Differential Equations Notes

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September 2023

MATH 4B

UC Santa Barbara

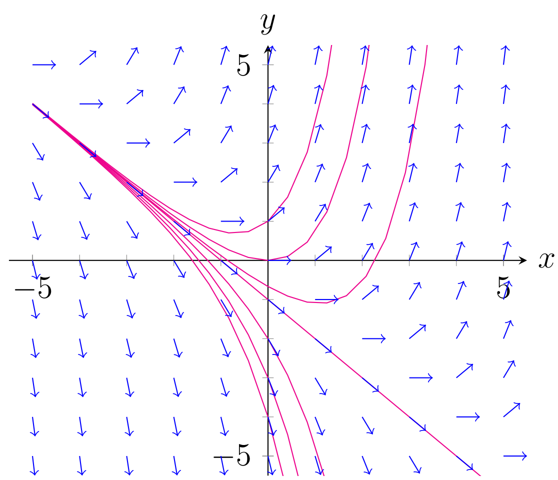
1 – Tools for Differential Equations

1.1 – Direction Fields

The graph below is a **direction field** (or **slope field**) of the differential equation $\frac{dy}{dx} = x + y$. The magenta-colored lines represent a set of **integral curves** (or **solution curves**).

Each blue arrow represents the slope, $\frac{dy}{dx}$, at its tail. In other words, the slope attained when plugging the point of the tail into the differential equation. This is useful because it showcases the behavior of the differential equation across a wide range of x and y values.

The integral curves represent specific solutions to the differential equation. In this case, the general solution is $y = -x - 1 + ce^x$, where c is a constant. Several solutions have been plotted, each with a different value of c. Note that the integral curves always flow tangent to the slope arrows.



1.2 – Integration by Parts

If you have to integrate the product of an algebraic and a transcendental function like xe^x, you will probably need to integrate by parts. *Choose the algebraic function for u, and the transcendental function for v.* Here's the formula:

$$\int uvdx = u\int vdx - \int \left(u'\int vdx\right)dx$$

Let's do a quick example. Integrate:

$$\int xe^x dx$$

Solution:

$$u = x,\ v = e^x$$

$$\int xe^x dx = x\int e^x dx - \int \left(1\int e^x dx\right)dx$$

$$= xe^x - \int (e^x)dx$$

$$= \boxed{xe^x - e^x + c}$$

2 – First-Order Differential Equations

2.1 – Linear Equations

In this section we are going to solve differential equations of the form:

$$\frac{dy}{dt} + p(t)y(t) = g(t)$$

Let's start by multiplying everything by a function $u(t)$, called the **integrating factor**:

$$u(t)\frac{dy}{dt} + u(t)p(t)y(t) = u(t)g(t)$$

We'll change $\frac{dy}{dt}$ to $y'(t)$, and we'll assume that $u(t)p(t) = u'(t)$:

$$y'(t)u(t) + u'(t)y(t) = u(t)g(t)$$

On the left side, we merely have the product rule in action. We can thus rewrite the equation:

$$(y(t)u(t))' = u(t)g(t)$$

Let's integrate:

$$\int (y(t)u(t))'dt = \int u(t)g(t)dt$$

$$y(t)u(t) = \int u(t)g(t)dt + c$$

$$y(t) = \frac{\int u(t)g(t)dt + c}{u(t)}$$

We now have a formula for $y(t)$. However, we still need to find the integrating factor $u(t)$. Remember:

$$u(t)p(t) = u'(t)$$

Let's divide both sides by $u(t)$:

$$p(t) = \frac{u'(t)}{u(t)}$$

Though it may be a little hard to recognize, the right side is equal to a simple derivative:

$$\frac{d}{dt}ln(u(t)) = \frac{1}{u(t)}u'(t)$$

We then have:

$$ln(u(t))' = p(t)$$

Integrating and simplifying:

$$\int ln(u(t))'dt = \int p(t)dt$$

$$ln(u(t)) = \int p(t)dt$$

$$u(t) = e^{\int p(t)dt}$$

This is important. Let's say that $\int p(t)dt = q(t) + c$. Here's how we need to handle the c:

$$u(t) = e^{q(t)+c} = e^{q(t)}e^c = ce^{q(t)}$$

Believe it or not, we are ready to tackle an example problem. Solve the following IVP:

$$ty' + 2y = t^2 - t + 1,\ y(1) = \frac{1}{2}$$

Solution:

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$u(t) = e^{\int \frac{2}{t}dt} = e^{2\int \frac{1}{t}dt} = e^{2ln(t)} = e^{ln(t)^2} = t^2$$

$$t^2y' + 2ty = t^3 - t^2 + t$$

$$\int (yt^2)'dt = \int (t^3 - t^2 + t)dt$$

$$yt^2 = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c$$

$$y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$$

$$\left(\frac{1}{2}\right) = \frac{1}{4}(1)^2 - \frac{1}{3}(1) + \frac{1}{2} + \frac{c}{(1)^2}$$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c$$

$$c = \frac{1}{12}$$

$$\boxed{y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}}$$

2.2 – Separable Equations

We are now going to look at equations of the form:

$$N(y)\frac{dy}{dx} = M(x)$$

Let's start by integrating both sides with respect to x:

$$\int N(y)\frac{dy}{dx}dx = \int M(x)dx$$

$y = y(x)$, so let's use a change of variable and then integrate:

$$u = y(x),\ du = \frac{dy}{dx}dx = dy$$

$$\int N(u)du = \int M(x)dx$$

The above process is the mathematically correct way of solving the equation, but we can simply write:

$$\int N(y)dy = \int M(x)dx$$

Time for an example. Solve the following IVP and determine the interval of validity for the solution:

$$y' = e^{-y}(2x - 4),\ y(5) = 0$$

Solution:

$$\frac{1}{e^{-y}}\frac{dy}{dx} = 2x - 4$$

$$\int e^{y}dy = \int (2x - 4)dx$$

$$e^{y} = x^2 - 4x + c$$

$$y = ln(x^2 - 4x + c)$$

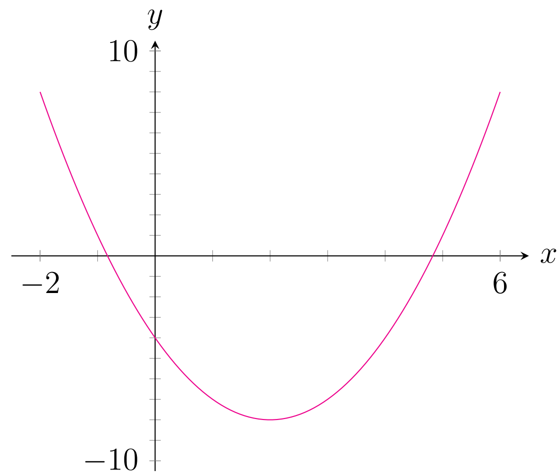
$$(0) = ln((5)^2 - 4(5) + c)$$

$$25 - 20 + c = 1$$

$$c = -4$$

$$\boxed{y(x) = ln(x^2 - 4x - 4)}$$

Here's what that quadratic $x^2 - 4x - 4$ looks like. The roots are $2 \pm 2\sqrt{2}$:



We can't take the natural log of 0 or a negative number. So the possible intervals of validity are $x = (-\infty, 2-2\sqrt{2})$ or $x = (2+2\sqrt{2}, \infty)$. We need to select the interval that contains our initial condition, so in this case, just $\boxed{x = (2+2\sqrt{2}, \infty)}$.

2.3 – Bernoulli Equations

This is a fun section. We are going to solve equations of the form:

$$y' + p(x)y = q(x)y^n$$

We'll start by dividing the equation by y^n, which gives us:

$$y^{-n}y' + p(x)y^{1-n} = q(x) \tag{1}$$

Let's use a substitution:

$$v(x) = y^{1-n},\ v'(x) = (1-n)y^{-n}y'$$

Here's how we obtained v' above:

$$v' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx}$$

$$\frac{dv}{dy} = (1-n)y^{-n},\ \frac{dy}{dx} = y'$$

Let's put our substitution to use, plugging it into (1):

$$\frac{v'}{1-n} + p(x)v = q(x)$$

We now have a linear differential equation, and after solving for v, we can plug it back into our substitution to solve for y.

Let's try an example. Solve the following IVP:

$$6y' - 2y = xy^4,\ y(0) = -2$$

Solution:

$$6y^{-4}y' - 2y^{-3} = x$$

$$v = y^{-3},\ v' = -3y^{-4}y'$$

$$-2v' - 2v = x$$

$$v' + v = -\frac{1}{2}x$$

$$u(x) = e^{\int 1dx} = e^x$$

$$e^x v' + ve^x = -\frac{1}{2}xe^x$$

$$\int (ve^x)'dx = -\frac{1}{2}\int xe^x dx$$

$$ve^x = -\frac{1}{2}(xe^x - e^x) + c$$

$$v = -\frac{1}{2}(x-1) + ce^{-x} = y^{-3}$$

$$y = \left(-\frac{1}{2}(x-1) + ce^{-x}\right)^{-\frac{1}{3}}$$

$$-2 = \left(-\frac{1}{2}((0)-1) + ce^{(0)}\right)^{-\frac{1}{3}}$$

$$-\frac{1}{8} = \frac{1}{2} + c$$

$$c = -\frac{5}{8}$$

$$\boxed{y(x) = \left(-\frac{1}{2}(x-1) - \frac{5}{8}e^{-x}\right)^{-\frac{1}{3}}}$$

2.4 – Substitutions, Part 1

We just learned about Bernoulli equations, where we used a substitution to convert the equation into something we could more easily solve. We are going to look at two more substitutions. The first one is for equations in the form:

$$y' = F\left(\frac{y}{x}\right) \tag{1}$$

We'll use the substitution:

$$v(x) = \frac{y}{x} \tag{2}$$

Rewriting it:

$$y(x) = xv(x)$$

Using the product rule, we find the derivative of y with respect to x:

$$\frac{dy}{dx} = x'v(x) + xv'(x)$$

We'll change $\frac{dy}{dx}$ to y', and we'll simplify the right side:

$$y' = v + xv' \tag{3}$$

Let's now plug our substitution into the original equation. (2) goes on the right side of (1), and (3) goes on the left side of (1):

$$v + xv' = F(v)$$

Though it may be hard to tell, we have separable equation. Let's solve it:

$$x\frac{dv}{dx} = F(v) - v$$

$$\int \frac{1}{F(v) - v} dv = \int \frac{1}{x} dx$$

We'll stop here because it is best to learn the rest of the process through an example. Solve the following IVP:

$$xy' = y(ln(y) - ln(x)),\ y(1) = 4,\ x > 0$$

Solution:

$$y' = \frac{y}{x} ln\left(\frac{y}{x}\right)$$

$$v = \frac{y}{x}$$

$$y = xv,\ y' = v + xv'$$

$$v + xv' = vln(v)$$

$$x\frac{dv}{dx} = vln(v) - v$$

$$\int \frac{1}{x} dx = \int \frac{1}{v(ln(v) - 1)} dv$$

$$u = ln(v) - 1,\ du = \frac{1}{v}$$

$$ln(x) = \int \frac{1}{u} du = ln(u) + c$$

$$ln(x) = ln(ln(v) - 1) + c$$

$$x = c(ln(v) - 1)$$

$$x = c\left(ln\left(\frac{y}{x}\right) - 1\right)$$

$$cx + 1 = ln\left(\frac{y}{x}\right)$$

$$\frac{y}{x} = e^{cx+1} = ee^{cx} = ee^{c^x} = ec^x$$

$$y = xec^x$$

$$(4) = (1)ec^1$$

$$c = \frac{4}{e}$$

$$\boxed{y(x) = xe\left(\frac{4}{e}\right)^x}$$

2.5 – Substitutions, Part 2

This is the last substitution we will learn. We'll solve equations of the form:

$$y' = G(ax + by)$$

We will use the substitution:

$$v = ax + by, \; v' = a + by'$$

Rewriting v':

$$y' = \frac{v' - a}{b}$$

Let's now plug this into the original equation and simplify things:

$$\frac{v' - a}{b} = G(v)$$

$$v' = bG(v) + a$$

Like we did in the last section, we have arrived at a separable differential equation:

$$\frac{dv}{dx} = bG(v) + a$$

$$\int dx = \int \frac{1}{bG(v) + a} dv$$