

User guide for the **SASfit** software package

A program for fitting elementary structural models
to small angle scattering data

April 14, 2018



SASfit: A program for fitting simple structural models to small angle scattering data

Joachim Kohlbrecher

PAUL SCHERRER INSTITUTE
LABORATORY FOR NEUTRON SCATTERING (LNS)
CH-5232 VILLIGEN PSI
JOACHIM.KOHLBRECHER@PSI.CH

APRIL 14, 2018

ABSTRACT. **SASfit** has been written for analyzing and plotting small angle scattering data. It can calculate integral structural parameters like radius of gyration, scattering invariant, Porod constant. Furthermore it can fit size distributions together with several form factors including different structure factors. Additionally an algorithm has been implemented, which allows to simultaneously fit several scattering curves with a common set of (global) parameters. This last option is especially important in contrast variation experiments or measurements with polarised neutrons. The global fit helps to determine fit parameters unambiguously which by analyzing a single curve would be otherwise strongly correlated. The program has been written to fulfill the needs at the small angle neutron scattering facility at PSI (<http://kur.web.psi.ch>). The numerical routines have been written in C whereas the menu interface has been written in tcl/tk and the plotting routine with the extension blt. The newest **SASfit** version can be downloaded from <http://kur.web.psi.ch/sans1/SANSSoft/sasfit.html>.

Contents

Chapter 1. Introduction to the data analysis program SASfit	13
1.1. Overview	13
1.2. System Requirements And Software Installation	14
1.3. Installation Procedure	14
Chapter 2. Quick Start Tour	17
2.1. User Interface Window	17
2.2. Importing data files for a single data set	18
2.3. Importing data files for multiple data sets	20
2.4. Reducing the number of data points	20
2.5. Simulating scattering curves	22
2.6. Fitting	23
2.6.1. Model Independent Fitting (Integral parameters)	23
2.6.2. Model dependent analysis	25
2.6.2.1. Modeling a single data set	25
2.6.2.2. Modeling multiple data sets	26
2.7. Criteria for goodness-of-fit	30
2.7.1. chi square test	30
2.7.2. R-factor	31
2.7.3. Confidence interval of the fitting parameter	32
2.8. Fitting strategies	34
2.9. Data I/O Formats	35
2.9.1. Input Format	35
2.9.2. Error bar	37
2.9.3. Export Format	38
2.10. Scattering length density calculator	39
2.11. Resolution Function	41
Chapter 3. Form Factors	43
3.1. Spheres & Shells	43
3.1.1. Sphere	43
3.1.2. Spherical Shell i	45
3.1.3. Spherical Shell ii	47
3.1.4. Spherical Shell iii	49
3.1.5. Bilayered Vesicle	51
3.1.6. Multi Lamellar Vesicle	53
3.1.7. RNDMultiLamellarVesicle	55
3.1.8. Vesicle with aligned flat capped ends [108, 109]	57

3.2. Polymers and Micelles	60
3.2.1. Gaussian chain	60
3.2.1.1. Gauss [43]	62
3.2.1.2. Gauss2 [43]	63
3.2.1.3. Gauss3 [43]	63
3.2.1.4. Polydisperse flexible polymers with Gaussian statistics [165]	64
3.2.1.5. generalized Gaussian coil [80, 78, 76, 79]	65
3.2.1.6. generalized Gaussian coil 2 [80, 78]	66
3.2.1.7. generalized Gaussian coil 3 [80, 78]	66
3.2.2. Star polymer with Gaussian statistic according to Benoit [13]	68
3.2.3. Polydisperse star polymer with Gaussian statistics [28]	70
3.2.4. Star polymer according to Dozier [47]	72
3.2.4.1. Dozier	72
3.2.4.2. Dozier2	74
3.2.5. Flexible Ring Polymer [30]	76
3.2.6. m -membered twisted ring [30]	78
3.2.7. Daisy-like Ring [30]	80
3.2.8. Unified Exponential Power Law according to Beaucage [10, 11]	82
3.2.8.1. Beaucage	82
3.2.8.2. Beaucage2	85
3.2.9. WormLikeChainEXV [168]	87
3.2.10. KholodenkoWorm	89
3.2.11. Diblock copolymer micelles	92
3.2.11.1. Micelles with a homogeneous core and Gaussian chains on the surface	92
3.2.11.2. Spherical core:	93
3.2.11.3. ellipsoidal core with semi-axis ($R, R, \epsilon R$):	96
3.2.11.4. cylindrical core with radius R_{core} and height H:	99
3.2.11.5. wormlike micelles with cylindrical cross-section with radius R_{core}, Kuhn-length l and contour length L:	102
3.2.11.6. micelles with rod-like core:	105
3.2.11.7. Micelles with a homogeneous core and a corona with decaying density profile . of the form $\varphi(r) \propto r^{-\alpha}$	107
3.2.11.8. spherical core:	108
3.2.11.9. rodlike core:	112
3.2.11.10. spherical Micelles with a homogeneous core and a corona of semi-flexible interacting self-avoiding chains	115
3.2.12. Sphere with Gaussian chains attached	115
3.2.13. Sphere with Gaussian chains attached (block copolymer micelle)	117
3.3. Clustered Objects	118
3.3.1. Mass Fractal [198, 197, 97, 130, 131, 132, 133]	118
3.3.2. Stacked Discs [120, 83]	120
3.3.3. DumbbellShell	122
3.3.4. DoubleShellChain	123
3.3.5. simplified cluster	125
3.4. Cylindrical Objects	126
3.4.1. Disc	126

3.4.2. Rod	127
3.4.3. Porod's approximation for a long cylinder [173]	128
3.4.4. Porod's approximation for a flat cylinder [173]	130
3.4.5. Porod's approximations for cylinder [173]	132
3.4.6. Cylinder of length L , radius R and scattering contrast $\Delta\eta$	134
3.4.7. Random oriented cylindrical shell with circular cross-section	135
3.4.8. Random oriented cylindrical shell with elliptical cross-section	138
3.4.9. Torus with elliptical shell cross-section [107, 61]	141
3.4.10. stacked tori with elliptical shell cross-section	143
3.5. Local Planar Objects	145
3.5.1. Shape factors $P'(Q)$	145
3.5.1.1. Polydisperse infinitesimal thin discs	145
3.5.1.2. Infinitesimal thin spherical shell	145
3.5.1.3. Infinitesimal thin elliptical shell	146
3.5.1.4. Infinitesimal thin cylindrical shell	146
3.5.2. Cross-section form factors $P_{cs}(Q)$	147
3.5.2.1. homogeneousXS	147
3.5.3. TwoInfinitelyThinPlates	148
3.5.4. LayeredCentroSymmetricXS	149
3.5.5. BiLayerGauss [162]	150
3.6. Magnetic Scattering	151
3.6.1. Magnetic Saturation	153
3.6.1.1. MagneticShellAniso	153
3.6.1.2. MagneticShellCrossTerm	154
3.6.1.3. MagneticShellPsi	155
3.6.2. Superparamagnetic Particles (like ferrofluids)	156
3.6.2.1. SuperparamagneticFFpsi	156
3.6.2.2. SuperparamagneticFFAniso	156
3.6.2.3. SuperparamagneticFFIso	156
3.6.2.4. SuperparamagneticFFCrossTerm	156
3.7. Lorenz-Mie Form Factors for Static Light Scattering	157
3.7.1. MieSphere	157
3.7.2. MieShell	159
3.8. Other functions	161
3.8.1. DLS_Sphere.RDG	161
3.8.2. Langevin	162
3.8.3. SuperParStroboPsi	163
3.8.4. SuperParStroboPsiSQ	170
3.8.5. SuperParStroboPsiSQBt	171
3.8.6. SuperParStroboPsiSQLx	171
3.8.7. SuperParStroboPsiSQL2x	171
Chapter 4. Analytical Solutions for Structure factors	173
4.1. Methods to include structure factors	173
4.1.1. Monodisperse approach	173
4.1.2. Decoupling approximation	173

4.1.3. Local monodisperse approximation	174
4.1.4. partial structure factors	174
4.1.5. Scaling approximation	175
4.1.6. van der Waals one-fluid approximation	175
4.2. Hard & Sticky Hard Sphere	177
4.2.1. Hard Sphere [169, 221]	177
4.2.2. Sticky Hard Sphere	178
4.2.3. Sticky Hard Sphere (2 nd version [180, 181])	180
4.2.4. Square Well Potential [194]	182
4.2.5. Square Well Potential 2	184
4.3. Multi Lamellar Structures	185
4.3.1. Multi-Lamellar Structures, Thermal Disorder	185
4.3.2. Multi-Lamellar Structures, Paracrystalline Theory	187
4.3.3. Multi-Lamellar Structures, Modified Caillé Theory	190
4.4. Mass Fractal	193
4.4.1. Mass Fractal (Exp Cut-Off)	195
4.4.2. Mass Fractal (Exp(-x ^a) Cut-Off)	196
4.4.3. Mass Fractal (Gaussian Cut-Off)	197
4.4.4. Mass Fractal (OverlapSph Cut-Off)	198
4.5. Other Structure Factors	199
4.5.1. Hayter-Penfold RMSA [86, 84]	199
4.5.2. MacroIon	200
4.5.3. Critical Scattering	201
4.5.4. Cylinder(PRISM)	201
4.5.5. Voigt Peak	202
 Chapter 5. Numerical solutions of the Ornstein Zernike equations	203
5.1. Background	203
5.2. Numerical implementation of the iterative algorithm in SASfit	205
5.3. Thermodynamic Parameters and Consistency Tests	208
5.4. Closures	211
5.4.1. Hypernetted-chain(HNC) approximation	211
5.4.2. Reference hypernetted chain closure (RHNC)	211
5.4.3. modified hypernetted chain closure (MHNC)	212
5.4.4. Kovalenko-Hirata or partially linearized hypernetted chain closure (KH, PLHNC)	212
5.4.5. Percus-Yevick (PY) approximation	212
5.4.6. Mean Spherical Approximation (MSA)	213
5.4.7. Rescaled Mean Spherical Approximation (RMSA)	213
5.4.8. "Soft core" MSA (SMSA) Approximation	213
5.4.9. Verlet Approximation	214
5.4.10. Choudhury-Gosh (CG) Approximation	214
5.4.11. Duh-Haymet (DH) Approximation	214
5.4.12. Martynov-Sarkisov (MS) Approximation	215
5.4.13. Ballone, Pastore, Galli, and Gazzillo (BPGG) approximations	215
5.4.14. Vompe-Martynov (VM) Approximation	215

5.4.15. Bomont-Bretonnet (BB) Approximation	215
5.4.16. Charpentier-Jakse' semiempirical extention of the VM Approximation (CJ-VM)	216
5.4.17. HNC-SMSA (HMSA) Approximation	216
5.4.18. Roger-Young (RY) closure	216
5.4.19. Zero separation theorem based closure (ZSEP)	217
5.5. Interaction Potentials	217
5.5.1. Hard Sphere Potential	218
5.5.2. Sticky Hard Sphere Potential (SHS)	219
5.5.3. Soft Sphere Potential	221
5.5.4. Penetrable Sphere Model	222
5.5.5. Generalized Gaussian Core Model Potential - GGCM-n Potential	223
5.5.6. Fermi Distribution Model	224
5.5.7. Star Polymer Potential	225
5.5.8. Lennard-Jones Potential	227
5.5.9. Depletion Potential	228
5.5.10. Ionic Microgel Potential	233
5.5.11. DLVO Potential	234
5.5.12. Hard Core 3Yukawa Potential	236
5.6. GUI interface for the Ornstein Zernike solver	237
 Chapter 6. Size Distributions	241
6.1. Delta	241
6.2. Uniform distribution	242
6.3. Triangular distribution	243
6.4. Log-Normal distribution	244
6.5. Schulz-Zimm (Flory) distribution	246
6.6. Gamma distribution	248
6.7. PearsonIII distribution	250
6.8. Gauss distribution	251
6.9. Generalized exponential distribution (GEX)	252
6.10. Generalized extreme value distribution (GEV)	253
6.11. Maxwell distribution	255
6.12. Weibull distribution	256
6.13. fractal size distribution	257
 Chapter 7. Peak functions	259
7.1. Beta	260
7.1.1. Beta (Amplitude)	260
7.1.2. Beta (Area)	260
7.2. Chi-Squared	262
7.2.1. Chi-Squared (Amplitude)	262
7.2.2. Chi-Squared (Area)	263
7.3. Erfc peak	265
7.3.1. Erfc (Amplitude)	265
7.3.2. Erfc (Area)	266

7.4.	Error peak	267
7.4.1.	Error (Amplitude)	267
7.4.2.	Error (Area)	268
7.5.	Exponentially Modified Gaussian	269
7.5.1.	Exponentially Modified Gaussian (Amplitude)	269
7.5.2.	Exponentially Modified Gaussian (Area)	271
7.6.	Extreme Value	272
7.6.1.	Extreme Value (Amplitude)	272
7.6.2.	Extreme Value (Area)	273
7.7.	F-Variance	274
7.7.1.	F-Variance (Amplitude)	274
7.7.2.	F-Variance (Area)	275
7.8.	Gamma	277
7.8.1.	Gamma (Amplitude)	277
7.8.2.	Gamma (Area)	278
7.9.	Gaussian or Normal distribution	280
7.9.1.	Gaussian (Amplitude)	280
7.9.2.	Gaussian (Area)	281
7.10.	Gaussian-Lorentzian cross product	282
7.10.1.	Gaussian-Lorentzian cross product (Amplitude)	283
7.10.2.	Gaussian-Lorentzian cross product (Area)	284
7.11.	Gaussian-Lorentzian sum	285
7.11.1.	Gaussian-Lorentzian sum (Amplitude)	286
7.11.2.	Gaussian-Lorentzian sum (Area)	287
7.12.	generalized Gaussian 1	288
7.12.1.	generalized Gaussian 1 (Amplitude)	289
7.12.2.	generalized Gaussian 1 (Area)	290
7.13.	generalized Gaussian 2	291
7.13.1.	generalized Gaussian 2 (Amplitude)	292
7.13.2.	generalized Gaussian 2 (Area)	293
7.14.	Giddings	294
7.14.1.	Giddings (Amplitude)	295
7.14.2.	Giddings (Area)	296
7.15.	Haarhoff - Van der Linde (Area)	297
7.16.	Half Gaussian Modified Gaussian (Area)	299
7.17.	Inverted Gamma	301
7.17.1.	Inverted Gamma (Amplitude)	302
7.17.2.	Inverted Gamma (Area)	303
7.18.	Kumaraswamy	304
7.18.1.	Kumaraswamy (Amplitude)	304
7.19.	Kumaraswamy (Area)	306
7.20.	Laplace	307
7.20.1.	Laplace (Amplitude)	307
7.20.2.	Laplace (Area)	308
7.21.	Logistic	309
7.21.1.	Logistic (Amplitude)	309

7.21.2. Logistic (Area)	310
7.22. LogLogistic	311
7.22.1. LogLogistic (Amplitude)	311
7.22.2. LogLogistic (Area)	313
7.23. Lognormal 4-Parameter	314
7.23.1. Lognormal 4-Parameter (Amplitude)	314
7.23.2. Lognormal 4-Parameter (Area)	315
7.24. LogNormal	316
7.24.1. LogNormal (Amplitude)	317
7.24.2. LogNormal (Area)	318
7.25. Lorentzian or Cauchy distribution	319
7.25.1. Lorentzian (Amplitude)	319
7.25.2. Lorentzian (Area)	320
7.26. Maxwell-Boltzmann distribution	321
7.26.1. Maxwell (Amplitude)	322
7.26.2. Maxwell (Area)	323
7.26.3. generalized Maxwell (Amplitude)	324
7.26.4. generalized Maxwell (Area)	325
7.27. Pearson-IV	326
7.27.1. Pearson-IV (Amplitude)	326
7.27.2. Pearson-IV (Area)	327
7.28. Pearson-VII	328
7.28.1. Pearson-VII (Amplitude)	330
7.28.2. Pearson-VII (Area)	331
7.29. Pulse Peak	332
7.29.1. Pulse Peak (Amplitude)	332
7.29.2. Pulse Peak (Area)	333
7.30. Pulse Peak with 2nd Width Term	334
7.30.1. Pulse Peak with 2nd Width Term (Amplitude)	334
7.30.2. Pulse Peak with 2nd Width Term (Area)	335
7.31. Pulse Peak with Power Term	336
7.31.1. Pulse Peak with Power Term (Amplitude)	336
7.31.2. Pulse Peak with Power Term (Area)	337
7.32. Student-t	338
7.32.1. Student-t (Amplitude)	338
7.32.2. Student-t (Area)	339
7.33. Voigt	340
7.33.1. Voigt (Amplitude)	341
7.33.2. Voigt (Area)	342
7.33.3. Weibull	343
7.33.4. Weibull (Amplitude)	343
7.33.5. Weibull (Area)	344
Chapter 8. Plugin functions for form factors	345
8.1. Non spherical-symmetric randomly oriented particles	345
8.1.1. Rectangular parallelepiped	345

8.2. Very anisotropic particles (local planar & local cylindrical objects)	348
8.2.1. $P_{cs}(Q)$ for planar obj.	349
8.2.1.1. $P_{cs}(Q)$ for a homogeneous cross-section	350
8.2.1.2. $P_{cs}(Q)$ for two infinitely thin parallel layers	352
8.2.1.3. $P_{cs}(Q)$ for a layered centro symmetric cross-section structure	354
8.2.1.4. $P_{cs}(Q)$ for a bilayer with a Gaussian electron density profile [162, 161]	356
8.2.2. $P_{cs}(Q)$ for cylindrical obj.	358
8.2.2.1. $P_{cs}(Q)$ for homogeneous cross-section of a cylinder	359
8.2.2.2. $P_{cs}(Q)$ for cross-section of a cylindrical shell with elliptical cross section	361
8.2.3. $P'(Q)$ for local planar obj.	362
8.2.3.1. $P'(Q)$: thin discs	363
8.2.3.2. $P'(Q)$: thin spherical shell	365
8.2.3.3. $P'(Q)$: thin ellipsoidal shell	367
8.2.3.4. $P'(Q)$: thin hollow cylinder	369
8.2.4. $P'(Q)$ for local cylindrical obj.	371
8.2.4.1. $P'(Q)$: rods	372
8.2.4.2. $P'(Q)$: Kholodenko's worm	373
8.2.4.3. $P'(Q)$: wormlike PS1	375
Without Excluded Volume Effects.	375
With Excluded Volume Effects.	376
8.2.4.4. $P'(Q)$: wormlike PS2	379
8.2.4.5. $P'(Q)$: wormlike PS3	380
8.2.5. local planar obj.	380
8.2.6. local cylindrical obj.	380
8.3. Spherical core-shell structures with smooth or fuzzy interfaces	381
8.3.1. JuelichCoreShell	382
8.3.2. Fuzzy Sphere	385
8.3.3. CoreShellMicrogel	388
8.3.4. Spherical shell with linear varying contrast profile (LinShell)	391
8.3.5. LinShell2	394
8.3.6. ExpShell	396
8.3.7. BoucherSphere, radial profile for a sphere resulting in a Porod law both below and above q^{-4}	399
8.4. Ellipsoidal Objects	403
8.4.1. Ellipsoid of revolution or spheroid	403
8.4.2. Ellipsoid with two equal equatorial semi-axis R and volume V	407
8.4.3. Ellipsoidal core shell structure	408
8.4.4. triaxial ellipsoidal core shell structure	411
8.5. Cylindrical objects	414
8.5.1. Helical structures	414
8.5.1.1. Fanlike helix	414
8.5.1.2. Helix with round cross-section	416
8.5.1.3. Beads model of a single helical strand	419
8.5.1.4. straight superhelix	421
8.5.1.5. coiled superhelix	423
8.6. Sheared and deformed objects	426

8.7.	Non-equilibrium static form factor of a reptating chain	426
8.7.1.	ShearedCylinderHayterPenfold	428
8.7.2.	partly aligned cylindrical shell	431
8.7.2.1.	Maier-Saupe orientation distribution	434
8.7.2.2.	Onsager orientation distribution	435
8.7.2.3.	Boltzmann orientation distribution	436
8.7.2.4.	Gaussian orientation distribution	437
8.7.2.5.	ShearedCylinderHeaviside	438
8.8.	Form factors of magnetic objects	439
8.8.1.	Magnetic spin misalignment	440
8.8.2.	Ferrofluids	442
8.8.3.	Langevin statistics for averages of the form factor and averages of the squared form factor	445
8.9.	Bi-continuous and non-particulate structures	449
8.9.1.	Teubner-Strey	449
8.9.2.	Debye-Anderson-Brumberger(DAB)	452
8.9.3.	Generalization of the Debye-Anderson-Brumberger model (gDAB)	453
8.9.4.	Spinodal	454
8.9.5.	OrnsteinZernike	456
8.9.6.	Broad-Peak	457
8.9.7.	generalized Guinier approximation [52, 92, 90, 91]	459
8.9.8.	generalized Guinier-Porod law	461
8.10.	Functions for analysing azimuthal averaged data	463
8.10.1.	$\sin^2\text{-}\sin^4$ azimuthal analysis	463
8.10.2.	Maier-Saupe azimuthal analysis	463
8.10.3.	Ellipsoidal azimuthal analysis	463
8.11.	fractal size distribution of particles	464
8.12.	Plugin functions for SESANS	468
8.12.1.	$G(z)$ of a sphere	470
8.13.	$G(z)$ of a randomly distributed, two-phase system (DAB)	471
8.14.	$G(z)$ of a self-affine random density distribution (gDAB)	472
8.15.	$G(z)$ for a generalized Gaussian coil (gGc)	472
Chapter 9.	Plugin functions for size distributions	475
9.1.	LogNorm.fp	475
Chapter 10.	Plugin functions for structure factors	479
10.1.	Structure factor of a random flight model	479
10.2.	ordered particle systems	480
10.2.1.	Domains of ordered particle systems isotropically oriented	481
10.2.2.	Domains of ordered particle systems with preferred orientation	484
Chapter 11.	Absolute intensities, moments and volume fractions	487
11.1.	Fitting absolute intensities	487
11.2.	Contrast - Concentration - Forward Scattering - Particle Volume - Absolute Scale	489
11.3.	Moments of scattering curves and size distribution	491

11.4. Volume fractions	494
Chapter 12. Basic Analysis of Dynamic Light Scattering Data	497
12.1. Cumulant Analysis	498
12.2. Double Decay Cumulant Analysis	499
12.3. Fit of Double Stretched Exponentials	500
12.3.1. The least squares minimiser and the robust least squares procedure	501
Chapter 13. Scattering Theory	503
13.1. Scattering Cross-Section	503
13.1.1. Scattering of neutrons on atoms	504
13.1.1.1. Nuclear scattering	505
13.1.1.2. Magnetic Scattering	506
13.1.2. Scattering of x-ray at atoms	508
13.1.2.1. Anomalous scattering of x-rays	509
13.2. Small angle scattering	510
13.2.1. Autocorrelation function $\Gamma(\mathbf{r})$ and $\gamma(\mathbf{r})$	511
13.2.1.1. Isotropic averages	511
13.2.1.2. Absence of long range order	512
13.2.1.3. Limits $r = 0$ and $r = \infty$	512
13.2.2. Volume fraction	513
13.2.3. Interparticle interferences	513
13.2.3.1. Isotropic ensemble of particles	515
13.2.4. Influence of the relative arrangement of scatterers on interparticle interferences	515
13.2.4.1. Formula from Prins and Zernicke	516
13.2.4.2. Isolate particles	517
13.2.4.3. Polydisperse System of isolated particles	517
13.2.5. Influence of $N(R)$ and $F(Q, R)$ on interparticle interferences	518
13.2.6. Scattering laws and structural parameter	519
13.2.6.1. Porod volume	519
13.2.6.2. Radius of gyration and Guinier approximation	520
13.2.6.3. Correlation length	520
13.2.6.4. Porod law and specific surfaces	521
Chapter 14. Experimental Setup of a Small Angle Scattering Instrument	523
14.1. SANS-Camera	523
14.2. SAXS-Camera	523
Chapter 15. Data Reduction in SAS	525
15.1. Correction and Normalization of SANS-Raw data	525
15.1.1. Contribution of the isolated sample	525
15.1.2. Correction for sample holder and background noise	526
15.2. Correction and normalization of SAXS raw data	527
Appendix. Bibliography	537

CHAPTER 1

Introduction to the data analysis program **SASfit**

Small-angle scattering (SAS) is one of the powerful techniques to investigate the structure of materials on a mesoscopic length scale (10 - 10000 Å). It is used to study the shapes and sizes of the particles dispersed in a homogenous medium. The materials could be a macromolecule (biological molecule, polymer, micelle, etc) in a solvent, a precipitate of material A in a matrix of another material B, a microvoid in certain metal or a magnetic inhomogeneity in a non-magnetic material. This technique is also used to study the spatial distribution of particles in a medium, thus providing the information about the inter-particle interactions. The small angle scattering methods includes small angle neutron, x-ray or light scattering. The type of samples that can be studied by scattering techniques, the sample environment that can be applied, the actual length scale probed and the information that can be obtained, all depend on the nature of the radiation employed. The advantage of small-angle neutron scattering (SANS) over other SAS methods is the deuteration method. This consists in using deuterium labeled components in the sample in order to enhance their contrast. Whereas the disadvantage of SANS over small-angle x-ray scattering (SAXS) is the intrinsically low flux of neutron sources compared to the orders of magnitude higher flux at x-ray sources. Neutron scattering in general is sensitive to fluctuations in the density of nuclei in the sample. X-ray scattering is sensitive to inhomogeneities in electron densities whereas light scattering is sensitive to fluctuations in polarizability (refractive index). In general, irrespective of the type of radiation, they also share several similarities. Perhaps the most important of these is the fact that, with minor adjustments to account for the different types of radiation, the same basic equations and laws can be used to analyze data from these techniques. The small-angle scattering data can contain information concerning both the structure and interaction within the system. This information can be obtained by either performing model-independent analysis or detailed model dependent analysis. **SASfit** is such a software package built originally for analysis of small-angle scattering data in the field of soft matter. The main emphasis of the software is to provide easy to use graphical user interface. The software package contains most of the tools to treat the basic scientific problems for data produced on a SAS instrument. It allows users to derive useful information from the SAS scattering data.

1.1. Overview

SASfit is designed for analysing small angle scattering data and also supplies some simple analysis for dynamic light scattering data. At the moment it performs the following analysis routines to both single 1D data sets as well as multiple 1D data sets, where one like to simultaneously fit several data sets, like from concentration series or contrast

variation experiments. In principle by this way also 2D data sets can be analysed by loading segmentationally averaged 2D data.

- determining integral structural parameters by simple Guinier and Porod analysis
- fitting and simulating single or multiple data sets in terms of size distribution, form factor and structure factors
- analysing DLS data in terms of stretched exponentials or cumulants.
- calculating scattering length densities for x-rays and neutrons
- solving numerically the Ornstein-Zernike equations for any combination of supplied pair interaction potential and closure relation
- allows to read, transforms, merge, clip and scale data files. The number of formats is limited.

1.2. System Requirements And Software Installation

SASfit is a program for analyzing small angle scattering data. The numerical fitting routines are written in C and the menu interface in **tcl/tk**. For the plotting of the data the **tcl** extension **blt** has been used. The last version 0.85 of **SASfit** has been tested with the **tcl/tk** version 8.3 and the **blt** version 2.4s.

SASfit is available for users analysing data taken at PSI.

SASfit has been developed at the Paul Scherrer Institute (PSI) and remains © of PSI.

SASfit is provided to users of the PSI facilities.

SASfit is provided "as is", and with no warranty.

1.3. Installation Procedure

SASfit has been compiled with **tcl/tk** 8.4 and **Blt** 2.4. To install the **SASfit** package one has to do the following:

- (1) Download the zip-file "sasfit.zip" from the **SASfit**-home page
<http://kur.web.psi.ch/sans1/SANSSoft/sasfit.html>
- (2) extract the contents of the zip file. A new subdirectory called **sasfit** will be generated, which contains all required files.
- (3) Execute the program **./sasfit/sasfit.exe**



FIGURE 1.1. Mind map about the tools in SASfit

CHAPTER 2

Quick Start Tour

2.1. User Interface Window

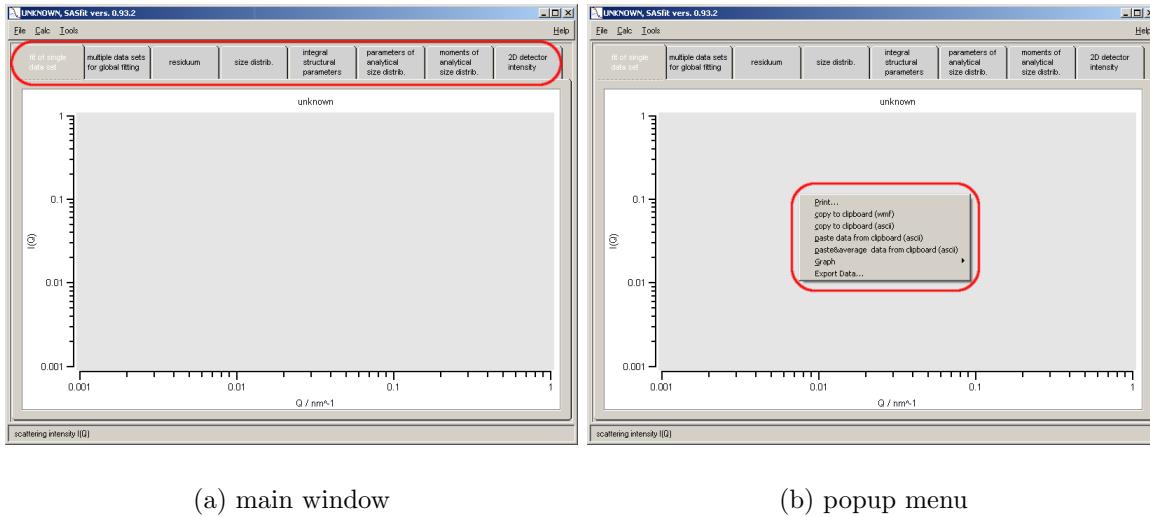


FIGURE 2.1. Main SASfit graphical user interface window

The core SASfit window consists of various tabs (shown in the oval marking in the figure 2.1(a), they are "fit of single data set", "multiple data sets for global fitting", "residuum", "size distrib.", "integral structural parameters", "parameters of analytical size distrib.", "moments of analytical size distrib.", and "2D detector intensity" as shown in the red oval selection. The tabs for single data set and multiple data sets are used to plot single or multiple data files and view the plotted graphs along with the operations to perform during fitting. Residuum shows the difference between the experimental and theoretical fits. Size distributions give the plotted view of the number density v/s radius of the particle. Integral structural parameters are obtained using model independent fitting, such as Guinier approximation, Porod law etc. Parameters of analytical size distributions provides with details of size distributions used and the numbers obtained, whereas moments of analytical size distribution shows the contribution of scattering from different size distribution. The final tab 2D detector intensity is used in case of anisotropic scattering data. The window where the graphs are generated has options of printing the graph plotted view, copying the data in the ASCII format or as an image (wmf) format for further processing or presenting (figure 2.1(b)). SASfit accepts the

isotropic data in the ASCII format. The data can be imported as a single data set or for multiple data sets (several scattering curves).

2.2. Importing data files for a single data set



FIGURE 2.2. Menu interface for input single data set

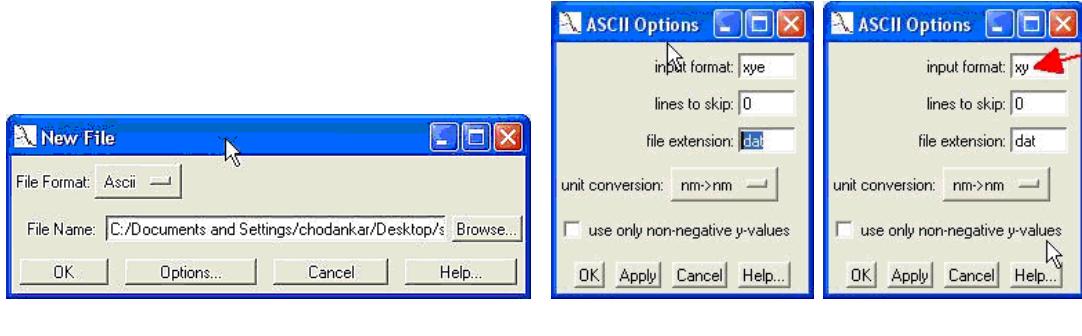
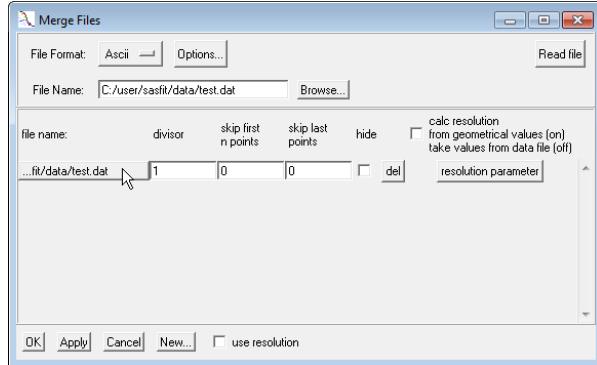


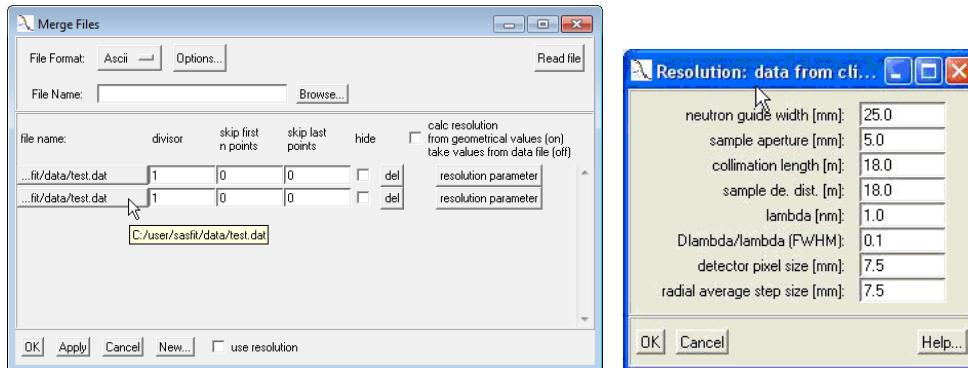
FIGURE 2.3. Importing single data sets

The single data set option allows the user to perform operation on a single data file only. The file is imported via, [File|Single Data Set...] (Figure 2.2). This will open a new file window as shown in figure 2.3(a). The location of the file could be browsed and respectively selected. The options buttons is supplied to select the input format, which is performed by supplying a string such as xye . Where x , y and e stands for the scattering vector Q , scattering intensity $I(Q)$, e signifies the error bar $\Delta I(Q)$ on the measured scattering intensity. The error bars are required during the fitting operation and for files which do not contain the error bar column it would be calculated by default from the smoothing of the curve. There is an option to skip lines at the beginning of the data, which is intended to be used to skip header information in a data file , which could be misinterpreted as data. The number n specified in the menu defines the number

of lines skipped at the beginning of the data file. Furthermore a file extension can be provided, unit conversions can be performed as well as only non-negative y -values could be selected for plotting and performing further analysis. On pressing ok the data is loaded and the graph is plotted, with a new window labeled merge files being opened.



(a) Merge window for merging different Q scales into a single profile



(b) An example showing merged data files

(c) resolution parameter interface

FIGURE 2.4. Merging many data files to one data set

In SAS, data can be collected at different collimator and sample to detector distances to correspond for a wide Q scale. Thus for a single sample at a given condition there can be more than one data files, to merge all of them together for completing the scattering profile, the above shown window comes into play. As shown in the merge files window, the new file could be browsed and selected; it has to be read using the read file button. The newly read file is listed below the first file, if it's a wrong selection it could be deleted back, also one can scale the different files measured at different Q windows, using the divisor column to have a continuous scattering profile. After scaling all the data profiles into one single profile, the statistically bad and unwanted data points can be removed by skipping the points at the beginning and at the end of the data files. The resolution parameters can be provided by pressing the resolution button and the

required information such as sample to detector distance, collimation distance, cross section of the guide etc as shown in figure 2.4(c) has to be entered to use resolution smearing during the fitting. The new button is use to discard all the current selections and plotted data files and starts a new session. The file could also be imported by pasting the clipboard data on the graph view as shown in the figure below. The conditions for columns are same as that for reading the file via browse method.

2.3. Importing data files for multiple data sets

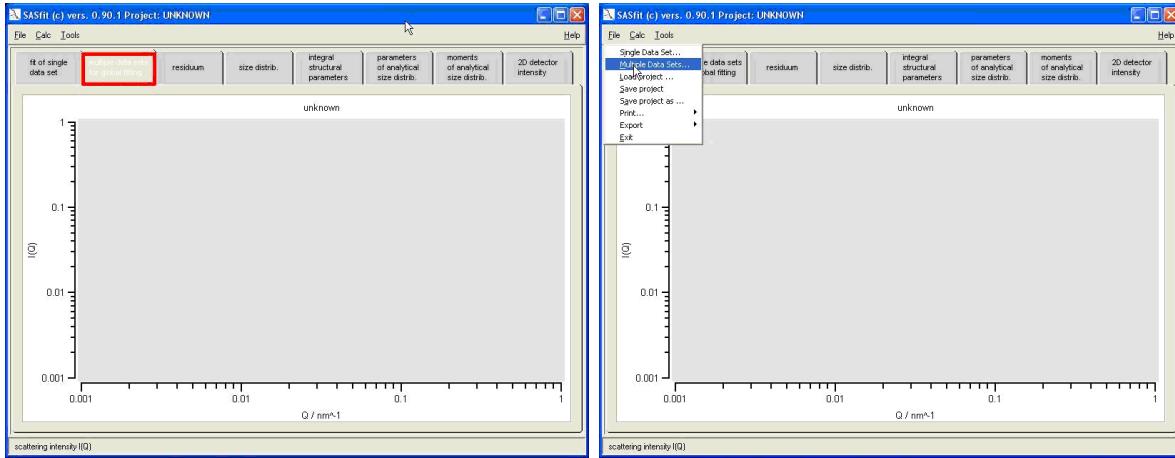


FIGURE 2.5. Procedure for importing data files for multiple data sets

The multiple data set option allows the user to perform operation on multiple data files by using common set of (global) parameters. This option is especially important in contrast variation experiments or measurements with polarized neutrons. The global fit helps to determine the fit parameters unambiguously which by analyzing a single curve would be otherwise strongly correlated. The file is imported by first going to multiple data sets for global fitting (red box in Figure 2.5) and then via, [File|Multiple Data Set...]. This will open a new window as was the case for importing single data set as shown below. The procedure for importing the first file is same as was the case for single file. On reading the first file the merge file window opens, which has additional buttons as compared to the merge file window in single data set (shown in the red rectangular box in figure 2.6(c)). In multiple data fitting, almost any number of data files could be loaded. The present active number of data is shown next to the data set in the merge file window. One can switch over from one data file to another by clicking previous or next. Add and remove buttons are used to add or remove another file. The addition of curves for different Q scales is performed similar to as mentioned in the Input single data set

2.4. Reducing the number of data points

SASfit is calculating the intensity for each data point. To make the minimization faster it can be useful to reduce the number of data points. A reduction of data points make sense if they are oversampled, i.e. within the distance of neighbouring q-values



FIGURE 2.6. Procedure for importing data files for multiple data sets

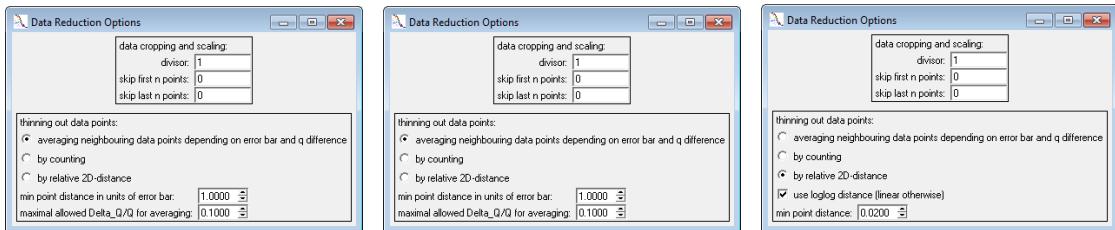
the intensity is only changing little. **SASfit** supplies two algorithms for reducing the number of data points and one algorithm which is averaging data points. After reading a data file a new entry is generated in the GUI, see e. g. 2.6(c). The filename button activates the GUI (see fig. 2.7) for choosing an algorithm to reduce optionally the number of data points. Three procedures are supplied at the moment:

- (1) averaging neighbouring data points depending on error bar and q difference
- (2) skipping equally data points to reduce their numbers to a certain percentage
- (3) skipping nearby data points

By using the first option an averaging of data points $I_k(q_k) \dots I_l(q_l)$ is done as long as they fulfill the conditions that

$$\forall n \in (k, l] : |I_k - I_n| < D_{\min}(\Delta I_k + \Delta I_n) \quad \wedge \quad |q_k - q_n|/\bar{q} < \Delta q_{\max} \quad (2.1)$$

where D_{\min} and Δq_{\max} are user supplied values. The averaging of data points therefore depends on a user-defined maximum allowed q -smearing Δq_{\max} and a user-defined maximum distance in intensity in units of the error bar (D_{\min}), so that an averaging is only performed, if the intensities look similar with D_{\min} -times the intensity error bars and are not too far away from each other on the q -axes. The second option simply skips data points equally to reduce the amount data. The parameter "percentage/100 to load" defines the amount of data to be kept. A value of 1 means all data are kept and a value of 0.5 means every second data point will be kept. In the third option one can define a distance in the $q : I(q)$ -plot or $\log q : \log I(q)$ plot. The next data point, which will be plotted must have either in the linear or logarithmic plot a distance exceeding a user defined value.



(a) Averaging data points depending on the error bar and resulting resolution.
(b) skipping data points to reduce the number of points to a certain amount.
(c) reducing the number of data points by defining a minimum distance between them to be plotted.

FIGURE 2.7. GUI for input values of the two algorithms for reducing the number of data points and for one algorithm which is averaging data points.

2.5. Simulating scattering curves

In addition to reading and loading data set, one can also have a realistic view of the experimental scattering data for a known structure by simulating the scattering profile beforehand to get a feel of the actual experiment scattering profile. The simulation can be performed either for a single data set or for multiple data sets using global parameters. To generate theoretical scattering profile, follow [Calc|Single Data Set...|simulate] or [Calc|Multiple Data Sets...|simulate], either of them to generate a single data set or multiple data sets varied by changing the global parameter. The data can be

generated for vast number of form factors and structure factor included in the software. The simulation is calculated using physically relevant parameters, this is useful to plan the experiment and to know whether a given concentration and contrast would produce a measurable signal.



FIGURE 2.8. Procedure for simulating data profiles for single as well as multiple data files

2.6. Fitting

SASfit can analyse the data using both model-independent analysis and using a non-linear least square method to fit models. The model-independent analysis is a preliminary process of analyzing SAS data and does not require any advanced knowledge of the system to extract structural information this includes fittings (Guinier, Kratky, Porod, power-laws, etc.). On the other hand in case of non-linear least square methods a detailed fitting to the experimental data is performed using a wide variety of form factors and structure factors. The **SASfit** model library consists of large number of such functions, which can be readily used for the analysis. Moreover it can also fit different size distributions.

2.6.1. Model Independent Fitting (Integral parameters).

Model independent analysis requires no advanced knowledge about the sample and most importantly no experimental bias of assumed structure. It includes linearized fitting (Guinier, Porod and Zimm plot) to extract structural information. Model independent analysis are performed via [Cals|integral parameters]. In **SASfit** there are basically three functions available to do the analysis; they are Guinier, Zimm and Porod approximations (shown in the blue box in Figure 2.9(b)). The number of data points to be included in the analysis can be accordingly varied and the resulting fit and the available parameters can be viewed instantaneously. For a large number of data a small script can be return to automate the process. This is performed by using the lower

section of the integral structural parameters window (shown in red box). Plename indicates a character or string of characters with which all the data file names to be analysed starts with followed by certain numbers. The number of digits/characters in the filename could be given in the digits submission box, whereas the start number and the last file number are provided in their respective submission boxes. The step box indicates the incremental step of the file names which has to be analysed. The fitted parameters can be saved in the custom file to be viewed later. Load next file does step by step analysis of different files, whereas Do all would perform calculations on the entire file list, to be saved in the custom file for later viewing.



FIGURE 2.9. Integral fit parameters

A set of valuable size and integrated parameters that can be calculated directly from the scattering curves $I(Q)$ [39, 195, 38, 224, 150, 73] consists of

$$\tilde{Q}_{\text{inv}} = \int_0^{\infty} Q^2 I(Q) dQ \text{ (scattering invariant)} \quad (2.2a)$$

$$\frac{S}{V} = \frac{\pi}{\tilde{Q}_{\text{inv}}} \lim_{Q \rightarrow \infty} \{Q^4 I(Q)\} \text{ (specific surface)} \quad (2.2b)$$

$$\langle R_G \rangle^2 = 3 \left(- \lim_{Q \rightarrow 0} \left\{ \frac{d[\ln I(Q)]}{d(Q^2)} \right\} \right) \text{ (squared Guinier radius)} \quad (2.2c)$$

$$\langle d \rangle = \frac{4}{\pi} \frac{\int_0^{\infty} Q^2 I(Q) dQ}{\lim_{Q \rightarrow \infty} \{Q^4 I(Q)\}} \text{ (average intersection length)} \quad (2.2d)$$

$$\langle l \rangle = \frac{\pi}{\tilde{Q}_{\text{inv}}} \int_0^{\infty} Q I(Q) dQ \text{ (correlation length)} \quad (2.2e)$$

$$\langle A \rangle = \frac{2\pi}{\tilde{Q}_{\text{inv}}} \int_0^{\infty} I(Q) dQ \text{ (correlation surface)} \quad (2.2f)$$

$$\langle V \rangle = \frac{2\pi^2}{\tilde{Q}_{\text{inv}}} I(0) \text{ (correlation volume, Porod volume)} \quad (2.2g)$$

2.6.2. Model dependent analysis.

2.6.2.1. Modeling a single data set.

For Modeling a SANS data set the **SASfit** -programm allows to describe experimental data with an arbitrary number of scattering objects types. Each of them can have a size distribution, whereby the user can choose over which parameter a_i of the form factor the integration will be performed. For example in case of a spherical shell with a core radius of R and a shell thickness of ΔR **SASfit** allows to integrate either over the core radius $x = R$ or the shell thickness $x = \Delta R$ by marking the corresponding parameter (see option **distr** in Fig . 2.10(b)). Furthermore an additional structure factor can be included for each scattering object. Several ways to account for the structure factor have been implemented like the monodisperse approximation (4.1.1), decoupling approach (4.1.2), local monodisperse approximation (4.1.3), partial structure factor (4.1.4) and scaling approximation of partial structure factors (4.1.5). The details are described in chapter 4.

Implemented size distributions, form factors and structure factors are described in chapters 3, 4 and 6. Optional an additional smearing of this function with the instrument resolution function $R_{av}(Q, \langle Q \rangle)$ can be activated so that

$$I(\langle Q \rangle) = \int_0^{\infty} R_{av}(Q, \langle Q \rangle) \frac{d\sigma}{d\Omega}(Q) dQ \quad (2.3)$$



(a) Menu through which fitting procedure is initiated



(b) User interface for fitting, containing different form factors and structure factor

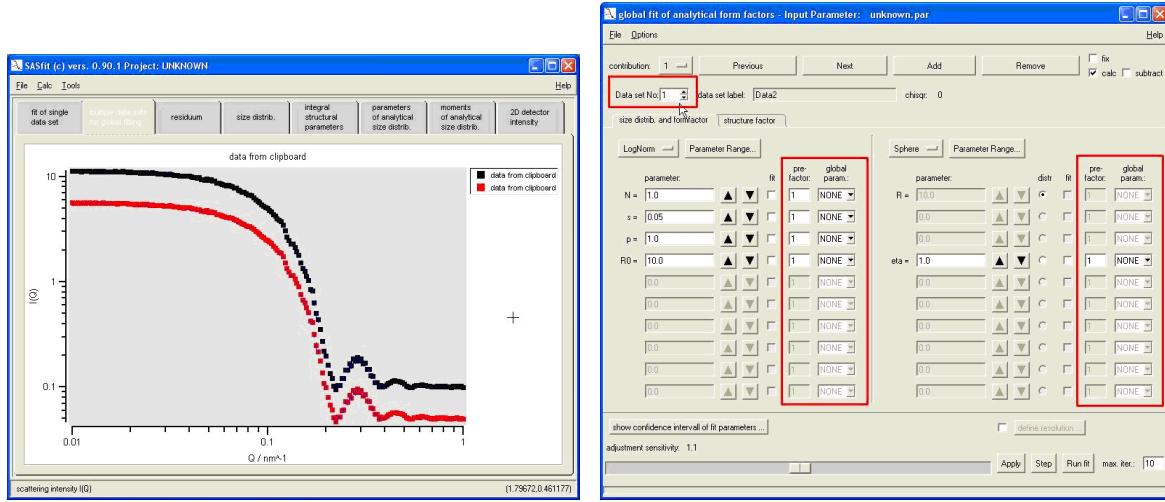
(c) Tab summarizing the analyzed parameters

FIGURE 2.10. Menus for fitting a single data set

A user interface shown in Fig. 2.10 is supplied to choose between the number of scattering objects and to define parameter for each of them. Next to varying the different parameters one can mark those to be fixed or fitted (see option **fit** in Fig. 10(b)(b)). Model dependent analysis for single files are performed via [Calc|Single Data Set|Fit...].

2.6.2.2. Modeling multiple data sets.

The multiple data set option allows the user to perform operation on multiple data file by using common set (global) parameters. This option is especially important in



(a) Imported multiple data sets

(b) User interface for fitting multiple data files

FIGURE 2.11. Menus for fitting a simultaneously multiple data sets

contrast variation experiments or measurements with polarized neutrons. The global fitting helps the user to analyse large number of data which has a similar form or structure factor however different scaling constant. The data shown in the figure below is for a spherical monodispersed system both the data profile has identical features, except that the scaling factor between the two is of a factor of two. The data are called by the procedure explained in the input multiple data file section. The fitting of the data is performed by calling the fitting function via [Calc|Multiple Data Sets|Fit...]. The user interface for multiple data fitting has additional feature than to that for single data fitting, they are pre-factor and global parameters as shown in figure 2.11(b) red markings.

The procedure for fitting the data set is similar to that mentioned in the earlier section. The only difference is to include the global parameters for each function included. The scattering profile shown in the figure 2.11(a) is for a spherical monodispersed particle, both the profiles have identical features with a scaling factor of two. The user interface for fitting shows the following window in figures 2.12. The data set number shows the active data file, whereas contribution indicates the number of scattering objects. We have selected the form factor for a spherical particle. In the global parameter a new variable is produced for both radius and η (scattering contrast). The pre-factor is kept constant at 1. A second contribution is added to the data set one by pressing add. In this case it is a background contribution, a new global parameter is introduced for it ($P3$). A similar procedure is done for second data set where the global parameter for the radius is kept same to that for data set one, whereas new global parameters are defined for scattering contrast and background. The fitting procedure can then be started by pressing Run fit. The figures 2.13 show the graphs during the fitting process.

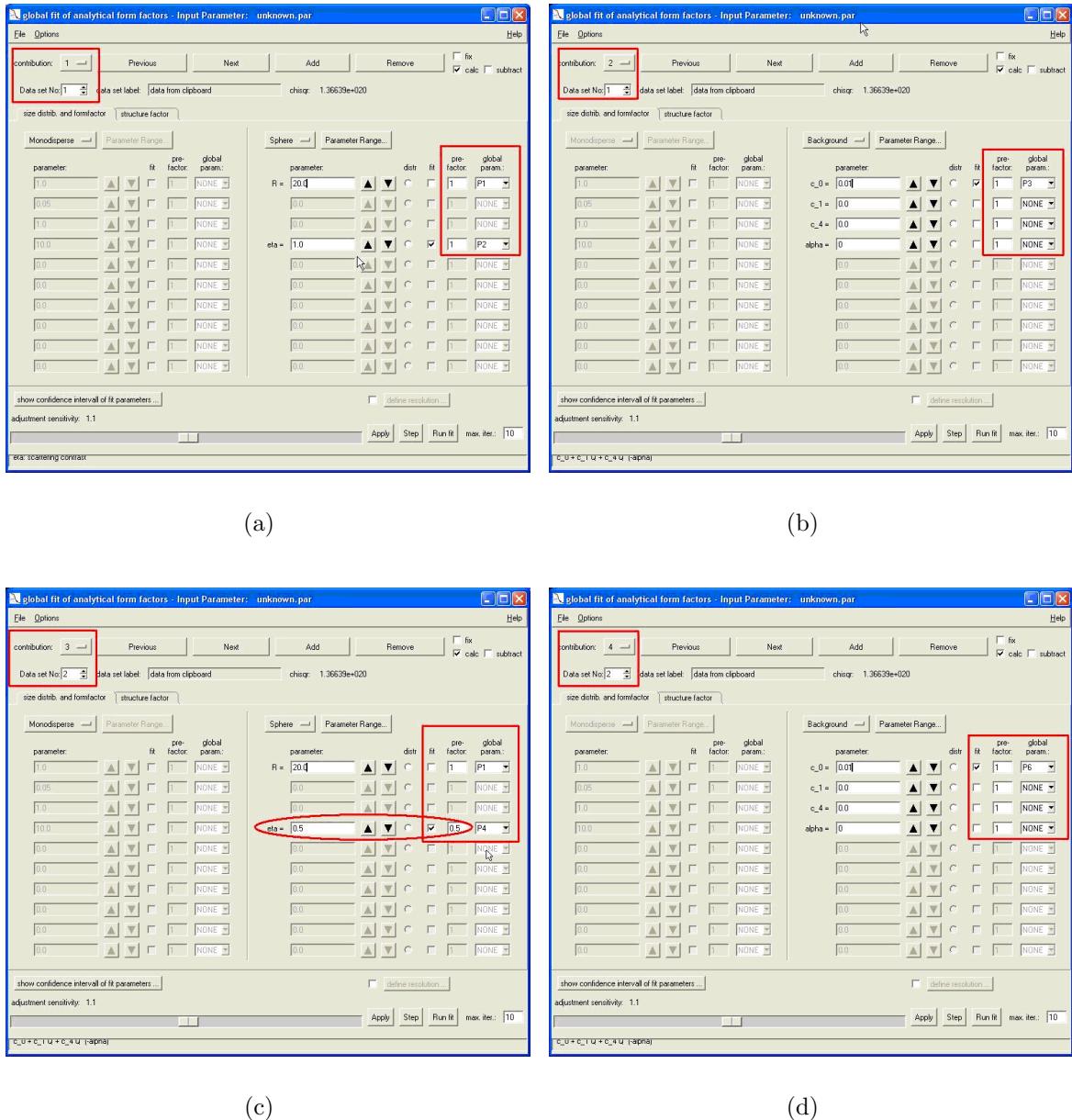


FIGURE 2.12. Different windows showing different controlling parameters during multiple data fitting.

The parameters of fitting for all the contribution can be viewed by pressing parameters of analytical size distributions (figure 2.13(d)).



FIGURE 2.13. The scattering data profile and the analytical parameters obtained during the fitting process.

2.7. Criteria for goodness-of-fit

All criteria shown below for testing the goodness of a fit should be considered with caution [17, 174]. When you get data on a SAS instrument the measured intensities are measured with some statistical uncertainties. Normally one assumes Poisson statistics to determine the uncertainty in the counting statistics. The data reduction software should then perform a proper error propagation analysis for all succeeding data treatment operations. However, by this procedure only statistical uncertainties are taken into account. All systematic uncertainties are then hopefully covered during the data treatment, as for example background correction, transmission correction etc 15.

2.7.1. chi square test.

The method of least squares is built on the hypothesis that the optimum description of a set of data is one which minimizes the weighted sum of squares of deviations, between the data, $I_{\text{exp}}(q_i)$, and the fitting function $I_{\text{th}}(q_i)$:

$$\chi^2 = \sum_{i=1}^N \left(\frac{I_{\text{exp}}(q_i) - I_{\text{th}}(q_i)}{\Delta I(q_i)} \right)^2 \quad (2.4)$$

As a rule of thumb for chi-square fitting is the statement that a good fit is achieved when the reduced chi-square equals one. The reduced chi-square value, which equals the residual sum of square divided by the degree of freedom can be computed by

$$\chi_\nu^2 = \frac{1}{N-m} \sum_{i=1}^N \left(\frac{I_{\text{exp}}(q_i) - I_{\text{th}}(q_i)}{\Delta I(q_i)} \right)^2 = \frac{\chi^2}{N-m} \quad (2.5)$$

where N is the number of data points and m the number of fit parameters. $\nu = N - m$ is called the "number of degree of freedom". The reduced chi-square is closely related to the variance of the fit s^2 by

$$s^2 = \chi_\nu^2 \left(\frac{1}{N} \sum_{i=1}^N \frac{1}{(\Delta I_i^{\text{exp}})^2} \right)^{-1} \quad (2.6)$$

In the theory of hypothesis testing χ^2 can be used to test for goodness of a fit. The probability that a random set of N data points would yield a value of χ^2 equal or greater than the measured one is given by

$$Q_{\text{factor}} = Q \left(\frac{N-m}{2}, \frac{\chi^2}{2} \right) = \frac{\Gamma \left(\frac{N-m}{2}, \frac{\chi^2}{2} \right)}{\Gamma \left(\frac{N-m}{2} \right)} \quad \text{with } \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \quad (2.7)$$

For a fitting function being a good approximation to the data the experimental value of χ_ν^2 should be close to one and the probability Q_{factor} somewhere between 0.01 and 0.5. For probability values close to one the fit seems to be too good to be true.

The interpretation of the parameter Q_{factor} can be summarized as:

goodness-of-fit parameter: Q_{factor}

$Q_{\text{factor}} > 0.99$: too good

$Q_{\text{factor}} > 0.1$: believable

$Q_{\text{factor}} > 0.001$: may be acceptable

$Q_{\text{factor}} < 0.001$: questionable

The Q_{factor} -parameter is calculated after each calculation of the scattering curve together with χ^2_ν and s^2 . They are displayed and updated in the red marked area of the screen shot shown in fig. 2.14.

2.7.2. R-factor.

The crystallographers have introduced another parameter for the goodness of a fit. They use the R factor [157, 75] as a measure of model quality which is defined as

$$R = \frac{\sum_{i=1}^N ||I_{\text{exp}}(q_i) - |I_{\text{th}}(q_i)||}{\sum_{i=1}^N |I_{\text{exp}}(q_i)|} \quad (2.8)$$

Theoretical values of R range from zero (perfect agreement of calculated and observed intensities) to infinity. R factors greater than 0.5 indicate in crystallography very poor agreement between observed and calculated intensities. Models refining to $R < 0.05$ are often considered to be good. However, the R factor must always be treated with caution, only as an indicator of precision and not accuracy. In Crystallography partially incorrect structures have been reported with R values below 0.1; many imprecise but essentially correct structures have been reported with higher R values.

In practice, weighted R factors R_w are more often used to track least-squares refinement, since the functions minimized are weighted according to estimates of the precision of the measured quantity. The weighted residuals are defined as:

$$R_w = \sqrt{\frac{\sum_{i=1}^N \left(\frac{|I_{\text{exp}}(q_i)| - |I_{\text{th}}(q_i)|}{\Delta I(q_i)} \right)^2}{\sum_{i=1}^N \frac{I_{\text{exp}}^2(q_i)}{\Delta I^2(q_i)}}} \quad (2.9)$$

The interpretation of the parameters R and R_w can be summarized as:

goodness-of-fit parameters: R, R_w

$R, R_w > 0.3$: questionable

$0.1 > R, R_w > 0.3$: may be acceptable

$R, R_w < 0.1$: believable

The parameters R and R_w are calculated like the Q_{factor} -parameter after each calculation of the scattering curve. They are displayed and updated together with other goodness-of-fit parameters in the red marked area of the screen shot shown in fig. 2.14.

The goodness-of-fit parameters are not minimized. The parameter which is minimized is the χ^2 -value, defined in eq. 2.4. The other goodness-of-fit parameters are than calculated after each fitting step or after evaluating the theoretical scattering curve via the "Apply"-button.

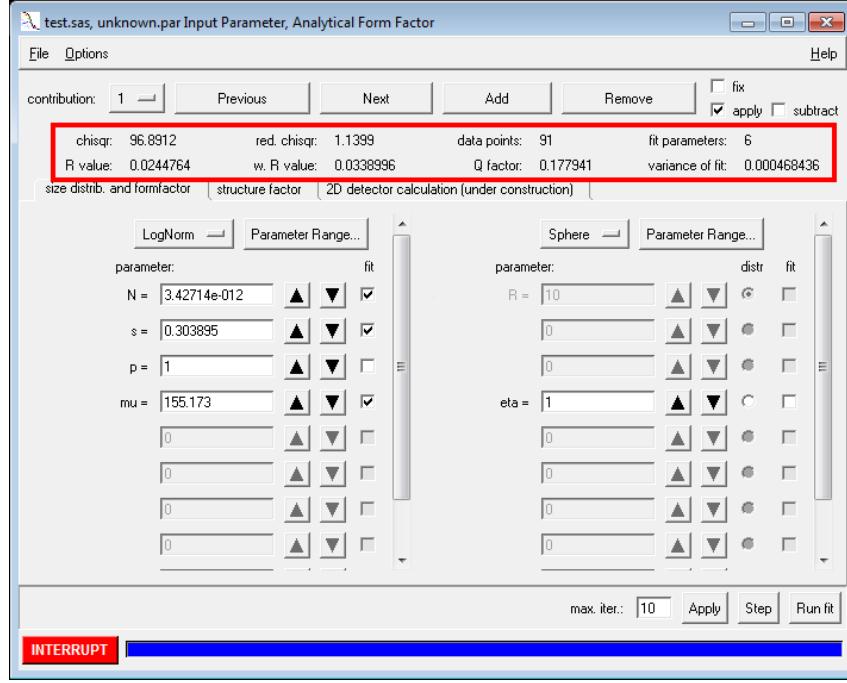


FIGURE 2.14. The parameters Q, R, R_w , describing the goodness of the fit, are displayed in the marked area of the GUI

2.7.3. Confidence interval of the fitting parameter.

The confidence intervals of the fitting parameters are calculated at the minimum of the χ^2 . At the minimum first the partial derivatives according to the fit parameters a_i are calculated to get the matrix elements A_{kl} via

$$A_{kl} = \sum_{i=1}^N \frac{1}{(\Delta I_i^{\text{th}})^2} \frac{\partial I_i^{\text{th}}(q_i, \mathbf{a})}{\partial a_k} \frac{\partial I_i^{\text{th}}(q_i, \mathbf{a})}{\partial a_l} \quad (2.10)$$

The inversion of this matrix yield the covariance matrix

$$\mathbf{C} = \mathbf{A}^{-1} \quad (2.11)$$

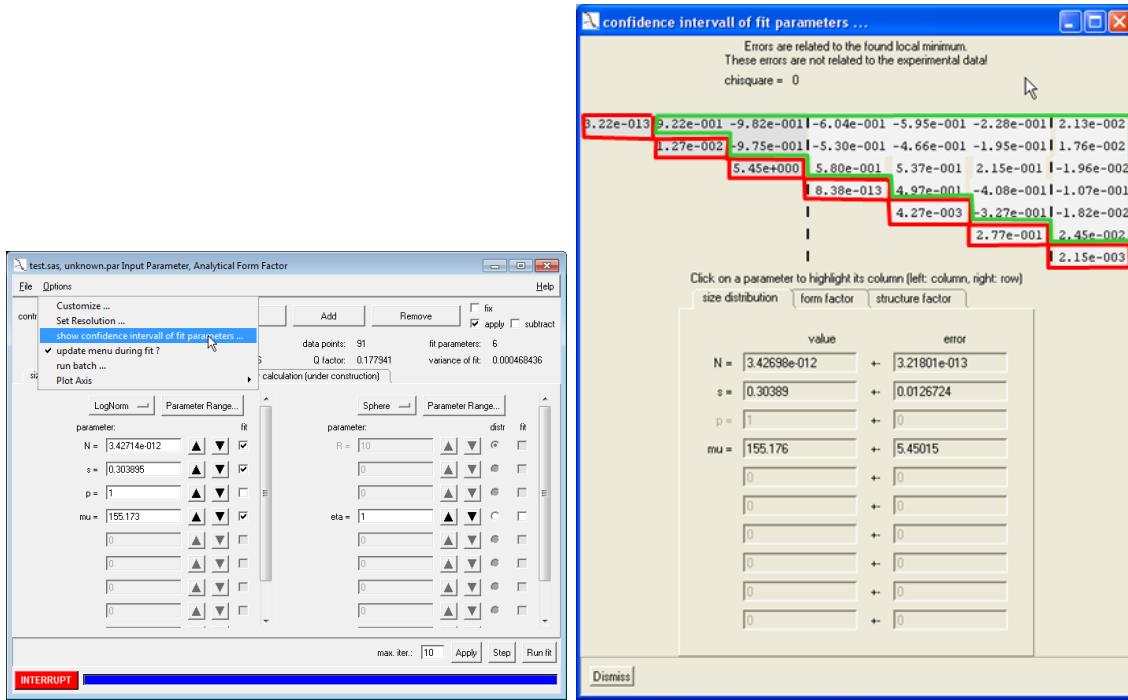
The standard deviations Δa_i of the best-fit parameters are given by the square root of the corresponding diagonal elements of the covariance matrix

$$\Delta a_i = \sqrt{C_{ii}} \quad (2.12)$$

The correlation coefficient of the fit parameters a_k and a_l are given by

$$r_{kl} = \frac{C_{kl}}{\sqrt{C_{kk}C_{ll}}} = \frac{C_{kl}}{\Delta a_k \Delta a_l} \quad (2.13)$$

Both the non-diagonal elements of the correlation matrix $r_{k>l}$ and the confidence intervals of the fitting parameters Δa_k are calculated when the fit converges and displayed in a GUI (fig. 2.15(b)) which can be opened via a menu button as shown in fig. 2.15(a).



