We consider the parabolic approximation of a sinusoid of amplitude 1 and frequency f_0:

$$x(t) \approx \sin(2\pi f_0 t)$$

$$x(t) \xleftrightarrow{\text{F.S.}} a_k^s$$

$$a_k^s = -\frac{16j}{\pi^3 k^3}$$

Delay property:

$$x(t+d) \xleftrightarrow{\text{F.S.}} a_k^s e^{jk2\pi f_0 d}$$

$$x(t+T_0/4) \approx \cos(2\pi f_0 t)$$

$$x(t+T_0/4) \xleftrightarrow{\text{F.S.}} a_k^c$$

$$\begin{aligned} a_k^c &= -\frac{16j}{\pi^3 k^3} e^{jk2\pi f_0 T_0/4} \\ &= -\frac{16j}{\pi^3 k^3} e^{jk\pi/2} \end{aligned}$$

$$e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t)$$

$$\text{Parabolic approximation of } e^{j2\pi f_0 t} \xleftrightarrow{\text{F.S.}} a_k$$

$$\begin{aligned} a_k &= a_k^c + j a_k^s \\ &= -\frac{16j}{\pi^3 k^3} e^{jk\pi/2} - \frac{16j^2}{\pi^3 k^3} \\ &= \frac{16}{\pi^3 k^3}(-je^{jk\pi/2} + 1) \end{aligned}$$

We now consider a sinusoid of amplitude 1 and frequency f_0, sampled at rate f_s for a duration of ℓ seconds. This gives a total number of samples $N = f_s \ell$. We can write the sampled signal as

$$x[n] = \sin(2\pi f_0 n/N)$$

For discrete-time, length-N signals, we define the inner product as

$$\langle x_1, x_2 \rangle = \sum_{k=0}^{N-1} x_1[k] x_2^*[k]$$

Let us compute the inner product of the sinusoids

$$x_1[n] = \sin(2\pi f_1 n/N),\ x_2[n] = \sin(2\pi f_2 n/N)$$

We have

$$\begin{aligned} \langle x_1, x_2 \rangle &= \sum_{k=0}^{N-1} x_1[k] x_2^*[k] \\ &= \sum_{k=0}^{N-1} x_1[k] x_2^*[k] \end{aligned}$$