Differential Equations Notes

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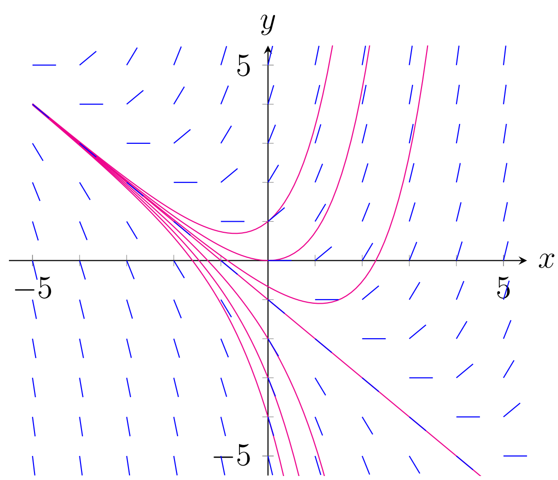
1 – Tools for Differential Equations

1.1: Direction Fields

The graph below is a **direction field** (or **slope field**) of the differential equation $\frac{dy}{dx} = x + y$. The magenta-colored lines represent a set of **integral curves** (or **solution curves**).

Each blue arrow represents the slope, $\frac{dy}{dx}$, at its tail. In other words, the slope attained when plugging the point of the tail into the differential equation. This is useful because it showcases the behavior of the differential equation across a wide range of x and y values.

The integral curves represent specific solutions to the differential equation. In this case, the general solution is $y = -x - 1 + ce^x$, where c is a constant. Several solutions have been plotted, each with a different value of c. Note that the integral curves always flow tangent to the slope arrows.



1.2: Integration by Parts

If you have to integrate the product of an algebraic and a transcendental function like xe^x, you will probably need to integrate by parts. *Choose the algebraic function for u, and the transcendental function for v.* Here's the formula, where u' and v' are the derivatives of u and v with respect to t:

$$\int uv' dt = uv - \int u'v dt$$

Let's do a quick example. Integrate:

$$\int xe^x dx$$

Solution:

$$u = x,\ v = e^x;\ u' = 1,\ v' = e^x$$

$$(x)(e^x) - \int (1)(e^x)dt$$

$$= \boxed{xe^x - e^x + c}$$

2 – First-Order Differential Equations

2.1: Linear Equations

In this section we are going to solve differential equations of the form:

$$\frac{dy}{dt} + p(t)y(t) = g(t)$$

Let's start by multiplying everything by a function $u(t)$, called the **integrating factor**:

$$u(t)\frac{dy}{dt} + u(t)p(t)y(t) = u(t)g(t)$$

We'll change $\frac{dy}{dt}$ to $y'(t)$, and we'll assume that $u(t)p(t) = u'(t)$:

$$y'(t)u(t) + u'(t)y(t) = u(t)g(t)$$

On the left side, we merely have the product rule in action. We can thus rewrite the equation:

$$(y(t)u(t))' = u(t)g(t)$$

Let's integrate:

$$\int (y(t)u(t))'dt = \int u(t)g(t)dt$$

$$y(t)u(t) = \int u(t)g(t)dt + c$$

$$y(t) = \frac{\int u(t)g(t)dt + c}{u(t)}$$

We now have a formula for $y(t)$. However, we still need to find the integrating factor $u(t)$. Remember:

$$u(t)p(t) = u'(t)$$

Let's divide both sides by $u(t)$:

$$p(t) = \frac{u'(t)}{u(t)}$$

Though hard to recognize, the right side is equal to a simple derivative (the chain rule is used):

$$\frac{d}{dt}ln(u(t)) = \frac{1}{u(t)}u'(t)$$

We then have:

$$ln(u(t))' = p(t)$$

Integrating and simplifying:

$$\int ln(u(t))'dt = \int p(t)dt$$

$$ln(u(t)) = \int p(t)dt$$

$$u(t) = e^{\int p(t)dt}$$

For the integrating factor only, we can ignore the typical + c constant. However, when solving for $y(t)$ we will often need to exponentiate functions of the form $a(t)+c$. The constant falls out of the exponential and instead goes beside it. Here's what that looks like:

$$e^{a(t)+c} = e^{a(t)}e^c = ce^{a(t)}$$

Believe it or not, we are ready to tackle an example problem. Solve the following IVP:

$$ty' + 2y = t^2 - t + 1,\ y(1) = \frac{1}{2}$$

Solution:

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$u(t) = e^{\int \frac{2}{t}dt} = e^{2\int \frac{1}{t}dt} = e^{2ln(t)} = e^{ln(t)^2} = t^2$$

$$t^2y' + 2ty = t^3 - t^2 + t$$

$$\int (yt^2)'dt = \int (t^3 - t^2 + t)dt$$

$$yt^2 = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c$$

$$y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$$

$$\left(\frac{1}{2}\right) = \frac{1}{4}(1)^2 - \frac{1}{3}(1) + \frac{1}{2} + \frac{c}{(1)^2}$$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c$$

$$c = \frac{1}{12}$$

$$\boxed{y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}}$$

2.2: Separable Equations

We are now going to look at equations of the form:

$$N(y)\frac{dy}{dx} = M(x)$$

Let's start by integrating both sides with respect to x:

$$\int N(y)\frac{dy}{dx}dx = \int M(x)dx$$

$y = y(x)$, so let's use a change of variable and then integrate:

$$u = y(x),\ du = \frac{dy}{dx}dx = dy$$

$$\int N(u)du = \int M(x)dx$$

The above process is the mathematically correct way of solving the equation, but we can simply write:

$$\int N(y)dy = \int M(x)dx$$

Time for an example. Solve the following IVP and determine the interval of validity for the solution:

$$y' = e^{-y}(2x-4),\ y(5) = 0$$

Solution:

$$\frac{1}{e^{-y}}\frac{dy}{dx} = 2x - 4$$

$$\int e^{y}dy = \int (2x-4)dx$$

$$e^{y} = x^2 - 4x + c$$

$$y = ln(x^2 - 4x + c)$$

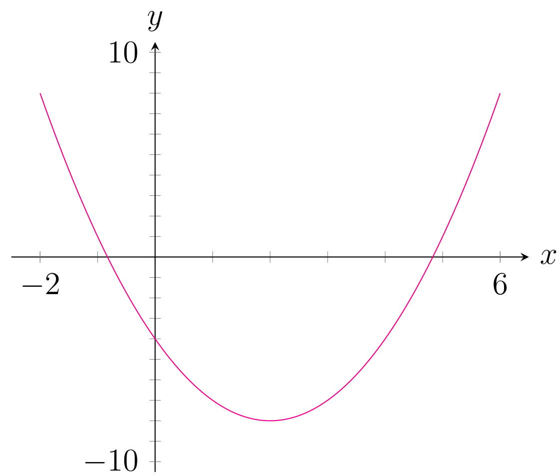
$$(0) = ln((5)^2 - 4(5) + c)$$

$$25 - 20 + c = 1$$

$$c = -4$$

$$\boxed{y(x) = ln(x^2 - 4x - 4)}$$

Here's what that quadratic $x^2 - 4x - 4$ looks like. The roots are $2 \pm 2\sqrt{2}$:



We can't take the natural log of 0 or a negative number. So the possible intervals of validity are $x = (-\infty, 2-2\sqrt{2})$ or $x = (2+2\sqrt{2}, \infty)$. We need to select the interval that contains our initial condition, so in this case, just $\boxed{x = (2+2\sqrt{2}, \infty)}$.

2.3: Bernoulli Equations

This is a fun section. We are going to solve equations of the form:

$$y' + p(x)y = q(x)y^n$$

We'll start by dividing the equation by y^n, which gives us:

$$y^{-n}y' + p(x)y^{1-n} = q(x) \tag{1}$$

Let's use a substitution:

$$v(x) = y^{1-n},\ v'(x) = (1-n)y^{-n}y'$$

Here's how we obtained v' above:

$$v' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx}$$

$$\frac{dv}{dy} = (1-n)y^{-n},\ \frac{dy}{dx} = y'$$

Let's put our substitution to use, plugging it into (1):

$$\frac{v'}{1-n} + p(x)v = q(x)$$

We now have a linear differential equation, and after solving for v, we can plug it back into our substitution to solve for y.

Let's try an example. Solve the following IVP:

$$6y' - 2y = xy^4,\ y(0) = -2$$

Solution:

$$6y^{-4}y' - 2y^{-3} = x$$

$$v = y^{-3},\ v' = -3y^{-4}y'$$

$$-2v' - 2v = x$$

$$v' + v = -\frac{1}{2}x$$

$$u(x) = e^{\int dx} = e^x$$

$$e^x v' + ve^x = -\frac{1}{2}xe^x$$

$$\int (ve^x)' dx = -\frac{1}{2}\int xe^x dx$$

$$ve^x = -\frac{1}{2}(xe^x - e^x) + c$$

$$v = -\frac{1}{2}(x-1) + ce^{-x} = y^{-3}$$

$$y = \left(-\frac{1}{2}(x-1) + ce^{-x}\right)^{-\frac{1}{3}}$$

$$-2 = \left(-\frac{1}{2}((0)-1) + ce^{(0)}\right)^{-\frac{1}{3}}$$

$$-\frac{1}{8} = \frac{1}{2} + c$$

$$c = -\frac{5}{8}$$

$$\boxed{y(x) = \left(-\frac{1}{2}(x-1) - \frac{5}{8}e^{-x}\right)^{-\frac{1}{3}}}$$

2.4: Substitutions, Part 1

We just learned about Bernoulli equations, where we used a substitution to convert the equation into something we could more easily solve. We are going to look at two more substitutions. The first one is for equations in the form:

$$y' = F\left(\frac{y}{x}\right) \tag{1}$$

We'll use the substitution:

$$v(x) = \frac{y}{x} \tag{2}$$

Rewriting it:

$$y(x) = xv(x)$$

Using the product rule, we find the derivative of y with respect to x:

$$\frac{dy}{dx} = x'v(x) + xv'(x)$$

We'll change $\frac{dy}{dx}$ to y', and we'll simplify the right side:

$$y' = v + xv' \tag{3}$$

Let's now plug our substitution into the original equation. (2) goes on the right side of (1), and (3) goes on the left side of (1):

$$v + xv' = F(v)$$

Though it may be hard to tell, we have separable equation. Let's solve it:

$$x\frac{dv}{dx} = F(v) - v$$

$$\int \frac{1}{F(v) - v} dv = \int \frac{1}{x} dx$$

It is best to learn the rest of the process through an example. Solve the following IVP:

$$xy' = y(ln(y) - ln(x)),\ y(1) = 4,\ x > 0$$

Solution:

$$y' = \frac{y}{x} ln\left(\frac{y}{x}\right)$$

$$v = \frac{y}{x}$$

$$y = xv,\ y' = v + xv'$$

$$v + xv' = vln(v)$$

$$x\frac{dv}{dx} = vln(v) - v$$

$$\int \frac{1}{x} dx = \int \frac{1}{v(ln(v) - 1)} dv$$

$$u = ln(v) - 1,\ du = \frac{1}{v}$$

$$ln(x) = \int \frac{1}{u} du = ln(u) + c$$

$$ln(x) = ln(ln(v) - 1) + c$$

$$x = c(ln(v) - 1)$$

$$x = c\left(ln\left(\frac{y}{x}\right) - 1\right)$$

$$cx + 1 = ln\left(\frac{y}{x}\right)$$

$$\frac{y}{x} = e^{cx+1} = ee^{cx} = ee^{c^x} = ec^x$$

$$y = xec^x$$

$$(4) = (1)ec^1$$

$$c = \frac{4}{e}$$

$$\boxed{y(x) = xe\left(\frac{4}{e}\right)^x}$$

2.5: Substitutions, Part 2

This is the last substitution we will learn. We'll solve equations of the form:

$$y' = G(ax + by)$$

We will use the substitution:

$$v = ax + by,\ v' = a + by'$$

Rewriting v':

$$y' = \frac{v' - a}{b}$$

We'll plug this into the original equation and simplify things:

$$\frac{v' - a}{b} = G(v)$$

$$v' = bG(v) + a$$

Like we did in the last section, we have arrived at a separable differential equation:

