DESIGN OF COMPOUND PARAMETRICAL SURFACES BY MEANS OF CENTRAL AND ORTHOGONAL PROJECTIONS

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ABSTRACT In this work we consider a problem of representation of three-dimensional object surface as a vector function of rectangular range of definition. We consider different combinations of base elements of the form (BEF). For combination of separate segments of BEF surfaces we develop a method of construction of parametrical lines on the surface of object. This method provides continuity of a parametrical line on combining objects surfaces. It can be used in CAD-systems. The method let solve different geometric problems of compound objects such as central and orthogonal projections, construction of sections of objects by flats, their intersections with different parametrical surfaces, and so on.
Keywords: 3D Modeling, Geometry, Computer Graphics, Central and Orthogonal Projections, CAD-systems

1. INTRODUCTION

Suppose we know the equation of some surface $\vec{R}(u,v)$, let name this surface the "basic" one. However, a question about choice of the "basic" surface deserves special research. We also know the equations $\vec{R}_i(u_i,v_i)$ of surfaces segments, which we want to combine with the "basic" surface. Each segment intersects the "basic" surface in the closed line Γ_i. Area Ω_i corresponds to points of the "basic" surface in the plane of parameters (u,v). This surface is bounded by the intersection line Γ_i. Area Ψ_i corresponds to points of the surface $\vec{R}_i(u_i,v_i)$, which has the same bound, in the plane of parameters (u_i,v_i). In this paper, we consider a problem related to determination of a bijective map $\mathrm{P}_i:\Omega_i \to \Psi_i$. Let a surface $\vec{P}(u,v)$ be a result of superposition of $\vec{R}(u,v)$ and all $\vec{R}_i(u_i,v_i)$; then we can use the following formula for its description:

$$\vec{P}(u,v) = \left(\prod_i f_{1i}(u,v)\right)\cdot \vec{R}(u,v) + \\ +\sum_i \left(f_{2i}(u,v)\cdot \vec{R}_i\left(u_i(u,v),v_i(u.v)\right)\right), \quad (1)$$

where $f_{1i}(u,v)=\begin{cases}0,(u,v)\in\Omega_i\\1,(u,v)\notin\Omega_i\end{cases}$, $f_{2i}(u,v)=\begin{cases}1,(u,v)\in\Omega_i\\0,(u,v)\notin\Omega_i\end{cases}$.

There exist many bijective transformations $\mathrm{P}_i:\Omega_i \to \Psi_i$, but we can use orthogonal or central projection of $\vec{R}_i(u_i,v_i)$ points onto the surface $\vec{R}(u,v)$, all this surfaces are bounded by the intersection line Γ_i. In some cases, it is possible to get a biunique map. The criterion of getting a map of this kind can be a theme of special research, but in this article we don't consider this question.

2. CONSTRUCTION OF SURFACES SEGMENTS BIJECTIVE CORRESPONDENCE BY MEANS OF ORTHOGONAL PROJECTION

Construction of bijective correspondence between segments of surfaces by means of orthogonal projection is convenient when one of combined surfaces is a plane. Consider a plane in parametrical form $\vec{P}(t_1,t_2)=\vec{C}+t_1\cdot\vec{l}+t_2\cdot\vec{m}$. The other surface is set by the parametrical equation $\vec{S}(u_1,u_2)$ too, coordinate functions are $S_1(u_1,u_2)$, $S_2(u_1,u_2)$, $S_3(u_1,u_2)$. Points of the surface $\vec{S}(u_1,u_2)$ can be orthogonally projected on the plane $\vec{P}(t_1,t_2)$. Then solving the system of equations

$$\left[\left(\vec{S}(u_1,u_2)-\vec{P}(t_1,t_2)\right)\times\left[\vec{l}\times\vec{m}\right]\right]=\vec{\theta} \quad (2)$$

where $\vec{\theta}$ is a zero vector, we can find correspondence between points of the plane and a segment of the surface. To tell more precisely, it is correspondence between points of some area on the plane of parameters (u_1,u_2) and similar area on the plane of parameters (t_1,t_2) under the condition that the system (2) has a decision**.** Borders of this areas are defined by the line of coinciding surfaces crossing. At this stage the question about which surface is "basic" must be solved. It is supposed that there is an interactive environment of surfaces designing; then the decision on this question depends on a developer.

