c = (\alpha_c - \alpha_s) t L$$

$$\therefore \frac{P_s L}{A_s E_s} + \frac{P_c L}{A_c E_c} = (\alpha_c - \alpha_s) t L$$

But  $P_s = 2P_c$

$$\therefore \frac{2P_c}{500 \times 2 \times 10^5} + \frac{P_c}{200 \times 1 \times 10^5} = (17 - 12) \times 10^{-6} \times 80$$

$$\therefore P_c = 5714.28 \text{ N}$$

$$\therefore P_s = 11428.56 \text{ N}$$

$$\therefore \text{Stress in copper} = \frac{P_c}{A_c} = \frac{5714.28}{200} = 28.57 \text{ N/mm}^2$$

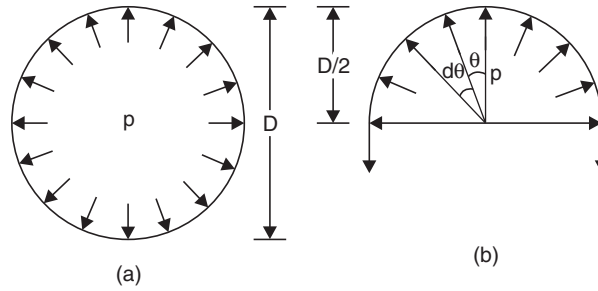
$$\text{Stress in steel} = \frac{P_s}{A_s} = \frac{5714.28 \times 2}{500} = 22.86 \text{ N/mm}^2$$

Changed length if compound bar is represented by line  $CC$  in the figure.

$$\begin{aligned}
 \therefore \text{Change in length} &= \alpha_s t L + \delta_s \\
 &= \alpha_s t L + \frac{P_s L_s}{A_s E_s} \\
 &= 12 \times 10^{-6} \times 80 \times 1000 + \frac{11428.56 \times 1000}{500 \times 2 \times 10^5} \\
 &= 1.07 \text{ mm}
 \end{aligned}$$

## 8.22 HOOP STRESSES

The internal or external pressure applied to thin cylinders is resisted by stresses developed in the circumferential direction of the cylinder. This type of stress is called hoop stress. Consider a thin cylinder of diameter subject to internal press  $p$  as shown in Fig. 8.49(a).



**Fig. 8.49**

Consider half the section shown in Fig. 8.49(b). The normal pressure on the element of unit length as shown in Fig. 8.49(b)

$$= p \times \frac{D}{2} d\theta$$

$$\text{Its vertical component} = \frac{pD}{2} d\theta \cos \theta$$

$\therefore$  Bursting force normal to the horizontal section

$$\begin{aligned}
 &= 2 \int_0^{\pi/2} p \frac{d}{2} \cos \theta d\theta = pd [\sin \theta]_0^{\pi/2} \\
 &= pd.
 \end{aligned}$$

This bursting force is resisted by the hoop stresses  $f$ . Let ' $t$ ' be the thickness of cylinder. Then resisting force per unit length of cylinder.

$$= 2 \times ft$$

Equating resisting force to bursting force, we get

$$2ft = pd$$

$$\text{or} \quad f = \frac{pd}{2t} \quad \dots(8.29)$$



**Example 8.30.** A thin cylinder of internal diameter  $D = 1$  m and thickness  $t = 12$  mm is subjected to internal pressure of  $2$  N/mm<sup>2</sup>. Determine the hoop stress developed.

**Solution:** Hoop stress  $f = \frac{pd}{2t} = \frac{2 \times 1000}{2 \times 12} = 83.33$  N/mm<sup>2</sup>

**IMPORTANT FORMULAE**

1. If stress is uniform

$$p = \frac{P}{A}$$

2. (i) Linear strain =  $\frac{\text{Change in length}}{\text{Original length}}$

(ii) Lateral strain =  $\frac{\text{Change in lateral dimension}}{\text{Original lateral dimension}}$

3. Poisson's ratio =  $\frac{\text{Lateral strain}}{\text{Linear strain}}$ , within elastic limit.

4. Percentage elongation =  $\frac{L' - L}{L} \times 100$ .

5. Percentage reduction in area =  $\frac{A - A'}{A} \times 100$ .

6. Nominal stress =  $\frac{\text{Load}}{\text{Original cross-sectional area}}$ .

7. True stress =  $\frac{\text{Load}}{\text{Actual cross-sectional area}}$ .

8. Factor of safety =  $\frac{\text{Ultimate stress}}{\text{Working stress}}$

However in case of steel, =  $\frac{\text{Yield stress}}{\text{Working stress}}$ .

9. Hooke's Law,  $p = Ee$ .

10. Extension/shortening of bar =  $\frac{PL}{AE}$ .

11. Extension of flat bar with linearly varying width and constant thickness =  $\frac{PL}{tE(b_1 - b_2)} \log \frac{b_1}{b_2}$ .

12. Extension of linearly tapering rod =  $\frac{4PL}{\pi E d_1 d_2} = \frac{PL}{(\pi/4 d_1 d_2) E}$ .

13. Direct shear stress =  $\frac{Q}{A}$ .

14. Volumetric strain  $e_v = \frac{\delta V}{V} = e_x + e_y + e_z$ .

15.  $E = 2G(1 + \mu) = 3K(1 - 2\mu)$

or  $\frac{9}{E} = \frac{3}{G} + \frac{1}{K}$ .

16. Extension due to rise in temperature:

$$\Delta = \alpha tL$$

17. Thermal force,  $P$  is given by

$$\frac{PL}{AE} = \text{extension prevented.}$$

### THEORY QUESTIONS

1. Explain the terms: stress, strain, modulus of elasticity.
2. Draw a typical stress-strain curve for mild steel, indicate salient point and define them.
3. Distinguish between
  - (i) Elastic limit and yield point.
  - (ii) Nominal stress and true stress.
4. Explain the term factor of safety.
5. A bar of uniform thickness  $t$  tapers uniformly from a width  $b_1$  at one end to  $b_2$  at the other end in a length  $L$ . Find the expression for its extension under an axial pull  $P$ .
6. A tapering rod has diameter  $d_1$  at one end and it tapers uniformly to a diameter  $d_2$  at the other end in a length  $L$ . If the modulus of elasticity is  $E$ , find the change in length when subjected to an axial force  $P$ .
7. Show that pure shear gives rise to tensile and compressive stresses across the planes inclined at  $45^\circ$  to the shearing planes, the intensity of the direct stresses being of the same magnitude as shearing stress.
8. Show that volumetric strain in a bar is equal to the sum of strains in three mutually perpendicular directions.
9. Explain the terms:
  - (i) Modulus of elasticity
  - (ii) Modulus of rigidity and
  - (iii) Bulk modulus.
10. Derive the relationship between
  - (i) Modulus of elasticity and modulus of rigidity
  - (ii) Modulus of elasticity and bulk modulus.
11. Show that in a compound bar of length  $L$ , when temperature increases by  $t$ , the force ' $P$ ' developed is given by

$$\frac{PL}{A_1 E_1} + \frac{PL}{A_2 E_2} = (\alpha_1 - \alpha_2) tL$$

where  $A_1, A_2$ —Cross-sectional areas of bar 1 and bar 2 respectively  
 $E_1, E_2$ —Young's moduli of bar 1 and bar 2 respectively and  
 $\alpha_1, \alpha_2$ —are coefficient of thermal expansion of bars 1 and 2 respectively.

12. Show that in a thin cylinder subject to internal pressure ' $p$ ' is given by

$$f = \frac{pd}{2t}$$

where  $d$ —diameter of the cylinder  
 $t$ —thickness of cylinder.

### PROBLEMS FOR EXERCISE

1. Tension test was conducted on a specimen and the following readings were recorded.  
 Diameter = 25 mm  
 Gauge length of extensometer = 200 mm  
 Least count of extensometer = 0.001 mm  
 At a load of 30 kN, extensometer reading = 60  
 At a load of 50 kN, extensometer reading = 100  
 Yield load = 160 kN  
 Maximum load = 205 kN  
 Diameter neck = 17 mm  
 Final extension over 125 mm original length = 150 mm  
 Find Young's Modulus, yield stress, ultimate stress, percentage elongation and percentage reduction in area.  
 [Ans.  $E = 2.0372 \times 10^5 \text{ N/mm}^2$ ,  $p_y = 325.95 \text{ N/mm}^2$ . Ultimate stress =  $417.62 \text{ N/mm}^2$ ,  
 % elongation = 20% reduction in area = 53.76]
2. The composite bar shown in Fig. 8.50 is subjected to a tensile force of 30 kN. The extension observed is 0.44. Find the Young's modulus of brass, if Young's modulus of steel is  $2 \times 10^5 \text{ N/mm}^2$ .  
 [Ans.  $99745 \text{ N/mm}^2$ ]

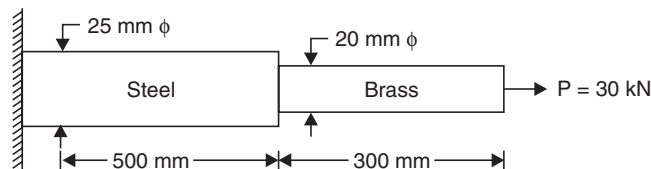
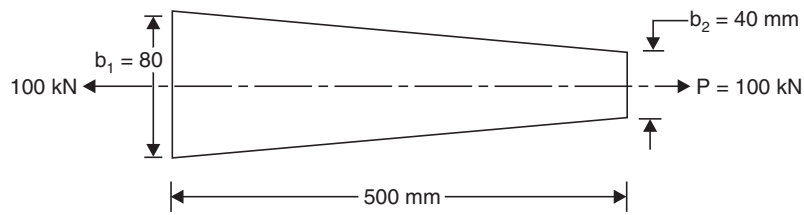


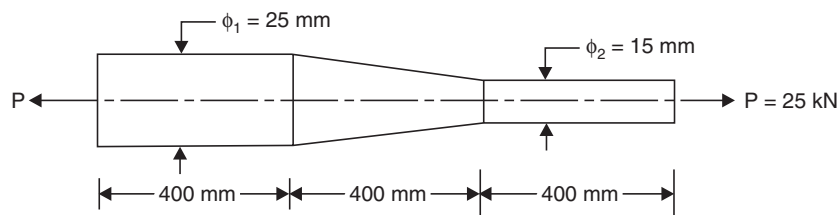
Fig. 8.50

3. The steel flat shown in Fig. 8.51 has uniform thickness of 20 mm. Under an axial load of 100 kN, its extension is found to be 0.21 mm. Determine the Young's modulus of the material.  
 [Ans.  $E = 2.06293 \times 10^5 \text{ N/mm}^2$ ]

**Fig. 8.51**

4. Find the extension of the bar shown in Fig. 8.52 under an axial load of 25 kN.

[Ans.  $\Delta = 0.555$  mm]

**Fig. 8.52**

5. The compound bar shown in Fig. 8.54 consists of three materials and supports a rigid horizontal platform. Find the stresses developed in each bar, if the platform remains horizontal even after the loading with 100 kN. Given

	Bar 1	Bar 2	Bar 3
Length	1200 mm	1500 mm	1800 mm
C.S. Area	800 mm <sup>2</sup>	1000 mm <sup>2</sup>	1200 mm <sup>2</sup>
Young's modulus	$2 \times 10^5$ N/mm <sup>2</sup>	$1.2 \times 10^5$ N/mm <sup>2</sup>	$1.0 \times 10^5$ N/mm <sup>2</sup>

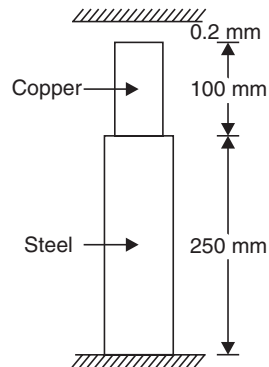
[Hint:  $\Delta_1 = \Delta_2 = \Delta_3 = \Delta$ , Find  $P_2 - P_3$  in terms of  $P_1$ .  $P_1 + P_2 + P_3 = 100 \times 10^3$ ]

[Ans.  $p_1 = 59.52$  N/mm<sup>2</sup>,  $p_2 = 28.57$  N/mm<sup>2</sup>,  $p_3 = 19.84$  N/mm<sup>2</sup>]

6. The composite bar shown in Fig. 8.53 is 0.20 mm short of distance between the rigid supports at room temperature. What is the maximum temperature rise which will not produce stresses in the bar? Find the stresses induced in the bar when temperature rise is 60°. Given:

$$\begin{aligned} \alpha_s &= 12 \times 10^{-6}/^\circ\text{C} & \alpha_c &= 17.5 \times 10^{-6}/^\circ\text{C} \\ E_s &= 2 \times 10^5 \text{ N/mm}^2 & E_c &= 1.2 \times 10^5 \text{ N/mm}^2 \\ A_s &= 400 \text{ mm}^2 & A_c &= 300 \text{ mm}^2 \end{aligned}$$

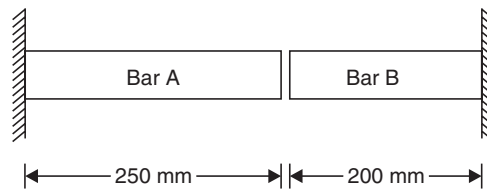
[Ans. (a) 42.10 °C, (b)  $p_s = 36$  N/mm<sup>2</sup>,  $p_c = 48$  N/mm<sup>2</sup>]

**Fig. 8.53**

7. At room temperature the gap between bar A and bar B shown in Fig. 8.54 is 0.2 mm. What are the stresses developed in the bars, if temperature rise is  $30^\circ\text{C}$  ? Given:

$$\begin{aligned} A_a &= 800 \text{ mm}^2 & A_b &= 600 \text{ mm}^2 \\ E_a &= 2 \times 10^5 \text{ N/mm}^2 & E_b &= 1 \times 10^5 \text{ N/mm}^2 \\ \alpha_a &= 12 \times 10^{-6} / ^\circ\text{C} & \alpha_b &= 23 \times 10^{-6} / ^\circ\text{C} \\ L_a &= 250 \text{ mm} & L_b &= 200 \text{ mm}. \end{aligned}$$

[Ans.  $p_a = 7.15 \text{ N/mm}^2$  ;  $p_b = 9.53 \text{ N/mm}^2$ ]

**Fig. 8.54**

### 9.1 INTRODUCTION

A beam may be defined as a structural element which has one dimension considerably larger than the other two dimensions, namely breadth and depth, and is supported at few points. The distance between two adjacent supports is called span. It is usually loaded normal to its axis. The applied loads make every cross-section to face bending and shearing. The load finally get transferred to supports. The system of forces consisting of applied loads and reactions keep the beam in equilibrium. The reactions depend upon the type of supports and type of loading. In this chapter type of supports, types of beams and types of loading are first explained and then the methods of finding reactions, bending moment and shear forces are illustrated for the following determinate beams:

- (a) Simply supported beams
- (b) Cantilever beams and
- (c) Overhanging beams.

### 9.2 TYPES OF SUPPORTS

Various types of supports and reactions developed are listed below:

**Simple Support:** If the beam rests simply on a support it is called a simple support. In such case the reaction at the support is at right angles to the support and the beam is free to move in the direction of its axis and also it is free to rotate about the support (Fig. 9.1).

**Roller Support:** In this case, beam end is supported on rollers. In such cases, reaction is normal to the support since rollers can be treated as frictionless. [Fig. 9.2 (a)]. Many mechanical components are having roller supports which roll between guides. In such cases, reaction will be normal to the guides, in both the direction (Fig. 9.2). At roller support beam is free to move along the support. It can rotate about the support also.

**Hinged Support:** At a hinged end, a beam cannot move in any direction. However, it can rotate about the support (Fig. 9.3). Hence the support will not develop any resisting moment, but it can develop reaction in any direction to keep the end stationary. The reaction  $R$  can be split into its horizontal and vertical components for the purpose of analysis.

Thus, in Fig. 9.3,

$$V_A = R \sin \theta$$

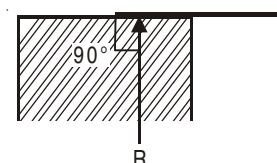


Fig. 9.1

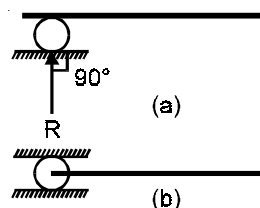


Fig. 9.2

$$H_A = R \cos \theta$$

**Fixed Support:** At such supports, the beam end is not free to translate or rotate. Translation is prevented by developing support reaction in any required direction.

Referring to Fig. 9.4 the support reaction  $R$  which is at an angle  $\theta$  to  $x$  axis may be represented by  $H_A$  and  $V_A$ , where

$$V_A = R \sin \theta$$

$$H_A = R \cos \theta$$

Rotation is prevented by developing support moment  $M_A$  as shown in Fig. 9.4. Thus at fixed support  $A$ , there are three reactions  $H_A$ ,  $V_A$  and  $M_A$ .

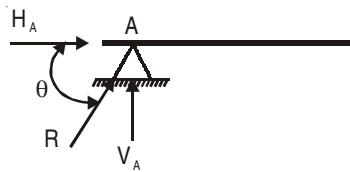


Fig. 9.3

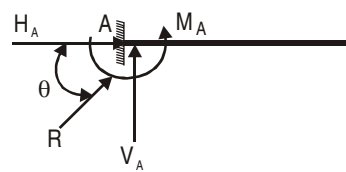


Fig. 9.4

### 9.3 TYPES OF BEAMS

**Simply Supported Beam:** When both end of a beam are simply supported it is called simply supported beam (Fig. 9.5). Such a beam can support load in the direction normal to its axis.

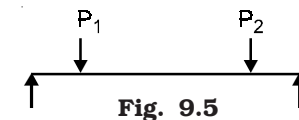


Fig. 9.5

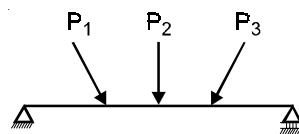


Fig. 9.6

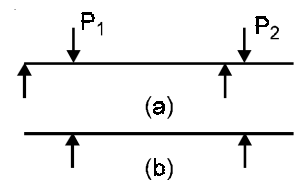


Fig. 9.7

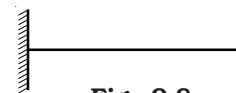


Fig. 9.8

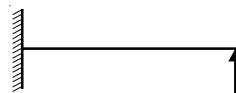


Fig. 9.9

**Beam with One End Hinged and the Other on Rollers:** If one end of a beam is hinged and other end is on rollers, the beam can resist load in any direction (see Fig. 9.6).

**Over-hanging Beam:** If a beam is projecting beyond the support. It is called an over-hanging beam (Fig. 9.7). The overhang may be only on one side as in Fig. 9.7(a) or may be on both sides as in Fig. 9.7(b).

**Cantilever Beam:** If a beam is fixed at one end and is free at the other end, it is called cantilever beam (Fig. 9.8).

**Propped Cantilever:** It is a beam with one end fixed and the other end simply supported (Fig. 9.9).

**Both Ends Hinged:** In these beams both ends will be having hinged supports (Fig. 9.10).

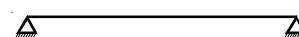


Fig. 9.10

**Continuous Beam:** A beam is said to be continuous, if it is supported at more than two points (Fig. 9.11).

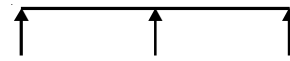


Fig. 9.11

In the case of simply supported beams, beams with one end hinged and the other on rollers, cantilever and over-hanging beams, it is possible to determine the reactions for given loadings by using the equations of equilibrium only. In the other cases, the number of independent equilibrium equations are less than the number of unknown reactions and hence it is not possible to analyse them by using equilibrium equations alone. The beams which can be analysed using only equilibrium equations are known as *Statically Determinate* beams and those which cannot be analysed are known as *Statically Indeterminate* beams. The latter beams can be analysed using the conditions of continuity in deformations in addition to equilibrium equations. Such cases will not be treated in this book.

#### 9.4 TYPES OF LOADING

Usual types of loadings on the beams are discussed here.

**Concentrated Loads:** If a load is acting on a beam over a very small length, it is approximated as acting at the mid point of that length and is represented by an arrow as shown in Fig. 9.12.

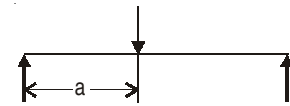


Fig. 9.12

**Uniformly Distributed Load (UDL):** Over considerably long distance such load has got uniform intensity. It is represented as shown in Fig. 9.13 (a) or as in (b). For finding reaction, this load may be assumed as total load acting at the centre of gravity of the loading (middle of the loaded length). For example, in the beam shown in Fig. 9.13, the given load may be replaced by a  $20 \times 4 = 80$  kN concentrated load acting at a distance 2 m from the left support.

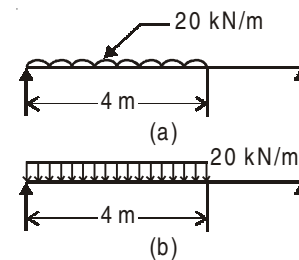


Fig. 9.13

**Uniformly Varying Load:** The load shown in Fig. 9.14 varies uniformly from *C* to *D*. Its intensity is zero at *C* and is 20 kN/m at *D*. In the load diagram, the ordinate represents the load intensity and the abscissa represents the position of load on the beam.

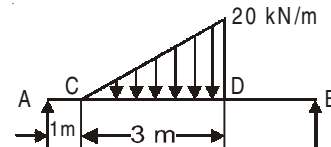


Fig. 9.14

Hence the area of the triangle represents the total load and the centroid of the triangle represents the centre of gravity of the load. Thus, total load in this case is  $\frac{1}{2} \times 3 \times 20 = 30$  kN and the centre of gravity of this loading is at  $\frac{1}{3} \times 3 = 1$  m from *D*, i.e.,  $1 + 3 - 1 = 3$  m from *A*. For finding the reactions, we can assume that the given load is equivalent to 30 kN acting at 3 m from *A*.

**General Loadings:** Figure 9.15 shows a general loading. Here the ordinate represents the intensity of loading and abscissa represents position of the load on the beam. For simplicity in analysis such loadings are replaced by a set of equivalent concentrated loads.

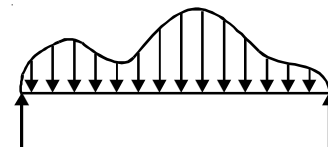


Fig. 9.15



**External Moment:** A beam may be subjected to external moment at certain points. In Fig. 9.16, the beam is subjected to clockwise moment of 30 kN-m at a distance of 2 m from the left support.

In this chapter the beams subjected to concentrated loads, *udl* and external moments are dealt with.

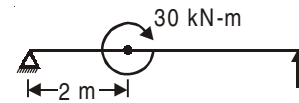


Fig. 9.16

### 9.5 REACTIONS FROM SUPPORTS OF BEAMS

A beam is in equilibrium under the action of the loads and the reactions. Hence the equilibrium may be written for the system of forces consisting of reactions and the loads. Solutions of these equations give the unknown reactions.

**Example 9.1.** The beam AB of span 12 m shown in Fig. 9.17 (a) is hinged at A and is on rollers at B. Determine the reactions at A and B for the loading shown in the Figure.

**Solution:** At A the reaction can be in any direction. Let this reaction be represented by its components  $V_A$  and  $H_A$  as shown in Fig. 9.17 (b). At B the reaction is in vertical direction only. The beam is in equilibrium under the action of system of forces shown in Fig. 9.17 (b).

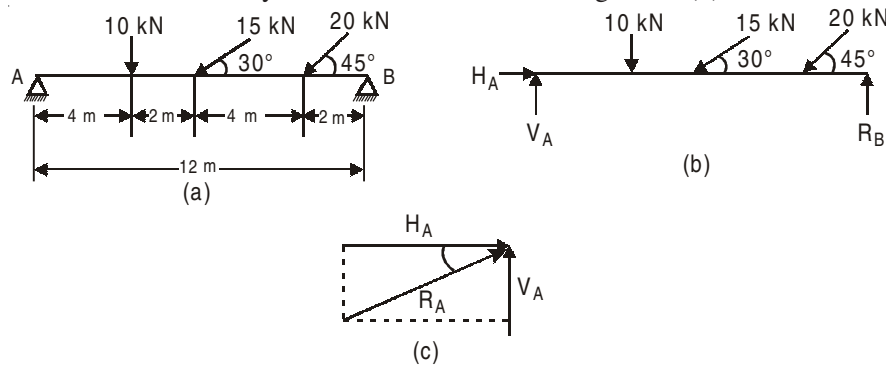


Fig. 9.17

Now,

$$\sum H = 0, \text{ gives}$$

$$H_A - 15 \cos 30^\circ - 20 \cos 45^\circ = 0$$

$$H_A = 27.1325 \text{ kN}$$

$$\sum M_A = 0, \text{ gives}$$

$$R_B \times 12 - 10 \times 4 - 15 \sin 30^\circ \times 6 - 20 \sin 45^\circ \times 10 = 0$$

$$R_B = 18.8684 \text{ kN.}$$

$$\sum V = 0, \text{ gives}$$

$$V_A + 18.8684 - 10 - 15 \sin 30^\circ - 20 \sin 45^\circ = 0$$

$$\therefore V_A = 12.7737 \text{ kN}$$

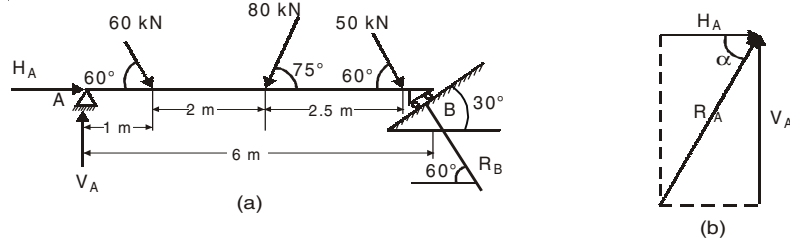
$$\therefore R_A = \sqrt{H_A^2 + V_A^2} = \sqrt{27.1325^2 + 12.7737^2}$$

$$\text{i.e., } R_A = 29.989 \text{ kN.}$$

$$\text{and } \alpha = \tan^{-1} \left( \frac{12.7737}{27.1325} \right) = 25.21^\circ.$$

[**Note:** For finding moments, inclined loads are split into their vertical and horizontal components. Horizontal components do not produce moment about A.]

**Example 9.2.** Find the reactions at supports A and B in the beam AB shown in Fig. 9.18 (a).



**Fig. 9.18**

**Solution:** The reaction at B will be at right angles to the support, i.e., at  $60^\circ$  to horizontal as shown in the figure. Let the components of the reactions at A be  $H_A$  and  $V_A$ . Then

$$\sum M_A = 0 \text{ gives}$$

$$R_B \sin 60^\circ \times 6 - 60 \sin 60^\circ \times 1 - 80 \times \sin 75^\circ \times 3 - 50 \times \sin 60^\circ \times 5.5 = 0$$

$$\therefore \quad R_B = 100.4475 \text{ kN.}$$

$$\sum H = 0, \text{ gives}$$

$$H_A + 60 \cos 60^\circ - 80 \cos 75^\circ + 50 \cos 60^\circ - R_B \cos 60^\circ = 0$$

$$H_A = -60 \cos 60^\circ + 80 \cos 75^\circ - 50 \cos 60^\circ + 100.4475 \cos 60^\circ \\ = 15.9293 \text{ kN}$$

$$\sum V = 0, \text{ gives}$$

$$V_A + R_B \sin 60^\circ - 60 \sin 60^\circ - 80 \sin 75^\circ - 50 \sin 60^\circ = 0$$

$$V_A = -100.4475 \sin 60^\circ + 60 \sin 60^\circ + 80 \sin 75^\circ + 50 \sin 60^\circ \\ = 85.5468 \text{ kN}$$

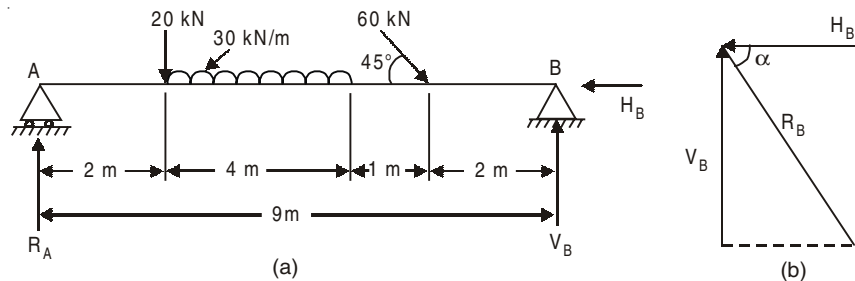
$$\therefore \quad R_A = \sqrt{15.9293^2 + 85.5468^2}$$

$$i.e., \quad R_A = 87.0172 \text{ kN.}$$

$$\alpha = \tan^{-1} \frac{85.5468}{15.9293}$$

$$i.e., \quad \alpha = 79.45^\circ, \text{ as shown in Fig. 9.18(b).}$$

**Example 9.3.** Find the reactions at supports A and B of the loaded beam shown in Fig. 9.19(a).



**Fig. 9.19**

**Solution:** The reaction at A is vertical. Let  $H_B$  and  $V_B$  be the components of the reaction at B.

$$\sum M_B = 0, \text{ gives}$$

$$R_A \times 9 - 20 \times 7 - 30 \times 4 \times 5 - 60 \sin 45^\circ \times 2 = 0$$

$\therefore$

$$R_A = 91.6503 \text{ kN.}$$

$$\sum H_A = 0, \text{ gives}$$

$$H_B - 60 \cos 45^\circ = 0$$

$\therefore$

$$H_B = 42.4264 \text{ kN.}$$

$$\sum V_A = 0$$

$$91.6503 + V_B - 20 - 30 \times 4 - 60 \sin 45^\circ = 0$$

$$V_B = 90.7761 \text{ kN.}$$

$\therefore$

$$R_B = \sqrt{42.4264^2 + 90.7761^2}$$

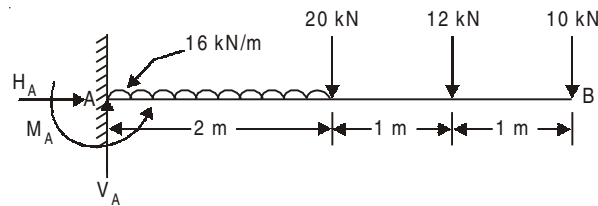
$\therefore$

$$R_B = 100.2013 \text{ kN.}$$

$$\alpha = \tan^{-1} \frac{90.7761}{42.4264}$$

$$\alpha = 64.95^\circ, \text{ as shown in Fig. 9.19(b).}$$

**Example 9.4.** The cantilever shown in Fig. 9.20 is fixed at A and is free at B. Determine the reactions when it is loaded as shown in the Figure.