(a) menu button showing the GUI for the confidence intervals and correlation matrix

(b) GUI displaying the fitting parameters including confidence intervals and correlation matrix

FIGURE 2.15. The confidence intervals are calculated at the end of the minimization procedure.

The routine can give nonphysical confidence intervals if

- no proper error bar for the experimental data are supplied
- a wrong physical model for describing the data is used
- the physical model has strongly correlated parameters

$$r_{kl} \approx 0, a_k \text{ and } a_l \text{ are uncorrelated parameters} \quad (2.14a)$$

$$r_{kl} \approx \pm 1, a_k \text{ and } a_l \text{ are strongly correlated parameters} \quad (2.14b)$$

The given value for the confidence interval of the fit parameters should not be used in the cases

- where the error is not normal distributed
- where the data is given without error bar
- where parameters are strongly correlated

2.8. Fitting strategies

2.9. Data I/O Formats

2.9.1. Input Format.

SASfit supports a simple ASCII format. Options for reading ASCII data can be set in the corresponding menu, where one can set an input format and the number of lines to be skipped at the beginning of the data file. To set an input format one has to supply a string like "xyer". Each line which does not contain valid float numbers are skipped automatically. Each line further should at least contain as many valid numbers as the supplied format string characters. That means if the line contains only three numbers but the format string is 4 or more characters long the line will be ignored. Separators between numbers can be "white space", "tabulator", or ",". For identifying the columns the characters and their position in the string are interpreted. **x**, **y** and **e** stands for the scattering vector Q , scattering intensity $I(Q)$ and its error $\Delta I(Q)$, respectively. **r** defines the column for a resolution parameter σ . The position of the character in the string defines which data column is assigned to Q , $I(Q)$, $\Delta I(Q)$, and σ . In case of double occurrence of a character the position of the last one is the significant position. Any characters not belonging to {x,y,e,r} can be used to skip a column. A definition string need to contain at least the two characters **x** and **y**.

Example 1 (BerSANS format):

```
%File
FileName=D0002831.200 FileDate=28-Jun-99 FileTime=11:57:16
Type=SANSDIso Title=IMF
%Counts
 2.651E-02, 2.372E+02, 4.650E+00
 3.240E-02, 2.170E+02, 2.291E+00
 3.829E-02, 1.898E+02, 1.713E+00
 4.418E-02, 1.743E+02, 1.479E+00
 5.007E-02, 1.528E+02, 1.318E+00
 5.596E-02, 1.361E+02, 1.153E+00
  :
```

As the first lines start with a string, they will be automatically ignored. To interpret the three columns as Q , $I(Q)$, $\Delta I(Q)$ the format string should be simply **xyz**. The BerSANS format can also be read in by explicitly selecting the "BerSANS"-format button in the menu instead of the "ASCII"-format.

Example 2:

	19	0	0	0	0	0	0	6
1	0.100000E+01	0.100000E+04	0.000000E+00	0.100000E+01	0.120000E+01			
2	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00			
teflon		instrument	tests					
1	2.617993E-04	3.700000E+01	4.301163E+00					
2	1.062462E-03	6.412500E+01	1.634587E+00					
3	2.107973E-03	1.410135E+03	5.207492E+00					
4	3.167636E-03	1.752197E+03	4.801586E+00					
5	4.189463E-03	7.581771E+02	2.810281E+00					
	:							

```

45  1.255376E-02   1.486688E+01   2.197023E-01
46  1.360724E-02   1.204012E+01   1.927716E-01
47  1.466810E-02   1.026648E+01   1.679423E-01

```

A definition string `ixye` would ignore the leading line number at the beginning of each data line, but in the present example also the first 3 lines would also be interpreted as data points. To skip them one has to use the option for skipping leading lines in a data file. In the above case the number should be set to 3 or 4. As the 4th line is anyway ignored a value of 3 is sufficient to skip non data points.

Example 3: ILL data files from regrouped treatment (`gnnnnnn.eee`).

```

[ 37      0      0      0      0      0      0      0      6
  0.100000E+01  0.250000E+03  0.000000E+00  0.100000E+01  0.105400E+01
  0.000000E+00  0.000000E+00  0.000000E+00  0.000000E+00  0.000000E+00
  0.000000E+00  0.000000E+00  0.000000E+00 ]
  2.194656E-03  3.442688E-01  8.329221E-02
  5.466116E-03  3.000000E-01  5.008947E-02
  8.480323E-03  3.877941E-01  4.232426E-02
  1.189216E-02  6.498784E-01  1.519078E-02
  1.497785E-02  7.493181E-01  1.173622E-02
  :

```

To read in a regrouped ILL data file one has to use the definition string `xye` and secondly one has to skip the first 44 lines in the data file to ignore also the lines marked with `[]`. If one does not skip the first 44 lines the marked lines are interpreted erroneously also as data points. The other lines at the beginning of the data file are ignored as they do not fulfill the condition that they have 3 columns containing only valid numbers.

2.9.2. Error bar.

In case no error bar is supplied **SASfit** will try to guess one. To do this an polynomial $y_p(Q)$ of degree p

$$y_p(Q) = \sum_{k=0}^p c_k Q^k \quad (2.15)$$

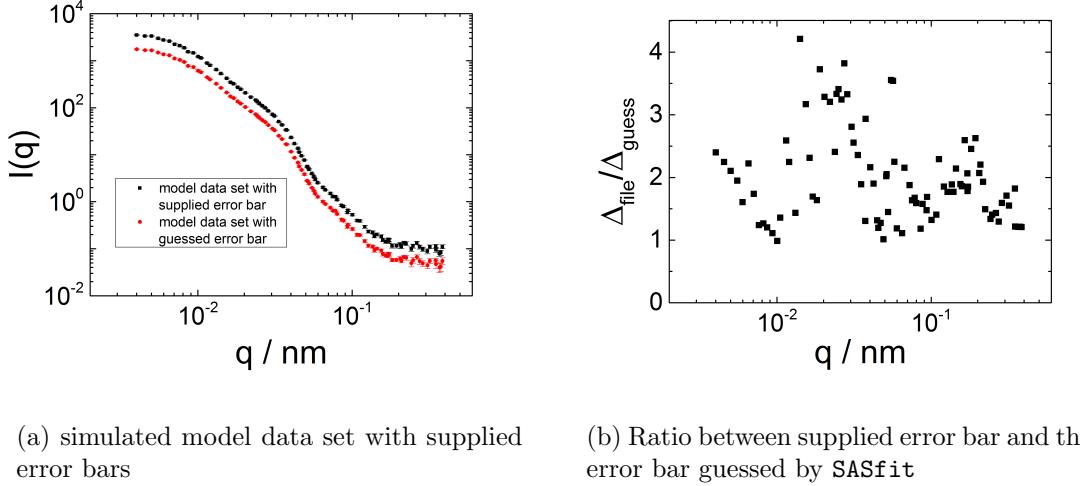
is fitted to the data point i and its n^{th} neighbors to the left and right, i.e. is fitted to $2n + 1$ points from $I_{i-n}(Q_{i-n})$ to $I_{i+n}(Q_{i+n})$. After the fit χ_i^2 is calculated

$$\chi_i^2 = \sum_{j=i-n}^{i+n} (I_j(Q_j) - y_p(Q_j))^2 \quad (2.16)$$

The error bar ΔI_i for Q_i is than defined as

$$\Delta I_i = \sqrt{\frac{\chi_i^2}{2n - p}} \quad (2.17)$$

SASfit is using two nearest neighbors $n = 2$ and fitting a polynomial of degree $p = 2$ to it to guess an error bar. This procedure gives reasonable error bars as long as the data are oversampled and do not show sharp features within the data points $i - n$ and $i + n$. A diffraction peak or a minimum in a form factor like for monodisperse particle might not reasonable well described by the polynomial and consequently the resulting error bar will get large. Furthermore systematic errors in the absolute intensity like an overall scaling factor will not be recognized. The procedure to guess the error bar is basing on the assumption that the scattering curve behaves locally approximately like a polynomial function of degree $p = 2$.



(a) simulated model data set with supplied error bars

(b) Ratio between supplied error bar and the error bar guessed by SASfit

FIGURE 2.16. Comparison of a simulated model data set with supplied error bar and an error bar guessed by SASfit .

2.9.3. Export Format.

Example for an exported data file:

```

0.00401571,      3497.47,      90.7282,      0,      0.00401571,      260294,      -1,      0,
0.00454087,      3340,        84.9531,      0,      0.00454087,      254548,      -1,      0,
0.0050096,       3322.47,      79.6313,      0,      0.0050096,      248833,      -1,      0,
0.00552335,      2983.23,      73.7254,      0,      0.00552335,      241949,      -1,      0,
0.00598495,      2737.17,      68.4395,      0,      0.00598495,      235226,      -1,      0,
0.0065309,       2598.76,      62.3109,      0,      0.0065309,      226647,      -1,      0,
0.00706977,      2233.9,       56.4829,      0,      0.00706977,      217551,      -1,      0,
0.00764207,      2080.96,      50.6186,      0,      0.00764207,      207264,      -1,      0,
0.00815988,      1882.88,      45.6557,      0,      0.00815988,      197459,      -1,      0,
          :           :           :           :           :           :
          ,           ,           ,           ,           0.0445634,      1535.14,      -1,      0,
          ,           ,           ,           ,           0.0453557,      1473.71,      -1,      0,
          ,           ,           ,           ,           0.0470219,      1340.34,      -1,      0,
          ,           ,           ,           ,           0.0490017,      1192.64,      -1,      0,
          ,           ,           ,           ,           0.0510837,      1055.44,      -1,      0,

```

If one like to export the data of an xy -plot all curves are stored in a single data file. Each curve will occupy four columns (Q , $I(Q)$, $\Delta I(Q)$, σ). If an error $\Delta I(Q)$ is not available, e.g. for theoretical data curves, the corresponding column will be filled with -1. Similar is valid for the resolution parameter σ which will be set to 0 in case it is not available. The individual columns are separated by ", ". If the curve have different amount of data points the column will be filled with empty space for the missing data. This comma separated data format has been chosen as it can be imported easily by many commercial plotting softwares. The drawback of this format is, however, that SASfit cannot read it correctly, if the individual curves are of different length.

2.10. Scattering length density calculator

The SLD calculator is using thermal neutron cross-sections only to calculate neutron scattering length density. For x-rays the energy dependent scattering coefficients f' and f'' are derived using the theoretical approximation developed by Cromer and Liberman. This theory gives accurate values far from an absorption edge but does not account for the effects of neighboring atoms, which can be very substantial near an absorption edge. Before conducting an anomalous scattering experiment close to an absorption edge it is therefore advisable to determine the actual scattering behavior of the sample. The x-ray data have been taken from http://skuld.bmsc.washington.edu/scatter/AS_periodic.html and those for neutrons from <http://www.ncnr.nist.gov/resources/n-lengths/list.html>. The menu in-

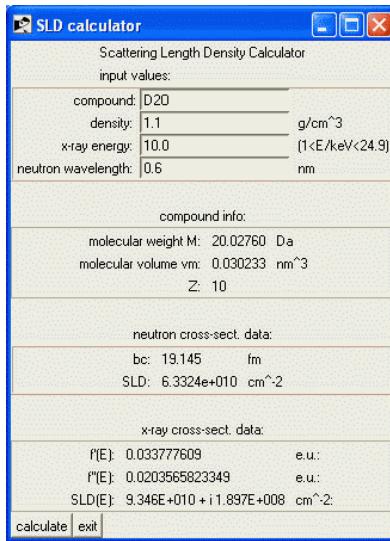


FIGURE 2.17. Input menu for the scattering length density calculator

terface in Fig. 2.17 has for input parameters, the sum formulae of the compound, its mass density in g/cm^3 , the x-ray energy in keV and the neutron wave length in nm. In the compound name non-integer stoichiometry is supported, e.g. H_{0.2}O_{0.1} and H₂O will calculate the same scattering length density. However, the molecular volume v_m and the molecular weight M of cause depend on such differences. The elements in the compound name are case sensitive. Therefore you have to use SiO₂ instead of SI02. Also isotopes are handled like C(13) (Carbon-13), H(2) (Deuterium), or O(18) (Oxygen-18). For Deuterium next to H(2) also D can be used.

Examples of how to format the compound name:

- Magnetite: Fe₃O₄, 5.15 g/cm^3
- Eucryptite: LiAlSiO₄, 2.67 g/cm^3
- protonated toluene, C₇H₈, 0.865 g/cm^3
- deuterated toluene, C₇D₈ or C₇H(2)₈, 0.94 g/cm^3
- mixture of 65/35 heavy water/light water, (D₂O)_{0.65}(H₂O)_{0.35}, 1.065 g/cm^3

From the compound name and the density first the molecular weight M , molecular volume v_m , and total number of electrons Z are calculated. Together with

the tabulated neutron scattering length and tabulated energy dependent scattering coefficient $f'(E)$ and $f''(E)$ the corresponding coherent neutron scattering length $b_c = \sum_i b_i$, coherent neutron scattering length density $\eta_{n,SLD} = b_c/v_m$ and for x-rays the complex energy dependent scattering scattering length density $\eta_{x,SLD} = (Z - (Z/82.5)^{2.37} + f'(E) + i f''(E)) / v_m$ of the compound are calculated.

2.11. Resolution Function

The resolution function is taken from [167].

$$\langle k \rangle = 2\pi/\lambda \quad (2.18)$$

$$\langle \theta \rangle = \arcsin(\langle Q_{av} / (2\langle k \rangle) \rangle) \quad (2.19)$$

$$a_1 = \frac{r_1}{L + l/\cos^2(2\langle \theta \rangle)} \quad (2.20)$$

$$a_2 = r_2 \cos^2(2\langle \theta \rangle)/l \quad (2.21)$$

$$\Delta\beta_1 = \begin{cases} a_1 \geq a_2 : & \frac{2r_1}{L} - \frac{r_2^2}{2r_1} \frac{\cos^4(2\langle \theta \rangle)}{l^2 L} \left(L + \frac{l}{\cos^2(2\langle \theta \rangle)} \right)^2 \\ a_1 < a_2 : & 2r_2 \left(\frac{1}{L} + \frac{\cos^2(2\langle \theta \rangle)}{l} \right) - \frac{r_1^2}{2r_2} \frac{l}{L} \\ & \times \frac{1}{\cos^2(2\langle \theta \rangle) \left(L + \frac{l}{\cos^2(2\langle \theta \rangle)} \right)} \end{cases} \quad (2.22)$$

$$\sigma_W = \langle Q \rangle \frac{\Delta\lambda}{\lambda} \frac{1}{2\sqrt{2\ln(2)}} \quad (2.23)$$

$$\sigma_{C1} = \langle k \rangle \cos(\langle \theta \rangle) \frac{\Delta\beta_1}{2\sqrt{2\ln(2)}} \quad (2.24)$$

$$\sigma_{D1} = \langle k \rangle \cos(\langle \theta \rangle) \cos^2(2\langle \theta \rangle) \frac{D}{l 2\sqrt{2\ln(2)}} \quad (2.25)$$

$$\sigma_{av} = \langle k \rangle \cos(\langle \theta \rangle) \cos^2(2\langle \theta \rangle) \frac{\Delta D}{l 2\sqrt{2\ln(2)}} \quad (2.26)$$

$$\sigma = \sqrt{\sigma_W^2 + \sigma_{C1}^2 + \sigma_{D1}^2 + \sigma_{av}^2} \quad (2.27)$$

$$R_{av}(Q, \langle Q \rangle) = \frac{Q}{\sigma^2} \exp\left(-\frac{1}{2} (Q^2 + \langle Q \rangle^2) / \sigma^2\right) I_0(Q\langle Q \rangle / \sigma^2) \quad (2.28)$$

$$I(\langle Q \rangle) = \int_0^\infty R_{av}(Q, \langle Q \rangle) \frac{d\sigma}{d\Omega}(Q) dQ \quad (2.29)$$

$$\frac{d\sigma}{d\Omega}(Q) = \int_0^\infty N(R) F^2(Q, R) dR \quad (2.30)$$

CHAPTER 3

Form Factors

3.1. Spheres & Shells

3.1.1. Sphere.

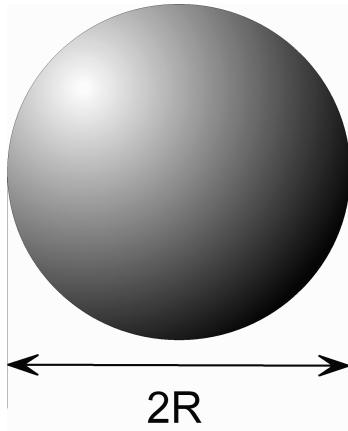


FIGURE 3.1. Sphere with diameter $2R$

The scattering intensity and scattering amplitude of a homogeneous sphere with a small relative refraction index is given by [179]

$$I_{\text{Sphere}}(Q, R) = K^2(Q, R, \Delta\eta) \quad (3.1a)$$

with

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.1b)$$

The forward scattering for $Q = 0$ is given by

$$\lim_{Q=0} I_{\text{Sphere}}(Q, R) = \left(\frac{4}{3}\pi R^3 \Delta\eta\right)^2$$

Input Parameters for model Sphere:

R: radius of sphere R

- - -: not used

- - -: not used

eta: scattering length density difference between particle and matrix $\Delta\eta$

Note:

- The parameters `param.p[1]` and `param.p[2]` are not used.

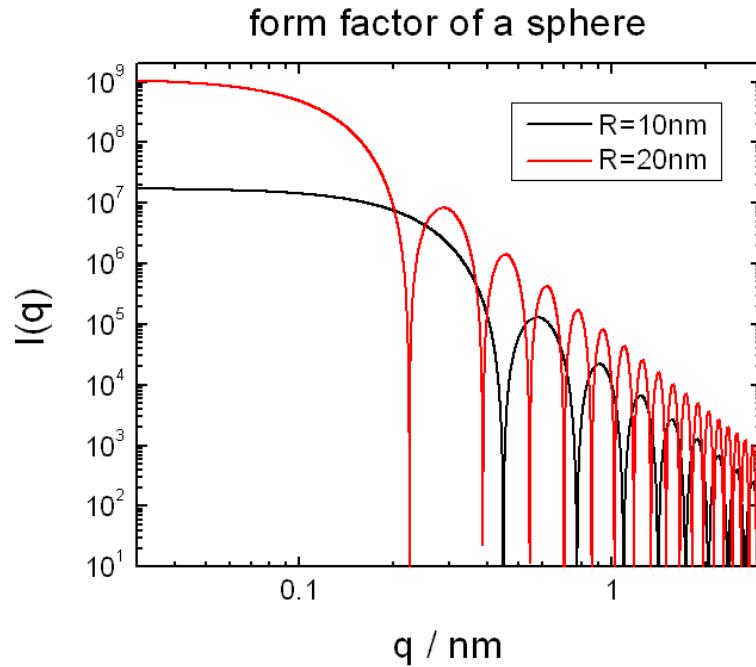


FIGURE 3.2. Scattering intensity of spheres with radii $R = 10\text{nm}$ and $R = 20\text{nm}$. The scattering length density contrast is set to 1.

3.1.2. Spherical Shell i.



FIGURE 3.3. Spherical Shell i

This implementation of a spherical shell is parametrised with an inner radius R_2 and outer radius R_1 . The scattering contrast relative to the matrix of the core is $\mu\Delta\eta$ and the one of the shell $\Delta\eta$.

$$I_{\text{Shell1}}(Q, R_1, R_2, \Delta\eta, \mu) = [K(Q, R_1, \Delta\eta) - K(Q, R_2, \Delta\eta(1 - \mu))]^2 \quad (3.2)$$

with

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta^3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.3)$$

The forward scattering for $Q = 0$ is given by

$$\lim_{Q=0} I_{\text{Shell1}}(Q, R_1, R_2, \Delta\eta, \mu) = \left(\frac{4}{3}\pi \Delta\eta [R_1^3 - R_2^3(1 - \mu)] \right)^2$$

Input Parameters for model Spherical Shell i:

R1: overall radius of spherical shell R_1

R2: radius of core R_2

eta: scattering length density difference between shell and matrix $\Delta\eta$

mu: scattering length density difference between core and matrix relative to the shell contrast μ

Note:

None

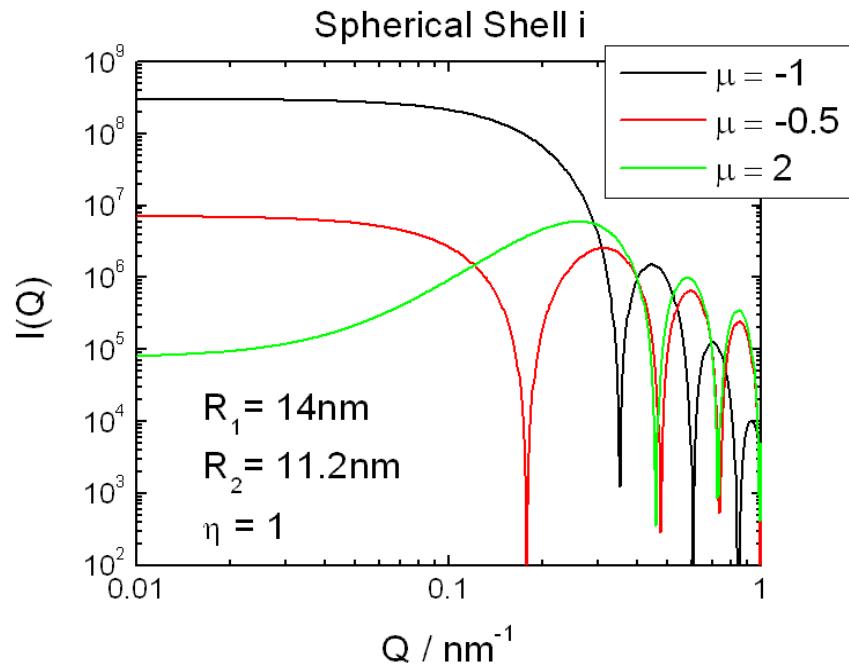


FIGURE 3.4. Scattering intensity of spherical shell with outer radius of $R_1 = 14\text{nm}$ and inner radius of $R_2 = 11.2\text{nm}$. The scattering length density contrast the shell is set to 1 and the one of the core to -1, -0.5, and 2.

3.1.3. Spherical Shell ii.



FIGURE 3.5. Spherical Shell ii

This implementation of a spherical shell is parametrised with an outer radius R and an inner radius νR . The scattering contrast relative to the matrix of the core is $\mu\Delta\eta$ and the one of the shell $\Delta\eta$.

$$I_{\text{Shell2}}(Q, R, \nu, \Delta\eta, \mu) = (K(Q, R, \Delta\eta) - K(Q, \nu R, \Delta\eta(1 - \mu)))^2 \quad (3.4)$$

with

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.5)$$

The forward scattering for $Q = 0$ is given by

$$\lim_{Q=0} I_{\text{Shell2}}(Q, R, R, \Delta\eta, \mu) = \left(\frac{4}{3}\pi \Delta\eta [R^3 - \nu^3 R^3 (1 - \mu)] \right)^2$$

Input Parameters for model Spherical Shell ii:

R: overall radius of spherical shell R

nu: the radius of the core is only the fraction ν of the overall radius R

eta: scattering length density difference between shell and matrix $\Delta\eta$

mu: scattering length density difference between core and matrix relative to the shell contrast μ

Note:

None



FIGURE 3.6. Scattering intensity of spherical shell with outer radius of $R = 14\text{nm}$ and inner radius of $\nu R = 11.2\text{nm}$. The scattering length density contrast the shell is set to 1 and the one of the core to -1, -0.5, and 2.

3.1.4. Spherical Shell iii.



FIGURE 3.7. Spherical Shell iii

This implementation of a spherical shell is parametrised with an inner radius R and a shell thickness ΔR . The scattering contrast relative to the matrix of the core is $\Delta\eta_1$ and the one of the shell $\Delta\eta_2$.

$$I_{\text{Shell3}}(Q, R, \Delta R, \Delta\eta_1, \Delta\eta_2) = [K(Q, R + \Delta R, \Delta\eta_2) - K(Q, R, \Delta\eta_2 - \Delta\eta_1)]^2 \quad (3.6)$$

with

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.7)$$

The forward scattering for $Q = 0$ is given by

$$\lim_{Q=0} I_{\text{Shell3}}(Q, R, \Delta R, \Delta\eta_1, \Delta\eta_2) = \left(\frac{4}{3}\pi [(R + \Delta R)^3 \Delta\eta_2 - R^3 (\Delta\eta_2 - \Delta\eta_1)] \right)^2$$

Input Parameters for model Spherical Shell iii:

R: radius of core R

dR: thickness of the shell ΔR

eta1: scattering length density difference between core and matrix $\Delta\eta_1$

eta2: scattering length density difference between shell and matrix $\Delta\eta_2$

Note:

None

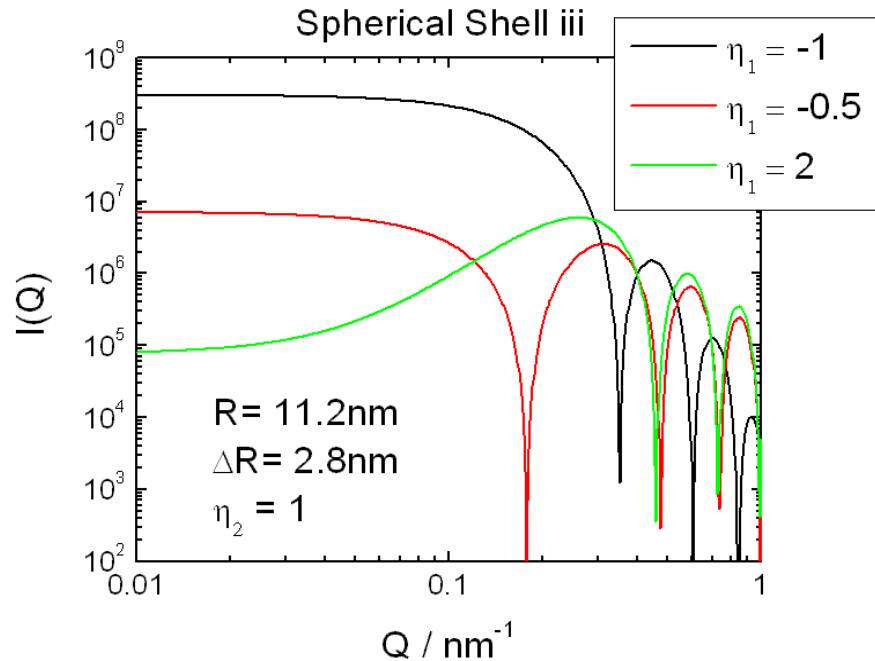


FIGURE 3.8. Scattering intensity of spherical shell with core radius of $R = 11.2\text{nm}$ and shell thickness of $\Delta R = 2.8\text{nm}$. The scattering length density contrast the shell is set to 1 and the one of the core to -1, -0.5, and 2.

3.1.5. Bilayered Vesicle.

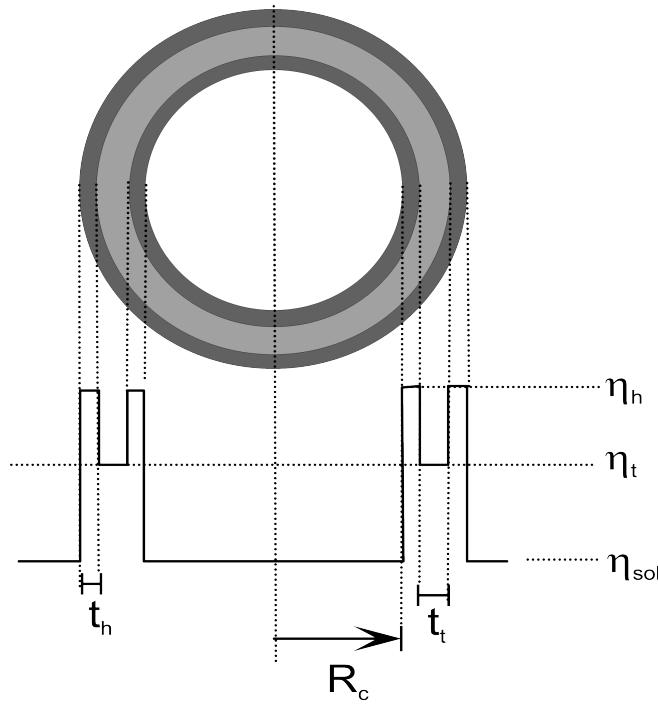


FIGURE 3.9. BiLayeredVesicle

$$I_{BLV}(Q) = \left(+ K(Q, R_c, \eta_{sol} - \eta_t) + K(Q, R_c + t_t, \eta_t - \eta_h) \right. \\ \left. + K(Q, R_c + t_t + t_h, \eta_h - \eta_t) + K(Q, R_c + 2t_t + t_h, \eta_t - \eta_{sol}) \right)^2 \quad (3.8)$$

with

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.9)$$

Input Parameters for model BilayeredVesicle:

R_c: radius of core R_c which consists of solvent

t_h: thickness of outer part of bilayer (in contact with solvent, head group) t_h

t_t: thickness of inner part of bilayer (tail group) t_t

eta_sol: scattering length density of solvent η_{sol}

eta_h: scattering length density of outer part of bilayer η_h

eta_t: scattering length density of inner part of bilayer η_t

Note:

None



FIGURE 3.10. Scattering intensity of a bilayered vesicle. The scattering intensity has been calculated with a lognormal [LogNorm($N = 1, \sigma = 0.05, p = 1, R = 30$)] size distribution for the vesicle radius R_c .

3.1.6. Multi Lamellar Vesicle.



FIGURE 3.11. MultiLamellarVesicle

$$I_{MLV}(Q) = \left(\sum_{i=0}^{n-1} \left[K(Q, R_c + it_{sh} + it_{sol}, \eta_{sol} - \eta_{sh}) \right. \right. \\ \left. \left. + K(Q, R_c + (i+1)t_{sh} + it_{sol}, \eta_{sh} - \eta_{sol}) \right] \right)^2 \quad (3.10)$$

with

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.11)$$

Input Parameters for model MultiLamellarVesicle:

R_c: radius of core R_c which consists of solvent

t_sh: surfactant layer thickness t_{sh}

t_sol: thickness of solvent layer t_{sol}

eta_sh: scattering length density of surfactant layer η_{sh}

eta_sol: scattering length density of solvent η_{sol}

n: total number of surfactant layers n

Note:

None



FIGURE 3.12. Scattering intensity of a multilamellar vesicle. The scattering intensities has been calculated for a&b) a distribution of the core radius R_c by $\int \text{LogNorm}(R_c; N = 1, \sigma = 0.3, p = 1, R = 10) I(q, R_c) dR_c$ and c) for a distribution of the distances between the lamellars $\int \text{LogNorm}(t_{\text{sol}}; N = 1, \sigma = 0.3, p = 1, R = 10) I(q, t_{\text{sol}}) dt_{\text{sol}}$.

3.1.7. RNDMultiLamellarVesicle.



FIGURE 3.13. randomMultiLamellarVesicle

$$\begin{aligned} I_{\text{RndMLV}}(Q) = & \Delta\eta^2 \sum_{i=1}^N F_i^2(q, R_i, t_{sh,i}) \\ & + \Delta\eta^2 \sum_{i < j}^N 2F_i(q, R_i, t_{sh,i})F_j(q, R_j, t_{sh,j}) \frac{\sin qr_{ij}}{qr_{ij}} \end{aligned} \quad (3.12)$$

with

$$r_{ij} = |\mathbf{R}_i - \mathbf{R}_j| \quad (3.13a)$$

$$F_i(q, R_i, t_{sol,i}) = K(q, R_i + t_{sol,i}, \Delta\eta) - K(q, R_i, \Delta\eta) \quad (3.13b)$$

$$K(q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin qR - qR \cos qR}{(qR)^3} \quad (3.13c)$$

$$R_1 = \text{ran}_{lognormal}(\log(R_c), \sigma_{R_c}) \quad (3.14a)$$

$$\Delta R_i = \text{ran}_{gaussian}(\sigma_{t_{sol}}) \quad (3.14b)$$

$$R_i = R_{i-1} + t_{sh,i-1} + \Delta R_i \quad (3.14c)$$

$$\mathbf{R}_i = R_i \text{ ran}_{dir,3D} \text{ ran}_{uniform} \Delta t_{sol} \quad (3.14d)$$

Input Parameters for model RNDMultiLamellarVesicle:

- t_{sh} : average surfactant layer thickness t_{sh}
- s_{sh} : Gaussian thickness distribution of surfactant layer with a width of σ_{sh}
- R_c : average radius of core R_c which consists of solvent
- s_c : lognormal size distribution of core radius R_c with a width of σ_c
- n : average number of surfactant layers n
- s_n : lognormal distribution of the number of surfactant layers with a width of σ_n
- t_{sol} : average thickness of solvent layer t_{sol}
- s_{sol} : lognormal thickness distribution of solvent layer with a width of σ_{sol}
- $\Delta\eta_{sh}$: scattering length density contrast $\Delta\eta$ between surfactant layer and solvent

Note:

The number of Monte Carlo iterations can be set via the menu [Options|Customize...]



FIGURE 3.14. Scattering intensity of a multilamellar vesicle where several distribution of parameters within a single vesicle are calculated via a Monte Carlo algorithm. .

3.1.8. Vesicle with aligned flat capped ends [108, 109].



FIGURE 3.15. Sketch of a vesicle with horizontally aligned flat capped ends perpendicular to the incoming neutron beam

The shape of this form factor consist of spherical vesicle containing two flat domains. The flat thought to be aligned in a horizontal magnetic field perpendicular to the incoming neutron beam. The size of the domains are characterized by the angles θ_1 and θ_2 . The thicknesses t_{c1} and t_{c2} can be different than the thickness t_{sh} of the spherical part of the vesicles. The same hold for the scattering length densities η_{c1} , η_{c2} and η_{sh} . The form factor $F_{cv}(Q)$ of this object can be calculated by performing the Fourier transformation of the scattering length density in separate steps. First one calculates the Fourier transformation of a sphere F_{cSph} with flat capped ends on each side in cylinder coordinates.

$$F_{cSph}(Q, R, \psi, \theta_1, \theta_2, \Delta\eta) = \Delta\eta \int_{-R\cos\theta_2}^{R\cos\theta_1} dz \int_0^{\sqrt{R^2-z^2}} d\rho \int_0^{2\pi} d\phi e^{i\mathbf{Q}\cdot\mathbf{r}} \rho \quad (3.15a)$$

$$\text{with } \mathbf{Q} = Q \begin{pmatrix} 0 \\ \sin\psi \\ \cos\psi \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} \rho \cos\phi \\ \rho \sin\phi \\ z \end{pmatrix} \quad (3.15b)$$

The form factor of vesicle $F_{cv}(Q)$ with a layer thickness of t_{sh} can than be calculated by

$$F_{cv}(Q, R, t_{sh}, \theta_1, \theta_2, \Delta\eta_{sh}) = + F_{cSph}(Q, R + t_{sh}, \Theta_1, \Theta_2, \Delta\eta_{sh}) - F_{cSph}(Q, R, \theta_1, \theta_2, \Delta\eta_{sh}) \quad (3.16a)$$

with

$$\Theta_1 = \arcsin \left(\frac{R_{c1}}{R + t_{sh}} \right), \quad R_{c1} = R \sin(\theta_1), \quad (3.16b)$$

$$\Theta_2 = \arcsin \left(\frac{R_{c2}}{R + t_{sh}} \right), \quad R_{c2} = R \sin(\theta_2). \quad (3.16c)$$

As the flat capped ends are allowed to have independent thicknesses t_{c1}, t_{c2} and scattering length densities η_1, η_2 the scattering amplitude contribution of the flat capped ends, which have the shape of a disc, need to be corrected. Their contribution can be calculated by

$$\begin{aligned} F_c(Q, R, \theta_1, \theta_2, \dots) &= F_{c1}(Q, R, \theta_1, \Delta\eta_{c1}) - F_{d1}(Q, R, t_{d1}, \Delta\eta_{sh}) \\ &\quad + F_{c2}(Q, R, \theta_2, \Delta\eta_{c2}) - F_{d2}(Q, R, t_{d2}, \Delta\eta_{sh}) \\ &= \Delta\eta_{c1} \int_{l_{c1}}^{l_{c1}+t_{c1}} dz \int_0^{R_{c1}} d\rho \int_0^{2\pi} d\phi e^{i\mathbf{Q}\cdot\mathbf{r}} \rho - \Delta\eta_{sh} \int_{l_{c1}}^{l_{c1}+t_{d1}} dz \int_0^{R_{c1}} d\rho \int_0^{2\pi} d\phi e^{i\mathbf{Q}\cdot\mathbf{r}} \rho \\ &\quad + \Delta\eta_{c2} \int_{-(l_{c2}+t_{c2})}^{-l_{c2}} dz \int_0^{R_{c2}} d\rho \int_0^{2\pi} d\phi e^{i\mathbf{Q}\cdot\mathbf{r}} \rho - \Delta\eta_{sh} \int_{-(l_{c2}+t_{d2})}^{-l_{c2}} dz \int_0^{R_{c2}} d\rho \int_0^{2\pi} d\phi e^{i\mathbf{Q}\cdot\mathbf{r}} \rho \end{aligned} \quad (3.17)$$

with

$$\Delta\eta_{sh} = \eta_{sh} - \eta_{sol}, \quad \Delta\eta_{c1} = \eta_1 - \eta_{sol}, \quad \Delta\eta_{c2} = \eta_2 - \eta_{sol} \quad (3.18a)$$

$$l_{c1} = R \cos \theta_1, \quad l_{c2} = R \cos \theta_2 \quad (3.18b)$$

$$R_{c1} = R \sin \theta_1, \quad R_{c2} = R \sin \theta_2 \quad (3.18c)$$

$$t_{d1} = \sqrt{(R + t_{sh})^2 - R_{c1}^2} - \sqrt{R^2 - R_{c1}^2} \quad (3.18d)$$

$$t_{d2} = \sqrt{(R + t_{sh})^2 - R_{c2}^2} - \sqrt{R^2 - R_{c2}^2} \quad (3.18e)$$

$$(3.18f)$$

The solution of the integrals in eq. 3.15a and 3.17 are

$$\begin{aligned} F_{cSph}(Q, R, \psi, \theta_1, \theta_2, \Delta\eta) &= \Delta\eta \int_{-R \cos \theta_2}^{R \cos \theta_1} dz \int_0^{\sqrt{R^2 - z^2}} d\rho \int_0^{2\pi} d\phi e^{i\mathbf{Q}\cdot\mathbf{r}} \rho \\ &\quad \Delta\eta \int_{-R \cos \theta_2}^{R \cos \theta_1} dz \exp(iQz \cos \psi) 2\pi (R^2 - z^2) \frac{J_1(Q\sqrt{R^2 - z^2} \sin \psi)}{Q\sqrt{R^2 - z^2} \sin \psi} \end{aligned} \quad (3.19a)$$

and

$$\begin{aligned} F_{c_i,d_i}(Q, R_{c_i,d_i}, \psi, \Delta\eta) &= \Delta\eta \int_a^b dz \int_0^{R_{c_i}} d\rho \int_0^{2\pi} d\phi \ e^{i\mathbf{Q}\cdot\mathbf{r}} \rho \\ &= 4\pi R \frac{i(\exp(i\alpha Q \cos \psi) - \exp(ibQ \cos \psi)) J_1(QR \sin \psi)}{\sin(2\psi) Q^2} \end{aligned} \quad (3.19b)$$

whereby J_1 the regular cylindrical Bessel function of first order. The overall scattering intensity $I_{\text{alignedVes}}(Q, \psi, \dots)$ is finally given by

$$I_{\text{alignedVes}}(Q, \psi, \dots) = |F_{\text{cv}}(Q, R, \psi, t_{\text{sh}}, \theta_1, \theta_2, \Delta\eta_{\text{sh}}) + F_c(Q, R, \psi, \theta_1, \theta_2, \dots)|^2 \quad (3.20)$$

Input Parameters for the models of MagneticFieldAlignedVesicle:

Rsh: radius of spherical vesicle shell
theta1: angle to describe size of first capped side
theta2: angle to describe size of second capped side
t_sh: thickness of spherical vesicle shell
t_c1: thickness of first flat capped side
t_c2: thickness of second flat capped side
eta_sh: scattering length density of spherical vesicle shell
eta_1: scattering length density of first capped side
eta_2: scattering length density of second capped side
eta_sol: scattering length density of solvent

Note:

None

3.2. Polymers and Micelles

3.2.1. Gaussian chain.

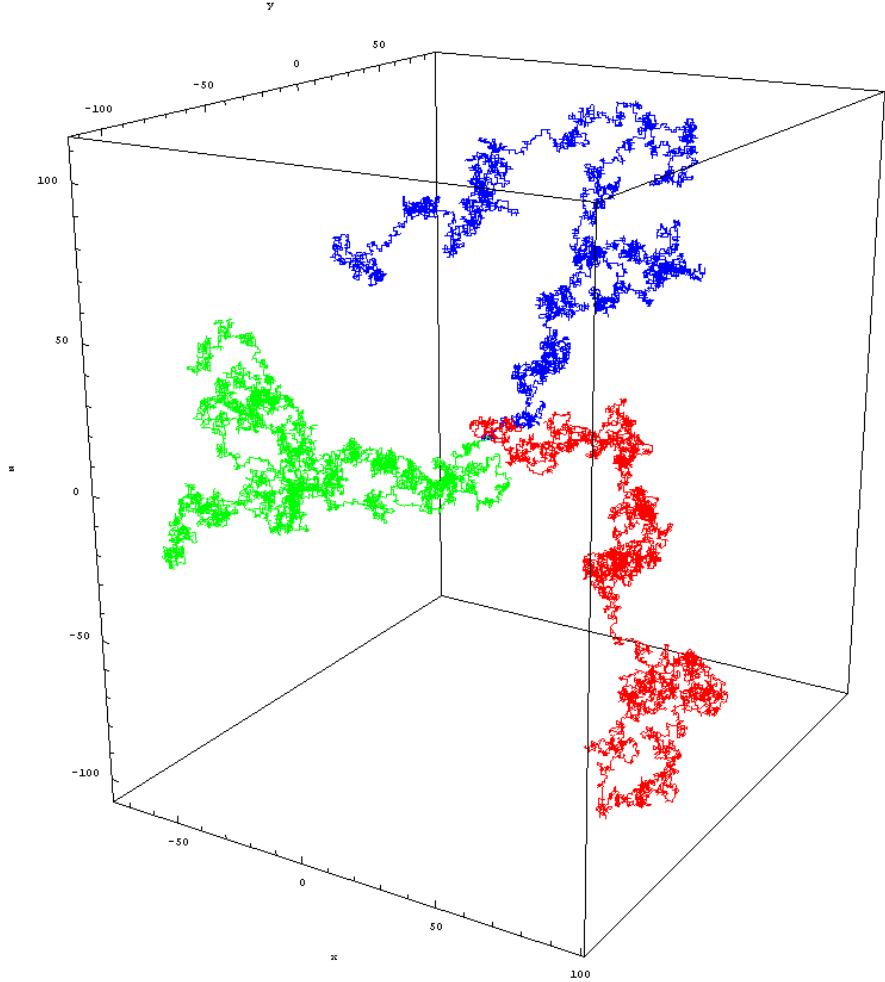


FIGURE 3.16. The underlying model for a polymer chain is an isotropic random walk on the Euclidean lattice \mathbb{Z}^3 . This picture shows three different walks after 10 000 unit steps, all three starting from the origin.

Consider a flexible polymer coil where each monomer located at a distance \mathbf{R}_m its scattering field amplitude is given by

$$F(\mathbf{q}, t) = \sum_{m=1}^N e^{-i\mathbf{q} \cdot \mathbf{R}_m(t)}. \quad (3.21)$$

The scattering intensity averaged over all molecule configurations reads

$$\langle |F(\mathbf{q})|^2 \rangle = \sum_{m,n} \langle e^{-i\mathbf{q} \cdot (\mathbf{R}_m - \mathbf{R}_n)} \rangle \quad (3.22)$$

As the monomer segments $\mathbf{R}_m - \mathbf{R}_n$ are Gaussian distributed the averages $\langle \dots \rangle$ can be written as

$$\langle e^{-\imath \mathbf{q} \cdot (\mathbf{R}_m - \mathbf{R}_n)} \rangle = e^{\frac{q^2}{6} \langle (\mathbf{R}_m - \mathbf{R}_n)^2 \rangle} \quad (3.23a)$$

$$= e^{-\frac{q^2 b^2}{6} |m-n|^{2\nu}} \quad (3.23b)$$

Here b is the statistical segment length and the contour length L equals $L = Nb$. The average of the segment inter-distances squares is kept in the general form

$$\langle (\mathbf{R}_m - \mathbf{R}_n)^2 \rangle = b^2 |m - n|^{2\nu}. \quad (3.24)$$

ν is the excluded volume parameter from the Flory mean field theory¹² of polymer solutions. The radius of gyration R_G is given by

$$R_G^2 = \frac{1}{2N^2} \sum_{m,n}^N \langle (\mathbf{R}_m - \mathbf{R}_n)^2 \rangle \quad (3.25a)$$

$$= \frac{1}{2N^2} \sum_{m,n}^N b^2 |m - n|^{2\nu} \quad (3.25b)$$

$$= \frac{b^2}{N} \sum_k^N \left(1 - \frac{k}{N}\right) k^{2\nu} \quad (3.25c)$$

$$= \frac{b^2}{(2\nu + 1)(2\nu + 2)} N^{2\nu} \quad (3.25d)$$

Three cases are relevant:

- (1) Self-avoiding walk corresponds to swollen chains with $\nu = 3/5$, for which $R_G^2 = \frac{25}{176} b^2 N^{6/5}$.
- (2) Pure random walk corresponds to chains in Θ -conditions (where solvent-solvent, monomer-monomer and solvent-monomer interactions are equivalent) with $\nu = 1/2$, for which $R_G^2 = \frac{1}{6} b^2 N$.
- (3) Self attracting walk corresponds to collapsed chains with $\nu = 1/3$, for which $R_G^2 = \frac{9}{40} b^2 N^{2/3}$.

Using the general identity

$$\sum_{i,j}^N y(|i - j|) = N + 2 \sum_{k=1}^N (N - k) y(k) \quad (3.26)$$

the form factor reads

$$P(q) = \frac{1}{N^2} |F(q)|^2 = \frac{1}{N^2} \left\{ N + 2 \sum_{k=1}^N (N - k) e^{-\frac{q^2 b^2}{6} k^{2\nu}} \right\} \quad (3.27)$$

¹P.J. Flory, "Statistical Mechanics of Chain Molecules", Interscience Publishers (1969)

²Boualem Hammouda, [the_SANS_toolbox.pdf](#)

Going to the continuous limit ($N \gg 1$), one obtains:

$$P(q) = 2 \int_0^1 dx (1-x) e^{-\frac{q^2 b^2}{6} N^{2\nu} x^{2\nu}} \quad (3.28a)$$

$$= \frac{U^{\frac{1}{2\nu}} \Gamma(\frac{1}{2\nu}) - \Gamma(\frac{1}{\nu}) - U^{\frac{1}{2\nu}} \Gamma(\frac{1}{2\nu}, U) + \Gamma(\frac{1}{\nu}, U)}{\nu U^{1/\nu}} \quad (3.28b)$$

$$= \frac{1}{\nu U^{\frac{1}{2\nu}}} \gamma\left(\frac{1}{2\nu}, U\right) - \frac{1}{\nu U^{\frac{1}{\nu}}} \gamma\left(\frac{1}{\nu}, U\right) \quad (3.28c)$$

with the modified variable

$$U = \frac{q^2 b^2 N^{2\nu}}{6} = (2\nu + 1)(2\nu + 2) \frac{q^2 R_G^2}{6} \quad (3.29)$$

and the upper incomplete Gamma Function $\Gamma(a, x) = \int_x^\infty dt t^{a-1} \exp(-t)$ and lower incomplete Gamma Function $\gamma(a, x) = \int_0^x dt t^{a-1} \exp(-t)$ for a real and $x \geq 0$ and the Gamma function $\Gamma(a) = \Gamma(a, 0) = \gamma(a, \infty) = \Gamma(a, x) + \gamma(a, x) = \int_0^\infty dt t^{a-1} \exp(-t)$. Polymer chains follow Gaussian statistics in polymer solutions: they are swollen in good solvents $\nu = 3/5$, are thermally relaxed in "theta"-solvents $\nu = 1/2$ and partially precipitate in poor solvents $\nu = 1/3$. The familiar Debye function is recovered when $\nu = 1/2$. The asymptotic limit at large q -values of the generalized Gaussian chain is dominated by the $\frac{1}{\nu U^{\frac{1}{2\nu}}} \Gamma(\frac{1}{2\nu})$ term which varies like $U^{-1/(2\nu)} \sim q^{-1/\nu}$. For $\nu = 1$ we get the limit of an infinitesimal thin rod and for $\nu = 1/4$ a compact object with a Porod law of q^{-4} .

SASfit has implemented the generalized form of a Gaussian (**generalized Gaussian coil**) coil and the standard Debye formula **Gauss**. In both cases three version are implemented which only differ in their parametrization of the forward scattering. In case of the Debye-formula also the polydisperse **GaussPoly** is implemented.

3.2.1.1. *Gauss* [43].

Flexible polymer chains which are not self-avoiding and obey Gaussian statistics. Debye (1947) has calculated the form factor of such chains:

$$I_{\text{Gauss}}(q) = I_0 2 \frac{\exp(-u) + u - 1}{u^2} \quad (3.30)$$

$$u = q^2 R_g^2 \quad (3.31)$$

Input Parameters for model Gauss:

Rg: radius of gyration R_g

I0: forward scattering I_0 for $q = 0$

3.2.1.2. *Gauss2* [43].

This form factor differs only by the parametrization for the forward scattering $I_0 = (b_p - V\eta_s)^2$ from the Debye formula in eq. 3.30

$$\begin{aligned} I_{\text{Gauss2}}(q) &= \beta^2 2 \frac{\exp(-u) + u - 1}{u^2} \\ u &= q^2 R_g^2 \\ \beta &= b_p - V\eta_s, \end{aligned} \quad (3.32)$$

where b_p is the scattering length of a polymer molecule of molecular volume V dissolved in a solvent of scattering length density η_s from which the excess scattering length of a polymer molecule β can be calculated. Combining this form factor with a **Delta** size distribution 6.1 is needed to scale the scattering intensity. With proper values for the form factor the parameter N of the **Delta**-distribution yields the particle number density.

Input Parameters for model **Gauss2**:

Rg: radius of gyration R_g

b_p: scattering length of polymer b_p in [cm]

V: molecular volume of a single polymer molecule V in [cm^3]

eta_s: scattering length density of solvent η_s in [cm^{-2}]

3.2.1.3. *Gauss3* [43].

This form factor differs only by the parametrization for the forward scattering $I_0 = (b_p - \frac{M_w}{N_a \rho_p} \eta_s)^2$ from the Debye formula in eq. 3.30

$$I_{\text{Gauss3}}(q) = \beta^2 2 \frac{\exp(-u) + u - 1}{u^2} \quad (3.33)$$

with

$$u = q^2 R_g^2$$

$$\beta = b_p - V\eta_s$$

$$V = \frac{M_w}{N_a \rho_p}$$

N_a = Avogadro number

Input Parameters for model **Gauss3**:

Rg: radius of gyration R_g

b_p: scattering length of polymer b_p in [cm]

M_w: molecular weight of polymer M_w in [g/mol]

rho_p: mass density of polymer ρ_p in [g cm^{-3}]

eta_s: scattering length density of solvent η_s in [cm^{-2}]

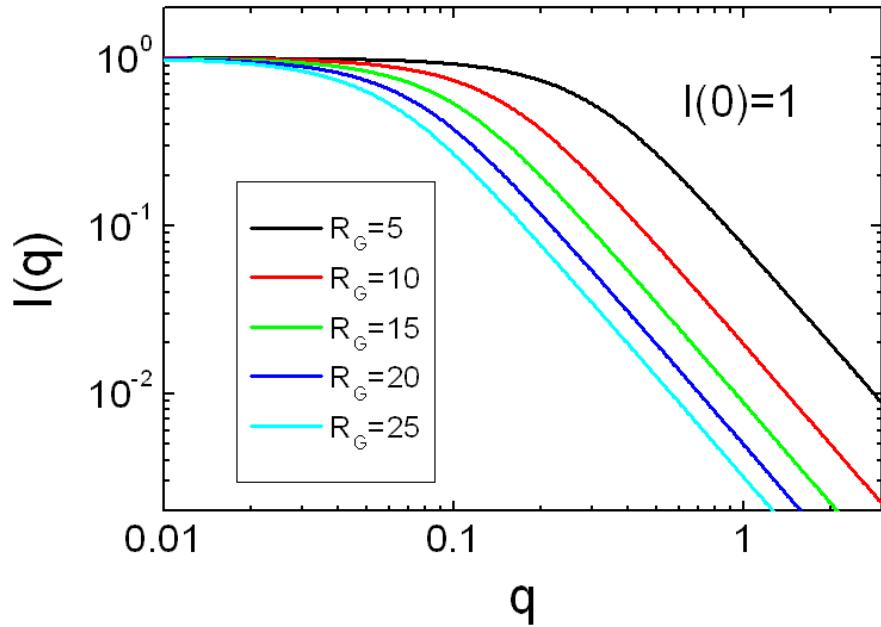


FIGURE 3.17. Scattering function of Gaussian coils plotted for several radii of gyration.

3.2.1.4. Polydisperse flexible polymers with Gaussian statistics [165].

Polydispersity has been included in terms of a SchulzZimm mass distribution by Zimm (1948) [236] and Greschner (1973) [68]

$$I_{\text{GaussPoly}}(q) = I_0 2 \frac{(1 + Ux)^{-1/U} + x - 1}{(1 + U)x^2} \quad (3.34)$$

$$x = q^2 R_g^2 / (1 + 2U)$$

$$U = \frac{M_w}{M_n} - 1$$

Input Parameters for model GaussPoly:

Rg: radius of gyration R_g

M_w: weight averaged molecular weight M_w

M_n: number averaged molecular weight M_n

I0: forward scattering I_0 for $q = 0$

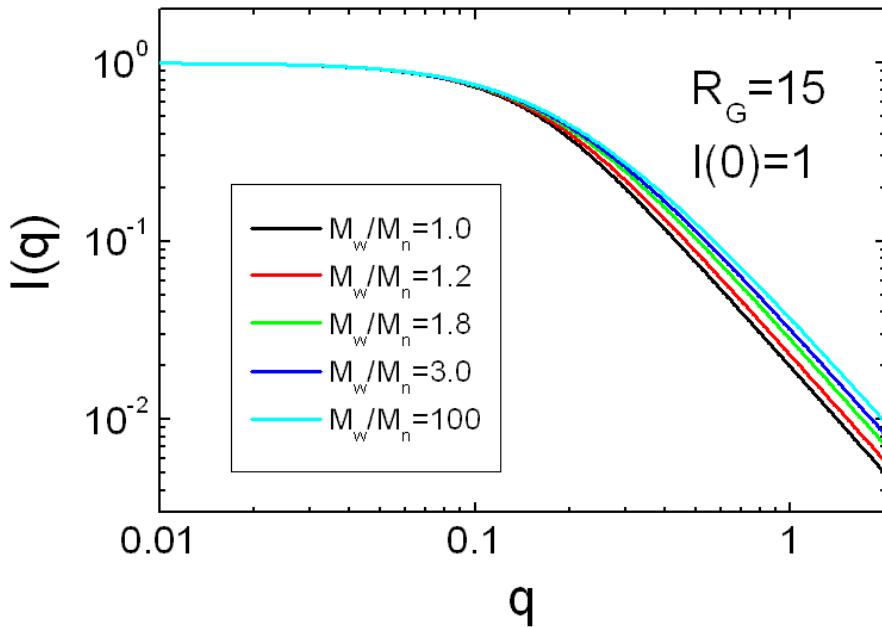


FIGURE 3.18. Scattering function of polydisperse Gaussian coil plotted for several ratios of M_w/M_n .

3.2.1.5. generalized Gaussian coil [80, 78, 76, 79].

The scattering function for the generalized Gaussian coil is according to eq. 3.28c

$$I_{\text{gGc}}(q) = I_0 \left(\frac{1}{\nu U^{\frac{1}{2\nu}}} \gamma \left(\frac{1}{2\nu}, U \right) - \frac{1}{\nu U^{\frac{1}{\nu}}} \gamma \left(\frac{1}{\nu}, U \right) \right) \quad (3.35)$$

with the modified variable

$$U = (2\nu + 1)(2\nu + 2) \frac{q^2 R_G^2}{6} \quad (3.36)$$

and the lower incomplete Gamma Function $\gamma(a, x) = \int_0^x dt t^{a-1} \exp(-t)$. ν is the excluded volume parameter from the Flory mean field theory and typical values for them are

- $\nu = 1/3$: partially precipitate in poor solvents
- $\nu = 1/2$: thermally relaxed in "theta"-solvents
- $\nu = 3/5$: swollen in good solvents

Input Parameters for model generalized Gaussian coil:

- Rg: radius of gyration R_g
- nu: excluded volume parameter $\nu \in [1/2; 1]$
- I0: forward scattering I_0 for $q = 0$

3.2.1.6. *generalized Gaussian coil 2* [80, 78].

The scattering function for the generalized Gaussian coil is according to eq. 3.28c and differs only by the parametrization for the forward scattering $I_0 = (b_p - V\eta_s)^2$ from the formula in eq. 3.35

$$I_{gGc2}(q) = (b_p - V\eta_s)^2 \left(\frac{1}{\nu U^{\frac{1}{2\nu}}} \gamma \left(\frac{1}{2\nu}, U \right) - \frac{1}{\nu U^{\frac{1}{\nu}}} \gamma \left(\frac{1}{\nu}, U \right) \right) \quad (3.37)$$

with the modified variable

$$U = (2\nu + 1)(2\nu + 2) \frac{q^2 R_G^2}{6} \quad (3.38)$$

Input Parameters for model **generalized Gaussian coil 2**:

Rg: radius of gyration R_g

b_p: scattering length of polymer b_p in [cm]

V: molecular volume of a single polymer molecule V in [cm^3]

eta_s: scattering length density of solvent η_s in [cm^{-2}]

3.2.1.7. *generalized Gaussian coil 3* [80, 78].

The scattering function for the generalized Gaussian coil is according to eq. 3.28c and differs only by the parametrization for the forward scattering $I_0 = (b_p - \frac{M_w}{N_a \rho_p} \eta_s)^2$ from the formula in eq. 3.35

$$I_{gGc3}(q) = \left(b_p - \frac{M_w}{N_a \rho_p} \eta_s \right)^2 \left(\frac{1}{\nu U^{\frac{1}{2\nu}}} \gamma \left(\frac{1}{2\nu}, U \right) - \frac{1}{\nu U^{\frac{1}{\nu}}} \gamma \left(\frac{1}{\nu}, U \right) \right) \quad (3.39)$$

with the modified variable

$$U = (2\nu + 1)(2\nu + 2) \frac{q^2 R_G^2}{6} \quad (3.40)$$

Input Parameters for model **generalized Gaussian coil 3**:

Rg: radius of gyration R_g

b_p: scattering length of polymer b_p in [cm]

M_w: molecular weight of polymer M_w in [g/mol]

rho_p: mass density of polymer ρ_p in [g cm^{-3}]

eta_s: scattering length density of solvent η_s in [cm^{-2}]

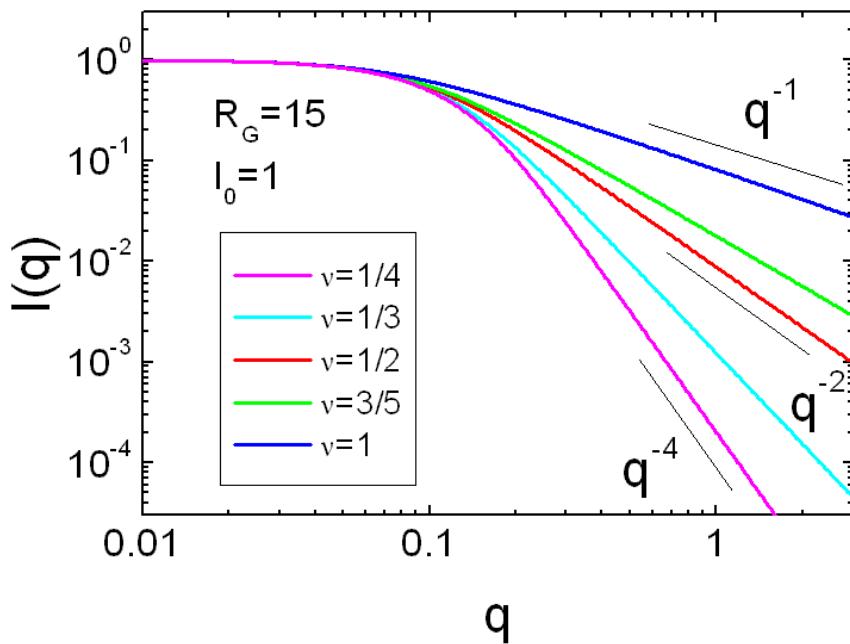


FIGURE 3.19. Scattering function of the generalized Gaussian coil plotted for several excluded volume parameters.

3.2.2. Star polymer with Gaussian statistic according to Benoit [13].

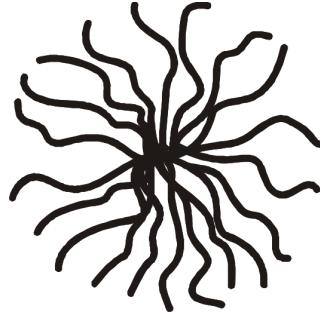


FIGURE 3.20. Sketch of a branched or star polymers with f number of arms

Benoit [13] derived an expression for the scattering from branched or star polymers with a number of arms f , which can be expressed in the following way:

$$I_{\text{Star}}(Q, R_G, f) = I_0 \frac{2}{f\nu^2} \left(\nu - [1 - e^{-\nu}] + \frac{f-1}{2} [1 - e^{-\nu}]^2 \right) \quad (3.41)$$

with $u = R_G^2 Q^2$, $\nu = \frac{uf}{3f-2}$ and $\lim_{Q=0} I_{\text{Star}}(Q, R_G, f) = I_0$. f denotes the number of arms and R_G the Guinier radius of a single arm.