$$\frac{dv}{dx} = bG(v) + a$$

$$\int dx = \int \frac{1}{bG(v) + a} dv$$

Let's do an example. Solve the following IVP:

$$y' - (4x - y + 1)^2 = 0,\ y(0) = 2$$

Solution:

$$y = (4x - y + 1)^2$$

$$v = 4x - y + 1,\ v' = 4 - y'$$

$$y' = 4 - v'$$

$$4 - v' = v^2$$

$$v' = \frac{dv}{dx} = 4 - v^2$$

$$\int dx = \int \frac{1}{4 - v^2} dv$$

We need to do a partial fraction decomposition:

$$\frac{1}{4 - v^2} = \frac{a}{2+v} + \frac{b}{2-v} = \frac{a}{2+v} \cdot \frac{2-v}{2-v} + \frac{b}{2-v} \cdot \frac{2+v}{2+v}$$

$$2a - av + 2b + bv = 1$$

$$bv - av = 0,\ 2a + 2b = 1$$

$$a = b$$

$$2a + 2(a) = 1$$

$$a = b = \frac{1}{4}$$

$$\frac{1}{4 - v^2} = \frac{1}{8 + 4v} + \frac{1}{8 - 4v}$$

Resuming our original problem:

$$x = \int \frac{1}{8 + 4v} dv + \int \frac{1}{8 - 4v} dv$$

$$u = 8 + 4v,\ du = 4;\ w = 8 - 4v,\ dw = -4$$

$$x = \frac{1}{4} \int \frac{1}{u} du - \frac{1}{4} \int \frac{1}{w} dw$$

$$4x = ln(u) - ln(w) + c = ln\left(\frac{u}{w}\right) + c$$

$$e^{4x} = e^{ln(\frac{u}{w}) + c} = e^c e^{ln(\frac{u}{w})} = ce^{ln(\frac{u}{w})} = c\frac{u}{w}$$

$$e^{4x} = c\frac{8 + 4v}{8 - 4v} = c\frac{8 + 4(4x - y + 1)}{8 - 4(4x - y + 1)}$$

$$ce^{4x} = \frac{8 + 4(4x - y + 1)}{8 - 4(4x - y + 1)}$$

$$8ce^{4x} - 4ce^{4x}(4x - y + 1) = 8 + 4(4x - y + 1)$$

$$8ce^{4x} - 16cxe^{4x} + 4yce^{4x} - 4ce^{4x} = 8 + 16x - 4y + 4$$

$$4yce^{4x} + 4y = -4ce^{4x} + 16cxe^{4x} + 16x + 12$$

$$y(4ce^{4x} + 4) = -4ce^{4x} + 16cxe^{4x} + 16x + 12$$

$$y = \frac{-4ce^{4x} + 16cxe^{4x} + 16x + 12}{4ce^{4x} + 4}$$

$$y = \frac{-ce^{4x} + 4cxe^{4x} + 4x + 3}{ce^{4x} + 1}$$

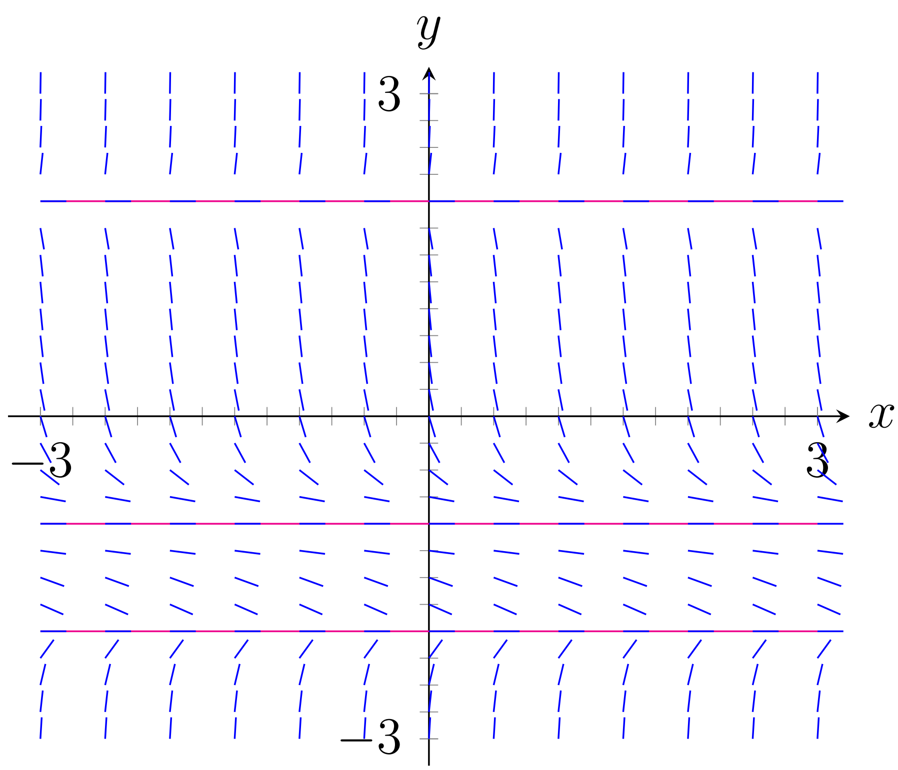
$$2 = \frac{-ce^{4(0)} + 4c(0)e^{4(0)} + 4(0) + 3}{ce^{4(0)} + 1}$$

