Trigonometric integrals and trig substitution — Solutions

Question 1: Trigonometric Integrals

(a) $\int \sin^3 x \cos^2 x \, dx$.

Write $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$. Let $u = \cos x$, $du = -\sin x \, dx$. Then

$$\int \sin^3 x \cos^2 x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx = -\int (1 - u^2) u^2 \, du = -\int (u^2 - u^4) \, du.$$

Integrate:

$$-\Big(\frac{u^3}{3} - \frac{u^5}{5}\Big) + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

(b) $\int \cos^3 x \, dx$.

Write $\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x$. Let $u = \sin x$, $du = \cos x \, dx$. Then

$$\int (1 - u^2) \, du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

(c) $\int \sin^2 x \cos^2 x \, dx$.

Use the double-angle identities:

$$\sin^2 x \cos^2 x = \frac{1}{4} \sin^2(2x) = \frac{1}{4} \cdot \frac{1 - \cos(4x)}{2} = \frac{1}{8}\big(1 - \cos 4x\big).$$

Thus

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8} \int (1 - \cos 4x) \, dx = \frac{1}{8}\Big(x - \frac{\sin 4x}{4}\Big) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

(d) $\int \tan^3 x \sec^4 x \, dx$.

Write $\tan^3 x = (\tan^2 x) \tan x = (\sec^2 x - 1) \tan x$. Then

$$\int \tan^3 x \sec^4 x \, dx = \int (\sec^2 x - 1) \tan x \sec^4 x \, dx = \int \big(\tan x \sec^6 x - \tan x \sec^4 x \big) \, dx.$$

Factor $\sec x \tan x \, dx$ and set $u = \sec x$, $du = \sec x \tan x \, dx$:

$$\int (u^5 - u^3) \, du = \frac{u^6}{6} - \frac{u^4}{4} + C = \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C.$$

(e) $\int \sec x \, dx$.

Standard antiderivative:

$$\int \sec x \, dx = \ln \big| \sec x + \tan x \big| + C.$$

(f) $\displaystyle\int \sin(3x)\cos(5x)\,dx$.

Use the product-to-sum formula $\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$:

$$\sin(3x)\cos(5x) = \tfrac{1}{2}\big(\sin 8x + \sin(-2x)\big) = \tfrac{1}{2}(\sin 8x - \sin 2x).$$

Integrate:

$$\int \sin(3x)\cos(5x)\,dx = \tfrac{1}{2}\Big(-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\Big) + C = -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

Question 2: Trigonometric Substitution

(a) $\displaystyle\int \sqrt{9-x^2}\,dx$.

Use $x = 3\sin\theta$ $(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$, so $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Then

$$\int 3\cos\theta\cdot 3\cos\theta\,d\theta = 9\int \cos^2\theta\,d\theta = 9\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{9}{2}\theta + \frac{9}{4}\sin\theta\cos\theta + C.$$

Return to x: $\theta = \arcsin(x/3)$, $\sin\theta = \frac{x}{3}$, $\cos\theta = \frac{\sqrt{9-x^2}}{3}$. Hence

$$\boxed{\int \sqrt{9-x^2}\,dx = \frac{9}{2}\arcsin\frac{x}{3} + \frac{x\sqrt{9-x^2}}{4} + C}.$$

(b) $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$.

Standard result (use $x = 4\tan\theta$ or recognize form):

$$\int \frac{dx}{\sqrt{x^2+16}} = \ln\left|x + \sqrt{x^2+16}\right| + C.$$

(Equivalent form: $\operatorname{arsinh}(x/4) + C$.)

(c) $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$.

Use $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, $\sqrt{x^2+4} = 2\sec\theta$. Then

$$\int \frac{4\tan^2\theta\cdot 2\sec^2\theta\,d\theta}{2\sec\theta} = 4\int \tan^2\theta\sec\theta\,d\theta = 4\int (\sec^3\theta - \sec\theta)\,d\theta.$$

Use $\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$ and $\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$. Thus

$$4\Big(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\Big) + C = 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C.$$

Back-substitute: $\tan\theta = \frac{x}{2}$, $\sec\theta = \frac{\sqrt{x^2+4}}{2}$. So

$$\boxed{\int \frac{x^2}{\sqrt{x^2+4}}\,dx = \frac{x\sqrt{x^2+4}}{2} - 2\ln\left|x + \sqrt{x^2+4}\right| + C}.$$

(d) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-25}}$.

Use $x = 5\sec\theta$ (for $|x| > 5$), $dx = 5\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-25} = 5\tan\theta$. Then

$$\int \frac{5\sec\theta\tan\theta\,d\theta}{25\sec^2\theta\cdot 5\tan\theta} = \int \frac{5\sec\theta\tan\theta}{125\sec^2\theta\tan\theta}\,d\theta = \frac{1}{25}\int\cos\theta\,d\theta = \frac{1}{25}\sin\theta + C.$$

Return to x: $\sin\theta = \dfrac{\sqrt{x^2-25}}{x}$. Therefore

$$\boxed{\int \frac{dx}{x^2\sqrt{x^2-25}} = \frac{1}{25}\cdot\frac{\sqrt{x^2-25}}{x} + C}.$$

(e) $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$.

Write $4x^2-1 = (2x)^2-1$. Set $2x = \sec\theta$ so $x = \frac{1}{2}\sec\theta$, $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$, and $\sqrt{4x^2-1} = \tan\theta$. Then

$$\int \frac{\frac{1}{2}\sec\theta\tan\theta\,d\theta}{\tan\theta} = \frac{1}{2}\int\sec\theta\,d\theta = \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C.$$

Back-substitute $\sec\theta = 2x$, $\tan\theta = \sqrt{4x^2-1}$:

$$\boxed{\int \frac{dx}{\sqrt{4x^2-1}} = \frac{1}{2}\ln\left|2x+\sqrt{4x^2-1}\right| + C}.$$

(f) $\displaystyle\int \frac{dx}{x^2+4x+8}$.

Complete the square: $x^2+4x+8 = (x+2)^2+4$. Let $u = x+2$, $du = dx$. Then

$$\int \frac{du}{u^2+2^2} = \frac{1}{2}\arctan\frac{u}{2} + C = \frac{1}{2}\arctan\frac{x+2}{2} + C.$$

So

$$\boxed{\int \frac{dx}{x^2+4x+8} = \frac{1}{2}\arctan\frac{x+2}{2} + C}.$$