Input Parameters for model Benoit:

RG: radius of gyration of the star polymer R_g

f: number of arms f

I0: forward scattering I_0 for $q = 0$

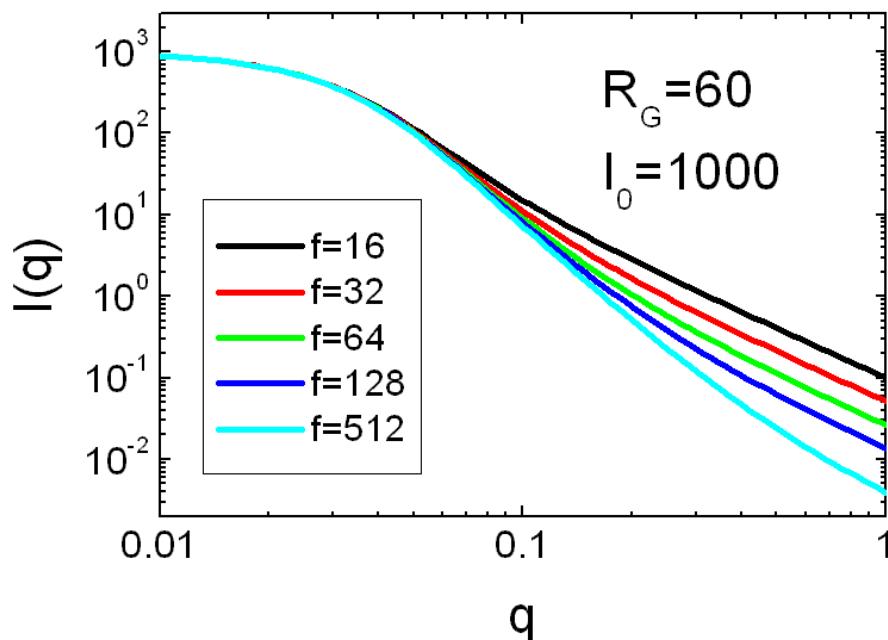


FIGURE 3.21. Scattering function of a star polymer according to Benoit.

3.2.3. Polydisperse star polymer with Gaussian statistics [28].

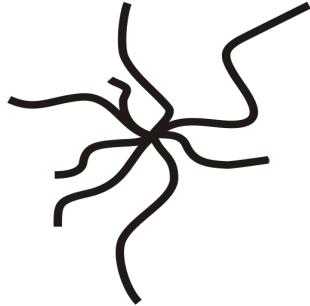


FIGURE 3.22. Polydisperse star polymer with Gaussian statistics

For a SchulzFlory (most probable) distribution (SchulzZimm distribution with $z = 1$) for the mass distribution of the arms, Burchard [28] has given the form factor:

$$I_{\text{PolydisperseStar}}(Q) = I_0 \frac{1 + \frac{u^2}{3f}}{\left(1 + \frac{u^2(f+1)}{6f}\right)^2} \quad (3.42)$$

where f is the number of arms and $u^2 = \langle R_g^2 \rangle_z Q^2$, where $\langle R_g^2 \rangle_z$ is the z -average radius of gyration squared of an arm.

Input Parameters for model PolydisperseStar:

R_G: radius of gyration R_G

f: number of arms f

I0: forward scattering I_0

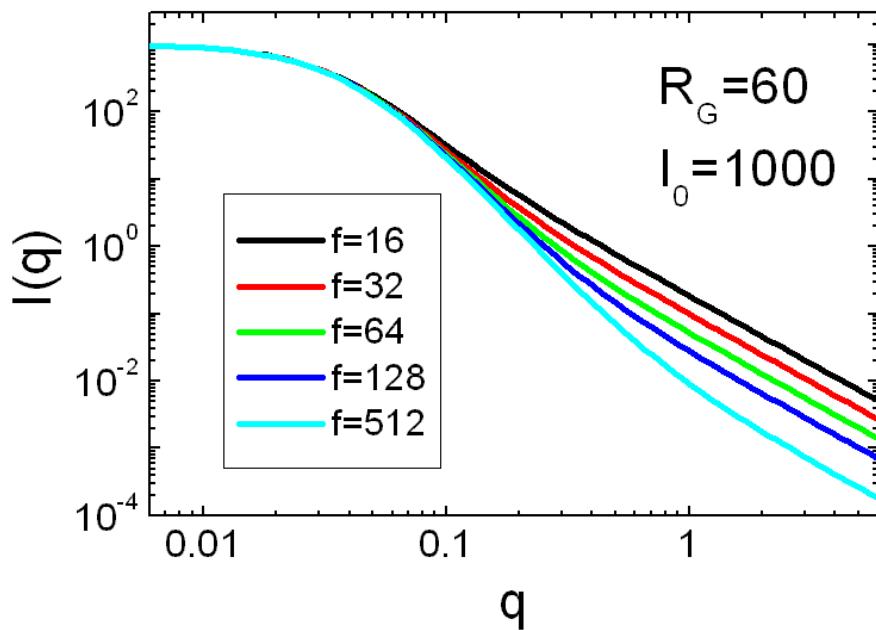


FIGURE 3.23. Scattering function of a polydisperse star polymer with Gaussian statistics.

3.2.4. Star polymer according to Dozier [47].

3.2.4.1. Dozier.

Branched polymers having all branches emanating from the center of the macromolecule

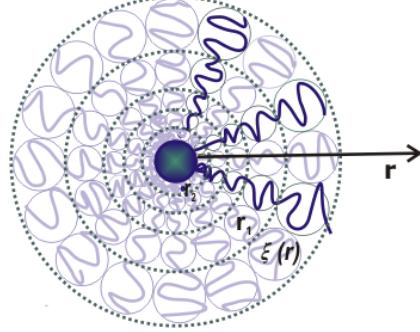


FIGURE 3.24. Star polymer according to Dozier

are commonly called star polymers. For a star polymer Dozier [47] has developed a scattering function which reads:

$$I_{\text{DozierStar}}(Q, I_0, R_G, \alpha, \nu, \xi) = I_0 \exp\left(-\frac{Q^2 R_G^2}{3}\right) + \frac{4\pi\alpha}{Q\xi} \Gamma(\mu) \frac{\sin(\mu \arctan(Q\xi))}{(1 + Q^2\xi^2)^{\mu/2}} \quad (3.43)$$

with $\mu = 1/\nu - 1$

R_G : radius of gyration

I_0 : scale parameter

α : scale parameter for fractal term

ξ : exponential damping length in mass fractal

ν : Flory exponent, 3/5 in good solvent, 1/2 in theta solvent (i.e. $\mu = 2/3$ to 1)

Input Parameters for model Dozier:

R_G: radius of gyration R_G

I_0: scale parameter I_0

alpha: scale parameter for fractal term α

xi: exponential damping length in mass fractal ξ

nu: excluded volume parameter or Flory exponent ν

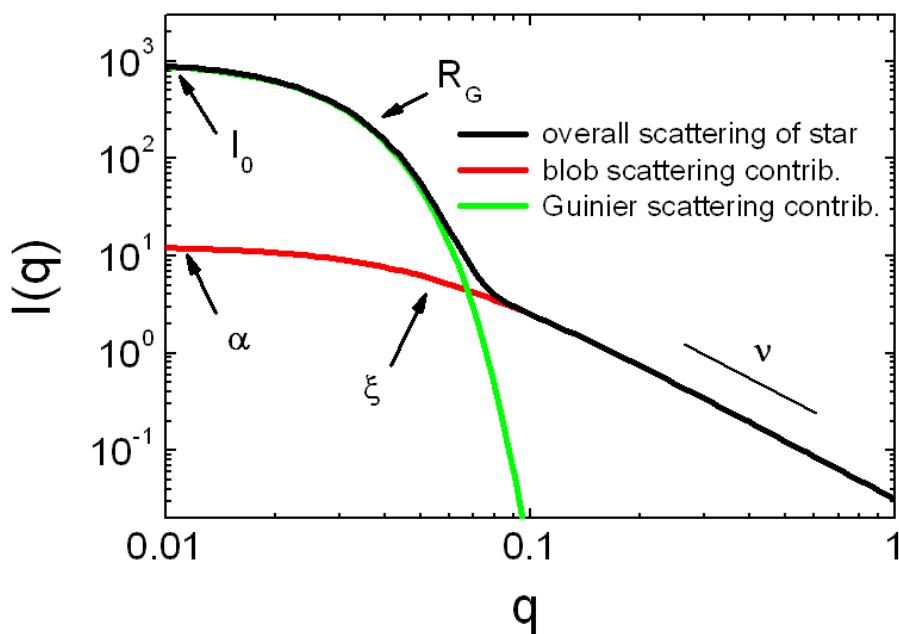


FIGURE 3.25. Scattering function of a star polymer according to Dozier:
 $I_0 = 10^3$, $R_g = 60$, $\alpha = 1$, $\xi = 20$, $\nu = 1/2$

3.2.4.2. *Dozier2*.

This is a re-parametrization of the **Dozier** form factor to scale the scattering of the overall star to the local scattering of the individual arms.

$$I_{\text{DozierStar2}}(Q, I_0, R_G, N_{\text{agg}}, \nu, \xi) = \frac{I_0}{N_{\text{agg}}} \left((N_{\text{agg}} - 1) \exp \left(-\frac{Q^2 R_G^2}{3} \right) + \frac{\Gamma(\mu)}{Q\xi} \frac{\sin(\mu \arctan(Q\xi))}{(1 + Q^2 \xi^2)^{\mu/2}} \right) \quad (3.44)$$

with $\mu = 1/\nu - 1$

R_G : radius of gyration of the star

I_0 : scale parameter

N_{agg} : number of arms in the star

ξ : exponential damping length in mass fractal

ν : Flory exponent, 3/5 in good solvent, 1/2 in theta solvent (i.e. $\mu = 2/3$ to 1)

Input Parameters for model **Dozier2**:

R_G: radius of gyration of the star R_G

I_0: scale parameter I_0

Nagg: number of arms N_{agg} in the star from which the scale parameter for fractal term is calculated

xi: exponential damping length in mass fractal ξ

nu: Flory exponent, $\nu = 3/5$ in good solvent, $\nu = 1/2$ in theta solvent (i.e. $\mu = 2/3$ to 1)

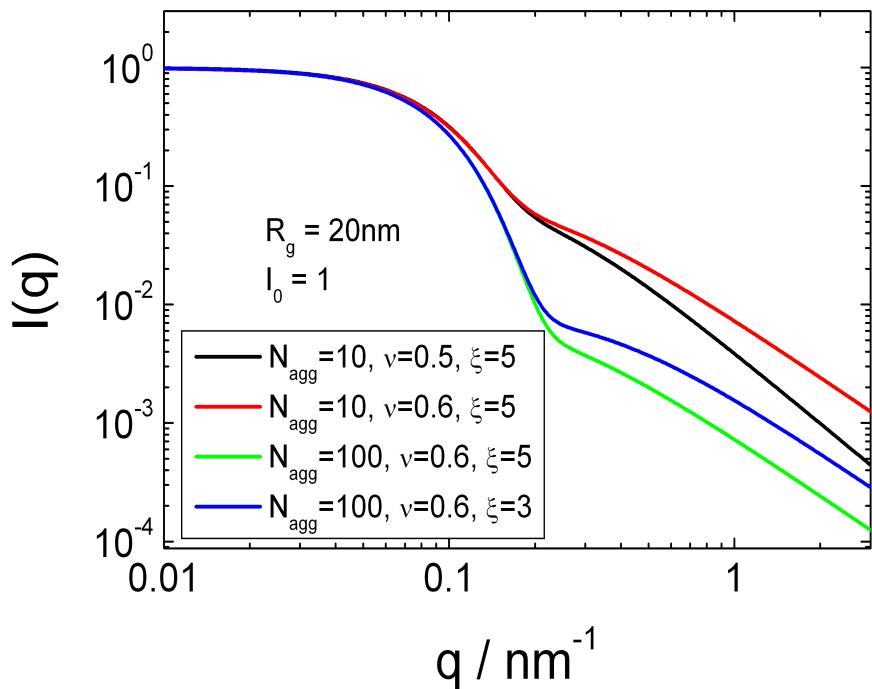


FIGURE 3.26. Scattering function of a star polymer according to Dozier but modified to scale the scattering of the overall star to the local scattering of the individual arms by the number of arms

3.2.5. Flexible Ring Polymer [30].

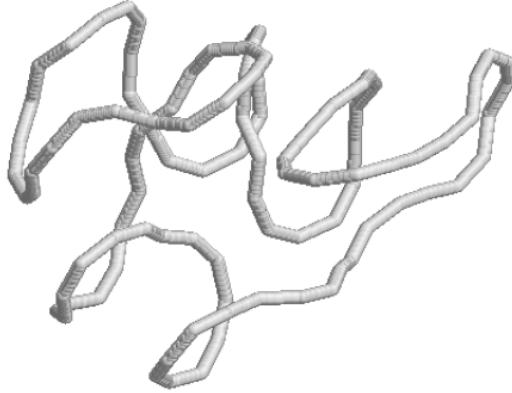


FIGURE 3.27. Sketch of a flexible ring polymer.

$$P_{1r}(q) = \sqrt{\frac{2}{u_{1r}^2}} D \left[\sqrt{\frac{u_{1r}^2}{2}} \right] \quad (3.45)$$

$$u_{1r}^2 = q^2 R_{g,1r}^2 \quad (3.46)$$

$$R_{g,1r}^2 = \sqrt{\frac{b^2 N}{12}} \quad (3.47)$$

$$D(X) = \exp(X^2) \int_0^X \exp(t^2) dt \quad (3.48)$$

Input Parameters for model **FlexibleRingPolymer**:

Rg: radius of gyration R_G

I0: forward scattering I_0

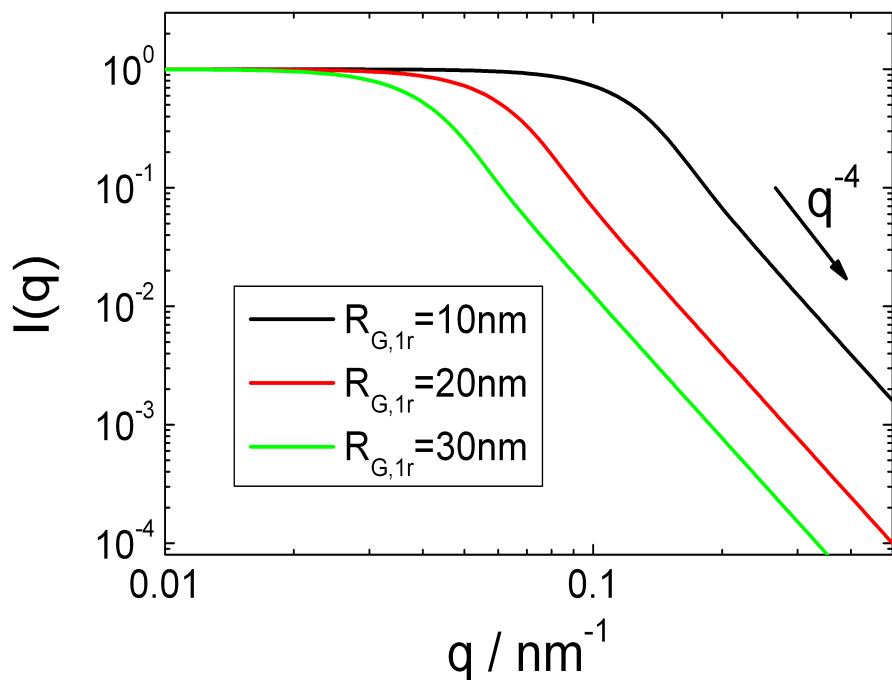


FIGURE 3.28. Scattering intensity of ring polymers of different radius of gyration.

3.2.6. *m*-membered twisted ring [30].

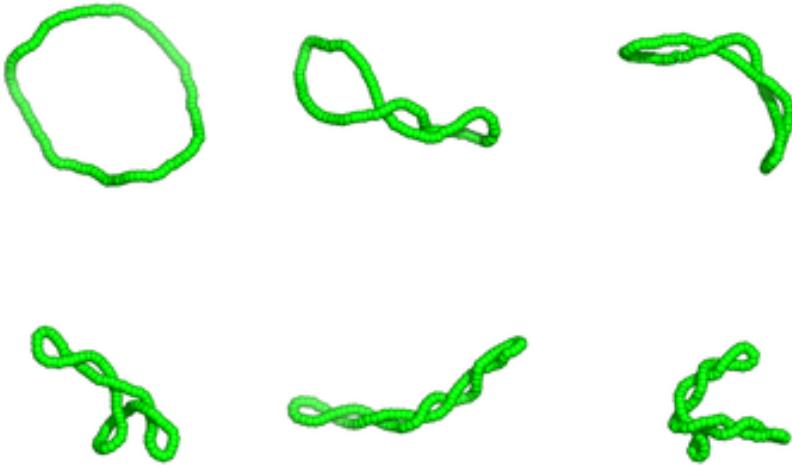


FIGURE 3.29. Sketch of ring polymers which different degree of twisting

$$P_{mr}(q) = I_0 \left(\frac{P_{1r}(q)}{m} + \frac{2}{m^2} P_{1r}^2(q) \sum_{j=1}^{m-1} (m-j) \exp\left(-\frac{q^2 R_{g,1r}^2}{2}(j-1)\right) \right) \quad (3.49)$$

$$P_{1r}(q) = \sqrt{\frac{2}{u_{1r}^2}} D \left[\sqrt{\frac{u_{1r}^2}{2}} \right] \quad (3.50)$$

$$u_{1r}^2 = q^2 R_{g,1r}^2 \quad (3.51)$$

$$R_{g,1r}^2 = \sqrt{\frac{b^2 N}{12}} \quad (3.52)$$

$$D(X) = \exp(X^2) \int_0^X \exp(t^2) dt \quad (3.53)$$

Input Parameters for model `mMemberedTwistedRing`:

`R_G,1r`: radius of gyration $R_{G,1r}$ of one of m loop

`m`: number of twists m

`I0`: forward scattering I_0

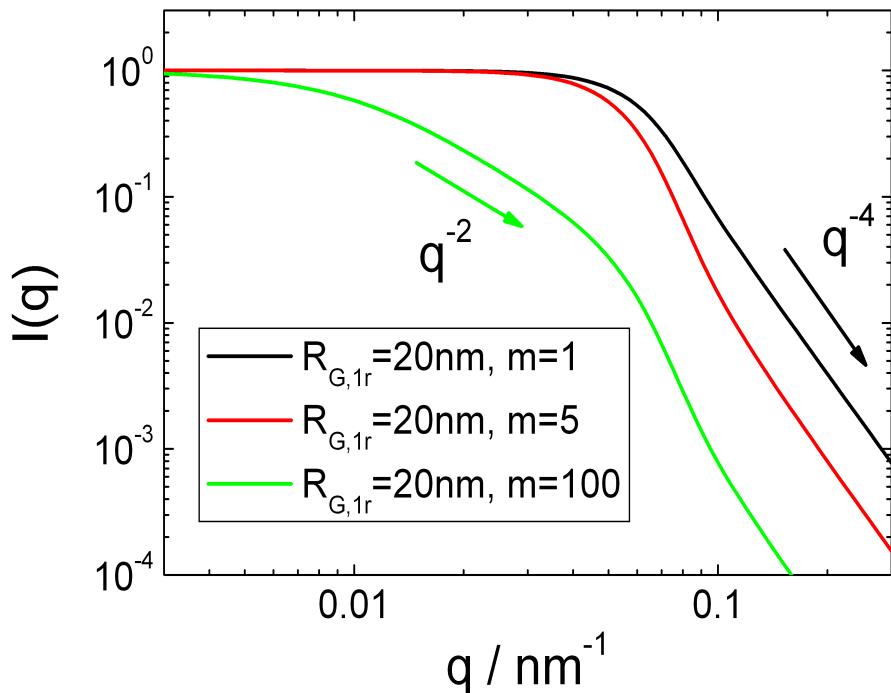


FIGURE 3.30. Scattering intensity of an m -membered twisted ring polymers with different values for m .

3.2.7. Daisy-like Ring [30].

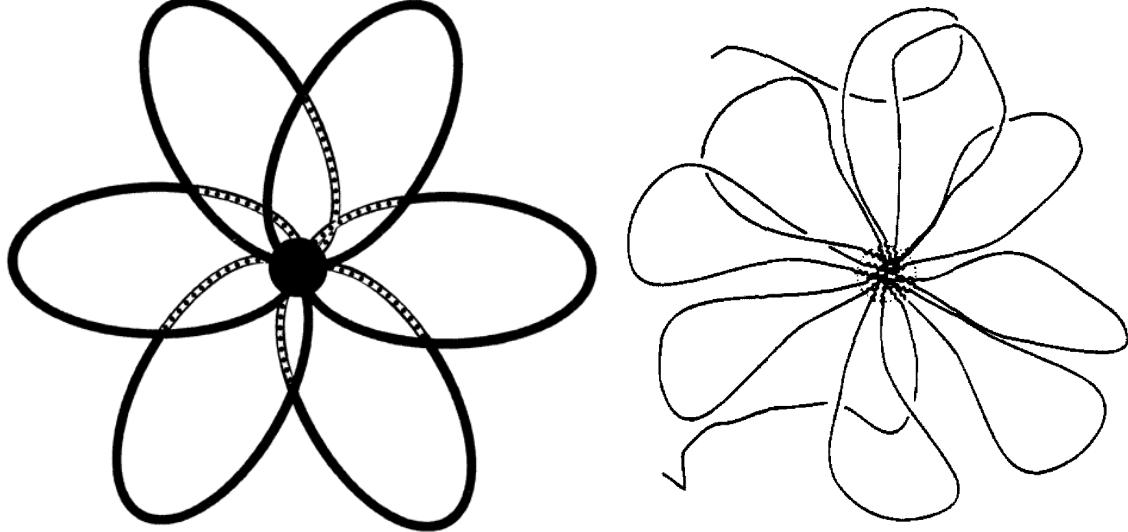


FIGURE 3.31. Sketch of a Daisy-like polymer.

$$P_{mr}(q) = \frac{I_0}{m} (P_{1r}(q) + (m-1)P_{1r}^2(q)) \quad (3.54)$$

$$P_{1r}(q) = \sqrt{\frac{2}{u_{1r}^2}} D \left[\sqrt{\frac{u_{1r}^2}{2}} \right] \quad (3.55)$$

$$u_{1r}^2 = q^2 R_{g,1r}^2 \quad (3.56)$$

$$R_{g,1r}^2 = \sqrt{\frac{b^2 N}{12}} \quad (3.57)$$

$$D(X) = \exp(X^2) \int_0^X \exp(t^2) dt \quad (3.58)$$

Input Parameters for model `DaisyLikeRing`:

`R_G,1r`: radius of gyration $R_{G,1r}$ of one of m loop

`m`: number of loops m

`I0`: forward scattering I_0

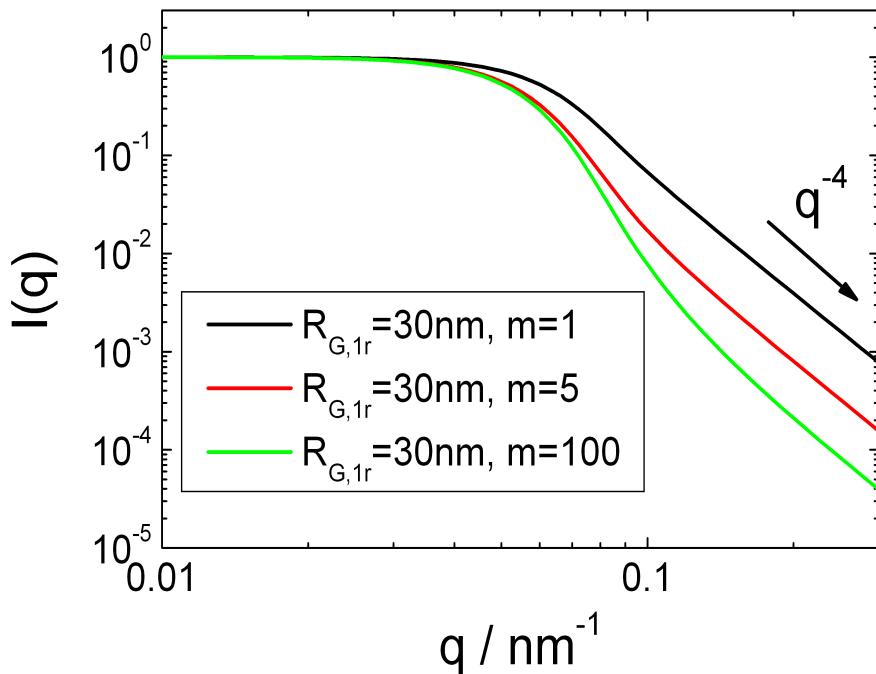


FIGURE 3.32. Scattering intensity of a Daisy-like ring polymers with different number of loops.

3.2.8. Unified Exponential Power Law according to Beaucage [10, 11].

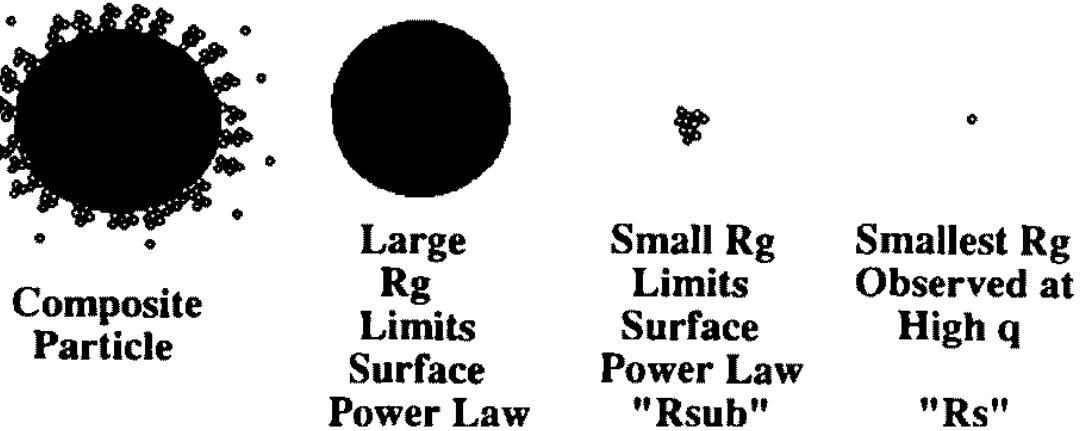


FIGURE 3.33. A typical case in which two R_g 's are observed. Particles composed of sub-particles where a radius of gyration for the entire particle, R_g , and a radius of gyration for the sub-particles, R_s , are observed. The surface-fractal cut-off radius of gyration, R_{sub} , differs from the high- Q radius of gyration, R_s , in this case. Generally, $R_s = R_{sub}$

3.2.8.1. Beaucage.

$$\begin{aligned}
 I_{\text{Beaucage}}(Q) \simeq & G \exp \left(-\frac{Q^2 R_g^2}{3} \right) \\
 & + B \exp \left(-\frac{Q^2 R_{sub}^2}{3} \right) \left(\frac{\left[\operatorname{erf}(QkR_g/\sqrt{6}) \right]^3}{Q} \right)^P \\
 & + G_s \exp \left(-\frac{Q^2 R_s^2}{3} \right) \\
 & + B_s \left(\frac{\left[\operatorname{erf}(Qk_s R_s/\sqrt{6}) \right]^3}{Q} \right)^{P_s}
 \end{aligned} \tag{3.59}$$

The first term in eq. 3.59 describes the large-scale structure of size R_g composed of small subunits of size R_s , captured in the third term. The second term describes the mass-fractal regime with two structural limits. The low- Q limit is at R_g and is described by the error function. The high- Q limit is at R_{sub} and is described by the exponential pre-factor [10]. The final two terms are for the sub-structural mer unit. Using eq. 3.59, scattering from a system with multiple-size-scale features is parameterized. Generally, the high- Q cutoff for the intermediate power law, R_{sub} , is identical to the sub-structural radius of gyration, R_s . The assumption that $R_{sub} = R_s$ should always be true for typical

mass fractals. It should be noted that, although eq. 3.59 appears cumbersome, no new parameters have been introduced over local fits using exponentials and power laws.

G is the Guinier pre-factor defined above and B is a pre-factor specific to the type of power-law scattering: B is defined according to the regime in which the exponent P falls. Generally, for surface fractals $4 > P > 3$, for mass fractals $P < 3$ and for diffuse interfaces $P > 4$. For Porod's law, $P = 4$ and $B = N_p 2\pi \rho_c^2 p S_p$, where S_p , is the particulate surface area. For a Gaussian polymer, $P = 2$, and B is given by $2G/R_g^2$, through a comparison with the Debye form factor 3.2.1 at the high- Q limit as discussed below. The constant, k in 3.59, accounts for an approximation involved in the description of the low- Q power-law limit [10]. This is an empirical constant that has a value of 1 for steep power-law decays, $P > 3$. For weak power-law decays, k deviates slightly from 1. For polymeric mass fractals of fractal dimension d_f close to 2 (1.5 to 3), k is empirically found to be close to 1.06. Weak deviations are observed between the scattered intensity as calculated using 3.59 and exact calculations for values of Q between $2\pi/R_g$ and π/R_g in these cases when $k = 1$. These deviations are reduced to less than 3% of the calculated intensity using $k = 1.06$.

Input Parameters for model Beaucage:

- G:** G is the Guinier pre-factor of the larger structure
- B:** B is a pre-factor specific to the type of power-law scattering: B is defined according to the regime in which the exponent P falls.
- Gs:** G_s is the Guinier pre-factor of the smaller structure
- Bs:** B_s is a pre-factor specific to the type of power-law scattering: B_s is defined according to the regime in which the exponent P_s falls.
- Rg:** large-scale structure
- Rsub:** surface-fractal cut-off radius of gyration, R_{sub} defines the high- Q cutoff for the intermediate power law
- Rs:** size R_s of small subunits
- P:** scaling exponent of the power law assigned to the larger structure R_g
- Ps:** scaling exponent of the power law assigned to the smaller structure R_s

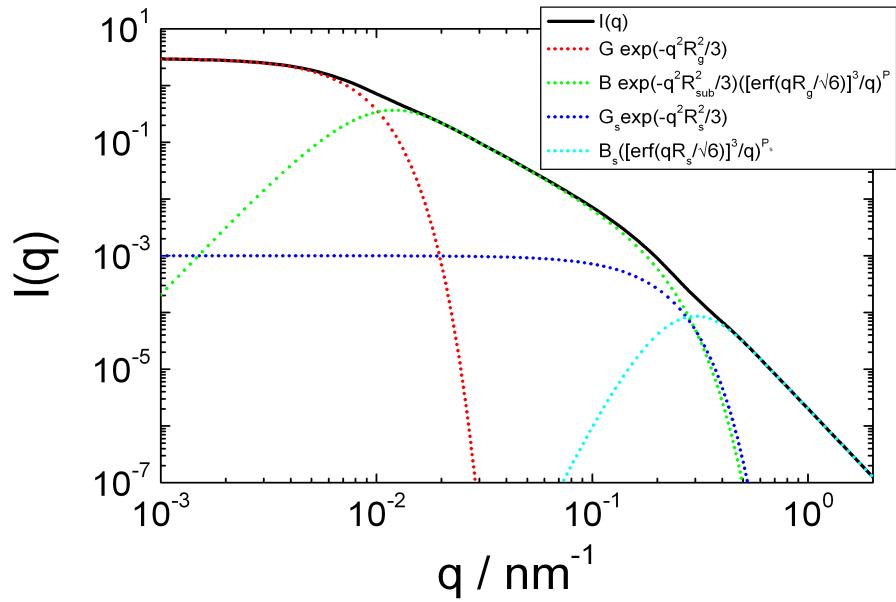


FIGURE 3.34.

3.2.8.2. *Beaucage2.*

Equation 3.59 can be extended to describe an arbitrary number of interrelated structural levels under the generally applicable assumption that $R_{\text{sub}} = R_s$,

$$I_{\text{Beaucage}}(Q) \simeq \sum_{i=1}^n G_i \exp\left(-\frac{Q^2 R_{g,i}^2}{3}\right) + B_i \exp\left(-\frac{Q^2 R_{g,i+1}^2}{3}\right) \left(\frac{\left[\text{erf}(Qk_i R_{g,i}/\sqrt{6})\right]^3}{Q}\right)^{P_i} \quad (3.60)$$

In 3.60, $i = 1$ refers to the largest-size structural level. Extensions, such as eq. 3.60, can only be justified when data extend over many decades in Q . Eq. 3.60 introduces no new parameters over local Guinier and power-law fits.

Input Parameters for model *Beaucage2*:

G: Guinier pre-factor G_i

B: pre-factor B_i , specific to the type of power-law scattering: B_i is defined according to the regime in which the exponent P_i falls.

R_i: large-scale structure $R_{g,i}$

R_i+1: size $R_{g,i+1}$ of smaller subunits

k: This is an empirical constant that has a value of 1 for steep power-law decays, $P > 3$. For weak power-law decays, k deviates slightly from 1

P: scaling exponent of the power law assigned to the larger structure $R_{g,i}$

Note: For calculating a summation over n interrelated structural levels one has to use global fitting. For R_i and R_{i+1} one has to define global parameters, let say P1 and P2 in the first scattering contribution. In the second scattering contribution one has to use for R_i the same global parameter P2 as for R_{i+1} in the previous scattering contribution. The global parameters for R_i in one scattering contribution and the global parameter for R_{i+1} in the previous scattering contribution have to be the same. By this an arbitrary number of interrelated structural levels can be calculated.

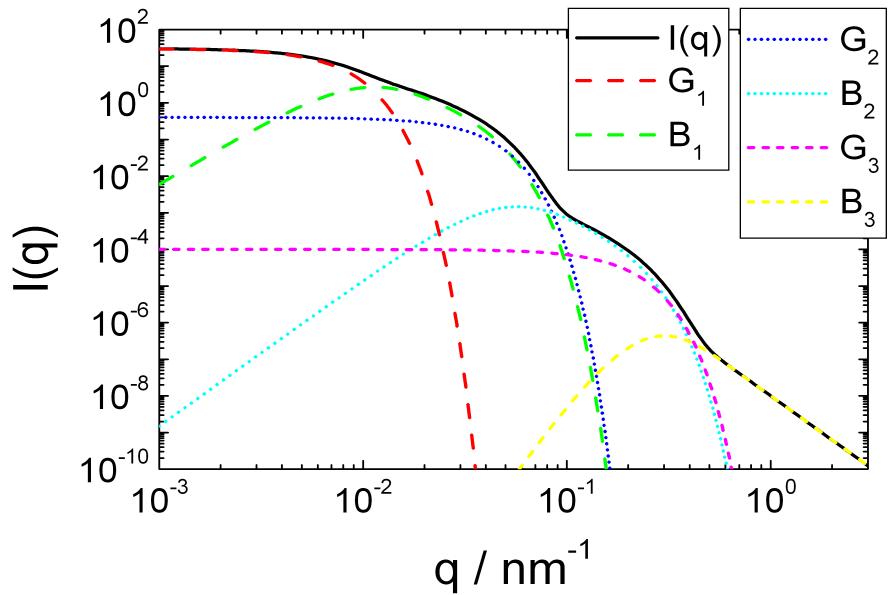


FIGURE 3.35.

3.2.9. WormLikeChainEXV [168].

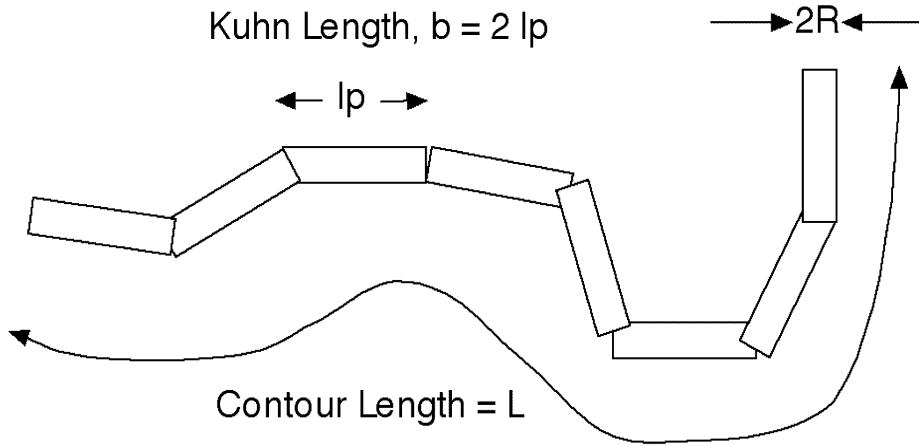


FIGURE 3.36. The chain of contour length, L , (the total length) can be described a chain of some number of locally stiff segments of length l_p . The persistence length, l_p , is the length along the cylinder over which the flexible cylinder can be considered a rigid rod. The Kuhn length (b) used in the model is also used to describe the stiffness of a chain, and is simply $b = 2l_p$.

This form factor calculates the form factor for a flexible cylinder with a circular cross section and a uniform scattering length density. The non-negligible diameter of the cylinder is included by accounting for excluded volume interactions within the walk of a single cylinder. Inter-cylinder interactions are NOT included. The function calculated has been given by Pedersen et al. [168]. The model "Method 3 With Excluded Volume" is used, which is a parametrization of simulations of a discrete representation of the worm-like chain model of Kratky and Porod applied in the pseudo-continuous limit.

Input Parameters for model WormLikeChainEXV:

- R: radius R of cylindrical core with uniform scattering length density
- 1: Kuhn length³ l of semi-flexible worm-like structure
- L: contour length L of semi-flexible worm-like structure

³The Kuhn length l is related to the length a of locally stiff segment simply via $l = 2a$

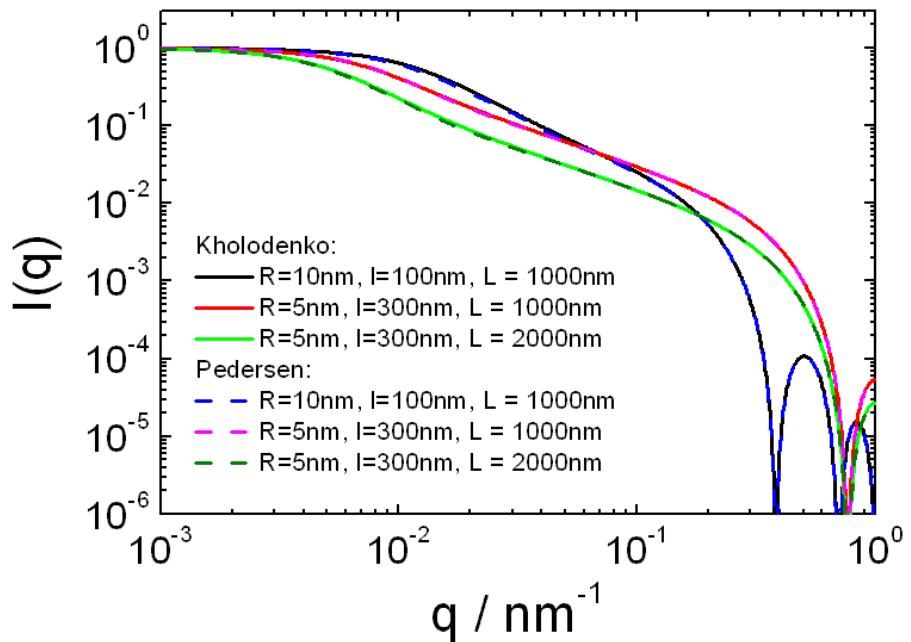


FIGURE 3.37. Comparison of wormlike micelles according to Pedersen [168] and Kholodenko [112]

3.2.10. KholodenkoWorm.

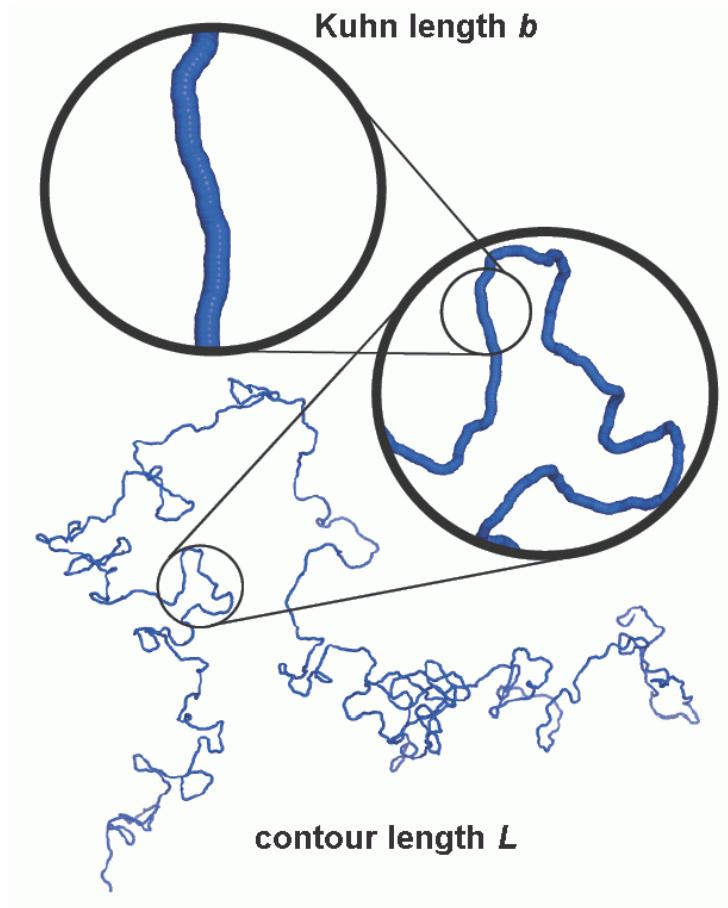


FIGURE 3.38.

Kholodenko [112] presented a new approach using the analogy between Diracs fermions and semi-flexible polymers. The form factor $P_0(Q)$ resulting from Kholodenkos approach is designed to reproduce correctly the rigid-rod limit and the random-coil limit. Defining $x = 3L/l$ (L : contour length, l : Kuhn length), it is given by

$$P_0(Q, L, l) = \frac{2}{x} \left[I_{(1)} - \frac{1}{x} I_{(2)} \right] \quad (3.61)$$

where

$$I_{(n)}(x) = \int_0^x f(z) z^{n-1} dz \quad (3.62)$$

together with

$$f(z) = \begin{cases} \frac{1}{E} \frac{\sinh(Ez)}{\sinh(z)} & \text{for } Q \leq \frac{3}{l} \\ \frac{1}{F} \frac{\sin(Fz)}{\sinh(z)} & \text{for } Q > \frac{3}{l} \end{cases} \quad (3.63)$$

and

$$E = \sqrt{1 - \left(\frac{lQ}{3}\right)^2} \quad \text{and} \quad F = \sqrt{\left(\frac{lQ}{3}\right)^2 - 1} \quad (3.64)$$

For flexible cylinders with a circular cross section and a uniform scattering length density the cross section form factor is given by

$$P_{cs} = \left(2 \frac{J_1(QR)}{QR}\right)^2 \quad (3.65)$$

so that the overall form factor is given by

$$P(Q, L, l, R) = P_0(Q, L, l) P_{cs}(Q, R) \quad (3.66)$$

Input Parameters for model KholodenkoWorm:

R: radius R of cylindrical core with uniform scattering length density

l: Kuhn length⁴ l of semi-flexible worm-like structure

L: contour length L of semi-flexible worm-like structure

⁴The Kuhn length l is related to the length a of locally stiff segment simply via $l = 2a$

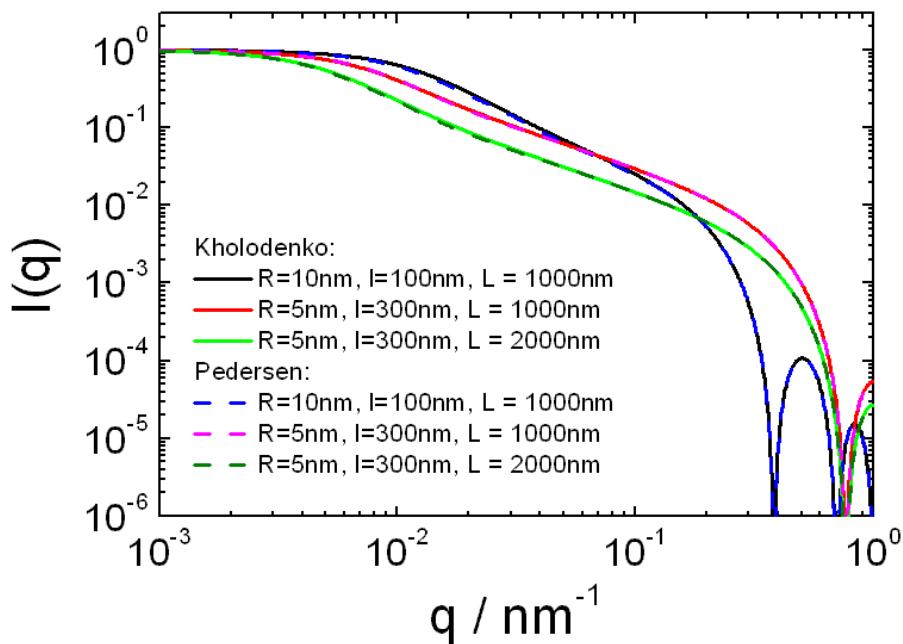


FIGURE 3.39. Comparison of wormlike micelles according to Pedersen [168] and Kholodenko [112].

3.2.11. Diblock copolymer micelles.

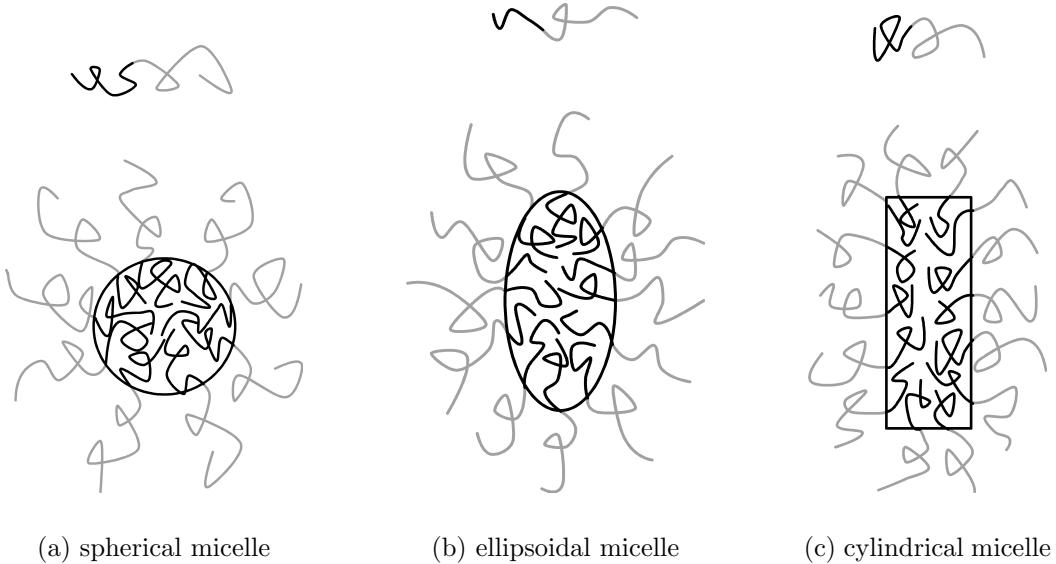


FIGURE 3.40. Block copolymer forming micelles of different shapes

For block copolymers, which form micelles, several form factor have been implemented [166, 164, 204] for spherical, ellipsoidal, and cylindrical shapes. It has been assumed that one unit is forming the core of the micelles and the other the corona. The core is assumed to have a homogeneous scattering length density, but may contain some amount of solvent. For the polymer chains in the corona either a model where Gaussian chains are attached to the core or a corona model of semi-flexible interacting self-avoiding chains (only for spherical core) or a continuous model, where a radial profile of the form $\Phi(r) \propto r^{-\alpha}$ has been assumed. The form factors have been parameterized such, that the excess scattering of the corona and the core are consistent with the composition and density of the two separate block units of the copolymer.

3.2.11.1. Micelles with a homogeneous core and Gaussian chains on the surface.

It is assumed that the diblock copolymer consist of a block unit for which the solvent is poor and a block unit with is good. The insoluble blocks form a relatively compact core whereas the soluble blocks form a diffuse corona surrounding the core. The form factor of a micelle contains four different terms: the self-correlation term of the core $N_{\text{agg}}^2 \beta_{\text{core}}^2 P_{\text{core}}(q)$, the self-correlation term of the chains $N_{\text{agg}} \beta_{\text{brush}}^2 P_{\text{brush}}(q)$, the cross-term between the core and chains $2N_{\text{agg}}^2 \beta_{\text{core}} \beta_{\text{brush}} S_{\text{brush-core}}(q)$, and the cross term between different chains $N_{\text{agg}}(N_{\text{agg}} - 1) \beta_{\text{brush}}^2 S_{\text{brush-brush}}(q)$. It can be written (Pedersen & Gerstenberg, 1996)

$$\begin{aligned}
 I_{\text{mic}} = & N_{\text{agg}}^2 \beta_{\text{core}}^2 P_{\text{core}}(q) + N_{\text{agg}} \beta_{\text{brush}}^2 P_{\text{brush}}(q) \\
 & + 2N_{\text{agg}}^2 \beta_{\text{core}} \beta_{\text{brush}} S_{\text{brush-core}}(q) + N_{\text{agg}}(N_{\text{agg}} - 1) \beta_{\text{brush}}^2 S_{\text{brush-brush}}(q)
 \end{aligned} \tag{3.67}$$

N_{agg} is the aggregation number of diblock polymers forming the micelle and $\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}})$ and $\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}})$ the excess scattering length of a block in the corona and in the core, respectively. V_{brush} and V_{core} are the total volume of a block in the corona and in the core. η_{brush} and η_{core} are the corresponding scattering length densities and η_{solv} is the scattering length density of the surrounding solvent. The functions $P_{\text{core}}(q)$, $P_{\text{brush}}(q)$, $S_{\text{brush-core}}(q)$, and $S_{\text{brush-brush}}(q)$ are all 1 for $q = 0$. The definitions of these four functions depend on the shape of the core and are given below.

3.2.11.2. Spherical core:

$$P_{\text{core}}(q, R_{\text{core}}) = \Phi^2(qR_{\text{core}}) \quad (3.68)$$

$$\Phi(qR) = 3 \frac{\sin(qR) - qR \cos(qR)}{(qR)^3} \quad (3.69)$$

$$P_{\text{brush}}(q, R_g) = 2 \frac{\exp(-x) - 1 + x}{x^2} \text{ with } x = R_g^2 q^2 \quad (3.70)$$

$$S_{\text{brush-core}}(q, R_{\text{core}}, Rg, d) = \Phi(qR_{\text{core}})\psi(qR_g) \frac{\sin(q[R_{\text{core}} + dR_g])}{q[R_{\text{core}} + dR_g]} \quad (3.71)$$

$$\psi(qRg) = \frac{1 - \exp(-x)}{x} \text{ (form factor amplitude of the chain)}$$

$$S_{\text{brush-brush}}(q, R_{\text{core}}, d, R_g) = \psi^2(qR_g) \left[\frac{\sin(q[R_{\text{core}} + dR_g])}{q[R_{\text{core}} + dR_g]} \right]^2 \quad (3.72)$$

For micelles with a spherical core a few different parameterizations have been implemented `SPHERE+Chains(RW)`, `SPHERE+Chains(RW)_Rc` and `SPHERE+Chains(RW)_Nagg`. The parameters they all have in common are:

V_{brush} : molecular volume the diblock copolymer part forming the corona

η_{core} : scattering length density of the diblock copolymer part forming the core

η_{brush} : scattering length density of the diblock copolymer part forming the corona

η_{solv} : scattering length density of the solvent

$x_{\text{solv,core}}$: volume fraction of solvent in the micellar core

R_g : radius of gyration of the block unit in the corona

d : non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

For the model `SPHERE+Chains(RW)` the other parameters are

R_{core} : radius of the micellar core

n_{agg} : grafting density (number of copolymer molecules N_{agg} per surface area S , $n_{\text{agg}} = N_{\text{agg}}/S$)

In contrast to the form factor `SPHERE+Chains(RW)_Rc` and `SPHERE+Chains(RW)_Nagg` this one does not necessarily consist of copolymers. The excess scattering lengths and

aggregation number needed in eq. 3.67 are than given by

$$N_{\text{agg}} = n_{\text{agg}} S \quad (3.73)$$

where the surface of the core is given by $S = 4\pi R_{\text{core}}^2$. Together with the core volume $V = \frac{4}{3}\pi R_{\text{core}}^3$ one gets for the excess scattering lengths

$$\beta_{\text{core}} = \frac{V(1 - x_{\text{solv,core}})}{N_{\text{agg}}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.74)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.75)$$

Input Parameters for model SPHERE+Chains(RW):

R_core: core radius

n_agg: specific aggregation number (number of chains per surface area)

V_brush: molecular volume of a block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

For the model SPHERE+Chains(RW)_Rc the other parameters are

R_core: core radius

V_core: molecular volume of single block unit in the micellar core

The excess scattering lengths and aggregation number in eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.76)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.77)$$

$$N_{\text{agg}} = (1 - x_{\text{solv,core}})\frac{4}{3}\pi R_{\text{core}}^3/V_{\text{core}} \quad (3.78)$$

Input Parameters for model SPHERE+Chains(RW)_Rc:

R_core: core radius

V_core: molecular volume of single block unit in the micellar core

V_brush: molecular volume of single block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

For the model **SPHERE+Chains(RW)_Nagg** the other parameters are

N_{agg}: aggregation number

V_{core}: molecular volume of single block unit in the micellar core

The excess scattering lengths and the core radius R_{core} needed in eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.79)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.80)$$

$$R_{\text{core}} = \left(\frac{N_{\text{agg}} V_{\text{core}}}{1 - x_{\text{solv,core}}} \frac{3}{4\pi} \right)^{1/3} \quad (3.81)$$

Input Parameters for model SPHERE+Chains(RW)_Nagg:

N_{agg}: aggregation number

V_{core}: molecular volume of single block unit in the micellar core

V_{brush}: molecular volume of single block unit in the micellar corona

eta_{core}: scattering length density of spherical core

eta_{brush}: scattering length density of the block unit in the corona

eta_{solv}: scattering length density of solvent

x_{solv_core}: amount of solvent in core

R_g: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

3.2.11.3. ellipsoidal core with semi-axis ($R, R, \epsilon R$):

$$P_{\text{core}}(q, R_{\text{core}}, \epsilon) = \int_0^{\pi/2} \Phi^2[qr(R_{\text{core}}, \epsilon, \alpha)] \sin \alpha d\alpha \quad (3.82)$$

with $r(R_{\text{core}}, \epsilon, \alpha) = R_{\text{core}} \sqrt{\sin^2 \alpha + \epsilon^2 \cos^2 \alpha}$

$$P_{\text{brush}}(q, R_g) = 2 \frac{\exp(-x) - 1 + x}{x^2} \text{ with } x = R_g^2 q^2 \quad (3.83)$$

$$S_{\text{brush-core}}(q, R_{\text{core}}, \epsilon, Rg, d) = \psi(qR_g) \int_0^{\pi/2} \Phi(qr(\dots)) \frac{\sin(q[r(\dots) + dR_g])}{q[r(\dots) + dR_g]} \sin \alpha d\alpha \quad (3.84)$$

$$S_{\text{brush-brush}}(q, R_{\text{core}}, d, R_g) = \psi^2(qR_g) \int_0^{\pi/2} \left[\frac{\sin(q[r(\dots) + dR_g])}{q[r(\dots) + dR_g]} \right]^2 \sin \alpha d\alpha \quad (3.85)$$

As for micelles with spherical core also for those with an ellipsoidal core several parameterizations have been implemented **ELL+Chains(RW)**, **ELL+Chains(RW)_Rc** and **ELL+Chains(RW)_Nagg**. The parameters they all have in common are:

- V_{brush} : molecular volume the diblock copolymer part forming the corona
- η_{core} : scattering length density of the diblock copolymer part forming the core
- η_{brush} : scattering length density of the diblock copolymer part forming the corona
- η_{solv} : scattering length density of the solvent
- $x_{\text{solv,core}}$: volume fraction of solvent in the micellar core
- R_g : radius of gyration of the block unit in the corona
- d : non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$
- ϵ : eccentricity of the ellipsoidal micelle ($R_{\text{core}}, R_{\text{core}}, \epsilon R_{\text{core}}$)

For the model **ELL+Chains(RW)** the other parameters are

- R_{core} : radius of the micellar core ($R_{\text{core}}, R_{\text{core}}, \epsilon R_{\text{core}}$)
- n_{agg} : grafting density (number of copolymer molecules N_{agg} per surface area S , $n_{\text{agg}} = N_{\text{agg}}/S$)

In contrast to the form factor **ELL+Chains(RW)_Rc** and **ELL+Chains(RW)_Nagg** this one does not necessarily consist of copolymers. The excess scattering lengths and aggregation number needed in eq. 3.67 are given by

$$N_{\text{agg}} = n_{\text{agg}} S \quad (3.86)$$

where the surface of the core is given by

$$S = \begin{cases} 2\pi R_{\text{core}}^2 \left(1 + \frac{\operatorname{arctanh}(\sin(\alpha))}{\sin(\alpha)} \right) & \text{for } \epsilon < 1 \\ 2\pi R_{\text{core}}^2 \left(1 + \frac{\alpha}{\tan(\alpha)} \right) & \text{for } \epsilon \geq 1 \end{cases} \quad (3.87)$$

$$\alpha = \begin{cases} \arccos(\epsilon) & \text{for } \epsilon < 1 \\ \arccos(1/\epsilon) & \text{for } \epsilon \geq 1 \end{cases}$$

Together with the core volume $V = \frac{4}{3}\pi\epsilon R_{\text{core}}^3$ one gets for the excess scattering lengths

$$\beta_{\text{core}} = \frac{V(1 - x_{\text{solv,core}})}{N_{\text{agg}}} (\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.88)$$

$$\beta_{\text{brush}} = V_{\text{brush}} (\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.89)$$

Input Parameters for model ELL+Chains(RW):

R_core: core radius

n_agg: specific aggregation number (number of chains per surface area)

V_brush: molecular volume of a block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

epsilon: eccentricity of the ellipsoidal micelle ($R_{\text{core}}, R_{\text{core}}, \epsilon R_{\text{core}}$)

For the model **ELL+Chains(RW)_Rc** the other parameters are

R_core: core radius

V_core: molecular volume of single block unit in the micellar core

The excess scattering lengths and aggregation number in eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}} (\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.90)$$

$$\beta_{\text{brush}} = V_{\text{brush}} (\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.91)$$

$$N_{\text{agg}} = (1 - x_{\text{solv,core}}) \frac{4}{3}\pi\epsilon R_{\text{core}}^3 / V_{\text{core}} \quad (3.92)$$

Input Parameters for model ELL+Chains(RW)_Rc:

R_core: core radius

V_core: molecular volume of single block unit in the micellar core

V_brush: molecular volume of single block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

epsilon: eccentricity of the ellipsoidal micelle ($R_{\text{core}}, R_{\text{core}}, \epsilon R_{\text{core}}$)

For the model **ELL+Chains(RW)_Nagg** the other parameters are

Nagg: aggregation number

Vcore: molecular volume of single block unit in the micellar core

The excess scattering lengths and the core radius R_{core} needed in eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.93)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.94)$$

$$R_{\text{core}} = \left(\frac{N_{\text{agg}} V_{\text{core}}}{1 - x_{\text{solv,core}}} \frac{3}{4\pi\epsilon} \right)^{1/3} \quad (3.95)$$

Input Parameters for model **ELL+Chains(RW)_Nagg:**

N_agg: aggregation number

V_core: molecular volume of single block unit in the micellar core

V_brush: molecular volume of single block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

epsilon: eccentricity of the ellipsoidal micelle ($R_{\text{core}}, R_{\text{core}}, \epsilon R_{\text{core}}$)

3.2.11.4. cylindrical core with radius R_{core} and height H :

$$P_{\text{core}}(q, R_{\text{core}}, H) = \int_0^{\pi/2} \Psi^2(q, R_{\text{core}}, H, \alpha) \sin \alpha d\alpha \quad (3.96)$$

$$\text{with } \Psi(q, R_{\text{core}}, H, \alpha) = \frac{2J_1(qR_{\text{core}} \sin \alpha)}{qR_{\text{core}} \sin \alpha} \frac{\sin(qH/2 \cos \alpha)}{qH/2 \cos \alpha} \quad (3.97)$$

and $J_1(x)$ is the first order Bessel function of the first kind.

$$P_{\text{brush}}(q, R_g) = 2 \frac{\exp(-x) - 1 + x}{x^2} \text{ with } x = R_g^2 q^2 \quad (3.98)$$

$$S_{\text{brush-core}}(q, R_{\text{core}}, H, R_g, d) = \psi(qR_g) \times \int_0^{\pi/2} \Psi(q, R_{\text{core}}, H, \alpha) \Xi(q, R + dR_g, H + 2dR_g, \alpha) \sin \alpha d\alpha \quad (3.99)$$

where $\Xi(q, R_{\text{core}}, H, \alpha)$ is the form factor amplitude of the shell:

$$\begin{aligned} \Xi(q, R_{\text{core}}, H, \alpha) = & \left[\frac{R}{R_{\text{core}} + H} \frac{2J_1(qR_{\text{core}} \sin \alpha)}{qR_{\text{core}} \sin \alpha} \cos(qH/2 \cos \alpha) \right. \\ & \left. + \frac{H}{R_{\text{core}} + H} J_0(qR_{\text{core}} \sin \alpha) \frac{\sin(qH/2 \cos \alpha)}{qH/2 \cos \alpha} \right] \end{aligned} \quad (3.100)$$

where $J_0(x)$ is the zeroth order Bessel function of the first kind.

$$S_{\text{brush-brush}}(q, R_{\text{core}}, H, d, R_g) = \psi^2(qR_g) \times \int_0^{\pi/2} \Xi^2(q, R_{\text{core}} + dR_g, H + 2dR_g, \alpha) \sin \alpha d\alpha \quad (3.101)$$

As for micelles with spherical core also for those with a cylindrical core several parameterizations have been implemented **CYL+Chains(RW)**, **CYL+Chains(RW)_Rc** and **CYL+Chains(RW)_Nagg**. The parameters they all have in common are:

V_{brush} : molecular volume the diblock copolymer part forming the corona

η_{core} : scattering length density of the diblock copolymer part forming the core

η_{brush} : scattering length density of the diblock copolymer part forming the corona

η_{solv} : scattering length density of the solvent

$x_{\text{solv,core}}$: volume fraction of solvent in the micellar core

R_g : radius of gyration of the block unit in the corona

d : non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

H : height of the cylindrical core of the micelle

For the model **CYL+Chains(RW)** the other parameters are

R_{core} : radius of the micellar core ($R_{\text{core}}, R_{\text{core}}, \epsilon R_{\text{core}}$)

n_{agg} : grafting density (number of copolymer molecules N_{agg} per surface area S , $n_{\text{agg}} = N_{\text{agg}}/S$)

In contrast to the form factor **CYL+Chains(RW)_Rc** and **CYL+Chains(RW)_Nagg** this one does not necessarily consist of copolymers. The excess scattering lengths and aggregation number needed in eq. 3.67 are then given by

$$N_{\text{agg}} = n_{\text{agg}} S \quad (3.102)$$

where the surface of the core is given by

$$S = 2\pi R_{\text{core}} H \quad (3.103)$$

Together with the core volume $V = \pi R_{\text{core}}^2 H$ one can calculate the excess scattering lengths by

$$\beta_{\text{core}} = \frac{V(1 - x_{\text{solv},\text{core}})}{N_{\text{agg}}} (\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.104)$$

$$\beta_{\text{brush}} = V_{\text{brush}} (\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.105)$$

Input Parameters for model **CYL+Chains(RW)**:

R_core: core radius

n_agg: specific aggregation number (number of chains per surface area)

V_brush: molecular volume of a block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

x_solv_core: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

H: height of the cylindrical core of the micelle

For the model **CYL+Chains(RW)_Rc** the other parameters are

R_core: core radius

V_core: molecular volume of single block unit in the micellar core

The excess scattering lengths and aggregation number in eq. 3.67 are then given by

$$\beta_{\text{core}} = V_{\text{core}} (\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.106)$$

$$\beta_{\text{brush}} = V_{\text{brush}} (\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.107)$$

$$N_{\text{agg}} = (1 - x_{\text{solv},\text{core}}) \pi R_{\text{core}}^2 H / V_{\text{core}} \quad (3.108)$$