**Fig. 9.20**

**Solution:** Let the reactions at A be  $H_A$ ,  $V_A$  and  $M_A$  as shown in the figure

Now

$$\sum H = 0, \text{ gives}$$

$$H_A = 0.$$

$$\sum V = 0, \text{ gives}$$

$$V_A - 16 \times 2 - 20 - 12 - 10 = 0$$

$\therefore$

$$V_A = 74 \text{ kN.}$$

$$\sum M = 0, \text{ gives}$$

$$M_A - 16 \times 2 \times 1 - 20 \times 2 - 12 \times 3 - 10 \times 4 = 0$$

$\therefore$

$$M_A = 148 \text{ kN-m.}$$

**Example 9.5.** Compute the reaction developed at support in the cantilever beam shown in Fig. 9.21.

**Solution:** Let the vertical reaction be  $V_A$  and moment be  $M_A$ . There is no horizontal component of reactions, since no load is having horizontal component

$$\sum V = 0, \text{ gives}$$

$$V_A - 20 \times 2 - 15 \times 10 = 0$$

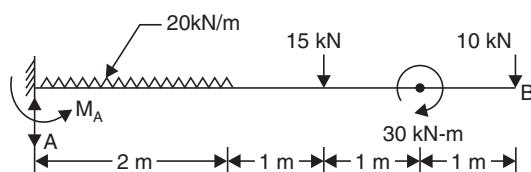


Fig. 9.21

$$\therefore V_A = 65 \text{ kN.}$$

$$\Sigma M = 0, \text{ gives}$$

$$M_A - 20 \times 2 \times 1 - 15 \times 3 - 30 - 10 \times 5 = 0$$

$$M_A = 165 \text{ kN-m}$$

**Example 9.6.** Determine the reactions at supports A and B of the overhanging beam shown in Fig. 9.22.

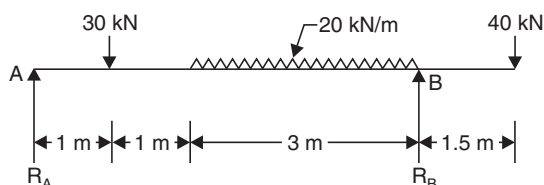


Fig. 9.22

**Solution:** As supports A and B are simple supports and loading is only in vertical direction, the reactions  $R_A$  and  $R_B$  are in vertical directions only.

$$\Sigma M_A = 0, \text{ gives}$$

$$R_B \times 5 - 30 \times 1 - 20 \times 3 \times (2 + 1.5) - 40 \times 6.5 = 0$$

$$\therefore R_B = 100 \text{ kN.}$$

$$\Sigma V = 0, \text{ gives}$$

$$R_A + R_B - 30 - 20 \times 3 - 40 = 0$$

$$\therefore R_A = 130 - R_B = 130 - 100 = 30 \text{ kN.}$$

**Example 9.7.** Find the reactions at supports A and B of the beam shown in Fig. 9.23.

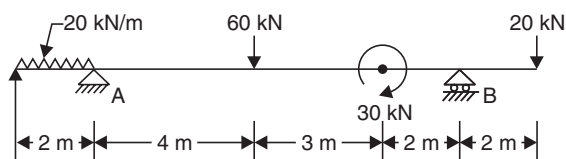


Fig. 9.23

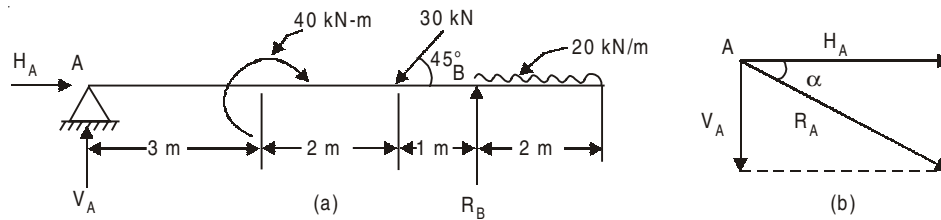
**Solution:** Let  $V_A$  and  $H_A$  be the vertical and the horizontal reactions at A and  $V_B$  be vertical reaction at B.

$$\Sigma H = 0, \text{ gives}$$

$$H_A = 0.$$

$$\begin{aligned} \Sigma M_A &= 0, \text{ gives} \\ -20 \times 2 \times 1 + 60 \times 4 + 30 + 20 \times 11 - V_B \times 9 &= 0 \\ \therefore V_B &= 50 \text{ kN.} \\ \Sigma V &= 0, \text{ gives} \\ -20 \times 2 + V_A - 60 + V_B - 20 &= 0 \\ V_A &= 120 - V_B = 120 - 50 \\ V_A &= 70 \text{ kN.} \end{aligned}$$

**Example 9.8.** Determine the reactions at A and B of the overhanging beam shown in Fig. 9.24(a).



**Fig. 9.24**

**Solution:**

$$\begin{aligned} \Sigma M_A &= 0 \\ R_B \times 6 - 40 - 30 \sin 45^\circ \times 5 - 20 \times 2 \times 7 &= 0 \\ R_B &= 71.0110 \text{ kN.} \\ \Sigma H &= 0 \\ H_A &= 30 \cos 45^\circ = 21.2132 \text{ kN} \\ \Sigma V &= 0 \\ V_A - 30 \sin 45^\circ + R_B - 20 \times 2 &= 0 \\ V_A &= 30 \sin 45^\circ - R_B + 40 \\ V_A &= -9.7978 \end{aligned}$$

(Negative sign show that the assumed direction of  $V_A$  is wrong. In other words,  $V_A$  is acting vertically downwards).

$$\begin{aligned} R_A &= \sqrt{V_A^2 + H_A^2} \\ R_A &= 23.3666 \text{ kN.} \end{aligned}$$

$$\alpha = \tan^{-1} \frac{V_A}{H_A}$$

$\alpha = 24.79^\circ$ , as shown in Fig. 9.24(b).

**Example 9.9.** A beam AB 20 m long supported on two intermediate supports 12 m apart, carries a uniformly distributed load of 6 kN/m and two concentrated loads of 30 kN at left end A and 50 kN at the right end B as shown in Fig. 9.25. How far away should the first support C be located from the end A so that the reactions at both the supports are equal ?

**Solution:** Let the left support C be at a distance  $x$  metres from A.

Now,  $R_C = R_D$  (given)

$$\Sigma V = 0, \text{ gives}$$

$$R_C + R_D - 30 - 6 \times 20 - 50 = 0$$

or  $2R_C = 30 + 120 + 50$  since  $R_C = R_D$

or  $R_C = 100 \text{ kN}$

$\therefore R_D = 100 \text{ kN}$

$\Sigma M_A = 0$ , gives

$$100x + 100(12 + x) - 6 \times 20 \times 10 - 50 \times 20 = 0$$

$$200x = 1000$$

$$x = 5 \text{ m.}$$

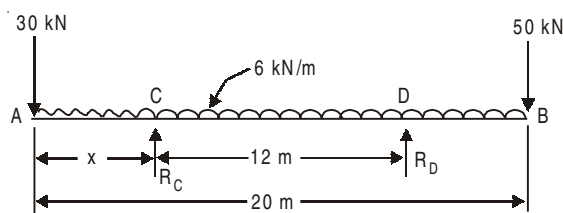


Fig. 9.25

## 9.6 SHEAR FORCE AND BENDING MOMENT

The load applied on a beam gets transferred to supports. To see how this transfer takes place, consider a simply supported beam subject to the loads as shown in Fig. 9.26.

$\Sigma M_B = 0$ , gives

$$R_A \times 7 = 20 \times 5 + 40 \times 3 + 60 \times 1$$

$$R_A = 40 \text{ kN}$$

$$R_B = (20 + 40 + 60) - 40$$

$$= 80 \text{ kN}$$

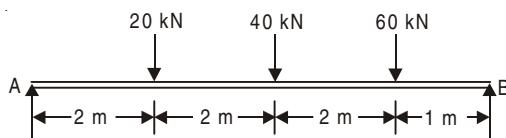


Fig. 9.26

Now to find what is happening at a section, consider the section at C which is at a distance of 3 m from A. Imagining a cut at this section, left hand side portion and right hand side portions are drawn separately in Fig. 9.27.

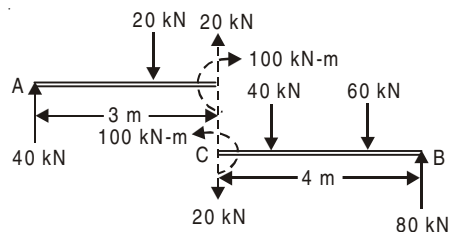


Fig. 9.27

Considering the algebraic sum of vertical forces acting on left hand side portion, it is found that a net vertical force of  $40 - 20 = 20$  kN is experienced at the section. This effect is shown in Fig. 9.27 by dotted line. Again considering the portion on left hand side, the moment experienced at  $C$  is given by

$$\begin{aligned} M_C &= 40 \times 3 - 20 \times 1 \\ &= 100 \text{ kN-m clockwise.} \end{aligned}$$

*This moment is also shown on left hand side portion of the beam at  $C$  by dotted line.*

Now, considering the right hand side portion:

Force experienced at  $C$

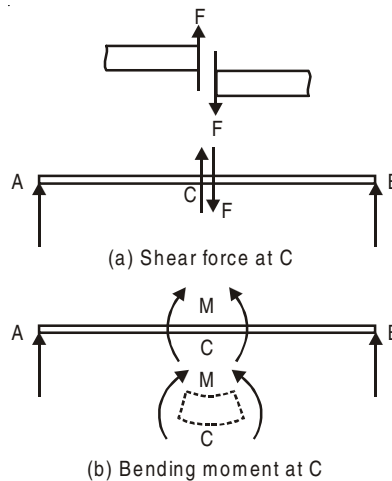
$$\begin{aligned} &= 80 - 60 - 40 = -20 \text{ kN} \\ &= 20 \text{ kN. downward} \end{aligned}$$

and the moment experienced is  $M = 80 \times 4 - 60 \times 3 - 40 \times 1$

$$= 100 \text{ kN-m. anticlockwise}$$

These forces and moments are also shown in Fig. 9.27 on right hand side portion of the beam at  $C$ .

Thus the section  $C$  is subjected to a force of 20 kN, which is trying to shear off the beam as shown in Fig. 9.28(a), and is also subjected to a moment of 100 kN-m which is trying to bend the beam as shown in Fig. 9.28(b). Since this force of 20 kN is trying to shear off the section, it is called as shear force at section  $C$ . The moment is trying to bend the beam at  $C$  and hence it is called as bending moment at that section. The shear force and bending moment at a section in a beam may be defined as follows:



**Fig. 9.28**

“*Shear Force* at a section in a beam (or any structural member) is the force that is trying to shear off the section and is obtained as algebraic sum of all the forces acting normal to the axis of beam either to the left or to the right of the section”.

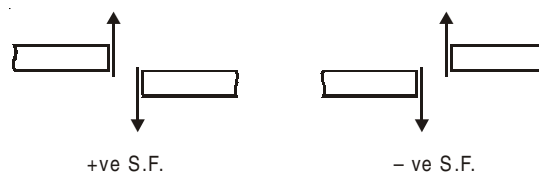
“*Bending Moment* at a section in a beam is the moment that is trying to bend the beam and is obtained as algebraic sum of moment of all the forces about the section, acting either to the left or to the right of the section”.

Hence to find shear force or bending moment at a section, a cut at the section is to be imagined and any one portion with all the forces acting on that portion, is to be considered. It may be noted that for finding bending moment at a section, the moment of the forces are to be found about the section considered.

### 9.7 SIGN CONVENTION

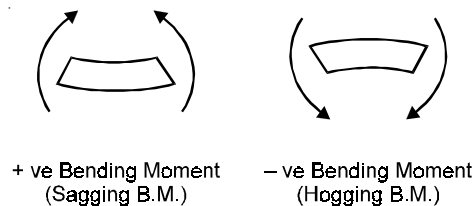
Although different sign conventions may be used, most of the engineers use the following sign conventions for shear forces and bending moment.

(a) The shear force that tends to move left portion upward relative to the right portion shall be called as positive shear force (Fig. 9.29).



**Fig. 9.29**

(b) The bending moment that is trying to sag the beam shall be taken as positive bending moment. If left portion is considered positive bending moment comes out to be clockwise moment (Fig. 9.30).



**Fig. 9.30**

To decide the sign of moment due to a force about a section, assume the beam is held tightly at that section and observe the deflected shape. Then looking at the shape sign can be assigned. Thus in the problem shown in Fig. 9.26 and 9.27, 40 kN reaction at A produces positive moment at C and 20 kN load produces negative moment.

### 9.8 RELATIONSHIP BETWEEN LOAD INTENSITY, SHEAR FORCE AND BENDING MOMENT

Consider the beam  $AB$  subject to a general loading as shown in Fig. 9.31(a). The free body diagram of a segment of beam at a distance  $x$  from A and of length  $\delta x$  is shown in Fig. 9.31(b). The intensity of loading on this elemental length may be taken as constant. Let it be  $w$ /unit length.

Let shear force and bending moment acting on the section at a distance  $x$  be  $F$  and  $M$  respectively. At section at a distance  $x + \delta x$ , these values be  $F + \delta F$  and  $M + \delta M$  respectively. Now from the equilibrium of the element.



$$\sum V = 0 \text{ leads to}$$

$$-F + F + \delta F - w\delta x = 0$$

or  $\frac{\delta F}{\delta x} = w$

In the limiting case as  $\delta x \rightarrow 0$ ,

$$\frac{dF}{dx} = w \quad \dots(9.1)$$

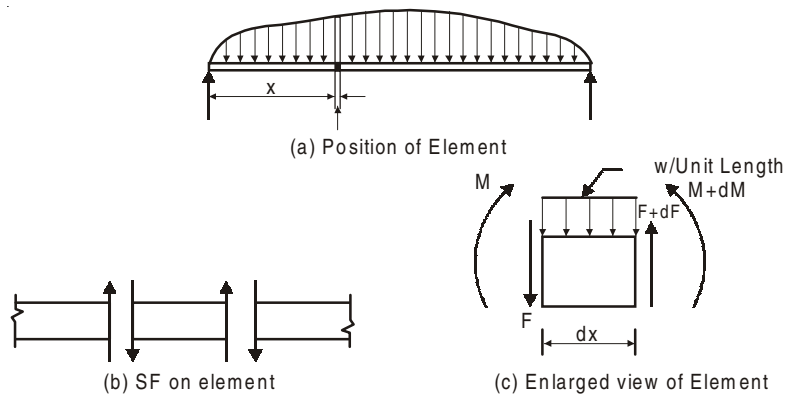


Fig. 9.31

The moment equilibrium condition at section  $x + dx$  leads to

$$M - F\delta x - w\delta x \times \frac{\delta x}{2} - (M + \delta M) = 0$$

Neglecting the small quantity of higher order, we get

$$F\delta x + \delta M = 0$$

or  $\frac{\delta M}{\delta x} = -F$

In the limiting case as  $\delta x \rightarrow 0$ , we get  $\dots(9.2)$

$$\frac{dM}{dx} = -F$$

## 9.9 SHEAR FORCE AND BENDING MOMENT DIAGRAMS

Shear force and bending moment in a beam vary from section to section. The graphical representation of shear force in which ordinate represents shear force and the abscissa represents the position of the section is called *Shear Force Diagram (SFD)*. The diagram in which the ordinate represent bending moment the abscissa represent the position of the section is called *Bending Moment Diagram (BMD)*. In drawing *SFD* and *BMD*, the sign conventions explained earlier are used. These diagrams are usually located below the load diagram.

From equations 1 and 2, it may be concluded that the rate of change of shear force (slope of shear force diagram curve) at any section is equal to the intensity of loading at that section and the rate

of change of bending moment (*i.e.*, shape of bending moment diagram curve) is equal to the shear force at that section. From equation 2, it can also be concluded that the bending moment will be maximum/minimum where shear force ( $dM/dx$ ) is zero.

At any section, if moment changes its sign the point representing that section is called the point of contraflexure. Obviously, the moment at that section is zero.

### 9.10 SFD AND BMD FOR A FEW STANDARD CASES

The methods of drawing shear force and bending moment diagrams have been explained here in case of the following beams subjected to standard loading conditions.

- (a) Cantilever beams
- (b) Simply supported beams and
- (c) Overhanging beams

#### 9.10.1 Cantilever Subject to a Concentrated Load at Free End

Consider the section X–X at a distance  $x$  from free end in a cantilever beam shown in Fig. 9.32(a).

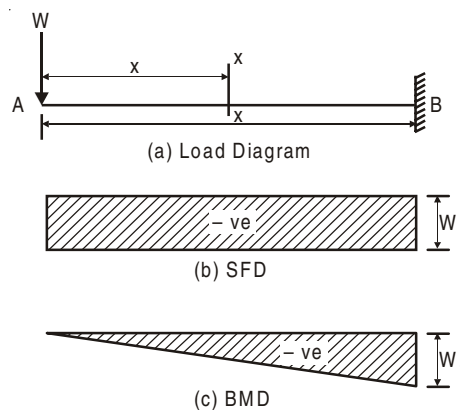
From left hand side segment of beam,

$$F = -W$$

Thus shear force is constant *i.e.*, it will not vary with  $x$ . Hence the *SFD* is as shown in Fig. 9.32(b).

$$M = -Wx, \text{ linear variation.}$$

At  $x = 0, M_A = 0$



**Fig. 9.32**

At  $x = l, M_B = -Wl$

Hence *BMD* is as shown in Fig. 9.32(c).

#### 9.10.2 A Cantilever Subject to UDL Over its Entire Span

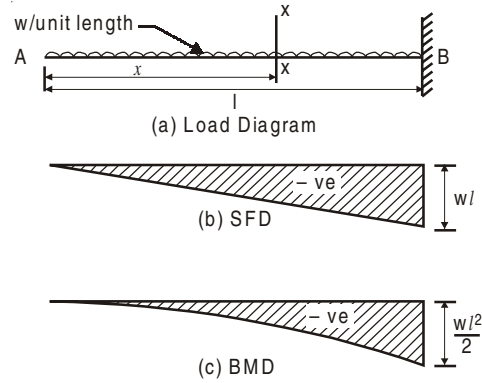
Consider the beam shown in Fig. 9.33(a).

Considering the left hand side portion of the beam from the section X-X which is at a distance  $x$  from the free end A,

$$F = -Wx, \text{ linear variation}$$

At  $x = 0, F_A = 0$

At  $x = l, F_B = -wl$



**Fig. 9.33**

Hence *SFD* is as shown in Fig. 9.33(b).

$$M = -wx \frac{x}{2} = -\frac{wx^2}{2}$$

This is parabolic variation. As magnitude increases at a faster rate with  $x$ , it is concave parabola as shown in Fig. 9.33(c), with extreme values as given below:

At  $x = 0, M_A = 0,$

At  $x = l, M_B = -\frac{wl^2}{2}$

### 9.10.3 Simply Supported Beam Subjected to a Concentrated Load

Let  $W$  be the concentrated load acting on beam  $AB$  at a distance ' $a$ ' from the end  $A$  as shown in Fig. 9.34(a).

Now  $R_A = \frac{Wb}{l}$  and  $R_B = \frac{Wa}{l}$

Consider the portion  $AC$ . At any distance  $x$  from  $A$ ,

$$F = R_A = \frac{Wb}{l}, \text{ constant}$$

$$M = R_A x = \frac{Wb}{l} x, \text{ linear variation.}$$

At  $x = 0, M_A = 0$

At  $x = a, M_C = \frac{Wab}{l}$

For portion *AC* *SFD* and *BMD* can be drawn.

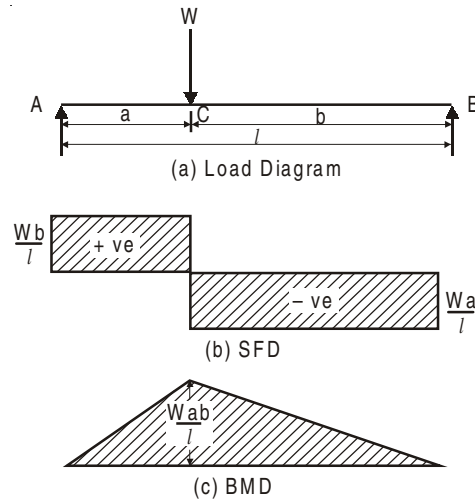
Consider portion *CB*. The expression derived for portion *AC* will not hold good for this portion. Taking a section at a distance  $x$  from *B* and considering the right hand side segment of the beam,

$$F = -R_B = -\frac{Wa}{l}, \text{ constant}$$

$$M = R_B x = \frac{Wa}{l} x, \text{ linear variation.}$$

At  $x = 0, M_B = 0$

At  $x = b, M_B = \frac{Wab}{l}$



**Fig. 9.34**

*SF* and *BM* diagrams for this portion can now be drawn. Hence *SFD* and *BMD* for the beam is as shown in Fig. 9.34(b) and 9.34(c) respectively.

Particular case:

When  $a = b = l/2$

$$F = \frac{Wb}{l} = \frac{W(l/2)}{l} = \frac{W}{2}$$

and moment under the load (centre of span since  $a = b = l/2$ )

$$M = \frac{Wab}{l} = \frac{W l/2 \times l/2}{l} = \frac{Wl}{4}$$

#### 9.10.4 A Simply Supported Beam Subjected to UDL

Let the beam *AB* of span  $l$  be subjected to uniformly distributed load of intensity  $w$ /unit length as shown in Fig. 9.35(a).

$$R_A = \frac{wl \cdot l/2}{l} = \frac{wl}{2}$$

$$R_B = \frac{wl}{2}$$

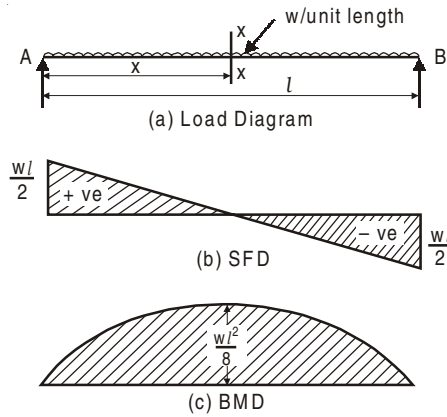
At a section X-X which is at a distance  $x$  from A,

$$F = R_A - wx = \frac{wl}{2} - wx, \text{ linear variation}$$

At  $x = 0, F_A = \frac{wl}{2}$

At  $x = l, F_B = -\frac{wl}{2}$

$\therefore$  SFD is as shown in Fig. 9.35(b).



**Fig. 9.35**

Shear force is zero at  $x$  given by

$$0 = \frac{wl}{2} - wx$$

$$x = \frac{l}{2}$$

$\therefore$  Maximum moment occurs at this point.

At section X-X

$$M = R_A x - wx \cdot \frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2}, \text{ parabolic variation.}$$

As  $x$  increases rate of reduction in the value of  $M$  is faster. Hence it is convex parabola.

At  $x = 0, M_A = 0$

At  $x = l, M_B = 0$

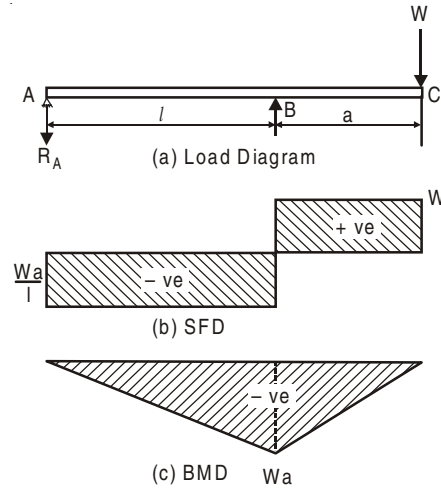
Maximum moment occurs at  $x = \frac{l}{2}$  where shear force  $\left( i.e., \frac{dM}{dx} \right) = 0$

$$M_{\max} = \frac{wl}{2} \cdot \frac{l}{2} - \frac{w(l/2)^2}{2} = \frac{wl^2}{8}$$

Hence BMD is as shown in Fig. 9.35(c).

### 9.10.5 Overhanging Beam Subjected to a Concentrated Load at Free End

Consider the overhanging beam  $ABC$  of span  $AB = l$  and overhang  $BC = a$ , subjected to a concentrated load  $W$  at free end as shown in Fig. 9.36(a).



**Fig. 9.36**

$$R_A = \frac{Wa}{l}, \text{ downward}$$

$\therefore$

$$R_B = W + \frac{Wa}{l} = W \left( l + \frac{a}{l} \right)$$

Portion  $AB$ :

Measuring  $x$  from  $A$  and considering left side of the section.

$$F = -R_A = -\frac{Wa}{l}, \text{ constant}$$

$$M = -R_A x = -\frac{Wax}{l}, \text{ linear variation.}$$

At  $x = 0, M_A = 0$

At  $x = l, M_B = -Wa$

Portion  $BC$ :

Measuring  $x$  from  $C$ , shear force and bending moments at that section are given by

$$F = W, \text{ constant}$$

$$M = -Wx, \text{ linear variation.}$$

At  $x = 0, M_C = 0$

At  $x = a, M_B = -Wa.$

$SFD$  and  $BMD$  for the entire beam is shown in Fig. 9.36(b) and 9.36(c) respectively.

**Example 9.10.** Draw shear force and bending moment diagram for the cantilever beam shown in Fig. 9.37(a).

**Solution:** Portion *AB*:

At distance  $x$ , from *A*,

$$F = -20 - 20x, \text{ linear variation.}$$

At  $x = 0$ ,  $F_A = -20 \text{ kN}$

At  $x = 1$ ,  $F_B = -20 - 20 \times 1 = -40 \text{ kN}$ .

$$M = -20x - 20x \cdot \frac{x}{2}, \text{ parabolic variation}$$

At  $x = 0$ ,  $M_A = 0$

At  $x = 1 \text{ m}$ ,  $M_B = -20 - 20 \times 1 \times \frac{1}{2} = -30 \text{ kN-m}$ .

Portion *BC*:

Measuring  $x$  from *A*,

$$F = -20 - 40 - 20x, \text{ linear variation.}$$

At  $x = 1 \text{ m}$ ,  $F_B = -80 \text{ kN}$

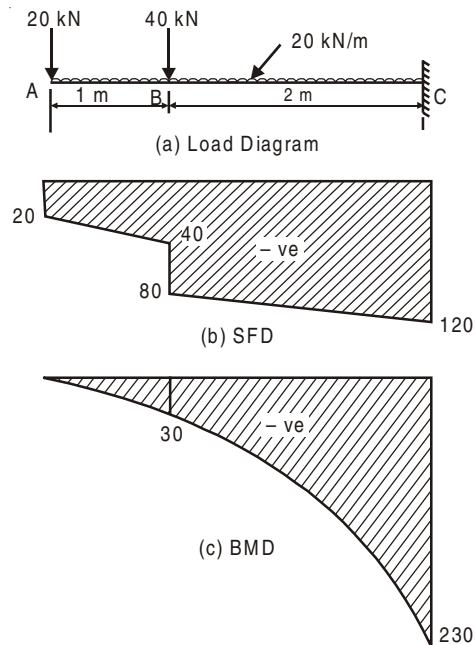
At  $x = 3 \text{ m}$ ,  $F_C = -120 \text{ kN}$ .

$$M = -20x - 40(x - 1) - 20x \cdot \frac{x}{2}, \text{ parabolic variation;}$$

At  $x = 1 \text{ m}$ ,  $M = -30 \text{ kN-m}$

At  $x = 3 \text{ m}$ ,  $M = -60 - 40 \times 2 - 20 \times 3 \times \frac{3}{2}$   
 $= -230 \text{ kN-m}$

Hence *SFD* and *BMD* are shown in Fig. 9.37(b) and 9.37(c) respectively.



**Fig. 9.37**

**Example 9.11.** Draw the SF and BM diagrams for the beam shown in Fig. 9.38(a) and find out the position and the magnitude of maximum moment.