2.1 Combination of conic surfaces and planes

The elementary example is combining of a conic segment and a plane. It is supposed that at development of the automated environment of technical surfaces designing there will be a library of base elements of form (BEF).

Let a plane be given by equation $\vec{P}(t_1,t_2)=\vec{C}+t_1\cdot\vec{l}+t_2\cdot\vec{m}$ where $\vec{C}=(0,0,0),\vec{l}=(1;0;0),\vec{m}=(0;1;0)$.A conic surface is set by the equation $S(u_1,u_2)=(1-u_2)\cdot\vec{V}+u_2\cdot(\vec{N}(u_1)-\vec{V})$ such that $\vec{V}=(0;0;1)$ is the top and $\vec{N}(u_1)=(\cos(u_1),\sin(u_1),0)$ is a directing line to the conic surface, it is a line of crossing with the plane (a circle of unit radius with the center in the point $\vec{C}$). In this case the system of equations (2) results in following dependences of parameters u_1, u_2 on parameters t_1, t_2: $u_2(t_1,t_2)=\sqrt{t_1^2+t_2^2}$, $\sin(u_1)={}^{t_2}\!/\!_{\sqrt{t_1^2+t_2^2}}$, $\cos(u_1)={}^{t_1}\!/\!_{\sqrt{t_1^2+t_2^2}}$. After substitution we get the equation of the conic surface, it is defined by parameters t_1, t_2: $\vec{S}(t_1,t_2)=\left(t_1,t_2,1-\sqrt{t_1^2+t_2^2}\right)$. It is the equation of a cone in Cartesian coordinates. Points of area Ω on the plane of parameters t_1, t_2 satisfy the inequality $\sqrt{t_1^2+t_2^2}\le 1$. In this case, the plane is the "basic" surface. Now we can use the formula (1). The result is shown in fig. 1.

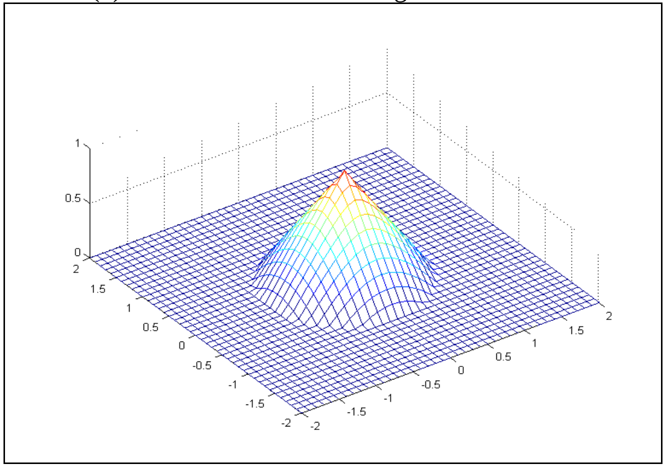

Figure 1. The conic surface and the plane

Note that at parametrization of a cone with its base (a segment of the plane that is bounded by the circle) it is necessary to use other method [4] than is described in this paragraph.

2.2 A sphere and a plane

Consider a combination of a plane and a sphere. The plane is set as well as in the previous subsection and the sphere is set by the equation

$\vec{S}(u_1,u_2)=(\cos(u_1)\cdot\sin(u_2);\sin(u_1)\cdot\sin(u_2);0.5+\cos(u_2))$.

We place the center of the unit sphere on height 0.5 on purpose. The plane dissects the sphere on two unequal parts. If we want to combine the top (big) part of the sphere then use of the orthogonal projection will not give one-to-one correspondence of the points of spherical segments and the plane. The solution for this case will be shown in § 4.2. Here we find the combination of the bottom (smaller) part of the sphere and the plane. In this case, the system of equations (2) is transformed into $\begin{cases}\cos(u_1)\cdot\sin(u_2)-t_1=0\\ \sin(u_1)\cdot\sin(u_2)-t_2=0\end{cases}$. The system for the case with the bottom part of the sphere has following solution: $u_1(t_1,t_2)=\arcsin\left(\frac{t_2}{\sqrt{t_1^2+t_2^2}}\right)$ or $u_1(t_1,t_2)=\arccos\left(\frac{t_1}{\sqrt{t_1^2+t_2^2}}\right)$, $u_2(t_1,t_2)=\pi-\arcsin\left(\sqrt{t_1^2+t_2^2}\right)$. The equation of the bottom part of the sphere can be expressed by parameters t_1, t_2: $\vec{S}(t_1,t_2)=\left(t_1,t_2,0.5-\sqrt{1-t_1^2-t_2^2}\right)$. Points of the area Ω on the plane of parameters t_1, t_2 satisfy the inequality $t_1^2+t_2^2\le{}^3\!/\!_4$. Now we can use the formula (1). (Fig. 2).

