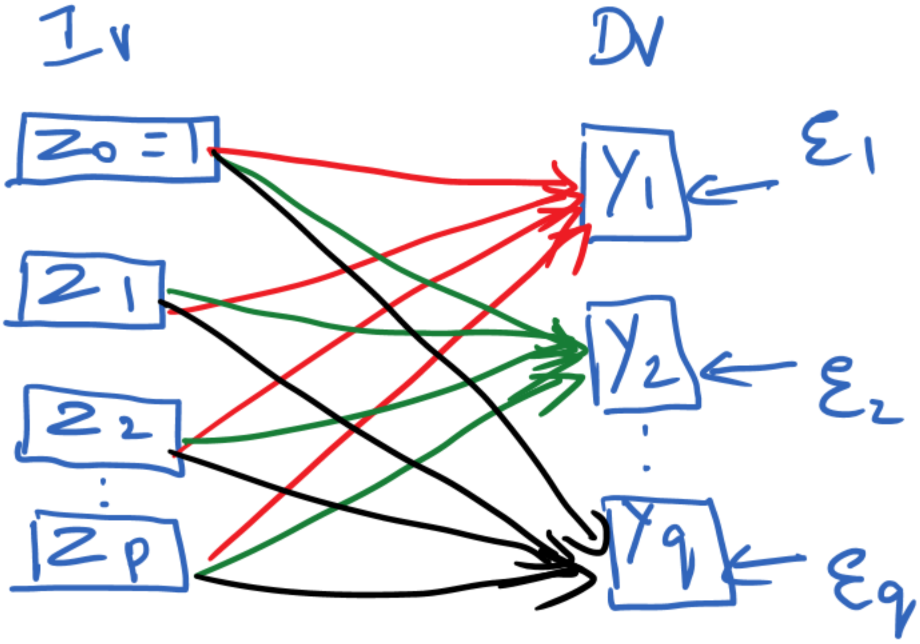
Multi Variate Linear Regression

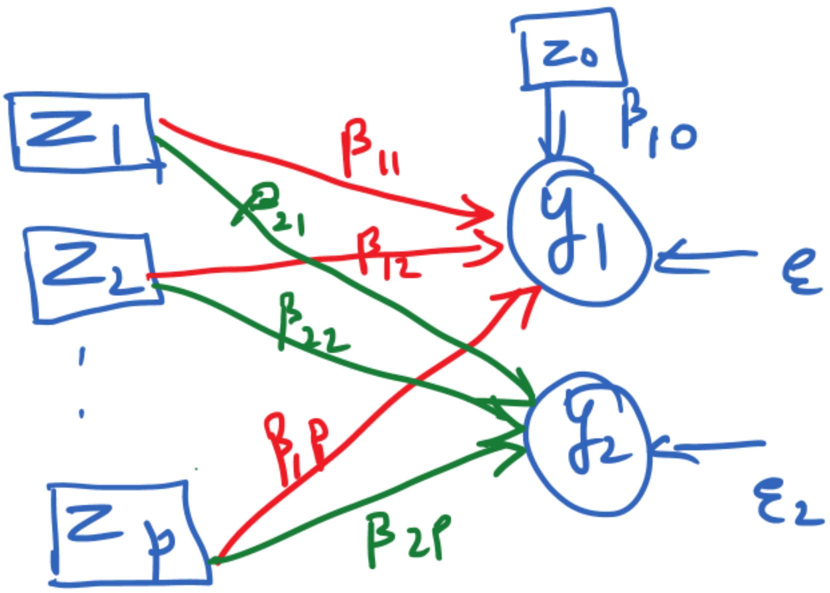
SLR: one dependent variable, One independent variable

MLR: one dependent variable, Many independent variable

MVLR: Many dependent variable, Many independent variable

(For differentiating MLR and MVLR, instead of X, Z is used in this MVLR explanation)





Conceptual model:

$$y_1 = \beta_{10} + \beta_{11} z_1 + \beta_{12} z_2 + \dots \beta_{1p} z_p + \varepsilon_1$$

$$y_2 = \beta_{20} + \beta_{21} z_1 + \beta_{22} z_2 + \dots \beta_{2p} z_p + \varepsilon_2$$

$$\vdots$$

$$y_q = \beta_{q0} + \beta_{q1} z_1 + \beta_{q2} z_2 + \dots \beta_{qp} z_p + \varepsilon_q$$

matrix:

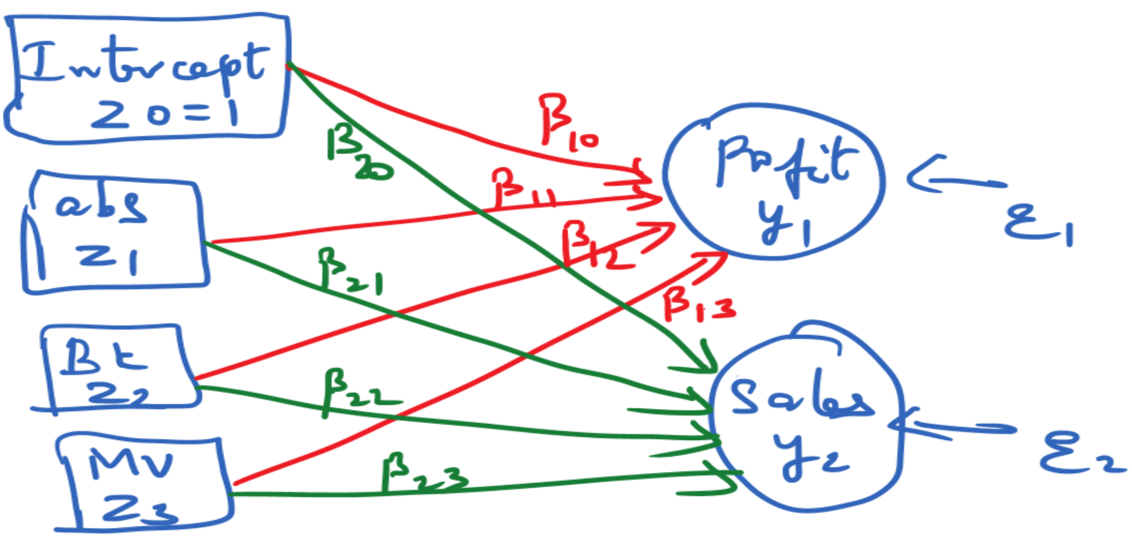
$$Y = \underset{q \times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}} ; \quad Z = \underset{p \times 1}{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{bmatrix}} ; \quad \varepsilon = \underset{q \times 1}{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_q \end{bmatrix}} ; \quad \beta = \underset{(p+1) \times q}{\begin{bmatrix} \beta_{10} & \beta_{20} & \cdots & \beta_{q0} \\ \beta_{11} & \beta_{21} & \cdots & \beta_{q1} \\ \vdots & & & \vdots \\ \beta_{1p} & \beta_{2p} & - & \beta_{qp} \end{bmatrix}}$$

DV   IV   Errors   Reg. coeff.

Example:

DV: profit (y_1), sales (y_2)

IV: abs (z_1), brakedown (z_2), Mv (z_3)



$$y_1 = \beta_{10} + \beta_{11} z_1 + \beta_{12} z_2 + \beta_{13} z_3 + \varepsilon_1$$

$$y_2 = \beta_{20} + \beta_{21} z_1 + \beta_{22} z_2 + \beta_{23} z_3 + \varepsilon_2$$

$$Y_{n \times q} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1q} \\ y_{21} & y_{22} & \cdots & y_{2q} \\ \vdots & & & \\ y_{i1} & y_{i2} & \cdots & y_{iq} \\ y_{n1} & y_{n2} & \cdots & y_{nq} \end{bmatrix}_{n \times q}$$

$$\begin{array}{c} Z = \begin{bmatrix} 1 & z_{11} & \cdots & z_{1p} \\ 1 & z_{21} & & z_{2p} \\ \vdots & & & \vdots \\ 1 & z_{i1} & & z_{ip} \\ 1 & z_{n1} & & z_{np} \end{bmatrix}_{n \times (p+1)} \\ \begin{array}{cccc} \quad\quad\uparrow z_0 & \uparrow z_1 & \quad\quad & \uparrow z_p \end{array} \end{array}$$

$$y_{i1} = \beta_{10} + \beta_{11} z_{i1} + \cdots + \beta_{1p} z_{ip} + \varepsilon_{i1}$$

$$y_{i2} = \beta_{20} + \beta_{21} z_{i1} + \cdots + \beta_{2p} z_{ip} + \varepsilon_{i2}$$

$$\vdots$$

$$y_{ik} = \beta_{k0} + \beta_{k1} z_{i1} + \cdots + \beta_{kp} z_{ip} + \varepsilon_{ik}$$

$$\vdots$$

$$y_{iq} = \beta_{q0} + \beta_{q1} z_{i1} + \cdots + \beta_{qp} z_{ip} + \varepsilon_{iq}$$

$$i = 1, 2 \ldots n$$

MVLR:

$$\boxed{y_{n\times q} = z_{n\times(p+1)} \, \beta_{(p+1)\times q} + \varepsilon_{n\times q}}$$

Assumptions:

1. Errors $\varepsilon_{n\times q}$ are multivariate normal.
2. Error variances are equal (homogeneous) across observations, conditional on predictors
3. Error have common covariance structure across observations & independent observations.

$$\operatorname{Cov}(y) = E\left[(y-\bar{y})'_{q\times n}(y-\bar{y})_{n\times q}\right]$$

$$= \Sigma_{q\times q}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1q} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2q} \\ \vdots & & & \\ \sigma_{1q} & \cdot & \cdots & \sigma_q^2 \end{pmatrix}$$

$$E(\varepsilon) = 0$$

$$Cov(\varepsilon) = E(\varepsilon^T \varepsilon) = \Sigma$$

$$Cov(\varepsilon_j, \varepsilon_k) = 0$$

$$Cov(\varepsilon_k, \varepsilon_k) = \sigma_k^2$$

$$\therefore \quad \varepsilon_{n\times q} \sim N_q(\varepsilon, \Sigma)$$

$$\Sigma_1 = \Sigma_2 = \cdots = \Sigma_n$$

$$y_k = \begin{bmatrix} y_{1k} \\ y_{2k} \\ \vdots \\ y_{nk} \end{bmatrix} \qquad \varepsilon_k = \begin{bmatrix} \varepsilon_{1k} \\ \varepsilon_{2k} \\ \vdots \\ \varepsilon_{nk} \end{bmatrix}$$

$$var(\varepsilon_{1k}) = var(\varepsilon_{2k}) = \cdots = var(\varepsilon_{nk}) = \sigma_k^2$$

$$k = 1 \text{ to } q$$

Estimation of parameter

$$Y = Z\beta + \varepsilon$$

$$Y = [y_1, y_2 \cdots y_z]$$

$$= X[\beta_1 \cdots \beta_q] + [\varepsilon_1 \cdots \varepsilon_q]$$

$$\beta_{(p+1)\times q} = \begin{matrix} \beta_1 & (p+1)\times 1 \\ \beta_2 & (p+1)\times 1 \\ \vdots & \\ \beta_q & (p+1)\times 1 \end{matrix}$$

q no. of MLR

$$y_1 = z_1\beta_1 + \varepsilon_1 \leftarrow \text{MLR}_1$$

$$\vdots$$

$$y_q = z_q\beta_q + \varepsilon_q \leftarrow \text{MLR}_q$$

$$\text{MLR}_1 \longrightarrow y_1 = z_1\beta_1 + \varepsilon_1$$

$$\varepsilon_1 = y_1 - z_1\beta_1$$

$$\varepsilon = y - z\beta$$

$$SSE = \underset{1\times n}{\varepsilon'}\ \underset{n\times 1}{\varepsilon} = \underset{\substack{1\times 1 \\ \text{Scalar}}}{(y - Z\beta)'(y - Z\beta)}$$

$$\frac{\partial SSE}{\partial \beta} = 0 \Rightarrow \hat{\beta} = (Z'Z)^{-1}Z'y$$

$$\therefore \boxed{\hat{\beta}_1 = (Z'Z)^{-1}Z'y_1}$$

MvLR:

$$\underset{n\times q}{Y} = \underset{n\times(p+1)}{Z} \cdot \underset{(p+1)\times q}{\beta} + \underset{n\times q}{\varepsilon}$$

$$\underset{n\times q}{\varepsilon} = \underset{n\times q}{Y} - \underset{n\times q}{Z\beta}$$

Sum sq. cross product:

$$\underbrace{\varepsilon' \cdot \varepsilon}_{q\times q} = \begin{bmatrix} \sum_{i=1}^{n} \varepsilon_{i1}^2 & & \\ & \sum_{i=1}^{n} \varepsilon_{i2}^2 & \\ & & \ddots & \sum_{i=1}^{n} \varepsilon_{in}^2 \end{bmatrix}$$

$$\text{trace}(SSCP_{\varepsilon}) = tr(\varepsilon' \varepsilon)$$

$$= \sum \varepsilon_{i1}^2 + \sum \varepsilon_{i2}^2 \cdots + \sum \varepsilon_{iq}^2$$

Minimize this to estimate β

$$\underset{(p+1)\times q}{\beta} = [\underset{(p+1)\times 1}{\beta_1} \quad \beta_2 \quad \cdots \quad \beta_q]$$

$$\frac{\partial\, tr(\varepsilon' \varepsilon)}{\partial \beta_{kj}} = 0, \; j = 1 \text{ to } p$$

$$\varepsilon = y - z\beta$$

$$\varepsilon' \varepsilon = (y - z\beta)'(y - z\beta)$$

$$tr(\varepsilon' \varepsilon) = tr[(y - z\beta)'(y - z\beta)]$$

$$\hat{\beta} = (Z'Z)^{-1} Z' \underbrace{[\, y_1 \,\vdots\, y_2 \,\vdots\, y_q \,]}_{Y}$$

$$\boxed{\hat{\beta} = (Z'Z)^{-1} Z' Y},$$

Individual MLE

$$\hat{\beta}_1 = (Z'Z)^{-1} Z' y_1$$
$$\hat{\beta}_2 = (Z'Z)^{-1} Z' y_2$$
$$\vdots$$
$$\hat{\beta}_q = (Z'Z)^{-1} Z' y_q$$

$$\hat{\beta} = [\hat{\beta}_1 \quad \hat{\beta}_q]$$
$$= \begin{bmatrix} \hat{\beta}_{10} & \hat{\beta}_{q0} \\ \vdots & \vdots \\ \hat{\beta}_{1p} & \hat{\beta}_{2q} \end{bmatrix}$$

$$= (Z'Z)^{-1} Z' [y_1 \cdots y_q]$$

MVLR Estimation of parameters

$$\underset{n \times q}{Y} = \underset{n \times (p+1)}{Z} \; \underset{(p+1) \times q}{\beta} + \underset{n \times q}{\varepsilon}$$

$$\hat{\beta}_1, \hat{\beta}_2 \cdots \hat{\beta}_q \rightarrow \boxed{\hat{\beta} = (Z'Z)^{-1} Z' Y}$$

Example Problem: Calculate $\hat{\beta}$

$$\underset{\substack{3 \times 2 \\ n \quad q}}{Y} = \begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix}$$

IVs $P = 2$; $X = \begin{bmatrix} 9 & 62 \\ 8 & 58 \\ 7 & 64 \end{bmatrix}$

Step1: Compute $z'z$

$$\begin{pmatrix} 1 & 1 & 1 \\ 9 & 8 & 7 \\ 62 & 58 & 64 \end{pmatrix} \begin{pmatrix} 1 & 9 & 62 \\ 1 & 8 & 58 \\ 1 & 7 & 64 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 24 & 184 \\ 24 & 194 & 1420 \\ 184 & 1420 & 11304 \end{pmatrix}$$

Step 2: Compute $(z'z)^{-1}$

$$(z'z)^{-1} = \frac{1}{|z'z|} Adj(z'z)$$

$$= \begin{pmatrix} 320.7 & -8.16 & -4.16 \\ -8.16 & 0.56 & 0.06 \\ -4.16 & 0.06 & 0.06 \end{pmatrix}$$

Step 3: compute $Z'Y$

$$Z'Y = \begin{pmatrix} 1 & 1 & 1 \\ 9 & 8 & 7 \\ 62 & 58 & 64 \end{pmatrix} \begin{pmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{pmatrix}$$

$$= \begin{pmatrix} 33 & 315 \\ 263 & 2515 \\ 2020 & 19300 \end{pmatrix}$$

Step 4: compute $(Z'Z)^{-1}Z'Y$

$(Z'Z)^{-1}Z'Y =$

$$\begin{pmatrix} 320.7 & -8.16 & -4.16 \\ -8.16 & 0.02 & 0.06 \\ -4.16 & 0.06 & 0.06 \end{pmatrix} \begin{bmatrix} 33 & 315 \\ 263 & 2515 \\ 2020 & 19300 \end{bmatrix}$$

$$
\begin{array}{cc}
\hat{\beta}_1 & \hat{\beta}_2 \\
\end{array}
$$

$$
= \begin{pmatrix} 35.8 & 229 \\ -0.80 & -4 \\ -0.3 & -1.5 \end{pmatrix}
$$

$$
\hat{\beta} = \begin{pmatrix} \hat{\beta}_{10} & \hat{\beta}_{20} \\ \hat{\beta}_{11} & \hat{\beta}_{21} \\ \hat{\beta}_{12} & \hat{\beta}_{22} \end{pmatrix}
$$

$$
Y = Z \beta + \varepsilon
$$

$$
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}' \begin{bmatrix} \hat{\beta}_{10} & \hat{\beta}_{20} \\ \hat{\beta}_{11} & \hat{\beta}_{21} \\ \hat{\beta}_{12} & \hat{\beta}_{22} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}
$$

$$
y_1 = \hat{\beta}_{10} + \hat{\beta}_{11} z_1 + \hat{\beta}_{12} z_2 + \varepsilon_1
$$

$$
y_2 = \hat{\beta}_{20} + \hat{\beta}_{21} z_1 + \hat{\beta}_{22} z_2 + \varepsilon_2
$$

$$
y_1 = 35.80 - 0.80 z_1 - 0.30 z_2 + \varepsilon_1
$$

$$
y_2 = 229 - 4 z_1 - 1.5 z_2 + \varepsilon_2
$$

Now $\hat{\varepsilon} = ?$

$$\hat{y} = z\hat{\beta}$$

$$y = z\hat{\beta} + \hat{\varepsilon}$$

$$\hat{\varepsilon} = y - \hat{y}$$

$$= \begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix} - \begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix} = 0$$

$$\hat{y} = z\hat{\beta} = \begin{pmatrix} 1 & 9 & 62 \\ 1 & 8 & 58 \\ 1 & 7 & 64 \end{pmatrix} \begin{pmatrix} 35.8 & 229 \\ -0.8 & -4 \\ -0.3 & -1.8 \end{pmatrix} = \begin{pmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{pmatrix}$$

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

$$= 1 - \frac{(n-p-1)^2 s_e^2}{(n-1) s_e^2} = 1$$

Sampling distribution of $\hat{\beta}$:

$E(\hat{\beta})$; $Cov(\hat{\beta})$ - are to be estimated.

$$E(\hat{\beta}) = E\left[(Z'Z)^{-1} Z'Y\right]$$
$$= (Z'Z)^{-1} Z' E[Y]$$
$$= \underline{(Z'Z)^{-1}}\ \underline{Z'Z}\beta$$
$$= I \beta$$
$$E(\hat{\beta}) = \beta$$

unbiased estimator.

$$Cov(\hat{\beta}) = E\left[\{\hat{\beta} - E(\hat{\beta})\}\{\hat{\beta} - E(\hat{\beta})\}'\right]$$
$$\hat{\beta} - E(\hat{\beta}) = \hat{\beta} - \beta$$
$$= (Z'Z)^{-1}Z'Y - \beta$$
$$= (Z'Z)^{-1}Z'(Z\beta + \varepsilon) - \beta$$
$$= \underbrace{(Z'Z)^{-1}Z'Z}_{I}\beta + (Z'Z)^{-1}Z'\varepsilon - \beta$$
$$= \cancel{\beta} + (Z'Z)^{-1}Z'\varepsilon \cancel{-\beta}$$

$$\boxed{\hat{\beta} - E(\hat{\beta}) = (\cancel{z}'z)^{-1} z' \varepsilon}$$

$$(\hat{\beta} - \beta)' = ((z'z)^{-1} z' \varepsilon)'$$

$$= \varepsilon' z (z'z)^{-1}$$

$$Cov(\hat{\beta}) = E\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'\right]$$

$$= E\left[(\cancel{z}'z)^{-1} z' \varepsilon . \varepsilon' z (z'z)^{-1}\right]$$

$$= (z'z)^{-1} z' E(\varepsilon\varepsilon') \, z (z'z)^{-1}$$

$$= (z'z)^{-1} z' \, I \otimes \Sigma \, z (z'z)^{-1}$$

$\otimes$ -> kroneker product

$$= (z'z)^{-1} z' z (z'z)^{-1} \otimes \Sigma$$

$$= (z'z)^{-1} \otimes \Sigma$$

$Y_1 : (Z'Z)^{-1}\sigma_1^2$

$Y_2 : (Z'Z)^{-1}\sigma_2^2$

$\vdots$

$Y_q : (Z'Z)^{-1}\sigma_q^2$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1q} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2q} \\ \vdots & & & \vdots \\ \sigma_{1q} & \cdots & \cdots & \sigma_q^2 \end{bmatrix}$$

$\sigma_k^2 \div, \ k = 1$ to q

$Cov(\hat{\beta}_k) = (Z'Z)^{-1}\sigma_k^2$

but Σ is not known.

$\hat{\varepsilon}^T \varepsilon = SSE \rightarrow$ Scalar in MLR

$\underset{q \times n}{\hat{\varepsilon}'} \ \underset{n \times q}{\hat{\varepsilon}} = \underset{q \times q}{SSCP_{\varepsilon}} \rightarrow$ vector in MVLR

$SSCP_{\varepsilon} = (n-p-1)\hat{\Sigma}$

$$= (n-p-1)\begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots & \hat{\sigma}_{1q} \\ \hat{\sigma}_{12} & \sigma_2^2 & \cdots & \hat{\sigma}_{2q} \\ \vdots & & \sigma_k^2 \cdots & \hat{\sigma}_{kq} \\ \hat{\sigma}_{1q} & \cdots & & \hat{\sigma}_q^2 \end{bmatrix}$$

$$\therefore \; s_k^2 = (n-p-1)\hat{\sigma}_k^2$$

We know that

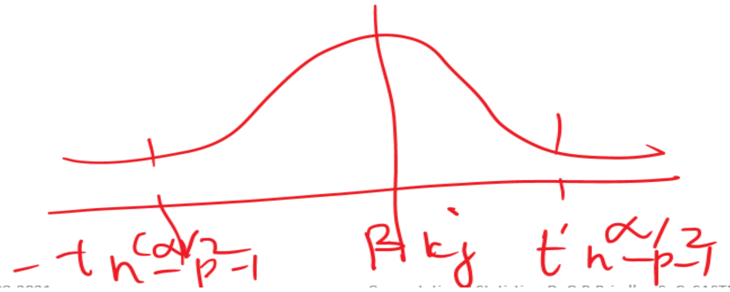
$\hat{\beta}_{(p+1)\times q} \rightarrow$ point estimate

$$\boxed{\hat{\beta}_k \sim N_{p+1}\left(\beta_k, (x'x)^{-1}\hat{\sigma}_k^2\right)}$$

$$\hat{\sigma}_k^2 = \frac{s_k^2}{n-p-1}$$

Confidence interval

$$\frac{\hat{\beta}_{kj} - E(\hat{\beta}_{kj})}{SE(\hat{\beta}_{kj})} \sim t_{n-p-1}$$



$$-t_{n-p-1}^{\alpha/2} < \frac{\hat{\beta}_{kj} - \beta_{kj}}{SE(\hat{\beta}_{kj})} < t_{n-p-1}^{\alpha/2}$$

$$\hat{\beta}_{kj} - t_{n-p-1} SE(\hat{\beta}_{kj}) < \beta_{kj} < \hat{\beta}_{kj} + t_{n-p-1}^{\alpha/2} SE(\hat{\beta}_{kj})$$

$$SE(\beta_{kj}) = (X^T X)^{-1} \sigma_k^2 \sim (X^T X)^{-1} \frac{s_k^2}{n-p-1}$$

Multivariate multiple Regression:

Modeling the relationship between m responses $y_1, y_2, \ldots y_m$ and a single set of prediction variables $z_1, z_2 \ldots z_r$.

$$y_1 = \beta_{01} + \beta_{11} z_1 + \ldots + \beta_{r1} z_r + \varepsilon_1$$

$$y_2 = \beta_{02} + \beta_{12} z_2 + \ldots + \beta_{r2} z_r + \varepsilon_2$$

$$\vdots$$

$$y_m = \beta_{0m} + \beta_{1m} z_1 + \ldots + \beta_{rm} z_m + \varepsilon_m$$

Error term $\varepsilon' = [\varepsilon_1, \varepsilon_2 \ldots \varepsilon_m]$

$E(\varepsilon) = 0$

$var(\varepsilon) = \Sigma$

Design matrix Z

$$\underset{n\times(r+1)}{Z} = \begin{bmatrix} z_{10} & z_{11} & \cdots & z_{1r} \\ \vdots & & & \\ z_{n0} & z_{n1} & \cdots & z_{nr} \end{bmatrix}$$

$$\underset{n\times m}{Y} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ \vdots & & & \\ y_{n1} & y_{n2} & \cdots & y_{nm} \end{bmatrix}$$

$$= \left[y_{(1)} \quad y_{(2)} \quad \cdots \quad y_{(m)} \right]$$

$$\underset{(r+1)\times m}{\beta} = \begin{bmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0m} \\ \vdots & & & \\ \beta_{r1} & \beta_{r2} & \cdots & \beta_{rm} \end{bmatrix}$$

$$= \left[\beta_{(1)} \,\vdots\, \beta_{(2)} \,\vdots\, \beta_{(m)} \right]$$

$$\underset{n\times m}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1m} \\ \vdots & & & \\ \varepsilon_{n1} & \varepsilon_{n2} & \cdots & \varepsilon_{nm} \end{bmatrix}$$

$$= \left[\varepsilon_{(1)} \,\vdots\, \varepsilon_{(2)} \,\vdots\, \cdots\, \varepsilon_{(m)} \right]$$

$$= \begin{bmatrix} \varepsilon_1' \\ \hline \varepsilon_2' \\ \vdots \\ \varepsilon_m' \end{bmatrix}$$

Multivariate linear regression model is

$$Y_{n\times m} = Z_{n\times(r+1)} \beta_{(r+1)\times m} + \varepsilon_{n\times m}$$

with

$$E(\varepsilon_{(i)}) = 0 \quad \text{and}$$

$$Cov(\varepsilon_{(i)}, \varepsilon_{(k)}) = \sigma_{ik} I \quad , i,k = 1-m$$

here β and σ_{ik} are unknown

Note:

i) $\hat{\beta} = (Z'Z)^{-1} Z'Y$

Predicted value

ii) $\hat{Y} = Z\hat{\beta} = Z(Z'Z)^{-1}Z'Y$

Residuals:

iii) $\hat{\varepsilon} = Y - \hat{Y} = [I - Z(Z'Z)^{-1}Z']Y$

$$Z'\hat{\varepsilon} = Z'[I - Z(Z'Z)^{-1}Z']Y = 0$$

$$\hat{Y}'\hat{\varepsilon} = (Z\hat{\beta})'[I - Z(Z'Z)^{-1}Z']Y$$

$$= \hat{\beta}'Z'[I - Z(Z'Z)^{-1}Z']Y$$

$$= 0$$

Predicted values $\hat{y}_{(i)}$ are

$\perp$r to all residual

vectors $\hat{\ }_{(k)}$

$$\boxed{Y = \hat{y} + \hat{\varepsilon}}$$

$$Y'Y = (y' + \hat{\varepsilon})'(\hat{y} + \hat{\varepsilon})$$

$$= \hat{y}'\hat{y} + \hat{\varepsilon}'\hat{\varepsilon} + \hat{y}'\hat{\varepsilon} + \hat{\varepsilon}'\hat{y}$$

$$\boxed{\therefore Y'Y = \hat{y}'\hat{y} + \hat{\varepsilon}'\hat{\varepsilon}}$$

Total sum of squares and cross products= Predicted sum of squares and cross products+ residual sum of squares and cross products

$$\boxed{\hat{\varepsilon}'\hat{\varepsilon} = y'y - \hat{y}'y}$$

Fitting a Multivariate Straight-line regression model.

z_1	0	1	2	3	4
y_1	1	4	3	8	9
y_2	-1	-1	2	3	2

Design matrix Z.

$$Z' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$$(Z'Z)^{-1} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.1 \end{bmatrix}$$

$$Z'y_1 = \underset{2\times5}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}} \underset{5\times1}{\begin{bmatrix} 1 \\ 4 \\ 3 \\ 8 \\ 9 \end{bmatrix}}$$

$$Z'y_1 = \begin{bmatrix} 25 \\ 70 \end{bmatrix}$$

$$z'y_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}_{2\times5} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 3 \\ 2 \end{bmatrix}_{5\times1}$$

$$z'y_2 = \begin{bmatrix} 5 \\ 20 \end{bmatrix}$$

compute $\hat{\beta}$ ie $\hat{\beta}_1$ $\hat{\beta}_2$

$$\hat{\beta}_1 = (z'z)^{-1} z' y_1 = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 25 \\ 70 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{\beta}_2 = (z'z)^{-1} z' Y_2$$

$$= \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore \hat{\beta} = [\hat{\beta}_1 \,\vdots\, \hat{\beta}_2] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\hat{y} = X\hat{\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}_{5\times 2} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}_{2\times 2}$$

$$\hat{y} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 5 & 1 \\ 7 & 2 \\ 9 & 3 \end{bmatrix}_{5\times 2}$$

$$\hat{\varepsilon} = y - \hat{y}$$

$$= \begin{bmatrix} 1 & -1 \\ 4 & -1 \\ 3 & 2 \\ 8 & 3 \\ 9 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 5 & 1 \\ 7 & 2 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -2 & 1 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\hat{\Sigma}' \hat{Y} = \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 5 & 1 \\ 7 & 2 \\ 9 & 3 \end{bmatrix}$$

$$\hat{\Sigma}' \hat{Y} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$y' y = \begin{bmatrix} 1 & 4 & 3 & 8 & 9 \\ -1 & -1 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 4 & -1 \\ 3 & 2 \\ 8 & 3 \\ 9 & 2 \end{bmatrix}$$

$$y' y = \begin{bmatrix} 171 & 43 \\ 43 & 19 \end{bmatrix}$$

$$\hat{y}'\hat{y} = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ -1 & 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 5 & 1 \\ 7 & 2 \\ 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 165 & 45 \\ 45 & 15 \end{bmatrix}$$

$$\hat{\varepsilon}'\hat{\varepsilon} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$$

Actual Predicted Error

$$\therefore \boxed{y'y = \hat{y}'\hat{y} + \hat{\varepsilon}'\hat{\varepsilon}}$$

$$\begin{bmatrix} 171 & 43 \\ 43 & 19 \end{bmatrix} = \begin{bmatrix} 165 & 45 \\ 45 & 15 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$$

LHS = RHS.