**Solution:**

$$\sum M_A = 0 \rightarrow$$

$$R_B \times 10 = 20 \times 5 \times 2.5 + 20 \times 5 + 40 \times 7.5 + 20 \times 8.5$$

$$\therefore R_B = 82 \text{ kN.}$$

$$\sum V = 0 \rightarrow$$

$$\therefore R_A = 20 \times 5 + 20 + 40 + 20 - 82 = 98 \text{ kN.}$$

Portion AC:

Measuring  $x$  from A,

$$F = 98 - 20x, \text{ linear variation}$$

$$\text{At } x = 0, F_A = 98 \text{ kN}$$

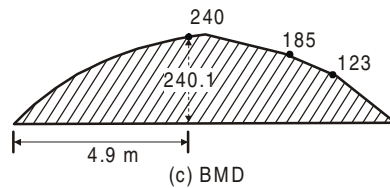
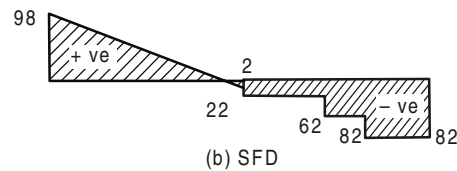
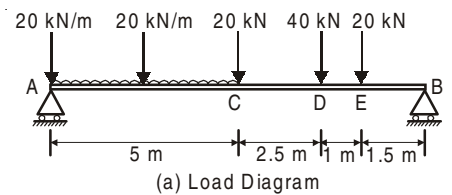
$$\text{At } x = 5 \text{ m, } F_B = 98 - 100 = -2 \text{ kN}$$

Points where shear force is zero is given by,

$$0 = 98 - 20x$$

or

$$x = 4.9 \text{ m}$$



**Fig. 9.38**

Moment is given by

$$M = 98x - 20x \frac{x}{2}, \text{ parabolic variation.}$$

$$\text{At } x = 0, M_A = 0$$

$$\text{At } x = 5 \text{ m, } M_B = 98 \times 5 - \frac{20 \times 5^2}{2} = 240 \text{ kN-m}$$



Maximum moment occurs at  $x = 4.9$  m where  $F = \frac{dM}{dx} = 0$

$$M_{\max} = 98 \times 4.9 - 20 \times \frac{4.9^2}{2} = 240.1 \text{ kN-m}$$

Portion *CD*:

Measuring  $x$  from  $B$  and considering right hand side segment of the beam,

$$F = -82 + 20 + 40 = -22 \text{ kN, constant}$$

$$M = 82x - 20(x - 1.5) - 40(x - 2.5) \\ = 22x + 130, \text{ linear variation.}$$

At  $x = 2.5$  m,  $M_D = 22 \times 2.5 + 130 = 185 \text{ kN-m}$

At  $x = 5$  m,  $M_C = 22 \times 5 + 130 = 240 \text{ kN-m.}$

Portion *DE*:

Measuring  $x$  from  $B$  and considering the portion of the beam on the right side of the section,

$$F = -82 + 20 = -62 \text{ kN, constant}$$

$$M = 82x - 20(x - 1.5) \text{ linear variation}$$

At  $x = 1.5$  m,

$$M_E = 82 \times 1.5 = 123 \text{ kN-m.}$$

At  $x = 2.5$ ,  $M = 82 \times 2.5 - 20 \times 1 = 185 \text{ kN-m.}$

Portion *EB*:

Measuring  $x$  from  $B$  and considering the right side segment,

$$F = -82 \text{ kN, constant}$$

$$M = 82x, \text{ linear variation.}$$

At  $x = 0$ ,  $M_B = 0$

At  $x = 1.5$  m,  $M_E = 82 \times 1.5 = 123 \text{ kN-m.}$

*SFD* and *BMD* are shown in Fig. 9.38(b) and 9.38(c) respectively, for the entire beam.

**Example 9.12.** A beam of span 8 m has roller support at  $A$  and hinge support at  $B$  as shown in Fig. 9.39(a). Draw *SF* and *BM* diagrams when the beam is subjected to udl, a concentrated load and an externally applied moment as shown in the Figure.

**Solution:**

$$\sum M_A = 0 \rightarrow$$

$$R_B \times 8 - 10 \times 4 \times 2 - 20 \times 4 + 240 = 0$$

$$R_B = -10 \text{ kN (upwards)}$$

$$= 10 \text{ kN. (downwards)}$$

$$\sum V = 0 \rightarrow$$

$$R_A = 10 \times 4 + 20 + 10 = 70 \text{ kN.}$$

Portion *AC*:

Measuring  $x$  from  $A$  and considering left hand side segment of the beam,

$$F = 70 - 10x, \text{ linear variation}$$

At  $x = 0$ ,  $F_A = 70 \text{ kN}$

At  $x = 4$  m,  $F_C = 70 - 40 = 30 \text{ kN}$

$$M = 70x - 10x \frac{x}{2}, \text{ parabolic variation}$$

At  $x = 0, M_A = 0$

At  $x = 4 \text{ m}, M_C = 70 \times 4 - 10 \times 4 \times \frac{4}{2} = 200 \text{ kN-m.}$

Portion *CD*: Measuring  $x$  from *B*,

$$F = 10 \text{ kN, constant}$$

$$M = -10x + 240, \text{ linear variation}$$

$$x = 4 \text{ m}, M_C = 200 \text{ kN-m}$$

At  $x = 2 \text{ m}, M_D = -10 \times 2 + 240 = 220 \text{ kN-m.}$

Portion *DB*:

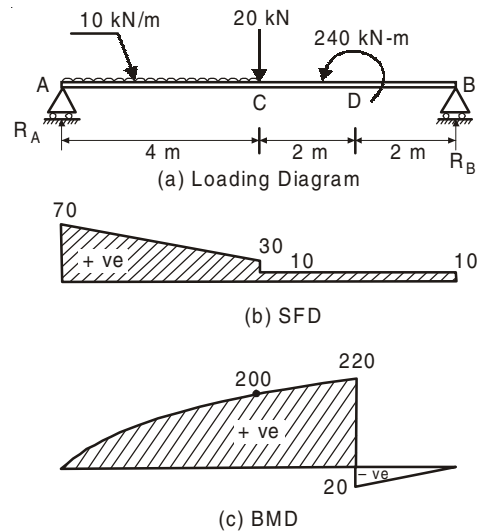
Measuring  $x$  from *B*,

$$F = 10 \text{ kN, constant}$$

$$M = -10x, \text{ linear variation}$$

At  $x = 0, M_B = 0$

At  $x = 2 \text{ m}, M_D = -10 \times 2 = -20 \text{ kN-m.}$



**Fig. 9.39**

*SFD* and *BMD* are shown in Fig. 9.39(b) and 9.39(c).

**Note:** The bending moment value will be the same at a point, whether calculated by considering left hand side or the right hand side segment of the beam, if there is no external moment acting at that point. If the external moment is acting at a point, there will be sudden change in *BMD* at that point to the extent equal to the magnitude of external bending moment.

**Example 9.13.** The overhanging beam *ABC* is supported at *A* and *B*, the span *AB* being 6 m. The overhang *BC* is 2 m (see Fig. 3.42 a). It carries a uniformly distributed load of 30 kN/m over a length of 3 m from *A* and concentrated load of 20 kN at free end. Draw *SF* and *BM* diagrams.

**Solution:**

$$\Sigma M_A = 0 \rightarrow$$

$$R_B \times 6 = 20 \times 8 + 30 \times 3 \times 1.5$$

$\therefore$

$$R_B = 49.167 \text{ kN.}$$

$$\Sigma V = 0 \rightarrow$$

$$R_A = 30 \times 3 + 20 - 49.167 = 60.833 \text{ kN.}$$

Portion AD:

Measuring  $x$  from A and considering left side segment,

$$F = 60.833 - 30x, \text{ linear variation}$$

At  $x = 0, F_A = 60.833 \text{ kN.}$

At  $x = 3 \text{ m,}$

$$F_D = 60.833 - 30 \times 3 = -29.167 \text{ kN.}$$

The point of zero shear is given by

$$0 = 60.833 - 30x$$

or,

$$x = 2.028 \text{ m.}$$

At section X-X, the moment is given by

$$M = 60.833x - 30 \frac{x^2}{2}, \text{ parabolic variation}$$

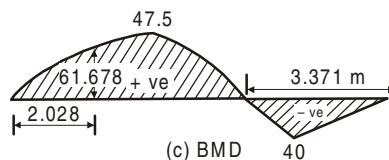
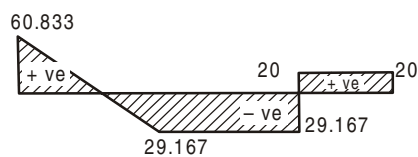
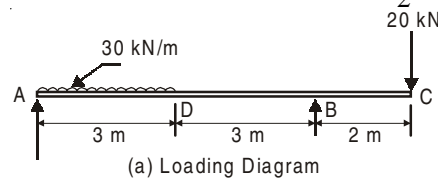
At  $x = 0, M_A = 0$

At  $x = 3 \text{ m,}$

$$M_D = 60.833 \times 3 - 30 \times \frac{9}{2} = 47.5 \text{ kN-m.}$$

Maximum moment occurs at  $x = 2.028 \text{ m.}$  since here  $F = \frac{dM}{dx} = 0$

$$M_{\max} = 60.833 \times 2.028 - 30 \times \frac{2.028^2}{2} = 61.678 \text{ kN-m.}$$



**Fig. 9.40**

Portion *DB*:

Measuring  $x$  from the free end  $C$  and considering right hand side segment,

$$F = 20.0 - 49.167 \\ = 29.167 \text{ kN, constant.}$$

$$M = -20x + 49.167 (x - 2), \text{ linear variation}$$

At  $x = 5 \text{ m}, M_D = -20 \times 5 + 49.167 \times 3 = 47.5 \text{ kN-m.}$

At  $x = 2 \text{ m}, M_B = -20 \times 2 = -40 \text{ kN-m.}$

In this portion the bending moment changes the sign. The point of contraflexure is given by the expression

$$0 = -20x + 49.167 (x - 2)$$

i.e.,  $x = 3.371 \text{ m from free end.}$

Portion *BC*:

Measuring  $x$  from free end,

$$F = 20 \text{ kN, constant}$$

$$M = -20x, \text{ linear variation}$$

At  $x = 0, M_C = 0$

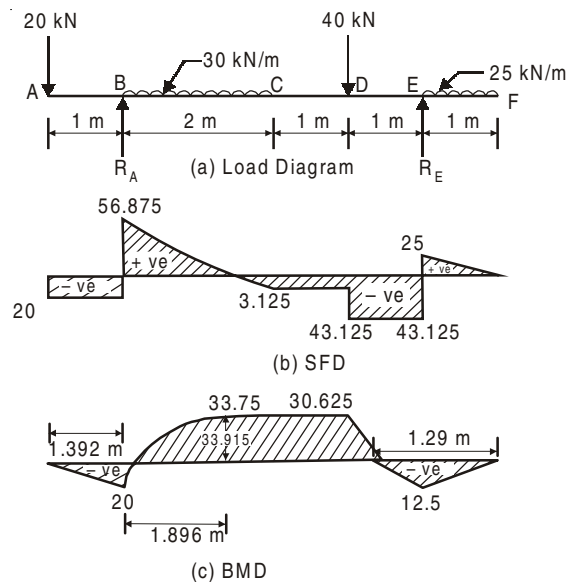
At  $x = 2 \text{ m}, M_B = -40 \text{ kN-m.}$

Hence *SF* and *BM* diagrams are as shown in Fig. 9.40(b) and 9.40(c) respectively.

**Example 9.14.** Draw *BM* and *SF* diagrams for the beam shown in Fig. 9.41(a), indicating the values at all salient points.

**Solution:**

$$\sum M_B = 0 \rightarrow \\ R_E \times 4 + 20 \times 1 - 30 \times 2 \times 1 - 40 \times 3 - 25 \times 1 \times 4.5 = 0$$



**Fig. 9.41**

$$R_E = 68.125 \text{ kN.}$$

$$\Sigma V = 0 \rightarrow$$

$$R_B = 20 + 30 \times 2 + 40 + 25 \times 1 - 68.125 \\ = 76.875 \text{ kN.}$$

Portion *AB*:

Measuring  $x$  from  $A$ ,

$$F = -20 \text{ kN, constant}$$

$$M = -20x, \text{ linear variation}$$

At  $x = 0, M_A = 0$

At  $x = 1 \text{ m}, M_B = -20 \text{ kN-m}$

Portion *BC*:

Measuring  $x$  from  $B$ ,

$$F = -20 + 76.875 - 30x, \text{ linear variation}$$

At  $x = 0, F = 56.875 \text{ kN.}$

At  $x = 2 \text{ m}, F = -3.125 \text{ kN-m.}$

The point of zero shear force is given by

$$0 = -20 + 76.875 - 30x$$

$$x = 1.896 \text{ m from } B.$$

At distance  $x$  from  $B$  the moment is given by

$$M = -20(x + 1) + 76.875x - 30x \frac{x}{2} \\ = -20 + 56.875x - 15x^2, \text{ parabolic variation}$$

At  $x = 0, M_B = -20 \text{ kN-m.}$

At  $x = 2 \text{ m}, M_C = -20 + 56.875 \times 2 - 15 \times 4 \\ = 33.75 \text{ kN-m.}$

Maximum moment occurs where  $SF = 0$ . *i.e.*, at  $x = 1.896 \text{ m}$ .

$$\therefore M_{\max} = -20 + 56.875 \times 1.896 - 15 \times 1.896^2 \\ = 33.913 \text{ kN-m.}$$

The bending moment is changing its sign in this portion. Hence the point of contraflexure exists in this portion. It is given by

$$0 = -20 + 56.875x - 15x^2$$

$$\therefore x = 0.392 \text{ m.}$$

*i.e.*, the point of contraflexure is at 1.392 m from the free end  $A$ .

Portion *CD*:

Measuring  $x$  from  $F$ ,

$$\text{Shear force} = 25 \times 1 - 68.125 - 40 \\ = -3.125 \text{ kN, constant}$$

$$M = -25 \times 1 \times (x - 0.5) + 68.125 (x - 1) - 40 (x - 2), \text{ linear variation}$$

At  $x = 3 \text{ m}, M_C = 33.75 \text{ kN-m.}$

At  $x = 2 \text{ m}, M_D = -25 \times 1.5 + 68.125 \\ = 30.625 \text{ kN-m.}$

Portion *DE*:

Measuring  $x$  from free end  $F$ ,

$$\text{Shear force} = 25 - 68.125 = -43.125 \text{ kN, constant}$$

$$M = -25(x - 0.5) + 68.125(x - 1), \text{ linear variation}$$

At  $x = 2 \text{ m}$ ,  $M_D = 30.625 \text{ kN-m}$ .

At  $x = 1 \text{ m}$ ,  $M_E = -12.5 \text{ kN-m}$ .

Portion *EF*:  $M = 0$ , at  $x = 1.29 \text{ m}$

Measuring  $x$  from free end  $F$ ,

$$\text{Shear force } F = 25x, \text{ linear constant}$$

At  $x = 0$ ,  $F_F = 0$

At  $x = 1 \text{ m}$ ,  $F_E = 25 \text{ kN}$

$$M = -25x \cdot \frac{x}{2}, \text{ parabolic variation}$$

At  $x = 0$ ,  $M_F = 0$

At  $x = 1 \text{ m}$ ,  $M_E = -12.5 \text{ kN-m}$ .

*SF* and *BM* diagrams are as shown in Fig. 9.41(*b*) and 9.41(*c*) respectively.

### 9.11 SHORT-CUT PROCEDURE

To save time, one can calculate the values only at salient points and draw shear force and bending moment diagrams, noting the following points:

(a) Wherever concentrated load is acting (including support reaction), there will be sudden change in *SF* value to the extent of that force at that point, in the direction of that force (while proceeding from left to right).

(b) If an external moment is acting at a point on the beam, there will be sudden change in the value of bending moment to the extent of that external bending moment. While proceeding from left, it will be increase in value, if the moment is clockwise. There will be drop in bending moment value if the external bending moment is anticlockwise. Exactly opposite phenomenon will be observed while proceeding from right to left.

(c) Since  $\frac{dF}{dx} = w$  and  $\frac{dM}{dx} = F$ , the shear force curve will be one degree higher than that of load curve and bending moment curve will be one degree higher than that of shear force curve. The nature of variation, for various loads is listed in Table 9.1.

**Table 9.1** Nature of *SF* and *BM* Variation

<i>Load</i>	<i>SF</i>	<i>BM</i>
No load	Constant	Linear
Uniformly distributed load	Linear	Parabolic
Uniformly varying load	Parabolic	Cubic

The following points also may be noted while drawing shear force and bending moment diagrams:

(a) The bending moment is maximum when shear force is zero. The location and the value of maximum moment should always be indicated in bending moment diagram.

(b) The point of contraflexure is an important point in a *BMD*, hence if it exists, its location should be indicated.

This method of drawing *SFD* and *BMD* is illustrated with two problems below:

**Example 9.15.** Determine the reactions and construct the shear force and bending moment diagrams for the beam shown in Fig. 9.42(a). Mark the salient points and the values at those points.

**Solution:**

$$\sum M_A = 0 \rightarrow$$

$$R_B \times 6 + 120 - 60 \times 4 - 60 \times 7 = 0$$

$$\therefore R_B = 90 \text{ kN.}$$

$$\sum V = 0 \rightarrow R_A = 60 + 60 - 90 = 30 \text{ kN.}$$

*SFD*:

In portion *ADE*,  $F = 30 \text{ kN.}$

In portion *EB*,  $F = 30 - 60 = -30 \text{ kN.}$

In portion *BC*,  $F = 60 \text{ kN.}$

*SFD* is as shown in Fig. 9.42(b).

*BMD*:

At *A*,  $M_A = 0$

Just to the left of *D*,  $M = 30 \times 2 = 60 \text{ kN-m.}$

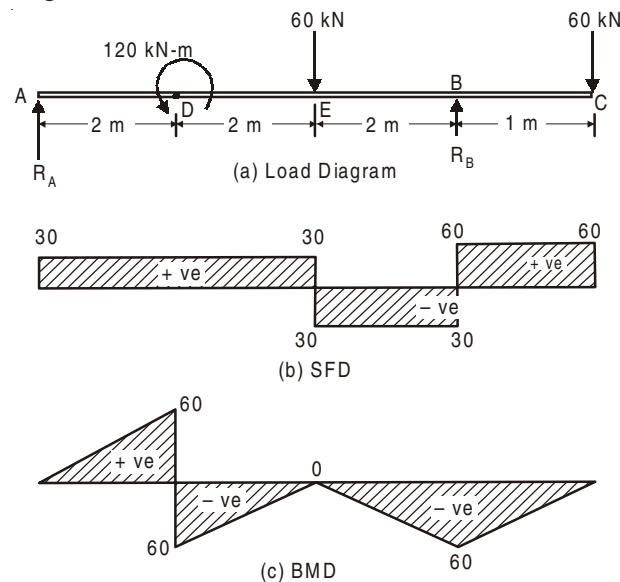
Just to the right of *D*,  $M = 30 \times 2 - 120 = -60 \text{ kN-m.}$

At *E*,  $M = -60 \times 3 + 90 \times 2 = 0$

At *B*,  $M = -60 \times 1 = -60 \text{ kN-m.}$

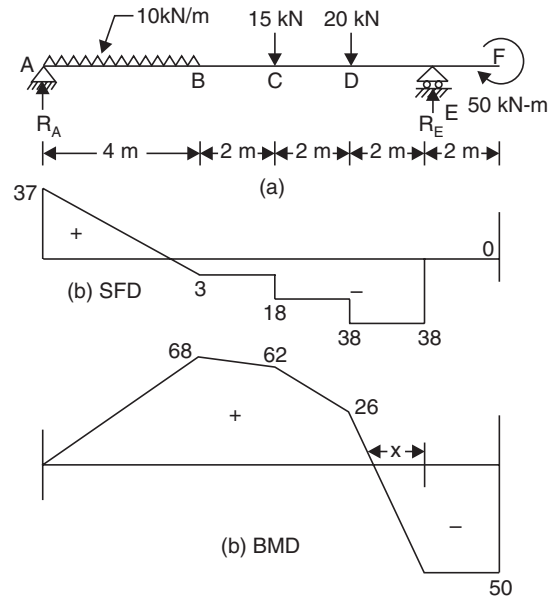
At *C*,  $M = 0$

*BMD* is as shown in Fig. 9.42(c).



**Fig. 9.42**

**Example 9.16.** Draw the bending moment and shear force diagram for the beam loaded as shown in Fig. 9.43. (a). Mark the values at the salient points. Determine the point of contraflexure also.



**Fig. 9.43**

**Solution:**

$$\Sigma M_A = 0, \text{ gives}$$

$$R_E \times 10 - 10 \times 4 \times 2 - 15 \times 6 - 20 \times 8 - 50 = 0$$

$\therefore$

$$R_E = 38 \text{ kN}$$

$$\Sigma V = 0, \text{ gives}$$

$$R_A + R_E - 10 \times 4 - 15 - 20 = 0$$

$$R_A = 75 - R_E = 75 - 38 = 37 \text{ kN}$$

SFD:

Values at salient points are:

$$F_A = R_A = 37 \text{ kN}$$

$$\text{In portion } AB, \quad F_B = 37 - 10 \times 4 = -3 \text{ kN}$$

$$\text{In portion } BC, \quad F = 37 - 40 = -3 \text{ kN}$$

$$\text{In portion } CD, \quad F = 37 - 40 - 15 = -18 \text{ kN}$$

$$\text{In portion } DE, \quad F = 37 - 40 - 15 - 20 = -38 \text{ kN}$$

In portion EF,  $F = 0$ .

$\therefore$  SFD is as shown in Fig. 9.43 (b)

BMD:

$$\text{At A,} \quad M_A = 0$$

$$\text{At B,} \quad M_B = 37 \times 4 - 10 \times 4 \times 2 = 68 \text{ kN-m}$$

$$\text{At C,} \quad M_C = 37 \times 6 - 10 \times 4 \times 4 = 62 \text{ kN-m}$$



At  $D$ , calculating from right hand side  $M_D = -50 + 38 \times 2 = 26 \text{ kN-m}$

At  $E$ ,  $M_E = -50 \text{ kN-m}$

Point of contraflexure is between  $D$  and  $E$ . Let it be at distance  $x$  from support  $E$ . Then

$$-50 + 38x = 0$$

$\therefore$

$x = 1.32 \text{ m from E as shown in Fig. 9.43 (c)}$

BMD is as shown in Fig. 9.43 (c)

### IMPORTANT FORMULAE

1. Shear force at a section of the beam =  $\Sigma$  all forces to the left or right of the section.
2. Bending moment at a section of the beam =  $\Sigma$  moments of all forces to the left or to the right of the section, moment centre being the section.
3.  $\frac{dF}{dx} = w$  and  $\frac{dM}{dx} = -F$

### THEORY QUESTIONS

1. Explain the following types of supports to beams:
  - (i) Simple support
  - (ii) Hinged support
  - (iii) Fixed support
2. Bring out the difference between statically determinate and indeterminate beams.
3. Derive the relationship between,
  - (i) Shear force and load intensity
  - (ii) Bending moment and shear force

### PROBLEMS FOR EXERCISE

1. Determine the reaction at  $A$  and  $B$  on the overhanging beam shown in Fig. 9.44.

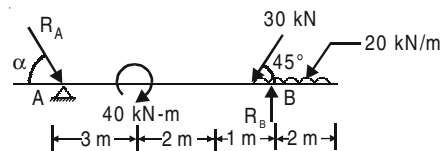


Fig. 9.44

[Ans.  $R_A = 23.3666 \text{ kN}$ ;  $a = 24.79^\circ$ ;  $R_B = 71.011 \text{ kN}$ ]

2. An overhanging beam is on rollers at  $A$  and is hinged at  $B$  and is loaded as shown in Fig. 9.45. Determine the reactions at  $A$  and  $B$ .

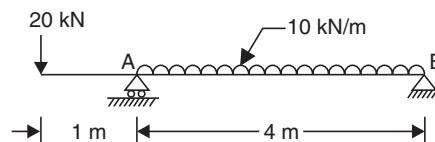


Fig. 9.45

[Ans.  $R_A = 45 \text{ kN}$ ;  $R_B = 15 \text{ kN}$ ]

3. In all the problems given below draw *SFD* and *BMD* indicating values at salient point. (See Fig. 9.46 to 9.50).

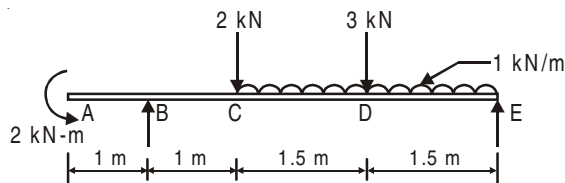


Fig. 9.46

[Ans.  $F_B = 4.25$ ,  $F_E = 3.75$ ,  $M_{\max} = 4.5$  kN-m at D  
Point of contraflexure at 1.47 m from A]

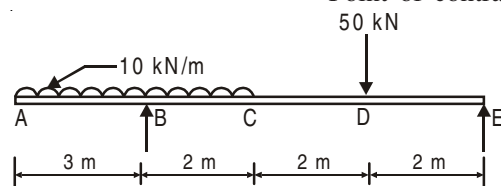


Fig. 9.47

[Ans.  $F_B = 70.83$ ,  $F_E = 29.17$ ,  $M_{\max} = 45$  kN-m at B  
Point of contraflexure at 4.313 m from A]

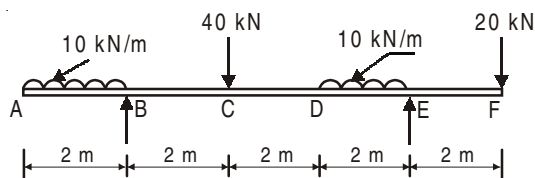


Fig. 9.48

[Ans.  $R_B = 46.67$  kN,  $R_E = 53.33$  kN,  $M_B = -20$  kN-m  
 $M_E = -40$  kN-m,  $M_C = 33.33$  kN-m,  
Point of contraflexure : 2.75 m from A and 3.57 m from F]

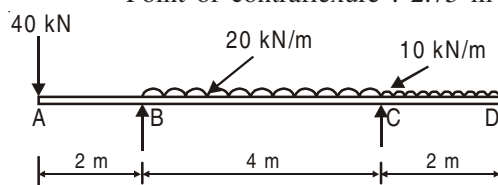


Fig. 9.49

[Ans.  $R_B = 95$ ,  $R_C = 45$ ,  $M_{\max} = -4.375$  at 4.75 m from A  
No +ve moment anywhere]

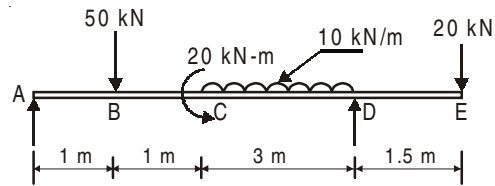


Fig. 9.50

[Ans.  $R_A = 47$ ,  $R_D = 53$ ,  $M_B = 47$ ,  $M_C = 44$  to  $24$ ,  $M_D = 30$

Point of contraflexure 2.59 m from F]

4. Determine load  $P$  such that reactions at supports  $A$  and  $B$  are equal in the beam shown in Fig. 9.51. Draw  $SF$  and  $BM$  diagram marking the values at salient point.

[Ans.  $P = 20$  kN,  $R_A = R_B = 50$  kN,  $M_{\max} = 62.5$  kN-m at 2.5 m from  $A$ ,

Point of contraflexure 1.667 m from  $C$ ]

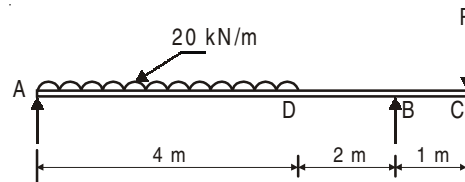


Fig. 9.51

5. A bar of length ' $l$ ' is supported at  $A$  and  $B$  which are at distances ' $a$ ' from the ends as shown in Fig. 9.52. Find the distance ' $a$ ' such that maximum moment is least. Using the above results find the most economical length of railway sleeper if the rails are 1.6 m apart.

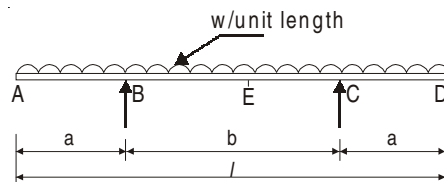


Fig. 9.52

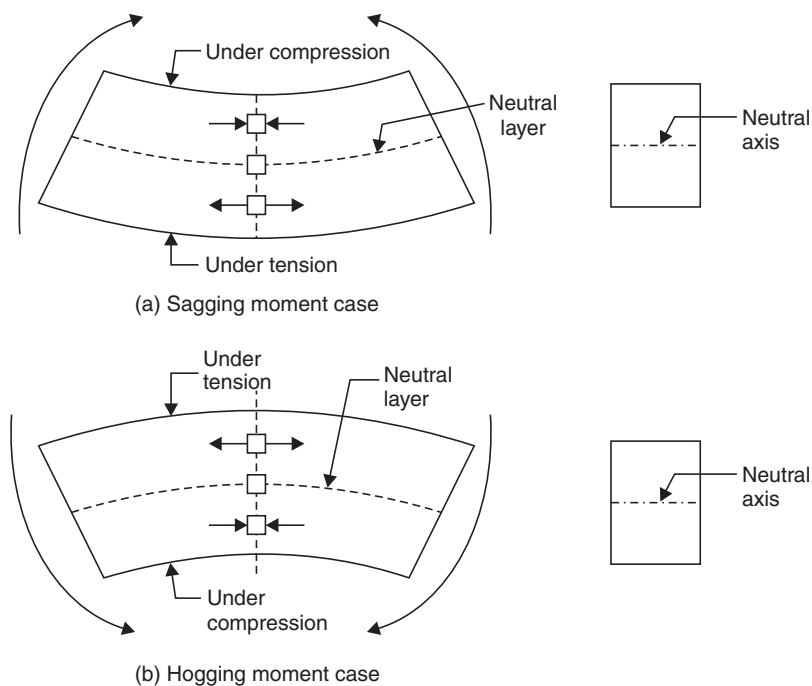
[Hint: Moment at  $B$  = Moment at central point  $E$ ]

[Ans.  $a = 0.2071$

Economical length of railway sleeper = 2.73 m]

# Stresses in Beams

As seen in the last chapter beams are subjected to bending moment and shear forces which vary from section to section. To resist them stresses will develop in the materials of the beam. For the simplicity in analysis, we consider the stresses due to bending and stresses due to shear separately.