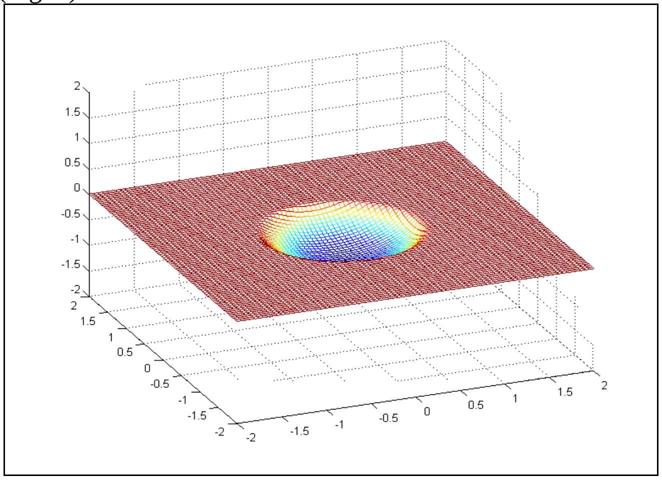

Figure 2. The sphere and the plane

2.3 A spherical surface bounded by planes

In two previous examples we use orthogonal projection to construct bijective correspondence between points of the plane and combined surface and get as a result well-known equations of a cone and a segment of sphere in Cartesian coordinates. Consider the following example to estimate opportunities and constraints of application of orthogonal projection for construction of bijective correspondence.

Consider a sphere of unit radius and three planes (fig. 3), surfaces are set by equations

$$\vec{S}(u_1,u_2)=\begin{cases}\sin(u_1)\cdot\sin(u_2)\\ \cos(u_1)\cdot\sin(u_2)\\ \cos(u_2)\end{cases},\ \vec{P}_1(t_{11},t_{12})=\begin{cases}t_{12}\\ 0.75\\ t_{11}\end{cases},$$

$$\vec{P}_2(t_{21},t_{22})=\begin{cases}t_{21}\\ t_{22}\\ 0.75\end{cases},\ \vec{P}_3(t_{31},t_{32})=\begin{cases}0.75\\ t_{31}\\ t_{32}\end{cases}.$$

Let combine segments of the planes and the surface of the sphere. Solving the system of equations (2) for each plane we get $\begin{matrix}t_{11}(u_1,u_2)=\cos(u_2)\\ t_{12}(u_1,u_2)=\cos(u_1)\cdot\sin(u_2)\end{matrix}$,

$$\begin{matrix}t_{21}(u_1,u_2)=\cos(u_1)\cdot\sin(u_2)\\ t_{22}(u_1,u_2)=\sin(u_1)\cdot\sin(u_2)\end{matrix},\ \begin{matrix}t_{31}(u_1,u_2)=\sin(u_1)\cdot\sin(u_2)\\ t_{32}(u_1,u_2)=\cos(u_2)\end{matrix}.$$

On the plane of parameters (u_1,u_2) there are areas Ω_1, Ω_2 and Ω_3, their points correspond to the cut off segments of the sphere. The areas Ω_1, Ω_2, Ω_3 are defined by inequalities $\sin(u_1)\cdot\sin(u_2)\geq 0.75$, $\cos(u_2)\geq 0.75$ and $\cos(u_1)\cdot\sin(u_2)\geq 0.75$. Let define functions for each segment of the plane

$$f_{i1}(u_1,u_2)=\begin{cases}0,(u_1,u_2)\in\Omega_i\\ 1,(u_1,u_2)\notin\Omega_i\end{cases}\text{ and } f_{i2}(u_1,u_2)=\begin{cases}1,(u_1,u_2)\in\Omega_i\\ 0,(u_1,u_2)\notin\Omega_i\end{cases}.$$

