CSE211 – Formal Languages and Automata Theory

U2L8_Problems in Equivalence of PDA and CFL

Dr. P. Saravanan

School of Computing

SASTRA Deemed University

Equivalence of PDA's and CFG's

- Equivalences to be proved:
  1) CFL's defined by CFG's
  2) Languages accepted by final state by some PDA
  3) Languages accepted by empty stack by some PDA

- – Equivalence of 2) and 3) above have been proved.

Equivalence of PDA's and CFG's

- From Grammars to PDA's
  - Given a CFG $G = (V, T, Q, S)$, construct a PDA P that accepts $L(G)$ by empty stack in the following way:
  - $P = (\{q\}, T, V \cup T, \delta, q, S)$ where the transition function δ is defined by:
    - for each nonterminal A,
      - $\delta(q, \varepsilon, A) = \{(q, \beta) \mid A \rightarrow \beta$ is a production of $G\}$;
    - for each terminal a,
      - $\delta(q, a, a) = \{(q, \varepsilon)\}$.

Equivalence of PDA's and CFG's

- From Grammars to PDA's
  - **Theorem 6.13**

    If PDA P is constructed from CFG G by the construction above, then $N(P) = L(G)$.

Equivalence of PDA's and CFG's

- From Grammars to PDA's
  - **Example 6.12** - Construct a PDA from the expression grammar

$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1;$$
$$E \rightarrow I \mid E*E \mid E+E \mid (E).$$

The transition function for the PDA is as follows:

a) $\delta(q, \varepsilon, I) = \{(q, a), (q, b), (q, Ia), (q, Ib), (q, I0), (q, I1)\}$

b) $\delta(q, \varepsilon, E) = \{(q, \mathrm{I}), (q, E+E), (q, E*E), (q, (E))\}$

c) $\delta(q, d, d) = \{(q, \varepsilon)\}$ where d may any of the terminals $a, b, 0, 1, (,), +, *$.

Equivalence of PDA's and CFG's

- From PDA's to Grammars
  - **Theorem 6.14**

    Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDA. Then there is a context-free grammar G such that $L(G) = N(P)$.

Proof. Construct $G = (V, T, P, S)$ where the set of nonterminals consists of:

- the special symbol S as the start symbol;
- all symbols of the form $[pXq]$ where p and q are states in Q and X is a stack symbol in Γ.

Equivalence of PDA's and CFG's

The productions of G are as follows.

(a) For all states p, G has the production $S \rightarrow [q_0 Z_0 p]$.

(b) Let $\delta(q, a, X)$ contain the pair $(r, Y_1 Y_2 \ldots Y_k)$, where

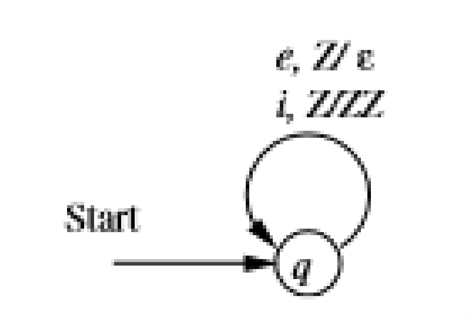
- a is either a symbol in Σ or $a = \varepsilon$;
- k can be any number, including 0, in which case the pair is (r, ε).

Then for all lists of states $r_1, r_2, \ldots, r_k$, G has the production

$$[qXr_k] \rightarrow a[rY_1r_1][r_1Y_2r_2]\ldots[r_{k-1}Y_kr_k].$$

Equivalence of PDA's and CFG's

- From PDA's to Grammars: **Example ---** Convert the following PDA to a grammar.



Nonterminals include only two symbols, S and $[qZq]$.

Productions:

1. $S \rightarrow [qZq]$ (for the start symbol S);
2. $[qZq] \rightarrow i[qZq][qZq]$ (from $\delta_N(q, i, Z) \vdash (q, ZZ)$)
3. $[qZq] \rightarrow e$ (from $\delta_N(q, e, Z) \vdash (q, \varepsilon)$)

Example: Grammar to PDA

- Convert the grammar to PDA

$$S \rightarrow 0S1 \mid A$$
$$A \rightarrow 1A0 \mid S \mid \epsilon$$

> for each nonterminal A,
> - $\delta(q, \varepsilon, A) = \{(q, \beta) \mid A \rightarrow \beta$ is a production of $G\}$;
>
> for each terminal a,
> - $\delta(q, a, a) = \{(q, \varepsilon)\}$.

- We add the following transitions for our variables.

  - $\delta(q, \epsilon, S) = \{(q, 0S1), (q, A)\}$
  - $\delta(q, \epsilon A) = \{(q, 1A0), (q, S)$ $,(q,\varepsilon)\}$

- we add a transition for each terminal.

  - $\delta(q, 0, 0) = \{(q, \epsilon)\}$
  - $\delta(q, 1, 1) = \{q, \epsilon)\}$

Example: Grammar to PDA

- Convert the grammar to PDA
  - $S \rightarrow aAA$
  - $A \rightarrow aS \mid bS \mid a$

- for each nonterminal A,
  - □ $\delta(q, \varepsilon, A) = \{(q, \beta) \mid A \rightarrow \beta$ is a production of $G\}$;
- for each terminal a,
  - □ $\delta(q, a, a) = \{(q, \varepsilon)\}$.

The PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is defined as

$$
\begin{aligned}
Q &= \{q\} \\
\Sigma &= \{a, b\} \\
\Gamma &= \{a, b, S, A\} \\
q_0 &= q \\
Z_0 &= S \\
F &= \{\}
\end{aligned}
$$

And the transition function is defined as:

$$
\begin{aligned}
\delta(q, \epsilon, S) &= \{(q, aAA)\} \\
\delta(q, \epsilon, I) &= \{(q, aS), (q, bS), (q, a)\} \\
\delta(q, a, a) &= \{(q, \epsilon)\} \\
\delta(q, b, b) &= \{(q, \epsilon)\}
\end{aligned}
$$

Example: PDA to Grammar

- Convert the PDA $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$ to a CFG

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$.
2. $\delta(q, 1, X) = \{(q, XX)\}$.
3. $\delta(q, 0, X) = \{(p, X)\}$.
4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$.
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$.
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$.

The productions of G are as follows.

(a) For all states p, G has the production $S \rightarrow [q_0 Z_0 p]$.

(b) Let $\delta(q, a, X)$ contain the pair $(r, Y_1 Y_2 \ldots Y_k)$, where

– a is either a symbol in Σ or $a = \varepsilon$;

– k can be any number, including 0, in which case the pair is (r, ε).

Then for all lists of states $r_1, r_2, \ldots, r_k$, G has the production

$$[qXr_k] \rightarrow a[rY_1r_1][r_1Y_2r_2]\ldots[r_{k-1}Y_kr_k].$$

For start Symbol

1. $S \rightarrow [qZ_0q] \mid [qZ_0p]$

Example: PDA to Grammar

- Convert the PDA $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$ to a CFG

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$.
2. $\delta(q, 1, X) = \{(q, XX)\}$.
3. $\delta(q, 0, X) = \{(p, X)\}$.
4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$.
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$.
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$.

The productions of G are as follows.

(a) For all states p, G has the production $S \rightarrow [q_0Z_0p]$.

(b) Let $\delta(q, a, X)$ contain the pair $(r, Y_1Y_2 \ldots Y_k)$, where

- a is either a symbol in Σ or $a = \varepsilon$;
- k can be any number, including 0, in which case the pair is (r, ε).

Then for all lists of states $r_1, r_2, \ldots, r_k$, G has the production

$$[qXr_k] \rightarrow a[rY_1r_1][r_1Y_2r_2]\ldots[r_{k-1}Y_kr_k].$$

From Rule (1)

2. $[qZ_0q] \text{ -> } 1[qXq][qZ_0q]$
3. $[qZ_0q] \text{ -> } 1[qXp][pZ_0q]$
4. $[qZ_0p] \text{ -> } 1[qXq][qZ_0p]$
5. $[qZ_0p] \text{ -> } 1[qXp][pZ_0p$

Example: PDA to Grammar

- Convert the PDA $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$ to a CFG

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$.
2. $\boxed{\delta(q, 1, X) = \{(q, XX)\}}$.
3. $\delta(q, 0, X) = \{(p, X)\}$.
4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$.
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$.
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$.