Input Parameters for model **CYL+Chains(RW)_Rc**:

R_core: core radius

V_core: molecular volume of single block unit in the micellar core

V_brush: molecular volume of single block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

H: height of the cylindrical core of the micelle

For the model **CYL+Chains(RW)_Nagg** the other parameters are

N_{agg}: aggregation number

V_{core}: molecular volume of single block unit in the micellar core

The excess scattering lengths and the core radius R_{core} are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.109)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.110)$$

$$R_{\text{core}} = \sqrt{\frac{N_{\text{agg}} V_{\text{core}}}{1 - x_{\text{solv,core}}} \frac{1}{\pi H}} \quad (3.111)$$

Input Parameters for model CYL+Chains(RW)_Nagg:

N_{agg}: aggregation number

V_{core}: molecular volume of single block unit in the micellar core

V_{brush}: molecular volume of single block unit in the micellar corona

eta_{core}: scattering length density of spherical core

eta_{brush}: scattering length density of the block unit in the corona

eta_{solv}: scattering length density of solvent

xsolv_{core}: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

H: height of the cylindrical core of the micelle

3.2.11.5. wormlike micelles with cylindrical cross-section with radius R_{core} , Kuhn-length l and contour length L :

The form factors for a worm-like micelles are approximated by the form factor of the Kholodenko-worm according to section 3.2.10 where the scattering length density profile across the worm segments are described by those of a rod-like micelle 3.2.11.6. The corresponding function in eq. 3.67 are given by

$$P_{\text{core}}(q, R_{\text{core}}, l, L) = P_{\text{worm}}(q, l, L)P_{\text{cs}}(q, R_{\text{core}}, d, R_g) \quad (3.112)$$

the contributiuion of the worm-like conformation of the micelle $P_{\text{worm}}(q, l, L)$ is described by the formula of Kholodenko for worm-like structures given in eq. 3.61. The contribution of the cross-section P_{cs} is the same as for rod-like micelles and given by

$$P_{\text{cs}}(q, R_{\text{core}}, d, R_g) = \left[\frac{2J_1(qR_{\text{core}})}{qR_{\text{core}}} \right]^2 \quad (3.113)$$

$$\text{Si}(x) = \int_0^x t^{-1} \sin t dt \quad (3.114)$$

$$P_{\text{brush}}(q, R_g) = 2 \frac{\exp(-x) - 1 + x}{x^2} \text{ with } x = R_g^2 q^2 \quad (3.115)$$

$$S_{\text{brush-core}}(q, R_{\text{core}}, l, L, R_g, d) = \psi(qR_g) \times \quad (3.116)$$

$$\frac{2J_1(qR_{\text{core}})}{qR_{\text{core}}} J_0[q(r_{\text{core}} + dR_g)] P_{\text{worm}}(q, l, L)$$

$$S_{\text{brush-brush}}(q, R_{\text{core}}, l, L, d, R_g) = \psi^2(qR_g) J_0^2[q(r_{\text{core}} + dR_g)] P_{\text{worm}}(q, l, L) \quad (3.117)$$

As for micelles with spherical core also for those worm-like micelles several parameterizations have been implemented **WORM+Chains(RW)**, **WORM+Chains(RW)_Rc** and **WORM+Chains(RW)_nagg**. The parameters they all have in common are:

V_{brush} : molecular volume the diblock copolymer part forming the corona

η_{core} : scattering length density of the diblock copolymer part forming the core

η_{brush} : scattering length density of the diblock copolymer part forming the corona

η_{solv} : scattering length density of the solvent

$x_{\text{solv,core}}$: volume fraction of solvent in the micellar core

R_g : radius of gyration of the block unit in the corona

l : contour length of the worm-like of the micelle

L : contour length of the worm-like of the micelle

For the model **WORM+Chains(RW)** the other parameters are

R_{core} : radius of the micellar core

n_{agg} : grafting density (number of copolymer molecules N_{agg} per surface are S , $n_{\text{agg}} = N_{\text{agg}}/S$)

In contrast to the form factor **WORM+Chains(RW)_Rc** and **WORM+Chains(RW)_nagg** this one does not necessarily consist of copolymers. The excess scattering lengths and aggregation number needed in eq. 3.67 are then given by

$$N_{\text{agg}} = n_{\text{agg}} S \quad (3.118)$$

where the surface of the core is given by

$$S = 2\pi R_{\text{core}} L \quad (3.119)$$

Together with the core volume $V = \pi R_{\text{core}}^2 L$ one can calculate the excess scattering lengths by

$$\beta_{\text{core}} = \frac{V(1 - x_{\text{solv,core}})}{N_{\text{agg}}} (\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.120)$$

$$\beta_{\text{brush}} = V_{\text{brush}} (\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.121)$$

Input Parameters for model **WORM+Chains(RW)**:

- R_core:** core radius
- n_agg:** specific aggregation number (number of chains per surface area)
- V_brush:** molecular volume of a block unit in the micellar corona
- eta_core:** scattering length density of spherical core
- eta_brush:** scattering length density of the block unit in the corona
- eta_solv:** scattering length density of solvent
- xsolv_core:** amount of solvent in core
- Rg:** gyration radius of polymer chains in the corona
- l:** contour length of the worm-like of the micelle
- L:** contour length of the worm-like of the micelle

For the model **WORM+Chains(RW)_Rc** the other parameters are

- R_core:** core radius

- V_core:** molecular volume of single block unit in the micellar core

The excess scattering lengths and aggregation number in eq. 3.67 are then given by

$$\beta_{\text{core}} = V_{\text{core}} (\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.122)$$

$$\beta_{\text{brush}} = V_{\text{brush}} (\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.123)$$

$$N_{\text{agg}} = (1 - x_{\text{solv,core}}) \pi R_{\text{core}}^2 L / V_{\text{core}} \quad (3.124)$$

Input Parameters for model **CYL+Chains(RW)_Rc**:

- R_core:** core radius
- V_core:** molecular volume of single block unit in the micellar core
- V_brush:** molecular volume of single block unit in the micellar corona
- eta_core:** scattering length density of spherical core
- eta_brush:** scattering length density of the block unit in the corona
- eta_solv:** scattering length density of solvent
- xsolv_core:** amount of solvent in core

- Rg:** gyration radius of polymer chains in the corona
l: contour length of the worm-like of the micelle
L: contour length of the worm-like of the micelle

For the model **WORM+Chains(RW)_Nagg** the other parameters are

N_{agg}: aggregation number

V_{core}: molecular volume of single block unit in the micellar core

The excess scattering lengths and the core radius R_{core} are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.125)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.126)$$

$$R_{\text{core}} = \sqrt{\frac{N_{\text{agg}} V_{\text{core}}}{1 - x_{\text{solv,core}}} \frac{1}{\pi L}} \quad (3.127)$$

Input Parameters for model **CYL+Chains(RW)_Nagg**:

N_{agg}: aggregation number

V_{core}: molecular volume of single block unit in the micellar core

V_{brush}: molecular volume of single block unit in the micellar corona

eta_{core}: scattering length density of spherical core

eta_{brush}: scattering length density of the block unit in the corona

eta_{solv}: scattering length density of solvent

x_{solv_core}: amount of solvent in core

Rg: gyration radius of polymer chains in the corona

l: contour length of the worm-like of the micelle

L: contour length of the worm-like of the micelle

3.2.11.6. micelles with rod-like core:

The form factors for micelles with a rod-like core are an approximations of the form factors of micelles with a cylindrical core where $H \gg R_{\text{core}} + dR_g$. The corresponding function in eq. 3.67 are given by

$$P_{\text{core}}(q, R_{\text{core}}, H) = P_H(q, H)P_{\text{cs}}(q, R_{\text{core}}, d, R_g) \quad (3.128)$$

with

$$P_H(q, H) = 2\text{Si}(qH)/(qH) - 4\sin^2(qH/2)/(q^2H^2) \quad (3.129)$$

$$P_{\text{cs}}(q, R_{\text{core}}, d, R_g) = \left[\frac{2J_1(qR_{\text{core}})}{qR_{\text{core}}} \right]^2 \quad (3.130)$$

$$\text{Si}(x) = \int_0^x t^{-1} \sin t dt \quad (3.131)$$

$$P_{\text{brush}}(q, R_g) = 2 \frac{\exp(-x) - 1 + x}{x^2} \text{ with } x = R_g^2 q^2 \quad (3.132)$$

$$S_{\text{brush-core}}(q, R_{\text{core}}, H, Rg, d) = \psi(qR_g) \times \frac{2J_1(qR_{\text{core}})}{qR_{\text{core}}} J_0[q(r_{\text{core}} + dR_g)] P_H(q, H) \quad (3.133)$$

$$S_{\text{brush-brush}}(q, R_{\text{core}}, H, d, R_g) = \psi^2(qR_g) J_0^2[q(r_{\text{core}} + dR_g)] P_H(q, H) \quad (3.134)$$

As otherwise the definitions of the geometry for rod-like micelles are mainly the same than for cylindrical micelles only the list of input parameters are given here. There is only one difference in the model `ROD+Chains(RW)_Nagg` compared to the model `CYL+Chains(RW)_Nagg` and that is that for rod-like structures always the grafting density of polymer chains on the surface of the core is used, i.e. $n_{\text{agg}} = N_{\text{agg}}/S$ instead of N_{agg} . For the model `ROD+Chains(RW)_nagg` this means that the core radius has to be calculated by $R_{\text{core}} = 2n_{\text{agg}}V_{\text{core}}/(1 - x_{\text{solv,core}})$

Input Parameters for model `ROD+Chains(RW)`:

`R_core`: core radius

`n_agg`: specific aggregation number (number of chains per surface area)

`V_brush`: molecular volume of a block unit in the micellar corona

`eta_core`: scattering length density of spherical core

`eta_brush`: scattering length density of the block unit in the corona

`eta_solv`: scattering length density of solvent

`xsolv_core`: amount of solvent in core

`Rg`: gyration radius of polymer chains in the corona

`d`: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$

`H`: height of the rod-like core of the micelle

Input Parameters for model ROD+Chains(RW)_Rc:

R_core: core radius
V_core: molecular volume of single block unit in the micellar core
V_brush: molecular volume of single block unit in the micellar corona
eta_core: scattering length density of spherical core
eta_brush: scattering length density of the block unit in the corona
eta_solv: scattering length density of solvent
xsolv_core: amount of solvent in core
Rg: gyration radius of polymer chains in the corona
d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$
H: height of the rod-like core of the micelle

Input Parameters for model ROD+Chains(RW)_nagg:

n_agg: specific aggregation number (number of chains per surface area)
V_core: molecular volume of single block unit in the micellar core
V_brush: molecular volume of single block unit in the micellar corona
eta_core: scattering length density of spherical core
eta_brush: scattering length density of the block unit in the corona
eta_solv: scattering length density of solvent
xsolv_core: amount of solvent in core
Rg: gyration radius of polymer chains in the corona
d: This value should be around 1. Non-penetration of the chains into the core is mimicked by $d \sim 1$ for $R_{\text{core}} \gg R_g$
H: height of the rod-like core of the micelle

3.2.11.7. *Micelles with a homogeneous core and a corona with decaying density profile . of the form $\phi(r) \propto r^{-\alpha}$.*

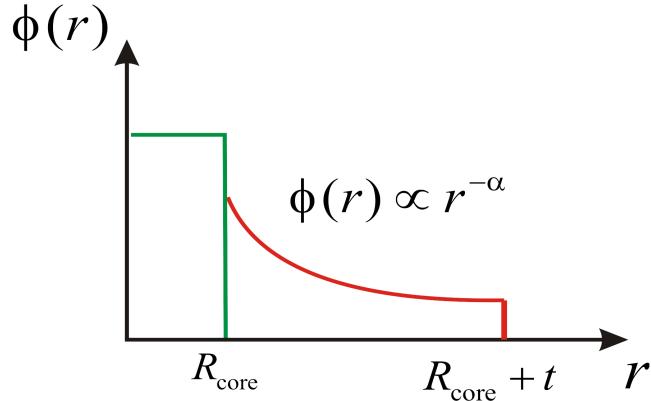


FIGURE 3.41. radial profile.

The structure of block copolymer micelles may be described in terms of the model of starlike polymer as proposed by Daoud and Cotton [40]. Starlike polymers consist of a homogeneous, dense polymer core surrounded by a polymer layer. As a consequence of the spherical or cylindrical geometry, the density profile $\phi(r)$ in the polymer layer decreases according to Wijmans & Zhulina [228] as

$$\phi(r) \begin{cases} \phi_{\text{core}} & \text{for } r < R_{\text{core}} \\ \phi_{\text{brush}} \left(\frac{r}{R_{\text{core}}} \right)^{-\alpha} & \text{for } R_{\text{core}} \leq r \leq R_{\text{core}} + t \\ 0 & \text{for } r > R_{\text{core}} + t \end{cases} \quad (3.135)$$

with $\alpha = (D - 1)(3\nu - 1)/(2\nu)$. D is determined by the dimension of the curvature of the grafted surface (spherical $D = 3$, cylindrical $D = 2$, planar $D = 1$). ν is the Flory exponent, which has characteristic values as given in Table 1. The corresponding density profile is schematically shown in Figure 3.41. Micelles consist of a well-defined micellar core with a radius R_{core} and a micellar shell or corona extending to the outer micellar radius $R_m = R_{\text{core}} + t$, where t is the thickness of the corona.

TABLE 1. Flory exponent ν and exponent α of the radial density profile for different thermodynamic states of the polymer chains

ν	α_{sphere}	α_{cylinder}	α_{planar}	remarks
$D = 3$	$D = 2$	$D = 1$		
1/3	0	0	0	collapsed polymer
1/2	1	1/2	0	polymer in Θ -solvent, semi-dilute solution
3/5	4/3	2/3	0	polymer in good solvent
1	2	1	0	polymer in stretched conformation, e.g. polyelectrolyte

For a given radial profile according to eq. 3.135 the form factor of spherical micelle can be calculated by

$$F_{\text{SPHERE}} = \int_0^{\infty} 2\pi r^2 \phi(r) \frac{\sin(qr)}{qr} dr \quad (3.136)$$

In case of a rod-like micelle the form factor can be separated in two terms $I(q) = P_H(q)P_{\text{cs}}(q)$ as already shown in eq. 3.128. The cross-term contribution to the scattering intensity is given by

$$\begin{aligned} P_{\text{cs}}(q) &= F_{\text{cs}}^2(q) \\ F_{\text{cs}}(q) &= \int_0^{\infty} 2\pi r \phi(r) J_0(q) dr \end{aligned} \quad (3.137)$$

However, as it is more convenient here to formulate the scattering intensity in terms of excess scattering length of the block units in the core β_{core} and the corona β_{brush} like in eq. 3.67 the form factor is split into two parts, the form factor of the homogeneous core $F_{\text{core}}(q)$ and the form factor of the corona $F_{\text{brush}}(q)$. The overall scattering intensity $I(q)$ is than given by

$$\begin{aligned} I(q) &= N_{\text{agg}}^2 \beta_{\text{core}}^2 F_{\text{core}}^2(q) + 2N_{\text{agg}}^2 \beta_{\text{core}} \beta_{\text{corona}} F_{\text{core}}(q) F_{\text{corona}}(q) \\ &\quad + N_{\text{agg}}(N_{\text{agg}} - 1) \beta_{\text{corona}}^2 F_{\text{corona}}^2(q) + N_{\text{agg}} P_{\text{brush}}(q) \end{aligned} \quad (3.138)$$

The excess scattering length of a block in the corona and in the core, respectively, $\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}})$ and $\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}})$ are defined in the same way than in eq. 3.67. V_{brush} and V_{core} are the total volume of a block in the corona and in the core. η_{brush} and η_{core} are the corresponding scattering length densities and η_{solv} is the scattering length density of the surrounding solvent. $F_{\text{core}}(q)$ is the form factor of the core and normalized to 1 for $q = 0$. Also the form factor of the corona $F_{\text{corona}}(q)$ and the form factor of the local fluctuations in the corona originating from the individual chains $P_{\text{brush}}(q)$ are normalized to 1 for $q = 0$. Similar to section 3.2.11.1 models for spherical and rod-like shapes have been implemented which are described in the following paragraphs.

3.2.11.8. spherical core:

$$F_{\text{core}}(q, R) = 3 \frac{\sin(qR) - qR \cos(qR)}{(qR)^3} \quad (3.139)$$

$$F_{\text{brush}}(q, R, t) = \frac{1}{C_{\text{norm}}} \int_{R_{\text{core}}}^{R_{\text{core}}+t} 2\pi r^2 r^{-\alpha} \frac{\sin(qr)}{qr} dr \quad (3.140)$$

with

$$C_{norm} = \begin{cases} \frac{4}{3-\alpha} \pi ((R_{core} + t)^{3-\alpha} - R_{core}^{3-\alpha}) & \text{for } \alpha \neq 2 \\ 4\pi \ln \left(\frac{R_{core} + t}{R_{core}} \right) & \text{for } \alpha = 2 \end{cases}$$

For the scattering contribution of the individual chains in the corona P_{brush} the scattering function for worm-like chains with excluded volume and negligible cross-section, contour length L and Kuhn-length b according to section 3.2.9 has been implemented.

For micelles with a spherical core a few different parameterizations have been implemented **SPHERE+R[^]-a**, **SPHERE+R[^]-a_Rc** and **SPHERE+R[^]-a_Nagg**.

The parameters they all have in common are:

V_{brush} : molecular volume the diblock copolymer part forming the corona

η_{core} : scattering length density of the diblock copolymer part forming the core

η_{brush} : scattering length density of the diblock copolymer part forming the corona

η_{solv} : scattering length density of the solvent

α : exponent of the radial scattering length density profile ($r^{-\alpha}$)

t : corona thickness

L : contour length of the chain in the corona

b : Kuhn-length of the chain in the corona

For the model **SPHERE+R[^]-a** the other parameters are

R_{core} : radius of the micellar core

n_{agg} : grafting density (number of copolymer molecules N_{agg} per surface area S , $n_{agg} = N_{agg}/S$)

In contrast to the form factor **SPHERE+R[^]-a_Rc** and **SPHERE+R[^]-a_Nagg** this one does not necessarily consist of copolymers. The excess scattering lengths and aggregation number are given by

$$N_{agg} = n_{agg} S \quad (3.141)$$

where the surface of the core is given by $S = 4\pi R_{core}^2$. Together with the core volume $V = \frac{4}{3}\pi R_{core}^3$ one gets for the excess scattering lengths

$$\beta_{core} = \frac{V(1 - x_{solv,core})}{N_{agg}} (\eta_{core} - \eta_{solv}) \quad (3.142)$$

$$\beta_{brush} = V_{brush} (\eta_{brush} - \eta_{solv}) \quad (3.143)$$

Input Parameters for model SPHERE+R[^]-a:

R_core: core radius

n_agg: specific aggregation number (number of chains per surface area)

V_brush: molecular volume of a block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

alpha: exponent of the radial scattering length density profile ($r^{-\alpha}$)

t: corona thickness

L: contour length of the chain in the corona

b: Kuhn-length of the chain in the corona

For the model SPHERE+R^{-a}Rc the other parameters are

R_{core} : core radius

V_{core} : molecular volume of single block unit in the micellar core

The excess scattering lengths and aggregation number for eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.144)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.145)$$

$$N_{\text{agg}} = (1 - x_{\text{solv},\text{core}}) \frac{4}{3} \pi R_{\text{core}}^3 / V_{\text{core}} \quad (3.146)$$

Input Parameters for model SPHERE+R^{-a}Rc:

R_{core} : core radius

V_{core} : molecular volume of single block unit in the micellar core

V_{brush} : molecular volume of single block unit in the micellar corona

η_{core} : scattering length density of spherical core

η_{brush} : scattering length density of the block unit in the corona

η_{solv} : scattering length density of solvent

α : exponent of the radial scattering length density profile ($r^{-\alpha}$)

t : corona thickness

L : contour length of the chain in the corona

b : Kuhn-length of the chain in the corona

For the model SPHERE+R^{-a}Nagg the other parameters are

N_{agg} : aggregation number

V_{core} : molecular volume of single block unit in the micellar core

The excess scattering lengths and the core radius R_{core} needed for eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.147)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.148)$$

$$R_{\text{core}} = \left(\frac{N_{\text{agg}} V_{\text{core}}}{1 - x_{\text{solv},\text{core}}} \frac{3}{4\pi} \right)^{1/3} \quad (3.149)$$

Input Parameters for model SPHERE+R^{-a}Nagg:

N_{agg} : aggregation number

V_{core} : molecular volume of single block unit in the micellar core

V_{brush} : molecular volume of single block unit in the micellar corona

η_{core} : scattering length density of spherical core

η_{brush} : scattering length density of the block unit in the corona

η_{solv} : scattering length density of solvent

α : exponent of the radial scattering length density profile ($r^{-\alpha}$)

t : corona thickness

L : contour length of the chain in the corona

b : Kuhn-length of the chain in the corona

3.2.11.9. rodlike core:

In case of a rod-like micelle the form factor can be separated in two terms $I(q) = P_H(q)P_{cs}(q)$ as already shown in eq. 3.128. The cross-term contribution to the scattering intensity is given by

$$\begin{aligned} P_{cs}(q) &= F_{cs}^2(q) \\ F_{cs}(q) &= \int_0^\infty 2\pi r \phi(r) J_0(qr) dr = F_{cs,\text{core}}(q) + F_{cs,\text{brush}}(q) \end{aligned} \quad (3.150)$$

The contribution of the homogeneous core is given by

$$F_{cs,\text{core}}(q) = \frac{2J_1(qR_c)}{qR_c} \quad (3.151)$$

and for the corona by

$$F_{cs,\text{brush}}(q) = \frac{1}{c_\alpha} \int_{R_c}^{R_c+t} 2\pi r r^{-\alpha} J_0(qr) dr \quad (3.152)$$

$$\begin{aligned} c_\alpha &= \int_{R_c}^{R_c+t} 2\pi r r^{-\alpha} dr \\ &= \begin{cases} 2\pi \ln\left(\frac{R_c+t}{R_c}\right) & \text{for } \alpha = 2 \\ \frac{2}{2-\alpha}\pi ((R_c+t)^{2-\alpha} - R_c^{2-\alpha}) & \text{for } \alpha \neq 2 \end{cases} \end{aligned} \quad (3.153)$$

For the scattering contribution of the individual chains in the corona P_{local} normally can be neglected for rod-like micelles in contrast to spherical structures as for structures with a lower dimension than spheres, this contribution becomes more and more negligible. To account for the scattering of the individual chains at least in first approximation and without introducing new parameters a form factor similar to the one of star polymers has been implemented.

$$\begin{aligned} P_{\text{local}}(q) &= \frac{\Gamma(\mu)}{qt} \frac{\sin(\mu \arctan(qt))}{(1+q^2t^2)^{\mu/2}} \\ \mu &= \frac{1}{\nu} - 1, \quad \alpha = \frac{3\nu - 1}{2\nu} \quad \Leftrightarrow \quad \mu = 2(1 - \alpha) \end{aligned} \quad (3.154)$$

The form factor to describe the scattering of the individual chains is identical to the blob scattering contribution in star-like polymers according to Dozier (3.2.4). The exponential damping length ξ in the definition of the star polymer has been set to the shell thickness t . In the original paper of Pedersen the P_{local} was described by the scattering of a semi-flexible chain with excluded volume according to section 3.2.9, which however would require to define two more parameters.

For micelles with a rod-like core a few different parameterizations have been implemented ROD+R[^]-a, ROD+R[^]-a_Rc and ROD+R[^]-a_Nagg.

The parameters they all have in common are:

V_{brush} : molecular volume the diblock copolymer part forming the corona
 η_{core} : scattering length density of the diblock copolymer part forming the core
 η_{brush} : scattering length density of the diblock copolymer part forming the corona
 η_{solv} : scattering length density of the solvent
 $x_{\text{solv,core}}$: amount of solvent in the core
 α : exponent of the radial scattering length density profile ($r^{-\alpha}$)
 t : corona thickness
 H : height of the cylinder

For the model ROD+R[^]-a the other parameters are

R_{core} : radius of the micellar core
 n_{agg} : grafting density (number of copolymer molecules N_{agg} per surface are S ,
 $n_{\text{agg}} = N_{\text{agg}}/S$)

In contrast to the form factor ROD+R[^]-a_Rc and ROD+R[^]-a_Nagg this one does not necessary consist of copolymers. The excess scattering lengths and aggregation number are given by

$$N_{\text{agg}} = n_{\text{agg}} S \quad (3.155)$$

where the surface of the core is given by $S = 2\pi R_{\text{core}} H$. Together with the core volume $V = \pi R_{\text{core}}^2 H$ one gets for the excess scattering lengths

$$\beta_{\text{core}} = \frac{V_{\text{core}}(1 - x_{\text{solv,core}})}{N_{\text{agg}}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.156)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.157)$$

Input Parameters for model ROD+R[^]-a:

R_{core} : core radius
 n_{agg} : specific aggregation number (number of chains per surface area)
 V_{brush} : molecular volume of a block unit in the micellar corona
 η_{core} : scattering length density of spherical core
 η_{brush} : scattering length density of the block unit in the corona
 η_{solv} : scattering length density of solvent
 $x_{\text{solv,core}}$: amount of solvent in the core
 α : exponent of the radial scattering length density profile ($r^{-\alpha}$)
 t : corona thickness
 H : rod height

For the model ROD+R[^]-a_Rc the other parameters are

R_{core} : core radius
 V_{core} : molecular volume of single block unit in the micellar core

The excess scattering lengths and aggregation number for eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.158)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.159)$$

$$N_{\text{agg}} = 2\pi R_{\text{core}}^2 H \frac{1 - x_{\text{solv,core}}}{V_{\text{core}}} \quad (3.160)$$

Input Parameters for model ROD+R⁺-a_Rc:

R_core: core radius

V_core: molecular volume of single block unit in the micellar core

V_brush: molecular volume of single block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in the core

alpha: exponent of the radial scattering length density profile ($r^{-\alpha}$)

t: corona thickness

H: rod height

For the model ROD+R⁺-a_Nagg the other parameters are

n_agg: specific aggregation number, aggregation number per surface area

V_core: molecular volume of single block unit in the micellar core

The excess scattering lengths and the core radius R_{core} needed for eq. 3.67 are given by

$$\beta_{\text{core}} = V_{\text{core}}(\eta_{\text{core}} - \eta_{\text{solv}}) \quad (3.161)$$

$$\beta_{\text{brush}} = V_{\text{brush}}(\eta_{\text{brush}} - \eta_{\text{solv}}) \quad (3.162)$$

$$R_{\text{core}} = \frac{2n_{\text{agg}} V_{\text{core}}}{1 - x_{\text{solv,core}}} \quad (3.163)$$

Input Parameters for model ROD+R⁺-a_nagg:

n_agg: specific aggregation number (number of chains per surface area)

V_core: molecular volume of single block unit in the micellar core

V_brush: molecular volume of single block unit in the micellar corona

eta_core: scattering length density of spherical core

eta_brush: scattering length density of the block unit in the corona

eta_solv: scattering length density of solvent

xsolv_core: amount of solvent in the core

alpha: exponent of the radial scattering length density profile ($r^{-\alpha}$)

t: corona thickness

H: rod height

REFERENCES:

[40, 60, 152, 228]

3.2.11.10. *spherical Micelles with a homogeneous core and a corona of semi-flexible interacting self-avoiding chains.*

3.2.12. Sphere with Gaussian chains attached.

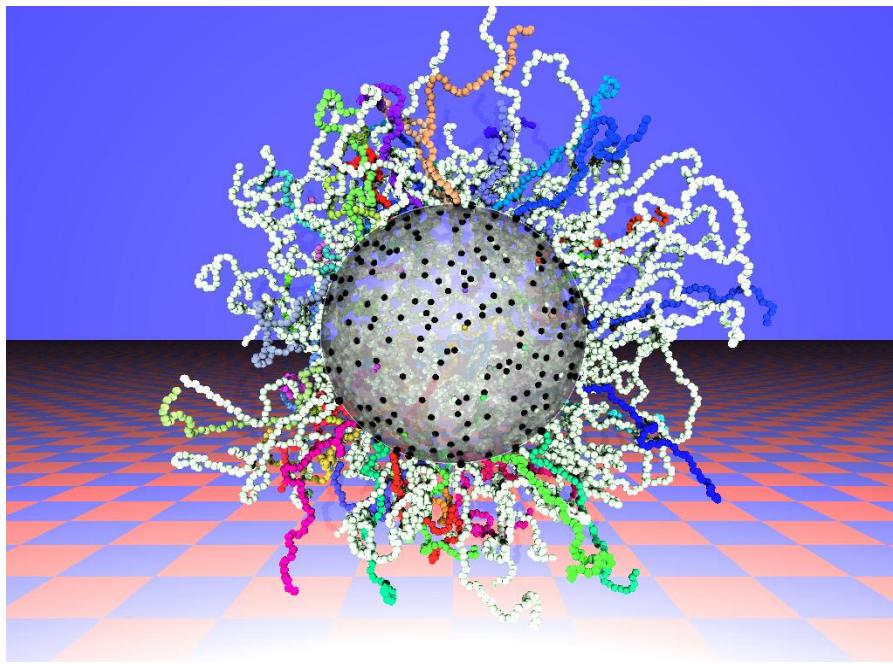


FIGURE 3.42. Block copolymer micelles.

The expressions have been derived by Pedersen and Gerstenberg [166, 164]. For a sphere with radius R and total excess scattering length ρ_s with N_{agg} attached chains

$$\begin{aligned} P_{\text{mic}}(Q) = & N_{\text{agg}}^2 \rho_s^2 P_s(Q, R) + N_{\text{agg}} \rho_c^2 P_c(Q, R_g) \\ & + N_{\text{agg}} (N_{\text{agg}} - 1) \rho_c^2 S_{cc}(Q) + 2 N_{\text{agg}}^2 \rho_s \rho_c S_{sc}(Q) \end{aligned} \quad (3.164)$$

with:

$$P_s = \Phi^2(Q, R) \quad (3.165)$$

$$\Phi(Q, R) = 3 \frac{\sin(QR) - QR \cos(QR)}{(QR)^3} \quad (3.166)$$

$$P_c(Q, R_g) = 2 \frac{\exp(-x) - 1 + x}{x^2} \quad (3.167)$$

$$x = R_g^2 Q^2 \quad (3.168)$$

$$\Psi(Q, R_g) = \frac{1 - \exp(-x)}{x} \quad (3.169)$$

$$S_{cc}(Q) = \Psi^2(Q, R_g) \left(\frac{\sin(Q(R + d R_g))}{Q(R + d R_g)} \right)^2 \quad (3.170)$$

$$S_{sc}(Q) = \Psi(Q, R_g) \Phi(QR_g) \frac{\sin(Q(R + d R_g))}{Q(R + d R_g)} \quad (3.171)$$

where R_g is the root-mean-square radius of gyration of a chain. ρ_c is the total excess scattering length of a single chain. Non-penetrating of the chains into the core region is mimicked by $d \approx 1$ for $R \gg R_g$.

Input Parameters for model `SphereWithGaussChains`:

`R`: radius of core R

`Rg`: gyration radius of chain R_g

`d`: for non-penetration of the chains into the core region $d \approx 1$.

`Nagg`: aggregation number N_{agg}

`rc`: excess scattering length of a block in the chains ρ_c

`rs`: excess scattering length of a block in the core ρ_s

3.2.13. Sphere with Gaussian chains attached (block copolymer micelle).

This form factor is the same than for `SphereWithGaussChains`. It has only been slightly re-parametrised. Instead of the core radius R and excess scattering lengths ρ_s and ρ_c the volumes $V_{\text{polym,c}}$ and $V_{\text{polym,sh}}$ of the block units building the core and the shell are required together with the corresponding scattering length densities $\eta_{\text{poly,c}}$, $\eta_{\text{poly,sh}}$ and that one of the solvent η_{solv} . Furthermore $x_{\text{solv,c}}$ is the amount of solvent in the core which takes account for a possible swelling of the core. These parameters allow one to calculate the core radius and excess scattering lengths by

$$R = \left(\left| \frac{N_{\text{agg}} V_{\text{polym,c}}}{1 - x_{\text{solv,c}}} \right| \frac{3}{4\pi} \right)^{1/3} \quad (3.172)$$

$$\rho_s = V_{\text{polym,c}} (\eta_{\text{poly,c}} - \eta_{\text{solv}}) \quad (3.173)$$

$$\rho_c = V_{\text{polym,sh}} (\eta_{\text{poly,sh}} - \eta_{\text{solv}}) \quad (3.174)$$

The volumes $V_{\text{polym,c}}$ and $V_{\text{polym,sh}}$ can be calculated by knowing the molecular weights⁵ of the block units of the polymer in the core $M_{\text{polym,c}}$ and in the shell $M_{\text{polym,sh}}$ together with their bulk mass densities $\rho_{\text{polym,c}}$ and $\rho_{\text{polym,sh}}$. The volumes are then given by

$$V_{\text{polym,c}} = \frac{M_{\text{polym,c}}}{N_a \rho_{\text{polym,c}}} \quad \text{and} \quad V_{\text{polym,sh}} = \frac{M_{\text{polym,sh}}}{N_a \rho_{\text{polym,sh}}} \quad (3.175)$$

whereby N_a is Avogadro's constant⁶. The units of the block units has to be supplied in units corresponding to the scattering vector Q , i.e. in nm³ in case Q is given in nm⁻¹ or in Å³ in case Q is given in Å⁻¹.

Input Parameters for model `BlockCopolymerMicelle`:

Vpolym_c: volume of a single block unit of the chains in the core $V_{\text{polym,c}}$, it should be given in units of nm³ in case Q is given in nm⁻¹ and in units of Å³ in case Q is given in Å⁻¹.

xsolv_c: amount of solvent in the core ($x_{\text{solv,c}} \neq 1$)

Vpolym_sh: volume of a single block unit of the chains in the shell $V_{\text{polym,sh}}$, it should be given in units of nm³ in case Q is given in nm⁻¹ and in units of Å³ in case Q is given in Å⁻¹.

eta_poly_c: scattering length density of the block units in the core η_c

eta_poly_sh: scattering length density of the block units in the chains η_{sh}

eta_solv: scattering length density of the solvent η_{solv}

Nagg: aggregation number N_{agg}

Rg: gyration radius of chain R_g

d: for non-penetration of the chains into the core region $d \approx 1$.

⁵ $u = 1.66053886 \times 10^{-27}$ kg

⁶ $N_a = 6.0221415 \times 10^{23}$ mol⁻¹

3.3. Clustered Objects

3.3.1. Mass Fractal [198, 197, 97, 130, 131, 132, 133].

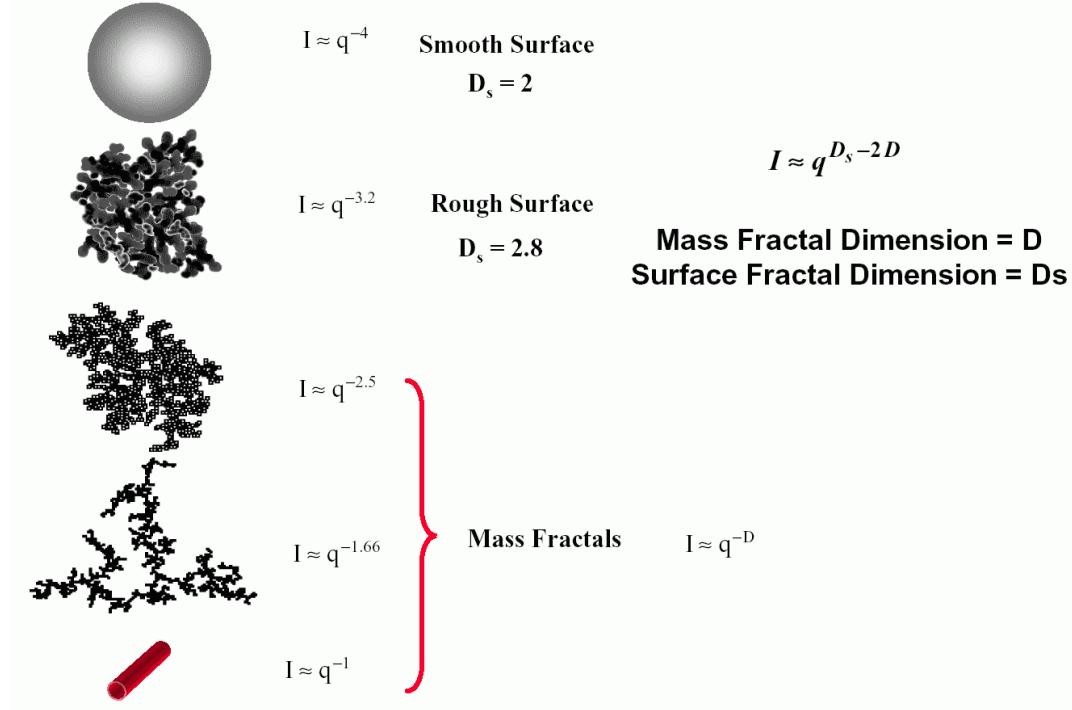


FIGURE 3.43.

Aggregates and clusters often have a fractal morphology. These self-similar clusters are well described by

$$N = k_0(R_g/r_0)^D \quad (3.176)$$

where N is the number of primary particles or monomers in the aggregate, k_0 is a constant of order unity, R_g is the radius of gyration of the aggregate, r_0 is the monomer radius, and D is the fractal dimension.

The scattering function and the density autocorrelation function of the aggregate are Fourier transform pairs; thus

$$I(q) = 4\pi \int_0^\infty g(r) r^2 \frac{\sin(qr)}{qr} dr \quad (3.177)$$

For a fractal aggregate the autocorrelation function has the form

$$g(r) \sim r^{D-d} h(r, \xi) \quad (3.178)$$

Here D is the fractal dimension, d the spatial dimension, and ξ a measure of the linear size of the aggregate proportional to the radius of gyration R_g . The function $h(r, \xi)$ is the cutoff function describing the perimeter of the aggregate. Its properties are that $h(r, \xi) \simeq 1$ for $r/\xi \lesssim 1$, but for large r/ξ it falls off faster than any power law.

TABLE 2. Scattering functions $I(q)$ for different cutoff functions $h(r, \xi)$.

SASfit-name	$h(r, \xi)$	ξ^2	$I(q)$	Ref.
Fischer-Burford	ca. $\exp\left[-\frac{r}{\xi}\right]$	$R_g^2/3$	$\left(1 + \frac{2}{3D}q^2R_g^2\right)^{-D/2}$	
MassFractExp	$\exp\left[-\frac{r}{\xi}\right]$	$\frac{2R_g^2}{D(D+1)}$	$\frac{\sin[(D-1)\arctan(q\xi)]}{(D-1)q\xi(1+q^2\xi^2)^{(D-1)/2}}$	
MassFractGauss	$\exp\left[-\left(\frac{r}{\xi}\right)^2\right]$	$\frac{4R_g^2}{D}$	$e^{-\frac{q^2R_g^2}{D}} {}_1F_1\left[\frac{3-D}{2}, \frac{3}{2}, \frac{q^2R_g^2}{D}\right]$	
Aggregate ($\text{Exp}(-x^\alpha)$ Cut-Off)	$\exp\left[-\left(\frac{r}{\xi}\right)^\alpha\right]$	—	numerical	
Aggregate (OverlapSph Cut-Off)	$\begin{cases} \left(1 + \frac{r}{4\xi}\right)\left(1 - \frac{r}{2\xi}\right)^2, & r < 2\xi \\ 0, & r \geq 2\xi \end{cases}$	$\frac{(D+2)(D+5)}{2D(D+1)}R_g^2$	numerical	
DLCAggregate	—	—	$\left[1 + \sum_{s=1}^4 C_s (qR_g)^{2s}\right]^{-D/8}$ $C_1 = \frac{8}{3}D, C_2 = 2.5$ $C_3 = -1.52, C_4 = 1.02$	
RLCAggregate	—	—	$C_1 = \frac{8}{3}D, C_2 = 3.13$ $C_3 = -2.58, C_4 = 0.95$	

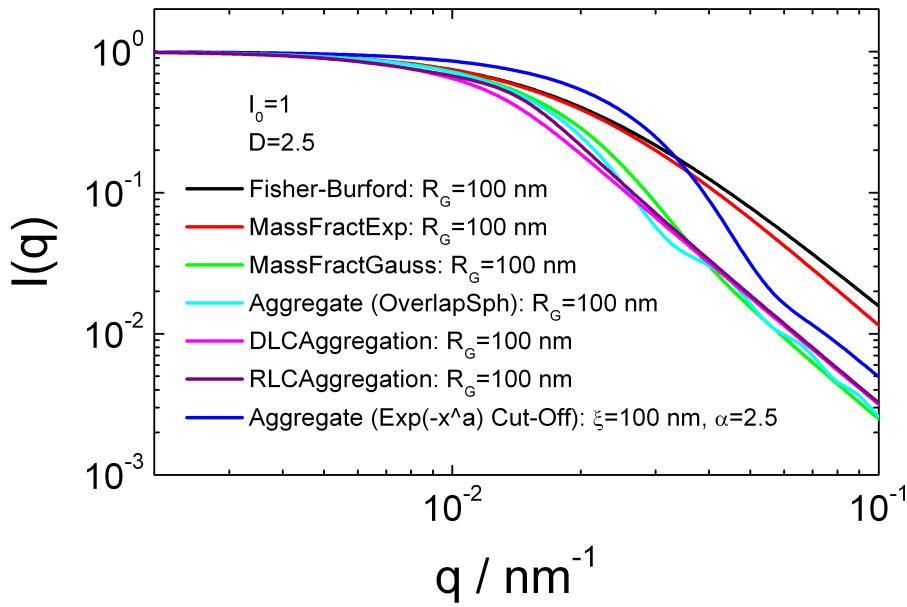


FIGURE 3.44. Form factor for the different types of mass fractals listed in 2.

3.3.2. Stacked Discs [120, 83].

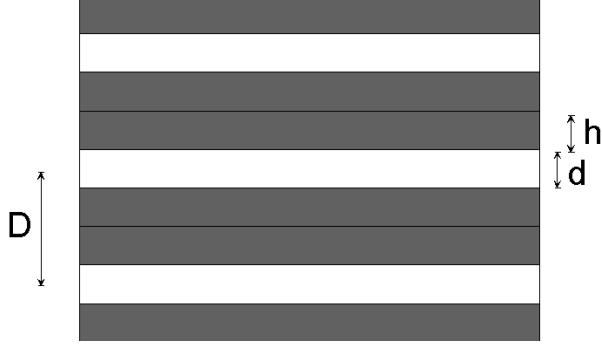


FIGURE 3.45. Sketch for a stack of discs with an additional surface layer

$$I_{\text{StackedDiscs}}(Q, R) = \int_0^{\pi/2} (\Delta\eta_l (V_t f_t - V_c f_c) + \Delta\eta_c V_c f_c)^2 S(Q, \Theta) \sin(\Theta) d\Theta \quad (3.179)$$

Here it is assumed that the nearest neighbor distance between the platelets obeys a Gaussian distribution and consider an internal structure factor, $S(Q, \Theta)$, first proposed by Kratky and Porod in 1949 [120]

$$S(Q, \Theta) = 1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \cos(kDQ \cos(\Theta)) \exp\left(-\frac{k}{2} (Q \cos(\Theta) \sigma_D)^2\right) \quad (3.180)$$

$$f_t = f_t = \frac{\sin(Q(\frac{d}{2} + h) \cos(\Theta))}{Q(\frac{d}{2} + h) \cos(\Theta)} - 2 \frac{J_1(QR \sin(\Theta))}{QR \sin(\Theta)} \quad (3.181)$$

$$f_c = f_c = \frac{\sin(Q\frac{d}{2} \cos(\Theta))}{Q\frac{d}{2} \cos(\Theta)} - 2 \frac{J_1(QR \sin(\Theta))}{QR \sin(\Theta)} \quad (3.182)$$

$$V_t = \pi R^2(d + 2h) \quad (3.183)$$

$$V_c = \pi R^2 d \quad (3.184)$$

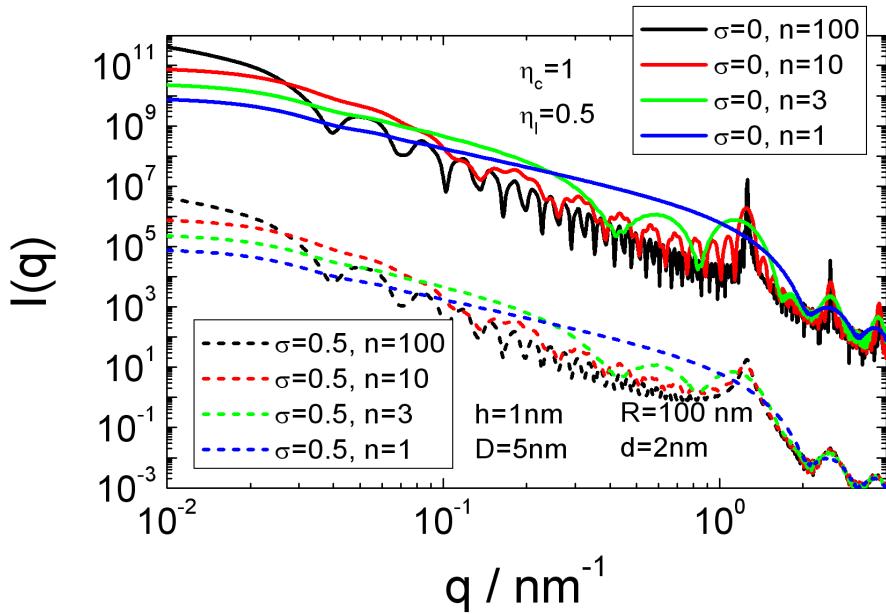


FIGURE 3.46. Scattering Intensity for a stack of discs with a layer.

3.3.3. DumbbellShell.

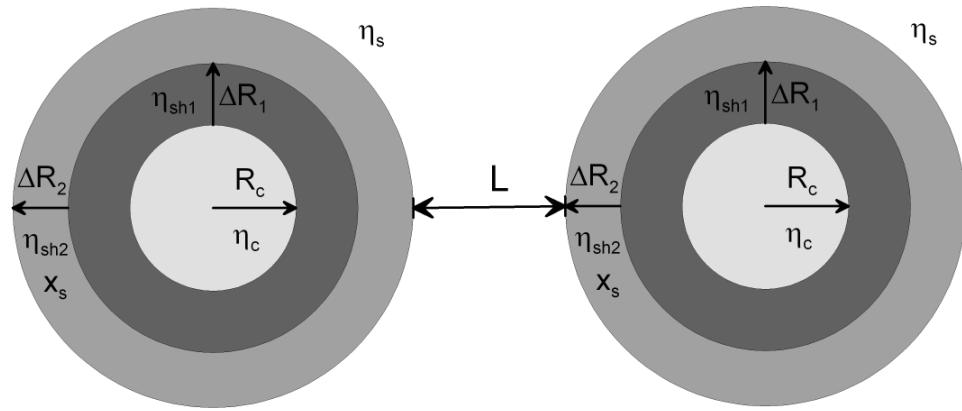


FIGURE 3.47.

3.3.4. DoubleShellChain.

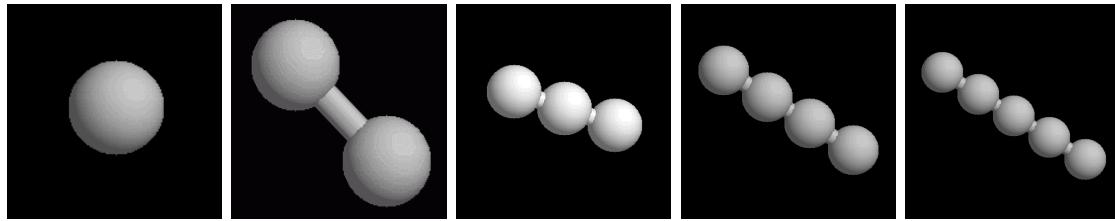


FIGURE 3.48.

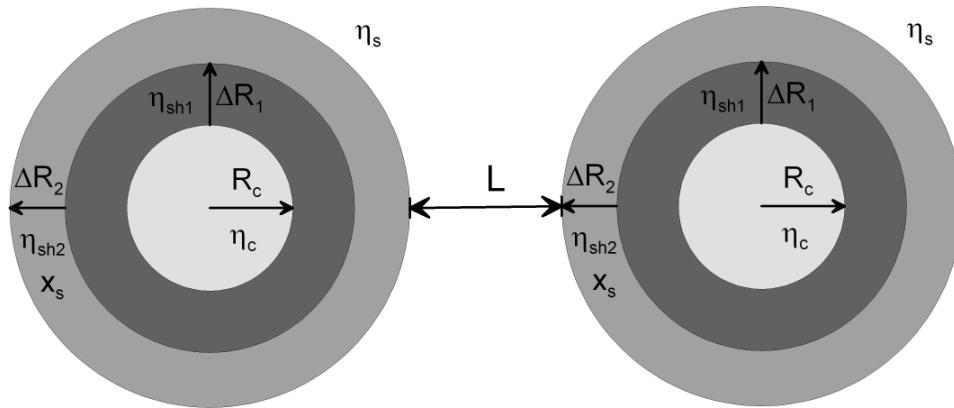


FIGURE 3.49.

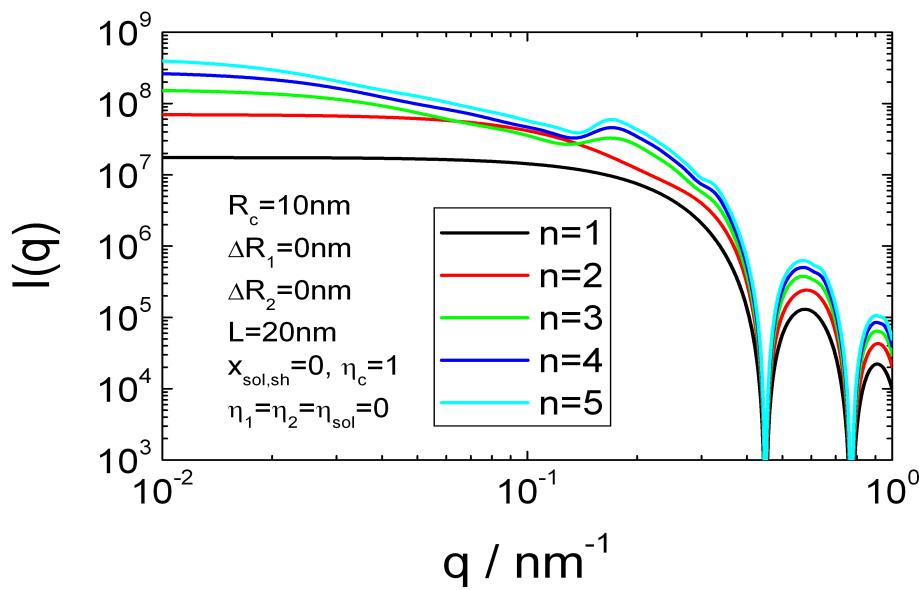


FIGURE 3.50.

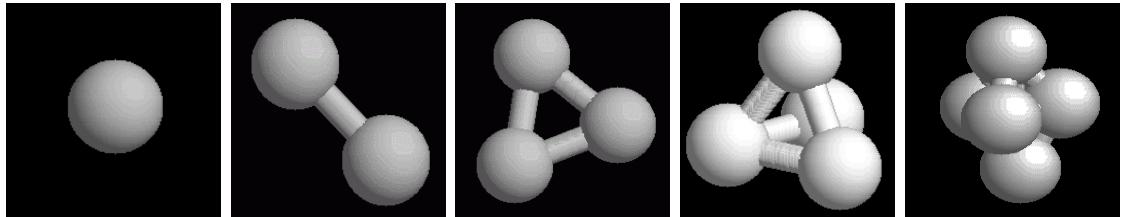


FIGURE 3.51.

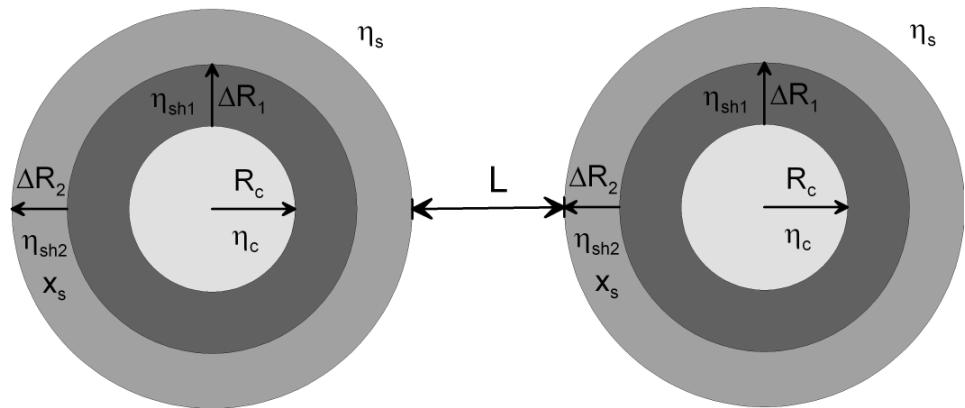


FIGURE 3.52.

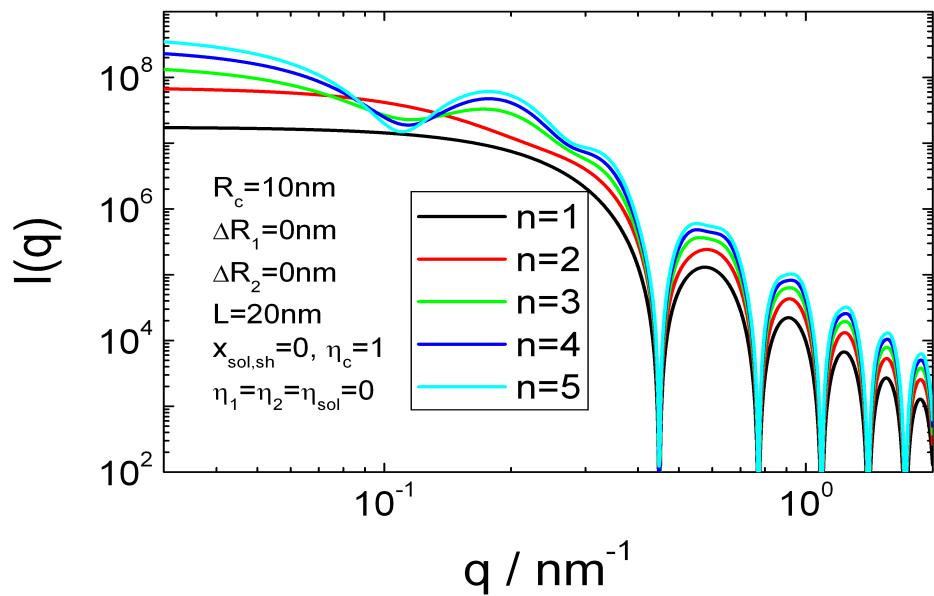


FIGURE 3.53.

3.3.5. simplified cluster.

3.4. Cylindrical Objects

3.4.1. Disc.



FIGURE 3.54.

$$I_{\text{Disc}}(q, R) = \pi^2 R^4 \Delta\eta^2 \frac{2}{(qR)^2} \left(1 - \frac{1}{qR} J_1(2qR) \right) \quad (3.185)$$

with $\lim_{q \rightarrow 0} I_{\text{Disc}}(q, R) = \pi^2 R^4 \Delta\eta^2$

Input Parameters for model Disc:

R: radius of disc R

eta: scattering contrast $\Delta\eta$

Note:

- none

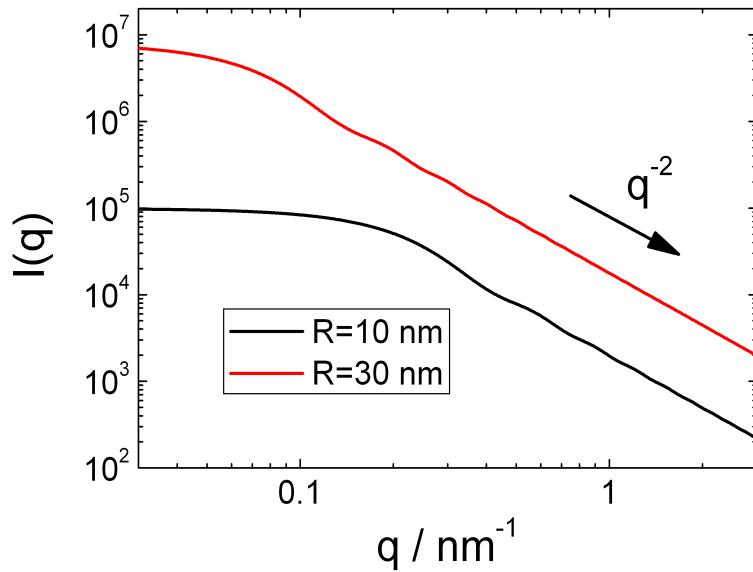


FIGURE 3.55. Scattering intensity of a disc with radii $R = 10$ nm and $R = 30$ nm. The scattering length density contrast is set to 1.

3.4.2. Rod.



FIGURE 3.56.

$$I_{\text{Rod}}(q, L) = \Delta\eta^2 L^2 \left(\frac{2}{qL} \text{Si}(qL) - \left(\frac{\sin(qL/2)}{qL/2} \right)^2 \right) \quad (3.186)$$

with $\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt$ and $\lim_{q=0} I_{\text{Rod}}(q, L) = \Delta\eta^2 L^2$

Input Parameters for model Rod:

L: length of rod L

eta: scattering contrast $\Delta\eta$

Note:

- none

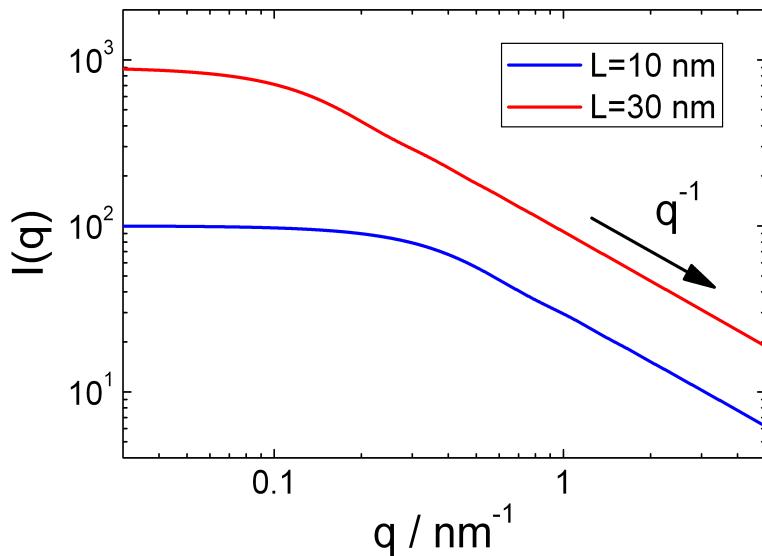


FIGURE 3.57. Scattering intensity of a rod of length $L = 10$ nm and $L = 30$ nm. The scattering length density contrast is set to 1.

3.4.3. Porod's approximation for a long cylinder [173].

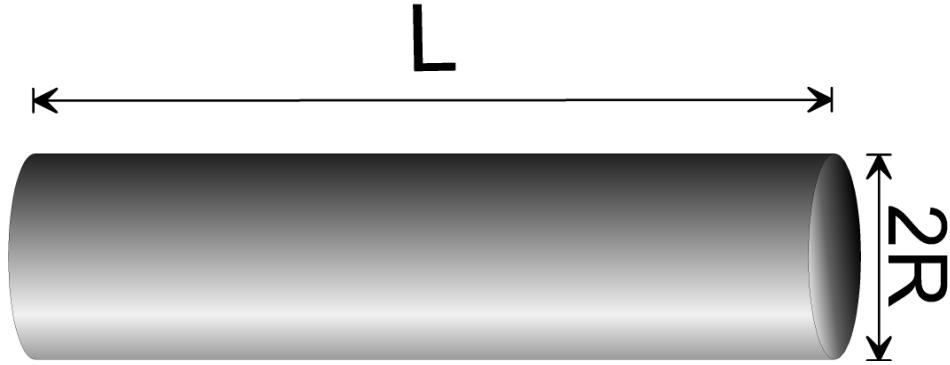


FIGURE 3.58.

$$\text{Si}_{\frac{\pi}{2}}(x) = \left(\text{Si}(x) + \frac{\cos x}{x} + \frac{\sin x}{x^2} \right) \xrightarrow{x \rightarrow \infty} \frac{\pi}{2} \quad (3.187)$$

$$\Lambda_1(x) = \frac{2}{x} J_1(x) \quad (3.188)$$

$$\Lambda_2(x) = \frac{8}{x^2} J_2(x) \quad (3.189)$$

$$\omega(x) = \frac{8}{x^2} (3J_2(x) + J_0(x) - 1) \quad (3.190)$$

$$\begin{aligned} \Phi_{\text{long}}(q, R, L) &= (\Delta\eta\pi R^2 L)^2 \frac{2}{QL} \\ &\times \left\{ \text{Si}_{\frac{\pi}{2}}(QL)\Lambda_1^2(QR) - \frac{\omega(2QR)}{QL} - \frac{\sin(QL)}{(QL)^2} \right\} \end{aligned} \quad (3.191)$$

$J_n(x)$ are the regular cylindrical Bessel function of order n .

Input Parameters for model LongCylinder:

R: radius of cylinder R

L: length of cylinder L

eta: scattering contrast $\Delta\eta$

Note:

- The approximation is valid for $L > 2R$

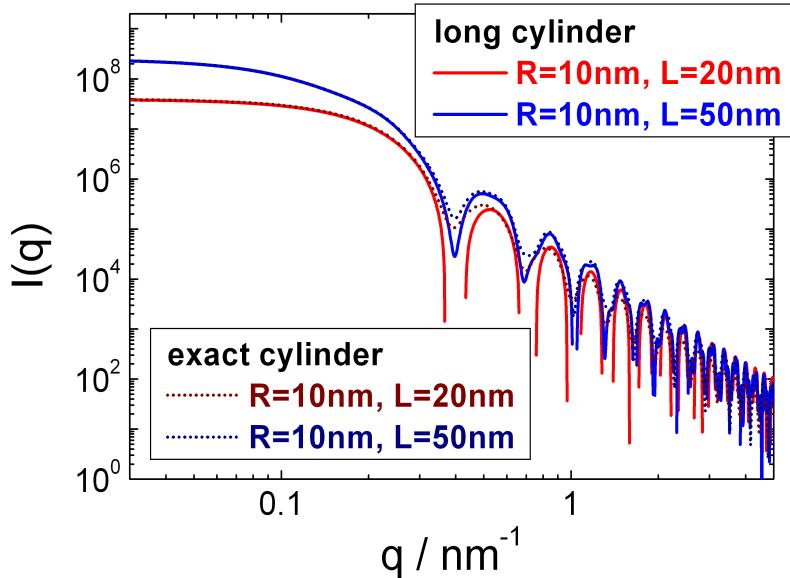


FIGURE 3.59. Scattering intensity of a cylinder with radius $R = 10$ nm and lengths of $L = 20$ nm and $L = 50$ nm. Next to Porod's approximation for long cylinders also the exact integral solution is shown for comparison. The scattering length density contrast is set to 1.