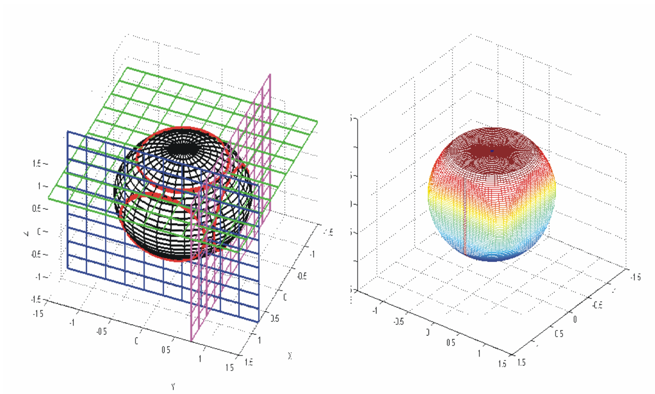

Figure 3 The bounded by planes sphere

After substitution this functions in the formula (1) we get the following result (Fig. 3). Note that lines of mutual crossing of the planes are outside the spherical surface. Otherwise, application of orthogonal projection is inconvenient for construction of bijective correspondences for a sphere and planes. In this case, on the plane of parameters (u_1,u_2) there are areas whose points should not be considered. The solution is reduced to designing a rectangular segment of a plane with several apertures of any form. This problem is considered in work [4]. In the following paragraph we consider application of central projection, which let avoid the specified computing difficulties.

3. BIJECTIVE CORRESPONDENCE OF SURFACES SEGMENTS BY MEANS OF CENTRAL PROJECTION

Let two surfaces be set by equations $\vec{S}(u_1,u_2)$ and $\vec{P}(t_1,t_2)$. Suppose the center of projection $\vec{C}$ is set also. Then it is possible to find correspondence between points of surfaces by solution of the vector equation (the homogeneous system of equations):

$$\left[\left(\vec{S}(u_1,u_2)-\vec{C}\right)\times\left(\vec{P}(t_1,t_2)-\vec{C}\right)\right]=\vec{\theta} \qquad (3)$$

The example from § 2.3 can be solved by means of central projection too. The center of projection must be placed in the center of the sphere. Solving the system of equations (3) we get:

$$\begin{matrix}t_{11}(u_1,u_2)=0.75\frac{\cos(u_2)}{\sin(u_1)\cdot\sin(u_2)}\\ t_{12}(u_1,u_2)=0.75\frac{\cos(u_1)}{\sin(u_1)}\end{matrix},$$

$$\begin{matrix}t_{21}(u_1,u_2)=0.75\frac{\cos(u_1)\cdot\sin(u_2)}{\cos(u_2)}\\ t_{22}(u_1,u_2)=0.75\frac{\sin(u_1)\cdot\sin(u_2)}{\cos(u_2)}\end{matrix},$$

$$\begin{matrix}t_{31}(u_1,u_2)=0.75\frac{\sin(u_1)}{\cos(u_1)}\\ t_{32}(u_1,u_2)=0.75\frac{\cos(u_2)}{\cos(u_1)\cdot\sin(u_2)}\end{matrix}.$$

Shape of this spherical surface is similar to fig. 3. Just character of parametrical lines varies a little.

3.1 Combination of spherical surfaces and dihedral angles

Consider a problem of parametrization of a spherical surface, from which a dihedral angle (fig. 4) is cut out.

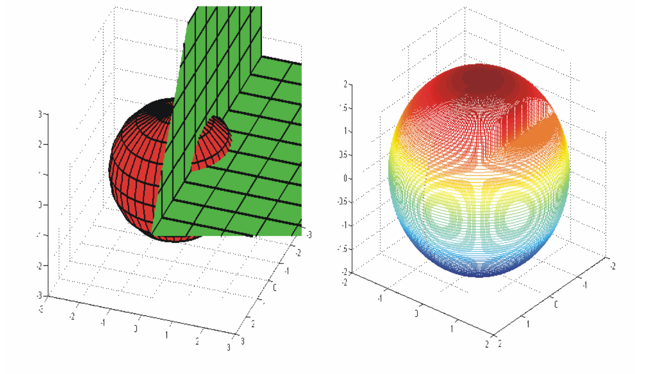

Figure 4 The sphere and the dihedral angle

The equation of the sphere is $\vec{S}(u_1,u_2)=\begin{cases}2\cos(u_1)\sin(u_2)\\2\sin(u_1)\sin(u_2)\\2\cos(u_2)\end{cases}$. The dihedral angle is set

$\vec{P}(t_1,t_2)=\begin{cases}-t_2\\1+f_1(t_1)\cdot t_1\\1-f_2(t_1)\cdot t_1\end{cases}$, $f_1(t_1)\begin{cases}1,t_1\geq 0\\0,t_1\prec 0\end{cases}$,

$f_2(t_1)=\begin{cases}0,t_1\geq 0\\1,t_1\prec 0\end{cases}$. Area Ω corresponds to the removed part of the sphere in the plane (u_1,u_2), this plane is a definitional domain of the sphere. Points of this area satisfy to the logic term

$$\left(\sin(u_1)\sin(u_2)\succ 0.5\right) and \left(\cos(u_2)\succ 0.5\right).$$

For each of half planes of the dihedral angle we solve the system of equations (3):

$$\begin{matrix}t_1=\frac{\sin(u_1)\sin(u_2)}{\cos(u_2)}-1\\t_2=-\frac{\cos(u_1)\sin(u_2)}{\cos(u_2)}\end{matrix},\cos(u_1)-\sin(u_1)\sin(u_2)\leq 0$$

and

$$\begin{matrix}t_1=1-\frac{\cos(u_2)}{\sin(u_1)\sin(u_2)}\\t_2=-\frac{\cos(u_1)}{\sin(u_1)}\end{matrix},\cos(u_2)-\sin(u_1)\sin(u_2)\succ 0.$$

To expression $\cos(u_2)-\sin(u_1)\sin(u_2)=0$ there corresponds a projection of the dihedral angle's edge (its part inside the sphere) on the sphere. The result of substitution of all expressions in the formula (1) is represented in fig. 4.

Use of the central projection at construction bijective conformitys of surfaces segments has limitations. In the previous example planes of the dihedral angle can't pass through the center of projection (the center of the sphere). But this center can be displaced. So there is a theme of special research.

Under some conditions use of the central projection cause indetermination of $0/0$ type. This complicates process of algorithmization. In this case, use of orthogonal projection is more convenient. However now there is no strict criterion what method to use. The choice of the method completely depends on a designer, who has to have good mathematical qualification**.** It is supposed that this person work with an interactive development environment**.**

4. INTERMEDIARY SURFACE AT CONSTRUCTION OF BIJECTIVE CORRESPONDENCES

In the previous examples we use orthogonal and central projections for construction of bijective correspondences. But this can be done not always. In some cases it is possible to apply projection by means of an intermediary surface.

4.1 Combination of a cylindrical surface and a plane

In Cartesian coordinates the explicit equation can be derived only for some part of a sphere or a cylindrical surface. This implies that it is difficult to construct combination of their segments with a plane. Consider the right circular cylinder (fig. 5).

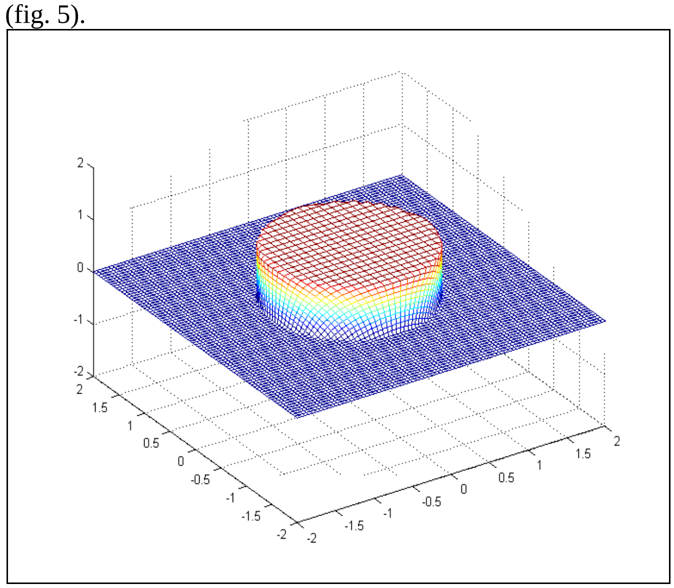

Figure 5 The cylindrical surface and the plane

The problem can be formulated as follows: each point of the plane segment, which is bounded by the line of crossing with the cylinder, must correspond to only one point on the cylindrical segment. It is obvious that there exists a solution to this problem and it is not unique. The main problem is to find the solution that let derivation of explicit formulas of coordinate transformation and makes algorithmization easier.

Let h be height of the combined with the plane segment of the cylinder and let r be radius of its base (fig. 6). Let choose a conic segment of the same height and radius of base as an intermediary surface. Let the plane be set $\vec{P}(t_1,t_2)=(t_1;t_2;0)$ and the segment of the conic surface be set: $\vec{S}(u_1,u_2)=(1-u_2)\cdot\vec{V}+u_2\cdot\vec{N}(u_1)$, where $\vec{V}=(0;0;h)$, $\vec{N}(u_1)=(r\cdot\cos(u_1);r\cdot\sin(u_1);0)$. The equation of the conic segment can be expressed by parameters (t_1,t_2): $\vec{S}(t_1,t_2)=\left(t_1;t_2;h-\frac{h}{r}\cdot\sqrt{t_1^2+t_2^2}\right)$.

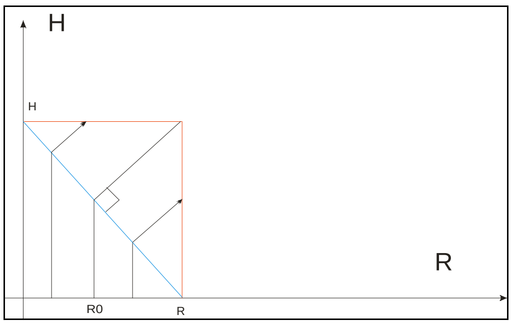

Figure 6 A cone as an intermediary surface

The area Ω is defined by an inequality $t_1^2+t_2^2\le r^2$. Let every point of the plane inside area Ω correspond to a point of the cone (orthogonal projection, fig. 6) and every point of the cone correspond to a point of the cylindrical surface. If $t_1^2+t_2^2\le r_0^2$, where $r_0=r\cdot\left(1-\frac{h\cdot r}{h^2+r^2}\right)$, then the cylindrical segment can be described

$$\vec{\tilde{S}}(t_1,t_2)=\left(\frac{t_1\cdot(h^2+r^2)}{r^2};\frac{t_2\cdot(h^2+r^2)}{r^2};h\right)$$

. If $t_1^2+t_2^2\succ r_0^2$, then we have

$$\vec{\tilde{S}}(t_1,t_2)=\left(r\frac{t_1}{\sqrt{t_1^2+t_2^2}};r\frac{t_2}{\sqrt{t_1^2+t_2^2}};\left(r-\sqrt{t_1^2+t_2^2}\right)\cdot\frac{h^2+r^2}{h\cdot r}\right).$$

The result after use of the formula (1) is shown in fig. 5.

Note that in this example we find parametrical representation of the cylinder with its top base. In this case, it is possible to construct bijective correspondence of points of the plane segment and the cylinder. If we need to describe only the lateral surface of the cylinder then it can be done by elimination of the point $(0,0)$ on the plane of parameters (t_1,t_2).

4.2 Combination of a sphere and a plane

In the example in § 2.2 we combine the bottom (small) segment of sphere and the plane. If we need to combine the top (big) segment and a plane (fig. 7) then we can use the bottom segment as an intermediary surface. The plane is set by the equation $\vec{P}(t_1,t_2)=(t_1;t_2;0)$. The spherical surface is set by the equation $\vec{S}(u_1,u_2)=(\cos(u_1)\cdot\sin(u_2),\sin(u_1)\cdot\sin(u_2),0.5+\cos(u_2))$.

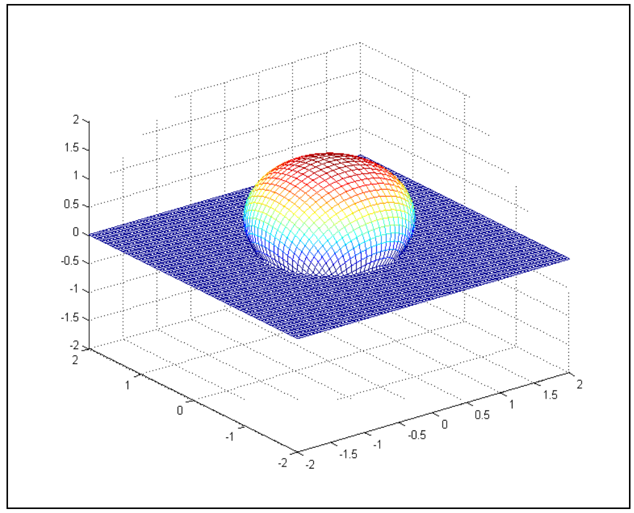

Figure 7 The bottom segment of the sphere as an intermediary surface

Let $\vec{S}_{bot}(\phi_1,\phi_2)$ be the equation of the bottom part of the sphere. Parameters (ϕ_1,ϕ_2) belong to the plane of parameters (u_1,u_2) and $u_1=\varphi_1, u_2=\phi_2$. Different symbols for designation are applied only to make this text more comprehensible. But the range of definition (u_1,u_2) is a rectangle $[0:2\pi,0:\pi]$ and the range of definition (ϕ_1,ϕ_2) is a rectangle $\left[0:2\pi,\frac{2\pi}{3}:\pi\right]$. From § 2.2 we have $\sin(\phi_1)=\frac{t_2}{\sqrt{t_1^2+t_2^2}}$ or $\cos(\phi_1)=\frac{t_1}{\sqrt{t_1^2+t_2^2}}$, $\phi_2=\pi-\arcsin\left(\sqrt{t_1^2+t_2^2}\right)$. Let $\vec{S}_{top}(\gamma_1,\gamma_2)$ be the equation of the top part of the sphere. Parameters (γ_1,γ_2) also belong to the plane (u_1,u_2), but their range of definition is a rectangle $\left[0:2\pi,0:\frac{2\pi}{3}\right]$. Let us construct bijective correspondence between rectangles $\left[0:2\pi,\frac{2\pi}{3}:\pi\right]$ and

$\left[0:2\pi,0:\frac{2\pi}{3}\right]$ under formulas $\gamma_1=\varphi_1$ и. $\gamma_2=2\cdot\left(\pi-\phi_2\right)$. This correspondence is represented in fig. 8. After substitution of this parameters in the equation of the sphere we have the equation of its top part, it is expressed by parameters t_1,t_2:

$$\vec{S}_{top}\left(t_1,t_2\right)=\begin{cases}\frac{t_1}{\sqrt{t_1^2+t_2^2}}\cdot\sin\left(2\arcsin\left(\sqrt{t_1^2+t_2^2}\right)\right)\\ \frac{t_2}{\sqrt{t_1^2+t_2^2}}\cdot\sin\left(2\arcsin\left(\sqrt{t_1^2+t_2^2}\right)\right)\\ 0.5+\cos\left(2\arcsin\left(\sqrt{t_1^2+t_2^2}\right)\right)\end{cases}.$$

This result is shown in fig. 7.

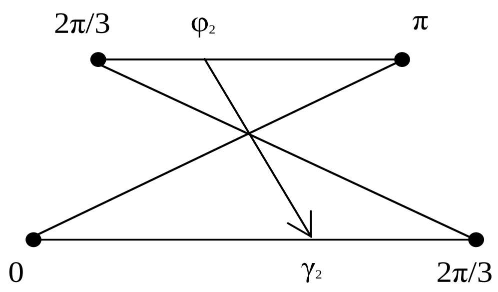

Figure 8 Bijective correspondence between parts of sphere

It is clear that the choice of the intermediary surface is individual in each case. But if we deal with a limited set of base elements of the form, then it is possible to create a library where all types of correspondences will be described.

5. CONCLUSION

If some geometrical object consists of combination of various base elements of the form such as segments of planes, linear surfaces, surfaces of the second order and other surfaces that are set by parametrical equations, then it is possible to create an interactive environment of compound surfaces designing.

It is logical to devide this environment into following components:

1. Closed compound curves design.
2. Base elements of the form design.
3. Combination of the base elements of the form.

Besides specified parts there has to be a graphic environment of visual modelling like CAD-systems (AutoCAD, ADEM, KOMPASS, etc.) with realization of the objective binding. Recall that the result of designer's work will be a parametrical equation of the geometrical object's surface.

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