**Fig. 10.1.** Nature of Stresses in Beams

Due to pure bending, beams sag or hog depending upon the nature of bending moment as shown in Fig. 10.1. It can be easily observed that when beams sag, fibres in the bottom side get stretched while fibres on the top side are compressed. In other words, the material of the beam is subjected to tensile stresses in the bottom side and to compressive stresses in the upper side. In case of hogging the nature of bending stress is exactly opposite, *i.e.*, tension at top and compression at bottom. Thus bending stress varies from compression at one edge to tension at the other edge. Hence somewhere in between the two edges the bending stress should be zero. The layer of zero stress due to bending is called **neutral layer** and the trace of neutral layer in the cross-section is called **neutral axis** [Refer Fig. 10.1].

### 10.1 ASSUMPTIONS

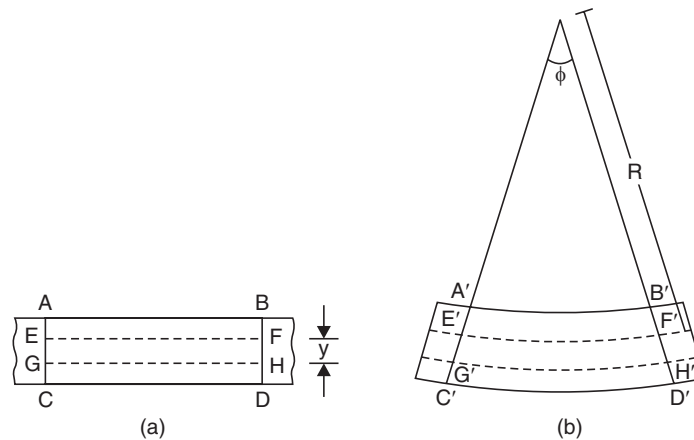
Theory of simple bending is developed with the following assumptions which are reasonably acceptable:

- (i) The material is homogeneous and isotropic.
- (ii) Modulus of elasticity is the same in tension and in compression.
- (iii) Stresses are within the elastic limit.
- (iv) Plane section remains plane even after deformations.
- (v) The beam is initially straight and every layer of it is free to expand or contract.
- (vi) The radius of curvature of bent beam is very large compared to depth of the beam.

### 10.2 BENDING EQUATION

There exists a definite relationship among applied moment, bending stresses and bending deformation (radius of curvature). This relationship can be derived in two steps:

- (i) Relationship between bending stresses and radius of curvature.
  - (ii) Relationship between applied bending moment and radius of curvature.
- (i) *Relationship between bending stresses and radius of curvature:* Consider an elemental length  $AB$  of the beam as shown in Fig. 10.2(a). Let  $EF$  be the neutral layer and  $CD$  the bottom most layer. If  $GH$  is a layer at distance  $y$  from neutral layer  $EF$ , initially  $AB = EF = GH = CD$ .



**Fig. 10.2**

Let after bending  $A, B, C, D, E, F, G$  and  $H$  take positions  $A', B', C', D', E', F', G'$  and  $H'$  respectively as shown in Fig. 10.2(b). Let  $R$  be the radius of curvature and  $\phi$  be the angle subtended by  $C'A'$  and  $D'B'$  at centre of radius of curvature. Then,

$$\begin{aligned} EF &= E'F', \text{ since } EF \text{ is neutral axis} \\ &= R\phi \end{aligned} \quad \dots(i)$$

$$\text{Strain in } GH = \frac{\text{Final length} - \text{Initial length}}{\text{Initial length}}$$

$$= \frac{G'H' - GH}{GH}$$

But  $GH = EF$  (The initial length)

$$= R\phi$$

and  $G'H' = (R + y)\phi$

$$\therefore \text{Strain in layer } GH = \frac{(R + y)\phi - R\phi}{R\phi}$$

$$= \frac{y}{R} \quad \dots(ii)$$

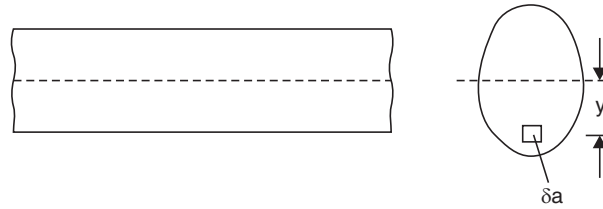
Since strain in  $GH$  is due to tensile forces, strain in  $GH = f/E$  ...(iii)  
where  $f$  is tensile stress and  $E$  is modulus of elasticity.

From eqns. (ii) and (iii), we get

$$\frac{f}{E} = \frac{y}{R}$$

$$\text{or } \frac{f}{y} = \frac{E}{R} \quad \dots(10.1)$$

(ii) *Relationship between bending moment and radius of curvature:* Consider an elemental area  $\delta a$  at distance  $y$  from neutral axis as shown in Fig. 10.3.



**Fig. 10.3**

From eqn. 10.1, stress on this element is

$$f = \frac{E}{R} y \quad \dots(i)$$

$\therefore$  Force on this element

$$= \frac{E}{R} y \delta a$$

Moment of resistance of this elemental force about neutral axis

$$= \frac{E}{R} y \delta a y$$

$$= \frac{E}{R} y^2 \delta a$$

∴ Total moment resisted by the section  $M'$  is given by

$$\begin{aligned} M' &= \sum \frac{E}{R} y^2 \delta a \\ &= \frac{E}{R} \sum y^2 \delta a \end{aligned}$$

From the definition of moment of inertia (second moment of area) about centroidal axis, we know

$$I = \sum y^2 \delta a$$

$$\therefore M' = \frac{E}{R} I$$

From equilibrium condition,  $M = M'$  where  $M$  is applied moment.

$$\therefore M = \frac{E}{R} I$$

$$\text{or } \frac{M}{I} = \frac{E}{R} \quad \dots(10.2)$$

From eqns. (10.1) and (10.2), we get

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R} \quad \dots(10.3)$$

where  $M$  = bending moment at the section

$I$  = moment of inertia about centroid axis

$f$  = bending stress

$y$  = distance of the fibre from neutral axis

$E$  = modulus of elasticity and

$R$  = radius of curvature of bent section.

Equation (10.3) is known as bending equation.

### 10.3 LOCATING NEUTRAL AXIS

Consider an elemental area  $\delta a$  at a distance  $y$  from neutral axis [Ref. Fig. 10.3].

If ' $f$ ' is the stress on it, force on it =  $f \delta a$

But  $f = \frac{E}{R} y$ , from eqn. (10.1).

$$\therefore \text{Force on the element} = \frac{E}{R} y \delta a$$

Hence total horizontal force on the beam

$$= \sum \frac{E}{R} y \delta a$$

$$= \frac{E}{R} \Sigma y \delta a$$

Since there is no other horizontal force, equilibrium condition of horizontal forces gives

$$\frac{E}{R} \Sigma y \delta a = 0$$

As  $\frac{E}{R}$  is not zero,

$$\Sigma y \delta a = 0 \quad \dots(i)$$

If  $A$  is total area of cross-section, from eqn. (i), we get

$$\Sigma \frac{y \delta a}{A} = 0 \quad \dots(ii)$$

Noting that  $\Sigma y \delta a$  is the moment of area about neutral axis,  $\frac{\Sigma y \delta a}{A}$  should be the distance of centroid of the area from the neutral axis. Hence  $\frac{\Sigma y \delta a}{A} = 0$  means the *neutral axis coincides with the centroid of the cross-section*.

#### 10.4 MOMENT CARRYING CAPACITY OF A SECTION

From bending equation, we have

$$\frac{M}{I} = \frac{f}{y}$$

$$i.e., \quad f = \frac{M}{I} y \quad \dots(i)$$

Hence bending stress is maximum, when  $y$  is maximum. In other words, maximum stress occurs in the extreme fibres. Denoting extreme fibre distance from neutral fibre as  $y_{\max}$  equation (i) will be

$$f_{\max} = \frac{M}{I} y_{\max} \quad \dots(ii)$$

In a design  $f_{\max}$  is restricted to the permissible stress in the material. If  $f_{\text{per}}$  is the permissible stress, then from equation (ii),

$$f_{\text{per}} = \frac{M}{I} y_{\max}$$

$$\therefore \quad M = \frac{I}{y_{\max}} f_{\text{per}}$$

The moment of inertia  $I$  and extreme fibre distance from neutral axis  $y_{\max}$  are the properties of section. Hence  $\frac{I}{y_{\max}}$  is the property of the section of the beam. This term is known as **modulus of section** and is denoted by  $Z$ . Thus



$$Z = \frac{I}{y_{\max}} \quad \dots(10.4)$$

and

$$M = f_{\text{per}} Z \quad \dots(10.5)$$

**Note :** If moment of inertia has unit  $\text{mm}^4$  and  $y_{\max}$  has mm,  $Z$  has the unit  $\text{mm}^3$ .

The eqn. (10.5) gives permissible maximum moment on the section and is known as **moment carrying capacity of the section**. Since there is definite relation between bending moment and the loading given for a beam it is possible to find the load carrying capacity of the beam by equating maximum moment in the beam to moment carrying capacity of the section. Thus

$$M_{\max} = f_{\text{per}} Z \quad \dots(10.6)$$

If permissible stresses in tension and compressions are different for a material, moment carrying capacity in tension and compression should be found separately and equated to maximum values of moment creating tension and compression separately to find the load carrying capacity. The lower of the two values obtained should be reported as the load carrying capacity.

### 10.5 SECTION MODULI OF STANDARD SECTIONS

Section modulus expressions for some of the standard sections are presented below:

(i) **Rectangular section:** Let width be ' $b$ ' and depth be ' $d$ ' as shown in Fig. 10.4.

Since  $N-A$  is in the mid depth

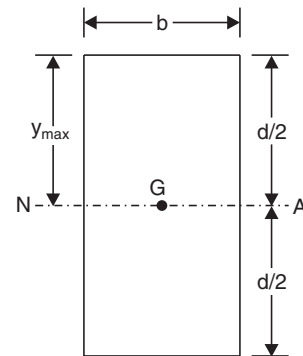
$$y_{\max} = d/2$$

$$I = \frac{1}{12} bd^3$$

$$\therefore Z = \frac{I}{y_{\max}} = \frac{1/12 bd^3}{d/2}$$

i.e.,

$$Z = 1/6 bd^2 \quad \dots(10.7)$$



**Fig. 10.4**

(ii) **Hollow rectangular section.** Figure 10.5 shows a typical hollow rectangular section with symmetric opening. For this,

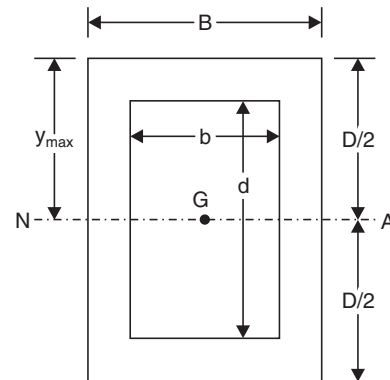
$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12} (BD^3 - bd^3)$$

$$y_{\max} = D/2$$

$$\therefore Z = \frac{I}{y_{\max}} = \frac{1}{12} \frac{(BD^3 - bd^3)}{D/2}$$

i.e.,

$$Z = \frac{1}{6} \frac{BD^3 - bd^3}{D} \quad \dots(10.8)$$



**Fig. 10.5**

(iii) **Circular section of diameter 'd'.** Typical section is shown in Fig. 10.6. For this,

$$I = \frac{\pi}{64} d^4$$

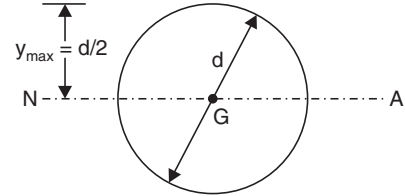
$$y_{\max} = d/2$$

$\therefore$

$$Z = \frac{I}{y_{\max}} = \frac{\pi/64 d^4}{d/2}$$

i.e.,

$$Z = \frac{\pi}{32} d^3$$



**Fig. 10.6**

(iv) **Hollow circular tube of uniform section.** Referring to Fig. 10.7,

$$I = \frac{\pi}{64} D^4 - \frac{\pi}{64} d^4$$

$$= \frac{\pi}{64} (D^4 - d^4)$$

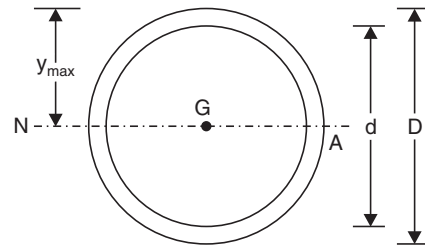
$$y_{\max} = D/2$$

$\therefore$

$$Z = \frac{I}{y_{\max}} = \frac{\pi}{64} \frac{(D^4 - d^4)}{D/2}$$

i.e.,

$$Z = \frac{\pi}{32} \frac{D^4 - d^4}{D} \quad \dots(10.9)$$



**Fig. 10.7**

(v) **Triangular section of base width b and height 'h'.** Referring to Fig. 10.8,

$$I = \frac{bh^3}{36}$$

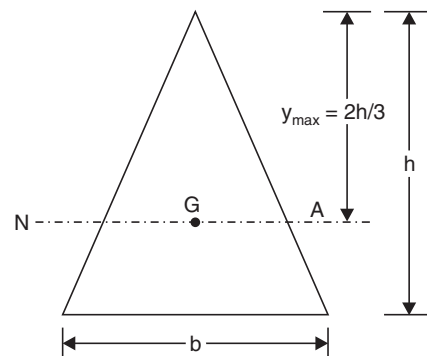
$$y_{\max} = \frac{2}{3} h$$

$\therefore$

$$Z = \frac{I}{y_{\max}} = \frac{bh^3/36}{2/3 h}$$

i.e.,

$$Z = \frac{bh^2}{24} \quad \dots(10.10)$$



**Fig. 10.8**

**Example 10.1.** A simply supported beam of span 3.0 m has a cross-section 120 mm × 180 mm. If the permissible stress in the material of the beam is 10 N/mm<sup>2</sup>, determine

(i) maximum udl it can carry

(ii) maximum concentrated load at a point 1 m from support it can carry.

Neglect moment due to self weight.

**Solution:**

Here  $b = 120 \text{ mm}$ ,  $d = 180 \text{ mm}$ ,  $I = \frac{1}{12} bd^3$ ,  $y_{\max} = \frac{d}{2}$

$$\begin{aligned} \therefore Z &= \frac{1}{6} bd^2 \\ &= \frac{1}{6} \times 120 \times 180^2 = 648000 \text{ mm}^3 \end{aligned}$$

$$f_{\text{per}} = 10 \text{ N/mm}^2$$

$$\begin{aligned} \therefore \text{Moment carrying capacity of the section} &= f_{\text{per}} \times Z \\ &= 10 \times 648000 \text{ N-mm} \end{aligned}$$

(i) Let maximum *udl* beam can carry be  $w$ /metre length as shown in Fig. 10.9.

In this case, we know that maximum moment occurs at mid span and is equal to  $M_{\max} = \frac{wL^2}{8}$ .

Equating it to moment carrying capacity, we get,

$$\frac{w \times 3^2}{8} \times 10^6 = 10 \times 648000$$

$$\therefore w = 5.76 \text{ kN/m.}$$

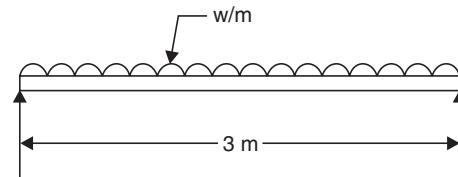
(ii) Concentrated load at distance 1 m from the support be  $P$  kN. Referring to Fig. 10.10.

$$\begin{aligned} M_{\max} &= \frac{P \times a \times b}{L} = \frac{P \times 1 \times 2}{3} \\ &= \frac{2P}{3} \text{ kN-m} \\ &= \frac{2P}{3} \times 10^6 \text{ N-mm} \end{aligned}$$

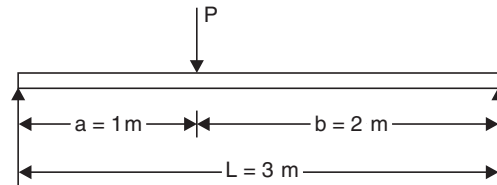
Equating it to moment carrying capacity, we get

$$\frac{2P}{3} \times 10^6 = 10 \times 648000$$

$$\therefore P = 9.72 \text{ kN-m.}$$



**Fig. 10.9**

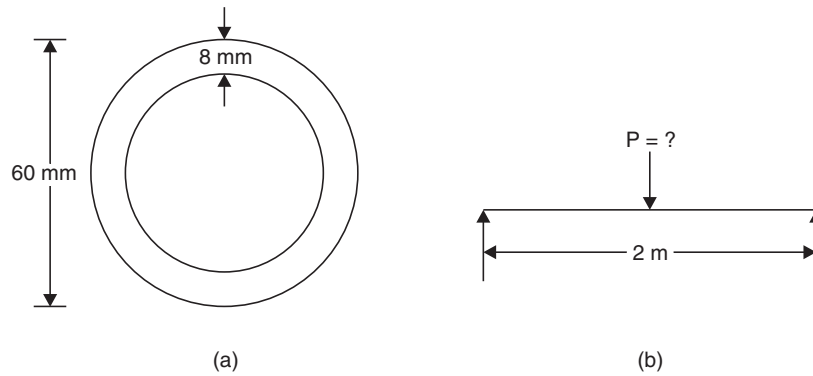


**Fig. 10.10**

**Example 10.2.** A circular steel pipe of external diameter 60 mm and thickness 8 mm is used as a simply supported beam over an effective span of 2 m. If permissible stress in steel is 150 N/mm<sup>2</sup>, determine the maximum concentrated load that can be carried by it at mid span.

**Solution:**

External diameter	$D = 60 \text{ mm}$
Thickness	$= 8 \text{ mm}$

**Fig. 10.11**

$$\therefore \text{Internal diameter} = 60 - 2 \times 8 = 44 \text{ mm.}$$

$$I = \frac{\pi}{64} (60^4 - 44^4) = 452188 \text{ mm}^4$$

$$y_{\max} = 30 \text{ mm.}$$

$$\therefore Z = \frac{I}{y_{\max}} = \frac{452188}{30} = 15073 \text{ mm}^3.$$

Moment carrying capacity

$$M = f_{\text{per}} Z = 150 \times 15073 \text{ N-mm.}$$

Let maximum load it can carry be  $P$  kN.

$$\text{Then maximum moment} = \frac{PL}{4}$$

$$= \frac{P \times 2}{4} \text{ kN-m}$$

$$= 0.5 P \times 10^6 \text{ N-mm.}$$

Equating maximum bending moment to moment carrying capacity, we get

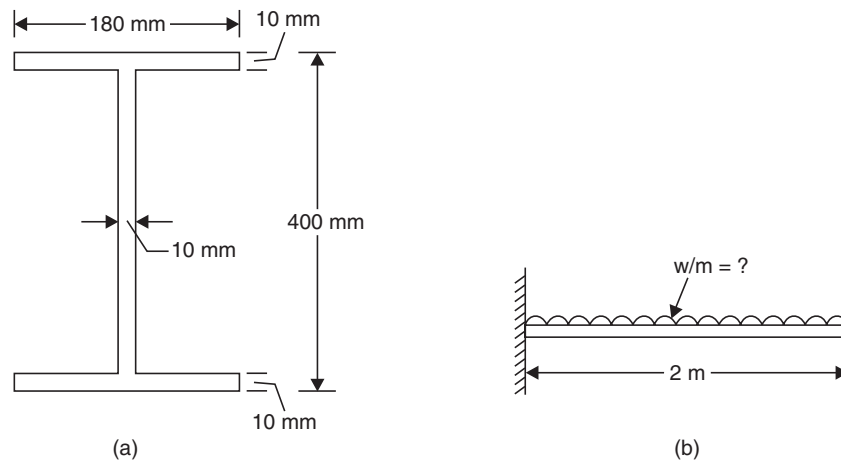
$$0.5P \times 10^6 = 150 \times 15073$$

$$\therefore P = 4.52 \text{ kN.}$$

**Example 10.3:** Figure 10.12 (a) shows the cross-section of a cantilever beam of 2.5 m span. Material used is steel for which maximum permissible stress is  $150 \text{ N/mm}^2$ . What is the maximum uniformly distributed load this beam can carry?

**Solution:** Since it is a symmetric section, centroid is at mid depth.

$$I = MI \text{ of 3 rectangles about centroid}$$

**Fig. 10.12**

$$\begin{aligned}
 &= \frac{1}{12} \times 180 \times 10^3 + 180 \times 10 (200 - 5)^2 \\
 &\quad + \frac{1}{12} \times 10 \times (400 - 20)^3 + 10 \times (400 - 20) \times 0^2 \\
 &\quad + \frac{1}{12} \times 180 \times 10^3 + 180 \times 10 (200 - 5)^2 \\
 &= 182.6467 \times 10^6 \text{ mm}^4
 \end{aligned}$$

[Note: Moment of above section may be calculated as difference between  $MI$  of rectangle of size  $180 \times 400$  and  $170 \times 380$ . i.e.,

$$\begin{aligned}
 I &= \frac{1}{12} \times 180 \times 400^3 - \frac{1}{12} \times 170 \times 380^3 \\
 y_{\max} &= 200 \text{ mm.}
 \end{aligned}$$

$$\therefore Z = \frac{I}{y_{\max}} = \frac{182.6467 \times 10^6}{200} = 913233 \text{ mm}^3.$$

$\therefore$  Moment carrying capacity

$$\begin{aligned}
 &= f_{\text{per}} \times Z \\
 &= 180 \times 913233 \\
 &= 136985000 \text{ N-mm.}
 \end{aligned}$$

If  $udl$  is  $w$  kN/m, maximum moment in cantilever

$$\begin{aligned}
 &= wL = 2w \text{ kN-mm} \\
 &= 2w \times 10^6 \text{ N-mm}
 \end{aligned}$$

Equating maximum moment to movement carrying capacity of the section, we get

$$2w \times 10^6 = 136985000$$

$$\therefore w = 68.49 \text{ kN/m}$$

**Example 10.4.** Compare the moment carrying capacity of the section given in example 10.3 with equivalent section of the same area but

- (i) square section
- (ii) rectangular section with depth twice the width and
- (iii) a circular section.

**Solution:**

$$\begin{aligned}\text{Area of the section} &= 180 \times 10 + 380 \times 10 + 180 \times 10 \\ &= 7400 \text{ mm}^2\end{aligned}$$

(i) Square section

If 'a' is the size of the equivalent square section,

$$a^2 = 7400 \quad \therefore a = 86.023 \text{ mm.}$$

Moment of inertia of this section

$$\begin{aligned}&= \frac{1}{12} \times 86.023 \times 86.023^3 \\ &= 4563333 \text{ mm}^4\end{aligned}$$

$$Z = \frac{I}{y_{\max}} = \frac{4563333}{86.023/2} = 106095.6 \text{ mm}^3$$

$$\begin{aligned}\text{Moment carrying capacity} &= fZ = 150 \times 106095.6 \\ &= 15.914 \times 10^6 \text{ N-mm}\end{aligned}$$

$$\therefore \frac{\text{Moment carrying capacity of I section}}{\text{Moment carrying capacity of equivalent square section}} = \frac{136985000}{15.914 \times 10^6} = 8.607.$$

(ii) Equivalent rectangular section of depth twice the width.

Let  $b$  be the width

$\therefore$  Depth  $d = 2b$ .

Equating its area to area of I-section, we get

$$b \times 2b = 7400$$

$$b = 60.8276 \text{ mm}$$

$$y_{\max} = d/2 = b = 60.8276$$

$$\begin{aligned}M &= f \frac{I}{y_{\max}} = 150 \times \frac{1}{12} \times \frac{b \times (2b)^3}{b} \\ &= 150 \times \frac{8}{12} b^3 = 150 \times \frac{8}{12} \times 60.8276^3 \\ &= 22506193 \text{ N-mm.}\end{aligned}$$

$$\therefore \frac{\text{Moment carrying capacity of I section}}{\text{Moment carrying capacity of this section}} = \frac{136985000}{22506193} = 6.086.$$

(iii) Equivalent circular section.

Let diameter be  $d$ .

Then, 
$$\frac{\pi d^2}{4} = 7400$$

$$d = 97.067$$

$$I = \frac{\pi}{64} d^4$$

$$y_{\max} = d/2$$

$$\therefore Z = \frac{I}{y_{\max}} = \frac{\pi}{32} d^3.$$

$$M = f_{\text{per}} Z = 150 \times \frac{\pi}{32} \times 97.067^3 = 13468024$$

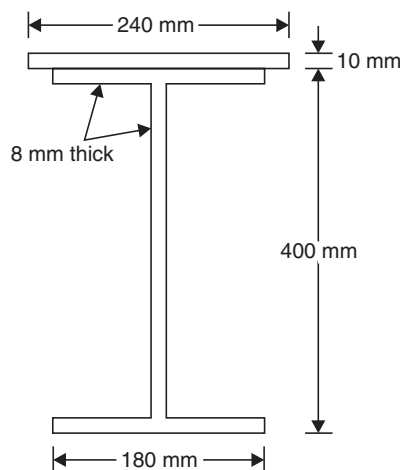
$$\therefore \frac{\text{Moment carrying capacity of I section}}{\text{Moment carrying capacity of circular section}} = \frac{136985000}{13468024} = 10.17.$$

[Note.  $I$  section of same area resists more bending moment compared to an equivalent square, rectangular or circular section. Reason is obvious because in  $I$ -section most of the area of material is in heavily stressed zone.]

**Example 10.5.** A symmetric  $I$ -section of size  $180 \text{ mm} \times 40 \text{ mm}$ ,  $8 \text{ mm}$  thick is strengthened with  $240 \text{ mm} \times 10 \text{ mm}$  rectangular plate on top flange as shown in Fig. 10.13. If permissible stress in the material is  $150 \text{ N/mm}^2$ , determine how much concentrated load the beam of this section can carry at centre of  $4 \text{ m}$  span. Given ends of beam are simply supported.

**Solution:** Area of section  $A$

$$= 240 \times 10 + 180 \times 8 + 384 \times 8 + 180 \times 8 = 8352 \text{ mm}^2$$



**Fig. 10.13**

Let centroid of the section be at a distance  $y$  from the bottom most fibre. Then

$$A \bar{y} = 240 \times 10 \times 405 + 180 \times 8 \times (400 - 4) + 384 \times 8 \times 200 + 180 \times 8 \times 4$$

$$\begin{aligned}
 \therefore \quad 8352 \bar{y} &= 2162400 \\
 \therefore \quad \bar{y} &= 258.9 \text{ mm} \\
 I &= \frac{1}{12} \times 240 \times 10^3 + 240 \times 10 (405 - 258.9)^2 \\
 &\quad + \frac{1}{12} \times 180 \times 8^3 + 180 \times 8 (396 - 258.9)^2 \\
 &\quad + \frac{1}{12} \times 8 \times 384^3 + 8 \times 384 (200 - 258.9)^2 \\
 &\quad + \frac{1}{12} \times 180 \times 8^3 + 180 \times 8 (4 - 258.9)^2 \\
 &= 220.994 \times 10^6 \text{ mm}^4 \\
 \therefore \quad y_{\text{top}} &= 405 - 258.9 = 146.1 \text{ mm} \\
 y_{\text{bottom}} &= 258.9 \text{ mm.} \\
 \therefore \quad y_{\text{max}} &= 258.9 \text{ mm} \\
 \therefore \quad Z &= \frac{I}{y_{\text{max}}} = \frac{220.994 \times 10^6}{258.9} = 853588.3 \\
 \therefore \quad \text{Moment carrying capacity of the section} \\
 &= f_{\text{per}} Z = 150 \times 853588.3 \\
 &= 128038238.7 \text{ N-mm} \\
 &= 128.038 \text{ kN-m.}
 \end{aligned}$$

Let  $P$  kN be the central concentrated load the simply supported beam can carry. Then max bending movement in the beam

$$= \frac{P \times 4}{4} = P \text{ kN-m}$$

Equating maximum moment to moment carrying capacity, we get

$$P = 128.038 \text{ kN.}$$

**Example 10.6.** The cross-section of a cast iron beam is as shown in Fig. 10.14(a). The top flange is in compression and bottom flange is in tension. Permissible stress in tension is  $30 \text{ N/mm}^2$  and its value in compression is  $90 \text{ N/mm}^2$ . What is the maximum uniformly distributed load the beam can carry over a simply supported span of 5 m?