$$c = \frac{1}{3}$$

$$\boxed{y(x) = \frac{-\frac{1}{3}e^{4x} + \frac{4}{3}xe^{4x} + 4x + 3}{\frac{1}{3}e^{4x} + 1}}$$

2.6: Equilibrium Solutions and Stability

Consider the slope field of $y' = (y^2 - 4)(y + 1)^2$ below:



Notice that y' (the slope of any solution) is zero at $y = 2$, $y = -2$, and $y = -1$. These points are known as **equilibrium solutions** as they will never change once they are reached – the slope is always zero.

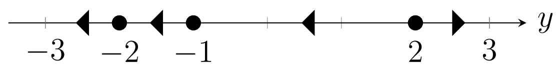
This raises an important question, however. What if we are stuck on an equilibrium solution and we receive a slight nudge upwards or downwards? Let's answer this for each equilibrium solution from above.

$\boldsymbol{y = 2}$: The slope above $y = 2$ is always positive, so if we are pushed upwards from $y = 2$ we will go flying up and away, never to return. Similarly, the slope below $y = 2$ is always negative, so if we are pushed slightly downwards off of $y = 2$ we will go down – again, never to return. Because we never return to $y = 2$ if we are nudged off in either direction, $y = 2$ is called an **unstable** equilibrium solution.

***y* = −2**: The slope above $y = -2$ is always negative, so if we are pushed upwards from $y = -2$ we will be pushed back down to $y = -2$. The slope below $y = -2$ is always positive, so if we are pushed below $y = -2$ we will be brought back up to $y = -2$. Because we always return to $y = -2$ if we are nudged off in either direction, $y = -2$ is called a **stable** equilibrium solution.

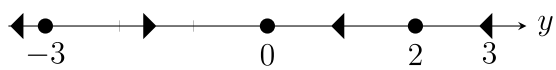
***y* = −1**: The slope above $y = -1$ is always negative, so if we are pushed upwards from $y = -1$ we will return down to $y = -1$. The slope below $y = -1$ is *also* always negative, so if we are pushed below $y = -1$ we will go down and never return. Because we return to $y = -1$ if we are nudged off in *only one* direction, $y = -1$ is called a **semi-stable** equilibrium solution.

We can represent all of this information in a neat and compact fashion using a **phase line**:



What does the phase line tell us? Notice the points at -2, -1, and 2. These are the equilibrium solutions of our differential equation. Then look at the arrows beside each point. They simply tell you which direction you would move if you were nudged towards the arrow.

Let's practice reading a phase line. Using the information from the phase line below, determine all critical points and indicate their stability:



Solution:

Critical points at $y = -3$, $y = 0$, and $y = 2$.

$y = -3$ is unstable, $y = 0$ is stable, and $y = 2$ is semi-stable.

Note that a differential equation need not contain equilibrium solutions of all kinds of stability, and it need not contain equilibrium solutions at all.

Let's do a more comprehensive example. For the differential equation below, determine all equilibrium points and classify them by their stability. Draw a phase line to represent your answer: