24. Primal: Max Z $= \frac{141}{5}$, $x_1 = \frac{9}{5}$, $x_2 = \frac{8}{5}$, $x_3 = 0$.

Dual : Min W $= \frac{141}{5}$, $y_1 = \frac{29}{5}$, $y_2 = \frac{-2}{5}$.

25. Min Z $= \frac{215}{23}$, $x_1 = \frac{65}{23}$, $x_2 = 0$, $x_3 = \frac{20}{23}$, $x_4 = 0$.

26. Cannot be solved by dual simplex method.

27. Min Z = 175.38, $x_1 = \frac{16}{13}$, $x_2 = \frac{6}{13}$, $x_3 = \frac{8}{13}$.

28. Max Z = –1, $x_1 = 0$, $x_2 = 1$.

29. Max Z $= \frac{-4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

30. Max Z = –36, $x_1 = 0$, $x_2 = 3$, $x_3 = 1$.

32. Min Z $= \frac{4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

2.2 TRANSPORTATION MODEL

2.2.1 Introduction

Transportation deals with the transportation of a commodity (single product) from 'm' sources (origins or supply or *capacity* centres) to 'n' destinations (sinks or demand or requirement centres). It is assumed that

i) Level of supply at each source and the amount of demand at each destination and

ii) The unit transportation cost of commodity from each source to each destination are known [given].

It is also assumed that the cost of transportation is linear.

The objective is to determine the amount to be shifted from each source to each destination such that the total transportation cost is minimum.

Note: The transportation model also can be modified to account for multiple commodities.

2.2.2 Mathematical Formulation of a Transportation Problem:

Let us assume that there are m sources and n destinations.

Let a_i be the supply (capacity) at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source i to destination j and x_{ij} be the number of units shifted from source i to destination j.

Then the transportation problem can be expressed mathematically as

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, 3 \ldots m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, 3 \ldots n.$$

and $x_{ij} \geq 0$, for all i and j.

24. Primal: Max Z $= \frac{141}{5}$, $x_1 = \frac{9}{5}$, $x_2 = \frac{8}{5}$, $x_3 = 0$.

Dual : Min W $= \frac{141}{5}$, $y_1 = \frac{29}{5}$, $y_2 = \frac{-2}{5}$.

25. Min Z $= \frac{215}{23}$, $x_1 = \frac{65}{23}$, $x_2 = 0$, $x_3 = \frac{20}{23}$, $x_4 = 0$.

26. Cannot be solved by dual simplex method.

27. Min Z $= 175.38$, $x_1 = \frac{16}{13}$, $x_2 = \frac{6}{13}$, $x_3 = \frac{8}{13}$.

28. Max Z $= -1$, $x_1 = 0$, $x_2 = 1$.

29. Max Z $= \frac{-4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

30. Max Z $= -36$, $x_1 = 0$, $x_2 = 3$, $x_3 = 1$.

32. Min Z $= \frac{4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

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subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, 3 \ldots m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, 3 \ldots n.$$

and $x_{ij} \geq 0$, for all i and j.

Note 1: The two sets of constraints will be ***consistent*** if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(total supply) *(total demand)*

which is the necessary and sufficient condition for a transportation problem to have a feasible solution. Problems satisfying this condition are called ***balanced transportation problems***.

Note 2: If $\sum a_i \neq \sum b_j$, then the transportation problem is said to be ***unbalanced.***

Note 3: For any transportation problem, the coefficients of all x_{ij} in the constraints are unity.

Note 4: The objective function and the constraints being all linear, the transportation problem is a special class of linear programming problem. Therefore it can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we can look for some other technique which would be simpler than the usual simplex method.

Standard transportation table:

Transportation problem is explicitly represented by the following transportation table.

Destination

		D_1	D_2	D_3	...	D_j	...	D_n	Supply
	S_1	c_{11}	c_{12}	c_{13}		c_{1j}		c_{1n}	a_1
Source	S_2	c_{21}	c_{22}	c_{23}		c_{2j}		c_{2n}	a_2
									⋮
	S_i	c_{i1}	c_{i2}			c_{ij}		c_{in}	⋮
	S_m	c_{m1}	c_{m2}			c_{mj}		c_{mn}	a_m
	Demand	b_1	b_2	b_3	...	...	...	b_n	$\Sigma a_i = \Sigma b_j$

The mn squares are called ***cells***. The unit transportation cost c_{ij} from the i^{th} source to the j^{th} destination is displayed in the ***upper left side of the (i, j)th cell.*** Any feasible solution is shown in the table by entering the value of x_{ij} ***in the centre of the*** $(i,j)^{th}$ cell. The various a's and b's are called ***rim requirements***. The feasibility of a solution can be verified by summing the values of x_{ij} along the rows and down the columns.

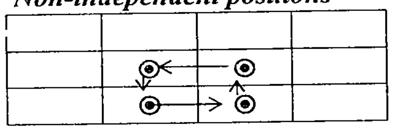
Definition 1: A set of non-negative values x_{ij}, $i = 1, 2, \ldots m$; $j = 1, 2\ldots n$. that satisfies the constraints (rim conditions and also the non-negativity restrictions) is called a ***feasible solution*** to the transportation problem.

Note: A balanced transportation problem will always have a feasible solution.

Definition 2: A feasible solution to a $(m \times n)$ transportation problem that contains no more than $m + n - 1$ non-negative allocations is called a ***basic feasible solution*** (BFS) to the transportation problem.

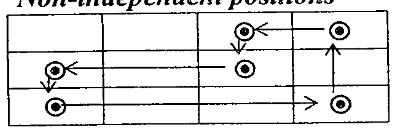
The allocations are said to be in ***independent positions*** if it is impossible to increase or decrease any allocation without either changing the position of the allocation or violating the rim requirements. A simple rule for allocations to be in independent positions is that it is impossible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of the route. Example

Non-independent positions



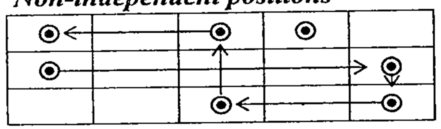
(i)

Non-independent positions



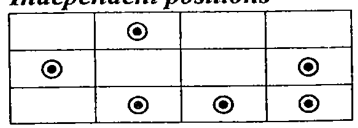
(ii)

Non-independent positions



(i)

Independent positions



(ii)

Definition 3: A basic feasible solution to a $(m \times n)$ transportation problem is said to be a ***non-degenerate basic feasible solution*** if it contains exactly $m + n - 1$ non-negative allocations in independent positions.

Definition 4: A basic feasible solution that contains less than $m + n - 1$ non-negative allocations is said to be a ***degenerate basic*** feasible solution.

Definition 5: A feasible solution (not necessarily basic) is said to be an ***optimal solution*** if it minimizes the total transportation cost.

Note: The number of basic variables in an $m \times n$ balanced transportation problem is atmost $m + n - 1$.

Note: The number of non–basic variables in an $m \times n$ balanced transportation problem is atleast $mn - (m + n - 1)$

2.2.3 Methods for finding initial basic feasible solution

The transportation problem has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given transportation problem is balanced. If not one has to balance the transportation problem first. The way of doing this is discussed in section 7.4 page 7.40 In this section all the given transportation problems are balanced.

Method 1: North west Corner Rule:

Step 1: The first assignment is made in the cell occupying the upper left–hand (north–west) corner of the transportation table. The maximum possible amount is allocated there. That is $x_{11} = \min\{a_1, b_1\}$.

Case (i) : If $\min\{a_1, b_1\} = a_1$, then put $x_{11} = a_1$, decrease b_1 by a_1 and move vertically to the 2nd row (*i.e.,*) to the cell (2,1) cross out the first row.

Case (ii) : If $\min\{a_1, b_1\} = b_1$, then put $x_{11} = b_1$, and decrease a_1 by b_1 and move horizontally right (*i.e.,*) to the cell (1,2) cross out the first column

Case (iii) : If $\min\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row and the first column.

Step 2: Repeat the procedure until all the rim requirements are satisfied.

Method 2: Least Cost method (or) Matrix minima method (or) Lowest cost entry method:

Step 1: Identify the cell with smallest cost and allocate $x_{ij} = \min\{a_i, b_j\}$.

Case (i) : If $\min\{a_i, b_j\} = a_i$, then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i, Go to step (2).

Case (ii) : If $\min\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ cross out the j^{th} column and decrease a_i by b_j Go to step (2).

Case (iii) : If $\min\{a_i, b_j\} = a_i = b_j$, then put $x_{ij} = a_i = b_j$, cross out either i^{th} row or j^{th} column but not both, Go to step (2).

Step 2: Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

Method 3: Vogel's approximation method (VAM) (or) Unit cost penalty method: ***[MU. MBA. Nov 96, Apr 95, Apr 97]***

Step 1: Find the difference (penalty) between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step 2: Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily. Choose the cell with smallest cost in that selected row or column and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3).

Step 3: Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Example 1 **Determine basic feasible solution to the following transportation problem using North West Corner Rule:**

		Sink A	B	C	D	E	Supply
Origin	P	2	11	10	3	7	4
	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

[MU. BE. Apr 94]

Solution: Since $\sum a_i = \sum b_j = 21$, the given problem is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

2 3	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

Following North West Corner rule, the first allocation is made in the cell (1,1).

Here $x_{11} = \min \{a_1, b_1\} = \min \{4, 3\} = 3$

$\therefore$ Allocate 3 to the cell (1,1) and decrease 4 by 3 *i.e.*, $4 - 3 = 1$

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

11 1	10	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

Here the North West Corner cell is (1,2).

So Allocate $x_{12} = \min \{1,3\} = 1$ to the cell (1,2) and move vertically to cell (2,2). The resulting reduced transportation table is

4 2	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate $x_{22} = \min \{8, 2\} = 2$ to the cell (2, 2) and move horizontally to the cell (2,3). The resulting transportation table is

7 4	2	1	6
4	8	12	9
4	5	6	

Allocate $x_{23} = \min \{6,4\} = 4$ and move horizontally to the cell (2,4).

The resulting reduced transportation table is

2 2	1	2
8	12	9
5	6	

Allocate $x_{24} = \{2,5\} = 2$ and move vertically to the cell (3,4). The resulting reduced transportation table is

8 3	12	9
3	6	

Allocate $x_{34} = \min \{9, 3\} = 3$ and move horizontally to the cell (3, 5), which is

12 6	6
6	

Allocate $x_{35} = \min \{6,6\} = 6$

Finally the initial basic feasible solution is as shown in the following table.

2 **3**	11 **1**	10	3	7
1	4 **2**	7 **4**	2 **2**	1
3	9	4	8 **3**	12 **6**

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 5 - 1 = 7$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $= \text{Rs. } 2 \times 3 + 11 \times 1 + 4 \times 2 + 7 \times 4 + 2 \times 2 + 8 \times 3 + 12 \times 6$

$= \text{Rs.}153/-$

Example 2 **Find the initial basic feasible solution for the following transportation problem by Least Cost Method.**

	To				Supply
From	**1**	**2**	**1**	**4**	**30**
	3	**3**	**2**	**1**	**50**
	4	**2**	**5**	**9**	**20**
Demand	**20**	**40**	**30**	**10**	

[MU. BE. Apr 95, MSU. BE. Nov 96]

Solution: Since $\Sigma a_i = \Sigma b_j = 100$, the given TPP is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

1 **20**	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	10	

By least cost method, min $c_{ij} = c_{11} = c_{13} = c_{24} = 1$

Since more than one cell having the same minimum c_{ij}, break the tie.

Let us choose the cell (1,1) and allocate $x_{11} = \min \{a_1, b_1\} = \min \{30, 20\} = 20$ and cross out the satisfied column and decrease 30 by 20.

The resulting reduced transportation table is

2	1 **10**	4	10
3	2	1	50
2	5	9	20
40	30	10	

Here min $c_{ij} = c_{13} = c_{24} = 1$

Choose the cell (1,3) and allocate $x_{13} = \min \{a_1, b_3\} = \min \{10, 30\} = 10$ and cross out the satisfied row.

The resulting reduced transportation table is

3	2	1 **10**	50
2	5	9	20
40	20	10	

Here min $c_{ij} = c_{24} = 1$,

$\therefore$ Allocate $x_{24} = \min\{a_2, b_4\} = \min(50, 10) = 10$ and cross out the satisfied column.

The resulting transportation table is

3	2 **20**	40
2	5	20
40	20	

Here min $c_{ij} = c_{23} = c_{32} = 2$. Choose the cell (2,3) and allocate $x_{23} = \min\{a_2, b_3\} = \min(40,20) = 20$ and cross out the satisfied column.

The resulting reduced transportation table is

3	20
2 20	20
40	

Here min $c_{ij} = c_{32} =$ 2. Choose the cell (3, 2) and allocate $x_{32} = \min\{a_3, b_2\} = \min(20,40) = 20$ and cross out the satisfied row.

The resulting reduced transportation table is

3 **20**	20
20	

Finally the initial basic feasible solution is as shown in the following table.

1 **20**	2	1 **10**	4
3	3 **20**	2 **20**	1 **10**
4	2 **20**	5	9

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $= \text{Rs. } 1 \times 20 + 1 \times 10 + 3 \times 20 + 2 \times 20 + 1 \times 10 + 2 \times 20$

$= 20 + 10 + 60 + 40 + 10 + 40$

$= \text{Rs.}180/-$

$\boxed{\textbf{Example 3}}$ **Find the initial basic feasible solution for the following transportation problem by VAM.**

Distribution Centres

		D_1	D_2	D_3	D_4	Availability
	S_1	11	13	17	14	250
Origin	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Requirements		200	225	275	250	

Solution: Since $\sum a_i = \sum b_j = 950$, the given problem is balanced.

$\therefore$ There exists a feasible solution to this problem.

11 200	13	17	14	250 (2)
16	18	14	10	300 (4)
21	24	13	10	400 (3)
200	225	275	250	
(5)	(5)	(1)	(0)	

First let us find the difference (penalty) between the smallest and next smallest costs in each row and column and write them in brackets against the respective rows and columns.

The largest of these differences is (5) and is associated with the first two columns of the transportation table. We choose the first column arbitrarily.

In this selected column, the cell (1,1) has the minimum unit transportation cost $c_{11} = 11$.

$\therefore$ Allocate $x_{11} = \min\{250, 200\} = 200$ to this cell (1,1) and decrease 250 by 200 and cross out the satisfied column.

The resulting reduced transportation table is

13 50	17	14	50 (1)
18	14	10	300 (4)
24	13	10	400 (3)
225	275	250	
(5)	(1)	(0)	

The row and column differences are now computed for this reduced transportation table. The largest of these is (5) which is associated with the second column. Since $c_{12} = 13$ is the minimum cost, we allocate $x_{12} = \min\{50,225\} = 50$ to the cell (1,2) and decrease 225 by 50 and cross out the satisfied row.

Continuing in this manner, the subsequent reduced transportation tables and the differences for the surviving rows and columns are shown below:

18 175	14	10	300 (4)
24	13	10	400 (3)
175	275	250	
(6)	(1)	(0)	

(i)

14	10 125	125 **(4)**
13	10	400 (3)
	250	
(1)	(0)	

(ii)

13	10 125	400
275	125	

(iii)

13 275	275
275	

(iv)

Finally the initial basic feasible solution is as shown in the following table.

11 **200**	13 **50**	17	14
16	18 **175**	14	10 **125**
21	24	13 **275**	10 **125**

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

∴ The initial transportation cost $= \text{Rs. } 11 \times 200 + 13 \times 50 + 18 \times 175$

$+ 10 \times 125 + 13 \times 275 + 10 \times 125$

$= \text{Rs. } 12075/-$

Example 4 **Find the starting solution of the following transportation model**

1	**2**	**6**	**7**
0	**4**	**2**	**12**
3	**1**	**5**	**11**
10	**10**	**10**	

using ***(i)*** ***North West Corner rule***

(ii) ***Least Cost method***

(iii) ***Vogel's approximation method.***

Solution: Since $\Sigma a_i = \Sigma b_j = 30$, the given Transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

(i) North West Corner rule: Using this method, the allocations are shown in the tables below:

1 7	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

(i)

0 **3**	4	2	12
3	1	5	11
3	10	10	

(ii)

4 **8**	2	9
1	5	11
10	10	

(iii)

1 **1**	5	11
1	10	

(iv)

5 **10**	10
10	

(v)

The starting solution is as shown in the following table:

1 **7**	2	6
0 **3**	4 **9**	2
3	1 **1**	5 **10**

∴ The initial transportation cost $= \text{Rs. } 1 \times 7 + 0 \times 3 + 4 \times 9 + 1 \times 1 + 5 \times 10$

$= \text{Rs. } 94/-$

(ii) Least Cost Method: Using this method, the allocations are as shown in the table below:

1	2	6 7	7
0 10	4	2	12
3	1	5	11
10	10	10	

(i)

2	6	7
4	2	2
1 10	5	11
10	10	

(ii)

6	7
2 **2**	2
5	1
10	

(iii)

6	7
5 **1**	1
8	

(iv)

6 **7**	7
7	

(v)

The starting solution is as shown in the following table:

1	2	6 **7**
0 **10**	4	2 **2**
3	1 **10**	5 **1**

∴ The initial transportation cost $= \text{Rs. } 6 \times 7 + 0 \times 10 + 2 \times 2 + 1 \times 10 + 5 \times 1$

$= \text{Rs. } 61/-$

(iii) Vogel's Approximation Method: Using this method, the allocations are shown in the tables below:

1	2	6	7	(1)
0	4	2 10	12	(2)
3	1	5	11	(2)
10	10	10		
(1)	(1)	(3)		

(i)

1	2	7	(1)
0 2	4	2	(4)
3	1	11	(2)
10	10		
(1)	(1)		

(ii)

1	2	7	(1)
3	1 10	11	(2)
8	10		
(2)	(1)		

(iii)

1 7	7
3	1
8	

(iv)

3 1	1
1	

(v)

The starting solution is as shown in the following table:

1 **7**	2	6
0 **2**	4	2 **10**
3 **1**	1 **10**	5

$\therefore$ The initial transportation cost $=$ Rs. $1 \times 7 + 0 \times 2 + 2 \times 10 + 3 \times 1 + 1 \times 10$

$=$ Rs. 40/-

Note: For the above problem, the number of positive allocations in independent positions is $(m + n - 1)$ (*i.e.*, $m + n - 1 = 3 + 3 - 1 = 5$). This ensures that the given problem has a non-degenerate basic feasible solution by using all the three methods. This need not be the case in all the problems.

2.2.4 Transportation Algorithm (or) MODI Method (modified distribution method) (Test for optimal solution).

[MU. MBA. Apr 96, Apr 97]

Step 1: Find the initial basic feasible solution of the given problem by Northwest Corner rule (or) Least Cost method or VAM.

Step 2: Check the number of occupied cells. If these are less than $m + n - 1$, there exists degeneracy and we introduce a very small positive assignment of $\in$ (≈ 0) in suitable independent positions, so that the number of occupied cells is exactly equal to $m + n - 1$.

Step 3: Find the set of values u_i, v_j $(i = 1,2,3, \ldots m \,; j = 1, 2, 3 \ldots n)$ from the relation $c_{ij} = u_i + v_j$ for each occupied cell (i, j), by starting initially with $u_i = 0$ or $v_j = 0$ preferably for which the corresponding row or column has maximum number of individual allocations.

Step 4: Find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding cell (i, j).