Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{64}{21}$, $x_1 = \frac{9}{7}$, $x_2 = \frac{10}{21}$, $x_3 = 0$.

1.4.5 The Two Phase Method

[BNU. BE. Nov 98]

The two phase method is another method to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I : In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 : Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function. Thus the new objective function is $Z^* = -R_1 - R_2 - R_3 - \ldots - R_n$

where R_i's are the artificial variables.

Step 2 : Construct the auxiliary LPP in which the new objective function Z^* is to be maximized subject to the given set of constraints.

Step 3 : Solve the auxiliary LPP by simplex method until either of the following three possibilities arise.

(i) Max $Z^* < 0$ and atleast one artificial variable appears in the optimum basis at a non–zero level. In this case the given LPP does not possess any feasible solution, stop the procedure.

(ii) Max $Z^* = 0$ and atleast one artificial variable appears in the optimum basis at zero level. In this case proceed to phase – II.

(iii) Max $Z^* = 0$ and no artificial variable appears in the optimum basis. In this case proceed to phase – II.

Phase II : Use the optimum basic feasible solution of Phase – I as a starting solution for the original LPP. Assign the actual costs to the variables in the objective function and a 0 cost to every artificial variable that appears in the basis at the zero level. Use simplex method to the modified simplex table obtained at the end of Phase – I, till an optimum basic feasible solution (if any) is obtained.

Note 1 : In Phase – I, the iterations are stopped as soon as the value of the new objective function becomes zero because this is its maximum value. There is no need to continue till the optimality is reached if this value becomes zero earlier than that.

Note 2 : The new objective function is always of maximization type regardless of whether the original problem is of maximization or minimization type.

Note 3 : Before starting phase – II, remove all artificial variables from the table which were non-basic at the end of phase – I.

Example 1 : **Use Two–phase simplex method to solve**

$$\text{Maximize } Z = 5x_1 + 8x_2$$

subject to the constraints

$$\begin{aligned} 3x_1 + 2x_2 &\geq 3 \\ x_1 + 4x_2 &\geq 4 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution: By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 - s_1 + 0s_2 + 0s_3 + R_1 &= 3 \\ x_1 + 4x_2 + 0s_1 - s_2 + 0s_3 + R_2 &= 4 \\ x_1 + x_2 + 0s_1 + 0s_2 + s_3 &= 5 \\ \text{and } x_1, x_2, s_1, s_2, s_3, R_1, R_2 &\geq 0. \end{aligned}$$

(***Here :*** s_1, s_2 – surplus, s_3 – slack, R_1, R_2 – artificials)

The initial basic feasible solution is given by

$R_1 = 3$, $R_2 = 4$, $s_3 = 5$ (basic) ($x_1 = x_2 = s_1 = s_2 = 0$, non-basic)

Phase–I : Assigning a cost -1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

subject to the given constraints.

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	0	−1	−1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
−1 R_1	3	3	2	−1	0	0	1	0	$\frac{3}{2}$
−1 R_2	4	1	(4)	0	−1	0	0	1	$\frac{4}{4}$
0 s_3	5	1	1	0	0	1	0	0	$\frac{5}{1}$
$Z_j^* - C_j$	−7	−4	−6	1	1	0	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop R_2.

	C_j	(0	0	0	0	0	−1	−1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
−1 R_1	1	$\left(\frac{5}{2}\right)$	0	−1	$\frac{1}{2}$	0	1	$\frac{-1}{2}$	$\frac{2}{5}$
0 x_2	1	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	0	0	$\frac{1}{4}$	4
0 s_3	4	$\frac{3}{4}$	0	0	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	$\frac{16}{3}$
$Z_J^* - C_J$	−1	$\frac{-5}{2}$	0	1	$\frac{-1}{2}$	0	0	$\frac{3}{2}$	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_1 and drop R_1.

	C_j	(0	0	0	0	0	−1	−1)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2
0 x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{-1}{5}$
0 x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	$\frac{-1}{10}$	$\frac{3}{10}$
0 s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	$\frac{-3}{10}$	$\frac{-1}{10}$
$Z_j^* - C_j$	0	0	0	0	0	0	1	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimum. Furthermore, no artificial variable appears in the optimum basis so we proceed to phase –II.

Phase – II :

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :

Initial iteration :

	C_j	(5	8	0	0	0	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	θ
5 x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$\left(\frac{1}{5}\right)$	0	2
8 x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	–
0 s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	37
$(Z_j - C_j)$	$\frac{46}{5}$	0	0	$\frac{-6}{5}$	$\frac{-7}{5}$	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce s_2 and drop x_1

	C_j	(5	8	0	0	0)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	θ
0 s_2	2	5	0	−2	1	0	–
8 x_2	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{-1}{2}$	0	0	–
0 s_3	$\frac{7}{2}$	$\frac{-1}{2}$	0	$\left(\frac{1}{2}\right)$	0	1	7
$(Z_j - C_j)$	12	7	0	−4	0	0	

Since there are some $(Z_j - C_j) < 0$, current basic feasible solution is not optimal.

Second iteration : Introduce s_1 and drop s_3

	C_j	(5	8	0	0	0)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0 s_2	16	3	0	0	1	4
8 x_2	5	1	1	0	0	1
0 s_1	7	–1	0	1	0	2
$(Z_J - C_J)$	40	3	0	0	0	8

Since all $(Z_J - C_J) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $z = 40$, $x_1 = 0$, $x_2 = 5$.

Example 2 : Solve by two phase simplex method

$$\text{Maximize } X_0 = -4x_1 - 3x_2 - 9x_3$$

Subject to

$$2x_1 + 4x_2 + 6x_3 - s_1 + R_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + R_2 = 12$$
$$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0.$$

[MU. BE. Nov 92]

Solution : The initial basic feasible solution is given by $R_1 = 15$, $R_2 = 12$, (basic) ($x_1 = x_2 = x_3 = s_1 = s_2 = 0$, non-basic)

Phase–I : Assigning a cost –1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	0	–1	–1)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2	θ
–1 R_1	15	2	4	6	–1	0	1	0	$\frac{15}{6}$
–1 R_2	12	6	1	(6)	0	–1	0	1	$\frac{12}{6}$
$Z_J^* - C_J$	–27	–8	–5	–12	1	1	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_3 and drop R_2.

	C_j	(0	0	0	0	0	–1	–1)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2	θ
–1 R_1	3	–4	(3)	0	–1	1	1	–1	$\frac{3}{3}$
0 x_3	2	1	$\frac{1}{6}$	1	0	$\frac{-1}{6}$	0	$\frac{1}{6}$	12
$Z_j^* - C_j$	–3	4	–3	0	1	–1	0	2	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_2 and drop R_1.

	C_j	(0	0	0	0	0	–1	–1
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2
0 x_2	1	$\frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$
0 x_3	$\frac{11}{6}$	$\frac{22}{18}$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{-1}{18}$	$\frac{4}{18}$
$Z_J^* - C_J$	0	0	0	0	0	0	1	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal. Further, no artificial variable appears in the basis, so we proceed to phase – II.

Phase – II :

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } X_0 = -4x_1 - 3x_2 - 9x_3 + 0s_1 + 0s_2$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :