GATE in Data Science and AI Study Materials
Calculus
By Piyush Wairale

Instructions:

- Kindly go through the lectures/videos on our website www.piyushwairale.com
- Read this study material carefully and make your own handwritten short notes. (Short notes must not be more than 5-6 pages)
- Attempt the question available on portal.
- Revise this material at least 5 times and once you have prepared your short notes, then revise your short notes twice a week
- **If you are not able to understand any topic or required detailed explanation, please mention it in our discussion forum on webiste**
- **Let me know, if there are any typos or mistake in study materials. Mail me at piyushwairale100@gmail.com**

1 Function:

Function of single variable:
A real valued function y = f(x) of a real variable x is a mapping whose domain S and co-domain R are sets of real numbers. The range of the function is the set $\{y = f(x) : x \in R\}$ which is a subset of R.

A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B. In other words, a function f is a relation such that no two pairs in the relation have the same first element.

The notation $f : X \rightarrow Y$ means that f is a function from X to Y. X is called the domain of f, and Y is called the co-domain of f.

Given an element $x \in X$, there is a unique element y in Y that is related to x. The unique element y to which f relates x is denoted by $f(x)$ and is called f of x, or the value of f at x, or the image of x under f.

The set of all values of $f(x)$ taken together is called the range of f or the image of X under f. Symbolically:

$$\text{range of } f = \{y \in Y \,|\, y = f(x), \text{ for some } x \in X\}$$

Function	Domain	Range
$y = x + 2$	$\mathbb{R}$	$\mathbb{R}$
$y = 3x^2 - 7$	$\mathbb{R}$	$\{y : y \geq -7\}$
$y = \sin x$	$\mathbb{R}$	$\{y : -1 \leq y \leq 1\}$
$y = 2^x$	$\mathbb{R}$	$\{y : y > 0\}$
$y = \frac{1}{x}$	$\{x : x \neq 0\}$	$\{y : y \neq 0\}$
$y = \log_2 x$	$\{x : x > 0\}$	$\mathbb{R}$

Types of functions:

- **Explicit Functions**

  Explicit functions are functions where the dependent variable (usually denoted as y) is expressed explicitly in terms of the independent variable (usually denoted as x), such as $y = f(x)$.
  Example :$y = f(x) = 2x + 3$

- **Implicit Functions**
  Implicit functions are functions where the relationship between the dependent and independent variables is defined implicitly, often by an equation involving both variables, like $x^2 + y^2 = 1$.

- **Composite Functions**

  Composite functions are formed by combining two or more functions, creating a new function. For example, if $f(x)$ and $g(x)$ are functions, the composite function $h(x) = f(g(x))$ or $h(x) = g(f(x))$
  Let $f(x) = 2x$ and $g(x) = x^2$. Then the composite function is $h(x) = f(g(x)) = 2x^2$.

- **Polynomial Functions**

  Polynomial functions are algebraic functions of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where a_i are constants, and n is a non-negative integer.
  Example: $f(x) = 3x^3 - 2x^2 + 5x - 1$
  Ex: $f(x) = 2 + 3x + 4^2$ is a polynomial function of 'x' with degree 2.
  Note:
  A polynomial function of degree '0' is called a constant polynomial function (or) simply constant function.

- **Rational Functions**

  Rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomial functions. Example: $f(x) = \frac{2x^2-3x+1}{x^2+4x+4}$

- **Algebraic Functions**

  Algebraic functions are functions that can be defined by algebraic equations involving polynomial, rational, and root functions
  Example: $f(x) = \sqrt{3x^3 - 2x^2 + 5x - 1}$
  If a relation arises due to performing a finite number of fundamental operations additions, subtraction, multiplication, division, root extraction etc. on polynomial functions then such a relation is also called an Algebraic function.
  1. All polynomial functions are algebraic but not the converse.

2. A function that is not algebraic is called transcendental function.

- **Logarithmic Functions**

  Logarithmic functions are functions of the form $f(x) = \log_b(x)$, where b is the base of the logarithm.
  Example: $f(x) = \log_{10}(x)$

- **Even and Odd Functions**

  Even functions are symmetric about the y-axis, and odd functions are symmetric about the origin.
  For even functions, $f(-x) = f(x)$, and for odd functions, $f(-x) = -f(x)$. Example:
  Even Function:
  $f(x) = x^2$ (Symmetric about the y-axis)
  Odd Function:
  $f(x) = x^3$ (Symmetric about the origin)

- **Exponential Functions**

  Exponential functions are functions of the form $f(x) = a^x$, where a is a positive constant.

  Example: $f(x) = 2^x$

- **Modulus Functions**

  Modulus functions, often denoted as $f(x) = |x|$, return the absolute value of x, making it always non-negative.

  Example: $f(x) = |x|$

- **Signum (Sign) Functions**

  The signum (sign) function is defined as $f(x) = \text{sgn}(x)$, where:

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

  Example: $f(x) = \text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \ 0 \\ \text{if } x = 0 \ 1 & \text{if } x > 0 \end{cases}$

Types of Functions (Important for GATE DA)

1. **One-One (Injective) Function**
   A function $f : X \to Y$ is defined to be one-one (or injective) if the images of distinct elements of X under f are distinct, i.e., for any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then it implies that $x_1 = x_2$.

2. **Onto (Surjective) Function**
   A function $f : X \to Y$ is said to be onto (or surjective) if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element $x \in X$ such that $f(x) = y$.

3. **One-One and Onto (Bijective) Function**
   A function $f : X \to Y$ is said to be one-one and onto (or bijective) if it is both one-one and onto.

Composition of Functions

- Let $f : A \to B$ and $g : B \to C$ be two functions. Then, the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : A \to C$ given by

$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in A.$$

- If $f : A \to B$ and $g : B \to C$ are one-one, then $g \circ f : A \to C$ is also one-one.
- If $f : A \to B$ and $g : B \to C$ are onto, then $g \circ f : A \to C$ is also onto.
- Let $f : A \to B$ and $g : B \to C$ be the given functions such that $g \circ f$ is one-one. Then f is one-one.
- Let $f : A \to B$ and $g : B \to C$ be the given functions such that $g \circ f$ is onto. Then g is onto.

Invertible Function

- A function $f : X \to Y$ is defined to be invertible if there exists a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.
- A function $f : X \to Y$ is invertible if and only if f is a bijective function.
- If $f : X \to Y$, $g : Y \to Z$, and $h : Z \to S$ are functions, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then $g \circ f$ is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2 Limit

In calculus, the concept of a limit is fundamental to understanding the behavior of functions as they approach specific points. A limit represents the value that a function approaches as its input (independent variable) gets arbitrarily close to a certain value. We denote the limit of a function $f(x)$ as x approaches a limit point c as follows:

$$\lim_{x \to c} f(x) = L$$

This means that as x gets very close to c, the values of $f(x)$ get arbitrarily close to L.

The function f is said to tend to the limit ℓ as $x \to a$, if for a given positive real number $\varepsilon > 0$ we can find a real number $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Symbolically we write $\lim_{x \to a} f(x) = \ell$

Left Hand and Right Hand Limits

Let $x < a$ and $x \to a$ from the left hand side.

If $|f(x) - \ell_1| < \varepsilon$, $a - \delta < x < a$ or $\lim_{x \to a^-} f(x) = \ell_1$

then ℓ_1 is called the left hand limit.

Let $x > a$ and $x \to a$ from the right hand side.

If $|f(x) - \ell_2| < \varepsilon$, $a < x < a + \delta$ or $\lim_{x \to a^+} f(x) = \ell_2$

then ℓ_2 is called the right hand limit.

If $\ell_1 = \ell_2$ then $\lim_{x \to a} f(x)$ exists. If the limit exists then it is unique.

Basic Limit Rules

There are several basic rules that help us evaluate limits:

1. The Limit of a Constant:
$$\lim_{x \to c} k = k$$
where k is a constant.

2. The Limit of a Sum or Difference:
$$\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

3. The Limit of a Product:

$$\lim_{x\to c}[f(x)\cdot g(x)] = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x)$$

4. The Limit of a Quotient:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \quad \text{if} \lim_{x\to c} g(x) \neq 0$$

Limits of Trigonometry Functions:

1. $\lim_{x\to 0}\sin x = 0$
2. $\lim_{x\to 0}\cos x = 1$
3. $\lim_{x\to 0}\frac{\tan x}{x} = 1$
4. $\lim_{x\to 0}\frac{\sin x}{x} = 1$
5. $\lim_{x\to 0}\frac{\sin^{-1} x}{x} = 1$
6. $\lim_{x\to 0}\frac{\tan^{-1} x}{x} = 1$
7. $\lim_{x\to \infty}\frac{\sin x}{x} = 0$
8. $\lim_{x\to 0}(cosx + asinbx)^{\frac{1}{x}} = e^{ab}$
8. $\lim_{x\to 0}(cosx + asinbx)^{\frac{1}{x}} = e^{ab}$
9. $\lim_{x\to 0}(\frac{1-cos(ax)}{x} = \frac{a^2}{2}$

Limits of form 1^∞:

1. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$
2. $\lim_{x\to 0}(1+ax)^{\frac{1}{x}} = e^a$
3. $\lim_{x\to \infty}(1+\frac{1}{x})^x = e$
3. $\lim_{x\to \infty}(1+\frac{a}{x})^x = e^a$

Limits of Log and Exponential Functions

1. $\lim_{x\to 0} e^x = 1$
2. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$
3. $\lim_{x\to 0} \frac{e^{mx}-1}{mx} = m$
4. $\lim_{x\to 0} \frac{a^x-1}{x} = log_e\, a$
5. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
6. $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$
7. $\lim_{x\to a} (\frac{a^x+b^x}{2})^{\frac{1}{x}} = \sqrt{ab}$

L'Hospital's Rule: (Very very important for GATE Exam)
We apply L'Hospital's Rule to the limit, if we get the limit in the following form (**Indeterminate forms**):
$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, \infty - \infty, 0^\infty, 1^\infty, \infty^0$

Try to convert all in indeterminate form into $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then only you can apply L'Hospital's Rule.

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is form $\frac{0}{0} or \frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$.
Note:
1. If $\lim_{x\to a} f(x)$exists then it is unique.
2. If f(x) is a polynomial function then $\lim_{x\to a} f(x) = f(a)$

3 Continuity

- **Continuity of a function at a point:**
  A function $f(x)$ is said to be continuous at $=a$ if it satisfies the following conditions
  (i) $f(a)$ is defined
  (ii) $\lim_{x \to a} f(x)$ exists i.e $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$
  (iii) $\lim_{x \to a^+} f(x) = f(a)$

- **Left continuous (or) continuity from the left at a point:**
  A function $f(x)$ is said to be continuous from the left (or) left continuous at $x = a$ if
  (i) $f(a)$ is defined
  (ii) $\lim_{x \to a^-} f(x) = f(a)$

- **Right continuous (or) continuity from the right at a point:**
  A function $f(x)$ is said to be continuous from the right (or) right continuous at $x = aif$
  (i) $f(a)$ is defined
  (ii) $\lim_{x \to a^+} f(x) = f(a)$

- **Continuity of a function in an open interval:**
  A function $f(x)$ is said to be continuous in an open interval (a, b) if $f(x)$ is continuous $\forall x \in (a, b)$ (or) $\lim_{x \to c} f(x) = f(c)$ $\forall c \in (a, b)$.

- **Continuity of a function on closed interval:**
  A function $f(x)$ is said to be continuous on closed interval $[a, b]$ if
  (i) $f(x)$ is continuous $\forall x \in (a, b)$
  (ii) $\lim_{x \to a^+} f(x) = f(a)$
  (iii)$\lim_{x \to b^-} f(x) = f(b)$

Important Points:

1. If f(x) and g(x) are two continuous functions then $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ and $\frac{f(x)}{g(x)}$($::$ g(x) $\neq 0$) are also continuous.
2. Polynomial function, exponential function, sine and cosine functions, and modulus function are continuous everywhere.
3. Logarithmic functions are continuous in $(0, \infty)$
4. Let the functions f and g be continuous at a point $x = x_0$ then,
   (i) $cf, f \pm g$ and $f.g$ are continuous at $x = x_0$, where c is any constant.
   (ii) $\frac{f}{g}$ is continuous at $x = x_0$, if $g(x_0) \neq 0$

5. If f is continuous at $x = x_0$ and g is continuous at $f(x_0)$ then the composite function $g(f(g))$ is continuous at $x = x_0$.

6. If f is continuous at an interior point c of a closed interval $[a, b]$ and $f(c) \neq 0$, then there exists a neighbourhood of c, throughout which $f(x)$ has the same sign as $f(c)$.

7. If f is continuous in a closed interval $[a, b]$ then it is bounded there and attains its bounds at least once in $[a, b]$.

8. If f is continuous in a closed interval $[a, b]$, and if $f(a) and f(b)$ are of opposite signs, then there exists at least one point $c \in [a, b]$ such that $f(c) = 0$.

9. If f is continuous in a closed interval $[a, b]$ and $f(a) \neq f(b)$ then it assumes every value between $f(a)$ and $f(b)$.

4 Differentiability

f(x) is said to be differentiable at the point x = a if the derivative f '(a) exists at every point in its domain. It is given by

$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Important Note:

1. If the derivative of $f(x)$ exists at $x = a$ then the function $f(x)$ is said to be differentiable function at $x = a$.
2. $f^l(a)$ exists at $x = a \iff \mathrm{L}f^l(a) = \mathrm{R}f^l(a)$.
3. If $f(x)$ and $g(x)$ are two differentiable functions then $f(x)+g(x)$, $f(x)-g(x)$, $f(x).g(x)$, $\frac{f(x)}{g(x)}..(g(x) \neq 0)$ are also differentiable.
4. Polynomial functions, exponential functions, sine and cosine functions are differentiable every where.
5. Every differentiable function is continuous but a continuous function need not be differentiable.

Derivability of a function in an open interval:
A function $f(x)$ is said to be derivable (or) differentiable in an open interval (a, b) if $f^l(c)$ exists $\forall c \in (a, b)$.

Derivability of a function on closed interval:
A function $f(x)$ is said to be derivable (or) differentiable on closed interval $[a, b]$ i. if $f^l(c)$ exists $\forall c \in (a, b)$
ii. $\mathrm{R}f^l(a)$ exists
iii. $\mathrm{L}f^l(b)$ exists.

5 Taylor Series

Let $f(x)$ be a function which is analytic at $x = a$ Then we can write $f(x)$ as the following power series, called the Taylor series of $f(x)$ at $x = a$:

$$f(x) = f(a)\frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots.$$

Maclaurin Series:

If the Taylor Series is centred at 0, then the series is known as the Maclaurin series. It means that,
If $a = 0$ in the Taylor series, then we get;

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots.$$

This is known as the Maclaurin series.

Some Standard series expansions:

01. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots$ $[-1 < x < 1]$
02. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots$ $[-1 < x < 1]$
03. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots$ $[-1 < x < 1]$
04. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots$ $[-1 < x < 1]$
05. $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots$ $[-1 < x < 1]$

06. $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots$ $[-1 < x < 1]$

07. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$ $[-1 < x < 1]$

08. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$ $[-1 < x < 1]$

09. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$ $[-\infty < x < \infty]$

10. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$ $[-\infty < x < \infty]$

11. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$ $[-\infty < x < \infty]$

12. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots$ $[-\infty < x < \infty]$

13. $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \ldots\ldots$ $[-1 < x < 1]$

14. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$ $[-\infty < x < \infty]$

15. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots$ $[-\infty < x < \infty]$

16. $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left\{x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \ldots\right\}$ $[-1 < x < 1]$

17. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$ $\left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$

18. $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots\ldots$ $\left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$

19. $$\tan^{-1}x = \begin{cases} x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\ldots & [-1 < x < 1] \\ \dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} - \dfrac{1}{5x^5} - \ldots\ldots\ldots & [x \geq 1] \\ \pi + \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} - \ldots\ldots & [x < 1] \end{cases}$$

20. $$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \ldots\ldots\ldots \qquad [0 < x < \pi]$$

21. $$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \ldots\ldots\ldots \qquad [0 < |x| < \pi]$$