The productions of G are as follows.

(a) For all states p, G has the production $S \rightarrow [q_0 Z_0 p]$.

(b) Let $\delta(q, a, X)$ contain the pair $(r, Y_1 Y_2 \ldots Y_k)$, where

- a is either a symbol in Σ or $a = \varepsilon$;
- k can be any number, including 0, in which case the pair is (r, ε).

Then for all lists of states $r_1, r_2, \ldots, r_k$, G has the production

$$[qXr_k] \rightarrow a[rY_1r_1][r_1Y_2r_2]\ldots[r_{k-1}Y_kr_k].$$

From Rule (2)

6. [qXq] -> 1[qXq][qXq]
7. [qXq] -> 1[qXp][pXq]
8. [qXp] -> 1[qXq][qXp]
9. [qXp] -> 1[qXp][pXp]

Example: PDA to Grammar

- Convert the PDA $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$ to a CFG

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$
2. $\delta(q, 1, X) = \{(q, XX)\}$
3. $\delta(q, 0, X) = \{(p, X)\}$.
4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$.
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$.
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$.

The productions of G are as follows.

(a) For all states p, G has the production $S \rightarrow [q_0 Z_0 p]$.

(b) Let $\delta(q, a, X)$ contain the pair $(r, Y_1 Y_2 \ldots Y_k)$, where

– a is either a symbol in Σ or $a = \varepsilon$;

– k can be any number, including 0, in which case the pair is (r, ε).

Then for all lists of states $r_1, r_2, \ldots, r_k$, G has the production

$$[qXr_k] \rightarrow a[rY_1r_1][r_1Y_2r_2]\ldots[r_{k-1}Y_kr_k].$$

From Rule (3)

10. [qXq] -> 0[pXq]

11. [qXp] -> 0[pXp]

From Rule (4)

12. [qXq] -> ε

Example: PDA to Grammar

- Convert the PDA $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$ to a CFG

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$.
2. $\delta(q, 1, X) = \{(q, XX)\}$.
3. $\delta(q, 0, X) = \{(p, X)\}$.
4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$.
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$.
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$.

The productions of G are as follows.

(a) For all states p, G has the production $S \rightarrow [q_0 Z_0 p]$.

(b) Let $\delta(q, a, X)$ contain the pair $(r, Y_1 Y_2 \ldots Y_k)$, where

- a is either a symbol in Σ or $a = \varepsilon$;
- k can be any number, including 0, in which case the pair is (r, ε).

Then for all lists of states $r_1, r_2, \ldots, r_k$, G has the production

$$[qXr_k] \rightarrow a[rY_1r_1][r_1Y_2r_2]\ldots[r_{k-1}Y_kr_k].$$

From Rule (5)

13. [pXp] -> 1

From Rule (6)

14. $[pZ_0q]$ -> $0[qZ_0q]$

15. $[pZ_0p]$ -> $0[qZ_0p]$

Try this...

Exercise 6.1.1: Suppose the PDA $P = (\{q, p\}, \{0, 1\}, \{Z_0, X\}, \delta, q, Z_0, \{p\})$ has the following transition function:

1. $\delta(q, 0, Z_0) = \{(q, XZ_0)\}$.
2. $\delta(q, 0, X) = \{(q, XX)\}$.
3. $\delta(q, 1, X) = \{(q, X)\}$.
4. $\delta(q, \epsilon, X) = \{(p, \epsilon)\}$.
5. $\delta(p, \epsilon, X) = \{(p, \epsilon)\}$.
6. $\delta(p, 1, X) = \{(p, XX)\}$.
7. $\delta(p, 1, Z_0) = \{(p, \epsilon)\}$.

References

- John E. Hopcroft, Rajeev Motwani and Jeffrey D. Ullman, *Introduction to Automata Theory,* Languages, and Computation, Pearson, 3rd Edition, 2011.
- Peter Linz, An Introduction to Formal Languages and Automata, Jones and Bartle Learning International, United Kingdom, 6th Edition, 2016.

Next Class:

Deterministic PDA

THANK YOU.