A two-dimensional r.v $\underset{=}{\overset{x}{}}$ (X, Y) have the bivariate distribution given by: $P(X=x, Y=y) = \dfrac{x^2+y}{32}$ for $x = 0, 1, 2, 3$ & $y = 0, 1$.

Find the marginal distribution of X & Y.

Soln

Y \ X	0	1	2	3	Distribution of Y
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$	$\frac{14}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$18/32$
Distribution of X	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{19}{32}$	1

Y:	0	1
f_Y:	$\frac{14}{32}$	$\frac{18}{32}$

X	0	1	2	3
f_X	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{19}{32}$

→ Two discrete random variables X and Y have the joint probability density function

$$P_{XY}(x,y) = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!} \quad \begin{cases} y = 0,1,2,\ldots,x \\ x = 0,1,2,\ldots \end{cases} \quad x \geq y$$

where λ, p are constants with $\lambda > 0$ & $0 < p < 1$.

find the (i) Marginal Probability density function of X & Y.

(ii) Conditional distribution of Y for a given X

(iii) Conditional distribution of X for a given Y.

Soln: Marginal density fn of X.

$$P_X(x) = \sum_{y=0}^{x} P_{XY}(x,y)$$

$$= \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{\lfloor y \; \lfloor x-y}$$

$$P_X(x) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} (1-p)^{x-y} p^y}{y!\ (x-y)!}$$

$$= \lambda^x e^{-\lambda} \sum_{y=0}^{x} \frac{(1-p)^{x-y} p^y}{\lfloor y \ \lfloor x-y}$$

$$= \frac{\lambda^x e^{-\lambda}}{\lfloor x} \sum_{y=0}^{x} \frac{\lfloor x \ p^y (1-p)^{x-y}}{\lfloor y \ \lfloor x-y}$$

$$= \frac{\lambda^x e^{-\lambda}}{\lfloor x} \sum_{y=0}^{x} {}^xc_y \, p^y (1-p)^{x-y}$$

$$= \frac{\lambda^x e^{-\lambda}}{\lfloor x} \{p+1-p\}^x = \frac{\lambda^x e^{-\lambda}}{\lfloor x}$$

$$\sum_{y=0}^{x} {}^xc_y \, p^y (1-p)^{x-y}$$

$${}^xc_0 \, p^0 (1-p)^{x-1} + {}^xc_1 \, p (1-p)^{x-2} + {}^xc_2 \, p^2 (1-p)^0 + \cdots + {}^xc_x \, p^x (1-p)$$

$$= p^x + {}^xc_{x-1} \, p^{x-1} (1-p) + {}^xc_{x-2} \, p^{x-2} (1-p)^2 + \cdots + (1-p)^x$$

$$= \{p+1-p\}^x$$

$$|p|<1$$

$$(a+b)^n$$

$$P_Y(y) = \sum_{x=y}^{\infty} P_{XY}(x,y)$$

$$= \sum_{x=y}^{\infty} \frac{e^{-\lambda}\lambda^{x} p^{y} (1-p)^{x-y}}{y!\,(x-y)!}$$

$$= \frac{e^{-\lambda} p^{y}}{y!} \sum_{x=y}^{\infty} \frac{\lambda^{x} (1-p)^{x-y}}{(x-y)!}$$

$$= \frac{\lambda^{y} e^{-\lambda} p^{y}}{y!} \sum_{x=y}^{\infty} \frac{\lambda^{x-y} (1-p)^{x-y}}{(x-y)!}$$

$$e^{\lambda(1-p)}$$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$P_Y(y) = \frac{\lambda^y \cancel{e^{-\lambda}} p^y}{y!} \cancel{e^{\lambda}} e^{-\lambda p} = \frac{e^{-\lambda p} (\lambda p)^y}{y!}$$

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$$

$$= \frac{\cancel{\lambda^x} \cancel{e^{-\lambda}} p^y (1-p)^{x-y} \, x!}{y! \, (x-y)! \, \cancel{\lambda^x} \cancel{e^{-\lambda}}} = \frac{p^y (1-p)^{x-y} \, x!}{y! \, (x-y)!}$$

$$= {}^xC_y \, p^y (1-p)^{x-y}$$

$x \geq y$ & $y = 0, 1, 2, \ldots x$

$$P_{X|Y}(x|y) = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y} \, \cancel{y!}}{\cancel{y!} \, (x-y)! \; e^{-\lambda p} (\lambda p)^y}$$

$x \geq y$ & $x = y, y+1, y+2, \ldots$

$$\left\{ \sum_{x=y}^{\infty} \frac{\lambda^{x-y} (1-p)^{x-y}}{\lfloor x-y} \right.$$

$$= \sum_{x=y}^{\infty} \frac{(\lambda(1-p))^{x-y}}{\lfloor x-y}$$

$$e^{\lambda(1-p)} = 1 + \frac{\lambda(1-p)}{\lfloor 1} + \frac{(\lambda(1-p))^2}{\lfloor 2} + \frac{(\lambda(1-p))^3}{\lfloor 3} + \cdots$$

$$\left. = \sum_{x=0}^{\infty} \frac{(\lambda(1-p))^{x}}{\lfloor x} \right\}$$

Suppose that 2-D r.v (continuous) has joint PDF

$$f(x,y) = \begin{cases} 6x^2 y & 0<x<1;\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

(i) Verify $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1.$ ✓

(ii) find $P\left(0<X<\frac{3}{4}, \frac{1}{3}<Y<2\right)$

(iii) $P(X+Y<1)$; (iv) $P(X>Y)$

(v) $P(X<1 \mid Y<2)$.

(i) $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 \int_0^1 6x^2 y\,dx\,dy$

$= \int_0^1 \left(\frac{{}^{2}\cancel{6}x^3}{\cancel{3}} \right)_0^1 y\,dy$

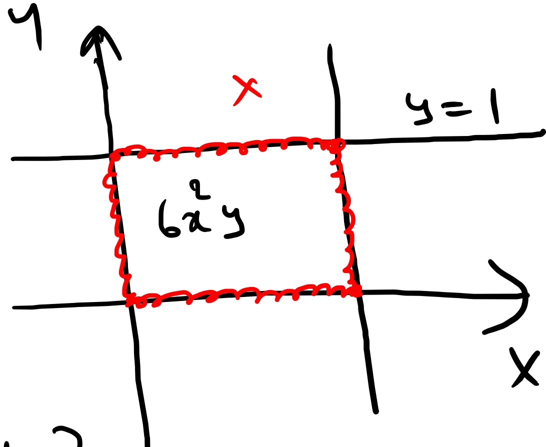
$= \int_0^1 2y\,dy = \left(\frac{2y^2}{2} \right)_0^1 = 1.$

$$f(x,y)=\begin{cases}6x^2y & 0<x,y<1\\ 0 & \text{otherwise}\end{cases}$$

(ii) $P\left(0<X<\frac{3}{4}, \frac{1}{3}<Y<2\right)$

$$\{0<y<1\}$$

$$= \int_0^{3/4}\int_{1/3}^{1} 6x^2y\,dy\,dx + \overset{0}{\cancel{\int_0^{3/4}\int_1^{2} f(x,y)\,dy\,dx}}$$



$$= \int_0^{3/4} 6x^2\left(\frac{y^2}{2}\right)_{1/3}^{1} dx$$

$$= \int_0^{3/4} 3x^2\left\{1-\frac{1}{9}\right\}dx$$

$$= \frac{24}{9}\left(\frac{x^3}{3}\right)_0^{3/4} = \frac{24}{27}\times\frac{3^3}{4^3}$$

(iii) $P(X+Y<1)$

$$= \int_0^1 \int_0^{1-x} 6x^2 y \, dy \, dx$$

$$= \int_0^1 \cancel{6}^{3} x^2 \left\{ \frac{y^2}{\cancel{2}} \right\}_0^{1-x} dx$$

$$= \int_0^1 3x^2 (1-x)^2 \, dx$$

$$= \int_0^1 3x^2 (1-2x+x^2) \, dx$$

$$= 3\left\{ \frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right\}_0^1$$

$$= 3\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{1}{10}$$

$$3\left(\frac{10-15+6}{30} \right) = \frac{3(1)}{30} = \frac{1}{10}$$