22. $$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x = \begin{cases} \dfrac{\pi}{2} - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\ldots\right) & [-1 < x < 1] \\ \dfrac{1}{x} - \dfrac{3}{3x^3} + \dfrac{1}{5x^5} - \ldots\ldots\ldots\ldots & [x \geq 1] \\ \pi + \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} - \ldots\ldots\ldots & [x < 1] \end{cases}$$

23. $$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \ldots\ldots \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$

24. $$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \ldots\ldots\ldots \qquad [0 < x < \pi]$$

25. $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \qquad [-1 < x < 1]$$

6 Maxima and Minima

Maxima and minima for functions of one variable:-

Local or relative maximum:
A function f(x) is said to have a Maximum at $x = c$ if there exists $\delta > 0$ such that $|x - c| < \delta \implies f(x) \leq f(c)$.

Local or relative minimum:
A function f(x) is said to have a minimum at $x = c$ if there exists $\delta > 0$ such that $|x - c| < \delta \implies f(x) \geq f(c)$.

Stationary points:
The values of x for which $f(x) = 0$ are called stationary points or turning points.

Stationary values:
A function $f(x)$ is said to be stationary at $x = a$ if $f'(a) = 0$ and $f(a)$ is a stationary value.

Extreme point:
The point at which the function has a maximum or a minimum is called an extreme point.

Extreme values:
The values of the function at extreme points are called extreme values (Extrema).

Point of inflection:
The point at which a curve crosses its tangents is called the point of inflection.
The function $f(x)$ has neither maximum nor minimum at the point of inflection.
Note:
1. A necessary condition for a function to have an extreme value at $x = a$ is $f'(a) = 0$.
2. $f'(a) = 0$ is only a necessary condition but not a sufficient condition for $f(a)$ to be an extreme value of $f(x)$.
3. Every extreme point is a stationary point but every static lary point need not be an extreme point.

Absolute or Global maximum/minimum :
The absolute maximum/minimum values of the function $f(x)$ in ie closed interval $[a, b]$ are given by

1. **Absolute maximum value**
   $=$ max $(f(a), f(b)$, all local maximum values of $f)$
   $=$ greatest value of $f(x)$ in $[a, b]$.
2. **Absolute minimum value**
   $=$ min $(f(a), f(b)$, all local minimum values of $f)$
   $=$ least value of $f(x)$ in $[a, b]$.

Working Rule to find maxima and minima:
Let $f(x)$ be the given function
Step 1: Find $f'(x)$
Step 2: Equate $f'(x)$ to zero to obtain the stationary points.
Step 3: Find $f''(x)$ at each stationary point.

- If $f(x_0) > 0$ then $f(x)$ has a minimum at $x = x_0$
- If $f''(x_0) < 0$ then $f(x)$ has a maximum at $x = x_0$
- If $f(x_0) = 0$ then $f(x)$ may (or) may not have extremum.

In this case, check for maxima and minima using the changes in sign of f(x) as given below.
1. For $x < x_0$ if $f'(x) < 0$ and $x > x_0$ if $f'(x) > 0$ then $f(x_0)$ is a minimum value of $f(x)$.
2. For $x < x_0$ if $f'(x) > 0$ and $x > x_0$ if $f(x) < 0$ then $f(x_0)$ is a maximum value of $f(x)$.
3. For $x < x_0$ and $x > x_0$ if $f'(x) > 0$ (or) $f'(x) < 0$ then $f(x_0)$ is not an extremum.

Maxima and minima for functions of two variables:
Let $z = f(x, y)$ be the function of two variables for which maxima or minima is to be obtained.

Working Rule:
Step1: Find p, q, r, s and t
Step2: Equate p and q to zero for obtaining stationary points.
Step3: Find r, s and t at each stationary point.
i) If $rt - s^2 > 0$ and $r > 0$ then $f(x, y)$ has a minimum at that stationary point.
ii) If $rt - s^2 > 0$ and $r < 0$ then $f(x, y)$ has a maximum at that stationary point.
iii) If $rt - s^2 < 0$ then $f(x, y)$ has no extremum at that stationary point and such points are called saddle points.
iv) If $rt - s^2 = 0$ then the case is undecided.