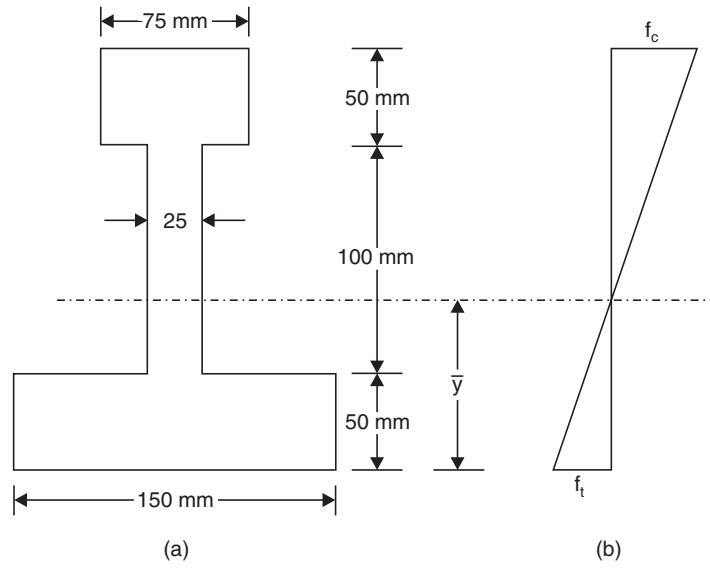
**Solution:**

$$\begin{aligned}
 \text{Cross-section area} \quad A &= 75 \times 50 + 25 \times 100 + 150 \times 50 \\
 &= 13750 \text{ mm}^2
 \end{aligned}$$

Let neutral axis lie at a distance  $\bar{y}$  from bottom most fibre. Then

$$\begin{aligned}
 A\bar{y} &= 75 \times 50 \times 175 + 25 \times 100 \times 100 + 150 \times 50 \times 25 \\
 13750 \times \bar{y} &= 1093750 \\
 \therefore \quad \bar{y} &= 79.54 \text{ mm}
 \end{aligned}$$



**Fig. 10.14**

$$\begin{aligned}
 \therefore I &= \frac{1}{12} \times 75 \times 50^3 + 75 \times 50 (175 - 79.54)^2 \\
 &\quad + \frac{1}{12} \times 25 \times 100^3 + 25 \times 100 (100 - 79.54)^2 \\
 &\quad + \frac{1}{12} \times 150 \times 50^3 + 150 \times 50 (25 - 79.54)^2 \\
 &= 61.955493 \times 10^6 \text{ mm}^4.
 \end{aligned}$$

Extreme fibre distances are

$$y_{\text{bottom}} = \bar{y} = 79.54 \text{ mm.}$$

$$y_{\text{top}} = 200 - \bar{y} = 200 - 79.54 = 120.46 \text{ mm.}$$

Top fibres are in compression. Hence from consideration of compression strength, moment carrying capacity of the beam is given by

$$\begin{aligned}
 M_1 &= f_{\text{per}} \text{ in compression} \times \frac{I}{y_{\text{top}}} \\
 &= 90 \times \frac{61.955493 \times 10^6}{120.46} \\
 &= 46.289178 \times 10^6 \text{ N-mm} \\
 &= 46.289178 \text{ kN-m.}
 \end{aligned}$$

Bottom fibres are in tension. Hence from consideration of tension, moment carrying capacity of the section is given by

$$M_2 = f_{\text{per}} \text{ in tension} \times \frac{I}{y_{\text{bottom}}}$$

$$\begin{aligned}
 &= \frac{30 \times 61.955493 \times 10^6}{79.54} \\
 &= 21.367674 \times 10^6 \text{ N-mm} \\
 &= 21.367674 \text{ kN-m.}
 \end{aligned}$$

Actual moment carrying capacity is the lower value of the above two values. Hence moment carrying capacity of the section is

$$= 21.367674 \text{ kN-m.}$$

Maximum moment in a simply supported beam subjected to *udl* of *w*/unit length and span *L* is

$$= \frac{wL^2}{8}$$

Equating maximum moment to moment carrying capacity of the section, we get maximum load carrying capacity of the beam as

$$w \times \frac{5^2}{8} = 21.367674$$

$$\therefore w = 6.838 \text{ kN/m.}$$

**Example 10.7.** The diameter of a concrete flag post varies from 240 mm at base to 120 mm at top as shown in Fig. 10.15. The height of the post is 10 m. If the post is subjected to a horizontal force of 600 N at top, find the section at which stress is maximum. Find its value also.

**Solution:** Consider a section *y* metres from top. Diameter at this section is

$$\begin{aligned}
 d &= 120 + \frac{y}{10} (240 - 120) \\
 &= 120 + 12y \text{ mm}
 \end{aligned}$$

$$\therefore I = \frac{\pi}{64} d^4$$

$$\begin{aligned}
 Z &= \frac{I}{d/2} = \frac{\pi}{32} d^3 \\
 &= \frac{\pi}{32} [120 + 12y]^3
 \end{aligned}$$

At this section, moment is given by

$$\begin{aligned}
 M &= 600 y \text{ N-m} \\
 &= 600000 y \text{ N-mm.}
 \end{aligned}$$

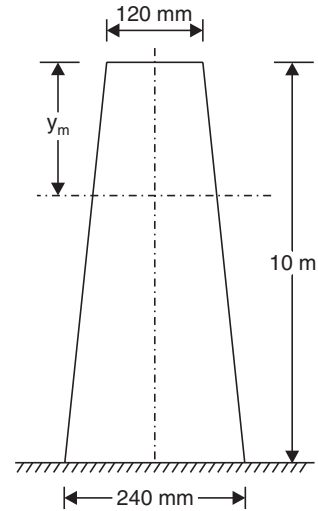
Equating moment of resistance to moment at the section, we get

$$f Z = M$$

where *f* is extreme fibre stress

$$\therefore f \cdot \frac{\pi}{32} [120 + 12y]^3 = 600000 y$$

$$\therefore f = 600000 \times 32 \frac{y}{\pi [120 + 12y]^3}$$



**Fig. 10.15**

For 'f' to be maximum,  $\frac{df}{dy} = 0$

$$600000 \times 32 [(120 + 12y)^{-3} + y(-3)(120 + 12y)^{-4} \times 12] = 0$$

$$\text{i.e.,} \quad (120 + 12y)^{-3} = 36 (120 + 12y)^{-4} y$$

$$\text{i.e.,} \quad 1 = 36 (120 + 12y)^{-1} y$$

$$\text{i.e.,} \quad 120 + 12y = 36y$$

$$\therefore \quad y = 5 \text{ m.}$$

Stress at this section  $f$  is given by

$$f = 600000 \times 32 \times \frac{5}{\pi (120 + 12 \times 5)^3}$$

$$f = 5.24 \text{ N/mm}^2.$$

**Example 10.8.** A circular log of timber has diameter  $D$ . Find the dimensions of the strongest rectangular section one can cut from this.

**Solution:** Let the width and depth of strongest section that can be cut from the log be 'b' and 'd' respectively. Then,

$$D^2 = b^2 + d^2$$

or

$$d^2 = D^2 - b^2$$

For rectangular section

$$I = \frac{1}{12} bd^3$$

$$y_{\max} = d/2.$$

$\therefore$

$$z = \frac{I}{y_{\max}} = \frac{1}{6} bd^2$$

$$= \frac{1}{6} b (D^2 - b^2) = \frac{1}{6} (bD^2 - b^3)$$

The beam is strongest if section modulus is maximum. Hence the condition is

$$\frac{dz}{db} = 0$$

$$\frac{1}{6} [D^2 - 3b^2] = 0$$

i.e.,

$$D^2 = 3b^2$$

or

$$b = \frac{D}{\sqrt{3}}.$$

$\therefore$

$$d = \sqrt{(D^2 - b^2)} = \sqrt{D^2 - \frac{D^2}{3}} = D \sqrt{2/3}$$

Thus the dimensions of strongest beam

$$= \frac{D}{\sqrt{3}} \text{ wide} \times \sqrt{2/3} D \text{ deep.}$$

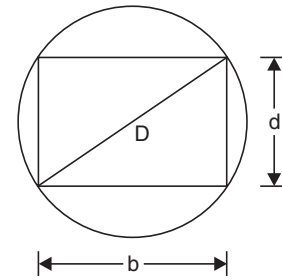


Fig. 10.16

### 10.6 PROPORTIONING SECTIONS

In designing beams, span of the beam is known and load expected on the beam can be estimated. Hence bending moment to be resisted by the beam can be calculated. A designer has to select suitable section of the beam of desirable materials. Theoretically speaking, the section required changes along the span. Usually uniform sections are used. Hence the section selected should be capable of resisting the maximum moment. In case of circular sections we may find the diameter required, since section modulus required depends only on diameter. In case of rectangular sections, section modulus depends upon width and depth. Hence usually width is assumed and depth is calculated or else ratio of width to depth is assumed and section is selected. For steel sections, Indian Standard Hand Book may be used to identify standard section that satisfies the required section modulus value. This process of proportioning sections is known as **Design**. The design process is illustrated with the following examples:

**Example 10.9.** Design a timber beam is to carry a load of 5 kN/m over a simply supported span of 6 m. Permissible stress in timber is 10 N/mm<sup>2</sup>. Keep depth twice the width.

**Solution:**

$$w = 5 \text{ kN/m}, \quad L = 6 \text{ m.}$$

∴ Maximum bending moment

$$\begin{aligned} &= \frac{wL^2}{8} = \frac{5 \times 6^2}{8} = 22.5 \text{ kN-m} \\ &= 22.5 \times 10^6 \text{ N-mm} \end{aligned}$$

Let  $b$  be the width and  $d$  the depth. Then in this problem  $d = 2b$

$$\begin{aligned} \therefore Z &= \frac{1}{6} bd^2 = \frac{1}{6} \times b (2b)^2 \\ &= \frac{2}{3} b^3 \\ f &= 10 \text{ N/mm}^2 \quad (\text{given}) \end{aligned}$$

Hence design requirement is

$$fZ = M$$

$$\text{i.e.,} \quad 10 \times \frac{2}{3} b^3 = 22.5 \times 10^6$$

$$\therefore b = 150 \text{ mm.}$$

$$\therefore d = 2b = 300 \text{ mm.}$$

Use 150 mm × 300 mm section.

**Example 10.10.** A cantilever of 3 m span, carrying uniformly distributed load of 3 kN/m is to be designed using cast iron rectangular section. Permissible stresses in cast iron are  $f = 30 \text{ N/mm}^2$  in tension and  $f_c = 90 \text{ N/mm}^2$  in compression. Proportion the section suitably.

**Solution:**

$$\text{Span of cantilever} = 3 \text{ m} \quad w = 3 \text{ kN/m}$$

$$\begin{aligned} \therefore \text{Maximum moment} &= \frac{wL^2}{2} = \frac{3 \times 3^2}{2} = 13.5 \text{ kN-m} \\ &= 13.5 \times 10^6 \text{ N-mm} \end{aligned}$$

Let  $b$  be the width and  $d$  the depth.

$$\therefore Z = \frac{1}{6} bd^2$$

Since it is rectangular section,  $N-A$  lies at mid-depth, and stresses at top and bottom are same. Hence, permissible tensile stress value is reached earlier and it governs the design.

$$f_{\text{per}} = 30 \text{ N/mm}^2$$

$\therefore$  Design condition is

$$fZ = M$$

$$30 \times \frac{1}{6} bd^2 = 13.5 \times 10^6$$

$$bd^2 = 2700000$$

Using  $b = 100 \text{ mm}$ , we get

$$d^2 = \frac{2700000}{100}$$

$$\therefore d = 164.3 \text{ mm}$$

Use 100 mm wide and 165 mm deep section.

**Example 10.11.** A circular bar of simply supported span 1 m has to carry a central concentrated load of 800 N. Find the diameter of the bar required, if permissible stress is 150 N/mm<sup>2</sup>.

**Solution:** Let the diameter of the bar be ' $d$ '. Now,  $W = 800 \text{ N}$   $L = 1 \text{ m} = 1000 \text{ mm}$ .

$$\begin{aligned} \therefore \text{Maximum moment} &= \frac{WL}{4} \\ &= \frac{800 \times 1000}{4} = 200000 \text{ N-mm} \end{aligned}$$

$$f = 150 \text{ N/mm}^2$$

$$Z = \frac{\pi}{32} d^3$$

$\therefore$  Design condition is,

$$150 \times \frac{\pi}{32} d^3 = 200000$$

$$\therefore d = 23.8 \text{ mm}$$

**Select 25 mm bar** (which is available in market)

## 10.7 SHEAR STRESS DISTRIBUTION

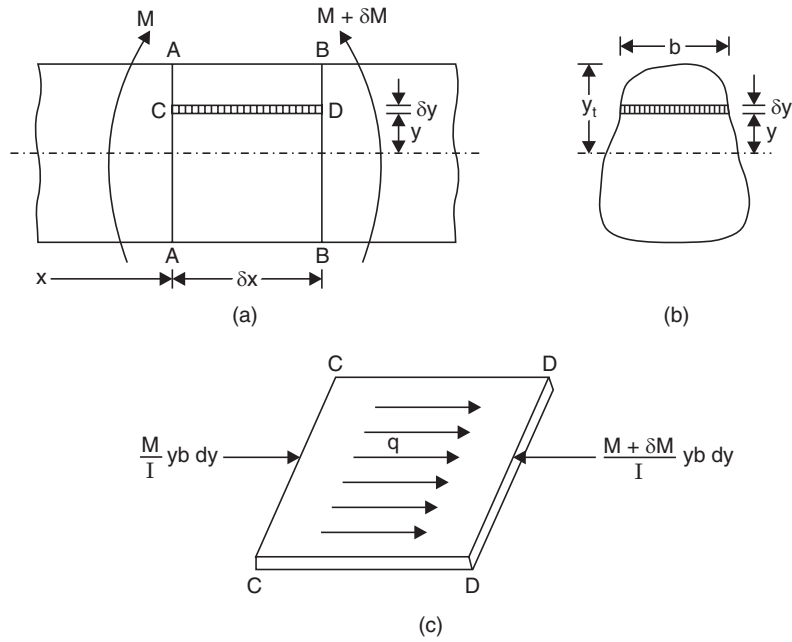
In the 9<sup>th</sup> chapter we have seen that in a beam bending moment as well as shearing forces act. Shear force gives rise to shearing stresses in the beam. In this article expression for shearing stress is derived and its variation across the section is discussed. A designer has to see that the beam is safe not only in bending but in shear also.

### 10.7.1 Expression for Shear Stress

Consider an elemental length ' $\delta x$ ' of beam shown in Fig. 10.17 (a). Let bending moment at section A-A be  $M$  and that at section B-B be  $M + \delta M$ . Let  $CD$  be an elemental fibre at distance  $y$  from neutral axis and its thickness be  $\delta y$ . Then,

Bending stress on left side of elemental fibre

$$= \frac{M}{I} y$$



**Fig. 10.17**

$\therefore$  The force on left side of element

$$= \frac{M}{I} y b \delta y$$

Similarly, force on right side on elemental fibre

$$= \frac{M + \delta M}{I} y b \delta y$$

$\therefore$  Unbalanced horizontal force on right side of elemental fibre

$$\begin{aligned} &= \frac{M + \delta M}{I} y b \delta y - \frac{M}{I} y b \delta y \\ &= \frac{\delta M}{I} y b \delta y \end{aligned}$$

There are a number of such elemental fibres above  $CD$ . Hence unbalanced horizontal force on section  $CD$

$$\begin{aligned}
 &= \int_y^{y_t} \frac{dM}{I} y b \delta y \\
 &= \int_y^{y_t} \frac{dM}{I} y b dy = \frac{\delta M}{I} \int_y^{y_t} yb dy
 \end{aligned}$$

Let intensity of shearing stress on element  $CD$  be  $q$ . [Refer Fig. 10.17 (c)]. Then equating resisting shearing force to unbalanced horizontal force, we get

$$q b \delta x = \frac{\delta M}{I} \int_y^{y_t} yb dy$$

$$\therefore q = \frac{\delta M}{\delta x} \frac{1}{bI} \int_y^{y_t} yb dy$$

$$\text{As } \delta x \rightarrow 0, \quad q = \frac{dM}{dx} \frac{1}{bI} (a\bar{y})$$

where  $a\bar{y}$  = Moment of area above the section under consideration about neutral axis.

$$\text{But we know } \frac{dM}{dx} = F$$

$$\therefore q = \frac{F}{bI} (a\bar{y}) \quad \dots(10.11)$$

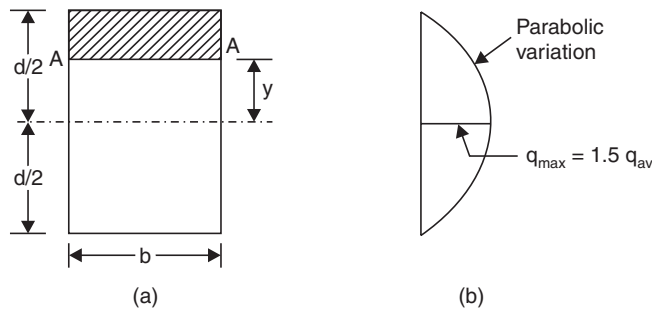
The above expression gives shear stress at any fibre  $y$  distance above neutral axis.

### 10.7.2 Variation of Shear Stresses Across Standard Sections

Variation of shear stresses across the following three cases are discussed below:

- (i) Rectangular
- (ii) Circular and
- (iii) Isosceles triangle.

- (i) **Rectangular section.** Consider the rectangular section of width ' $b$ ' and depth shown in Fig. 10.18(a). Let A-A be the fibre at a distance  $y$  from neutral axis. Let the shear force on the section be  $F$ .



**Fig. 10.18**

From equation (10.11), shear stress at this section is

$$q = \frac{F}{bI} (a\bar{y})$$

where  $(a\bar{y})$  is the moment of area above the section about the neutral axis. Now,

$$a = b(d/2 - y)$$

$$\bar{y} = y + \frac{1}{2} (d/2 - y) = \frac{1}{2} (d/2 + y)$$

$$\begin{aligned} \therefore a\bar{y} &= \frac{b}{2} (d/2 - y) \times \frac{1}{2} (d/2 + y) \\ &= \frac{b}{2} (d^2/4 - y^2) \end{aligned}$$

$$I = \frac{1}{12} bd^3$$

$$\begin{aligned} \therefore q &= \frac{F}{b \frac{1}{12} bd^3} \frac{b}{2} (d^2/4 - y^2) \\ &= \frac{6F}{bd^3} (d^2/4 - y^2) \end{aligned}$$

This shows shear stress varies parabolically.

When  $y = \pm d/2$ ,  $q = 0$

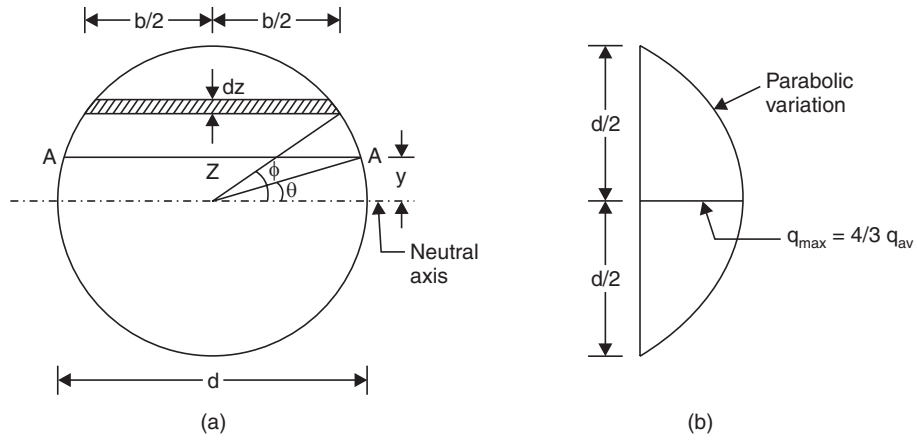
$$\begin{aligned} \text{At } y = 0, \quad q_{\max} &= \frac{6F}{bd^3} \frac{d^2}{4} = 1.5 \frac{F}{bd} \\ &= 1.5 q_{av} \end{aligned}$$

where  $q_{av} = \frac{F}{bd}$  is average shear stress.

Thus in rectangular section maximum shear stress is at neutral axis and it is 1.5 times average shear stress. It varies parabolically from zero at extreme fibres to  $1.5 q_{av}$  at mid depth as shown in Fig. 10.18(b).

(ii) **Circular section.** Consider a circular section of diameter ' $d$ ' as shown in Fig. 10.19 (a) on which a shear force  $F$  is acting. Let A-A be the section at distance ' $y$ ' from neutral axis at which shear stress is to be found. To find moment of area of the portion above A-A about neutral axis, let us consider an element at distance ' $z$ ' from neutral axis. Let its thickness be  $dz$ . Let it be at an angular distance  $\phi$  and A-A be at angular distance  $\theta$  as shown in figure.



**Fig. 10.19**

Width of element  $b = 2 \cdot \frac{d}{2} \cos \phi$

$$= d \cos \phi$$

$$z = \frac{d}{2} \sin \phi$$

$\therefore dz = \frac{d}{2} \cos \phi d\phi$

$\therefore$  Area of the element

$$a = b dz = d \cos \phi \cdot \frac{d}{2} \cos \phi d\phi$$

$$= \frac{d^2}{2} \cos^2 \phi d\phi$$

Moment of this area about neutral axis

$$= \text{area} \times z$$

$$= \frac{d^2}{2} \cos^2 \phi d\phi \cdot \frac{d}{2} \sin \phi$$

$$= \frac{d^3}{4} \cos^2 \phi \sin \phi d\phi$$

$\therefore$  Moment of area about section A-A about neutral axis

$$(a\bar{y}) = \int_{\theta}^{\pi/2} \frac{d^2}{4} \cos^2 \phi \sin \phi d\phi$$

$$= \frac{d^3}{4} \left[ \frac{-\cos^3 \phi}{3} \right]_{\theta}^{\pi/2}$$

[Since if  $\cos \phi = t$ ,  $dt = -\sin \phi d\phi$  and  $-t^3/3$  is integration]

$$\begin{aligned}\therefore (a\bar{y}) &= \frac{d^3}{4 \times 3} \left[ -\cos^2 \frac{\pi}{2} + \cos^3 \theta \right] \\ &= \frac{d}{12} \cos^3 \theta\end{aligned}$$

$$\text{Now } I = \frac{\pi d^4}{64}$$

$$\begin{aligned}\therefore q &= \frac{F}{bI} (a\bar{y}) \\ &= \frac{F}{d \cos \theta \frac{\pi}{64} d^4} \times \frac{d^3}{12} \cos^3 \theta \\ &= \frac{64}{12} \frac{F}{\pi d^2} \cos^2 \theta \\ &= \frac{16}{3} \frac{F}{\pi d^2} [1 - \sin^2 \theta] \\ &= \frac{16}{3} \frac{F}{\pi d^2} \left[ 1 - \left( \frac{y}{d/2} \right)^2 \right] \\ &= \frac{16}{3} \frac{F}{\pi d^2} \left[ 1 - \frac{4y^2}{d^2} \right]\end{aligned}$$

Hence shear stress varies parabolically.

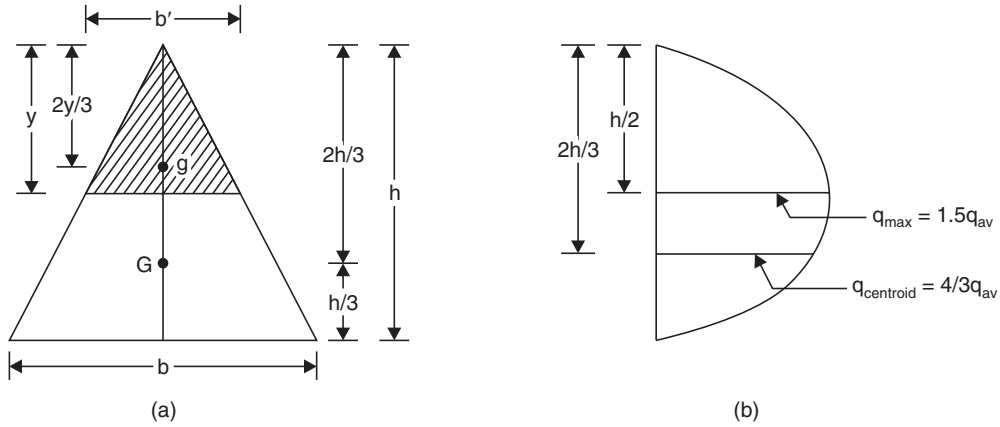
$$\text{At } y = \pm d/2, \quad q = 0$$

$$\begin{aligned}y = 0, \quad q &= q_{\max} = \frac{16}{3} \frac{F}{\pi d^2} \\ &= \frac{4}{3} \frac{F}{\pi/4 d^2} \\ &= \frac{4}{3} \frac{F}{\text{Area}} \\ &= \frac{4}{3} q_{\text{av}}.\end{aligned}$$

where  $q_{\text{av}}$  = average shear stress.

Thus in circular sections also shear stress varies parabolically from zero at extreme edges to the maximum value of  $\frac{4}{3} q_{\text{av}}$  at mid depth as shown in Fig. 10.19(b).

- (iii) **Isosceles triangular section.** Consider the isosceles triangular section of width ' $b$ ' and height ' $h$ ' as shown in Fig. 10.20(a). Its centroid and hence neutral axis is at  $\frac{2h}{3}$  from top fibre. Now shear stress is to be found at section A-A which is at a depth ' $y$ ' from top fibre.



**Fig. 10.20**

At A-A width  $b' = \frac{y}{h} b$

$$\begin{aligned} \text{Area above A-A} \quad a &= \frac{1}{2} b' y \\ &= \frac{1}{2} \frac{b}{h} y^2 \end{aligned}$$

Its centroid from top fibre is at  $\frac{2y}{3}$ .

$\therefore$  Distance of shaded area above the section A-A from neutral axis  $\bar{y} = \frac{2h}{3} - \frac{2y}{3}$ .

$$\begin{aligned} \therefore a \bar{y} &= \frac{1}{2} \frac{b}{h} y^2 \left( \frac{2h}{3} - \frac{2y}{3} \right) \\ &= \frac{1}{3} \frac{b}{h} y^2 (h - y) \end{aligned}$$

Moment of inertia of the section

$$I = \frac{bh^3}{36}.$$

∴ Shear stress at A-A

$$\begin{aligned}
 q &= \frac{F}{bI} a \bar{y} \\
 &= \frac{F}{\frac{y}{h} b \times \frac{bh^3}{36}} \times \frac{1}{3} \frac{b}{h} y^2 (h - y) \\
 &= \frac{12 F}{bh^3} y(h - y)
 \end{aligned}$$

Hence at  $y = 0$ ,  $q = 0$

At  $y = h$ ,  $q = 0$

At centroid,  $y = \frac{2h}{3}$

$$\begin{aligned}
 q &= \frac{12 F}{bh^3} \frac{2h}{3} (h - 2h/3) \\
 &= \frac{8}{3} \frac{F}{bh} = \frac{4}{3} \frac{F}{1/2 bh} \\
 &= \frac{4}{3} q_{av}
 \end{aligned}$$

where  $q_{av}$  is average shear stress.

For  $q_{max}$ ,  $\frac{dq}{dy} = 0$

$$i.e., \quad \frac{12 F}{bh^3} (h - 2y) = 0$$

$$i.e., \quad \text{at} \quad y = h/2$$

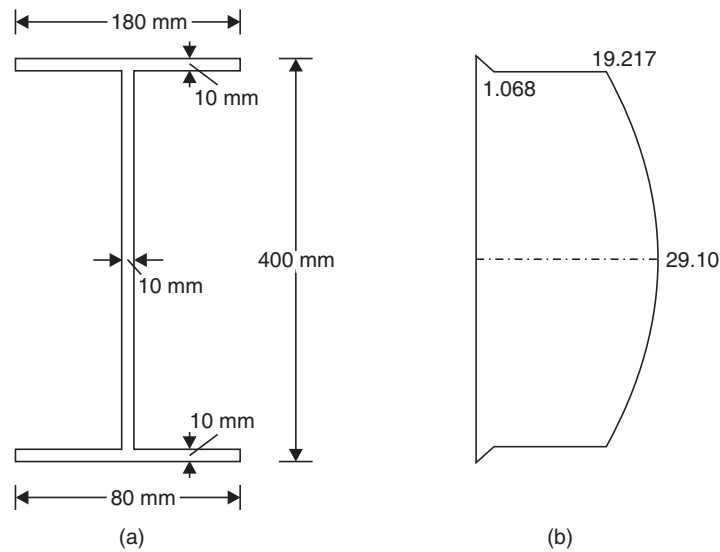
$$\begin{aligned}
 \text{and hence} \quad q_{max} &= \frac{12 F}{bh^3} \cdot \frac{h}{2} (h - h/2) \\
 &= \frac{12 F}{4 bh} = \frac{3F}{bh} \\
 &= \frac{1.5 F}{1/2 bh} \\
 &= 1.5 q_{av}
 \end{aligned}$$

Thus in isosceles triangular section shear stress is zero at extreme fibres, it is maximum of  $1.5 q_{av}$  at mid depth and has a value  $\frac{4}{3} q_{av}$  at neutral axis. The variation of shear stress is as shown in Fig. 10.20(b).

### 10.8 SHEAR STRESSES IN BUILT-UP SECTIONS

In sections like  $I$ ,  $T$  and channel, shear stresses at various salient points are calculated and the shear stress variation diagram across depth is plotted. It may be noted that at extreme fibres shear stress is zero since  $(a\bar{y})$  term works out to be zero. However it may be noted that the procedure explained below is for built up section with at least one symmetric axis. If there is no symmetric axis along the depth analysis for shear stress is complex, and that is treated beyond the scope to this book.

**Example 10.12.** Draw the shear stress variation diagram for the  $I$ -section shown in Fig. 10.21(a) if it is subjected to a shear force of 100 kN.



**Fig. 10.21**

**Solution:** Due to symmetry neutral axis is at mid depth.

$$\begin{aligned}
 I &= \frac{1}{12} \times 180 \times 10^3 + 180 \times 10 \times (200 - 5)^2 \\
 &\quad + \frac{1}{12} \times 10 \times 380^3 + 10 \times 380 \times (200 - 200)^2 \\
 &\quad + \frac{1}{12} \times 180 \times 10^3 + 180 \times 10 \times (200 - 5)^2 \\
 &= 182.646666 \times 10^6 \text{ mm}^4
 \end{aligned}$$

Shear stress at  $y = 200$  mm is zero since  $a\bar{y} = 0$ .