3.4.4. Porod's approximation for a flat cylinder [173].

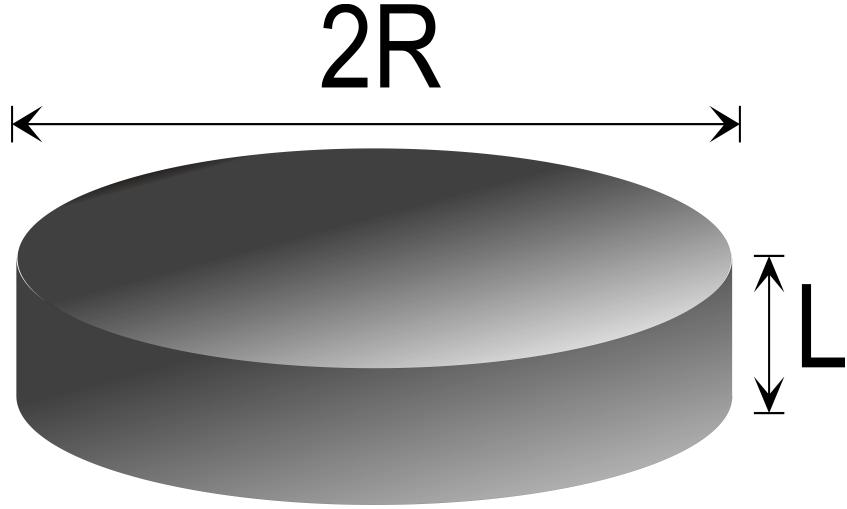


FIGURE 3.60.

$$\Lambda_1(x) = \frac{2}{x} J_1(x) \quad (3.192)$$

$$I_1(x) = \int_0^x \Lambda_1(x') dx' = \quad (3.193)$$

$$2xJ_0(x) - 2J_1(x) - \pi x [J_0(x)H_1(x) - J_1(x)H_0(x)]$$

$$I_0(x) = \frac{I_1(x) + x\Lambda_1(x)}{2} \quad (3.194)$$

$$\Omega(x) = \frac{2}{x} [I_0(x) - 2J_1(x)] \quad (3.195)$$

$$\chi(x) = \left(\frac{\sin(x/2)}{x/2} \right)^2 \quad (3.196)$$

$$\begin{aligned} \Phi_{\text{flat}}(q, R, L) &= (\Delta\eta\pi R^2 L)^2 \frac{8}{(2qR)^2} \\ &\times \left\{ \chi(qL) + \frac{I_1(2QR) \Omega(qL)}{2qR} - \Lambda_1(2qR) \right\} \end{aligned} \quad (3.197)$$

$H_\alpha(x)$ is the Struve function of order α and $J_n(x)$ are the regular cylindrical Bessel function of order n .

Input Parameters for model FlatCylinder:

R: radius of cylinder R

L: length of cylinder L

eta: scattering contrast $\Delta\eta$

Note:

- The approximation is valid for $L < 2R$

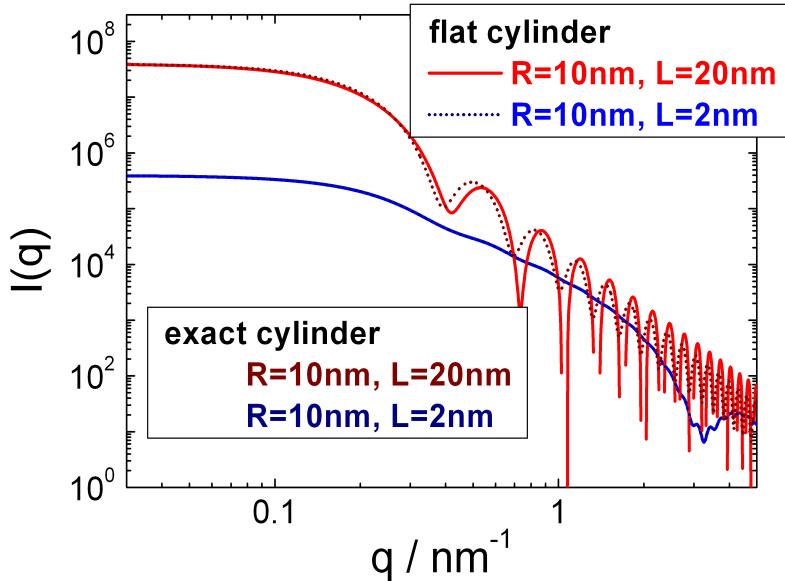


FIGURE 3.61. Scattering intensity of a cylinder with radius $R = 10$ nm and lengths of $L = 2$ nm and $L = 20$ nm. Next to Porod's approximation for flat cylinders also the exact integral solution is shown for comparison. The scattering length density contrast is set to 1.

3.4.5. Porod's approximations for cylinder [173].

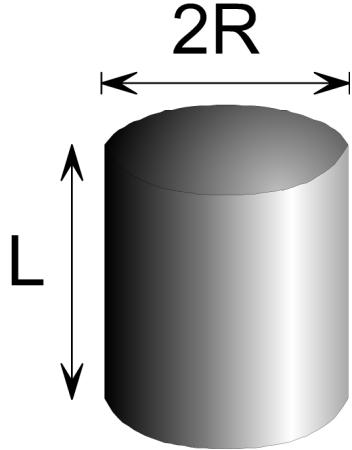


FIGURE 3.62.

This form factor combines the two solutions of Porod for a long $\Phi_{\text{long}}(q, R, L)$ (3.191) and a flat $\Phi_{\text{flat}}(q, R, L)$ (3.197) cylinder by a linear combination of both. A simple linear transition at $L = 2R$ is assumed.

$$\Phi_{\text{Porod}}(q, R, L) = p \left(\frac{2R}{L} \right) \Phi_{\text{flat}}(q, R, L) + \left(1 - p \left(\frac{2R}{L} \right) \right) \Phi_{\text{long}}(q, R, L) \quad (3.198)$$

$$p(x) = \begin{cases} 1 & \text{for } x > \frac{5}{4} \\ 2 \left(x - \frac{3}{4} \right) & \text{for } \frac{3}{4} \leq x \leq \frac{5}{4} \\ 0 & \text{for } x < \frac{3}{4} \end{cases} \quad (3.199)$$

Input Parameters for model PorodCylinder:

R: radius of cylinder R

L: length of cylinder L

eta: scattering contrast $\Delta\eta$

Note:

- less good approximation for $L \sim 2R$

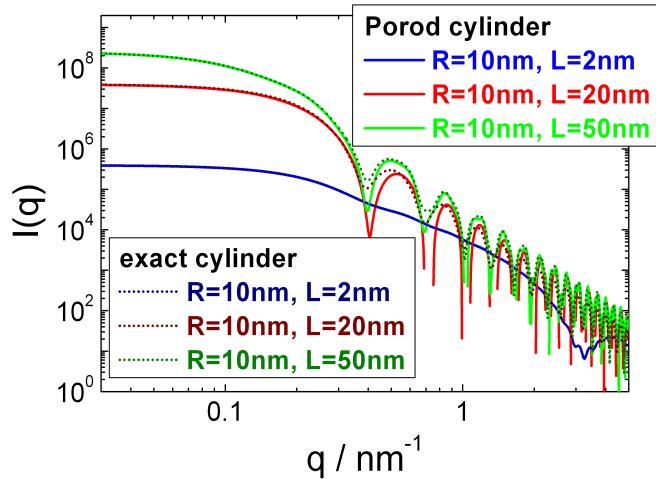


FIGURE 3.63. Scattering intensity of a cylinder with radius $R = 10$ nm and lengths of $L = 2$ nm, $L = 20$ nm, and $L = 50$ nm. Next to Porod's approximation for a cylinders also the exact integral solution is shown for comparison. The scattering length density contrast is set to 1.

3.4.6. Cylinder of length L , radius R and scattering contrast $\Delta\eta$.

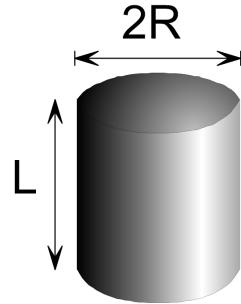


FIGURE 3.64.

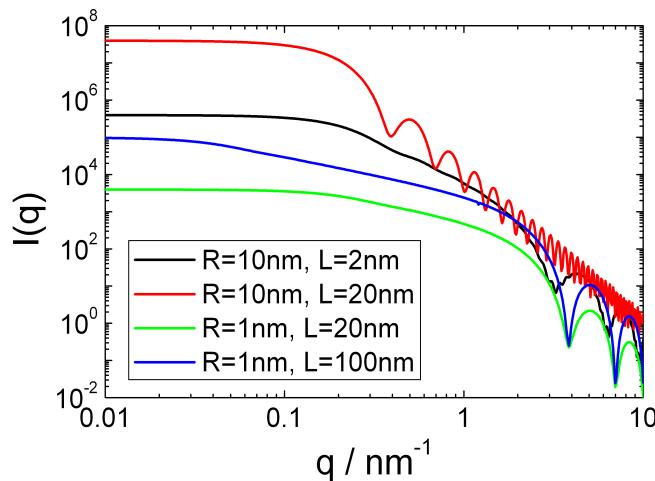
$$I_{\text{cyl}} = 16(\pi R^2 L)^2 \Delta\eta^2 \int_0^1 \left(\frac{J_1(QR\sqrt{1-x^2}) \sin(QLx/2)}{Q^2 R \sqrt{1-x^2} L x} \right)^2 dx \quad (3.200)$$

Input Parameters for model Cylinder:

- R: radius of cylinder R
- L: length of cylinder L
- eta: scattering contrast $\Delta\eta$

Note:

- None

FIGURE 3.65. Scattering intensity of a cylinder for different radii radius R nm and lengths L . The scattering length density contrast is set to 1.

3.4.7. Random oriented cylindrical shell with circular cross-section.

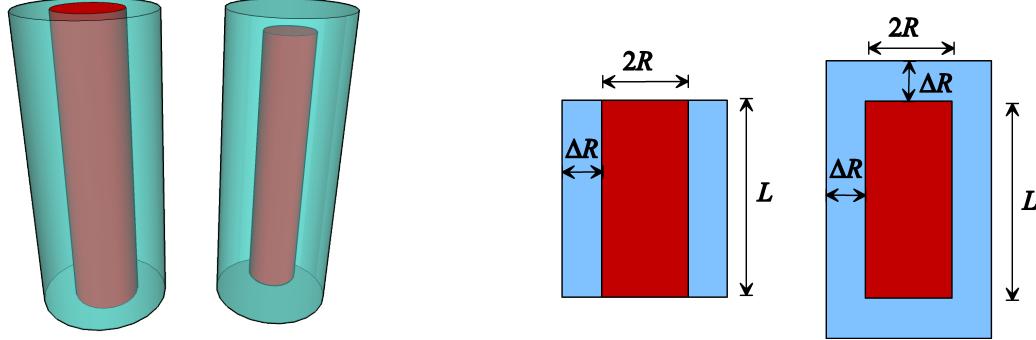


FIGURE 3.66. cylindrical shell with circular cross-section

To different versions for a random oriented cylindrical shell with a circular cross-section has been implemented. One without `CylShell1` and one with `CylShell2` capped ends. For very long cylinders a faster approximation for the uncapped version can be used `LongCylShell`

$$K_{\text{Cyl}}(Q, \Delta\eta, R, L, x) = 2\pi R^2 L \Delta\eta \frac{J_1(QR\sqrt{1-x^2})}{QR\sqrt{1-x^2}} \frac{\sin(QLx/2)}{QLx/2} \quad (3.201)$$

$$I_{\text{CylShell1}} = \int_0^1 \left(K_{\text{Cyl}}(Q, \eta_{\text{core}} - \eta_{\text{shell}}, R, L, x) + K_{\text{Cyl}}(Q, \eta_{\text{shell}} - \eta_{\text{solv}}, R + \Delta R, L, x) \right)^2 dx \quad (3.202)$$

$$I_{\text{CylShell2}} = \int_0^1 \left(K_{\text{Cyl}}(Q, \eta_{\text{core}} - \eta_{\text{shell}}, R, L, x) + K_{\text{Cyl}}(Q, \eta_{\text{shell}} - \eta_{\text{solv}}, R + \Delta R, L + 2\Delta R, x) \right)^2 dx \quad (3.203)$$

$$I_{\text{LongCylShell}}(Q) = P'(Q)P_{\text{cs}}(Q) \quad (3.204)$$

$$P'(Q) = 2 \frac{\text{Si}(QL)}{QL} - \left(\frac{\sin(QL/2)}{QL/2} \right)^2 \quad (3.205)$$

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad (3.206)$$

$$P_{\text{cs}}(Q) = \left(2 \frac{J_1(QR)}{QR} (\eta_{\text{core}} - \eta_{\text{shell}}) R^2 L \pi + 2 \frac{J_1(Q(R + \Delta R))}{Q(R + \Delta R)} (\eta_{\text{shell}} - \eta_{\text{solv}}) (R + \Delta R)^2 L \pi \right)^2 \quad (3.207)$$

Input Parameters for models CylShell1, CylShell2 and LongCylShell:

R: core radius R

DR: shell thickness ΔR

L: cylinder length L

eta_core: scattering length density η_{core} of cylinder core

eta_shell: scattering length density η_{shell} of cylinder shell

eta_solv: scattering length density η_{solv} of solvent

Note:

- The approximation for a long cylindrical shell (LongCylShell) only holds for $L \gg 2R$.

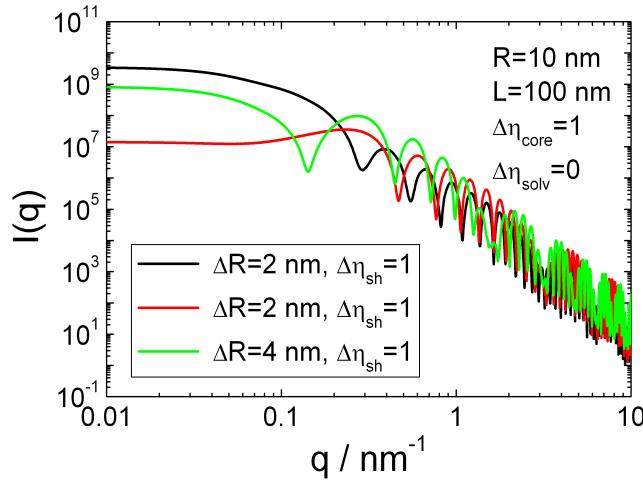


FIGURE 3.67. Scattering intensity of a cylinder shell CylShell1.

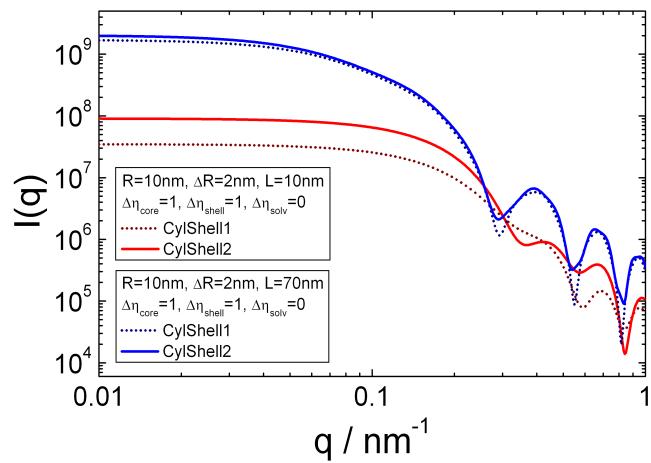


FIGURE 3.68. Scattering intensity of a cylinder shell CylShell12.

3.4.8. Random oriented cylindrical shell with elliptical cross-section.

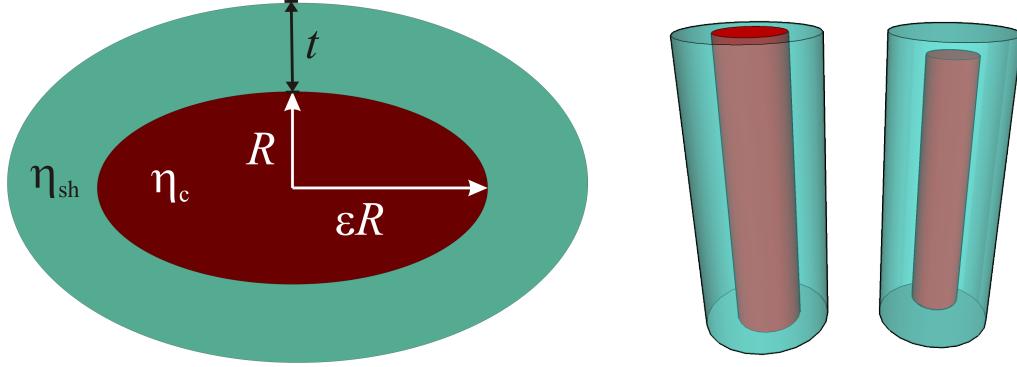


FIGURE 3.69. cylindrical shell with elliptical cross-section

To different versions for a random oriented cylindrical shell with an elliptical cross-section has been implemented. One without `ellCylShell1` and one with `ellCylShell2` capped ends.

$$K_{\text{ellCyl}}(q, \Delta\eta, R, \epsilon, L, t, \phi, \alpha) = \pi\epsilon R(\epsilon R + t)L\Delta\eta \quad (3.208)$$

$$\times \frac{2J_1(qr(R, \epsilon, \phi, \alpha))}{qr(R, \epsilon, \phi, \alpha)} \frac{\sin(q\frac{L}{2}\cos(\alpha))}{q\frac{L}{2}\cos(\alpha)}$$

$$r(R, \epsilon, t, \phi, \alpha) = \sqrt{R^2 \sin^2(\phi) + (\epsilon R + t)^2 \cos^2(\phi)} \sin(\alpha) \quad (3.209)$$

$$I_{\text{ellCylShell1}}(q) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(K_{\text{ellCyl}}(q, \eta_{\text{core}} - \eta_{\text{shell}}, R, \epsilon, L, 0, \phi, \alpha) \right. \quad (3.210)$$

$$\left. + K_{\text{ellCyl}}(q, \eta_{\text{shell}} - \eta_{\text{sol}}, R, \epsilon, L, t, \phi, \alpha) \right)^2 \sin(\alpha) d\alpha d\phi$$

$$I_{\text{ellCylShell2}}(q) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(K_{\text{ellCyl}}(q, \eta_{\text{core}} - \eta_{\text{shell}}, R, \epsilon, L, 0, \phi, \alpha) \right. \quad (3.211)$$

$$\left. + K_{\text{ellCyl}}(q, \eta_{\text{shell}} - \eta_{\text{sol}}, R, \epsilon, L+2t, t, \phi, \alpha) \right)^2 \sin(\alpha) d\alpha d\phi$$

Input Parameters for models `ellCylShell1` and `ellCylShell2`:

R: core radius R

epsilon: eccentricity ϵ of cross-section

L: cylinder length L

t : shell thickness t
 η_{core} : scattering length density η_{core} of cylinder core
 η_{shell} : scattering length density η_{shell} of cylinder shell
 η_{sol} : scattering length density η_{sol} of solvent

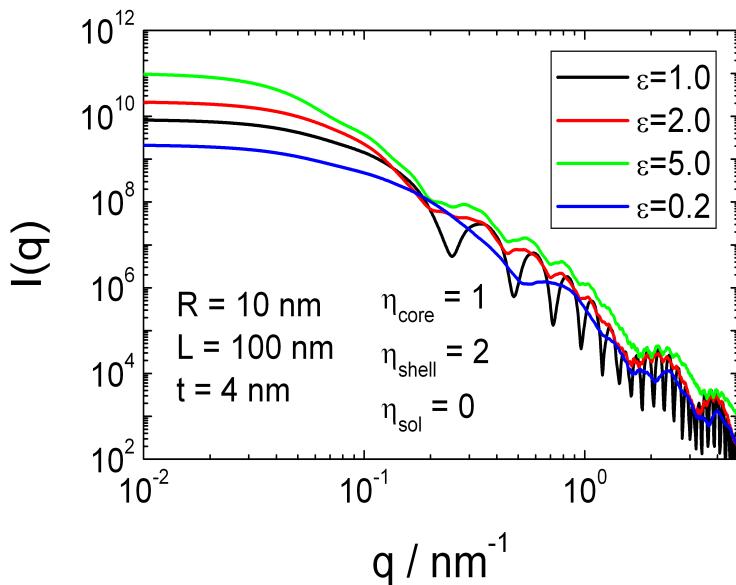


FIGURE 3.70. Scattering intensity of a cylinder with elliptical cross-section.

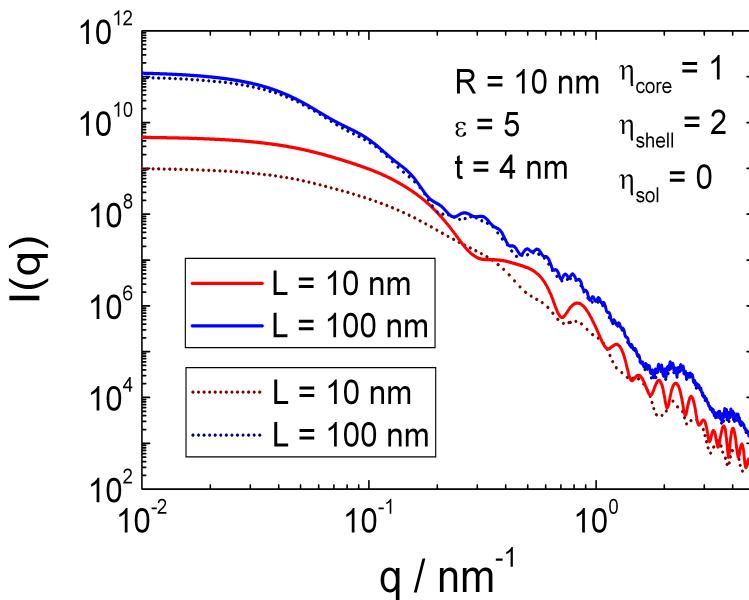


FIGURE 3.71. Scattering intensity of a cylinder with elliptical cross-section with and without capped ends.

For very long cylinders a faster approximation for the uncapped version can be used **Pcs:ellCylSh** combined with the structure factor $P'(Q)$:Rod. The implemented approximation is the following

$$I_{\text{Long-ellCylShell}}(q) = P'(q)P_{\text{cs}}(q) \quad (3.212)$$

$$P'(q) = 2 \frac{\text{Si}(qL)}{qL} - \left(\frac{\sin(qL/2)}{qL/2} \right) \quad (3.213)$$

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad (3.214)$$

$$r(R, \epsilon, \phi) = \sqrt{R^2 \sin^2(\phi) + (\epsilon R + t)^2 \cos^2(\phi)} \quad (3.215)$$

$$P_{\text{cs}}(q) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(\frac{2J_1(qr(R, \epsilon, \phi))}{qr(R, \epsilon, \phi)} (\eta_{\text{core}} - \eta_{\text{shell}}) \epsilon R^2 L \pi + \frac{2J_1(q(r(R, \epsilon, \phi) + t))}{Q(r(R, \epsilon, \phi) + t)} (\eta_{\text{shell}} - \eta_{\text{sol}}) (R + t)(\epsilon R + t) L \pi \right)^2 d\phi \quad (3.216)$$

Input Parameters for model **Pcs:ellCylSh**:

R: core radius R

epsilon: eccentricity ϵ of cross-section

t: shell thickness t

eta_core: scattering length density η_{core} of cylinder core

eta_shell: scattering length density η_{shell} of cylinder shell

eta_sol: scattering length density η_{sol} of solvent

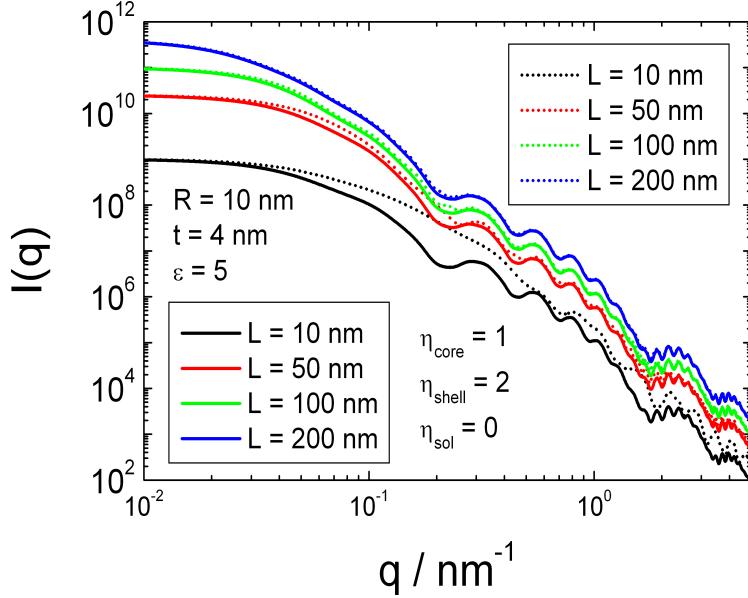


FIGURE 3.72. Scattering intensity of a cylinder with elliptical cross-section. The exact solutions `ellCylShe111` (dotted lines) are compared with `Pcs:ellCylSh` (solid lines), which is only valid for very long cylinders $L \gg 2R$.

3.4.9. Torus with elliptical shell cross-section [107, 61].

$$F_{\text{torus}}(Q, \Theta, R, x, \nu, \Delta\eta) = \int_{R-x}^{R+x} 4\pi r \Delta\eta \frac{J_0(Qr \sin \Theta) \sin(Q\gamma(r) \cos \Theta)}{Q \cos(\Theta)} dr \quad (3.217)$$

$$\text{with } \gamma(r) = \nu \sqrt{x^2 - (r - R)^2} \quad (3.218)$$

$$I_{\text{torus}}(Q, R, a, \nu, \Delta\eta) = \int_0^{\pi/2} |F_{\text{torus}}(Q, \Theta, R, a, \nu, \Delta\eta)|^2 \sin \Theta d\Theta \quad (3.219)$$

$$I_{\text{torus,sh}}(Q, R, a, \Delta a, \nu, \Delta\eta_{sh}, \Delta\eta_c) = \int_0^{\pi/2} \left| F_{\text{torus}}(Q, \Theta, R, a + \Delta a, \nu, \Delta\eta_{sh}) - F_{\text{torus}}(Q, \Theta, R, a, \nu, \Delta\eta_c) \right|^2 \sin \Theta d\Theta \quad (3.220)$$

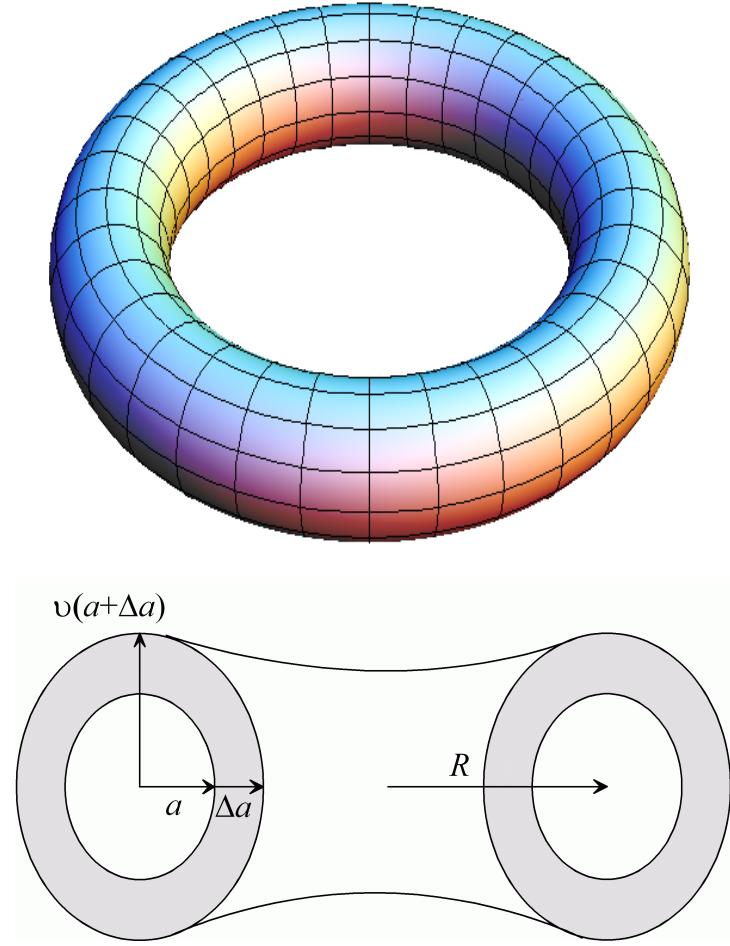


FIGURE 3.73.

An alternative form factor for F_{torus} following [61] is

$$F_{\text{torus}}(Q, \Theta, R, x, \nu, \Delta\eta) = 2\pi \int_{-x}^x \left[R_{(+)} J_1(QR_{(+)} \sin \theta) - R_{(-)} J_1(QR_{(-)} \sin \theta) \right] \frac{\cos(Qz \cos \theta)}{Q \sin \theta} dz \quad (3.221)$$

with $R_{(\pm)} = R \pm \nu \sqrt{x^2 - z^2}$

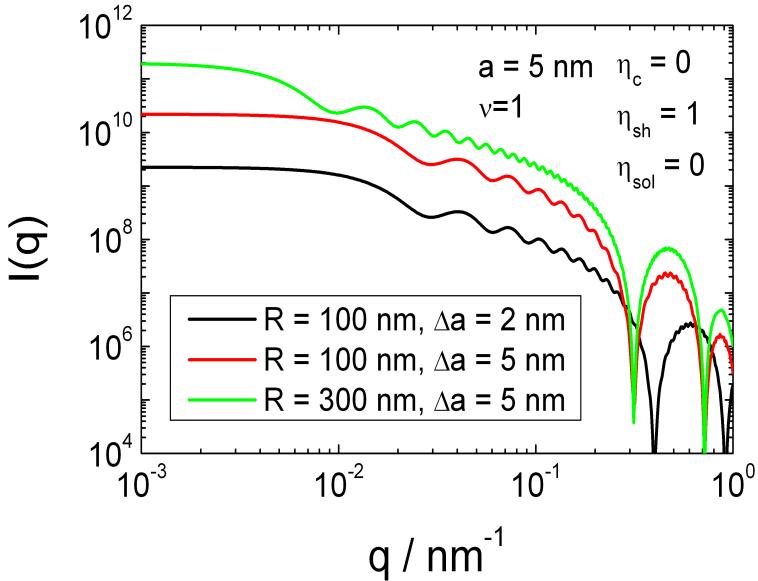


FIGURE 3.74. Scattering intensity of a torus.

3.4.10. stacked tori with elliptical shell cross-section.

$$F_{\text{stackedtori}}(Q, \Theta, R, x, \nu, \Delta\eta, \Delta D, N) = \quad (3.222)$$

$$\sum_{n=1}^N \int_{R-x}^{R+x} 4\pi r \Delta\eta \frac{J_0(Qr \sin \Theta) \sin(Q(\gamma(r) + \frac{2(n-1)-(N-1)}{4} \Delta D) \cos \Theta)}{Q \cos(\Theta)} dr$$

The scattering intensity is than calculated in the same way as for a single torus with an elliptical shell cross-section.

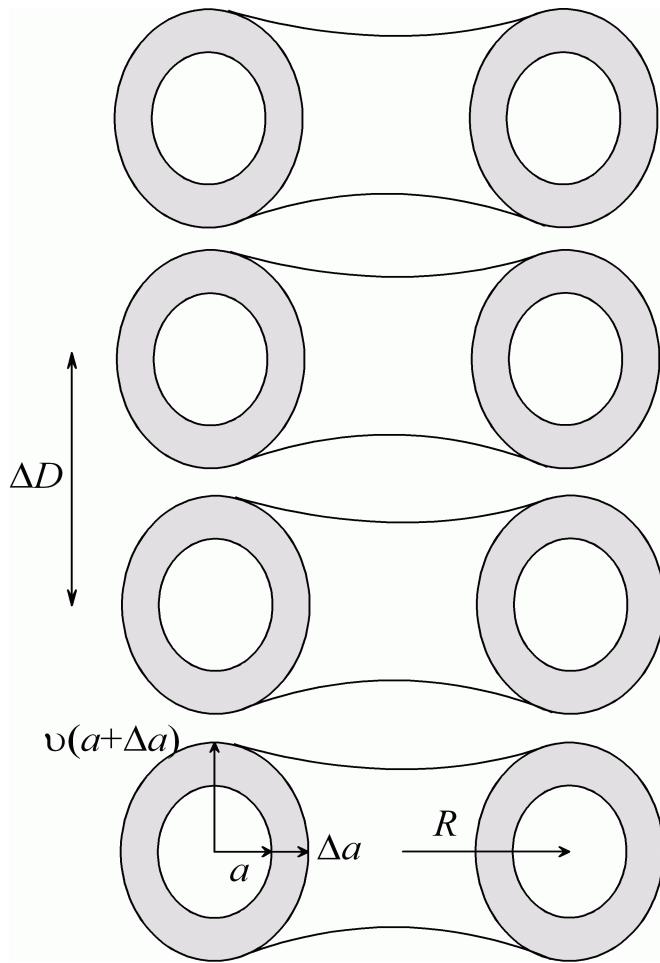


FIGURE 3.75.

3.5. Local Planar Objects

The form factor of very anisotropic particles with local planar geometry can be shown to factorize (see, e.g. [173]) into a cross-section form factor $P_{\text{cs}}(Q)$ for the shorter dimensions and a shape factor $P'(Q)$ for the larger dimension:

$$P_{\text{planar}}(Q) = P'(Q)P_{\text{cs}}(Q) \quad (3.223)$$

The factorization in the cross-section factor $P_{\text{cs}}(Q)$, which only depends on parameters describing the inner structure of the layer with short dimensions and the shape factor $P'(Q)$, describing the overall shape in the larger dimension has the big advantage when both the shorter dimension as well as the larger dimensions have a polydispersity. In this case we do not end up with a double integral but rather a product of two integrals when both a short and a large dimension have a polydispersity. This speeds up the numerical computation significantly. Therefore the following form factors already have a polydispersity parameter included.

3.5.1. Shape factors $P'(Q)$.

The shape form factor $P'(Q)$ are normalized for $Q \rightarrow 0$ on the squared surface area S ($\lim_{Q \rightarrow 0} P'(Q) = S^2$) and can be that of an infinitely thin disc, spherical shell, elliptical shell, or cylindrical shell: The shape factors are accessible as a structure factor under [anisotropic obj.| $P'(Q)$:local planar geometry| $P'(Q)$ xxx] and using the monodisperse approximation. Actually these shape factors are foreseen to be used with the cross-section factors available as form factors under [anisotropic obj.| $P_{\text{cs}}(Q)$ for planar obj.| $P_{\text{cs}}(Q)$ xxx]. The shape factors are also available in combination with some cross-section factors as form factors under [planar obj.].

3.5.1.1. Polydisperse infinitesimal thin discs.

$$P'_{\text{disc}}(Q, R) = \frac{2\pi^2 R^4}{(QR)^2} \left(1 - \frac{J_1(2QR)}{QR} \right) \quad (3.224)$$

The polydispersity is included as a LogNormal-distribution from section 6.4 by

$$P'_{\text{ThinDiscs}}(Q, R, \sigma) = \int_0^\infty \text{LogNorm}(R', R, \sigma, 1) P'_{\text{disc}}(Q, R') dR' \quad (3.225)$$

3.5.1.2. Infinitesimal thin spherical shell.

$$P'_{\text{sph. shell}}(Q, R) = \left(4\pi R^2 \frac{\sin QR}{QR} \right)^2 \quad (3.226)$$

$$(3.227)$$

3.5.1.3. *Infinitesimal thin elliptical shell.*

$$P'_{\text{ell. shell}}(Q, R, \epsilon) = S^2 \int_0^{\pi/2} \left(\frac{\sin(QR\sqrt{\sin^2 \alpha + \epsilon^2 \cos^2 \alpha})}{QR\sqrt{\sin^2 \alpha + \epsilon^2 \cos^2 \alpha}} \right)^2 \sin(\alpha) d\alpha \quad (3.228)$$

$$\text{with } S = \begin{cases} 4\pi R^2 & \text{for } \epsilon = 1 \\ 2\pi R^2 \left(1 + \epsilon \frac{\arccos(1/\epsilon)}{\tan(\arccos(1/\epsilon))} \right) & \text{for } \epsilon > 1 \\ 2\pi R^2 \left(1 + \epsilon \frac{\operatorname{arctanh}(\sin(\arccos(\epsilon)))}{\sin(\arccos(\epsilon))} \right) & \text{for } \epsilon < 1 \end{cases} \quad (3.229)$$

3.5.1.4. *Infinitesimal thin cylindrical shell.*

$$P'_{\text{closed cyl. sh.}}(Q, R, H) = \int_0^{\pi/2} (2\pi R^2 + 2\pi RH)^2 \left(\frac{R}{R+H} \frac{2J_1(QR \sin(\alpha))}{QR \sin(\alpha)} \cos(QH \cos(\alpha)/2) + \frac{H}{R+H} J_0(QR \sin \alpha) \frac{\sin(QH \cos(\alpha)/2)}{QH \cos(\alpha)/2} \right)^2 \sin(\alpha) d\alpha \quad (3.230)$$

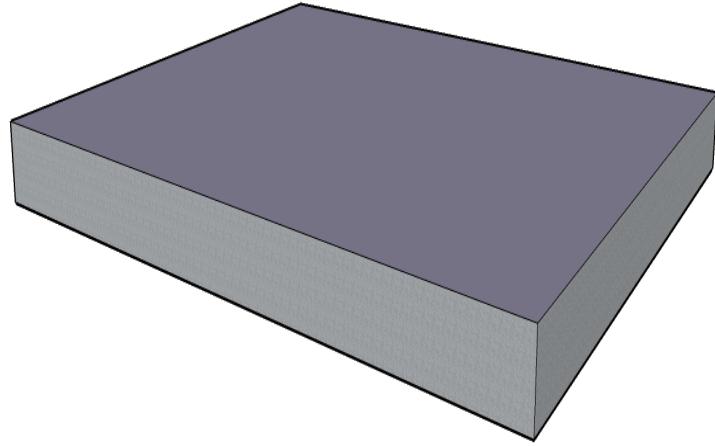
3.5.2. Cross-section form factors $P_{\text{cs}}(Q)$.3.5.2.1. *homogeneousXS*.

FIGURE 3.76. Planar object with homogeneous cross-section.

$$P_{\text{cs}}(Q, \eta, L) = \left(\eta L \frac{\sin(QL/2)}{QL/2} \right)^2 \quad (3.231)$$

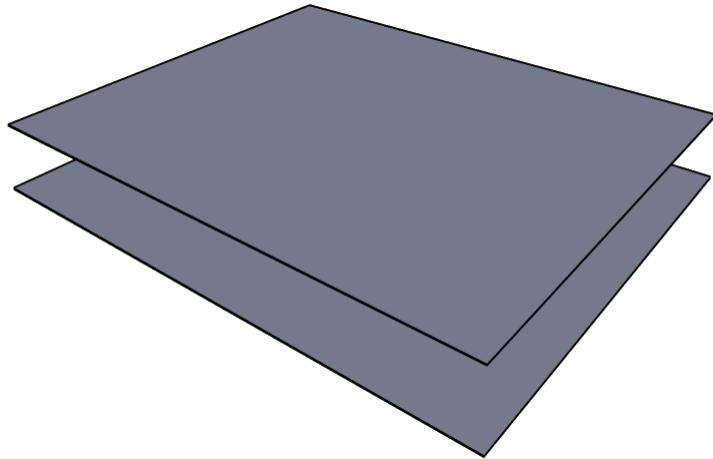
3.5.3. TwoInfinitelyThinPlates.

FIGURE 3.77. planar2thin.

$$P_{\text{cs}}(Q, \eta, L) = \eta^2 \cos^2(QL/2) \quad (3.232)$$

3.5.4. LayeredCentroSymmetricXS.

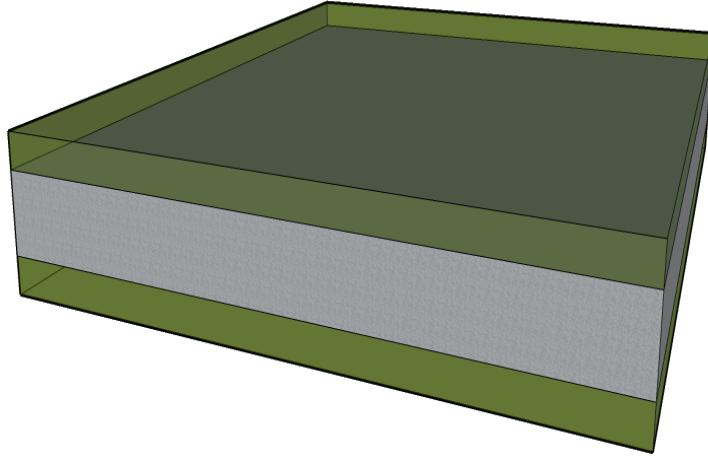


FIGURE 3.78. planar2centrosymHomo.

A layered centro symmetric cross-section structure with outer thickness L_{out} and a core of thickness L_c , where the core and the outer part have the scattering lengths density η_{out} and η_c , respectively, has

$$P_{\text{cs}}(Q, \eta_{\text{out}}, L_{\text{out}}, \eta_c, L_c) = \left(\frac{\eta_{\text{out}} L_{\text{out}} \sin\left(\frac{QL_{\text{out}}}{2}\right)}{QL_{\text{out}}/2} - \frac{(\eta_{\text{out}} - \eta_c)L_c \sin\left(\frac{QL_c}{2}\right)}{QL_c/2} \right)^2 \quad (3.233)$$

3.5.5. BiLayerGauss [162].

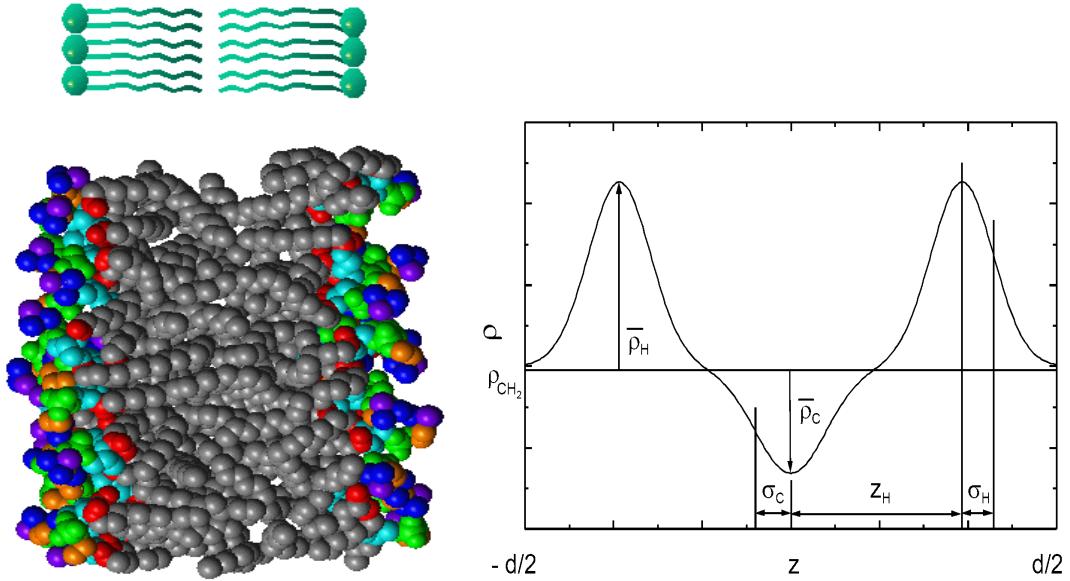


FIGURE 3.79. bilayerprof.

$$u_{\text{out}} = Q\sigma_{\text{out}} \quad (3.234)$$

$$u_{\text{core}} = Q\sigma_{\text{core}} \quad (3.235)$$

$$F_{\text{out}} = \sqrt{2\pi} \sigma_{\text{out}} b_{\text{out}} \exp(-u_{\text{out}}^2/2) \cos(Qt/2) \quad (3.236)$$

$$F_{\text{core}} = \sqrt{2\pi} \sigma_{\text{core}} b_{\text{core}} \exp(-u_{\text{core}}^2/2) \quad (3.237)$$

$$P_{\text{cs}} = (F_{\text{core}} + 2F_{\text{out}})^2 \quad (3.238)$$

3.6. Magnetic Scattering

In the case of magnetic moments in the sample, the neutron undergoes a magnetic interaction in addition to the nuclear interaction. The corresponding interaction potential is given by

$$V(\mathbf{r}) = -\boldsymbol{\mu}_N \cdot \mathbf{B}(\mathbf{r}) \quad \text{with} \quad \boldsymbol{\mu}_N = \gamma \frac{e\hbar}{2m_N c} \boldsymbol{\sigma}$$

where $\boldsymbol{\mu}_N = \gamma \frac{e\hbar}{2m_N c} \boldsymbol{\sigma}$ is the magnetic dipole moment of the neutron, $\boldsymbol{\sigma}$ the Pauli spin operator, $\gamma = -1.913$ the gyromagnetic ratio and $\mathbf{B}(\mathbf{r})$ the magnetic field induced by an atom at the position of the neutron. The latter has two components, one induced by the magnetic dipole moment $\boldsymbol{\mu}_S$ of the electrons, denoted $\mathbf{B}_S(\mathbf{r})$, and one by their orbital moment $\boldsymbol{\mu}_L$, denoted $\mathbf{B}_L(\mathbf{r})$. The (weak) magnetic interaction $V(\mathbf{r}) = -\boldsymbol{\mu}_N \cdot (\mathbf{B}_S(\mathbf{r}) + \mathbf{B}_L(\mathbf{r}))$ can as well be treated in first Born approximation, resulting in the magnetic scattering amplitude, in analogy to the nuclear scattering amplitude, given by the Fourier transform of the magnetic interaction potential $\mathcal{F}[V(\mathbf{r})]$:

$$b_M = -\frac{m_N}{2\pi\hbar^2} \boldsymbol{\mu}_N \cdot \int d^3r e^{i\mathbf{Q}\cdot\mathbf{r}} (\mathbf{B}_S(\mathbf{r}) + \mathbf{B}_L(\mathbf{r})). \quad (3.239)$$

An additional static magnetic field $\mathbf{H}(\mathbf{r})$ at the point of local magnetization $\mathbf{H}(\mathbf{r})$ (stemming from $\mathbf{B}_S(\mathbf{r}) + \mathbf{B}_L(\mathbf{r})$) induces a total local magnetic induction of

$$\mathbf{B}(\mathbf{r}) = \mu_0(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

and the Fourier transform of yields

$$\mathbf{B}(\mathbf{Q}) = \mu_0 \frac{\mathbf{Q} \times [\mathbf{M}(\mathbf{Q}) \times \mathbf{Q}]}{Q^2} = \mu_0 \mathbf{M}_\perp(\mathbf{Q}) = \mu_0 \mathbf{M}(\mathbf{Q}) \sin(\angle(\mathbf{Q}, \mathbf{M})) \quad (3.240)$$

where $\mathbf{M}(\mathbf{Q}) = \int d^3r \exp(i\mathbf{Q} \cdot \mathbf{r}) \mathbf{M}(\mathbf{r})$, with $\mathbf{M}(\mathbf{r})$ given in units of A/m. $\mathbf{M}_\perp(\mathbf{Q}) = \mathbf{Q} \times [\mathbf{M}(\mathbf{Q}) \times \mathbf{Q}]/Q^2$ is the magnetization component perpendicular to the scattering vector Q . The magnetic scattering length then is

$$b_M = D_M \mu_0 \boldsymbol{\sigma} \cdot \mathbf{M}_\perp(\mathbf{Q}) \quad \text{with} \quad D_M = \frac{m_N}{2\pi\hbar^2} \mu_N = 2.3161 \times 10^{14} \frac{1}{\text{Vs}}. \quad (3.241)$$

For the differential scattering cross section one finally obtains

$$\frac{d\sigma_M}{d\Omega}(\mathbf{Q}) = \frac{D_M^2}{N} |\mu_0 \mathbf{M}_\perp(\mathbf{Q})|^2 \quad (3.242)$$

In the presence of a preferred direction, for example induced by an external magnetic field, the magnetic scattering depends on the spin state of the neutrons. Let the z-axis be the preferred direction, and let (+) and (-) denote the neutron spin polarizations parallel and antiparallel to the z-axis, then the scattering is described by four scattering processes: two processes where the incident states (+) and (-) remain unchanged (++) and (--), the so-called 'non-spin-flip' processes, and two processes where the spin is flipped (+- and -+), the 'spin-flip' processes. Keeping in mind that the nuclear scattering does not flip the neutron spin, the four related scattering lengths are

$$b_{\pm\pm} = b_N \mp D_M \mu_0 M_{\perp x} \quad (3.243)$$

$$b_{\pm\mp} = -D_M \mu_0 (M_{\perp z} \pm i M_{\perp y}). \quad (3.244)$$

Hereby b_N is the nuclear scattering length. For an unpolarized neutron beam (which may be taken as composed of 50% (+) and 50% (-) polarization) the square of the modulus of the scattering length is

$$(b_{++}^2 + b_{--}^2 + b_{+-}^2 + b_{-+}^2)/2 = b_N^2 + D_M^2 \mu_0^2 M_{\perp}^2. \quad (3.245)$$

The differential cross section of the unpolarized neutron beam can therefore be described by the sum of the nuclear and the magnetic cross section, without any cross terms.

3.6.1. Magnetic Saturation.

3.6.1.1. MagneticShellAniso.

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.246)$$

$$K_{\text{sh}}(Q, R, \Delta R, \Delta\eta_{\text{sh}}, \Delta\eta_c) = K(Q, R + \Delta R, \Delta\eta_{\text{sh}}) - K(Q, R, \Delta\eta_c) \quad (3.247)$$

$$K_{\text{NUC}}(Q) = K_{\text{sh}}(Q, R, \Delta R, \eta_{\text{sh NUC}} - \eta_{\text{m NUC}}, \eta_{\text{c NUC}} - \eta_{\text{m NUC}}) \quad (3.248)$$

$$K_{\text{MAG}}(Q) = K_{\text{sh}}(Q, R, \Delta R, \eta_{\text{sh MAG}} - \eta_{\text{m MAG}}, \eta_{\text{c MAG}} - \eta_{\text{m MAG}}) \quad (3.249)$$

$$\begin{aligned} I(Q) &= \frac{1-p}{2} (K_{\text{MAG}}^2(Q) + 2K_{\text{NUC}}(Q)K_{\text{MAG}}(Q)) \\ &\quad + \frac{1+p}{2} (K_{\text{MAG}}^2(Q) - 2K_{\text{NUC}}(Q)K_{\text{MAG}}(Q)) \end{aligned} \quad (3.250)$$

$$= K_{\text{MAG}}^2(Q) - 2pK_{\text{NUC}}(Q)K_{\text{MAG}}(Q) \quad (3.251)$$

p : neutron polarization, $p \in [-1 : 1]$

R : radius of particle core

ΔR : thickness of particle shell

$\eta_{\text{sh NUC}}$: nuclear scattering length density of particle shell

$\eta_{\text{m NUC}}$: nuclear scattering length density of matrix

$\eta_{\text{c NUC}}$: nuclear scattering length density of particle core

$\eta_{\text{sh MAG}}$: magnetic scattering length density of particle shell

$\eta_{\text{m MAG}}$: magnetic scattering length density of matrix

$\eta_{\text{c MAG}}$: magnetic scattering length density of particle core

3.6.1.2. Magnetic Shell Cross Term.

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.252)$$

$$K_{\text{sh}}(Q, R, \Delta R, \Delta\eta_{\text{sh}}, \Delta\eta_c) = K(Q, R + \Delta R, \Delta\eta_{\text{sh}}) - K(Q, R, \Delta\eta_c) \quad (3.253)$$

$$K_{\text{NUC}}(Q) = K_{\text{sh}}(Q, R, \Delta R, \eta_{\text{sh NUC}} - \eta_{\text{m NUC}}, \eta_{\text{c NUC}} - \eta_{\text{m NUC}}) \quad (3.254)$$

$$K_{\text{MAG}}(Q) = K_{\text{sh}}(Q, R, \Delta R, \eta_{\text{sh MAG}} - \eta_{\text{m MAG}}, \eta_{\text{c MAG}} - \eta_{\text{m MAG}}) \quad (3.255)$$

$$I(Q) = 4pK_{\text{NUC}}(Q)K_{\text{MAG}}(Q) \quad (3.256)$$

p : neutron polarization, $p \in [-1 : 1]$

R : radius of particle core

ΔR : thickness of particle shell

$\eta_{\text{sh NUC}}$: nuclear scattering length density of particle shell

$\eta_{\text{m NUC}}$: nuclear scattering length density of matrix

$\eta_{\text{c NUC}}$: nuclear scattering length density of particle core

$\eta_{\text{sh MAG}}$: magnetic scattering length density of particle shell

$\eta_{\text{m MAG}}$: magnetic scattering length density of matrix

$\eta_{\text{c MAG}}$: magnetic scattering length density of particle core

3.6.1.3. MagneticShellPsi.

$$K(Q, R, \Delta\eta) = \frac{4}{3}\pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (3.257)$$

$$K_{\text{sh}}(Q, R, \Delta R, \Delta\eta_{\text{sh}}, \Delta\eta_c) = K(Q, R + \Delta R, \Delta\eta_{\text{sh}}) - K(Q, R, \Delta\eta_c) \quad (3.258)$$

$$K_{\text{NUC}}(Q) = K_{\text{sh}}(Q, R, \Delta R, \eta_{\text{sh NUC}} - \eta_{\text{m NUC}}, \eta_{\text{c NUC}} - \eta_{\text{m NUC}}) \quad (3.259)$$

$$K_{\text{MAG}}(Q) = K_{\text{sh}}(Q, R, \Delta R, \eta_{\text{sh MAG}} - \eta_{\text{m MAG}}, \eta_{\text{c MAG}} - \eta_{\text{m MAG}}) \quad (3.260)$$

$$I(Q) = K_{\text{NUC}}^2(Q) + (K_{\text{MAG}}^2(Q) - 2pK_{\text{NUC}}(Q)K_{\text{MAG}}(Q)) \sin^2 \Psi \quad (3.261)$$

p : neutron polarization, $p \in [-1 : 1]$

R : radius of particle core

Ψ : angle between \mathbf{Q} and \mathbf{H}

ΔR : thickness of particle shell

$\eta_{\text{sh NUC}}$: nuclear scattering length density of particle shell

$\eta_{\text{m NUC}}$: nuclear scattering length density of matrix

$\eta_{\text{c NUC}}$: nuclear scattering length density of particle core

$\eta_{\text{sh MAG}}$: magnetic scattering length density of particle shell

$\eta_{\text{m MAG}}$: magnetic scattering length density of matrix

$\eta_{\text{c MAG}}$: magnetic scattering length density of particle core

3.6.2. Superparamagnetic Particles (like ferrofluids).

$$\frac{I_{\pm\pm}(\mathbf{Q})}{N} = \left| F_N(\mathbf{Q}) \mp \tilde{F}_M(\mathbf{Q}) [L(\alpha) - \gamma] \sin^2 \epsilon \right|^2 + \left| \tilde{F}_M(\mathbf{Q}) \right|^2 \left(\frac{L(\alpha)}{\alpha} \sin^2 \epsilon - \mathcal{L}(\alpha) \sin^4 \epsilon \right) \quad (3.262)$$

$$\frac{I_{\mp\pm}(\mathbf{Q})}{N} = (\sin^2 \epsilon - \sin^4 \epsilon) [L(\alpha) - \gamma]^2 \left| \tilde{F}_M(\mathbf{Q}) \right|^2 + \left| \tilde{F}_M(\mathbf{Q}) \right|^2 \left((\sin^4 \epsilon - \sin^2 \epsilon) \mathcal{L}(\alpha) + (2 - \sin^2 \epsilon) \frac{L(\alpha)}{\alpha} \right) \quad (3.263)$$

$$\begin{aligned} I_{\text{unp}}(\mathbf{Q}) &= \frac{1}{2} (I_{++}(\mathbf{Q}) + I_{+-}(\mathbf{Q}) + I_{--}(\mathbf{Q}) + I_{-+}(\mathbf{Q})) \\ &= N \left(\left| \tilde{F}_M(\mathbf{Q}) \right|^2 [L(\alpha) - \gamma]^2 \sin^2 \epsilon + |F_N(\mathbf{Q})|^2 \right) \\ &\quad + N \left| \tilde{F}_M(\mathbf{Q}) \right|^2 \left(2 \frac{L(\alpha)}{\alpha} - \mathcal{L}(\alpha) \sin^2 \epsilon \right) \end{aligned} \quad (3.264)$$

Hereby $L(\alpha) = \coth \alpha - \frac{1}{\alpha}$ is the classical Langevin function with $\alpha = \mu_0(H + M_{\text{eff}})M_s^{cr}V_P/kT$.

Furthermore the following functions are defined as:

$$\mathcal{L}(\alpha) = L^2(\alpha) - 1 + 3 \frac{L(\alpha)}{\alpha},$$

$$F_N(\mathbf{Q}) = \Delta \eta V_N f_N(\mathbf{Q}),$$

$$\tilde{F}_M(\mathbf{Q}) = D_M M_s^{cr} V_M f_M(\mathbf{Q}) \text{ and}$$

$$\gamma = M_s^{am}/M_s^{cr}.$$

ϵ describes the angle between \mathbf{Q} and the applied magnetic field \mathbf{B} . If the magnetic field lies in the plane of the detector, i.e. perpendicular to the incoming beam direction, ϵ is in practice identical to Ψ so that $\cos \epsilon = \sin \delta \cos \Psi \simeq \cos \Psi$ for $\delta \simeq \pi/2$ (\mathbf{Q} in plane of detector for SANS, only for large scattering angle this will change).

3.6.2.1. *SuperparamagneticFFpsi*.

3.6.2.2. *SuperparamagneticFFAniso*.

3.6.2.3. *SuperparamagneticFFIso*.

3.6.2.4. *SuperparamagneticFFCrossTerm*.

3.7. Lorenz-Mie Form Factors for Static Light Scattering

Mie theory, also called Lorenz-Mie theory [147, 214, 19], is a complete mathematical-physical theory of the scattering of electromagnetic radiation by spherical particles. Mie theory is named after its developer German physicist Gustav Mie (1868 Rostock - 1957 Freiburg im Breisgau) and Danish physicist Ludvig Lorenz (1829-1891) who independently developed the theory of electromagnetic plane wave scattering by a dielectric sphere in 1908.

Mie scattering describes the scattering of electromagnetic radiation by spherical particles of any size r , relative to the wavelength, λ . Since the cases $r \ll \lambda$ and $r \gg \lambda$ are covered by Rayleigh (dipole) scattering and geometric scattering theories, respectively, Mie scattering often refers to the case of $r \sim \lambda$.

3.7.1. MieSphere.

The Mie scattering formulae are given in several books (Van de Hulst, 1957; Kerker, 1969; Deirmendjian, 1969) and by Dave (1968a, 1969a), although not always in the forms most suited to computation. The algorithm used here is based on the MIEV0 package described in [230, 231]. The following input values are used:

$$\Theta = 2 \arcsin(Q\lambda/(4\pi)) \text{ with } Q < Q_{\max} = 4\pi/\lambda$$

R = radius of scattering sphere

λ = wavelength of incident plane wave inside the solvent

m = complex refractive index of sphere relative to surrounding medium
 $= m_{\text{re}} - im_{\text{im}}$

$$|m| \geq 1 \text{ and } m_{\text{im}} \geq 0$$

or

$$|m| < 1 \text{ and } m_{\text{im}} = 0$$

$\text{pol} = 0$ unpolarized light

$\text{pol} = 1$ parallel to scattering plane polarized light

$\text{pol} = -1$ perpendicular to scattering plane polarized light

$|m| < 1$ would, for example, include visible light scattering from air bubbles in water.

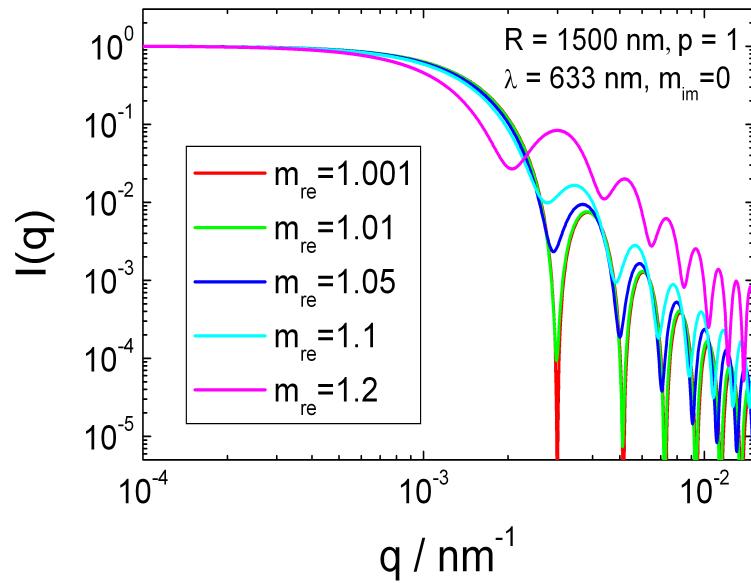


FIGURE 3.80. Scattering intensity of a sphere using the formalism for Mie scattering. The data are normalized to one for $q = 0$.

3.7.2. MieShell.

This form factor is basing on the version of MieLay, which computes electromagnetic scattering by a stratified sphere, i.e. a particle consisting of a spherical core surrounded by a spherical shell. The surrounding medium is assumed to have refractive index unity. The formulas, manipulated to avoid the ill-conditioning that plagued earlier formulations, were published in [208]. Further documentation, including definitions of input and output arguments, is inside the single precision version of this program (**SUBROUTINE** MieLay, available by anonymous ftp from `climate.gsfc.nasa.gov` in directory `pub/wiscombe`). The following input values are used:

$$\Theta = 2 \arcsin(Q\lambda/(4\pi)) \text{ with } Q < Q_{\max} = 4\pi/\lambda$$

R_c = radius of the core of scattering sphere

R_{sh} = thickness of the shell of scattering sphere

λ = wavelength of incident plane wave inside the solvent

m_c = complex refractive index of core relative to surrounding medium

$$= m_{c,re} - im_{c,im}$$

$$|m_c| \geq 1 \text{ and } m_{c,im} \geq 0$$

or

$$|m_c| < 1 \text{ and } m_{c,im} = 0$$

m_s = complex refractive index of core relative to surrounding medium

$$= m_{s,re} - im_{s,im}$$

$pol = 0$ unpolarized light

$pol = 1$ parallel to scattering plane polarized light

$pol = -1$ perpendicular to scattering plane polarized light

$|m| < 1$ would, for example, include visible light scattering from air bubbles in water.

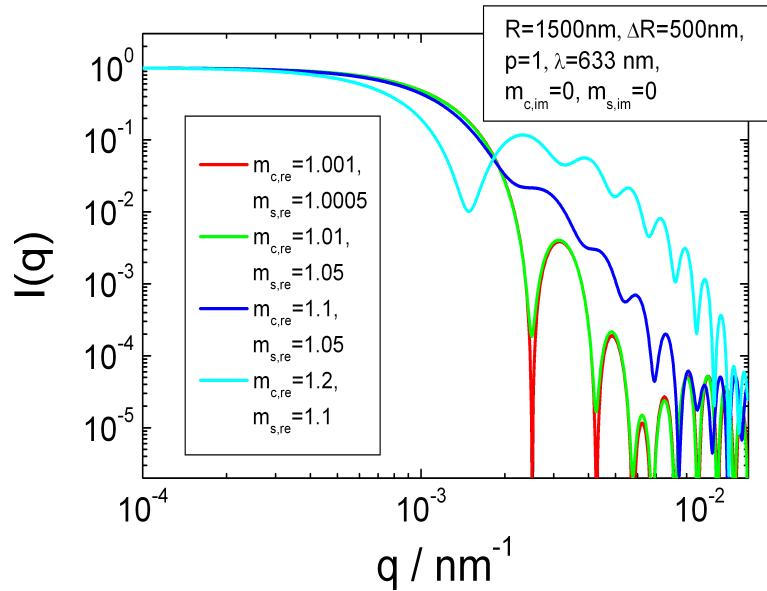


FIGURE 3.81. Scattering intensity of a spherical shell using the formalism for Mie scattering. The data are normalized to one for $q = 0$.

3.8. Other functions

3.8.1. DLS_Sphere_RDG.

This function has been implemented to simulate the relaxation signal $g_2(t) - 1 = g_1^2(t)$ of a DLS (dynamic light scattering) measurement. The Q dependent contribution to the relaxation signal by particles of different radius R is considered by weighting $g_2(t) - 1$ with the form factor of spherical particles in Rayleigh-Debye-Gans approximation:

$$I_{\text{DLS_Sphere_RDG}}(t, \eta, T, Q) = \int_0^{\infty} N(R) K_{\text{sp}}^2(Q, R) e^{-DQ^2 t} dR \quad (3.265)$$

with

$$\begin{aligned} D &= \frac{k_B T}{6\pi\eta R} \\ K_{\text{sp}}(Q, R) &= \frac{4}{3}\pi R^3 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \\ R &: \text{radius} \\ T &: \text{temperature} \\ \eta &: \text{viscosity} \\ Q &: \text{scattering vector} \end{aligned}$$

3.8.2. Langevin.

This function has been implemented to simulate the magnetisation curve $M(B)$ of superparamagnetic particles following the Langevin statistics with a size distribution $N(R)$: Magnetization curve of superparamagnetic particles with magnetization a M_s at temperature T

$$M(B) = \frac{\int_0^\infty N(R) \frac{4}{3}\pi R^3 M_p(B, R) dR}{\int_0^\infty N(R) \frac{4}{3}\pi R^3 dR} \quad (3.266)$$

$$M_p(B, R) = M_\infty \left(\coth(\alpha) - \frac{1}{\alpha} \right)$$

$$\alpha = \frac{BM_s \frac{4}{3}\pi R^3}{k_B T}$$

3.8.3. SuperParStroboPsi.

In the following the scattering of polarized incident neutrons with polarization analysis (POLARIS) is described. Experimental measured scattering signal are always a mixture of the spin dependent scattering cross-sections $I^{\pm\pm}(\mathbf{Q})$ and $I^{\pm\mp}(\mathbf{Q})$. The relative contribution of these cross-sections to the measured cross-section depend on the polarisation of the polarizer P_{pol} the efficiency of the spin flipper ϵ and the transmission values T_{\pm} of the polarization analyzer, which is assumed to be an opaque He-filter. If the neutron polarization of the polarizer is $P_{pol} \in [-1; 1]$ and the spin flipper is off the incident polarisation on the sample P_{in} is given by

$$P_{in} = P_{pol} = \frac{N_+ - N_-}{N_+ + N_-} = n_+ - n_- \quad (3.267a)$$

with

$$n_+ = \frac{N_+}{N_+ + N_-} \text{ and } n_- = \frac{N_-}{N_+ + N_-} \quad (3.267b)$$

$$\Rightarrow n_+ = \frac{1 + P_{pol}}{2} \text{ and } n_- = \frac{1 - P_{pol}}{2} \quad (3.267c)$$

N_{\pm} are the number of incident neutrons with spin polarizations up (+) and down (-). After switching on the spin flipper, which works with an efficiency of $\epsilon \in [0; 1]$ ($\epsilon = 0$: flipper off, $\epsilon \approx 1$: flipper on), one gets

$$\begin{aligned} n_+(P_{pol}, \epsilon) &= \epsilon \frac{1 - P_{pol}}{2} + (1 - \epsilon) \frac{1 + P_{pol}}{2} \\ &= \frac{1 - \epsilon P_{pol}}{2} \quad \text{and} \\ n_-(P_{pol}, \epsilon) &= \epsilon \frac{1 + P_{pol}}{2} + (1 - \epsilon) \frac{1 - P_{pol}}{2} \\ &= \frac{1 + \epsilon P_{pol}}{2} \end{aligned}$$

The efficiency of the analyzer to filter spin up (+) or spin down (-) neutrons is given in case of an opaque spin filter by its transmission $T_{\pm}(t) \in [0; 1]$, which can be a function of time, so that the measured scattering cross section $I_m(\mathbf{Q})$ is given by

$$\begin{aligned} I_m(\mathbf{Q}) &= n_+(P_{pol}, \epsilon) T_+(t) I_{++}(\mathbf{Q}) + n_+(P_{pol}, \epsilon) T_-(t) I_{+-}(\mathbf{Q}) \\ &\quad + n_-(P_{pol}, \epsilon) T_-(t) I_{--}(\mathbf{Q}) + n_-(P_{pol}, \epsilon) T_+(t) I_{-+}(\mathbf{Q}) \end{aligned} \quad (3.268)$$

The spin dependent scattering intensities of magnetic particles with an orientation distribution $f(\theta, \phi)$ of its magnetic moment can easily be calculated in terms of order parameters S_l if one assumes that the particle are spherical symmetric or the orientation of the magnetic moment of a particle is not correlated to the particle orientation. Furthermore it will be assumed that an external magnetic field is applied perpendicular to the incident neutron beam and that all orientations ϕ for a given angle θ , which is defined as the angle between the magnetisation vector of the particle and the direction of the external field \mathbf{B} have the same probability, i.e. the orientation distribution only depends on θ so that $f(\theta, \phi) = f(\theta)$. The orientation probability distribution function

can be expanded in terms of a complete set of orthogonal functions. Expanding it in terms of Legendre polynomials $P_l(\cos \theta)$ gives

$$f(\theta) = \sum_l c_l P_l(\cos \theta) = \sum_l \frac{2l+1}{2} S_l P_l(\cos \theta) \quad (3.269)$$

The expansion coefficients can be calculated by

$$c_l = \frac{2l+1}{2} \int_0^\pi f(\theta) P_l(\cos \theta) \sin \theta \, d\theta$$

or

$$(3.270)$$

$$S_l = \int_0^\pi f(\theta) P_l(\cos \theta) \sin \theta \, d\theta$$

The first three Legendre polynomials are defined by

$$P_0(\cos \theta) = 1 \quad (3.271a)$$

$$P_1(\cos \theta) = \cos \theta \quad (3.271b)$$

$$P_2(\cos \theta) = \frac{1}{2} (3 \cos^2 \theta - 1) \quad (3.271c)$$

For superparamagnetic particle the orientation probability distribution is given by

$$f(\theta) = \frac{\alpha}{4\pi \sinh \alpha} \exp(\alpha \cos \theta) \quad (3.272)$$

with $\alpha = BM_p V_p / (k_B T)$ being the Langevin parameter. For this orientation probability distribution the first order parameters can be calculated as

$$S_0 = 1 \quad (3.273a)$$

$$S_1 = L(\alpha) = \coth \alpha - \frac{1}{\alpha} \quad (3.273b)$$

$$S_2 = 1 - 3 \frac{L(\alpha)}{\alpha} \quad (3.273c)$$

The scattering from a system of many particles is obtained by summing up the scattering amplitudes of all precipitates weighted by the phase shift at each particle position. In the decoupling approach scattering intensity is given by

$$\frac{d\sigma_{\pm\pm}}{d\Omega}(\mathbf{Q}) = N \left\{ \left\langle \left| F_{\pm\pm}(\mathbf{Q}) \right|^2 \right\rangle + \left| \left\langle F_{\pm\pm}(\mathbf{Q}) \right\rangle \right|^2 (S(\mathbf{Q}) - 1) \right\} \quad (3.274)$$

which consists of two terms. The first one depends only on the particle structure and corresponds to the independent scattering of N particles, while the second one is also a function of their statistical distribution and reflects the interparticle interference, which is described by $S(\mathbf{Q})$. The $\langle \rangle$ indicates an average over all possible configurations, sizes and orientations of the magnetic moments of the particles. The spin dependent scattering

amplitudes $F_{\pm\pm}(\mathbf{Q})$ can be calculated from the nuclear and magnetic amplitudes

$$F_{\pm\pm}(\mathbf{Q}) = F_N(\mathbf{Q}) \mp F_{M_{\perp}x}(\mathbf{Q}) \quad (3.275)$$

$$F_{\pm\mp}(\mathbf{Q}) = - (F_{M_{\perp}y}(\mathbf{Q}) \pm i F_{M_{\perp}z}(\mathbf{Q})) \quad (3.276)$$

The nuclear scattering amplitude is proportional to the nuclear excess scattering $\beta_N = F_N(Q = 0)$ and the nuclear form factor $f_N(\mathbf{Q})$

$$F_N(\mathbf{Q}) = \beta_N f_N(\mathbf{Q}) \quad (3.277)$$

The magnetic scattering amplitude $\mathbf{F}_{M_{\perp}}(\mathbf{Q})$ can be described as a vector, with

$$\mathbf{F}_{M_{\perp}}(\mathbf{Q}) = \hat{\boldsymbol{\mu}}_{\perp} D_M \mu f_M(\mathbf{Q}) = \hat{\boldsymbol{\mu}}_{\perp} F_M(\mathbf{Q}) \quad (3.278)$$

where $f_M(Q)$ is the magnetic form factor, $\boldsymbol{\mu} = \mathbf{M}_p V_p$ the magnetic moment of the particle, $D_M = \frac{\gamma e}{2\pi\hbar}$, and $\hat{\boldsymbol{\mu}}_{\perp}$ the Halpern-Johnson vector defined as

$$\hat{\boldsymbol{\mu}}_{\perp} = \frac{\mathbf{Q}}{Q} \times \left(\frac{\boldsymbol{\mu}}{\mu} \times \frac{\mathbf{Q}}{Q} \right) \quad (3.279)$$

It is assumed here that the neutron spin polarization is parallel or antiparallel to the axes \mathbf{e}_x which is the direction perpendicular to the incoming neutron beam. If only the Halpern-Johnson vector $\hat{\boldsymbol{\mu}}_{\perp}$ depends on the orientation distribution $f(\theta)$ of the magnetic moments $\boldsymbol{\mu}$ of the particles but not the form factor $f_M(\mathbf{Q})$, which is valid for spherical symmetric particles or anisotropic shaped particles where the particle shape is not correlated to the direction of the magnetic moment, the averages in 3.274 can be written in terms of the order parameters S_1 and S_2

$$\langle F_{\pm\pm}(\mathbf{Q}) \rangle = F_N(\mathbf{Q}) + F_M(\mathbf{Q}) S_1 \sin^2 \psi \quad (3.280a)$$

$$\langle F_{\pm\mp}(\mathbf{Q}) \rangle = F_M(\mathbf{Q}) S_1 \sin \psi \cos \psi \quad (3.280b)$$

$$\begin{aligned} \langle |F_{\pm\pm}(\mathbf{Q})|^2 \rangle &= |F_N(\mathbf{Q})|^2 + |F_M(\mathbf{Q})|^2 \left[S_2 \sin^4 \psi + \frac{1 - S_2}{3} \sin^2 \psi \right] \\ &\mp [F_M(\mathbf{Q}) F_N^*(\mathbf{Q}) + F_N^*(\mathbf{Q}) F_M(\mathbf{Q})] S_1 \sin^2 \psi \end{aligned} \quad (3.280c)$$

$$\langle |F_{\pm\mp}(\mathbf{Q})|^2 \rangle = |F_M(\mathbf{Q})|^2 \left[2 \frac{1 - S_2}{3} - S_2 \sin^4 \psi + \frac{4S_2 - 1}{3} \sin^2 \psi \right] \quad (3.280d)$$

The spin-flip and spin-nonflip cross-section $\frac{d\sigma_{\pm\pm}}{d\Omega}(\mathbf{Q})$ can be obtained by combining 3.274 and 3.280. The cross-sections without polarization analysis $I_{\pm}(\mathbf{Q})$ and for unpolarized

neutrons $I_{unp}(\mathbf{Q})$ are given by

$$\begin{aligned} I_{\pm}(\mathbf{Q}) &= I_{\pm\pm}(\mathbf{Q}) + I_{\pm\mp}(\mathbf{Q}) \\ &= \left[|F_N(\mathbf{Q})|^2 + |F_M(\mathbf{Q})|^2 S_1^2 \sin^2 \psi \right. \\ &\quad \left. \mp [F_M(\mathbf{Q}) F_N^*(\mathbf{Q}) + F_M^*(\mathbf{Q}) F_N(\mathbf{Q})] S_1 \sin^2 \psi \right] S(\mathbf{Q}) \\ &\quad + |F_M(\mathbf{Q})|^2 \left(\frac{2}{3} (1 - S_2) + (S_2 - S_1^2) \sin^2 \psi \right) \end{aligned} \quad (3.281a)$$

$$\begin{aligned} I_{unp}(\mathbf{Q}) &= \frac{1}{2} (I_+(\mathbf{Q}) + I_-(\mathbf{Q})) \\ &= (|F_N(\mathbf{Q})|^2 + |F_M(\mathbf{Q})|^2 S_1^2 \sin^2 \psi) S(\mathbf{Q}) \\ &\quad + |F_M(\mathbf{Q})|^2 \left(\frac{2}{3} (1 - S_2) + (S_2 - S_1^2) \sin^2 \psi \right) \end{aligned} \quad (3.281b)$$

In the case of a Boltzmann orientation distribution $f(\theta) = \exp\left(\frac{\mathbf{B}\mu}{k_B T}\right) = \exp\left(\frac{B\mu \cos \theta}{k_B T}\right)$ the order parameter S_l already have been given in eq. 3.273 and the spin dependent intensities can be written as

$$\begin{aligned} \frac{I_{\pm\pm}(\mathbf{Q})}{N} &= |F_M(\mathbf{Q}) L(\alpha) \sin^2 \psi \pm F_N(\mathbf{Q})|^2 S(\mathbf{Q}) \\ &\quad + |F_M(\mathbf{Q})|^2 \left(\frac{L(\alpha)}{\alpha} \sin^2 \psi - \left(L^2(\alpha) - 1 + 3 \frac{L(\alpha)}{\alpha} \right) \sin^4 \psi \right) \end{aligned} \quad (3.282a)$$

$$\begin{aligned} \frac{I_{\mp\pm}(\mathbf{Q})}{N} &= (\sin^2 \psi - \sin^4 \psi) L^2(\alpha) |F_M(\mathbf{Q})|^2 S(\mathbf{Q}) \\ &\quad + |F_M(\mathbf{Q})|^2 \left((\sin^4 \psi - \sin^2 \psi) \left(L^2(\alpha) - 1 + 3 \frac{L(\alpha)}{\alpha} \right) + (2 - \sin^2 \psi) \frac{L(\alpha)}{\alpha} \right) \end{aligned} \quad (3.282b)$$

ψ is the angle between \mathbf{Q} and the horizontal axis in the plane of the detector. $L(\alpha) = \coth(\alpha) - \frac{1}{\alpha}$ is the Langevin function. In the case of a static field the superparamagnetic particle are thermodynamic equilibrium and α is given by

$$\alpha = \frac{BM_P V_P}{k_B T}, \quad (3.283)$$

with M_P being the particle magnetization, V_P the particle volume, T the temperature in Kelvin and k_B the Boltzmann constant.

In our experiments we applied an oscillating magnetic field to the sample described by:

$$B(t, \nu; d_{SD}, \lambda, \rho_0) = B_1 - B_0 \cos(\phi(t, \nu); \dots) \quad (3.284a)$$

$$\phi(t, \nu; d_{SD}, \lambda, \rho_0) = 2\pi\nu(t - d_{SD}\lambda/3956) + \rho_0 \quad (3.284b)$$

where t in [s] is the time between neutron detection and the trigger signal from the frequency generator, d_{SD} in [m] is the sample detector distance, λ in [\AA] the neutron

wavelength, and ν in [Hz] the frequency of the oscillating magnetic field. As t is defined here as the time of the neutron detection one has therefore to correct the phase in the argument for the magnetic field with an additional phase term. The term $t_{SD} = d_{SD}\lambda/3956$ takes into account the flight time t_{SD} of the neutrons between the sample and the detector. The term ρ_0 accounts for any other additional constant phase shift between trigger signal and the magnetic field due to phase shifts in the amplifier. If the neutron polarization can follow adiabatically the varying magnetic field needs to be verified experimentally. Therefore we introduce here a parameter $a_{ad} \in [0; 1]$ which takes into account whether ($a_{ad} = 1$) or not ($a_{ad} = 0$) the neutron spin adiabatically follows the change of magnetic field direction ($\text{sgn}(B(t))$).

$$\mathbf{r}_{ad} = \begin{pmatrix} a_{ad} \\ (1 - a_{ad}) \end{pmatrix} \text{ and } \mathbf{s}_{ad} = \begin{pmatrix} \text{sgn}(B(t)) \\ 1 \end{pmatrix} \quad (3.285)$$

The measured intensity than reads as

$$I_m(\mathbf{Q}, t) = \sum_{i=1,2} r_{ad,i} \left[n_+ (s_{ad,i} P_{pol}, \epsilon) A_+ I_{++}(\mathbf{Q}, t) + n_+ (s_{ad,i} P_{pol}, \epsilon) A_- I_{+-}(\mathbf{Q}, t) + n_- (s_{ad,i} P_{pol}, \epsilon) A_- I_{--}(\mathbf{Q}, t) + n_- (s_{ad,i} P_{pol}, \epsilon) A_+ I_{-+}(\mathbf{Q}, t) \right] \quad (3.286a)$$

$$= \sum_{i=1,2} \sum_{k,l=+,-} r_{ad,i} n_k (s_{ad,i} P_{pol}, \epsilon) A_l I_{kl}(\mathbf{Q}, t) \quad (3.286b)$$

with

$$A_{\pm} = \left(\frac{1 + s_{ad,2}}{2} r_{ad,1} + \frac{1 + s_{ad,1}}{2} r_{ad,2} \right) T_{\pm}(t) + \left(\frac{1 - s_{ad,2}}{2} r_{ad,1} + \frac{1 - s_{ad,1}}{2} r_{ad,2} \right) T_{\mp}(t) \quad (3.286c)$$

To calculate the time dependent scattering cross section a model for the time evolution of the orientation distribution of the magnetic moments $f(\theta, t)$ is needed, from which the time dependent order parameter $S_1(t)$ and $S_2(t)$ can be obtained, as well as a model

for the time evolution of the structure factor $S(\mathbf{Q}, t)$.

$$\frac{dM}{dt} = -\frac{M(t) - M_\infty L(\alpha'(t))}{\tau} \quad (3.287a)$$

$$M(t = 0) = M_0 \quad (3.287b)$$

$$\alpha'(t) = \alpha_0 \cos(\omega t + \phi_0) + \alpha_1 \quad (3.287c)$$

$$\alpha_0 = \frac{-B_0 \mu}{k_B T} \quad (3.287d)$$

$$\alpha_1 = \frac{B_1 \mu}{k_B T} \quad (3.287e)$$

$$\omega = 2\pi\nu \quad (3.287f)$$

$$\phi_0 = \varphi_0 - \frac{\omega d_{SD} \lambda}{3956} \quad (3.287g)$$

In the limit of small values for the Langevin parameter α the Langevin function can be approximated by

$$L(\alpha) \rightarrow \frac{\alpha}{3} \quad (3.288)$$

for which the differential equation has an analytical solution

$$M(t) = e^{-\frac{t}{\tau}} \left[M_0 - M_\infty \left(\frac{\alpha_0}{3} \frac{\cos(\phi_0) + \omega\tau \sin(\phi_0)}{1 + \omega^2\tau^2} + \frac{\alpha_1}{3} \right) \right] \\ + M_\infty \left(\frac{\alpha_0}{3} \frac{\cos(\omega t + \phi_0) + \omega\tau \sin(\omega t + \phi_0)}{1 + \omega^2\tau^2} + \frac{\alpha_1}{3} \right) \quad (3.289)$$

$$\underset{t \gg \tau}{\approx} M_\infty \left(\frac{\alpha_0}{3} \frac{\cos\left(\omega t + \phi_0 - \frac{\pi}{2} + \arcsin\left(\frac{1}{\sqrt{1+\omega^2\tau^2}}\right)\right)}{\sqrt{1+\omega^2\tau^2}} + \frac{\alpha_1}{3} \right) \quad (3.290)$$

Assuming that the system is at any time in thermodynamic equilibrium with the actual magnetic field $B(t)$ than the time dependent oscillating SANS signal can be described by simply introducing a time dependent value for $\alpha(t)$:

$$\alpha(t, \lambda, d_{SD}, \rho_0, \mu_{kT}) = B(t, \lambda, d_{SD}, \rho_0) \frac{M_P V_P}{k_B T} = B(\dots) \mu_{kT} \quad (3.291)$$

with

$$\mu_{kT} = \frac{M_P V_P}{k_B T} \quad (3.292)$$

In case that the magnetic moments can not follow the external magnetic field this could be described by an additional phase $\Delta\phi_\alpha$ between α and B and a damping factor d_α so that

$$\alpha(t, \lambda, d_{SD}, \rho_0 - \Delta\phi_\alpha, \mu_{kT}) = d_\alpha B(t, \lambda, d_{SD}, \rho_0 - \Delta\phi_\alpha) \frac{M_P V_P}{k_B T}. \quad (3.293)$$

In such a case of such

$$\mathbf{r}_{ad} = \begin{pmatrix} (1 - a_{ad}) \operatorname{sgn}(B(t)) \\ a_{ad} \operatorname{sign}(B(t)\alpha(t)) \end{pmatrix} \quad (3.294)$$

Furthermore we assume here that the size of the form factors of the magnetic and nuclear scattering are the same except the scattering contrast that means the ratio of magnetic to nuclear form factor is Q -independent and equal to the squared ratio of magnetic to nuclear scattering length density

$$\frac{F_N^2(Q)}{\tilde{F}_M^2(Q)} = \text{const} = \left(\frac{\Delta b_{\text{nuc}}}{\Delta b_{\text{mag}}} \right)^2 \quad (3.295)$$

Therefore the time dependent signal on the detector for a given direction ψ is given by

$$y(t, \dots) = \int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} p_\Delta(\lambda) I_m(t, \lambda, \psi, \dots) d\lambda \quad (3.296)$$

whereby $p_\Delta(\lambda)$ describes the triangular shaped wave length distribution of the neutron beam.

3.8.4. SuperParStroboPsiSQ.

The external applied field is given by:

$$B(t, d_{\text{SD}}, \lambda, \rho_0) = B_1 - B_0 \cos(2\pi\nu(t - d_{\text{SD}}\lambda/3956) + \rho_0) \quad (3.297)$$

where t in [s] is the time between neutron detection and the trigger signal from the frequency generator, d_{SD} in [m] is the sample detector distance, λ in [\AA] the neutron wavelength, and ν in [Hz] the frequency of the oscillating magnetic field. As t is defined here as the time of the neutron detection one has therefore to correct the phase in the argument for the magnetic field with an addition phase term. The term $t_{\text{SD}} = d_{\text{SD}}\lambda/3956$ takes account for the flight time t_{SD} of the neutrons between the sample and the detector. The term ρ_0 takes account for any other additional constant phase shift between trigger signal and the magnetic field due to phase shifts in the amplifier.

The scattering intensity of superparamagnetic particles including a structure factor $S(Q)$ is given by

$$\begin{aligned} I(Q) &= \underbrace{\tilde{F}_M^2(Q) 2 \frac{L(\alpha)}{\alpha} + F_N^2(Q) S(Q)}_{A(Q)} \\ &\quad + \underbrace{\tilde{F}_M^2(Q) \left[\left(1 - 3 \frac{L(\alpha)}{\alpha} - L^2(\alpha) \right) + L^2(\alpha) S(Q) \right]}_{B(Q)} \sin^2 \Psi \end{aligned} \quad (3.298)$$

Assuming that the system is in equilibrium faster than $1/\nu$ than the above equation can be used to describe the time dependent oscillating SANS signal simply by introducing a time dependent value for α :

$$\alpha(t, \lambda, d_{\text{SD}}, \rho_0, \mu_{\text{kT}}) = B(t, \lambda, d_{\text{SD}}, \rho_0) \frac{M_P V_P}{k_B T} = B(\dots) \mu_{\text{kT}} \quad (3.299)$$

with

$$\mu_{\text{kT}} = \frac{M_P V_P}{k_B T} \quad (3.300)$$

Furthermore we assume here that the size of the form factors of the magnetic and nuclear scattering are the same except the scattering contrast that means the ratio of magnetic to nuclear form factor is Q -independent and equal to the squared ratio of magnetic to nuclear scattering length density

$$\frac{F_N^2(Q)}{\tilde{F}_M^2(Q)} = \text{const} = \left(\frac{\Delta b_{\text{nuc}}}{\Delta b_{\text{mag}}} \right)^2 \quad (3.301)$$

Therefore the time dependent signal on the detector for a given direction Ψ and integrated over Q in this direction is given by

$$\begin{aligned} i(t, \dots) &= \int_{Q_{\min}}^{Q_{\max}} I(Q, t, \lambda, \Psi, d_{SD}, \rho_0) dQ \\ &= c \left[2 \frac{L(\alpha)}{\alpha} + \frac{F_N^2(Q)}{\tilde{F}_M^2(Q)} S(Q) + \left[\left(1 - 3 \frac{L(\alpha)}{\alpha} - L^2(\alpha) \right) + L^2(\alpha) S(Q) \right] \sin^2 \Psi \right] \\ &= c \left[2 \frac{L(\alpha)}{\alpha} + \left(\frac{\Delta b_{\text{nuc}}}{\Delta b_{\text{mag}}} \right)^2 S(Q) + \left[\left(1 - 3 \frac{L(\alpha)}{\alpha} - L^2(\alpha) \right) + L^2(\alpha) S(Q) \right] \sin^2 \Psi \right] \end{aligned} \quad (3.302)$$

$$y(t, \dots) = \int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} p_{\Delta}(\lambda) i(t, \lambda, \Psi, d_{SD}, \rho_0) d\lambda \quad (3.303)$$

whereby $p_{\Delta}(\lambda)$ describes the triangular shaped wave length distribution of the neutron beam.

3.8.5. SuperParStroboPsiSQBt.

The same as SuperParStroboPsiSQ except that the structure factor $S(Q, t)$ becomes field dependent:

$$S(Q, t) = 1 + [S(Q) - 1] \left| \frac{B(t, d_{SD}, \lambda, \rho_0)}{|B_1| + \frac{2}{\pi} |B_0|} \right| \quad (3.304)$$

3.8.6. SuperParStroboPsiSQLx.

The same as SuperParStroboPsiSQ except that the structure factor $S(Q, t)$ becomes field dependent:

$$S(Q, t) = 1 + [S(Q) - 1] |L(\alpha)| \quad (3.305)$$

3.8.7. SuperParStroboPsiSQL2x.

The same as SuperParStroboPsiSQ except that the structure factor $S(Q, t)$ becomes field dependent:

$$S(Q, t) = 1 + [S(Q) - 1] L^2(\alpha) \quad (3.306)$$

CHAPTER 4

Analytical Solutions for Structure factors

The different types of structure factors can be selected in the different submenus. Next top traditional structure factors also some functions, which are not structure factors at all have been implemented for some other purposes.

4.1. Methods to include structure factors

For each scattering object i next to a size distribution $N_i(x; \mathbf{l}_i)$ also a structure factor $S_i(Q; \mathbf{s}_i)$ can be included. When a structure factor is included several theoretical ways to account for it have been implemented like the monodisperse approximation (4.1.1), decoupling approach (4.1.2), local monodisperse approximation (4.1.3), partial structure factor (4.1.4) and scaling approximation of partial structure factors (4.1.5). At the moment it is assumed that there are no interactions between different species of scatterers so that the total scattering is given by the sum of the scattering of the individual species

$$\frac{d\sigma}{d\Omega}(Q) = \sum_{i=1}^N \frac{d\sigma_i}{d\Omega}(Q) \quad (4.1)$$

whereby different approaches to include structure factor effects in the differential scattering cross-sections $\frac{d\sigma_i}{d\Omega}(Q)$ of the species i of scatterer are defined below.

4.1.1. Monodisperse approach.

The monodisperse approach is the simplest way to include a structure factor in the analysis. This approach simply multiplies the size averaged form factor with the structure factor. Here it is assumed that the interaction potential between particles are spherical symmetric and independent of the particle size.

$$\frac{d\sigma_i}{d\Omega}(Q) = \left[\int_0^\infty N_i(x; \mathbf{l}_i) F_i^2(Q; \mathbf{a}_i, x) dx \right] S_i(Q; \mathbf{s}_i) \quad (4.2)$$

4.1.2. Decoupling approximation.

For systems with small polydispersities and small anisotropies leads to a decoupling approach of Kotlarchyk and Chen [117]. It is assumed that interactions are independent

of particle size and orientation.

$$\frac{d\sigma_i}{d\Omega}(Q) = \int_0^\infty N_i(x; \mathbf{l}_i) F_i^2(Q; \mathbf{a}_i, x) dx + \frac{1}{n_i} \left[\int_0^\infty N_i(x; \mathbf{l}_i) F_i(Q; \mathbf{a}_i, x) dx \right]^2 \times [S_i(Q; \mathbf{s}_i) - 1] \quad (4.3)$$

with

$$n_i = \int_0^\infty N_i(x; \mathbf{l}_i) dx. \quad (4.4)$$

The decoupling approximation can only be combined with those form factor, for which the scattering amplitude $F_i(Q; \mathbf{a}_i, x)$ has been implemented. However, for many form factors only the scattering intensity $F_i^2(Q; \mathbf{a}_i, x)$ is available. The combination of those form factors with the decoupling approach produces an error message in **SASfit**.

4.1.3. Local monodisperse approximation.

The opposite limit of the approximations as used for the decoupling approximation is used in the local monodisperse approximation [163]. In this approach it is assumed that a particle of a certain size is always surrounded by particles with the same size. Following this the scattering is approximated by that of monodisperse sub-systems weighted by the size distribution:

$$\frac{d\sigma_i}{d\Omega}(Q) = \int_0^\infty N_i(x; \mathbf{l}_i) F_i^2(Q; \mathbf{a}_i, x) S_i(Q; \mathbf{s}_i, R_i(\mathbf{a}_i, x)) dx \quad (4.5)$$

in which it has been indicated that the structure factor is for particles of size $R_i(\mathbf{a}_i, x)$. As the distribution $N_i(x; \mathbf{l}_i)$ does not necessarily describe the distribution of the overall size. **SASfit** assumes, that the radius of a particle with the form factor $F_i(Q; \mathbf{a}_i, x)$ used in the structure factor is given by the radius of a sphere with the same volume $V_i(\mathbf{a}_i, x)$

$$R_i(\mathbf{a}_i, x) = \sqrt[3]{\frac{3}{4\pi} V_i(\mathbf{a}_i, x)}. \quad (4.6)$$

This local monodisperse approximation works better than the decoupling approximation for systems with larger polydispersities and higher concentrations. As compared to the decoupling approximation and the scaling approximation described below, it has the advantage that the cross section is linear in the size distribution.

4.1.4. partial structure factors.

For polydisperse systems it is also not possible to write the scattering cross section as a product of a form factor and a structure factor. The scattering cross section has

the form

$$\begin{aligned} \frac{d\sigma_i}{d\Omega}(Q) = & \int_0^\infty N_i(x; \mathbf{l}_i) F_i^2(Q; \mathbf{a}_i, x) dx \\ & + \frac{1}{n_i} \int_0^\infty \int_0^\infty N_i(x; \mathbf{l}_i) N_i(x'; \mathbf{l}_i) F_i(Q; \mathbf{a}_i, x) F_i(Q; \mathbf{a}_i, x') \\ & \times [S_i(Q; \mathbf{s}_i, R_i(\mathbf{a}_i, x), R_i(\mathbf{a}_i, x')) - 1] dx dx' \end{aligned} \quad (4.7)$$

where monodisperse structure factor $S_i(Q; \mathbf{s}_i, \dots)$ is evaluated for the radius $(R_i(\mathbf{a}_i, x) + R_i(\mathbf{a}_i, x'))/2$. n_i and $R_i(\mathbf{a}_i, x)$ have the same definition as those in eq. 4.4 and 4.6.

4.1.5. Scaling approximation.

A scaling approximation has recently been introduced by Gazzillo et al. [65]. It is assumed that the pair correlation functions are identical except for a scaling constant. This leads to the following expression:

$$\begin{aligned} \frac{d\sigma_i}{d\Omega}(Q) = & \int_0^\infty N_i(x; \mathbf{l}_i) F_i^2(Q; \mathbf{a}_i, x) dx \\ & + \frac{1}{n_i} \int_0^\infty \int_0^\infty N_i(x; \mathbf{l}_i) N_i(x'; \mathbf{l}_i) F_i(Q; \mathbf{a}_i, x) F_i(Q; \mathbf{a}_i, x') \\ & \times \frac{\bar{V}_i(\mathbf{a}_i, x, x')}{V_{i,av}} [S_i(Q; \mathbf{s}_i, R_i(\mathbf{a}_i, x), R_i(\mathbf{a}_i, x')) - 1] dx dx' \end{aligned} \quad (4.8)$$

where $V_{i,av}$ is given by

$$V_{i,av} = \frac{\int_0^\infty N_i(x; \mathbf{l}_i) V_i(\mathbf{a}_i, x) dx}{\int_0^\infty N_i(x; \mathbf{l}_i) dx} \quad (4.9)$$

and $V_i(\mathbf{a}_i, x, x')$ by

$$\bar{V}_i(\mathbf{a}_i, x, x') = \frac{4}{3}\pi \left(\frac{1}{2} \left(\sqrt[3]{\frac{3}{4\pi} V_i(\mathbf{a}_i, x)} + \sqrt[3]{\frac{3}{4\pi} V_i(\mathbf{a}_i, x')} \right) \right)^3 \quad (4.10)$$

and the monodisperse structure factor is evaluated for the radius $(R_i(\mathbf{a}_i, x) + R_i(\mathbf{a}_i, x'))/2$. n_i and $R_i(\mathbf{a}_i, x)$ have the same definition as those in eq. 4.4 and 4.6.

Note that the expression is not linear in the size distribution and that it involves double integrations, which makes least-square fitting with this expression relatively slow.

4.1.6. van der Waals one-fluid approximation.

This approximation is similar to the scaling approximation introduced by Gazzillo et al.

[65]. The exact formula is given by

$$\begin{aligned} \frac{d\sigma_i}{d\Omega}(Q) = & \int_0^\infty N_i(x; \mathbf{l}_i) F_i^2(Q; \mathbf{a}_i, x) dx \\ & + \frac{1}{n_i} \int_0^\infty \int_0^\infty N_i(x; \mathbf{l}_i) N_i(x'; \mathbf{l}_i) F_i(Q; \mathbf{a}_i, x) F_i(Q; \mathbf{a}_i, x') \\ & \times \frac{\bar{V}_i(\mathbf{a}_i, x, x')}{V_{i,x}} [S_i(Q; \mathbf{s}_i, R_i(\mathbf{a}_i, x), R_i(\mathbf{a}_i, x')) - 1] dx dx' \end{aligned} \quad (4.11)$$

where $V_{i,x}$ is given by

$$V_{i,av} = \frac{\int_0^\infty N_i(x; \mathbf{l}_i) V_i(\mathbf{a}_i, x) dx}{\int_0^\infty N_i(x; \mathbf{l}_i) dx} \quad (4.12)$$

and $V_i(\mathbf{a}_i, x, x')$ by

$$\bar{V}_i(\mathbf{a}_i, x, x') = \frac{4}{3}\pi \left(\frac{1}{2} \left(\sqrt[3]{\frac{3}{4\pi} V_i(\mathbf{a}_i, x)} + \sqrt[3]{\frac{3}{4\pi} V_i(\mathbf{a}_i, x')} \right) \right)^3 \quad (4.13)$$

and the monodisperse structure factor is evaluated for the radius $(R_i(\mathbf{a}_i, x) + R_i(\mathbf{a}_i, x'))/2$. n_i and $R_i(\mathbf{a}_i, x)$ have the same definition as those in eq. 4.4 and 4.6.

Note that the expression is not linear in the size distribution and that it involves double integrations, which makes least-square fitting with this expression relatively slow.

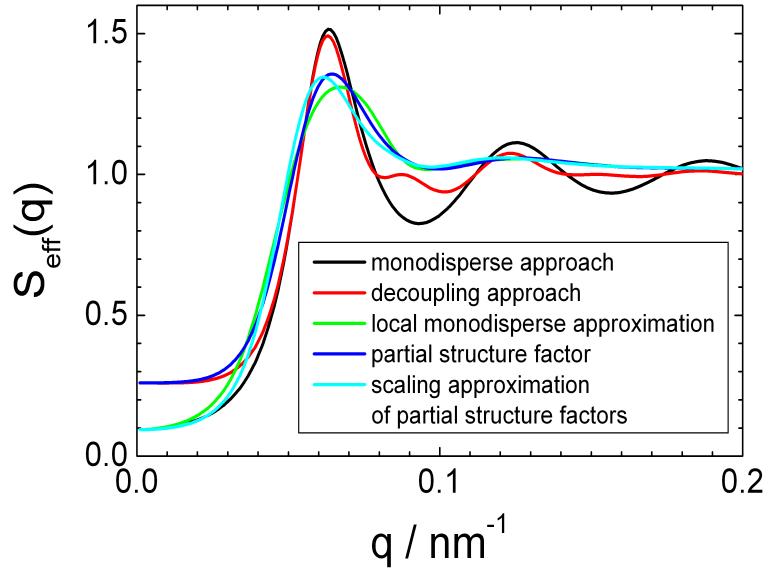


FIGURE 4.1. Effective structure factor $S_{\text{eff}}(q)$ for Spheres with Hard Sphere interaction potential. The fraction is assumed to be $\eta = 0.3$. A LogNormal distribution with $\sigma = 0.15$ and $R_0 = 50\text{nm}$ is assumed.

4.2. Hard & Sticky Hard Sphere

4.2.1. Hard Sphere [169, 221].

$$U(r) = \begin{cases} \infty & \text{for } 0 < r < \sigma \\ 0 & \text{for } r > \sigma \end{cases} \quad (4.14)$$

$$\alpha = \frac{(1 + 2f_p)^2}{(1 - f_p)^4} \quad (4.15a)$$

$$\beta = -6f_p \frac{(1 + f_p/2)^2}{(1 - f_p)^4} \quad (4.15b)$$

$$\gamma = \frac{f_p \alpha}{2} \quad (4.15c)$$

$$A = 2R_{\text{HS}}q \quad (4.15d)$$

$$G(q) = \alpha \frac{\sin A - A \cos A}{A^2} + \beta \frac{2A \sin A + (2 - A^2) \cos A - 2}{A^3} + \gamma \frac{-A^4 \cos A + 4[(3A^2 - 6) \cos A + (A^3 - 6A) \sin A + 6]}{A^5} \quad (4.15e)$$

$$S_{\text{HS}}(q, R_{\text{HS}}, f_p) = \frac{1}{1 + 24f_p \frac{G(f_p, A)}{A}} \quad (4.15f)$$

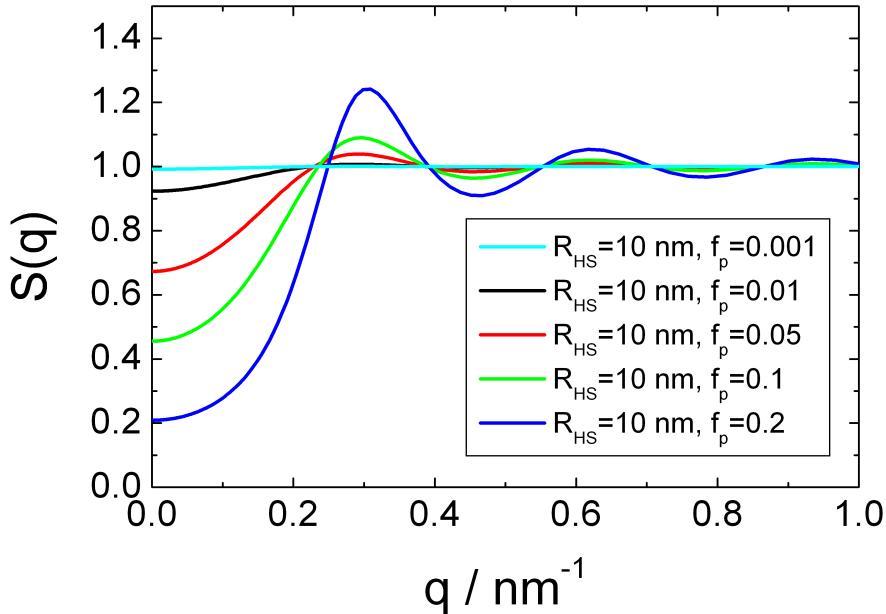


FIGURE 4.2. Structure factor $S(q)$ for a hard sphere interaction potential for the different volume fractions f_p .

4.2.2. Sticky Hard Sphere.

In Baxter's model [9, 183, 121, 8, 144, 145] of adhesive hard spheres the pair interaction potential $U(r)$ is replaced by

$$\frac{U(r)}{k_B T} = \begin{cases} \infty & \text{for } 0 < r < \sigma \\ \ln \frac{12\tau\Delta}{\sigma+\Delta} & \text{for } \sigma < r < \sigma + \Delta \\ 0 & \text{for } r > \sigma + \Delta \end{cases} \quad (4.16)$$

after which, when applied, the limit $\Delta \rightarrow 0$ is taken. Thus, only a single parameter, the so-called stickiness parameter τ , characterizes the adhesive strength.

$$\kappa = 2qR_{\text{HS}} \quad (4.17\text{a})$$

$$\eta = f_p \left(\frac{2R_{\text{HS}} + \Delta}{2R_{\text{HS}}} \right)^3 \quad (4.17\text{b})$$

$$\epsilon = \tau + \frac{\eta}{1 - \eta} \quad (4.17\text{c})$$

$$\gamma = f_p \frac{1 + \eta/2}{3(1 - \eta)^2} \quad (4.17\text{d})$$

$$\lambda = \frac{6}{\eta} \left(\epsilon - \sqrt{\epsilon^2 - \gamma} \right) \quad (4.17\text{e})$$

$$\mu = \lambda\eta(1 - \eta) \quad (4.17\text{f})$$

$$\beta = -\frac{3\eta(2 + \eta)^2 - 2\mu(1 + 7\eta + \eta^2) + \mu^2(2 + \eta)}{2(1 - \eta)^4} \quad (4.17\text{g})$$

$$\alpha = \frac{(1 + 2\eta - \mu)^2}{(1 - \eta)^4} \quad (4.17\text{h})$$

$$C(q) = 2\frac{\eta\lambda}{\kappa} \sin \kappa - 2\frac{\eta^2\lambda^2}{\kappa^2} (1 - \cos \kappa) - \quad (4.17\text{i})$$

$$\left\{ \alpha\kappa^3(\sin \kappa - \kappa \cos \kappa) + \beta\kappa^2(2\kappa \sin \kappa - (\kappa^2 - 2)\cos \kappa - 2) \right. \\ \left. + \frac{\eta\alpha}{2} ((4\kappa^3 - 24\kappa) \sin \kappa - (\kappa^4 - 12\kappa^2 + 24) \cos \kappa + 24) \right\} \times 24\frac{\eta}{\kappa^6}$$

$$S_{\text{sHS}}(q, R_{\text{HS}}, f_p, \tau) = \frac{1}{1 - C(q)} \quad (4.17\text{j})$$

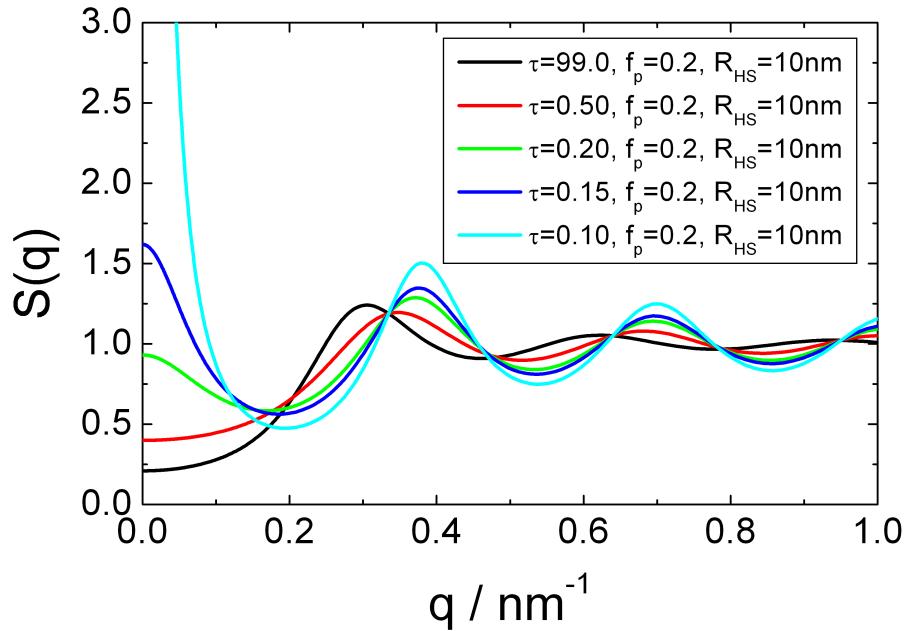


FIGURE 4.3. Structure factor $S(q)$ for a sticky hard sphere interaction potential for the different stickiness parameters τ .

4.2.3. Sticky Hard Sphere (2nd version [180, 181]).

In Baxter's model of adhesive hard spheres the pair interaction potential $U(r)$ is replaced by

$$\frac{U(r)}{k_B T} = \begin{cases} \infty & \text{for } 0 < r < \sigma - \Delta \\ \ln \frac{12\tau\Delta}{\sigma + \Delta} & \text{for } \sigma - \Delta < r < \sigma \\ 0 & \text{for } r > \sigma \end{cases} \quad (4.18)$$

$$\sigma = 2R_{\text{HS}} + \Delta \quad (4.19)$$

$$\kappa = q\sigma \quad (4.20)$$

$$\phi = f_p \left(\frac{\sigma}{2R_{\text{HS}}} \right)^3 \quad (4.21)$$

$$\lambda_{\pm} = 6 \left(\frac{\tau}{\phi} + \frac{1}{1-\phi} \right) \pm \sqrt{36 \left[\frac{\tau}{\phi} + \frac{1}{1-\phi} \right]^2 - \frac{12}{\phi} \frac{1 + \frac{\phi}{2}}{(1-\phi)^2}} \quad (4.22)$$

$$\lambda = \begin{cases} \lambda_+ & \text{for } \lambda_+ < |\lambda_-| \\ \lambda_- & \text{otherwise} \end{cases} \quad (4.23)$$

$$\mu = \lambda\phi(1-\phi) \quad (4.24)$$

$$A = \frac{1}{2} \frac{1 + 2\phi - \mu}{(1-\phi)^2} \quad (4.25)$$

$$B = \frac{\sigma}{2} \frac{\mu - 3\phi}{2(1-\phi)^2} \quad (4.26)$$

$$C = -A\sigma^2 - B\sigma + \lambda\sigma^2/12 \quad (4.27)$$

$$I_n(\kappa) = \int_0^1 x^n \cos(\kappa x) dx \quad (4.28)$$

$$J_n(\kappa) = \int_0^1 x^n \sin(\kappa x) dx \quad (4.29)$$

$$\alpha = 1 - 12f_p(C\sigma^{-2}I_0(\kappa) + B\sigma^{-1}I_1(\kappa) + AI_2(\kappa)) \quad (4.30)$$

$$\beta = 12f_p(C\sigma^{-2}J_0(\kappa) + B\sigma^{-1}J_1(\kappa) + AJ_2(\kappa)) \quad (4.31)$$

$$S(Q) = \frac{1}{\alpha^2 + \beta^2} \quad (4.32)$$

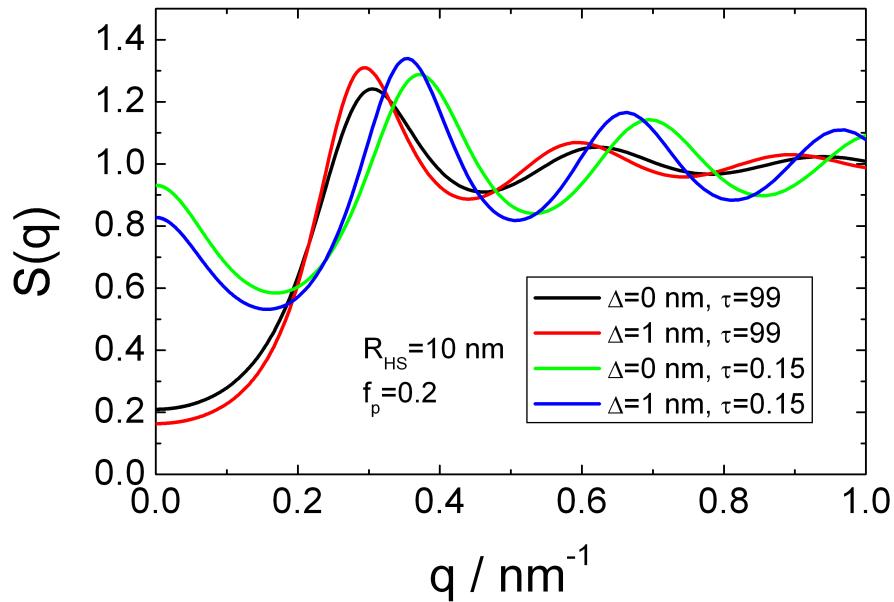


FIGURE 4.4. Structure factor $S(q)$ for a sticky hard sphere interaction potential for the different stickiness parameters τ .

4.2.4. Square Well Potential [194].

The Square well potential can be written as

$$U(r) = \begin{cases} \infty & \text{for } 0 < r < \sigma \\ -\epsilon & \text{for } \sigma < r < \lambda\sigma \\ 0 & \text{for } r > \lambda\sigma \end{cases} \quad (4.33)$$

where λ and ϵ correspond to the breadth and the depth of the square well potential. The structure factor $S(Q)$ is then given by the following relations:

$$S(Q) = \frac{1}{1 - C(Q)} \quad (4.34a)$$

$$\begin{aligned} C(Q) = & -\frac{24\eta}{(Q\sigma)^6} \left\{ \alpha(Q\sigma)^3 [\sin(Q\sigma) - Q\sigma \cos(Q\sigma)] \right. \\ & + \beta(Q\sigma)^2 [2Q\sigma \sin(Q\sigma) - (Q^2\sigma^2 - 2) \cos(Q\sigma) - 2] \\ & + \gamma [(4Q^3\sigma^3 - 24Q\sigma) \sin(Q\sigma) - (Q^4\sigma^4 - 12Q^2\sigma^2 + 24) \cos(Q\sigma) + 24] \\ & \left. - \frac{\epsilon}{k_B T} (Q\sigma)^3 [\sin(\lambda Q\sigma) - \lambda Q\sigma \cos(\lambda Q\sigma) + Q\sigma \cos(Q\sigma) - \sin(Q\sigma)] \right\} \end{aligned} \quad (4.34b)$$

where α , β and γ are given by

$$\alpha = \frac{(1+2\eta)^2 + \eta^3(\eta-4)}{(1-\eta)^4} \quad (4.34c)$$

$$\beta = -\frac{1}{3}\eta \frac{18 + 20\eta - 12\eta^2 + \eta^4}{(1-\eta)^4} \quad (4.34d)$$

$$\gamma = \frac{1}{2}\eta \frac{(1+2\eta)^2 + \eta^3(\eta-4)}{(1-\eta)^4} \quad (4.34e)$$

NOTE:

Values for the depth of $\epsilon > 1.5k_B T$ and for the volume fraction of $\eta > 0.08$ may give unphysical results when compared to Monte Carlo simulations according to [194].

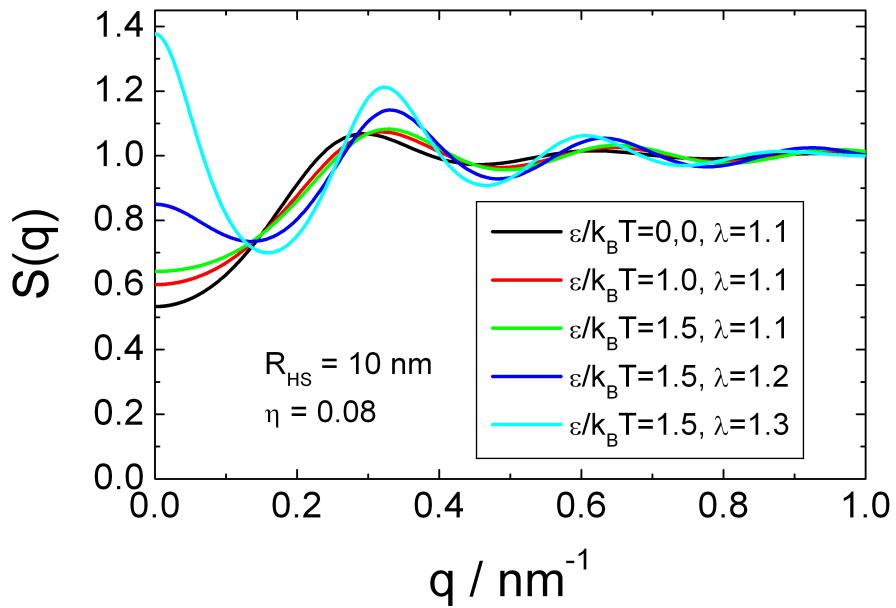


FIGURE 4.5. Structure factor $S(q)$ for a square well interaction potential.

4.2.5. Square Well Potential 2.

The Square well potential can be written as

$$U(r) = \begin{cases} \infty & \text{for } 0 < r < \sigma \\ -\epsilon & \text{for } \sigma < r < \sigma + \Delta \\ 0 & \text{for } r > \sigma + \Delta \end{cases} \quad (4.35)$$

where Δ and ϵ correspond to the width and the depth of the square well potential. The structure factor $S(Q)$ is then given by the following relations:

$$S(Q) = 1 - 4\pi\rho\sigma^3 \frac{\sin(Q\sigma) - Q\sigma \cos(Q\sigma)}{Q^3\sigma^3} + 4\pi\rho\sigma^2 \left[e^{-\frac{\epsilon}{k_B T}} - 1 \right] \frac{\sin(Q\sigma)}{Q\sigma} \Delta \quad (4.36)$$

where σ is the particle diameter ($R_{HS} = \sigma/2$: hard sphere radius is requested by software as input parameter), Δ the width of the square well potential, ϵ (input value in software is ϵ/k_B , i.e. in Kelvin), T (in Kelvin) the sample temperature, the depth and ρ the colloid concentration, which is related to the colloid volume fraction η by $\eta = \pi\rho\sigma^3/6$.

4.3. Multi Lamellar Structures

4.3.1. Multi-Lamellar Structures, Thermal Disorder.

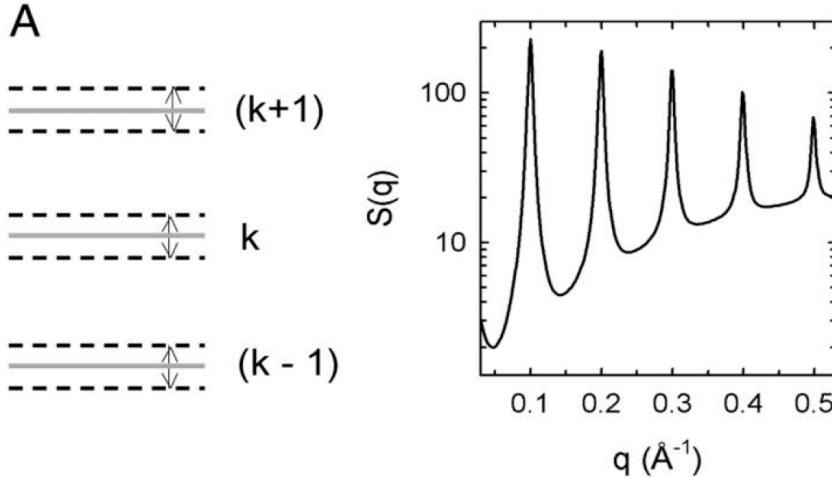


FIGURE 4.6. Thermal disorder, considering fluctuations of flat layers around well defined and evenly spaced equilibrium positions.

The first type describes thermal disorder (TD) caused by small fluctuations of the bilayers around well defined mean layer positions of equal separation (Fig. 4.6) [161, 54]. In such a crystal lattice the long-range order is preserved and the structure factor for a single domain of size $L = Nd$ is identical to that of a perfect finite crystal multiplied by the well known Debye-Waller temperature factor, where $\Delta = \langle (d_k - d)^2 \rangle$ denotes the mean square fluctuations of the bilayers. As shown in Fig. 4.6, $S_{\text{TD}}(Q)$ is characterized by a set of Bragg peaks of equal width, the diffraction order amplitudes of which decrease exponentially with the Debye-Waller factor. The lost intensity is found as a diffuse background scattering, which increases to the limit of N for large Q .

$$S_{\text{TD}}(Q, N, d, \Delta, N_{\text{diff}}) = N_{\text{diff}} + \sum_{N_k=N-2\sigma}^{N+2\sigma} x_k S_{k,\text{TD}} \quad (4.37)$$

with

$$S_{k,\text{TD}} = \left(N_k + 2 \exp\left(-\frac{Q^2 \Delta^2}{2}\right) \sum_{m=1}^{N_k-1} (N_k - m) \cos(mQd) \right) \quad (4.38)$$

N_{diff} accounts for an additional diffuse background, due to a number of uncorrelated scattering bilayers in $S_{\text{TD}}(Q, N, d, \Delta, N_{\text{diff}})$, which is not included in the TD theory. Its origin is attributed to bilayers with strong lattice defects or unilamellar vesicles, which display neither short-range nor (quasi-) long-range order.

The structure factors $S_{k,\text{TD}}(Q)$ with low, but fixed stacking numbers N show oscillations at low Q (as can be seen in Fig. 4.6), but no such oscillations are found in experimental data. This can be understood as the consequence of polydispersity in the

size of the different stacks. In order to eliminate these artifacts from strictly monodisperse systems, we use a ‘polydisperse’ structure factor, i.e. we use an average of a series of structure factors with varying numbers of bilayers [54]. The analytical form of the distribution is not known *a priori*. We use a Gaussian distribution approximated by a discrete series. The standard deviation σ for the Gaussian-weighted distribution is chosen as

$$\sigma = \begin{cases} \sqrt{N} & \text{for } N \geq 5, \\ 0.5(N-1) & \text{for } N < 5 \end{cases} \quad (4.39)$$

Therefore, N must be greater or equal to 2, which is a reasonable restriction for multilamellar stacks of bilayers. In the range of $N \pm 2\sigma$, structure factors weighted by

$$x_k = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(N_k - N)^2}{2\sigma^2}\right] \quad (4.40)$$

are calculated, where N is the mean number of stacks and N_k is one of the bilayers in the range $N \pm 2\sigma$. This polydispersity model does not introduce new free parameters and is symmetrical around the mean N .

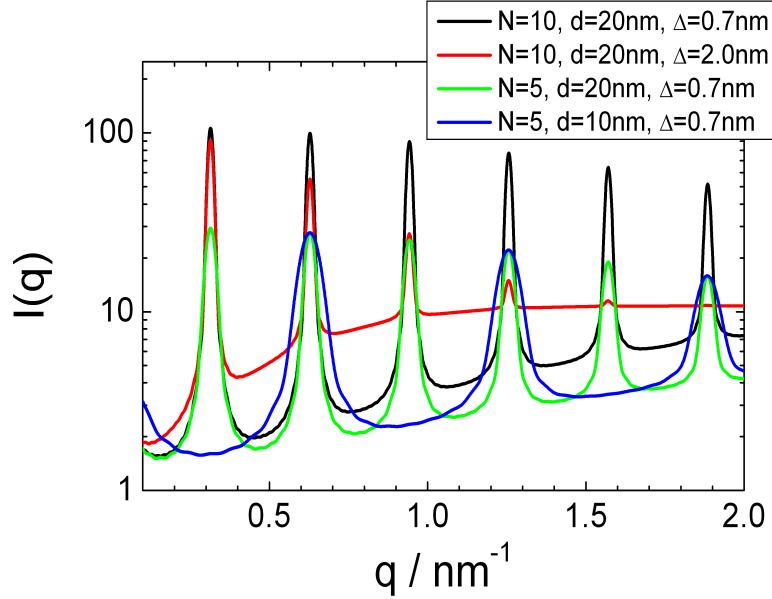


FIGURE 4.7. Structure factor of multi-lamellar structures with thermal disorder.

Input Parameters for model ThermalDisorder:

N: mean number of stacks N

d: stacking separation d

Delta: Debye-Waller disorder parameter Δ

Nu: number of uncorrelated scattering bilayers N_{diff}

Note:

- This structure factor is intended to be used with the `monodisperse` approximation.

4.3.2. Multi-Lamellar Structures, Paracrystalline Theory.

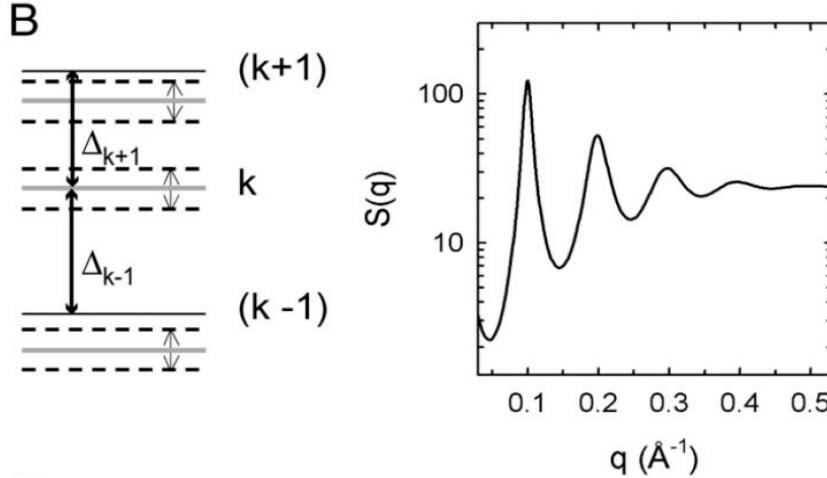


FIGURE 4.8. Stacking disorder as described within paracrystalline theory (PT) is due to displacements from the mean layer positions.

The second type of disorder accounts for the presence of small variations in the bilayer separations (Fig. 4.8), so-called stacking disorder, and is described within the paracrystalline theory (PT) [95, 72, 18, 161, 54]. As the position of an individual fluctuating layer in a paracrystal is determined solely by its nearestneighbour membranes, the crystalline long-range order is lost. Still, we are able to observe Bragg-peak scattering due to the fact that there is quasi long-range order. However, these quasi-Bragg peaks will display a typical line shape. In the case of disorder of the second kind, the structure factor derived from paracrystalline theory is given by [72]

$$S_{\text{PT}}(Q, N, d, \Delta, N_{\text{diff}}) = N_{\text{diff}} + \sum_{N_k=N-2\sigma}^{N+2\sigma} x_k S_{k,\text{PT}} \quad (4.41)$$

with

$$S_{k,\text{PT}} = \left(N_k + 2 \sum_{m=1}^{N_k-1} (N_k - m) \cos(mQd) \exp\left(-\frac{m^2 Q^2 \Delta^2}{2}\right) \right) \quad (4.42)$$

N_{diff} accounts for an additional diffuse background, due to a number of uncorrelated scattering bilayers in $S_{\text{PT}}(Q, N, d, \Delta, N_{\text{diff}})$, which is not included in the paracrystalline theory. Its origin is attributed to bilayers with strong lattice defects or unilamellar vesicles, which display neither short-range nor (quasi-) long-range order.

Fig. 4.8 shows that the quasi-Bragg peak intensity decreases for $S_{\text{PT}}(Q)$, as in the previous case of thermal disorder. However, the decrease in peak height is also accompanied by a progressive broadening proportional to the square of the diffraction order h [193]. The line shape of the tails is essentially Lorentzian with

$$S_{k,\text{PT}}(Q) \propto (Q - Q_h)^2,$$

where Q_h is the position of the h^{th} diffraction order in Q space. Again, the loss in intensity shows up as diffuse background scattering, which is stronger than from pure thermal disorder.

The structure factors $S_{k,\text{PT}}(Q)$ with low, but fixed stacking numbers N show oscillations at low Q (as can be seen in Fig. 4.8), but no such oscillations are found in experimental data. This can be understood as the consequence of polydispersity in the size of the different stacks. In order to eliminate these artifacts from strictly monodisperse systems, we use a ‘polydisperse’ structure factor, i.e. we use an average of a series of structure factors with varying numbers of bilayers [54]. The analytical form of the distribution is not known a priori. We use a Gaussian distribution approximated by a discrete series. The standard deviation σ for the Gaussian-weighted distribution is chosen as

$$\sigma = \begin{cases} \sqrt{N} & \text{for } N \geq 5, \\ 0.5(N-1) & \text{for } N < 5 \end{cases} \quad (4.43)$$

Therefore, N must be greater or equal to 2, which is a reasonable restriction for multilamellar stacks of bilayers. In the range of $N \pm 2\sigma$, structure factors weighted by

$$x_k = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(N_k - N)^2}{2\sigma^2}\right] \quad (4.44)$$

are calculated, where N is the mean number of stacks and N_k is one of the bilayers in the range $N \pm 2\sigma$. This polydispersity model does not introduce new free parameters and is symmetrical around the mean N .

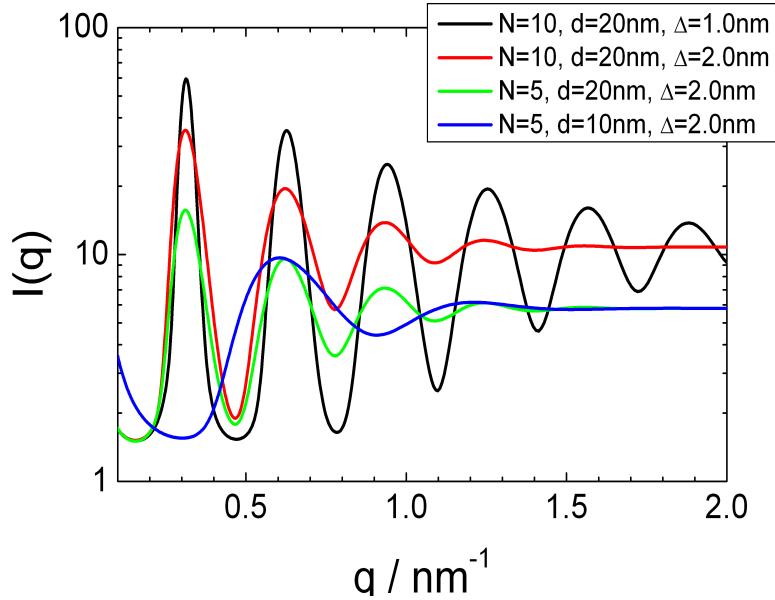


FIGURE 4.9. Structure factor of multi-lamellar structures with paracrystalline disorder.

Input Parameters for model Paracrystalline:

N: mean number of stacks N

d: stacking separation d

Delta: stacking disorder parameter Δ

Nu: number of uncorrelated scattering bilayers N_{diff}

Note:

- This structure factor is intended to be used with the **monodisperse approximation**.

4.3.3. Multi-Lamellar Structures, Modified Caillé Theory.

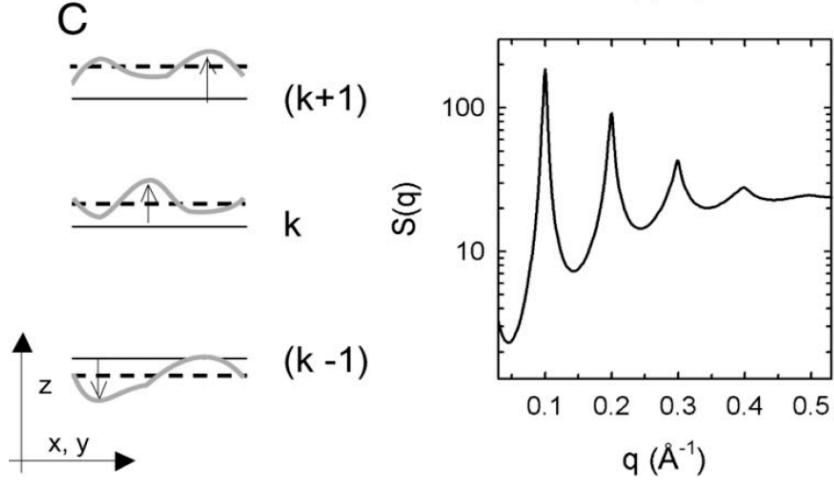


FIGURE 4.10. Bending fluctuation disorder is a particular feature of the L_α (smectic A) phase and is caused by bilayer undulations. The particular shape of the Bragg peaks is given by the modified Caillé theory (MCT).

There is another type of disorder when bilayer bending fluctuations are considered (Fig. 4.10) [161, 54]. Such fluctuations are particularly pronounced in the fluid L_α phase (equivalent to smectic A). Caillé Bending fluctuation disorder is a particular feature of the L_α (smectic A) phase and is caused by bilayer undulations. The particular shape of the Bragg peaks is given by the modified Caillé theory (MCT). [32] realized the impact on the structure factor, which in a modified version [235] of the Caillé theory (MCT) is

$$S_{\text{MC}}(Q, N, d, \eta_1, \gamma, N_{\text{diff}}) = N_{\text{diff}} + \sum_{N_k=N-2\sigma}^{N+2\sigma} x_k S_{k,\text{MC}} \quad (4.45)$$

with

$$S_{k,\text{MC}} = N_k + 2 \sum_{m=1}^{N_k-1} (N_k - m) \cos(mQd) e^{-\left(\frac{d}{2\pi}\right)^2 Q^2 \eta_1 \gamma} (\pi m)^{-(d/2\pi)^2 Q^2 \eta_1} \quad (4.46)$$

Here, γ is Euler's constant and

$$\eta = \pi k_B T / 2d^2 (BK_c)^{1/2}$$

is the Caillé parameter, which is a measure for the bilayer fluctuations and is inversely proportional to the square root of the bilayer bending rigidity K_c times the bulk modulus of compression B (De Gennes & Prost, 1993). Therefore, a lineshape analysis of the quasi-Bragg peaks opens an important experimental window on interbilayer interactions. Further, K_c and B can be decoupled as demonstrated recently by hydration studies [170, 161], or more elegantly by measuring multibilayers aligned on a solid substrate [137]. N_{diff} accounts for an additional diffuse background, due to a number of uncorrelated scattering bilayers in $S_{\text{MC}}(Q, N, d, \eta_1, \gamma, N_{\text{diff}})$, which is not included in the MCT. Its

origin is attributed to bilayers with strong lattice defects or unilamellar vesicles, which display neither short-range nor (quasi-) long-range order.

Fig. 4.10 shows a typical example of $S_{k,MCT}(Q)$, which is similar to $S_{PT}(Q)$ with respect to the progressive decrease in peak height and increase in peak width, but which differs significantly in line shape as

$$S_{k,MCT} \propto (Q - Q_h)^{-1+\eta h^2}$$

for randomly oriented scattering domains [186, 235].

The structure factors $S_{k,MCT}(Q)$ with low, but fixed stacking numbers N show oscillations at low Q (as can be seen in Fig. 4.10), but no such oscillations are found in experimental data. This can be understood as the consequence of polydispersity in the size of the different stacks. In order to eliminate these artifacts from strictly monodisperse systems, we use a ‘polydisperse’ structure factor, i.e. we use an average of a series of structure factors with varying numbers of bilayers [54]. The analytical form of the distribution is not known a priori. We use a Gaussian distribution approximated by a discrete series. The standard deviation σ for the Gaussian-weighted distribution is chosen as

$$\sigma = \begin{cases} \sqrt{N} & \text{for } N \geq 5, \\ 0.5(N-1) & \text{for } N < 5 \end{cases} \quad (4.47)$$

Therefore, N must be greater or equal to 2, which is a reasonable restriction for multilamellar stacks of bilayers. In the range of $N \pm 2\sigma$, structure factors weighted by

$$x_k = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(N_k - N)^2}{2\sigma^2}\right] \quad (4.48)$$

are calculated, where N is the mean number of stacks and N_k is one of the bilayers in the range $N \pm 2\sigma$. This polydispersity model does not introduce new free parameters and is symmetrical around the mean N .

Input Parameters for model ModifiedCaile:

N: mean number of stacks N

d: stacking separation d

eta: the Caillé parameter η is a measure for the bilayer fluctuations and inversely proportional to the square root of the bilayer bending rigidity

Nu: number of uncorrelated scattering bilayers N_{diff}

Note:

- This structure factor is intended to be used with the **monodisperse approximation**.

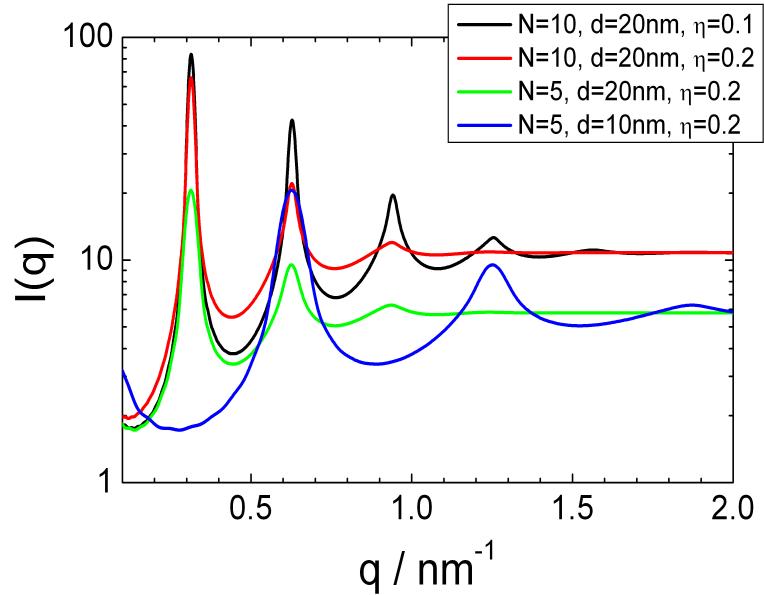


FIGURE 4.11. Structure factor of multi-lamellar structures according to the modified Caillé theory.

4.4. Mass Fractal

For a fractal object, the structure factor $S(q)$ can be calculated [198, 197] via the pair correlation function $g(r)$, which describes the total number of particles within a sphere of radius r centered in a central particle and is given (for $\text{dim} = 3$) by

$$N(r) = \Phi \int_0^r g(r) 4\pi r^2 dr \quad (4.49)$$

or

$$dN(r) = \Phi g(r) 4\pi r^2 dr \quad (4.50)$$

On the other hand, a fractal object is characterized by a spatial distribution of the individual scatterers given by

$$N(r) = \left(\frac{r}{r_0}\right)^D \quad (4.51)$$

where r_0 is the gauge of the measurement, which has the magnitude of the characteristic dimension of each individual scatterer. Differentiation of 4.51 and identification with 4.50 gives

$$\Phi g(r) = \frac{D}{4\pi r_0^D} r^{D-3} \quad (4.52)$$

Because D is smaller than 3, $g(r)$ goes to zero at large r . This is clearly unphysical. At some large scale, the sample will show a macroscopic density. A good knowledge of the sample allows in general a reasonable assumption for the large-scale behavior of $g(r)$. Therefore a cut-off function $h(r, \xi)$ has to be introduced, where ξ is a cut-off distance, to describe the behavior of the pair correlation function at large distances. To derive the analytical form of $S(q)$ within this assumption, one can use the general theory of liquids, where the uniform density is subtracted to avoid a divergence in the evaluation of $S(q)$. We then write

$$4\pi\Phi[g(r) - 1] = \frac{D}{r_0^D} r^{D-3} h(r, \xi) \quad (4.53)$$

The meaning of ξ is only qualitative and has to be made precise in any particular situation. Generally speaking, it represents the characteristic distance above which the mass distribution in the sample is no longer described by the fractal law. In practice, it can represent the size of an aggregate or a correlation length in a disordered material. For isotropic systems

$$S(q) = 1 + 4\pi\Phi \int_0^\infty [g(r) - 1] \frac{\sin(qr)}{qr} r^2 dr \quad (4.54)$$

Combined with 4.53 one gets

$$S(q) = 1 + \frac{D}{r_0^D} \int_0^\infty r^{D-3} h(r, \xi) \frac{\sin(qr)}{qr} r^2 dr \quad (4.55)$$

Several cut-off functions $h(r, \xi)$ have been discussed in the literature and compared by Sorensen et al. [198, 197].

$$h_{\text{Exp}}(r, \xi) = \exp\left[-\frac{r}{\xi}\right] \quad (4.56)$$

$$h_{\text{Gauss}}(r, \xi) = \exp\left[-\left(\frac{r}{\xi}\right)^2\right] \quad (4.57)$$

$$h_{\text{Exp}(-x^a)}(r, \xi, \alpha, D) = \exp\left[-\left(\frac{r}{\xi}\right)^\alpha\right] \quad (4.58)$$

$$h_{\text{OverlapSph}}(r, \xi) = \begin{cases} \left(1 + \frac{r}{4\xi}\right) \left(1 - \frac{r}{2\xi}\right)^2 & \text{for } r \leq 2\xi \\ 0 & \text{for } r > 2\xi \end{cases} \quad (4.59)$$

For the cut-off functions 4.56 and 4.57 the integral 4.55 can be solved analytically and the corresponding structure factors are given by

$$S_{\text{Exp}}(q, \xi, D, r_0) = 1 + \frac{D\Gamma(D-1) \sin([D-1] \arctan(q\xi))}{(qr_0)^D \left[1 + \frac{1}{q^2\xi^2}\right]^{(D-1)/2}} \quad (4.60)$$

$$S_{\text{Gauss}}(q, \xi, D, r_0) = 1 + \Gamma\left[\frac{D}{2}\right] \frac{D}{2} \left(\frac{\xi}{r_0}\right)^D {}_1F_1\left[\frac{D}{2}, \frac{3}{2}, -\frac{q^2\xi^2}{8}\right] \quad (4.61)$$

where D is the fractal dimension, ξ is a cut-off length for the fractal correlations, $\Gamma(x)$ is the gamma function. ${}_1F_1$ is the Kummer or hypergeometric function. For the cut-off functions 4.58 and 4.59 the integral 4.55 is solved numerically.

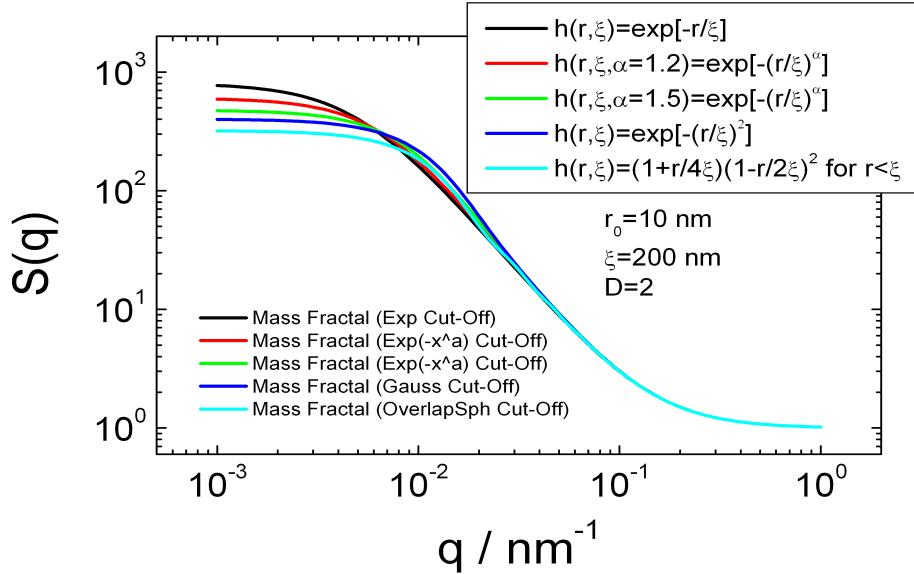


FIGURE 4.12. Comparison of the different structure factors for mass fractals.

4.4.1. Mass Fractal (Exp Cut-Off).

Input Parameters for model Mass Fractal (Exp Cut-Off):

r₀: characteristic dimension of individual scattering objects r₀

xi: cut-off length for the fractal correlations ξ

D: fractal dimension D

Note:

- D needs to be larger than 1 (D > 1). Physical values for D are between 1 and 3 (1 < D < 3).
- The fractal dimension needs to be large than the size of the individual scattering objects (r₀ < ξ).

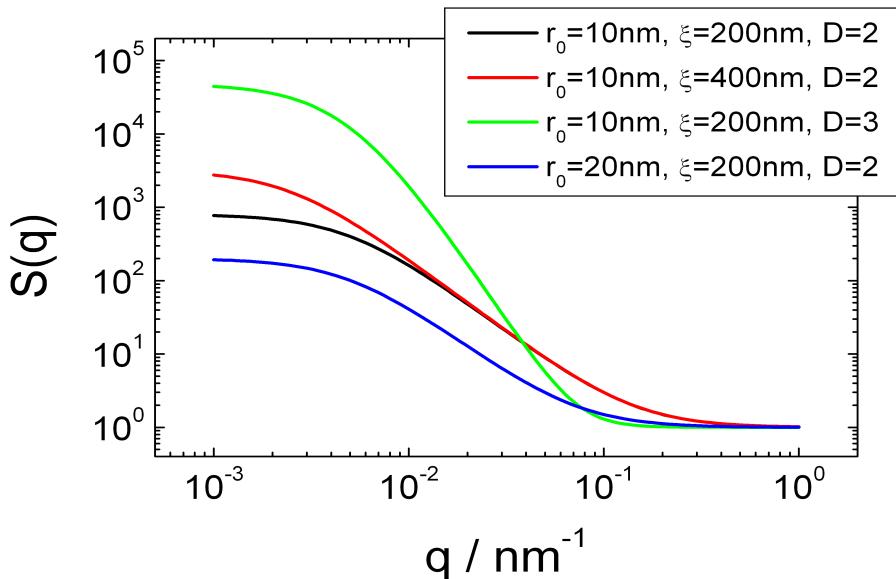


FIGURE 4.13. Structure factor of a mass fractal with an exponential cut-off function $h_{\text{Exp}}(r, \xi) = \exp \left[- \left(\frac{r}{\xi} \right)^\alpha \right]$.

4.4.2. Mass Fractal (Exp(-x^a) Cut-Off).

Input Parameters for model Mass Fractal (Exp(-x^a) Cut-Off):

r0: characteristic dimension of individual scattering objects r_0

xi: cut-off length for the fractal correlations ξ

D: fractal dimension D

Note:

- D needs to be larger than 1 ($D > 1$). Physical values for D are between 1 and 3 ($1 < D < 3$).
- The fractal dimension needs to be large than the size of the individual scattering objects ($r_0 < \xi$).
- the exponents α should be larger than 1, as otherwise the integral 4.55 for $S(q)$ does not converges.

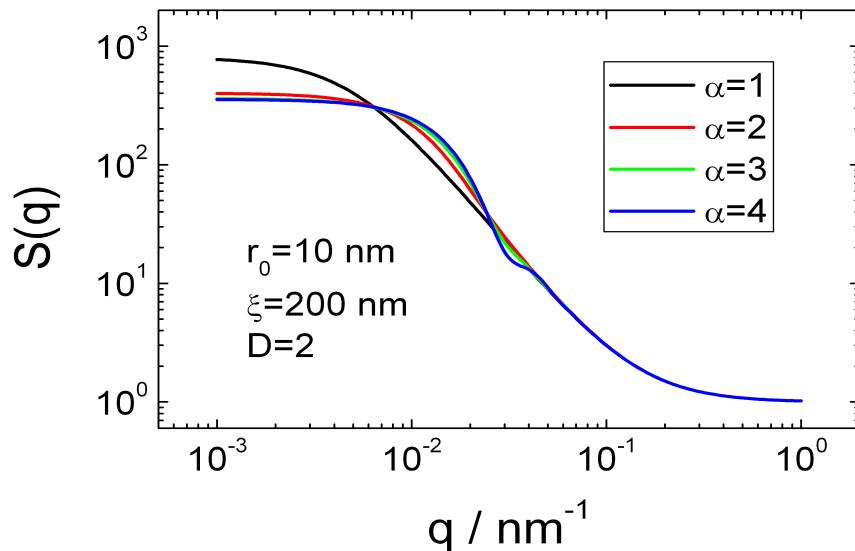


FIGURE 4.14. Structure factor of a mass fractal with a cut-off function $h_{\text{Exp}(-x^a)}(r, \xi, \alpha) = \exp \left[- \left(\frac{r}{\xi} \right)^\alpha \right]$.

4.4.3. Mass Fractal (Gaussian Cut-Off).

Input Parameters for model Mass Fractal (Gaussian Cut-Off):

r₀: characteristic dimension of individual scattering objects r_0

ξ : cut-off length for the fractal correlations ξ

D: fractal dimension D

Note:

- D needs to be larger than 1 ($D > 1$). Physical values for D are between 1 and 3 ($1 < D < 3$).
- The fractal dimension needs to be large than the size of the individual scattering objects ($r_0 < \xi$).

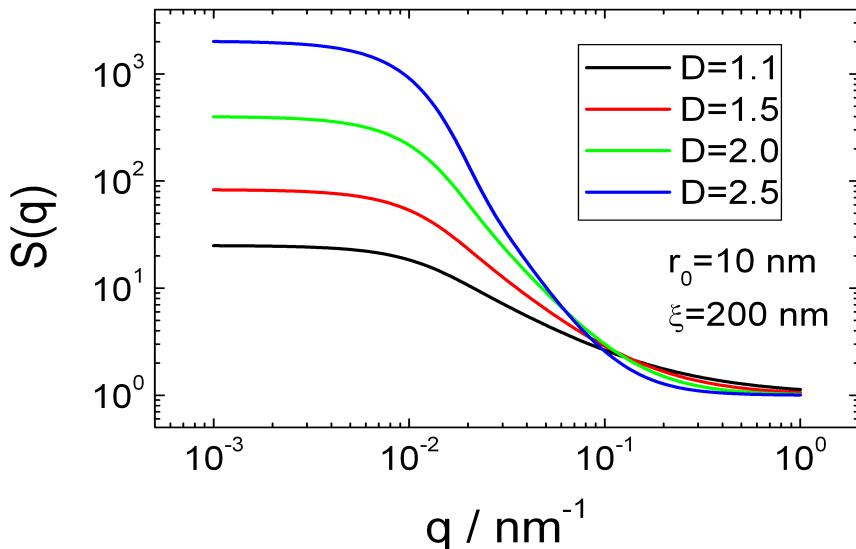


FIGURE 4.15. Structure factor of a mass fractal with an Gaussian cut-off function $h_{\text{Gauss}}(r, \xi) = \exp \left[- \left(\frac{r}{\xi} \right)^2 \right]$.

4.4.4. Mass Fractal (OverlapSph Cut-Off).

Input Parameters for model Mass Fractal (OverlapSph Cut-Off):

r0: characteristic dimension of individual scattering objects r_0

xi: cut-off length for the fractal correlations ξ

D: fractal dimension D

Note:

- D needs to be between 1 and 3 ($1 < D < 3$).
- The fractal dimension needs to be large than the size of the individual scattering objects ($r_0 < \xi$).

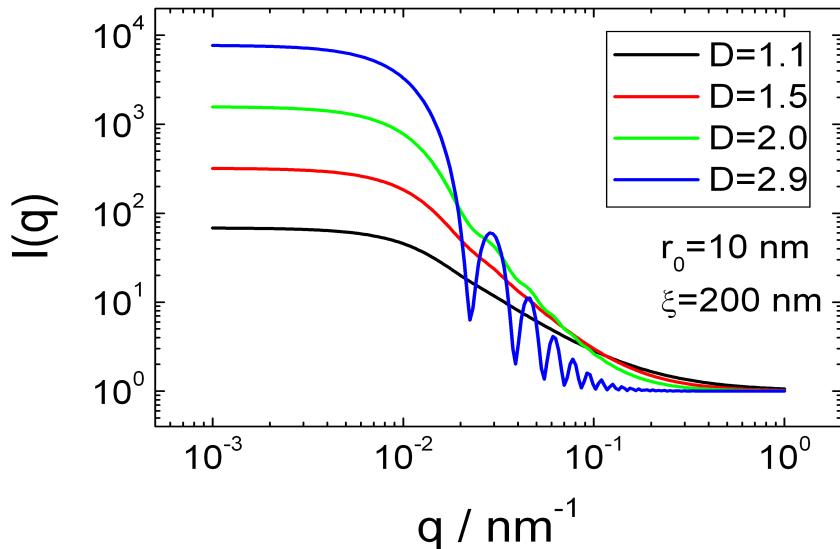


FIGURE 4.16. Structure factor of a mass fractal with a cut-off function $h_{\text{OverlapSph}}(r, \xi) = \left(1 + \frac{r}{4\xi}\right)^2 \left(1 - \frac{r}{2\xi}\right)^2$ for $r \leq 2\xi$.

4.5. Other Structure Factors

4.5.1. Hayter-Penfold RMSA [86, 84]. This is the structure factor for a system of charged, spheroidal objects in a dielectric medium. When combined with an appropriate form factor (such as sphere, core+shell, ellipsoid etc.), this allows for inclusion of the interparticle interference effects due to screened coulomb repulsion between charged particles. The salt concentration, is used to compute the ionic strength of the solution which in turn is used to compute the Debye screening length. At present there is no provision for entering the ionic strength directly nor for use of any multivalent salts. The counterions are also assumed to be monovalent.

Input Parameters for model Hayter Penfold RMSA:

RHS: hard sphere radius R_{HS} of particles in [nm].

Z: charge of particle in units of the charge of an electron $e = -1.6021766208 \times 10^{-19} \text{C}$

eta: volume fraction η of particles

T: sample temperature T in Kelvin

salt: monovalent salt concentration in [M]

eps_r: dielectric constant ϵ_r of solvent (vacuum:1, water at 25°: 78.2, water at 20°: 80.4, ethanol at 20°: 25.16, toluene at 23°: 2.38)

4.5.2. MacroIon.

$$ETA = \text{volume fraction} \quad (4.62\text{a})$$

$$AK = \kappa\sigma = \text{inv. screening length times diameter} \quad (4.62\text{b})$$

$$\kappa = \sqrt{e^2/(\epsilon\epsilon_0 k_B T) * (\rho_c + 2\rho_s)} \quad (4.62\text{c})$$

$$\rho_c = \text{density of counterions} = \rho_{\text{colloids}} Z \quad (4.62\text{d})$$

$$\rho_s = \text{density of salt cations or anions} \quad (4.62\text{e})$$

$$GEK = \text{charge}^2/(\pi k_B T \epsilon \epsilon_0 \sigma (2 + AK)^2) \quad (4.62\text{f})$$

$$\text{charge} = \text{Ladung eines Kolloids} = eZ \quad (4.62\text{g})$$

$$S = ETA^{1/3} = \text{scaling factor for rescaled MSA (RMSA)} \quad (4.62\text{h})$$

$$GAMK = 2 * S * GEK * \exp(AK - AK/S). \quad (4.62\text{i})$$

4.5.3. Critical Scattering.

$$S_{\text{crit}}(Q) = 1 + \frac{\kappa}{1 + \zeta^2 Q^2} \quad (4.63)$$

ζ : correlation length, κ : scaling factor

4.5.4. Cylinder(PRISM).

$$S_{\text{Cyl,PRISM}} = \frac{1}{1 + \nu C_q P_{15}} \quad (4.64)$$

$$x = 2QR$$

$$x_{P15} = Q(L - 2R)$$

$$P_{15} = 2 \frac{\text{Si}(x_{P15})}{x_{P15}} - 4 \frac{\sin^2(x_{P15}/2)}{x_{P15}^2}$$

$$C_q = 3 \frac{\sin(x) - x \cos(x)}{x^3}$$

4.5.5. Voigt Peak.

In spectroscopy, the Voigt profile is a spectral line profile named after Woldemar Voigt and found in all branches of spectroscopy in which a spectral line is broadened by two types of mechanisms, one of which alone would produce a Doppler profile, and the other of which would produce a Lorentzian profile. All normalized line profiles can be considered to be probability distributions. The Doppler profile is essentially a normal distribution and a Lorentzian profile is essentially a Cauchy distribution. Without loss of generality, we can consider only centered profiles which peak at zero. The Voigt profile is then the convolution of a Lorentzian profile and a Doppler profile:

$$V(x|\sigma, \gamma) = \int_{-\infty}^{\infty} D(x'|\sigma) L(x - x'|\gamma) dx' \quad (4.65a)$$

where x is frequency from line center, $D(x|\sigma)$ is the centered Doppler profile:

$$D(x|\sigma) = \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \quad (4.65b)$$

and $L(x|\gamma)$ is the centered Lorentzian profile:

$$L(x|\gamma) = \frac{\gamma}{\pi(x^2 + \gamma^2)}. \quad (4.65c)$$

The defining integral can be evaluated as:

$$V(x) = \frac{\Re[w(z)]}{\sigma\sqrt{2\pi}} \quad (4.65d)$$

where $\Re[w(z)]$ is the real part of the complex error function of z and

$$z = \frac{x + i\gamma}{\sigma\sqrt{2}} \quad (4.65e)$$

$$S_{\text{Voigt}}(Q, Q_m, A, \sigma, \gamma, c_0) = A V(Q - Q_m|\sigma, \gamma) + c_0 \quad (4.65f)$$

CHAPTER 5

Numerical solutions of the Ornstein Zernike equations

During an internship project in summer 2013 Evgeniy Ponomarev has implemented the core routine for solving the Ornstein Zernike equations. This algorithm numerically calculates structure factors $S(q)$, radial distribution functions $g(r)$ and direct correlation functions $c(r)$ of a systems with known pair interaction potential $U(r)$.

5.1. Background

The central part of this package is iterative solution of Ornstein-Zernike equation [159]. The theoretical background can be found in great detail in [154, 24, 85, 20, 31, 126]. The Ornstein-Zernike equation reads

$$h(\mathbf{r}_{12}) = c(\mathbf{r}_{12}) + \int c(\mathbf{r}_{13})\rho(\mathbf{r}_3)h(\mathbf{r}_{32})d\mathbf{r}_3 + \dots \quad (5.1)$$

where $h(\mathbf{r}_{12})$ is the total correlation function, $c(\mathbf{r}_{12})$ the direct correlation function (direct effect of particle 1 on particle 2), $c(\mathbf{r}_{13})$ the in direct correlation function describing the effect of particle 1 on particle 3 which influences particle 2, and ρ the particle number density of the colloids, molecules, or atoms that form the liquid. If the fluid is uniform and isotropic, the OrnsteinZernike relation becomes

$$h(r) = c(r) + \rho \int c(|\mathbf{r} - \mathbf{r}'|)h(r')d\mathbf{r}' \quad (5.2)$$

$$= c(r) + \gamma(r) \quad (5.3)$$

The basic idea of this equation is that total correlation between positions of two particles is a combination of their direct and indirect (through neighboring particles) interactions. The first contribution to $h(r)$ is the direct correlation function $c(r)$ that represents the correlation between a particle of a pair with its closest neighbor separated by a distance r . The second contribution is the indirect correlation function $\gamma(r)$, which represents the correlation between the selected particle of the pair with the rest of the fluid constituents.

$g(r)$ is known as the radial or pair distribution function, which measures the probability that given a particle at the origin, another particle of the fluid can be found at a distance r from it. When the distance separating a pair of particles tends to infinity, the correlations vanish and $g(r)$ tends to 1. This means that the total correlation function defined as $h(r) = g(r) - 1$ tends to 0. The Fourier transform $\mathcal{F}\{\}$ of $g(r)$ and $h(r)$ are directly related to the structure factor $S(q)$ that is experimentally measurable by x-ray or neutron scattering

$$S(q) = 1 + \rho \int [g(r) - 1] \exp(-i\mathbf{q} \cdot \mathbf{r}) d\mathbf{r} \quad (5.4)$$

and

$$S(q) = 1 + \rho \tilde{h}(q) \quad (5.5)$$

where $\tilde{h}(q) = \mathcal{F}\{h(r)\}$ is the Fourier transform of the total correlation function $h(r)$, so that

$$\tilde{h}(q) = \int h(r) \exp(-\imath \mathbf{q} \cdot \mathbf{r}) d\mathbf{r} \quad (5.6)$$

Via inverse Fourier transform $\mathcal{F}^{-1}\{\}$ of the structure factor the pair correlation function is obtained

$$\rho [g(r) - 1] = \mathcal{F}^{-1}\{S(q) - 1\} = \frac{1}{(2\pi)^3} \int (S(q) - 1) \exp(\imath \mathbf{q} \cdot \mathbf{r}) d\mathbf{q} \quad (5.7)$$

As for $g(r)$ also the direct correlation function $c(r)$ can be, in principle, derived from experiment as

$$\rho c(r) = \frac{1}{(2\pi)^3} \int \left(1 - \frac{1}{S(q)}\right) \exp(\imath \mathbf{q} \cdot \mathbf{r}) d\mathbf{q} \quad (5.8)$$

or

$$S(q) = \frac{1}{1 - \rho \tilde{c}(q)} \quad (5.9)$$

By Fourier transformation of eq. 5.2, one obtains

$$\tilde{h}(q) = \frac{\tilde{c}(q)}{1 - \rho \tilde{c}(q)} \quad (5.10)$$

where $\tilde{c}(q) = \mathcal{F}\{c(r)\}$ is the Fourier Transform of $c(r)$. In order to determine the two correlation functions $h(r)$ and $c(r)$ for a given pair potential $u(r)$, eq. 5.2 must be supplemented by an auxiliary closure relation. Furthermore the Fourier transform of eq. 5.3 in combination with eq. 5.10 allows to write the Fourier transform of the indirect correlation function as

$$\mathcal{F}\{\gamma(r)\} = \tilde{\gamma}(q) = \tilde{h}(q) - \tilde{c}(q) = \frac{\rho \tilde{c}^2(q)}{1 - \rho \tilde{c}(q)} \quad (5.11)$$

The closure is not complete as long the bridge function is unknown. Even though the bridge function is exactly defined the function is an unknown function of interparticle distance and only approximations can be given. The direct correlation function can be written in terms of the indirect correlation function, the interaction potential and the bridge function as

$$\begin{aligned} c(r) &= \exp[-\beta u(r) + \gamma(r) + B(r)] - \gamma(r) - 1 \\ &= g(r) - \gamma(r) - 1 \end{aligned} \quad (5.12)$$

$$\text{and } h(r) + 1 = g(r) = \exp[-\beta u(r) + \gamma(r) + B(r)] \quad (5.13)$$

where $u(r)$ is the pair interaction potential, $\beta = 1/(k_B T)$ the inverse temperature, with k_B being Boltzmanns constant, T is the absolute temperature, $\gamma(r)$ being the indirect correlation function and ρ is the number density of the molecules or atoms that form the liquid. $B(r)$ is the bridge function. The analytic expression of $B(r)$ is, in general, unknown. Unfortunately, the bridge function has to be approximated in

practice. Though a number of theoretical and simulation procedures have been derived in recent years in order to get an approximate estimate of $B(r)$ for different fluid models, the exact bridge function is not known for any system.

5.2. Numerical implementation of the iterative algorithm in SASfit

The easiest algorithm for solving the Ornstein Zernike (OZ) equation 5.2 together with the closure equation 5.12 is direct iteration using fast Fourier transformation (see e.g. [93]). The solution of the OZ equations can be treated either as a fixed point problem for the unknown function $\gamma(r)$, so that $T_f^{\text{OZ}}(\gamma) = \gamma$ or as a multidimensional root finding problem $F_r^{\text{OZ}}(\gamma) = T_f^{\text{OZ}}(\gamma) - \gamma = 0$. In SASfit the fix point problem $T_f^{\text{OZ}}(\gamma) = \gamma$ is implemented as:

- (1) using a starting value for $\gamma(r)$, e.g. set $\gamma(r) \equiv 0$
- (2) Calculate a bridge function. Many different theories how to calculate the bridge function have been published. A few of them are described in the next section. The bridge function is often expressed in terms of the known potential $u(r)$ and the indirect correlation function $\gamma(r)$, which needs to be determined.
- (3) Calculate the direct correlation function according to eq. 5.12.
- (4) Calculate the Fourier transform $\tilde{c}(q)$ of the direct correlation function.
- (5) Use eq. 5.11 to get $\tilde{\gamma}(q)$.
- (6) Calculate the inverse Fourier transform of $\tilde{\gamma}(q)$ to get the new guess for $\gamma(r)$.

In the above algorithm to iteratively solve the Ornstein-Zernike equation two Fourier transforms have to be performed:

$$c(r) \xrightarrow{\mathcal{F}\{\cdot\}} \tilde{c}(q) \quad (\text{in step 4}) \quad (5.14)$$

$$\tilde{\gamma}(q) \xrightarrow{\mathcal{F}^{-1}\{\cdot\}} \gamma(r) \quad (\text{in step 6}) \quad (5.15)$$

As we assume that the system is isotropic, i.e. $c(r)$ and $\gamma(r)$ are only functions of the modulus of $|\mathbf{r}| = r$ the Fourier transforms can be written as

$$\tilde{c}(q) = \int c(r) \exp(-i\mathbf{q} \cdot \mathbf{r}) dr = \int_0^\infty 4\pi r^2 c(r) \frac{\sin(qr)}{qr} dr \quad (5.16)$$

$$\gamma(r) = \frac{1}{(2\pi)^3} \int \tilde{\gamma}(q) \exp(i\mathbf{q} \cdot \mathbf{r}) d\mathbf{q} = \frac{1}{(2\pi)^3} \int_0^\infty 4\pi q^2 \tilde{\gamma}(q) \frac{\sin(qr)}{qr} dq \quad (5.17)$$

To calculate these transforms numerically on a discrete grid of points one has to substitute the integration by a summation. Herby $\int_0^\infty dr$ is replaced by $\sum_{j=0}^{N_p-1} \Delta r$, $r_j = (j+1)\Delta r$, $q_j = (j+1)\Delta q$, and $\int_0^\infty dq$ is replaced by $\sum_{j=0}^{N_p-1} \Delta q$. The step width Δr in real space has to be defined by the user as well as the number of grid points N_p . The step width in reciprocal space is than obtained by $\Delta q = \frac{\pi}{(N_p+1)\Delta r}$. It is important to mention that the values of the first elements of the arrays r_i and k_i equals Δr and

Δq , respectively ($r_0 = \Delta r, q_0 = \Delta q$).

$$\tilde{c}_i = \frac{4\pi\Delta r^2}{(i+1)\Delta q} \sum_{j=0}^{N_p-1} c_j(j+1) \sin\left(\frac{\pi(j+1)(i+1)}{N_p+1}\right) \quad (5.18)$$

$$\gamma_i = \frac{1}{(2\pi)^3} \frac{4\pi\Delta q^2}{(i+1)\Delta r} \sum_{j=0}^{N_p-1} \tilde{c}_j(j+1) \sin\left(\frac{\pi(j+1)(i+1)}{N_p+1}\right) \quad (5.19)$$

For calculating the Fourier transformation the library FFTW for fast fourier transformation [56, 55] has been used. The library (source code available at <http://www.fftw.org/>) supplies a discrete sin transformation (type-I DST) named FFTW_RODFT00 which is doing the transformation

$$Y_k = 2 \sum_{j=0}^{N_p-1} X_j \sin\left(\frac{\pi(j+1)(k+1)}{N_p+1}\right) \quad (5.20)$$

The unnormalized inverse of FFTW_RODFT00 is FFTW_RODFT00 itself. The same routine can be used for both transformations.

For the fix point problem $T_f^{\text{OZ}}(\gamma) = \gamma$ several iteration schemes are implemented. Tests have shown, that one can not easily make a prediction which one converges fastest or in case of e.g. very high volume or strong attraction converges at all. Next to some simple iteration schemes also a the very efficient Anderson Acceleration for fixed-point iteration [3, 223, 209]

The following iteration schemes are implemented:

Picard iteration [172]:

$$\gamma_{n+1} = T_f^{\text{OZ}}(\gamma_n) \quad (5.21)$$

Krasnoselskij iteration [119]:

$$\gamma_{n+1} = (1 - \alpha)\gamma_n + \alpha T_f^{\text{OZ}}(\gamma_n) \quad (5.22)$$

with $\alpha \in (0, 1]$ beeing a constant independent of n

Mann iteration [140]:

$$\gamma_{n+1} = (1 - \alpha_n)\gamma_n + \alpha_n T_f^{\text{OZ}}(\gamma_n) \quad (5.23)$$

with $\alpha_n \in (0, 1]$

Picard-Mann hybrid iteration (PMH) [111]:

$$y_n = (1 - \beta_n)\gamma_n + \beta_n T_f^{\text{OZ}}(\gamma_n) \quad (5.24)$$

$$\gamma_n = T_f^{\text{OZ}}(y_n) \quad (5.25)$$

with $\beta_n \in (0, 1]$

Mann II iteration [105]:

$$y_n = (1 - \beta_n)\gamma_n + \beta_n T_f^{\text{OZ}}(\gamma_n) \quad (5.26)$$

$$\gamma_n = (1 - \alpha_n)y_n + \alpha_n T_f^{\text{OZ}}(y_n) \quad (5.27)$$

with $\alpha_n, \beta_n \in (0, 1]$

S iteration [106, 1]:

$$y_n = (1 - \beta_n)\gamma_n + \beta_n T_f^{\text{OZ}}(\gamma_n) \quad (5.28)$$

$$\gamma_n = (1 - \alpha_n)T_f^{\text{OZ}}(\gamma_n) + \alpha_n T_f^{\text{OZ}}(y_n) \quad (5.29)$$

with $\alpha_n, \beta_n \in (0, 1]$

Ishikawa iteration [100]:

$$y_n = (1 - \beta_n)\gamma_n + \beta_n T_f^{\text{OZ}}(\gamma_n) \quad (5.30)$$

$$\gamma_n = (1 - \alpha_n)y_n + \alpha_n T_f^{\text{OZ}}(y_n) \quad (5.31)$$

with $\alpha_n, \beta_n \in (0, 1]$

SP iteration [171]:

$$z_n = (1 - \xi_n)x_n + \xi_n T_f^{\text{OZ}}(\gamma_n) \quad (5.32)$$

$$y_n = (1 - \beta_n)z_n + \beta_n T_f^{\text{OZ}}(z_n) \quad (5.33)$$

$$\gamma_n = (1 - \alpha_n)y_n + \alpha_n T_f^{\text{OZ}}(y_n) \quad (5.34)$$

with $\alpha_n, \beta_n, \xi_n \in (0, 1]$

Noor iteration [156]:

$$z_n = (1 - \xi_n)x_n + \xi_n T_f^{\text{OZ}}(\gamma_n) \quad (5.35)$$

$$y_n = (1 - \beta_n)x_n + \beta_n T_f^{\text{OZ}}(z_n) \quad (5.36)$$

$$\gamma_n = (1 - \alpha_n)y_n + \alpha_n T_f^{\text{OZ}}(y_n) \quad (5.37)$$

with $\alpha_n, \beta_n, \xi_n \in (0, 1]$

CR iteration [36]:

$$z_n = (1 - \xi_n)x_n + \xi_n T_f^{\text{OZ}}(\gamma_n) \quad (5.38)$$

$$y_n = (1 - \beta_n)T_f^{\text{OZ}}(\gamma_n) + \beta_n T_f^{\text{OZ}}(z_n) \quad (5.39)$$

$$\gamma_n = (1 - \alpha_n)y_n + \alpha_n T_f^{\text{OZ}}(y_n) \quad (5.40)$$

with $\alpha_n, \beta_n, \xi_n \in (0, 1]$

Picard-S iteration [74]:

$$z_n = (1 - \beta_n)x_n + \beta_n T_f^{\text{OZ}}(\gamma_n) \quad (5.41)$$

$$y_n = (1 - \alpha_n)T_f^{\text{OZ}}(\gamma_n) + \alpha_n T_f^{\text{OZ}}(z_n) \quad (5.42)$$

$$\gamma_n = T_f^{\text{OZ}}(y_n) \quad (5.43)$$

with $\alpha_n, \beta_n \in (0, 1]$

S* iteration [106]:

$$z_n = (1 - \xi_n)x_n + \xi_n T_f^{\text{OZ}}(\gamma_n) \quad (5.44)$$

$$y_n = (1 - \beta_n)T_f^{\text{OZ}}(\gamma_n) + \beta_n T_f^{\text{OZ}}(z_n) \quad (5.45)$$

$$\gamma_n = (1 - \alpha_n)T_f^{\text{OZ}}(\gamma_n) + \alpha_n T_f^{\text{OZ}}(y_n) \quad (5.46)$$

with $\alpha_n, \beta_n, \xi_n \in (0, 1]$

α_n , β_n , and ξ_n are series of constants between $(0; 1]$. In **SASfit** the parameters α_n , β_n , and ξ_n have been chosen to be the same for each iteration n . Furthermore two series schemes have been implemented. The starting values for $n = 0$ of the series is the modulus of the mixing parameter m_{mix} , which can be changed in the GUI. The default value is $+0.5$. As $\alpha_n, \beta_n, \xi_n \in (0, 1]$ the series is in **SASfit** is defined for positive mixing parameters as $\alpha_n = \beta_n = \xi_n = (1 - m_{\text{mix}}) \exp(-\log_{10}(\|T_f^{\text{OZ}}(\gamma_n) - \gamma_n\|/\Delta_{\text{rel}})) + m_{\text{mix}}$ where Δ_{rel} is a user defined relative iteration precision. The default value is $\Delta_{\text{rel}} = 10^{-12}$. If one supplies a negative mixing parameter the series is defined as $\alpha_n = \beta_n = \xi_n = -m_{\text{mix}} + (i_n/i_{\text{max}})(1 + m_{\text{mix}})$. **SASfit** uses for the moment $i_{\text{max}} = 20$. i_n is incremented by 1 one in case the relative error has been reduced and decremented if the relative error increased in one iteration step. Decrementation and incrementation is only performed in the interval $[0, i_{\text{max}}]$.

Instead of solving the OZ-fixed point problem $T_f^{\text{OZ}}(\gamma) = \gamma$ one can also use multi-dimensional root-finding algorithms for solving $F_r^{\text{OZ}}(\gamma) = T_f^{\text{OZ}}(\gamma) - \gamma = 0$. Some root finding algorithms are supplied from the gnu scientific library **gsl** for functions without derivatives namely the Hybrid algorithm which replaces calls to the Jacobian function by its finite difference approximation (**Hybrid**), a finite difference version of the Hybrid algorithm without internal scaling (**Hybrids (int. sc.)**), the discrete Newton algorithm (**dNewton**), and the Broyden algorithm (**Broyden**). All of them have the disadvantage to fill a Jacobian matrix and to calculate its inverse.

Next to the **gsl** root finding algorithms also several very efficient nonlinear system solver based on Newton-Krylov solver technology, like GMRES (Generalized Minimal RESidual)[**188, 110**], FGMRES (Flexible Generalized Minimal RESidual) [**187**], Bi-CGStab (Bi-Conjugate Gradient Stabilized) [**215, 110**], and TFQMR (Transpose-Free Quasi-Minimal Residual) [**53, 110**] are supplied. The nonlinear system solver are supplied using the "kinsol" library which is part of the SUNDIALS package [**88**] available at <https://computation.llnl.gov/casc/sundials/main.html>

Test have shown that either the Anderson acceleration, GMRES or FGMRES seems to be the most efficient algorithm to solve the Ornstein-Zernike with a minimum number of calls to the procedure to evaluating the self consistent operator $F_r^{\text{OZ}}(\gamma)$ or $T_f^{\text{OZ}}(\gamma)$. The best choice for the iteration method depends often on the problem, i.e. the type of closure and the potential and also on the volume fraction. Therefore there are still all methods mentioned above available, which might change after getting some more experience with them.

5.3. Thermodynamic Parameters and Consistency Tests

Thermodynamic consistencies have been an important tool finding a good bridge function. The Ornstein-Zenike equations in combination with a closure relation enable us to calculate the direct $c(r)$ and total $h(r)$ correlation function as well as the pair distribution function $g(r)$. From these functions thermodynamic properties via different routes can be calculated.

The thermodynamic routes are:

compressibility route:

$$\left(\frac{\beta \partial P}{\partial \rho}\right)_T = (\rho k_B T \kappa_T)^{-1} \Leftrightarrow \kappa_T = \frac{1}{\rho} \left(\frac{\partial P}{\partial \rho}\right)_T^{-1} \quad (5.47)$$

The compressibility can also be related to the static structure factor in the long-wavelength limit:

$$\begin{aligned} \kappa_T &= \frac{\beta}{\rho} \left[1 + \rho \int 4\pi r^2 h(r\rho, T) dr \right] \\ &= \frac{\beta}{\rho} \left[1 + \rho \int 4\pi r^2 [g(r; \rho, T) - 1] dr \right] \\ &= \frac{\beta}{\rho} \left[1 - \rho \int 4\pi r^2 c(r) dr \right]^{-1} \\ &= \frac{\beta}{\rho} S(q = 0, \rho, T) \end{aligned} \quad (5.48)$$

energy route:

The internal energy U is

$$U = U_{\text{id}} + U_{\text{ex}} \quad (5.49)$$

where $U_{\text{id}} = \frac{3}{2} N k_B T$ is the internal energy of an ideal gas and the excess internal energy per particle is given by

$$\frac{U_{\text{ex}}}{N} = 2\pi\rho \int g(r) u(r) r^2 dr \quad (5.50)$$

The excess internal energy U_{ex} is related to the excess Helmholtz free energy F_{ex} by

$$U_{\text{ex}} = \left(\frac{\partial \beta F_{\text{ex}}}{\partial \beta} \right)_V \quad (5.51)$$

The excess pressure P_{ex} ($P = P_{\text{id}} + P_{\text{ex}}$ and $P_{\text{id}} = \rho/\beta$) is than obtained as

$$P_{\text{ex}} = -\frac{\partial F_{\text{ex}}}{\partial V} \quad (5.52)$$

where $\beta F_{\text{ex}} = \int_0^\beta U(\beta', \rho) d\beta'$.

virial route:

The pressure or virial equation is given as

$$\frac{\beta P}{\rho} = 1 + \frac{2}{3} \beta \pi \rho \int g(r) r^3 \frac{du(r)}{dr} dr \quad (5.53)$$

where $\rho = N/V$ is the average number density of particles in the system. As for discontinuous potentials also $g(r)$ becomes discontinuous it might be an advantage to rewrite this equation in terms of $e(r) = \exp(-\beta u(r))$ and the cavity function $y(r) = (h(r) + 1)/e(r) = g(r) \exp(\beta u(r))$ where the cavity function is always a continuous function by using

$$g(r) \frac{du(r)}{dr} = y(r) e(r) \left(-\frac{1}{\beta e(r)} \frac{de(r)}{dr} \right) = \frac{y(r)}{\beta} \frac{de(r)}{dr} \quad (5.54)$$

so that

$$\frac{\beta P}{\rho} = 1 + \frac{2}{3}\pi\rho \int r^3 \frac{de(r)}{dr} y(r) dr \quad (5.55)$$

In case of potentials with a hard core or a step in the potential also $g(r)$ as well as $u(r)$ are discontinuous. The cavity function is a continuous function and helps to solve the integral. For discontinuous potentials the integration has to be done in parts and the discontinuity needs to be handled separately. It can be shown [196], that at a discontinuity of the potential, let say at $r = \sigma$, the discontinuity contributes to the integral with

$$\frac{2\pi}{3}\rho\sigma^3 (g(\sigma^+) - g(\sigma^-)) / \beta \quad (5.56)$$

To determine mixing parameters in various closures one often compare the pressures calculated via the three different routes leading to two consistency equations.

$P_v - \kappa_T$ consistency:

Via the virial route the pressure can be calculated by eq. 5.53 or 5.55. It's partial derivative according to the number density yield the isothermal compressibility (eq. 5.47), which is known from the structure factor in the long-wavelength limit at $q = 0$ according to eq. 5.48. This consistency condition can be written as an integral which has to be become zero [143, 219]

$$I(\rho, T) = 0 = \int \left[c(r) - \frac{r}{6} \frac{d(\beta u(r))}{dr} \left(2g(r) + \rho \frac{\partial g(r)}{\partial \rho} \right) \right] r^2 dr \quad (5.57)$$

$dU - dP$ consistency:

The Helmholtz free energy F obeys the Gibbs-Helmholtz relation

$$d(\beta F) = U d\beta - (\beta P) dV \quad (5.58)$$

In order for F to be an exact differential, the cross partial derivatives should be equal

$$\left(\frac{\partial U}{\partial V} \right)_T = \left(\frac{\partial(\beta P)}{\partial \beta} \right)_V = T^2 \frac{\partial}{\partial T} \left(\frac{P}{T} \right)_V \quad (5.59)$$

The internal energy U can be calculated via the energy route by eq. 5.50 and the pressure P again via the virial route by eq. 5.53 or 5.55. Also this consistency condition can be written in integral form as

$$J(\rho, T) = 0 = \int \left[u(r) \left(g(r) + \rho \frac{\partial g(r)}{\partial \rho} \right) + \frac{r}{3} \frac{du(r)}{dr} \left(g(r) - T \frac{\partial g(r)}{\partial T} \right) \right] r^2 dr \quad (5.60)$$

In practise the calculation of the integrals $I(\rho, T)$ and $J(\rho, T)$ involves the numerical partial derivation of the pair distribution function $g(r)$ with respect to the particle

number density ρ and the temperature T , which can be approximated by

$$\frac{\partial g(r, \rho, T)}{\partial \rho} \simeq \frac{g(r, \rho + \Delta\rho, T) - g(r, \rho - \Delta\rho, T)}{2\Delta\rho} \quad (5.61)$$

$$\frac{\partial g(r, \rho, T)}{\partial T} \simeq \frac{g(r, \rho, T + \Delta T) - g(r, \rho, T - \Delta T)}{2\Delta T} \quad (5.62)$$

The integral equations therefore needs to be solved for five different thermodynamic states, namely at (ρ, T) , $(\rho + \Delta\rho, T)$, $(\rho - \Delta\rho, T)$, $(\rho, T + \Delta T)$, and $(\rho, T - \Delta T)$.

5.4. Closures

The closures implemented in **SASfit** have been taken from the overview papers from Bomont [20] and Caccamo [31] and the references in there.

5.4.1. Hypernetted-chain(HNC) approximation.

The HNC is one of the earliest and most applied integral equation theory of fluids [85]. It simply assumes for completing the closure relation 5.12 a negligible bridge function [31, 20, 85].

$$B_{\text{HNC}}(r) \equiv 0 \quad (5.63)$$

$$g(r) = \exp [-\beta u(r) + \gamma(r)] \quad (5.64)$$

The HNC equation is a special case insofar as it corresponds to a well-defined free-energy functional, and differentiation of that free energy with respect to volume can be shown to give the same result as the virial equation. The energy and virial routes to the equation of state are therefore equivalent. The HNC closure is often used for long range repulsive potentials. For short range repulsive potentials the PY approximation yields somehow better results.

5.4.2. Reference hypernetted chain closure (RHNC).

This closure is an extension of the HNC closure, which requires that a reference system is known and only need to be corrected for a small perturbation. Therefore the potential needs to be written as a sum of a reference and perturbation potential $u(r) = u_{\text{ref}} + u_{\text{pert}}(r)$. By that the indirect correlation function is given by $\gamma(r) = \gamma_{\text{ref}}(r) + \gamma_{\text{pert}}(r)$ and the direct correlation function by

$$c(r) = g_{\text{ref}}(r) \exp (\gamma_{\text{pert}} - \beta u_{\text{pert}}) - \gamma(r) - 1 \quad (5.65)$$

with $g_{\text{ref}}(r)$ being the radial distribution function of the reference system. To solve the OZ equation for the reference system the Martynov-Sarkisov (MS) closure (sec. 5.4.12) is used in **SASfit** as it yields results with relative good thermodynamic consistency for hard sphere potentials, at least better than the PY closure.

5.4.3. modified hypernetted chain closure (MHNC).

The modified hypernetted chain closure was introduced by Resnfeld and Ashcroft [185]. They suggested to approximate the bridge $B_{\text{MHNC}}(r)$ function by the one of hard spheres $B_{\text{PY}}(r, \eta^*)$ however with an effective volume fraction η^* to get thermodynamic consistency. The bridge function in PY approximation they is given by

$$B_{\text{PY}}(r) = \begin{cases} c_{\text{PY}}(r) + 1 + \ln[-c_{\text{PY}}(r)] & \text{for } r < \sigma \\ -g_{\text{PY}}(r) + 1 + \ln[g_{\text{PY}}(r)] & \text{for } r > \sigma \end{cases} \quad (5.66)$$

The direct correlation function $c_{\text{PY}}(r)$ and radial distribution function $g_{\text{PY}}(r)$ are taken from the analytical solution for $c_{\text{PY}}(r)$ given by Wertheim [226] and $g_{\text{PY}}(r)$ by [210, 211]. SASfit is searching within the interval $\eta^* \in [0.02, 0.6]$ for a thermodynamic consistent solution.

5.4.4. Kovalenko-Hirata or partially linearized hypernetted chain closure (KH, PLHNC).

Kovalenko and Hirata [118] introduced a closure for studying self-consistent description of a metalwater interface and which has been used later on e.g. by [98] to study core-softened potentials. This closure is also named partially linearized hypernetted chain closure PLHNC. Their bridge function reads as

$$B_{\text{KH}}(r) = \begin{cases} 0 & \text{for } -\beta u(r) + \gamma(r) \leq 0 \\ \beta u(r) - \gamma(r) + \ln(-\beta u(r) + \gamma(r) + 1) & \text{for } -\beta u(r) + \gamma(r) > 0 \end{cases} \quad (5.67)$$

5.4.5. Percus-Yevick (PY) approximation.

This approximation has been successfully been used to find an analytical solution for the OZ equations for a Hard Sphere potential as well as a Sticky Hard Sphere potential. Also analytical solution for both Hard Sphere [169, 226] and Sticky Hard Sphere [9] with a size distribution has been found. The bridge function of this approximation reads as

$$B_{\text{PY}}(r) = \ln[\gamma(r) - 1] - \gamma(r) \quad (5.68)$$

$$g(r) = \exp[-\beta u(r)] (1 + \gamma(r)) \quad (5.69)$$

The PY equation proves to be more successful than the HNC approximation when the potential is strongly repulsive and short ranged. The PY equation is of particular interest

in the theory of simple liquids because it is soluble analytically in the important case of the hard-sphere fluid.

5.4.6. Mean Spherical Approximation (MSA).

This approximation has originally been proposed by Lebowitz and Percus [123]. It has been mainly successfully applied to spherical potentials interacting through an infinitely repulsive potential at short range. It uses the ansatz

$$g(r) = 0 \quad \text{for } \beta u(r) = +\infty \quad (5.70)$$

$$c(r) = -\beta u(r) \quad \text{for } \beta u(r) \neq +\infty \quad (5.71)$$

Analytical solutions have been obtained for charged hard spheres and hard core Yukawa potentials [31].

5.4.7. Rescaled Mean Spherical Approximation (RMSA).

The mean spherical approximation can especially for dilute cases result in a negative $g(r)$ at the contact point σ^+ , which is just out of the hard core radius, i.e. the first distance for which $\exp(-\beta u(r)) > 0$. The rescaled mean spherical approximation (RMSA) [84] is searching for the smallest apparent contact point $\tilde{\sigma}^+$ for which $g(\tilde{\sigma}^+) > 0$.

5.4.8. "Soft core" MSA (SMSA) Approximation.

This closure has been motivated because the Percus-Yevick (PY) approximation is not accurate for attractive potentials. The SMSA [138, 34] is a generalisation of the MSA closure. This has been done by incorporating the division scheme as of Weeks et al. [225] by dividing the potential in a entirely repulsive part and a attractive part $u(r) = u_R(r) + u_A(r)$. The SMSA bridge function reads as

$$B_{\text{SMSA}}(r) = \ln[1 + \gamma^*(r)] - \gamma^*(r) \quad (5.72)$$

with $\gamma^*(r) = \gamma(r) - \beta u_A(r)$. Normally $u_R(r)$ describes a short range repulsive interaction like hard sphere interaction or as in the division scheme from Weeks et al. the potential up to the minimum r_m . For $r > r_m$ or $r > \sigma^{\text{HS}}$ the closure reduces to the MSA approximation $c(r) = -\beta u(r)$. In a case of a purely repulsive interaction, one has $r_m \rightarrow \infty$ and $u_A(r) \equiv 0$. In this case the closure reduces to the PY closure. The SMSA interpolates between PY and MSA [20].

5.4.9. Verlet Approximation.

A semi-phenomenological equation for the radial distribution function was proposed by Verlet [216, 217] for the case of hard spheres with the following bridge function

$$B_{\text{Verlet}}(r) = \frac{-\gamma^2(r)}{2(1 + \frac{4}{5}\gamma(r))} \quad (5.73)$$

The approximation yield very good results for both thermodynamic as well as structural properties. This closure has been extended and refined by others and are listed below.

5.4.10. Choudhury-Gosh (CG) Approximation.

Choudhury and Gosh [35] proposed a bridge function based on the closure of Verlet which reads as

$$B_{\text{CG}}(r) = \frac{-\gamma^{*2}}{2(1 + \alpha\gamma^*(r))} \quad (5.74)$$

with $\alpha = 1.0175 - 0.275\rho\sigma^3$ and $\gamma^*(r) = \gamma(r) - \beta u_A(r)$.

5.4.11. Duh-Haymet (DH) Approximation.

Duh-Haymet [48] introduced a closure for the Lennard-Jones potential, where they partitioned the potential slightly differently than Weeks et al. [225]. As the perturbation potential they used

$$u_{\text{pert}}^{\text{LR,DH}}(r) = -4k_B T \epsilon \left(\frac{\sigma}{r}\right)^6 \times \exp\left(-\frac{k_B T \epsilon}{\rho^*} \left(\frac{\sigma}{r}\right)^6\right) \quad (5.75)$$

with the reduced density taken as $\rho^* = \rho\sigma^3$. Their bridge function reads

$$B_{\text{DH}}(r) = \frac{-\gamma^{*2}(r)}{2 \left[1 + \frac{5\gamma^*(r)+11}{7\gamma^*(r)+9} \gamma^*(r) \right]} \quad (5.76)$$

with $\gamma^*(r) = \gamma(r) - \beta u_A(r)$ or in case of the Lennard-Jones potential $\gamma^*(r) = \gamma(r) - \beta u_{\text{pert}}^{\text{LR,DH}}(r)$

5.4.12. Martynov-Sarkisov (MS) Approximation.

An approximation proposed by Martynov and Sarkisov [142] sets the bridge function as

$$B_{\text{MS}}(r) = [1 + 2\gamma(r)]^{1/2} - 1 - \gamma(r) \quad (5.77)$$

This approximation has been originally applied to hard spheres without an additional adjustable parameter. Compared to PY or HNC this closure reduces considerably the thermodynamic inconsistency.

5.4.13. Ballone, Pastore, Galli, and Gazzillo (BPGG) approximations.

Ballone, Pastore, Galli, and Gazzillo [7] improved the MS approximation by assuming a bridge function of the form

$$B_{\text{BPGG}}(r) = [1 + s\gamma(r)]^{1/s} - 1 - \gamma(r) \quad (5.78)$$

For $s = 1$ the HNC is recovered and for $s = 2$ the MS closure. s is used here to obtain a thermodynamic consistency condition.

5.4.14. Vompe-Martynov (VM) Approximation.

The MS bridge function [142] has been refined by Vompe and Martynov [219]. Their extended bridge function reads

$$B_{\text{VM}}(r) = \sqrt{1 + 2\gamma^*(r)} - 1 - \gamma^*(r) \quad (5.79)$$

with $\gamma^*(r) = \gamma(r) - \beta u_{\text{LR}}(r)$

5.4.15. Bomont-Bretonnet (BB) Approximation.

Bomont and Bretonnet [21, 22] have suggested an extension of the MS approximation with the bridge function

$$B_{\text{BB}}(r) = \sqrt{1 + 2\gamma^*(r) + f\gamma^{*2}(r)} - 1 - \gamma^*(r) \quad (5.80)$$

The adjustable parameter f interpolates between the HNC for $f = 1$ and the VM for $f = 0$.

5.4.16. Charpentier-Jakse' semiempirical extention of the VM Approximation (CJ-VM).

A similar semiempirical approach has been suggested by Charpentier and Jakse [33]

$$B_{\text{CJ-VM}}(r) = \frac{1}{2\alpha} \left[\sqrt{1 + 4\alpha\gamma^*(r)} - 1 - 2\alpha\gamma^*(r) \right] \quad (5.81)$$

Also here α can be chosen so that thermodynamic consistency is obtained.

5.4.17. HNC-SMSA (HMSA) Approximation.

The HNC-SMSA (HMSA) closure from Zerah and Hansen [233] has been proposed for the Lennard-Jones fluid. It interpolates between HNC and SMSA. The HMSA has strong theoretical basis since it can be derived from Percus functional expansion formalism and its bridge function expresses as a functional of the renormalised indirect correlation function $\gamma^*(r) = \gamma(r) - \beta u_R(r)$ so that

$$B_{\text{HMSA}}(r) = \ln \left[1 + \frac{\exp[f(r, \alpha)\gamma^*(r)] - 1}{f(r, \alpha)} \right] - \gamma^*(r) \quad (5.82)$$

where $f(r, \alpha)$ is the switching function defined as

$$f(r, \alpha) = 1 - \exp(-\alpha r) \quad (5.83)$$

The parameter α is varied in a self-consistent manner until virial and compressibility routes for the pressure coincide. The HMSA closure reduces to HNC for $f(r, \alpha) \xrightarrow{\alpha \rightarrow \infty} 1$ and to SMSA when $f(r, \alpha) \xrightarrow{\alpha \rightarrow 0} 0$.