Shear stress at bottom of top flange

$$= \frac{F}{bI} (a\bar{y})$$

$$= \frac{100 \times 1000}{180 \times 182.646666 \times 10^6} \times (180 \times 10 \times 195)$$

$$= 1.068 \text{ N/mm}^2$$

Shear stress in the web at the junction with flange

$$= \frac{100 \times 1000}{10 \times 182.646666 \times 10^6} (180 \times 10 \times 195)$$

$$= 19.217 \text{ N/mm}^2$$

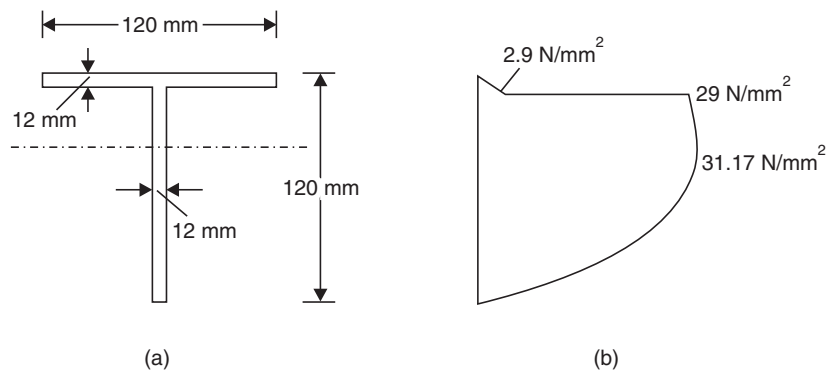
Shear stress at  $N-A$

$$= \frac{100 \times 1000}{10 \times 182.646666} \times \left[ 180 \times 10 \times 195 + 10 \times (200 - 10) \times \frac{190}{2} \right]$$

$$= 29.10 \text{ N/mm}^2.$$

Symmetric values will be there on lower side. Hence shear stress variation is as shown in Fig. 10.21(b).

**Example 10.13.** A beam has cross-section as shown in Fig. 10.22(a). If the shear force acting on this is 25 kN, draw the shear stress distribution diagram across the depth.



**Fig. 10.22**

**Solution:** Let  $\bar{y}$  be the distance of centroid of the section from its top fibre. Then

$$y' = \frac{\text{Moment of area about top fibre}}{\text{Total area}}$$

$$= \frac{120 \times 12 \times 6 + (120 - 12) \times 12 \times \left( 12 + \frac{120 - 12}{2} \right)}{120 \times 12 + (120 - 12) \times 12}$$

$$= 34.42 \text{ mm}$$

∴ Moment of inertia about centroid

$$\begin{aligned}
 I &= \frac{1}{12} \times 120 \times 12^3 + 120 \times 12 (34.42 - 6)^2 \\
 &\quad + \frac{1}{12} \times 12 \times 108^3 + 12 \times 108 \left( 34.42 - \frac{108}{2} \right)^2 \\
 &= 2936930 \text{ mm}^4
 \end{aligned}$$

Shear stresses are zero at extreme fibres.

Shear stress at bottom of flange:

Area above this level,  $a = 120 \times 12 = 1440 \text{ mm}^2$

Centroid of this area above  $N-A$

$$\bar{y} = 34.42 - 6 = 28.42 \text{ mm}$$

Width at this level  $b = 120 \text{ mm}$ .

$$\begin{aligned}
 \therefore q_{\text{bottom of flange}} &= \frac{25 \times 1000}{120 \times 2936930} \times 1440 \times 28.42 \\
 &= 2.90 \text{ N/mm}^2
 \end{aligned}$$

Shear stress at the same level but in web, where width  $b = 12 \text{ mm}$

$$\begin{aligned}
 &= \frac{25 \times 1000}{12 \times 2936930} \times 1440 \times 28.42 \\
 &= 29.0 \text{ N/mm}^2
 \end{aligned}$$

Shear stress at neutral axis:

For this we can consider  $a\bar{y}$  term above this section or below this section. It is convenient to consider the term below this level.

$$a = 12 \times (120 - 34.42) = 1026.96 \text{ mm}^2$$

The distance of its centroid from  $N-A$

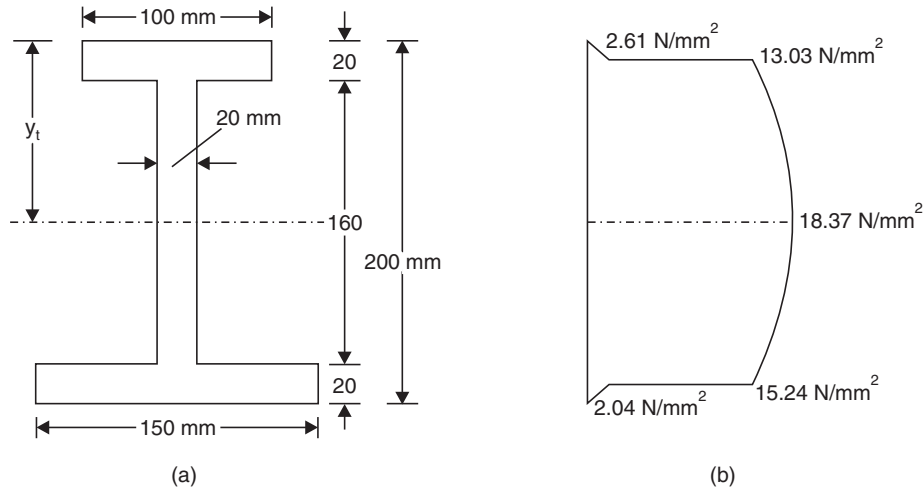
$$= \frac{120 - 34.42}{2} = 42.79 \text{ mm.}$$

Width at this section  $b = 12 \text{ mm}$ .

$$\begin{aligned}
 \therefore q &= \frac{25 \times 1000}{12 \times 2936930} \times 1026.96 \times 42.79 \\
 &= 31.17 \text{ N/mm}^2
 \end{aligned}$$

Hence variation of shear stress across the depth is as shown in Fig. 10.22(b).

**Example 10.14.** The unsymmetric I-section shown in Fig. 10.23(a) is the cross-section of a beam, which is subjected to a shear force of 60 kN. Draw the shear stress variation diagram across the depth.



**Fig. 10.23**

**Solution:** Distance of neutral axis (centroid) of the section from top fibre be  $y_t$ . Then

$$y_t = \frac{100 \times 20 \times 10 + (200 - 20 - 20) \times 20 \times \left(20 + \frac{160}{2}\right) + 150 \times 20 \times (200 - 10)}{100 \times 20 + 160 \times 20 + 150 \times 20}$$

$$= 111 \text{ mm}$$

$$I = \frac{1}{12} \times 100 \times 20^3 + 100 \times 20 (111 - 10)^2$$

$$+ \frac{1}{12} \times 20 \times 160^3 + 160 \times 20 (111 - 100)^2$$

$$+ \frac{1}{12} \times 150 \times 20^3 + 150 \times 20 (111 - 190)^2$$

$$= 46505533 \text{ mm}^4$$

Shear stress at bottom of top flange

$$= \frac{F}{bI} a \bar{y}$$

$$= \frac{60 \times 1000}{100 \times 46505533} \times 100 \times 20 \times (111 - 10)$$

$$= 2.61 \text{ N/mm}^2$$

$\therefore$  Shear stress at the same level, but in web

$$= \frac{60 \times 1000}{20 \times 46505533} \times 100 \times 20 (111 - 10)$$

$$= 13.03 \text{ N/mm}^2$$



Shear stress at neutral axis:

$$\begin{aligned} a\bar{y} &= a\bar{y} \text{ of top flange} + a\bar{y} \text{ of web above } N-A \\ &= 100 \times 20 \times (111 - 10) + 20 \times (111 - 20) \times \frac{111 - 20}{2} \\ &= 284810 \text{ mm}^3. \end{aligned}$$

$\therefore$  Shear stress at neutral axis

$$\begin{aligned} &= \frac{F}{bI} (a\bar{y}) \\ &= \frac{60 \times 1000}{20 \times 46505533} \times 284810 \\ &= 18.37 \text{ N/mm}^2. \end{aligned}$$

Shear stress at junction of web and lower flange:

Considering the lower side of the section for finding  $a\bar{y}$ , we get

$$a\bar{y} = 150 \times 20 \times (190 - 111) = 237000 \text{ mm}^3$$

$\therefore$

$$\begin{aligned} q &= \frac{60 \times 1000}{20 \times 46505533} \times 237000 \\ &= 15.28 \text{ N/mm}^2 \end{aligned}$$

At the above level but in web, shear stress

$$\begin{aligned} &= \frac{60 \times 1000}{150 \times 46505533} \times 237000 \\ &= 2.04 \text{ N/mm}^2 \end{aligned}$$

At extreme fibres shear stress is zero. Hence variation of shear across the depth of the section is as shown in Fig. 10.23.

#### IMPORTANT FORMULAE

1. Bending equation:  $\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$ .

2. Modulus of section  $Z = \frac{I}{y_{\max}}$ .

3. Moment carrying capacity of section  $= f_{\text{per}} Z$ .

4. Section modulus of various sections:

(i) Rectangular:  $\frac{1}{6} bd^2$

(ii) Hollow rectangular:  $\frac{1}{6} \frac{BD^3 - bd^3}{D}$

(iii) Solid circular section:  $\frac{\pi}{32} d^3$

(iv) Hollow circular section:  $\frac{\pi}{32} \frac{D^4 - d^4}{D}$

(v) Solid triangular section:  $\frac{bh^2}{24}$

5. Shear stress in a beam  $q = \frac{F}{bI}(a\bar{y})$

6. In rectangular sections,

$$q_{\max} = 1.5 q_{\text{av}}, \text{ at } y = d/2$$

In circular sections  $q_{\max} = \frac{4}{3} q_{\text{av}}, \text{ at centre}$

In triangular section,  $q_{\max} = 1.5 q_{\text{av}}, \text{ at } y = \frac{h}{2}$ .

### THEORY QUESTIONS

1. State the assumptions made in deriving bending equation.
2. Derive the bending equation  $\frac{M}{I} = \frac{F}{Y} = \frac{E}{R}$  with usual meaning.
3. Derive the expression for finding shear stress in a beam in the form  $q = \frac{F}{bI}(a\bar{y})$  with usual notations.
4. Show that maximum shear stress in a beam of rectangular section is  $1.5 q_{\text{average}}$ .
5. Show that in a beam of solid circular section maximum shear stress is  $\frac{4}{3} \times q_{\text{average}}$ .
6. Draw the variation of shear stresses across the depth of a beam of triangular section, finding the values at critical depths.

### PROBLEMS FOR EXERCISE

1. A *I*-section has flanges of size  $200 \times 12$  mm and its overall depth is 360 mm. Thickness of web is also 12 mm. It is used as a simply supported beam over a span of 4 m to carry a load of 60 kN/m over its entire span. Draw the variation of bending and shearing stresses across the depth.

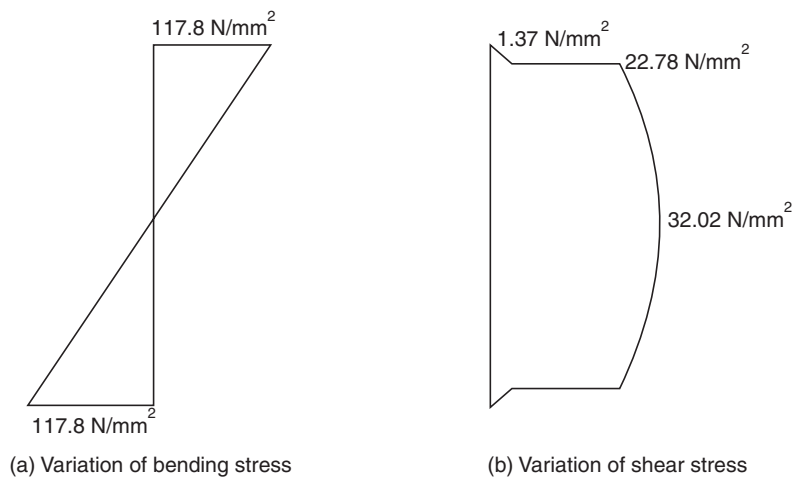
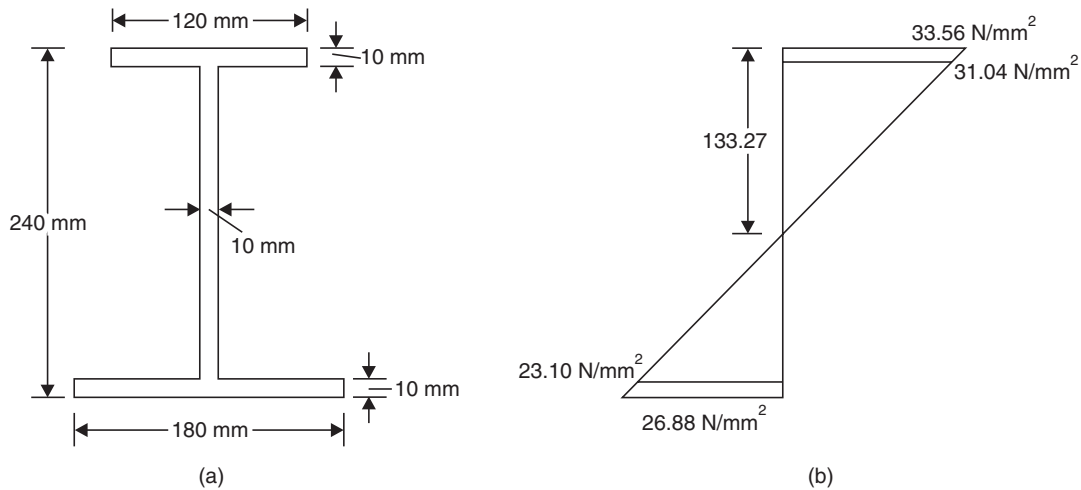


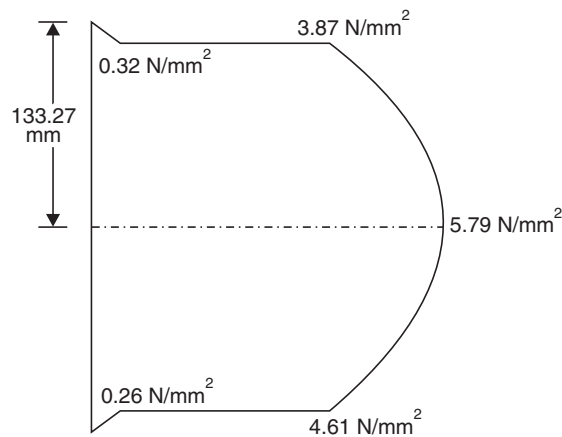
Fig. 10.24

2. Unsymmetric *I*-section shown in Fig. 10.25 is used as a cantilever of span 2 m to carry uniformly distributed load of 6 kN/m over entire span. Draw the variation of bending stress across the depth marking the values at salient point.



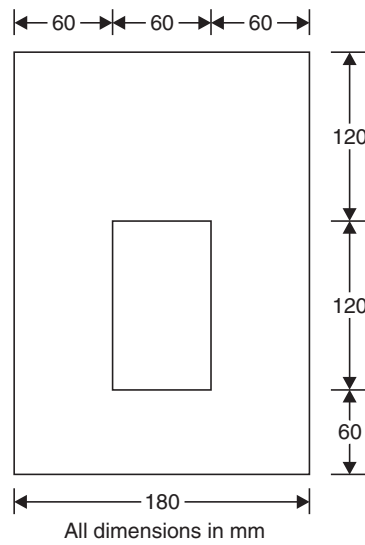
**Fig. 10.25**

3. Calculate the variation of shear stresses at various salient level near fixed support in the above problem and give the sketch. [Ans. See Fig. 10.26]



**Fig. 10.26**

4. A cantilever beam of 1.2 m span is having cross-section as shown in Fig. 10.27. The permissible stresses in tension and compressions are 20 N/mm<sup>2</sup> and 80 N/mm<sup>2</sup> respectively. Determine the maximum concentrated load *W* it can carry at the free end. [Ans. *W* = 44.576 kN]  
[Note: In cantilever tension is at top.]

**Fig. 10.27**

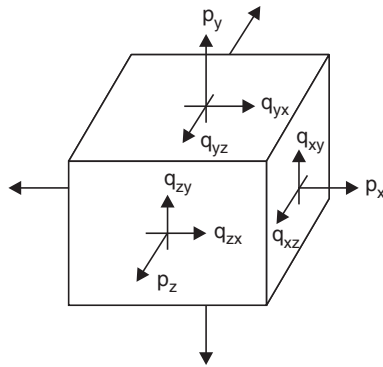
5. A timber beam is to be designed to carry a load of 6 kN/m over a simply supported span of 5 m. Permissible stress is 10 N/mm<sup>2</sup>. Keeping the depth twice the width, design the beam. If the permissible stress in shear is 1 N/mm<sup>2</sup>, check for shear.

[Ans. Required  $b = 141.18$ ,  $d = 282.31$ . Hence select 150 × 300 mm section]

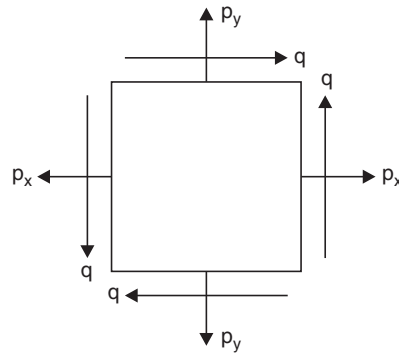
# Principal Stresses and Strains

A structural member need not be always under simple (only one type) of stress. It may be subjected to direct stresses in different directions and may be subjected to shear stresses also. A beam is usually subjected to axial stresses due to bending and also for shear stresses. A shaft is subjected to shear stresses due to torsion and axial stresses due to bending/direct thrust. The stresses may vary from point to point in the member.

In a three-dimensional system, stresses acting at a point may be represented as shown in Fig. 11.1.



**Fig. 11.1**



**Fig. 11.2**

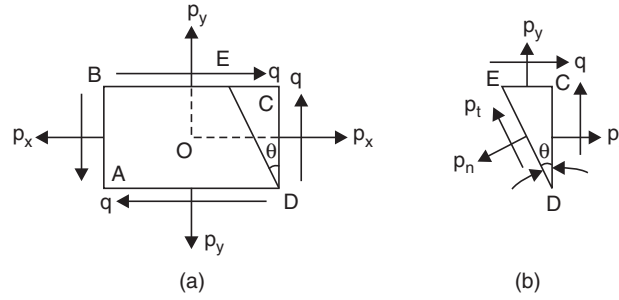
In many engineering problems, two dimensional idealization is made as shown in Fig. 11.2. In this chapter discussion is limited to two dimensional stress system. First general expression for stresses on a plane inclined at a selected axis is discussed. Then the terms principal stresses and planes are explained and the expressions to get them are presented. A number of problems are solved to make the concept clear. At the end analysis is given for principal strains also.

## 11.1 STRESSES ON INCLINED PLANES

Consider the element in a structural member which is under a general two dimensional state of stress as shown in Fig. 11.3(a). Note the sign convention:

+ve face +ve direction is +ve stress for  $p_x$  and  $p_y$ . In other words tensile stresses are +ve.

Shearing stress as shown in the figure is +ve stress.

**Fig. 11.3**

Now we are interested in finding state of stress on plane  $DE$  which makes anticlockwise angle  $\theta$  with the plane of stress  $p_x$ , in other words with  $y$ -axis.

For simplicity let us consider thickness of the element as unity. We are interested in finding normal and tangential stresses acting on the plane  $DE$ . Let normal stress be  $p_n$  and tangential stress  $p_t$  as shown in Fig. 11.3(b). Since the system is in equilibrium,

$\Sigma$  Forces normal to  $DE = 0$  gives

$$p_n \times DE \times 1 = p_x \times CD \times 1 \times \cos \theta + q \times CD \times 1 \times \sin \theta + p_y \times CE \times 1 \times \sin \theta + q \times CE \times 1 \times \cos \theta$$

$$\therefore p_n = p_x \frac{CD}{DE} \cos \theta + p_y \frac{CE}{DE} \sin \theta + q \frac{CD}{DE} \sin \theta + q \frac{CE}{DE} \cos \theta$$

Since  $\frac{CD}{DE} = \cos \theta$  and  $\frac{CE}{DE} = \sin \theta$ , we get

$$\begin{aligned} p_n &= p_x \cos^2 \theta + p_y \sin^2 \theta + q \cos \theta \sin \theta + q \sin \theta \cos \theta \\ &= p_x \left( \frac{1 + \cos 2\theta}{2} \right) + p_y \left( \frac{1 - \cos 2\theta}{2} \right) + 2q \sin \theta \cos \theta, \end{aligned}$$

since  $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$  and  $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$

Thus 
$$p_n = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\theta + q \sin 2\theta \quad \dots(11.1)$$

Similarly from equilibrium condition of forces tangential to plane  $DE$ , we get

$$p_t \times DE \times 1 = p_x \times CD \times 1 \times \sin \theta - q \times CD \times 1 \times \cos \theta - p_y \times CE \times 1 \times \cos \theta + q \times CE \times 1 \times \sin \theta$$

$$\therefore p_t = p_x \frac{CD}{DE} \sin \theta - p_y \frac{CE}{DE} \cos \theta - q \frac{CD}{DE} \cos \theta + q \frac{CE}{DE} \sin \theta$$

But  $\frac{CD}{DE} = \cos \theta$  and  $\frac{CE}{DE} = \sin \theta$ .

$$\therefore p_t = p_x \sin \theta \cos \theta - p_y \sin \theta \cos \theta - q \cos^2 \theta + q \sin^2 \theta$$

$$\begin{aligned}
 &= (p_x - p_y) \sin \theta \cos \theta - q(\cos^2 \theta - \sin^2 \theta) \\
 &= \frac{p_x - p_y}{2} \sin 2\theta - q \cos 2\theta,
 \end{aligned}$$

since  $2 \sin \theta \cos \theta = \sin 2\theta$   
 and  $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$

Thus

$$p_t = \frac{p_x - p_y}{2} \sin 2\theta - q \cos 2\theta \quad \dots(11.2)$$

Thus in case of a general two dimension stress system the normal and tangential stresses acting on a plane making anticlockwise angle  $\theta$  with the plane of  $p_x$  (y direction) are as given by equations (11.1) and (11.2).

### Angle of Obliquity of Resultant Stress

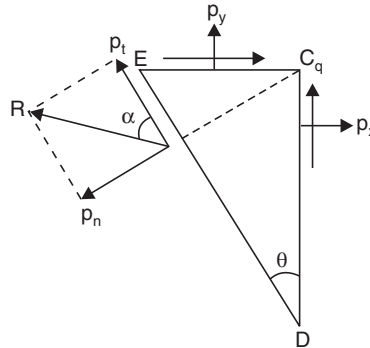
The resultant of the stresses on the above inclined plane can be found as

$$R = \sqrt{p_n^2 + p_t^2}$$

which is inclined at ' $\alpha$ ' to the plane (Ref. Fig. 11.3c).

$$\tan \alpha = \frac{p_n}{p_t}$$

In other words, the resultant is inclined at  $\theta + \alpha$  to the plane of  $p_x$ .



**Fig. 11.3(c)**

## 11.2 PRINCIPAL STRESSES AND PLANES

The planes on which shearing stresses are zero are called *principal planes* and the stresses normal to principal planes are known as *principal stresses*.

### Principal Plane

To locate principal plane

$$p_t = 0$$

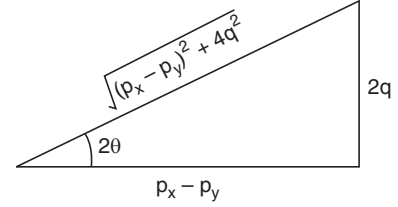
From eqn. (11.2),  $\frac{p_x - p_y}{2} \sin 2\theta - q \cos 2\theta = 0$

$$\therefore \tan 2\theta = \frac{q}{(p_x - p_y)/2} = \frac{2q}{p_x - p_y} \quad \dots(11.3)$$

There are two values for  $2\theta$  which differ by  $180^\circ$  for which eqn. (11.3) can be satisfied. Let  $2\theta_1$  and  $2\theta_2$  be the solution. Referring to Fig. 11.4, we find

$$\sin 2\theta_1 = \frac{2q}{\sqrt{(p_x - p_y)^2 + 4q^2}} \quad \dots(11.4a)$$

$$\cos 2\theta_1 = \frac{p_x - p_y}{\sqrt{(p_x - p_y)^2 + 4q^2}} \quad \dots(11.4b)$$



**Fig. 11.4**

Similarly,

$$\sin 2\theta_2 = \frac{-2q}{\sqrt{(p_x - p_y)^2 + 4q^2}} \quad \dots(11.5a)$$

$$\text{and} \quad \cos 2\theta_2 = \frac{-(p_x - p_y)}{\sqrt{(p_x - p_y)^2 + 4q^2}} \quad \dots(11.5b)$$

$2\theta_1$  and  $2\theta_2$  differ by  $180^\circ$ . Hence we can say  $\theta_1$  and  $\theta_2$  differ by  $90^\circ$ . Thus direction of principal planes to the plane of  $p_n$  are given by eqn. (11.4). Another principal plane is at right angles to it.

### Principal Stresses

Principal stresses are the normal stresses on principal planes. Hence the values of principal stresses may be obtained by substituting  $\theta_1$  and  $\theta_2$  values for  $\theta$  in the expression for  $p_n$ . Denoting the values as  $p_1$  and  $p_2$ , we get

$$\begin{aligned} p_1 &= p_n \text{ at } \theta = \theta_1 \\ &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\theta_1 + q \sin 2\theta_1 \\ &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \frac{p_x - p_y}{\sqrt{(p_x - p_y)^2 + 4q^2}} + q \frac{2q}{\sqrt{(p_x - p_y)^2 + 4q^2}} \\ &= \frac{p_x + p_y}{2} + \frac{1}{2} \frac{(p_x - p_y)^2 + 4q^2}{\sqrt{(p_x - p_y)^2 + 4q^2}} \\ &= \frac{p_x + p_y}{2} + \frac{1}{2} \sqrt{(p_x - p_y)^2 + 4q^2} \\ &= \frac{p_x + p_y}{2} + \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \end{aligned}$$

and

$$\begin{aligned} p_2 &= p_n \text{ at } \theta = \theta_2 \\ &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\theta_2 + q \sin 2\theta_2 \\ &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \frac{[-(p_x - p_y)]}{\sqrt{(p_x - p_y)^2 + 4q^2}} + \frac{q(-2q)}{\sqrt{(p_x - p_y)^2 + 4q^2}} \end{aligned}$$



$$\begin{aligned}
&= \frac{p_x + p_y}{2} - \frac{1}{2} \frac{(p_x - p_y)^2 + 4q^2}{\sqrt{(p_x - p_y)^2 + 4q^2}} \\
&= \frac{p_x + p_y}{2} - \frac{1}{2} \sqrt{(p_x - p_y)^2 + 4q^2} \\
&= \frac{p_x + p_y}{2} - \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}
\end{aligned}$$

Thus the principal stresses are given by

$$p_1 = \frac{p_x + p_y}{2} + \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \quad \dots(11.6a)$$

and

$$p_2 = \frac{p_x + p_y}{2} - \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \quad \dots(11.6b)$$

It can be proved that principal stresses are maximum and minimum stresses also. To find extreme value of normal stress  $p_n$ ,

$$\frac{dp_n}{d\theta} = 0$$

$$\text{i.e.,} \quad \frac{p_x - p_y}{2} 2(-\sin 2\theta) + q 2 \cos 2\theta = 0$$

$$\text{i.e.,} \quad \frac{p_x - p_y}{2} \sin 2\theta - q \cos 2\theta = 0$$

$$\text{i.e.,} \quad p_t = 0$$

Thus the *principal planes are the planes of maximum/minimum normal stresses also*. Plane corresponding to  $\theta_1$  gives the maximum value while plane corresponding to  $\theta_2$  gives minimum normal stress.

### Plane of Maximum Shear Stress

For maximum shear

$$\left. \frac{dp_t}{d\theta} \right|_{\theta=\theta'} = 0$$

$$\text{i.e.,} \quad \frac{p_x - p_y}{2} \cdot 2 \cos 2\theta' - q 2 (-\sin 2\theta') = 0$$

$$\text{or} \quad \tan 2\theta' = \frac{-(p_x - p_y)}{2q} \quad \dots(11.7)$$

From eqn. (11.3) and eqn. (11.7), we get

$$\tan 2\theta \times \tan 2\theta' = -1$$

Hence  $2\theta'$  and  $2\theta$  values differ by  $90^\circ$ . In other word *planes of extreme shearing stresses are at  $\frac{90}{2} = 45^\circ$  to the principal planes.*

Value of maximum shearing stress may be obtained by substituting  $\theta'$  for  $\theta$  in equation for  $p_t$  (eqn. 11.2)

Now 
$$\tan 2\theta' = \frac{-(p_x - p_y)}{2q}$$

From Fig. 11.5, we get

$$\sin 2\theta' = \frac{(p_x - p_y)}{\sqrt{(p_x - p_y)^2 + 4q^2}}$$

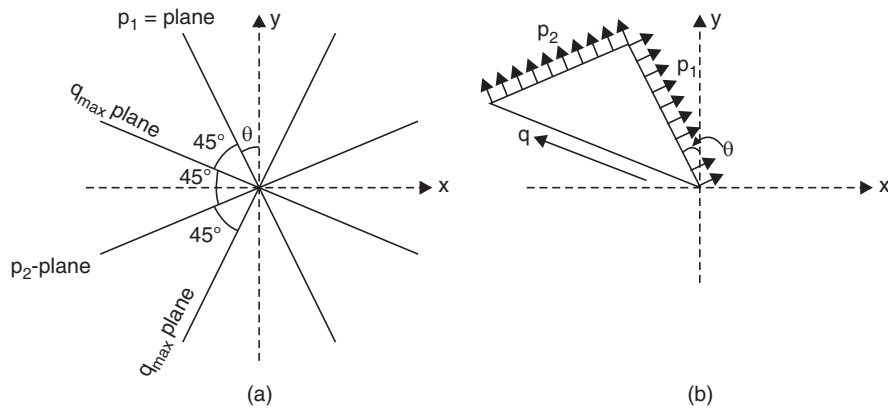
$$\cos 2\theta' = \frac{-2q}{\sqrt{(p_x - p_y)^2 + 4q^2}}$$

$\therefore$

$$\begin{aligned} p_{t \text{ (max)}} &= \frac{p_x - p_y}{2} \sin 2\theta' - q \cos 2\theta' \\ &= \frac{p_x - p_y}{2} \frac{(p_x - p_y)}{\sqrt{(p_x - p_y)^2 + 4q^2}} - \frac{q(-2q)}{\sqrt{(p_x - p_y)^2 + 4q^2}} \\ &= \frac{1}{2} \frac{(p_x - p_y)^2 + 4q^2}{\sqrt{(p_x - p_y)^2 + 4q^2}} \\ &= \frac{1}{2} \sqrt{(p_x - p_y)^2 + 4q^2} \\ &= \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \end{aligned} \quad \dots(11.8a)$$

$$\begin{aligned} &= \frac{1}{2} \text{ the difference between } p_1 \text{ and } p_2 \\ &= \frac{p_1 - p_2}{2} \end{aligned} \quad \dots(11.8b)$$

The direction of principal planes and the planes of maximum shearing stresses may be indicated as shown in Fig. 11.6(a) or 11.6(b).



**Fig. 11.6**

**Example 11.1.** Show that sum of normal stresses in any two mutually perpendicular directions is constant in case of a general two dimensional stress.

**Solution:** Let  $p_x$  and  $p_y$  as shown in Fig. 11.3(a) be normal stresses and  $q$  be the shearing stresses. Let the thickness of element be unity.

Taking element as shown in Fig. 11.3(b) and considering the equation of equilibrium in the direction normal to the plane  $DE$ , we get

$$p_n \times 1 \times DE = p_x \times 1 \times CD \cos \theta + p_y \times 1 \times CE \sin \theta + q \times 1 \times CD \sin \theta + q \times 1 \times CE \cos \theta$$

$$\therefore p_n = p_x \frac{CD}{DE} \cos \theta + p_y \frac{CE}{DE} \sin \theta + q \frac{CD}{DE} \sin \theta + q \frac{CE}{DE} \cos \theta$$

Noting that  $\frac{CD}{DE} = \cos \theta$  and  $\frac{CE}{DE} = \sin \theta$ , we get

$$p_n = p_x \cos^2 \theta + p_y \sin^2 \theta + q \cos \theta \sin \theta + q \sin \theta \cos \theta$$

$$\begin{aligned} \text{i.e., } p_n &= p_x \left( \frac{1 + \cos 2\theta}{2} \right) + p_y \left( \frac{1 - \cos 2\theta}{2} \right) + 2q \sin \theta \cos \theta \\ &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\theta + q \sin 2\theta \end{aligned} \quad \dots(1)$$

If  $p_n'$  is the stress on a plane at right angles to  $CD$ ,  $\theta' = \theta + 90^\circ$ . Hence from the above general expression for stress on an inclined plane, we get

$$\begin{aligned} p_n' &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2(\theta + 90) + q \sin 2(\theta + 90) \\ &= \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2} \cos 2\theta - q \sin 2\theta \end{aligned} \quad \dots(2)$$

Adding eqns. (1) and (2), we get

$$p_n + p_n' = p_x + p_y$$

Thus sum of normal stresses in any two mutually perpendicular planes is constant and is equal to  $p_x + p_y$ .

**Example 11.2.** A material has strength in tension, compression and shear as  $30 \text{ N/mm}^2$ ,  $90 \text{ N/mm}^2$  and  $25 \text{ N/mm}^2$ , respectively. If a specimen of diameter  $25 \text{ mm}$  is tested in tension and compression identify the failure surfaces and loads.

**Solution:**

(a) In tension: Let axial direction be  $x$  direction.

Since it is uniaxial loading,

$$p_y = 0, q = 0 \text{ and only } p_x \text{ exists.}$$

$\therefore$  when the material is subjected to full tensile stress,  $p_x = 30 \text{ N/mm}^2$ .

$$p_1 = \frac{30 + 0}{2} + \sqrt{\left( \frac{30 - 0}{2} \right)^2 + 0^2} = 30 \text{ N/mm}^2$$

$$p_2 = \frac{30+0}{2} - \sqrt{\left(\frac{30-0}{2}\right)^2 + 0^2} = 0$$

$$q_{\max} = \frac{p_x - p_y}{2} = \frac{30-0}{2} = 15 \text{ N/mm}^2 < 25 \text{ N/mm}^2.$$

Hence failure criteria is normal stress  $p_1 = 30 \text{ N/mm}^2$ .

Corresponding load  $P$  is obtained by

$$p = \frac{P}{A}$$

$$30 = \frac{P}{(\pi/4) \times 25^2} \quad \text{or} \quad P = 14726 \text{ N.}$$

Failure surface is given by plane of  $p_1$  which is as shown in Fig. 11.7.

(b) In case of compression test

$$p_x = -90 \text{ N/mm}^2, \quad p_y = q = 0$$

$\therefore$  At failure when  $p_x = -90 \text{ N/mm}^2$ , load is

$$-90 = \frac{P}{(\pi/4) \times 25^2}$$

or

$$P = -44178 \text{ N} \quad \text{i.e.,} \quad 44178 \text{ N compressive.}$$

At this stage

$$\begin{aligned} q_{\max} &= \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \\ &= \sqrt{\left(\frac{-90-0}{2}\right)^2 + 0^2} \\ &= 45 \text{ N/mm}^2 > 25 \text{ N/mm}^2. \end{aligned}$$

$\therefore$  Material fails because of maximum shear and not by axial compression.

$$\begin{aligned} q_{\max} &= \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} = \sqrt{\left(\frac{p_x - 0}{2}\right)^2 + 0^2} \\ &= \frac{p_x}{2} \end{aligned}$$

$$\therefore p_x = 2q_{\max} = 2 \times 25 = 50 \text{ N/mm}^2.$$

$\therefore$  Corresponding axial load is given by

$$50 = \frac{P}{(\pi/4) \times 25^2}$$

or

$$P = 24543 \text{ N}$$

The plane of  $q_{\max}$  is at  $45^\circ$  to the plane of  $p_x$ . This failure surface is shown in Fig. 11.8.

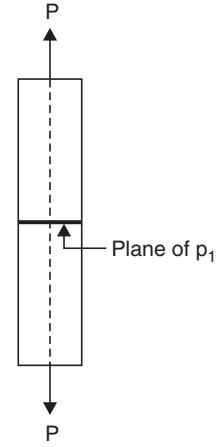


Fig. 11.7

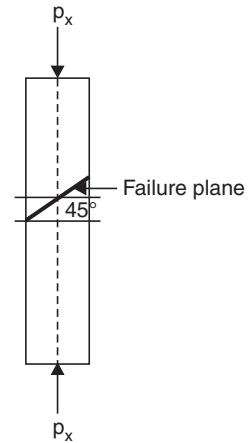
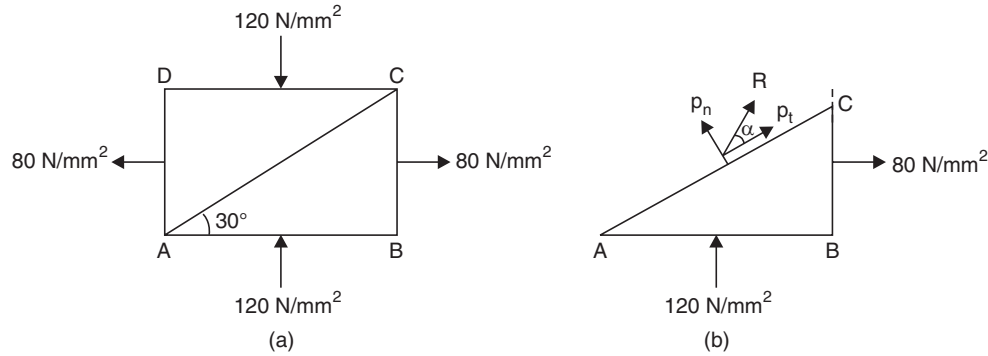


Fig. 11.8

**Example 11.3.** The direct stresses at a point in the strained material are  $120 \text{ N/mm}^2$  compressive and  $80 \text{ N/mm}^2$  tensile as shown in Fig. 11.9. There is no shear stress. Find the normal and tangential stresses on the plane AC. Also find the resultant stress on AC.



**Fig. 11.9**

**Solution:** The plane AC makes  $30^\circ$  (anticlockwise) to the plane of  $p_x$  (y-axis). Hence  $\theta = 30^\circ$ .

$$p_x = 80 \text{ N/mm}^2 \quad p_y = -120 \text{ N/mm}^2 \quad q = 0$$

$$\begin{aligned} \therefore p_n &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\theta + q \sin 2\theta \\ &= \frac{80 - 120}{2} + \frac{80 - (-120)}{2} \cos (2 \times 30) + 0 \\ &= -20 + 100 \cos 60 \end{aligned}$$

Thus

$$p_n = 30 \text{ N/mm}^2$$

$$\begin{aligned} p_t &= \frac{p_x - p_y}{2} \sin 2\theta - q \cos 2\theta \\ &= \frac{80 - (-120)}{2} \sin (2 \times 30) - 0 \end{aligned}$$

Thus

$$p_t = 86.6 \text{ N/mm}^2.$$

The resultant of  $p_n$  and  $p_t$  is given by

$$\begin{aligned} p &= \sqrt{p_n^2 + p_t^2} = \sqrt{30^2 + 86.6^2} \\ p &= 91.65 \text{ N/mm}^2 \end{aligned}$$

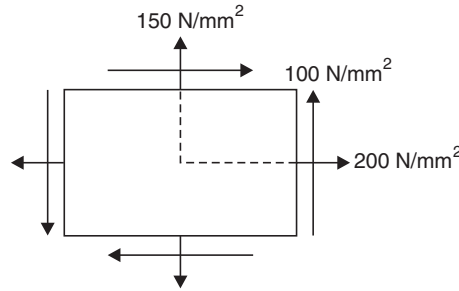
Angle made by the resultant stress with  $p_t$  is given by  $\tan \alpha = \frac{p_n}{p_t} = \frac{30}{86.6}$

$\therefore \alpha = 19.1^\circ$  as shown in Fig. 11.9(b).

**Example 11.4.** The state of stress at a point in a strained material is as shown in Fig. 11.10. Determine

- (i) the direction of principal planes
- (ii) the magnitude of principal stresses and
- (iii) the magnitude of maximum shear stress.

Indicate the direction of all the above by a sketch.



**Fig. 11.10**

**Solution:**  $p_x = 200 \text{ N/mm}^2$   $p_y = 150 \text{ N/mm}^2$   $q = 100 \text{ N/mm}^2$

Let the principal plane make anticlockwise angle  $\theta$  with the plane of  $p_x$  i.e. with y-axis. Then

$$\tan 2\theta = \frac{2q}{p_x - p_y} = \frac{2 \times 100}{200 - 150} = 4$$

$$\therefore 2\theta = 75.96 \quad \text{and} \quad 75.96 + 180$$

$$\therefore \theta = 37.98^\circ \quad \text{and} \quad 127.98^\circ$$

$$\begin{aligned} p_1 &= \frac{p_x + p_y}{2} + \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \\ &= \frac{200 + 150}{2} + \sqrt{\left(\frac{200 - 150}{2}\right)^2 + 100^2} \\ &= 175 + 103.08 \\ p_1 &= 278.08 \text{ N/mm}^2 \end{aligned}$$

and

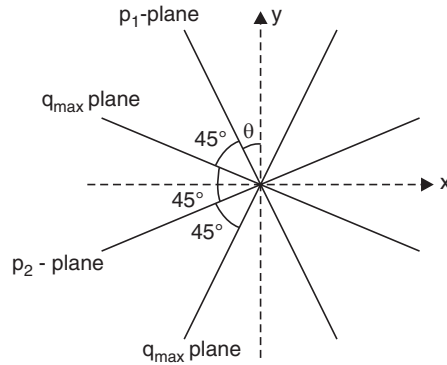
$$\begin{aligned} p_2 &= \frac{p_x + p_y}{2} - \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \\ \therefore p_2 &= 175 - 103.08 = 71.92 \text{ N/mm}^2. \end{aligned}$$

$$q_{\max} = \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

i.e.,

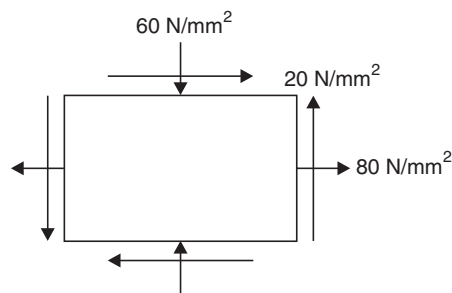
$$q_{\max} = 103.08 \text{ N/mm}^2$$

The planes of maximum shear are at  $45^\circ$  to the principal planes. Principal planes and planes of maximum shear are shown in Fig. 11.11.



**Fig. 11.11**

**Example 11.5.** The state of stress in a material stressed to two-dimensional state of stress is as shown in Fig. 11.12. Determine principal stresses and maximum shear stress and the planes on which they act.



**Fig. 11.12**

**Solution:**

$$p_{1,2} = \frac{p_x + p_y}{2} \pm \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

In this problem,

$$p_x = 80 \text{ N/mm}^2 \quad p_y = -60 \text{ N/mm}^2 \quad q = 20 \text{ N/mm}^2.$$

$$\therefore p_{1,2} = \frac{80 + (-60)}{2} \pm \sqrt{\left(\frac{80 - (-60)}{2}\right)^2 + 20^2}$$

$$= 10 \pm \sqrt{70^2 + 20^2}$$

$$= 10 \pm 72.8$$

$$\therefore p_1 = 82.8 \text{ N/mm}^2$$

and

$$p_2 = -62.8 \text{ N/mm}^2$$

$$q_{\max} = \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

$$= 72.8 \text{ N/mm}^2$$

Let  $\theta$  be the inclination of principal stress to the plane of  $p_x$ . Then,

$$\tan 2\theta = \frac{2q}{p_x - p_y} = \frac{2 \times 20}{80 - 60} = 2$$

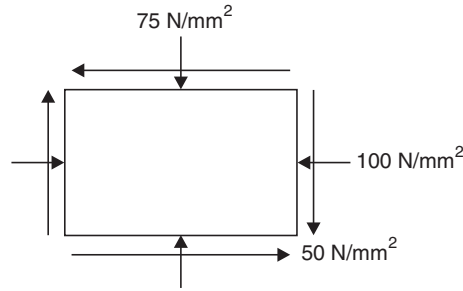
$$\therefore 2\theta = 63.44^\circ \text{ or } 63.44 + 180$$

$$\therefore \theta = 31.72^\circ \text{ or } 121.72^\circ$$

Planes of maximum shear make  $45^\circ$  to the above planes

$$\therefore \theta' = 15.86^\circ \text{ and } 60.86^\circ$$

**Example 11.6.** The state of stress in two-dimensionally stressed body at a point is as shown in Fig. 11.13(a). Determine the principal planes, principal stresses, maximum shear stress and their planes



**Fig. 11.13(a)**

**Solution:** Let  $x$  and  $y$  directions be selected as shown in the figure. Then

$$p_x = -100 \text{ N/mm}^2, \quad p_y = -75 \text{ N/mm}^2, \quad q = -50 \text{ N/mm}^2$$

$$\therefore p_1 = \frac{p_x + p_y}{2} + \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

$$= \frac{-100 - 75}{2} + \sqrt{\left(\frac{-100 + 75}{2}\right)^2 + (-50)^2}$$

$$= -87.5 + 51.54$$

$$\text{i.e., } p_1 = -35.96 \text{ N/mm}^2$$

$$p_2 = -\frac{p_x + p_y}{2} - \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$



$$= -87.5 - 51.54$$

i.e.,  $p_2 = -139.04 \text{ N/mm}^2$

$$q_{\max} = \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

i.e.,  $q_{\max} = 51.51.$

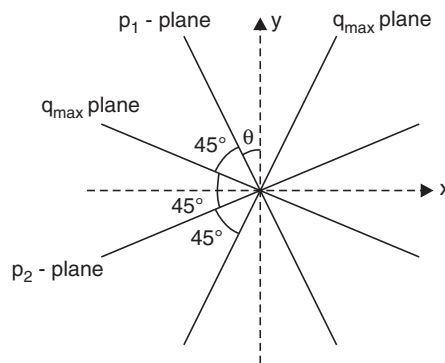
Let principal plane of  $p_1$  make angle  $\theta$  with  $x$ -axis. Then

$$\tan 2\theta = \frac{2q}{p_x - p_y} = \frac{2(-50)}{-100 + 75} = 4$$

$$\therefore 2\theta = 75.96 \text{ and } 75.96 + 180$$

or  $\theta = 37.98^\circ \text{ and } 127.98^\circ.$

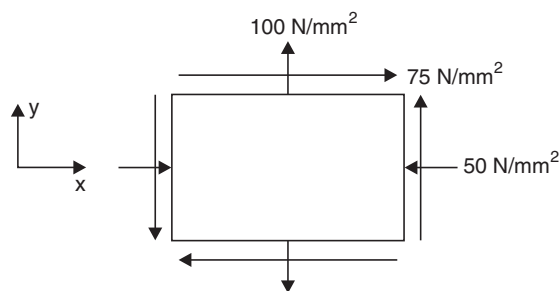
The planes of maximum shear stresses are at  $45^\circ$  to the principal planes. These planes are shown in Fig. 11.13(b).



**Fig. 11.13(b)**

**Example 11.7.** State of stress at a point in a material is as shown in the Fig. 11.14(a). Determine

- principal stresses
- maximum shear stress
- plane of maximum shear stress and
- the resultant stress on the plane of maximum shear stress.



**Fig. 11.14(a)**

**Solution:** Selecting  $x$  and  $y$ -axis as shown in figure,

$$p_x = -50 \text{ N/mm}^2, \quad p_y = 100 \text{ N/mm}^2, \quad \text{and } q = 75 \text{ N/mm}^2.$$

$$\begin{aligned} \therefore p_1 &= \frac{p_x + p_y}{2} + \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \\ &= \frac{-50 + 100}{2} + \sqrt{\left(\frac{-50 - 100}{2}\right)^2 + 75^2} \\ &= 25 + 106.07 \\ &= 131.07 \text{ N/mm}^2. \end{aligned}$$

$$\begin{aligned} p_2 &= \frac{p_x + p_y}{2} - \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \\ &= 25 - 106.07 \\ &= -81.07 \text{ N/mm}^2. \end{aligned}$$

$$\begin{aligned} q_{\max} &= \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \\ &= 106.07 \text{ N/mm}^2. \end{aligned}$$

The principal plane makes an angle  $\theta$  to  $y$ -axis in anticlockwise direction. Then

$$\tan 2\theta = \frac{2q}{p_x - p_y} = \frac{2 \times 75}{\left(\frac{-50 - 100}{2}\right)} = -2$$

$$\therefore 2\theta = -63.43^\circ$$

$$\text{or } \theta = -31.72^\circ = 31.72^\circ \text{ clockwise.}$$

Plane of maximum shear makes  $45^\circ$  to it

$$\theta = -31.72 + 45.00 = 13.28^\circ.$$

Normal stress on this plane is given by

$$\begin{aligned} p_x &= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\theta + q \sin 2\theta \\ &= \frac{-50 + 100}{2} + \frac{-50 - 100}{2} \cos 2(13.28) + 75 \sin (2 \times 13.28) \\ &= 25 - 67.08 + 33.54 \\ &= -8.54 \text{ N/mm}^2 \end{aligned}$$

$$p_t = q_{\max} = 106.07 \text{ N/mm}^2.$$

$$\begin{aligned} \therefore \text{Resultant stress } P &= \sqrt{(-8.54)^2 + 106.07^2} \\ &= 106.41 \text{ N/mm}^2 \end{aligned}$$

Let ' $p$ ' make angle  $\phi$  to tangential stress (maximum shear stress plane). Then referring to Fig. 11.14(b)

$$\tan \phi = \frac{p_n}{p_t} = \frac{8.54}{106.07}$$

$\therefore \phi = 4.6^\circ$  as shown in Fig. 11.14(b).

**Example 11.8.** Show that when a material is subjected to shearing stresses and unidirectional direct stress, the major and minor principal stresses are of opposite nature.

**Solution:** Let  $p_x$  be uniaxial stress

i.e.  $p_y = 0$

Let  $q$  be the shearing stresses.

Then from the equation

$$\begin{aligned} p_1 &= \frac{p_x + p_y}{2} + \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2} \\ &= \frac{p_x}{2} + \sqrt{\left(\frac{p_x}{2}\right)^2 + q^2} \\ \text{and} \quad p_2 &= \frac{p_x}{2} - \sqrt{\left(\frac{p_x}{2}\right)^2 + q^2} \end{aligned}$$

Since the second term is larger than the first term, naturally  $p_1$  is +ve and  $p_2$  is -ve. Thus major and minor principal stresses are having opposite nature.

### 11.3 PRINCIPAL STRESSES IN BEAMS

Figure 11.15 shows an element of a beam subjected to bending moment and shear force. We know, the bending stress at point A, in the beam,

bending stress 
$$p_x = \frac{M}{I} y$$

and shear stress 
$$q = \frac{F}{bI} (a\bar{y})$$

where  $M$ —Bending moment at the section

$F$ —Shear force

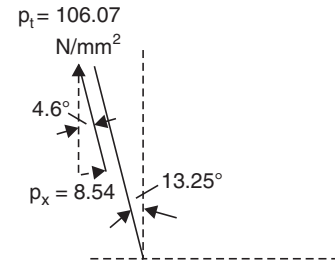
$I$ —Moment of inertia

$y$ —Distance of the point from neutral axis

$b$ —width

and  $a\bar{y}$ —moment of area above the level of fibre at A about neutral axis.

The state of stress at point A is as shown in the Fig. 11.16 knowing  $p_x$  and  $q$  the required principal stresses, maximum shear stress etc. may be found. The nature of bending stress (tensile or compressive) should be carefully noted.



**Fig. 11.14(b)**

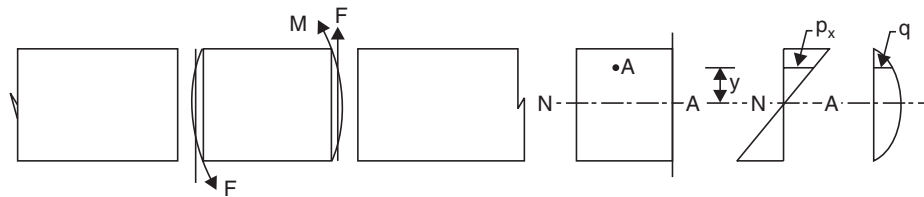


Fig. 11.15

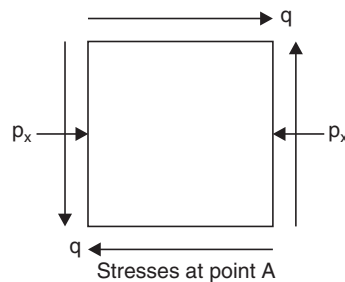


Fig. 11.16

**Example 11.9.** A shear force of 100 kN and a sagging moment of 80 kN-m act at a certain cross-section of rectangular beam 100 mm wide and 200 mm deep. Compute the principal stresses at a point 30 mm below the top surface.

**Solution:** Referring to Fig. 11.17,

$$I = \frac{1}{12} \times 100 \times 200^3 = 66.667 \times 10^6 \text{ mm}^4$$

At point A, which is at 30 mm below top fibre

$$y = 100 - 30 = 70 \text{ mm}$$

$$\therefore f_x = \frac{M}{I} y = \frac{80 \times 10^6}{66.667 \times 10^6} \times 70 = 84 \text{ N/mm}^2 \text{ (compressive)}$$

$$\therefore p_x = -84 \text{ N/mm}^2$$

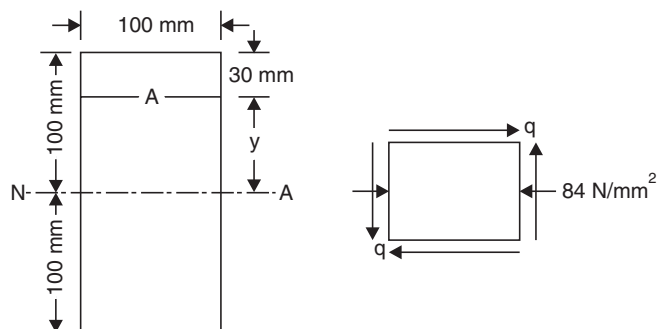


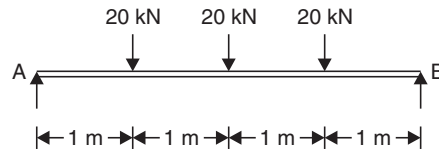
Fig. 11.17

$$\begin{aligned}
 \text{Shearing stress} \quad q &= \frac{F}{bI}(a\bar{y}) \\
 &= \frac{100 \times 10^3}{100 \times 66.667 \times 10^6} [100 \times 30 \times (100 - 15)] \\
 &= 3.82 \text{ N/mm}^2 \text{ as shown in Fig.}
 \end{aligned}$$

$$\text{Thus, } p_x = -84 \text{ N/mm}^2, p_y = 0, q = 3.82 \text{ N/mm}^2$$

$$\begin{aligned}
 \therefore p_{1,2} &= \frac{-84 + 0}{2} \pm \sqrt{\left(\frac{-84 + 0}{2}\right)^2 + (3.82)^2} \\
 &= -42 \pm 42.17 \\
 \therefore p_1 &= 0.17 \text{ N/mm}^2 \\
 p_2 &= -84.17 \text{ N/mm}^2
 \end{aligned}$$

**Example 11.10.** A simply supported beam of 4 m span carries loads as shown in Fig. 11.18. The cross-section of the beam is 100 mm wide and 180 mm deep. At a section 1.5 m from left support, calculate the bending and shearing stresses at distances 0, 45 mm and 90 mm above the neutral axis. Find the principal planes and principal stresses at these points.



**Fig. 11.18**

**Solution:** Due to symmetry,

$$R_A = R_B = \frac{20 + 20 + 20}{2} = 30 \text{ kN}$$

$\therefore$  At section 1.5 m from A,

$$F = 30 - 20 = 10 \text{ kN}$$

$$M = 30 \times 1.5 - 20 \times 0.5 = 35 \text{ kN-m}$$

$$I = \frac{1}{12} \times 100 \times 180^3 = 48.6 \times 10^6 \text{ mm}^4$$

$$\therefore \text{Bending stress} = f = \frac{M}{I}y = \frac{35 \times 10^6}{48.6 \times 10^6} y$$

It varies linearly across the depth.

when  $y = 0$ ,  $f_1 = 0$

$$y = 45 \text{ mm}, f_2 = 32.4 \text{ N/mm}^2 \text{ (comp).}$$

$$y = 90 \text{ mm}, f_3 = 64.8 \text{ N/mm}^2 \text{ (comp).}$$

Shearing stress at a fibre 'y' above  $N-A$  is

$$q = \frac{F}{bI}(a\bar{y})$$

$$= \frac{10 \times 1000}{100 \times 48.6 \times 10^6}(a\bar{y})$$

$$\therefore \text{ At } y = 0, \quad q_1 = \frac{10 \times 1000}{100 \times 48.6 \times 10^6} (100 \times 90 \times 45)$$

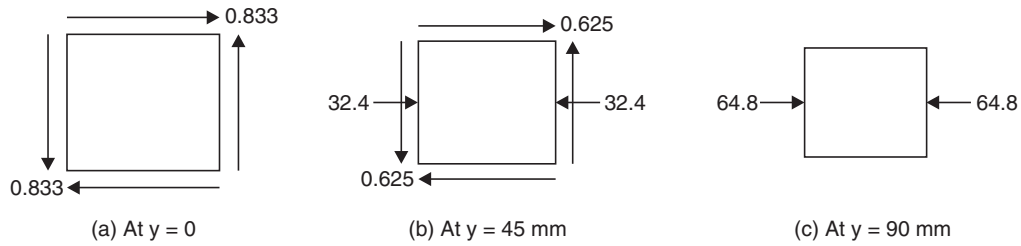
$$= 0.833 \text{ N/mm}^2$$

$$\text{ At } y = 45 \text{ mm}, \quad q_2 = \frac{10 \times 1000}{100 \times 48.6 \times 10^6} 100 \times 45 \times \left(90 - \frac{45}{2}\right)$$

$$= 0.625 \text{ N/mm}^2$$

$$\text{ At } y = 90 \text{ mm}, \quad q_3 = 0$$

The state of stresses on elements under considerations are as shown in Fig. 11.19.



**Fig. 11.19**

(a) **At neutral axis ( $y = 0$ ) :** The element is under pure shear

$$p_x = p_y = 0 ; q = 0.833 \text{ N/mm}^2$$

$$p_{1,2} = 0 \pm \sqrt{0^2 + (0.833)^2} = \pm 0.833 \text{ N/mm}^2$$

$$p_1 = 0.833 \text{ N/mm}^2$$

$$p_2 = -0.833 \text{ N/mm}^2.$$

$$\text{Inclination to the plane of } p_x, \quad \tan 2\theta = \frac{2 \times 0.833}{0} = \infty$$

$$\therefore 2\theta = 90^\circ \text{ and } 270^\circ$$

$$\therefore \theta = 45^\circ \text{ and } 135^\circ$$

(b) **At  $y = 45 \text{ mm}$**

$$p_x = -32.4 \text{ N/mm}^2, \quad p_y = 0, \quad q = 0.625 \text{ N/mm}^2$$

$$\therefore p_{1,2} = \frac{-32.4 + 0}{2} \pm \sqrt{\left(\frac{-32.4 - 0}{2}\right)^2 + (0.625)^2}$$

$$= -16.2 \pm 16.212$$

$$\therefore p_1 = 0.012 \text{ N/mm}^2$$

$$p_2 = -32.412 \text{ N/mm}^2$$

$$\tan 2\theta = \frac{2 \times 0.625}{-32.4 - 0} = 0.09876$$

$$2\theta = 5.64^\circ \text{ and } 185.64^\circ$$

$$\therefore \theta = 2.82^\circ \text{ and } 92.82^\circ$$

(c) At  $y = 90 \text{ mm}$ ,

$$p_x = -64.8 \text{ N/mm}^2, p_y = 0, q = 0$$

$$\therefore p_{1,2} = \frac{-64.8 + 0}{2} \pm \sqrt{\left(\frac{-64.8 - 0}{2}\right)^2 + 0^2}$$

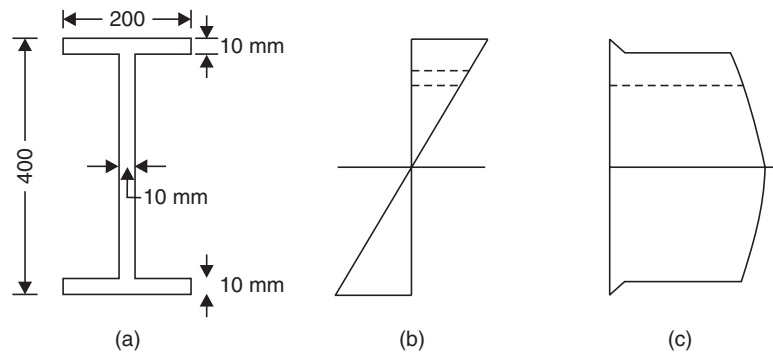
$$= -32.4 \pm 32.4$$

$$\therefore p_1 = 0 \text{ N/mm}^2$$

$$p_2 = -64.8 \text{ N/mm}^2$$

$$\tan 2\theta = 0 \therefore 2\theta = 0^\circ \text{ and } 180^\circ \text{ or } \theta = 0^\circ \text{ and } 90^\circ$$

**Example 11.11.** A simply supported beam of span 6 m has I-section as shown in Fig. 11.20(a). It carries uniformly distributed load (inclusive self weight) of 60 kN/m over entire span. Calculate the principal stresses and the maximum shearing stress at 100 mm above neutral axis of the beam at a section 1.5 m from support.



**Fig. 11.20**

**Solution:**  $L = 6 \text{ m}$ ,  $w = 60 \text{ kN/m}$

$$\therefore \text{Reaction at support} = \frac{6 \times 60}{2} = 180 \text{ kN}$$

$\therefore$  Moment at 1.5 m from support

$$M = 180 \times 1.5 - 60 \times \frac{1.5^2}{2} = 202.5 \text{ kN-m}$$

Shear force at 1.5 m from support

$$F = 180 - 1.5 \times 60 = 90 \text{ kN}$$

Moment of inertia of the I-section,

$$I = \frac{1}{12} \times 200 \times 400^3 - \frac{1}{12} \times 190 \times 380^3$$

$$= 197.86 \times 10^6 \text{ mm}^4$$

$\therefore$  Bending stress at 100 mm above N-A

$$f = \frac{M}{I} y = \frac{202.5 \times 10^6}{197.86 \times 10^6} \times 100 = 102.35 \text{ N/mm}^2 \text{ (compressive)}$$

Shear stress is given by

$$q = \frac{F}{bI} (a\bar{y}) = \frac{90 \times 1000}{10 \times 197.86 \times 10^6} \times (200 \times 10 \times 195 + 10 \times 90 \times 145)$$

$$= 23.68 \text{ N/mm}^2$$

Thus the state of stress on an element at  $y = 100 \text{ mm}$ , as  $p_x = f = -102.35 \text{ N/mm}^2$ ,  $p_y = 0$   
 $q = 23.68 \text{ N/mm}^2$

$$\therefore p_{1,2} = \frac{-102.35 + 0}{2} \pm \sqrt{\left(\frac{-102.35 - 0}{2}\right)^2 + (23.68)^2}$$

$$= -51.175 \pm 56.388$$

$$\therefore p_1 = 5.21 \text{ N/mm}^2 \quad p_2 = -107.56 \text{ N/mm}^2$$

$$q_{\max} = \sqrt{\left(\frac{-102.35 - 0}{2}\right)^2 + (23.68)^2} = 56.39 \text{ N/mm}^2$$

#### 11.4 PRINCIPAL STRAINS

Let  $e_x$  be the strain in  $x$ -direction and  $e_y$  be the strain in  $y$ -direction where  $x$  and  $y$  are cartesian coordinate directions. Let shearing strain with respect to  $x$ - $y$  coordinates be  $\gamma_{xy}$ . Then it is possible to find normal strain  $e_n$ , tangential strain  $e_t$  and shearing strain  $\gamma_{nt}$  on any plane inclined at  $\theta$  to the plane of  $p_x$ . Then defining the plane with zero shearing strain as principal shearing plane, we can find the principal planes and principal strains.

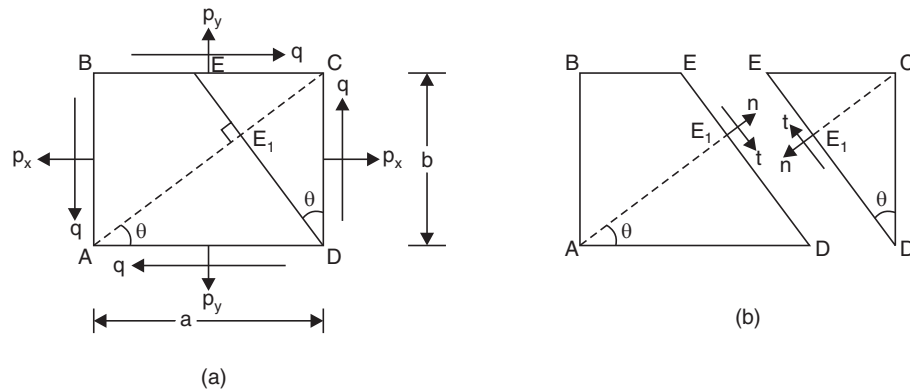


Fig. 11.21



Referring to Fig. 11.21(a), which represent state of stress at A with their positive senses it may be noted that direct strain  $e_x$  and  $e_y$  are tensile strains and shearing strain  $\gamma_{xy}$  reduces the angle  $\theta$ .

Figure 11.22 shows the element at A with its de-formed shape (shown with dotted lines) with  $e_x, e_y$  and  $\gamma_{xy}$  positive. Now we are interested in finding strains  $e_n, e_t$  and  $\gamma_{nt}$  on a plane at ' $\theta$ ' to the plane of  $p_x$  forces. The size of element ( $a \times b$ ) is so selected that the diagonal AE is normal to the plane DE i.e.  $\angle CAD$  is also  $\theta$ , for the convenience.

(a) To find  $e_n$

$$\begin{aligned} \text{Let} \quad AC &= l, \\ \therefore AD &= l \cos \theta = a \\ AB &= l \sin \theta = b \end{aligned}$$

Drop  $\perp C_1, P$  to  $AD$   
 $\perp BQ$  to  $C_1P$ .

Now,

$$\begin{aligned} AP &= AD + DD_1 + D_1P \\ &= a + ae_x + b\gamma_{xy} \\ &= l \cos \theta + l \cos \theta \times e_x + l \sin \theta \times \gamma_{xy} \\ &= l [\cos \theta + e_x \cos \theta + \gamma_{xy} \sin \theta] \\ PC &= PQ + QC \\ &= b + be_y \\ &= l \sin \theta + l \sin \theta \times e_y = l \sin \theta (1 + e_y) \end{aligned}$$

$$\begin{aligned} \therefore AC_1 &= \sqrt{AP^2 + PC^2} \\ &= l \sqrt{(\cos \theta + e_x \cos \theta + \gamma_{xy} \sin \theta)^2 + \sin^2 \theta (1 + e_y)^2} \end{aligned}$$

Neglecting small quantities of higher order,

$$\begin{aligned} AC_1 &= l \sqrt{\cos^2 \theta + 2e_x \cos^2 \theta + 2\gamma_{xy} \sin \theta \cos \theta + \sin^2 \theta + 2e_y \sin^2 \theta} \\ &= l \sqrt{1 + 2(e_x \cos^2 \theta + e_y \sin^2 \theta) + \gamma_{xy} \sin \theta \cos \theta} \end{aligned}$$

Expanding by the binomial theorem and neglecting small quantities of order 2 and above, we get

$$AC_1 = l [1 + e_x \cos^2 \theta + e_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta]$$

$$\therefore e_x = \frac{AC_1 - AC}{AC} = e_x \cos^2 \theta + e_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta \quad \dots(11.9a)$$

$$\text{But } \cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\text{and } \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

$$\therefore e_x = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos 2\theta + \frac{1}{2} \gamma_{xy} \sin 2\theta \quad \dots(11.9b)$$

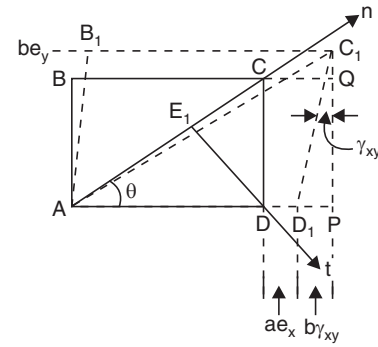
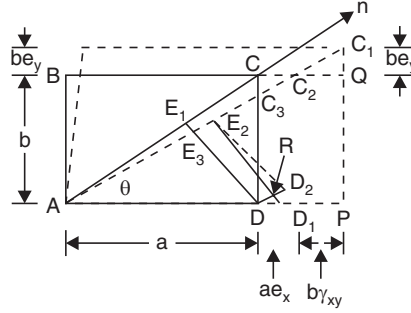


Fig. 11.22

Strain  $e_t$  can be found by replacing  $\theta$  in the above equation by  $90 + \theta$ .

$$\therefore e_t = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2} \cos 2\theta - \frac{1}{2} \gamma_{xy} \sin 2\theta \quad \dots(11.10)$$

To find  $\gamma_{nt}$  :



**Fig. 11.23**

After straining let point  $E_1$  move to  $E_2$ . Draw  $E_2R$  parallel to  $E_1D$  [Ref. Fig. 11.23]

$\therefore$  Total shearing strain

$$= \angle CAC_1 + \angle D_1E_2R = \phi_1 + \phi_2$$

To find  $\phi_1$  :

Now :  $CC_3 = CC_2 \sin \theta$

$$\begin{aligned} &= (CQ - C_2Q) \sin \theta \\ &= \left[ (ae_x + b\gamma_{xy}) - be_y \frac{1}{\tan \theta} \right] \sin \theta \\ &= (ae_x + b\gamma_{xy}) \sin \theta - be_y \cos \theta \end{aligned}$$

Noting that  $l \cos \theta = a$  and  $l \sin \theta = b$ , we get

$$\begin{aligned} CC_3 &= le_x \cos \theta \sin \theta + l \sin^2 \theta \gamma_{xy} - le_y \sin \theta \cos \theta \\ &= l (e_x - e_y) \sin \theta \cos \theta + l \gamma_{xy} \sin^2 \theta \end{aligned}$$

$$\therefore \phi_1 = \frac{CC_3}{l} = (e_x - e_y) \sin \theta \cos \theta + \gamma_{xy} \sin^2 \theta \quad \dots(1)$$

To find  $\phi_2$  :

$$\begin{aligned} E_3E_2 &= \text{Extension of } AE_1 \\ &= AE_1 e_n = a \cos \theta e_x = l \cos^2 \theta \times e_x \end{aligned} \quad \dots(2)$$

$$\begin{aligned} \phi_2 &= \frac{RD_2}{E_1D} = \frac{DD_2 - RD_2}{E_1D} = \frac{DD_2 - E_3E_2}{E_1D} \\ &= \frac{ae_x \cos \theta - l \cos^2 \theta e_n}{a \sin \theta} \\ &= \frac{le_x \cos^2 \theta - le_n \cos^2 \theta}{l \sin \theta \cos \theta} \\ &= (e_x - e_n) \cot \theta \end{aligned} \quad \dots(3)$$

Substituting the value of  $e_n$  from eqn. 11.9(a), we get

$$\begin{aligned}\phi_2 &= (e_x - e_x \cos^2 \theta - e_y \sin^2 \theta - \gamma_{xy} \sin \theta \cos \theta) \cot \theta \\ &= (e_x \sin^2 \theta - e_y \sin^2 \theta - \gamma_{xy} \sin \theta \cos \theta) \cot \theta \\ &= (e_x - e_y) \sin^2 \theta \cot \theta - \gamma_{xy} \sin \theta \cos \theta \cot \theta \\ &= (e_x - e_y) \sin \theta \cos \theta - \gamma_{xy} \cos^2 \theta\end{aligned}\quad \dots(4)$$

$$\begin{aligned}\therefore \gamma_{nt} &= \phi_1 + \phi_2 \\ &= (e_x - e_y) \sin \theta \cos \theta + \gamma_{xy} \sin^2 \theta + (e_x - e_y) \sin \theta \cos \theta - \gamma_{xy} \cos^2 \theta \\ &= 2(e_x - e_y) \sin \theta \cos \theta - \gamma_{xy} (\cos^2 \theta - \sin^2 \theta) \\ &= (e_x - e_y) \sin 2\theta - \gamma_{xy} \cos 2\theta \\ \therefore \frac{1}{2} \gamma_{nt} &= \frac{e_x - e_y}{2} \sin 2\theta - \frac{1}{2} \gamma_{xy} \cos 2\theta\end{aligned}\quad \dots(11.11)$$

**Note.** Equation 11.1 is analogous to eqn. 11.9(b) and eqn. 11.2 is analogous to eqn. 11.11 in which  $p_1$  or  $p_2$  are replaced by  $e_1$  and  $e_2$  and  $q$  has been replaced by  $\frac{1}{2} \gamma_{xy}$ .

Defining **principal strain** as the *normal strains on the plane where shearing strains is zero*, the direction of principal plane is obtained from 11.11 as

$$\tan 2\theta = \frac{2\left(\frac{1}{2}\gamma_{xy}\right)}{e_x - e_y} = \frac{\gamma_{xy}}{e_x - e_y}\quad \dots(11.12)$$

The magnitude of principal planes may be obtained exactly on the same line as the principal stresses were obtained. The final result will be

$$\begin{aligned}e_{1,2} &= \frac{1}{2}(e_x + e_y) \pm \sqrt{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2} \\ &= \frac{1}{2}(e_x + e_y) \pm \frac{1}{2}\sqrt{(e_x - e_y)^2 + \gamma_{xy}^2}\end{aligned}\quad \dots(11.13)$$

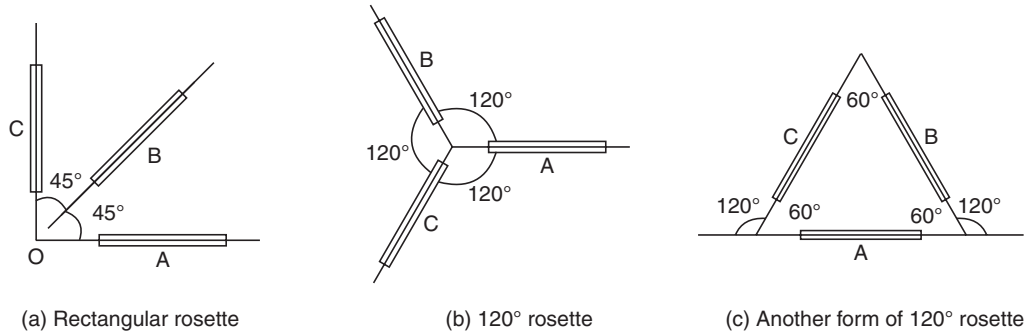
The maximum shearing strain occurs at  $45^\circ$  to the plane of principal plane and its magnitude is given by

$$\gamma_{\max} = \frac{1}{2}\sqrt{(e_x - e_y)^2 + \gamma_{xy}^2}\quad \dots(11.14)$$

### 11.5 MEASUREMENT OF STRAIN

Strain is a physical quantity while stress is a concept. Hence it is possible to measure the strains in an experimental investigation. For this electrical strain gauges are used. Electrical strain gauge consists of a set of fine wires fixed at a predefined angles and then glued to the experimental model. After loading the model is strained, resulting into changes in the length and diameter of the wire. Hence the resistance of the wire also changes. Electrical strain meters, which work on the principal of Wheatstone Bridge measure the change in resistance and converts it to strains and then displays the readings. The set of strain gauges fixed at a point in different direction is called 'strain rosettes'.

Commonly used strain rosettes are  $45^\circ$  rosette and  $120^\circ$  rosette which are as shown in Fig. 11.24.  $45^\circ$  rosette is also known as rectangular rosette.



**Fig. 11.24**

Treating one of the strain gauge direction, strains  $e_{\theta 1}$  and  $e_{\theta 2}$  of the other two gauges is known writing  $e_{\theta 1}$  and  $e_{\theta 2}$  in terms of  $e_x$ ,  $e_y$  and  $\gamma_{xy}$ , it is possible to find  $e_x$ ,  $e_y$  and  $\gamma_{xy}$ .

After finding  $e_x$ ,  $e_y$  and  $\gamma_{xy}$  it is possible to find the corresponding stresses. We know,

$$Ee_1 = p_1 - \mu p_2 \quad \dots(1)$$

$$Ee_2 = p_2 - \mu p_1 \quad \dots(2)$$

Multiplying Eqn. (2) with  $\mu$  and adding it to eqn.1, we get,

$$E(e_1 + \mu e_2) = p_1 - \mu^2 p_1 = p_1(1 - \mu^2)$$

$$\therefore p_1 = \frac{E(e_1 + \mu e_2)}{1 - \mu^2} \quad \dots(11.15a)$$

$$\text{Similarly} \quad p_2 = \frac{E(e_2 + \mu e_1)}{1 - \mu^2} \quad \dots(11.15b)$$

**Example 11.12.** At a point strains measured with rectangular rosettes are  $e_A = 600$  microns,  $e_B = 300$  microns and  $e_C = -200$  microns. Determine the principal strains. Find principal stresses also taking  $E = 2 \times 10^5$  and  $\mu = 0.3$ .

**Solution:** Taking the direction of strain gauge as  $x$ -axis,

$$e_x = 600 \text{ microns}$$

$$e_B = e_{45} = 300 \text{ microns} \quad \text{and} \quad e_C = e_{90} = -200 \text{ microns} = e_y$$

$$\begin{aligned} \therefore 300 &= \frac{1}{2}(e_x + e_y) + \frac{1}{2}(e_x - e_y) \cos 2 \times 45 + \frac{1}{2}\gamma_{xy} \sin 2 \times 45 \\ &= \frac{1}{2}(600 - 200) + \frac{1}{2}(600 + 200) \times 0 + \frac{1}{2}\gamma_{xy} \end{aligned}$$

$$\gamma_{xy} = 200 \text{ microns.}$$

$\therefore$  Principal strains are

$$e_{1,2} = \frac{1}{2}(e_x + e_y) \pm \sqrt{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2}$$

$$= \frac{1}{2}(600 - 200) \pm \sqrt{\left(\frac{600 + 200}{2}\right)^2 + \left(\frac{1}{2} \times 200\right)^2}$$

$$= 200 \pm 412.3$$

$\therefore$

$$e_1 = 612.3 \text{ micron} = 612.3 \times 10^{-6}$$

$$e_2 = -212.3 \text{ micron} = -212.3 \times 10^{-6}$$

[Note: 1 micron =  $1 \times 10^{-6}$ ]

$\therefore$

$$p_1 = \frac{E(e_1 + \mu e_2)}{1 - \mu^2} = \frac{2 \times 10^5 (612.3 - 0.3 \times 212.3) \times 10^{-6}}{1 - 0.3^2}$$

i.e.,

$$p_1 = 120.57 \text{ N/mm}^2.$$

$$p_2 = \frac{E(e_2 + \mu e_1)}{1 - \mu^2} = \frac{2 \times 10^5 (-212.3 + 0.3 \times 612.3)}{1 - 0.3^2}$$

i.e.,

$$p_2 = -6.29 \text{ N/mm}^2$$

**Example 11.13.** In an experimental investigation strains observed with a  $120^\circ$  rosette are,  $e_0 = 800$  microns,  $e_{120} = -600$  microns and  $e_{240} = 100$  microns. Determine the principal stresses, if  $E = 2 \times 10^5 \text{ N/mm}^2$  and  $\mu = 0.3$ .

**Solution:**

$$e_x = 800$$

$$-600 = e_{120} = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos 240 + \frac{1}{2} \gamma_{xy} \sin 240$$

i.e.,

$$-600 = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2} (0.5) - \frac{0.866}{2} \gamma_{xy} \quad \dots(1)$$

$$100 = e_{240} = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos 480 + \frac{1}{2} \gamma_{xy} \sin 480$$

$\therefore$

$$100 = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2} (0.5) + \frac{0.866}{2} \gamma_{xy} \quad \dots(2)$$

Adding eqns. (1) and (2), we get

$$-500 = e_x + e_y - \frac{e_x - e_y}{2}$$

i.e.,

$$-1000 = e_x + 3e_y$$

But

$$e_x = 800 \text{ microns}$$

$\therefore$

$$3e_y = -1000 - 800 = -1800$$

$\therefore$

$$e_y = -600 \text{ microns}$$

Substituting the values of  $e_x$  and  $e_y$  in eqn. (2), we get

$$100 = \frac{800 - 600}{2} - \frac{800 + 600}{2} (0.5) + \frac{0.866}{2} \gamma_{xy}$$

$$\gamma_{xy} = 808.3 \text{ microns}$$

Thus  $e_x = 800$  microns,  $e_y = -600$  micron and  $\gamma_{xy} = 808.3$  microns

$$\begin{aligned} \therefore e_{1,2} &= \frac{e_x + e_y}{2} \pm \sqrt{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2} \\ &= \frac{800 - 600}{2} \pm \sqrt{\left(\frac{800 + 600}{2}\right)^2 + \left(\frac{1}{2} \times 808.3\right)^2} \\ &= 100 \pm 808.29 \\ \therefore e_1 &= 908.29 \text{ microns} = 908.29 \times 10^{-6} \\ \text{and } e_2 &= -708.29 \text{ microns} = -708.29 \times 10^{-6} \\ \therefore p_1 &= \frac{E(e_1 + \mu e_2)}{1 - \mu^2} = \frac{2 \times 10^5 (908.29 - 0.3 \times 708.29) \times 10^{-6}}{1 - 0.3^2} \\ \text{i.e., } p_1 &= 152.9 \text{ N/mm}^2 \\ p_2 &= \frac{E(e_1 + \mu e_2)}{1 - \mu^2} = \frac{2 \times 10^5 (-708.29 + 0.3 \times 908.29) \times 10^{-6}}{1 - 0.3^2} \\ \text{i.e., } p_2 &= -95.8 \text{ N/mm}^2 \end{aligned}$$

**IMPORTANT FORMULAE**

1.  $p_n = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\theta + q \sin 2\theta.$
  2.  $p_t = \frac{p_x - p_y}{2} \sin 2\theta - q \cos 2\theta.$
  3. Principal planes are given by
 
$$\tan 2\theta = \frac{2q}{p_x - p_y}.$$
  4.  $p_{1,2} = \frac{p_x + p_y}{2} \pm \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}.$
  5. Maximum shear stress  $= \frac{p_1 - p_2}{2} = \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}.$
  6. Plane of maximum shear is at  $45^\circ$  to the principal planes.
  7. Obliquity of resultant stress on a plane at  $\theta$  to the plane of  $p_x$ 

$$= \theta + \alpha \text{ to the plane of } p_x$$
- where  $\alpha = \tan^{-1} \frac{p_n}{p_t}.$

$$8. \quad e_n = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos 2\theta + \frac{1}{2} \gamma_{xy} \sin 2\theta$$

$$\frac{1}{2} r_{nt} = \frac{(e_x - e_y)}{2} \sin 2\theta - \frac{1}{2\gamma_{xy}} \cos 2\theta$$

$$e_{1,2} = \frac{1}{2}(e_x + e_y) \pm \sqrt{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2}$$

$$\gamma_{\max} = \frac{e_1 - e_2}{2} = \sqrt{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2}.$$

### THEORY QUESTIONS

1. Derive the expressions for normal and tangential stresses on a plane inclined at  $\theta$  to the plane of  $p_x$  force. Take a general two dimensional state of stress.
2. State the equations for normal and tangential stresses on an inclined plane, in an element under general two dimensional stress system. Derive the expressions for principal planes, principal stresses and maximum shear stress.
3. Explain the terms principal stresses and principal strains.

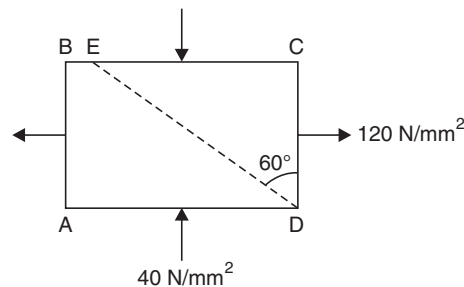
### PROBLEMS FOR EXERCISE

1. A point in a strained material is subjected to tensile stresses  $p_x = 180 \text{ N/mm}^2$  and  $p_y = 120 \text{ N/mm}^2$ . Determine the intensities of normal, tangential and resultant stresses on a plane inclined at  $30^\circ$  anticlockwise to the axis of minor stress.

[Note : Axis of minor stress means the plane of major stress]

[Ans.  $p_n = 165 \text{ N/mm}^2$ ,  $p_t = 25.98 \text{ N/mm}^2$ ,  $p = 167.03 \text{ N/mm}^2$ ,  $\alpha = 111.05^\circ$  anticlockwise to the axis of minor principal plane]

2. The state of stress at a point in a strained material is as shown in Fig. 11.25. Determine the normal, tangential and the resultant stress on plane  $DE$ . Determine the direction of resultant also.



**Fig. 11.25**

[Ans.  $p_n = 0$ ,  $p_t = 69.28 \text{ N/mm}^2$ ,  $p = 69.28 \text{ N/mm}^2$  acts tangential to  $DE$ .]

3. The state of stress at a point is as shown in Fig. 11.26. Determine the principal stresses and maximum shear stress. Indicate their planes on a separate sketch.

[Ans.  $p_1 = 133.13 \text{ N/mm}^2$ ,  $p_2 = -93.13 \text{ N/mm}^2$ ,  $q_{\max} = 113.13 \text{ N/mm}^2$ ,  $\theta = -22.5^\circ$  and  $67.5^\circ$ ,  $\theta' = 22.5$  and  $112.5^\circ$ ]

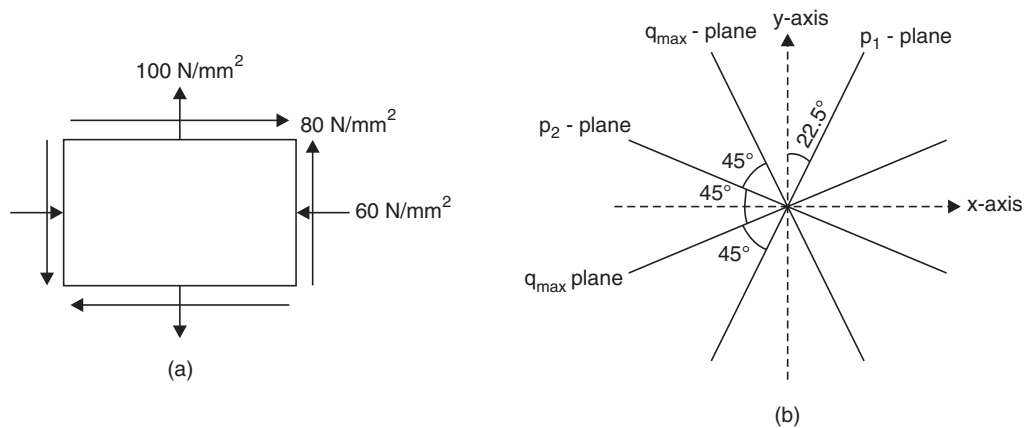


Fig. 11.26

4. The state of stress in a two dimensionally stressed material is as shown in Fig. 11.27. Determine the principal stresses, principal planes and the maximum shear stress. Determine normal and tangential stresses on plane AC also.

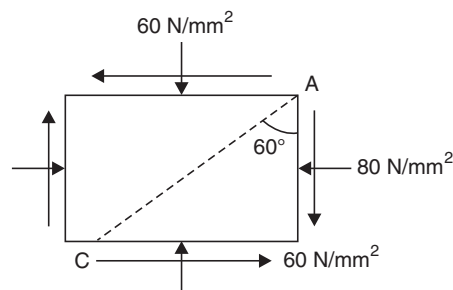


Fig. 11.27

[Ans.  $p_1 = -9.18 \text{ N/mm}^2$ ,  $p_2 = -130.82 \text{ N/mm}^2$ ,  $q_{\max} = 60.82 \text{ N/mm}^2$ ,  $\theta = -40.27^\circ$  and  $49.73^\circ$ ,  $p_n = -116.96 \text{ N/mm}^2$ ,  $p_t = 21.34 \text{ N/mm}^2$ ]