5.4.18. Roger-Young (RY) closure.

Roger and Young have introduced a closure to interpolate continuously between HNC and PY theory [184]

$$B_{\text{RY}}(r) = \ln \left[1 + \frac{\exp[f(r, \alpha)\gamma(r)] - 1}{f(r, \alpha)} \right] - \gamma(r) \quad (5.84)$$

where $f(r, \alpha)$ is a "switching function", which they defined as

$$f(r, \alpha) = 1 - \exp(-\alpha r) \quad (5.85)$$

For $\alpha \rightarrow \infty$ the switching function becomes $f(r, \alpha) = 1$ and the closure becomes the HNC closure. For $\alpha \rightarrow 0$ the switching function becomes $f(r, \alpha) = 0$ and the closure converges to the PY closure. The parameter α is varied until the virial-compressibility consistency condition is satisfied.

5.4.19. Zero separation theorem based closure (ZSEP).

The closure based on the zero separation theorem (ZSEP) has been introduced by Lloyd L. Lee [124, 135]. The closure is basing on the one from Verlet [216] and has the form

$$B_{\text{ZSEP}}(r) = -\frac{\zeta \gamma^*(r)^2}{2} \left[1 - \frac{\alpha \phi \gamma^*(r)}{1 + \alpha \gamma^*(r)} \right] \quad (5.86)$$

5.5. Interaction Potentials

Many of the interaction potentials described below are discussed in [126]. The different closures of the previous section require partly that the interaction potential $u(r, \dots)$ is partitioned in a reference $u_{\text{ref}}(r, \dots)$ and perturbation $u_{\text{pert}}(r, \sigma, \dots)$, or a short range $u_{\text{SR}}(r, \sigma, \dots)$ and long range $u_{\text{LR}}(r, \sigma, \dots)$, or an attractive $u_A(r, \sigma, \dots)$ and repulsive $u_R(r, \sigma, \dots)$ potential part. One famous partition has been introduced by Weeks-Chandler-Andersen (WCA) [225]. They partitioned the potential $u(r) = u_R(r) + u_A(r)$ in a repulsive part $u_R(r)$ with an additional condition that $u_R(r) \xrightarrow{r \rightarrow \infty} 0$, and an attractive part $u_A(r)$. In some other closures the partition is done in a reference and perturbation $u(r) = u_{\text{ref}}(r) + u_{\text{pert}}(r)$ or a short range and long range part $u(r) = u_{\text{SR}}(r) + u_{\text{LR}}(r)$. The definitions together with the different ways of partitioning the potentials are defined below for each potential.

5.5.1. Hard Sphere Potential.

Hard spheres are the simplest model which are defined simply as impenetrable spheres that cannot overlap in space. They mimic the extremely strong repulsion that atoms and spherical molecules experience at very close distances. Hard spheres of diameter σ are particles with the following pairwise interaction potential:

$$u^{\text{HS}}(r, \sigma) = \begin{cases} \infty & \text{if } r < \sigma \\ 0 & \text{if } r \geq \sigma \end{cases} \quad (5.87)$$

A splitting of the simple hard sphere potential is not required, so that

$$u_{\text{ref}}^{\text{HS}}(r, \sigma) = u_{\text{SR}}^{\text{HS}}(r, \sigma) = u_{\text{R}}^{\text{HS}}(r, \sigma) = \begin{cases} \infty & \text{if } r < \sigma \\ 0 & \text{if } r \geq \sigma \end{cases} \quad (5.88)$$

and

$$u_{\text{pert}}^{\text{HS}}(r, \sigma) = u_{\text{LR}}^{\text{HS}}(r, \sigma) = u_{\text{A}}^{\text{HS}}(r, \sigma) \equiv 0 \quad (5.89)$$

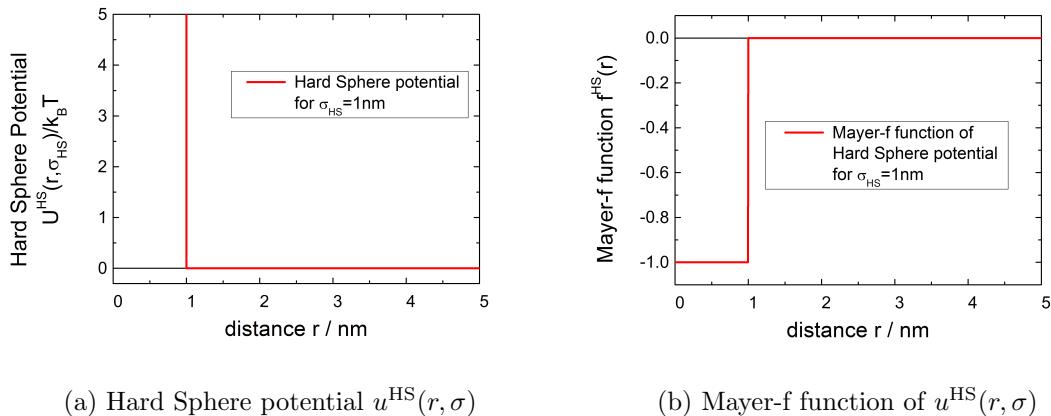


FIGURE 5.1. potential $u^{\text{HS}}(r, \sigma)$ and it's Mayer-f function $\exp(-u^{\text{HS}}(r, \sigma)/k_B T) - 1$

5.5.2. Sticky Hard Sphere Potential (SHS).

The original model of “adhesive” or “sticky” hard spheres potential was proposed by Baxter in 1968 [9]. This potential adds to a hard sphere interparticle repulsion an infinitely strong attraction when molecular surfaces come to contact. The adhesive surface contribution was defined by a particular limiting case of a square well tail, in which the well depth goes to infinity as the width goes to zero, in such a way that the contribution to the second osmotic virial coefficient B_2 remains finite but not zero (Baxter’s “sticky limit”). More explicitly, in the one-component case, the square well potential reads as

$$u^{\text{SHS}}(r) = \begin{cases} \infty, & 0 < r < \sigma \\ \epsilon, & \sigma \leq r \leq \sigma + \Delta \\ 0, & r > \sigma + \Delta \end{cases} \quad (5.90)$$

with

$$\epsilon = k_B T \ln \left(\frac{12\tau\Delta}{\sigma + \Delta} \right) \quad (5.91)$$

The hight of the square well ϵ is positive for $\tau > (\sigma + \Delta)/(12\Delta)$. Therefore the splitting in attractive and repulsive potential is done as

$$u_A^{\text{SHS}}(r) = \begin{cases} \epsilon & \tau < \frac{\sigma+\Delta}{12\Delta} \wedge r \leq \sigma + \Delta \\ 0 & \text{otherwise} \end{cases} \quad (5.92)$$

$$u_R^{\text{SHS}}(r) = \begin{cases} \infty & r < \sigma \\ \epsilon & \tau > \frac{\sigma+\Delta}{12\Delta} \wedge \sigma \leq r \leq \sigma + \Delta \\ 0 & \tau < \frac{\sigma+\Delta}{12\Delta} \wedge \sigma \leq r \leq \sigma + \Delta \\ 0 & \text{otherwise} \end{cases} \quad (5.93)$$

The partitioning in reference and perturbation is done by

$$u_{\text{ref}}^{\text{SHS}}(r) = \begin{cases} \infty & r < \sigma \\ 0 & \text{otherwise} \end{cases} \quad (5.94)$$

$$u_{\text{pert}}^{\text{SHS}}(r) = \begin{cases} \epsilon & r \leq \sigma + \Delta \\ 0 & \text{otherwise} \end{cases} \quad (5.95)$$

A partitioning in long range and short range is not done. The long range potential is set to be always $u_{\text{LR}}^{\text{SHS}}(r) \equiv 0$, so that $u_{\text{SR}}^{\text{SHS}}(r) = u^{\text{SHS}}(r)$.

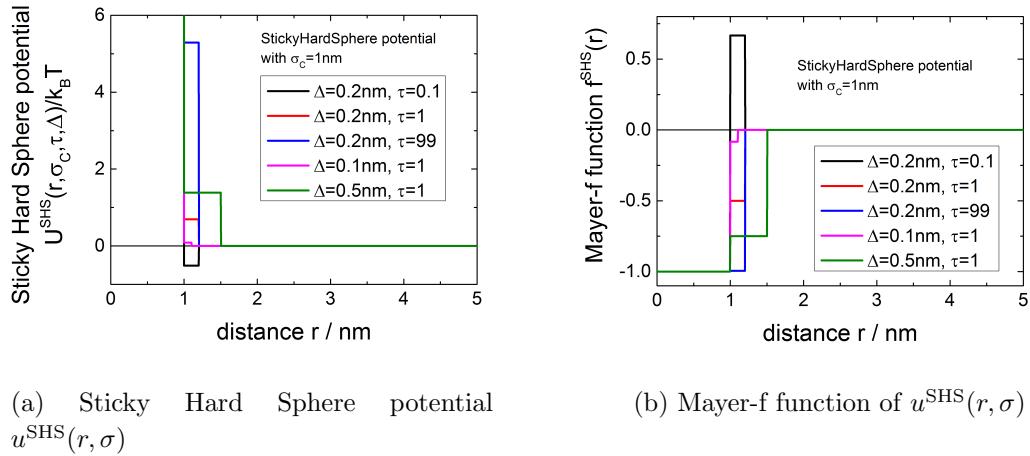


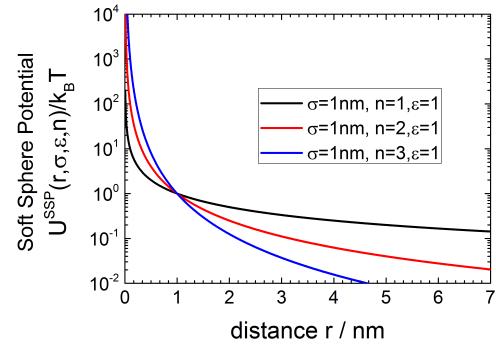
FIGURE 5.2. potential $u^{\text{SHS}}(r, \sigma)$ and it's Mayer-f function $\exp(-u^{\text{SHS}}(r, \sigma)/k_B T) - 1$

5.5.3. Soft Sphere Potential.

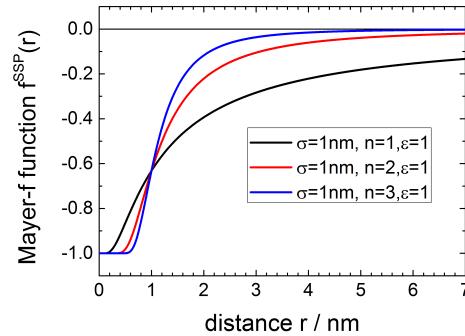
The soft sphere potential is defined as

$$u^{\text{SSP}}(r, \sigma, \epsilon) = k_B T \epsilon \left(\frac{\sigma}{r}\right)^n \quad (5.96)$$

σ is the nominal particle diameter, ϵ establishes the energy scale, and n controls the stiffness of the potential (the softness is $1/n$). Indeed for microgel particles this potential form has been used to interpret experimentally measured physical properties. $n \rightarrow \infty$ is the hard sphere limit, and this and intermediate n values (particularly $n = 12$) have been studied many times in the literature. For $n \leq 3$ this potential does not lead to a thermodynamically stable system as the volume integral of the potential diverges.



(a) Soft Sphere Potential $u^{\text{SSP}}(r, \sigma, \dots)$



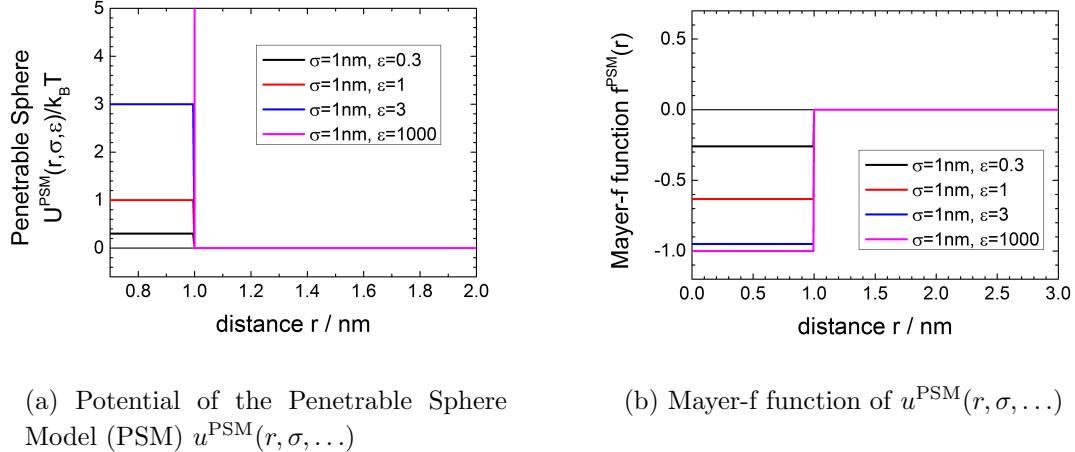
(b) Mayer-f function of $u^{\text{SSP}}(r, \sigma, \dots)$

FIGURE 5.3. potential $u^{\text{SSP}}(r, \sigma, \dots)$ and its Mayer-f function $\exp(-u^{\text{SSP}}(r, \sigma, \dots)/k_B T) - 1$

5.5.4. Penetrable Sphere Model.

The penetrable sphere model was used by Marqusee and Witten [141] who suggested it as a prototype for the interaction between micelles in a solvent. The potential is defined as [141, 218]

$$u^{\text{PSM}}(r, \sigma, \epsilon) = \begin{cases} k_B T \epsilon & \text{for } r \leq \sigma \\ 0 & \text{for } r > \sigma \end{cases} \quad (5.97)$$



(a) Potential of the Penetrable Sphere Model (PSM) $u^{\text{PSM}}(r, \sigma, \dots)$

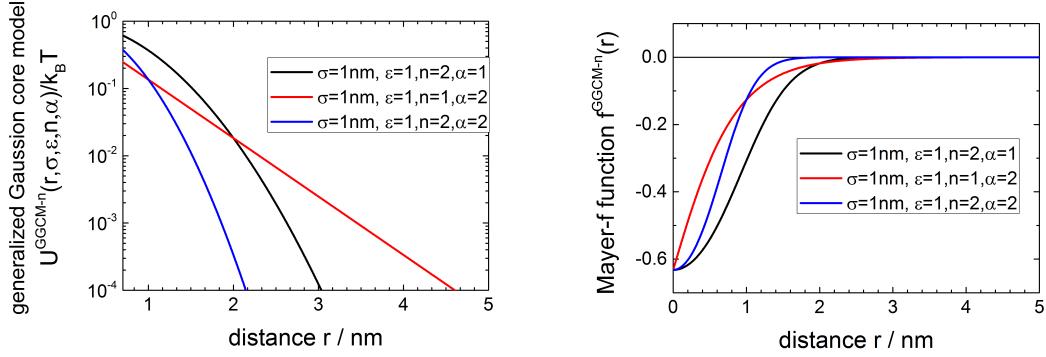
(b) Mayer-f function of $u^{\text{PSM}}(r, \sigma, \dots)$

FIGURE 5.4. potential $u^{\text{PSM}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u^{\text{PSM}}(r, \sigma, \dots)/k_B T) - 1$

5.5.5. Generalized Gaussian Core Model Potential - GGCM-n Potential.

The effective interaction between the centers of mass of two polymer chains in an athermal solvent can be described in very good approximation by an interaction of a Gaussian form $u^{\text{GCM}}(r) = k_B T \epsilon \exp(-(r/\sigma)^2)$. A generalisation of this form has been discussed in [149, 136] and has the form

$$u^{\text{GGCM-n}}(r) = k_B T \epsilon \exp\left(-\alpha \left(\frac{r}{\sigma}\right)^n\right) \quad (5.98)$$



(a) Potential of the Generalised Gaussian Code Model (GGCM-n)
 $u^{\text{GGCM-n}}(r, \sigma, \dots)$

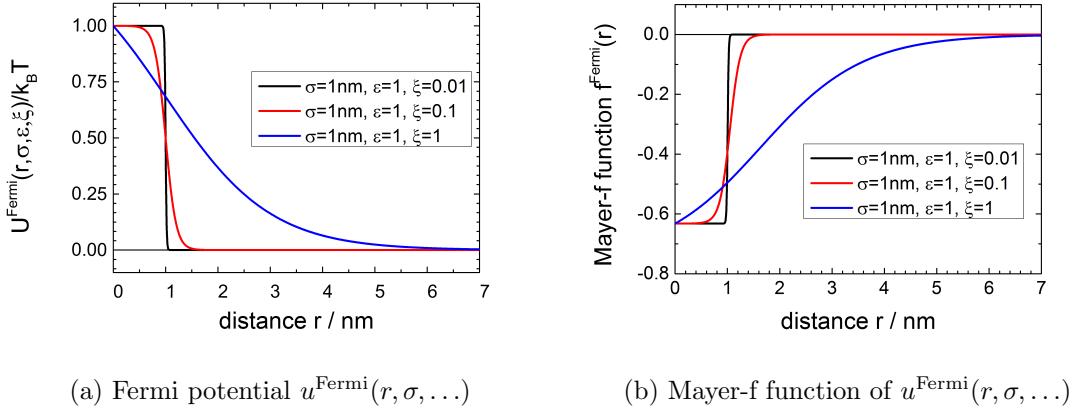
(b) Mayer-f function of $u^{\text{GGCM-n}}(r, \sigma, \dots)$

FIGURE 5.5. potential $u^{\text{GGCM-n}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u^{\text{GGCM-n}}(r, \sigma, \dots)/k_B T) - 1$

5.5.6. Fermi Distribution Model.

$$u^{\text{FDM}}(r, \sigma, \epsilon, \xi) = k_B T \epsilon \frac{1 + \exp(-\sigma/\xi)}{1 + \exp((r - \sigma)/\xi)} \quad (5.99)$$

For $\xi = 0$ the Fermi Potential goes over into the Penetrable Sphere Model.



(a) Fermi potential $u^{\text{Fermi}}(r, \sigma, \dots)$ (b) Mayer-f function of $u^{\text{Fermi}}(r, \sigma, \dots)$

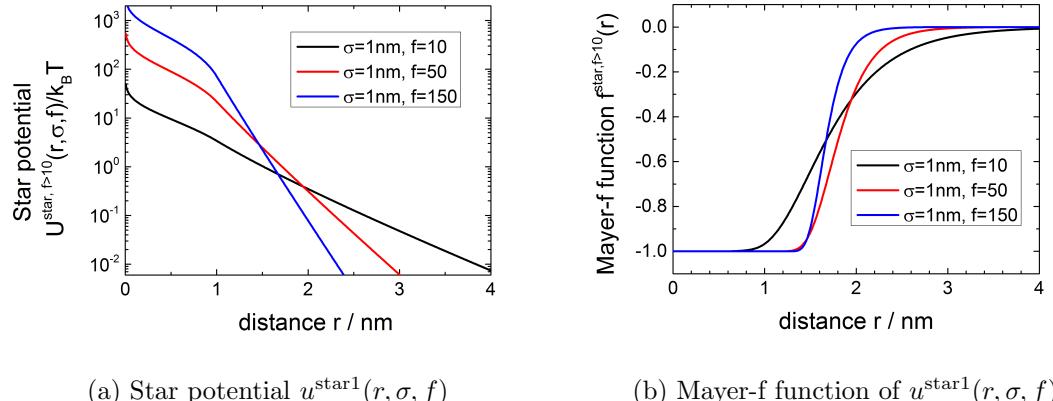
FIGURE 5.6. potential $u^{\text{Fermi}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u^{\text{Fermi}}(r, \sigma, \dots)/k_B T) - 1$

5.5.7. Star Polymer Potential.

Star polymers with tunable number f and size of arms, and thus interactions, represent ideal model systems for exploring the regime of soft material behaviour that interpolates between hard spheres and polymeric coils. Likos et al. [129, 126] have proposed an effective potential that describes the interaction between two stars. The potential is given by

$$u^{\text{star}1}(r, \sigma, f) = k_B T \frac{5}{8} f^{3/2} \begin{cases} \left(\frac{1}{1+\sqrt{f}} - \ln \left(\frac{r}{\sigma} \right) \right) & \text{for } r \leq \sigma \\ \frac{1}{1+\sqrt{f}} \frac{\sigma}{r} \exp \left(-\frac{\sqrt{f}(r-\sigma)}{2\sigma} \right) & \text{for } r > \sigma \end{cases} \quad (5.100)$$

where σ is the effective corona diameter (e.g., $\sigma \propto f^{1/5} N_a^{3/5}$ for good solvents, N_a being the arm degree of polymerization and f the functionality of the star). In equation 5.100 above, $\sigma/2$ is a length scale that extends from the star center to the middle of the outermost DaoudCotton blob [127]. Extensive comparisons with simulations have set this scale to $\sigma \cong 1.32R_g$, where R_g is the star radius of gyration. The effective potential shows a soft, logarithmic divergence at close approaches, followed by a crossover to a Yukawa form as the center-to-center separation r grows. The potential becomes harder with increasing f , tending eventually to a hard-sphere interaction formally obtained in the limit $f \rightarrow \infty$.



(a) Star potential $u^{\text{star}1}(r, \sigma, f)$

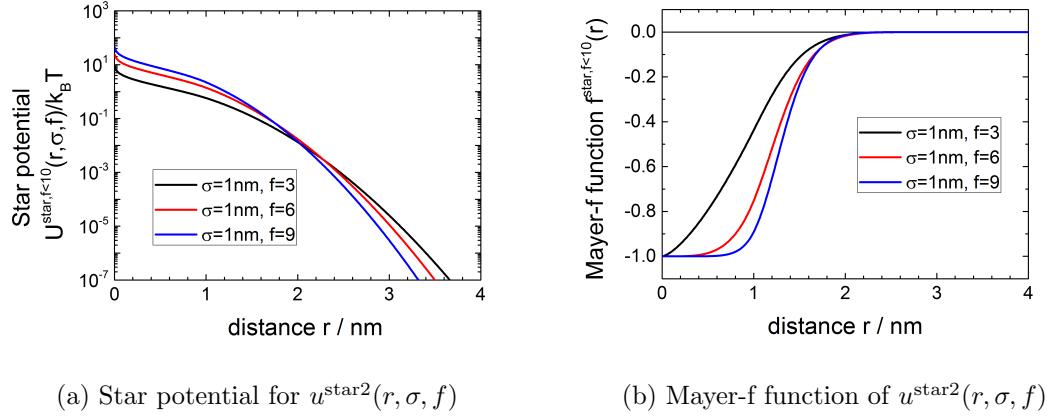
(b) Mayer-f function of $u^{\text{star}1}(r, \sigma, f)$

FIGURE 5.7. potential $u^{\text{star}1}(r, \sigma)$ and it's Mayer-f function $\exp(-u^{\text{star}1}(r, \sigma, f)/k_B T) - 1$

The effective potential valid for $f \leq 10$ has the form [104, 49, 126]:

$$u^{\text{star}2}(r, \sigma, f) = k_B T \frac{5}{8} f^{3/2} \begin{cases} \left(\frac{1}{2\tau^2 \sigma^2} - \ln \left(\frac{r}{\sigma} \right) \right) & \text{for } r \leq \sigma \\ \frac{1}{2\tau^2 \sigma^2} \exp \left(-\tau^2 (r^2 - \sigma^2) \right) & \text{for } r > \sigma \end{cases} \quad (5.101)$$

$$\text{with } \tau = \left(\frac{1.12}{3\sigma} - \frac{1.03}{3\sigma} \right) f + \frac{1.03}{\sigma} \quad (5.102)$$



(a) Star potential for $u^{\text{star}2}(r, \sigma, f)$ (b) Mayer-f function of $u^{\text{star}2}(r, \sigma, f)$

FIGURE 5.8. potential $u^{\text{star}2}(r, \sigma)$ and it's Mayer-f function $\exp(-u^{\text{star}2}(r, \sigma, f)/k_B T) - 1$

τ is a free parameter of the order and is obtained by fitting to computer simulation results [49].

5.5.8. Lennard-Jones Potential.

The Lennard-Jones potential is a mathematically simple model that approximates the interaction between a pair of neutral atoms or molecules. A form of the potential was first proposed in 1924 by John Lennard-Jones [103].

$$u^{\text{LJ}}(r, \sigma, \dots) = 4k_B T \epsilon \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^6 \right] \quad (5.103)$$

The Lennard-Jones (LJ) potential has a minimum at $r_m = 2^{1/6}\sigma$, so that the potential can be written in parts of

$$u_{\text{ref}}^{\text{LJ}}(r) \equiv 0 \quad (5.104)$$

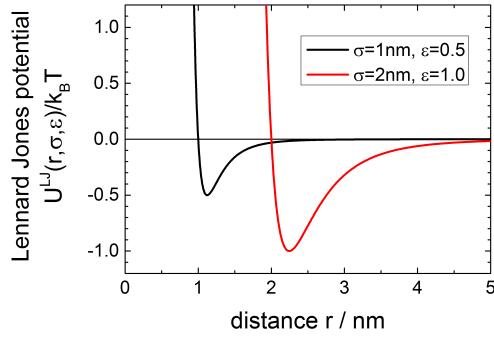
$$u_{\text{SR}}^{\text{LJ}}(r) = u_{\text{R}}^{\text{LJ}}(r) = \begin{cases} u^{\text{LJ}}(r) - u^{\text{LJ}}(r_m) & \text{if } r < r_m \\ 0 & \text{if } r \geq r_m \end{cases} \quad (5.105)$$

and

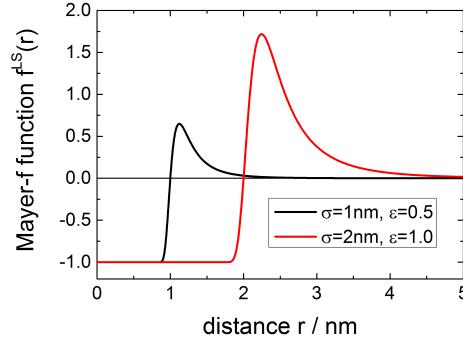
$$u_{\text{pert}}^{\text{LJ}}(r) = u^{\text{LJ}}(r) \quad (5.106)$$

$$u_{\text{LR}}^{\text{LJ}}(r) = u_{\text{A}}^{\text{LJ}}(r) = \begin{cases} u^{\text{LJ}}(r_m) & \text{if } r < r_m \\ u_{\text{LR}}^{\text{LJ}}(r) & \text{if } r \geq r_m \end{cases} \quad (5.107)$$

The LJ potential is not splitted in a reference and perturbation part, i.e. the reference part is set to 0.



(a) Lennard-Jones potential $u^{\text{LJ}}(r, \sigma, \dots)$



(b) Mayer-f function of $u^{\text{LJ}}(r, \sigma, \dots)$

FIGURE 5.9. potential $u^{\text{LJ}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u^{\text{LJ}}(r, \sigma, \dots)/k_B T) - 1$

5.5.9. Depletion Potential.

The pair interaction potential of spheres by a depleting agent is implemented depending on the shape of the depleting agent. Analytical expressions have been developed for spherical, disc-like or rod-like shapes [160, 125].

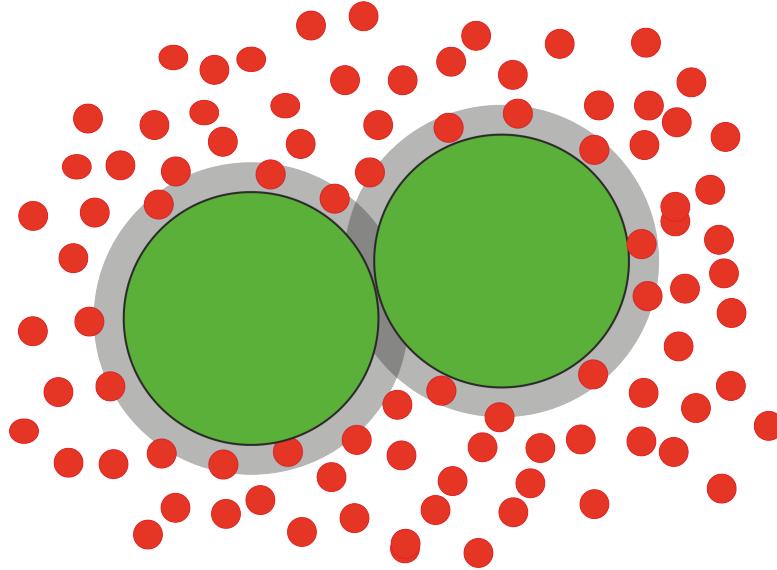


FIGURE 5.10. Attractive depletion forces between large spheres due to small spheres as depleting agent.

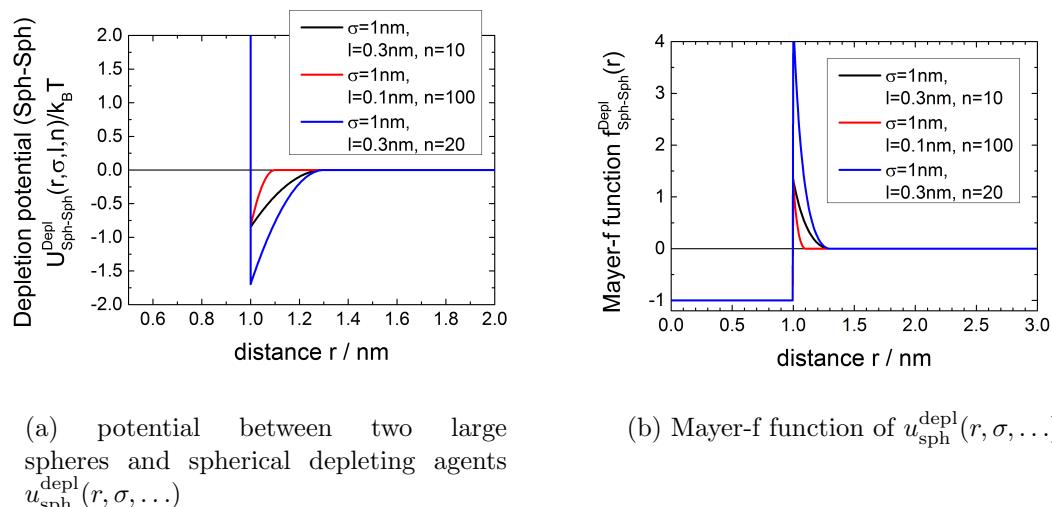


FIGURE 5.11. potential $u_{\text{sph}}^{\text{depl}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u_{\text{sph}}^{\text{depl}}(r, \sigma, \dots)/k_B T) - 1$

The potential between two large spheres and spherical depleting agents of diameter l is given by

$$u_{\text{sph}}^{\text{depl}}(r) = \begin{cases} \infty & \text{for } r < \sigma \\ -k_B T(N_I + N_{II} - N_{III}) & \text{for } r \geq \sigma \end{cases} \quad (5.108)$$

$$N_I = n_d \frac{2}{3} \left(\frac{l}{2} - \frac{r - \sigma}{2} \right)^2 \left(\frac{3\sigma}{2} + l + \frac{r - \sigma}{2} \right) \quad (5.109)$$

$$N_{II} = N_{III} = 0 \quad (5.110)$$

Also a splitting of the potential in repulsive-attractive, reference-perturbation, shortrange-longrange has been implemented as

$$u_{\text{sph,R}}^{\text{depl}}(r) = \begin{cases} u_{\text{sph}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.111a)$$

$$u_{\text{sph,A}}^{\text{depl}}(r) = \begin{cases} u_{\text{sph}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{sph}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.111b)$$

$$u_{\text{sph,ref}}^{\text{depl}}(r) = \begin{cases} u_{\text{sph}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.111c)$$

$$u_{\text{sph,pert}}^{\text{depl}}(r) = \begin{cases} u_{\text{sph}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{sph}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.111d)$$

$$u_{\text{sph,SR}}^{\text{depl}}(r) = \begin{cases} u_{\text{sph}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.111e)$$

$$u_{\text{sph,LR}}^{\text{depl}}(r) = \begin{cases} u_{\text{sph}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{sph}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.111f)$$

The potential between two large spheres and infinitesimal thin discs of diameter D as a depleting agents is given by

$$u_{\text{discs}}^{\text{depl}}(r) = -k_B T(N_I + N_{II} - N_{III}) \quad (5.112)$$

$$N_I = n_d \frac{\pi}{6} \sigma D^2 \left(1 - \frac{r - \sigma}{D} \right)^2 \left(\frac{3}{2} + \frac{D}{\sigma} + \frac{1}{2} \frac{r - \sigma}{\sigma} \right) \quad (5.113)$$

$$\begin{aligned} N_{II} - N_{III} = \frac{\pi}{6} n_d D^2 \sigma & \left\{ -\frac{3}{2} \left(1 - \frac{r - \sigma}{D} \right)^2 - \frac{3}{4} \pi \frac{r - \sigma}{D} \right. \\ & + \frac{3}{2} \frac{r - \sigma}{D} \arcsin \frac{r - \sigma}{D} \\ & \left. + \left(1 + \frac{1}{2} \left(\frac{r - \sigma}{D} \right)^2 \right) \sqrt{1 - \left(\frac{r - \sigma}{D} \right)^2} \right\} \end{aligned} \quad (5.114)$$

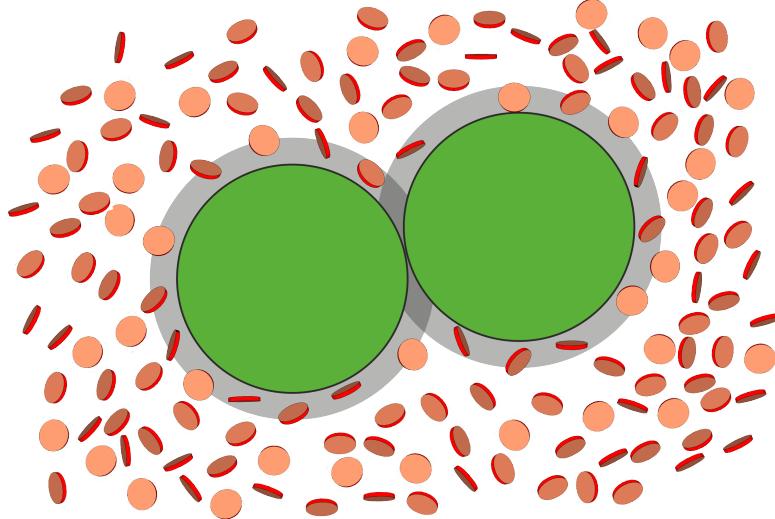


FIGURE 5.12. Attractive depletion forces between large spheres due to depletion forces of infinitesimal thin discs of diameter D as a depleting agents

Also a splitting of the potential in repulsive-attractive, reference-perturbation, shortrange-longrange has been implemented as

$$u_{\text{discs,R}}^{\text{depl}}(r) = \begin{cases} u_{\text{discs}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.115\text{a})$$

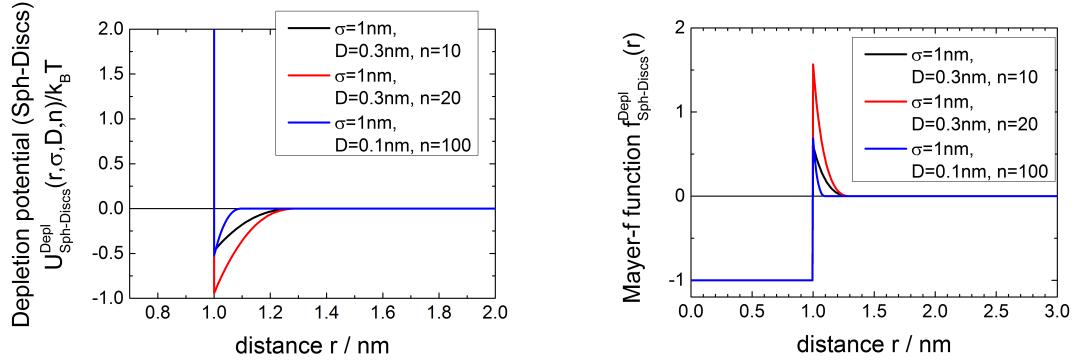
$$u_{\text{discs,A}}^{\text{depl}}(r) = \begin{cases} u_{\text{discs}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{discs}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.115\text{b})$$

$$u_{\text{discs,ref}}^{\text{depl}}(r) = \begin{cases} u_{\text{discs}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.115\text{c})$$

$$u_{\text{discs,pert}}^{\text{depl}}(r) = \begin{cases} u_{\text{discs}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{discs}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.115\text{d})$$

$$u_{\text{discs,SR}}^{\text{depl}}(r) = \begin{cases} u_{\text{discs}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.115\text{e})$$

$$u_{\text{discs,LR}}^{\text{depl}}(r) = \begin{cases} u_{\text{discs}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{discs}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.115\text{f})$$



(a) potential between two large spheres and dislike depleting agents $u_{\text{discs}}^{\text{depl}}(r, \sigma, \dots)$

(b) Mayer-f function of $u_{\text{discs}}^{\text{depl}}(r, \sigma, \dots)$

FIGURE 5.13. potential $u_{\text{discs}}^{\text{depl}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u_{\text{discs}}^{\text{depl}}(r, \sigma, \dots)/k_B T) - 1$

Another depletion potential of two hard spheres of diameter σ surrounded by infinitely thin hard rods of length L is given by

$$u_{\text{rods}}^{\text{depl}}(r) = -k_B T(N_{\text{I}} + N_{\text{II}} - N_{\text{III}}) \quad (5.116)$$

$$N_{\text{I}} = n_d \frac{\pi}{12} \sigma L^2 \left(1 - \frac{r - \sigma}{L}\right)^2 \left(3 + 2\frac{L}{\sigma} + \frac{r - \sigma}{\sigma}\right) \quad (5.117)$$

$$N_{\text{II}} - N_{\text{III}} = \frac{\pi}{12} n_d L^2 \sigma \left\{ -2 - \left(\frac{r - \sigma}{D}\right)^3 + 3\frac{r - \sigma}{D} \right\} \quad (5.118)$$

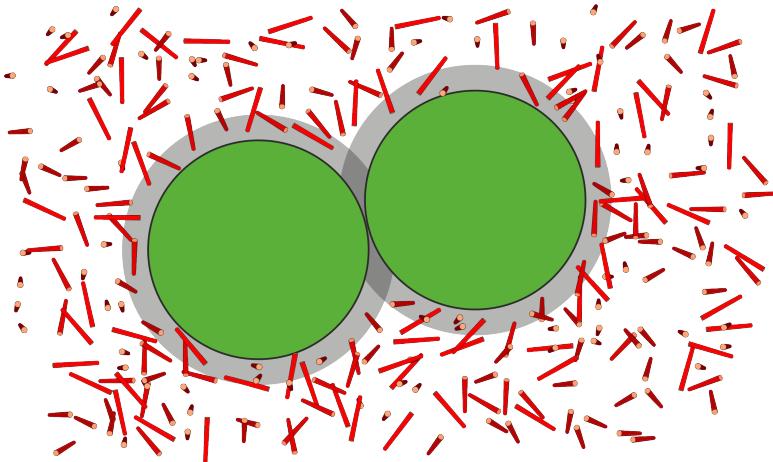


FIGURE 5.14. Attractive depletion forces between large spheres due to depletion forces of infinitely thin hard rods of length L as depleting agent.

Also a splitting of the potential in repulsive-attractive, reference-perturbation, shortrange-longrange has been implemented as

$$u_{\text{rods,R}}^{\text{depl}}(r) = \begin{cases} u_{\text{rods}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.119\text{a})$$

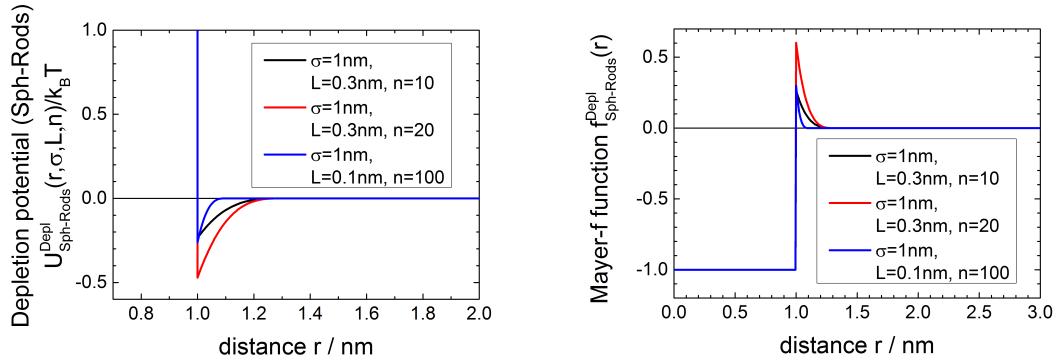
$$u_{\text{rods,A}}^{\text{depl}}(r) = \begin{cases} u_{\text{rods}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{rods}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.119\text{b})$$

$$u_{\text{rods,ref}}^{\text{depl}}(r) = \begin{cases} u_{\text{rods}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.119\text{c})$$

$$u_{\text{rods,pert}}^{\text{depl}}(r) = \begin{cases} u_{\text{rods}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{rods}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.119\text{d})$$

$$u_{\text{rods,SR}}^{\text{depl}}(r) = \begin{cases} u_{\text{rods}}^{\text{depl}}(r) & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.119\text{e})$$

$$u_{\text{rods,LR}}^{\text{depl}}(r) = \begin{cases} u_{\text{rods}}^{\text{depl}}(\sigma) & \text{for } r < \sigma \\ u_{\text{rods}}^{\text{depl}}(r) & \text{for } r \geq \sigma \end{cases} \quad (5.119\text{f})$$



(a) potential between two large spheres and rodlike depleting agents $u_{\text{rods}}^{\text{depl}}(r, \sigma, \dots)$

(b) Mayer-f function of $u_{\text{rods}}^{\text{depl}}(r, \sigma, \dots)$

FIGURE 5.15. potential $u_{\text{rods}}^{\text{depl}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u_{\text{rods}}^{\text{depl}}(r, \sigma, \dots)/k_B T) - 1$

5.5.10. Ionic Microgel Potential.

According to [128] the effective potential in ionic microgels can be partitioned in a contribution $\Phi_{r \leq \sigma}(r)$ for the overlap region $r \leq \sigma$ and one $\Phi_{r > \sigma}(r)$ for $r > \sigma$.

$$u^{\text{im}}(r) = k_B T \begin{cases} \Phi_{r \leq \sigma}(r) & \text{for } r \leq \sigma \\ \Phi_{r > \sigma}(r) & \text{for } r > \sigma \end{cases} \quad (5.120)$$

For the overlapping particle it has the form

$$\Phi_{r \leq \sigma}(r) = \frac{2Z^2 e^2}{\epsilon \sigma} \left[\frac{6}{5} - 2 \left(\frac{r}{\sigma} \right)^2 + \frac{3}{2} \left(\frac{r}{\sigma} \right)^3 - \frac{1}{5} \left(\frac{r}{\sigma} \right)^5 \right] - \frac{72 Z^2 e^2}{\epsilon \kappa^4 \sigma^4 r} \Phi_{\text{ind}}(r) \quad (5.121)$$

where $\Phi_{\text{ind}}(r)$ is given by the expression

$$\begin{aligned} \Phi_{\text{ind}}(r) = & \left(1 - e^{-\kappa r} + \frac{1}{2} \kappa^2 r^2 + \frac{1}{24} \kappa^4 r^4 \right) \left(1 - \frac{4}{\kappa^2 \sigma^2} \right) \\ & + \frac{4}{\kappa \sigma} e^{-\kappa \sigma} \sinh(\kappa r) \\ & + \left[e^{-\kappa \sigma} \sinh(\kappa \sigma) + \kappa^2 \sigma r + \frac{1}{6} \kappa^4 (\sigma^3 r + r^3 \sigma) \right] \left(1 + \frac{4}{\kappa^2 \sigma^2} \right) \\ & - \frac{4r}{\sigma} \left(1 + \frac{1}{2} \kappa^2 \sigma^2 + \frac{1}{30} \kappa^4 \sigma^4 \right) \\ & - \frac{8r^3}{3\sigma^3} \left(\frac{\kappa^2 \sigma^2}{4} + \frac{\kappa^4 \sigma^4}{12} \right) - \frac{1}{180} \frac{\kappa^4}{\sigma^2} r^6 \end{aligned} \quad (5.122)$$

For the nonoverlapping distances the effective potential becomes

$$\Phi_{r > \sigma}(r) = \frac{144 Z^2 e^2}{\epsilon \kappa^4 \sigma^4} \left[\cosh(\kappa \sigma / 2) - \frac{2 \sinh(\kappa \sigma / 2)}{\kappa \sigma} \right]^2 \frac{e^{-\kappa r}}{r} \quad (5.123)$$

5.5.11. DLVO Potential.

For the past half century or so, the stability of charged colloidal systems has generally been described in terms of the DLVO (Derjaguin-Landau-Verwey-Overbeek) potential between two colloidal particles. The DLVO potential is composed of a long-range screened Coulomb repulsion part $u^{\text{el}}(r)$ and a short-range van der Waals attraction part $u^{\text{vdW}}(r)$ [153].

$$u^{\text{DLVO}}(r) = \begin{cases} \infty & \text{if } r \leq \sigma \\ u^{\text{vdW}}(r) + u^{\text{el}}(r) & \text{if } r > \sigma \end{cases} \quad (5.124)$$

with

$$u^{\text{vdW}}(r) = -k_B T \frac{A_H}{12} \left(\frac{\sigma^2}{(r^2 - \sigma^2)} + \left(\frac{\sigma}{r} \right)^2 + 2 \ln \left(1 - \left(\frac{\sigma}{r} \right)^2 \right) \right) \quad (5.125)$$

$$u^{\text{el}}(r) = k_B T L_B Z_{\text{eff}}^2 \frac{1}{(1 + \kappa\sigma/2)^2} \frac{\exp(-\kappa(r - \sigma))}{r} \quad (5.126)$$

A_H is the Hamaker constant. $L_B = \frac{e^2}{4\pi\epsilon_r\epsilon_0 k_B T}$ is the so-called Bjerrum length¹. The vacuum permittivity is $\epsilon_0 = 8.854187817 \times 10^{-12} \frac{\text{F}}{\text{m}}$ and the elementary electric charge $e = 1.602176565(35) \times 10^{-19}$ coulombs². For water at 278 K with a relative permittivity of $\epsilon_r = 78.7$ the Bjerrum length is $L_B = 0.713\text{nm}$. The effective macroion valency is denoted Z_{eff} , and κ^{-1} is the Debye-Hückel screening length due to all macroions. This length characterize the typical distance at which the charge at the origin is screened by the other charges. It is given by $\kappa^2 = 4\pi L_B \sum_{i=1}^N n_i Z_i$, where n_i is the number density of ion i in the bulk solution, Z_i is the charge number or valency of that ion, and the sum is taken over all ions in the solution. A discussion about effective parameters can be found in [2, 212].

¹In Gaussian units, $4\pi\epsilon_0 = 1$ and the Bjerrum length has the simpler form $L_B = \frac{e^2}{\epsilon_r k_B T}$.

²The Gaussian units use the statcoulomb (statC) instead. The conversion between C and statC is different in different contexts. In case of electric charges $1\text{C} = 2997924580 \text{ statC} \approx 3.00 \times 10^9 \text{ statC}$.

Also a splitting of the potential in repulsive-attractive, reference-perturbation, shortrange-longrange has been implemented as

$$u_{\text{ref}}^{\text{DLVO}}(r, \dots) = \begin{cases} \infty & \text{for } r \leq \sigma \\ 0 & \text{for } r > \sigma \end{cases} \quad (5.127\text{a})$$

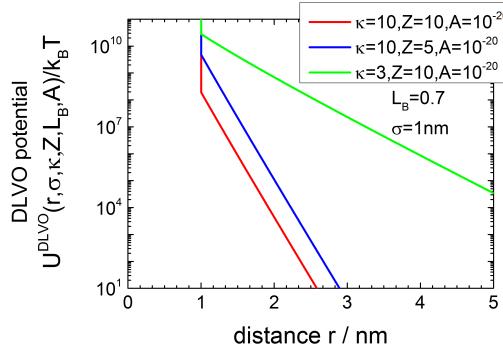
$$u_{\text{pert}}^{\text{DLVO}}(r, \dots) = \begin{cases} 0 & \text{for } r \leq \sigma \\ u^{\text{vdW}}(r) + u^{\text{el}}(r) & \text{for } r > \sigma \end{cases} \quad (5.127\text{b})$$

$$u_{\text{SR}}^{\text{DLVO}}(r, \dots) = \begin{cases} \infty & \text{for } r \leq \sigma \\ u^{\text{vdW}}(r) & \text{for } r > \sigma \end{cases} \quad (5.127\text{c})$$

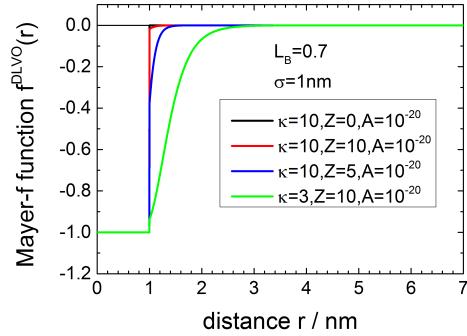
$$u_{\text{LR}}^{\text{DLVO}}(r, \dots) = \begin{cases} u^{\text{el}}(\sigma) & \text{for } r \leq \sigma \\ u^{\text{el}}(r) & \text{for } r > \sigma \end{cases} \quad (5.127\text{d})$$

$$u_{\text{R}}^{\text{DLVO}}(r, \dots) = \begin{cases} \infty & \text{for } r \leq \sigma \\ u^{\text{el}}(r) & \text{for } r > \sigma \end{cases} \quad (5.127\text{e})$$

$$u_{\text{A}}^{\text{DLVO}}(r, \dots) = \begin{cases} 0 & \text{for } r \leq \sigma \\ u^{\text{vdW}}(r) & \text{for } r > \sigma \end{cases} \quad (5.127\text{f})$$



(a) DLVO potential (Derjaguin and Landau, Verwey and Overbeek)
 $u^{\text{DLVO}}(r, \sigma, \dots)$



(b) Mayer-f function of $u^{\text{DLVO}}(r, \sigma, \dots)$

FIGURE 5.16. potential $u^{\text{DLVO}}(r, \sigma, \dots)$ and it's Mayer-f function
 $\exp(-u^{\text{DLVO}}(r, \sigma, \dots)/k_B T) - 1$

5.5.12. Hard Core 3Yukawa Potential.

This potential is useful for systems with complicated potentials. Next to the hard core repulsion three Yukawa contributions are allowed. The potential reads

$$u^{\text{HC3Y}}(r, \dots) = \begin{cases} \infty & \text{for } r < \sigma \\ k_B T \sum_{i=1}^{N=3} K_i \exp(-\lambda_i(r - \sigma)/\sigma) & \text{for } r \geq \sigma \end{cases} \quad (5.128)$$

The sign of K_i defines, if the potential is repulsive ($K_i > 0$) or attractive ($K_i < 0$) and the modulus of $|K_i|$ defines the strength of the interaction. λ_i defines the range of the interaction and should be always a positive number.

Also a splitting of the potential in repulsive-attractive, reference-perturbation, shortrange-longrange has been implemented as

$$u_{\text{ref}}^{\text{HC3Y}}(r, \dots) = \begin{cases} \infty & \text{for } r < \sigma \\ 0 & \text{for } r \geq \sigma \end{cases} \quad (5.129\text{a})$$

$$u_{\text{pert}}^{\text{HC3Y}}(r, \dots) = \begin{cases} u_{\text{ref}}^{\text{HC3Y}}(\sigma, \dots) & \text{for } r < \sigma \\ u_{\text{ref}}^{\text{HC3Y}}(r, \dots) & \text{for } r \geq \sigma \end{cases} \quad (5.129\text{b})$$

$$u_{\text{SR}}^{\text{HC3Y}}(r, \dots) = u_{\text{ref}}^{\text{HC3Y}}(r, \dots) \quad (5.129\text{c})$$

$$u_{\text{LR}}^{\text{HC3Y}}(r, \dots) = u_{\text{pert}}^{\text{HC3Y}}(r, \dots) \quad (5.129\text{d})$$

$$u_{\text{R}}^{\text{HC3Y}}(r, \dots) = \begin{cases} \infty & \text{for } r < \sigma \\ k_B T \sum_{\forall K_i > 0} K_i \exp(-\lambda_i(r - \sigma)/\sigma) & \text{for } r \geq \sigma \end{cases} \quad (5.129\text{e})$$

$$u_{\text{A}}^{\text{HC3Y}}(r, \dots) = \begin{cases} k_B T \sum_{\forall K_i < 0} K_i & \text{for } r < \sigma \\ k_B T \sum_{\forall K_i < 0} K_i \exp(-\lambda_i(r - \sigma)/\sigma) & \text{for } r \geq \sigma \end{cases} \quad (5.129\text{f})$$

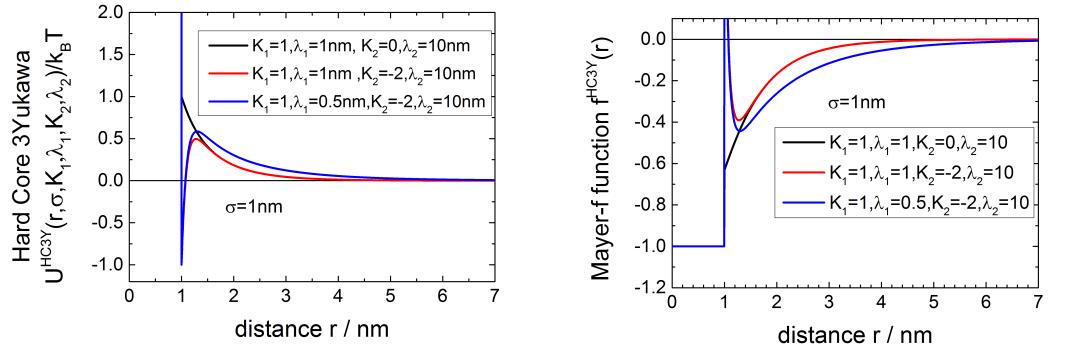


FIGURE 5.17. potential $u^{\text{HC3Y}}(r, \sigma, \dots)$ and it's Mayer-f function $\exp(-u^{\text{HC3Y}}(r, \sigma, \dots)/k_B T) - 1$

5.6. GUI interface for the Ornstein Zernike solver

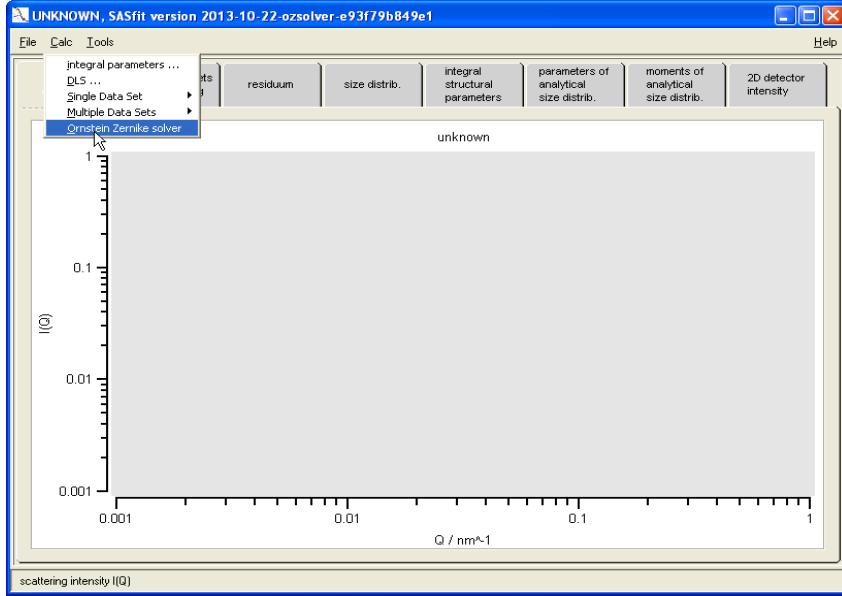


FIGURE 5.18. Starting the menu for OZ-solver.

The OZ-solver in **SASfit** does not allow at the moment an optimization of the parameters of the pair interaction potential in the fitting routine to fit the corresponding structure factor for single or multiple data sets. The main reason for that is, that in the implemented Levenberg-Marquardt-algorithmus the partial derivatives have to be calculated for every q-value. To do this with the implemented routine for the OZ-solver

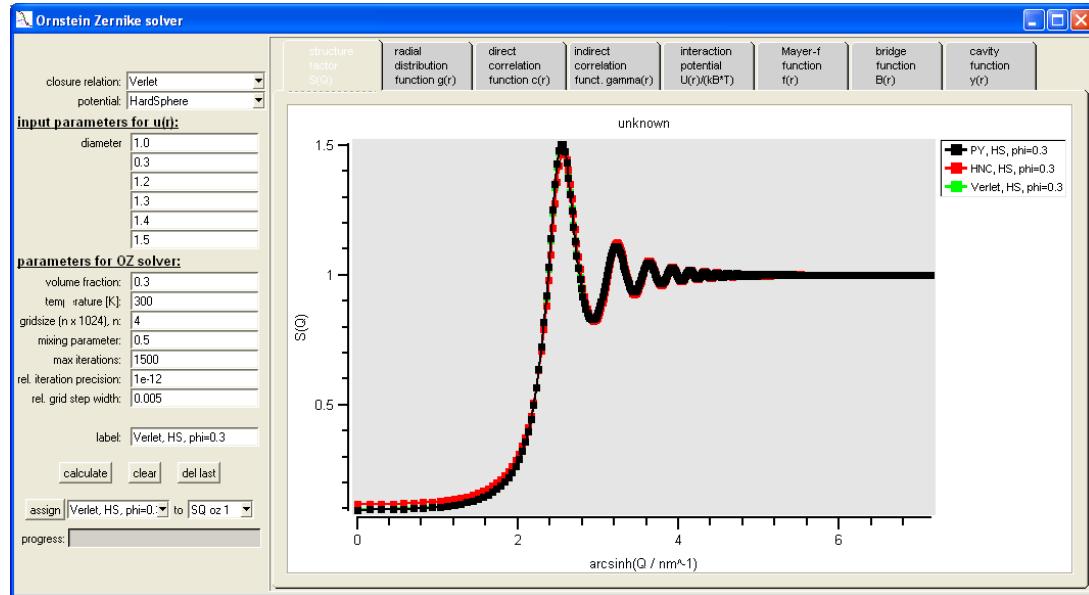


FIGURE 5.19. Main GUI for solving the Ornstein Zernike equation.

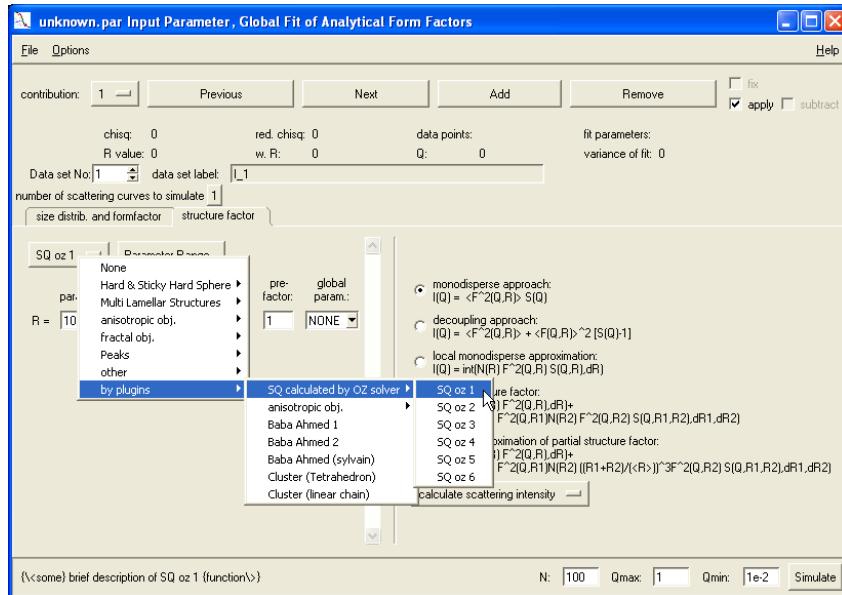


FIGURE 5.20. Access to the solutions of the OZ-solver.

would be too time consuming at the moment. Nevertheless, **SASfit** allows to have access up to 6 pre-calculated solutions of the OZ equations. These solutions can be stored as splines under the names "SQ oz 1", "SQ oz 2" to "SQ oz 6" in the menu interface of the OZ-solver. These splines functions are made available in the menu for calculating structure factors under the names [by plugins]SQ calculated by OZ solver[SQ oz 1] ... [by plugins]SQ calculated by OZ solver[SQ oz 6]. The extrapolation to $Q = 0$ is done by the OZ-solver by a quadratic extrapolation to zero of the first three

$S(Q)$. For $Q > Q_{\max}$ it is assumed that $S(Q > Q_{\max}) \equiv 1$. As at the moment all implemented potentials scale with a size parameter (diameter D), i.e. that the structure factor for different diameters $D = 2R$ scales as $S(QD) = S(Q2R)$ for all D . Therefore only the radius is an input values for the structure factors "SQ oz 1"- "SQ oz 6" and can be fitted.

CHAPTER 6

Size Distributions

6.1. Delta

Choosing **Delta** as a size distribution simply scales the form factor with a constant value N

Input Parameters for size distribution **Delta**:

N: particle number density N

6.2. Uniform distribution

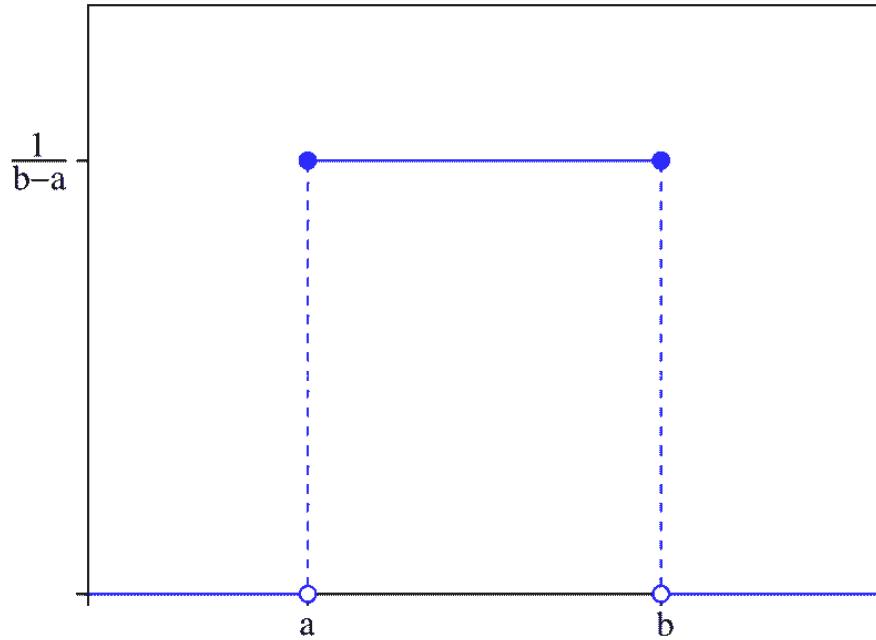


FIGURE 6.1. Uniform distribution function. $x_{\min}, x_{\max} \in (-\infty, \infty)$,
 $x_{\max} > x_{\min}$, $x_{\min} \leq x \leq x_{\max}$

The uniform distribution defines equal probability over a given range for a continuous distribution. The support is defined by the two parameters, x_{\min} and x_{\max} , which are its minimum and maximum values.

$$\text{Uniform}(x|N, x_{\min}, x_{\max}) = \begin{cases} \frac{N}{x_{\max} - x_{\min}} & \text{for } x_{\min} \leq x \leq x_{\max}, \\ 0 & \text{for } x < x_{\min} \text{ or } x > x_{\max} \end{cases} \quad (6.1a)$$

Input Parameters for size distribution Uniform:

N: particle number density N

Xmin: minimum value of the distribution (x_{\min})

Xmax: maximum value of the distribution (x_{\max})

6.3. Triangular distribution

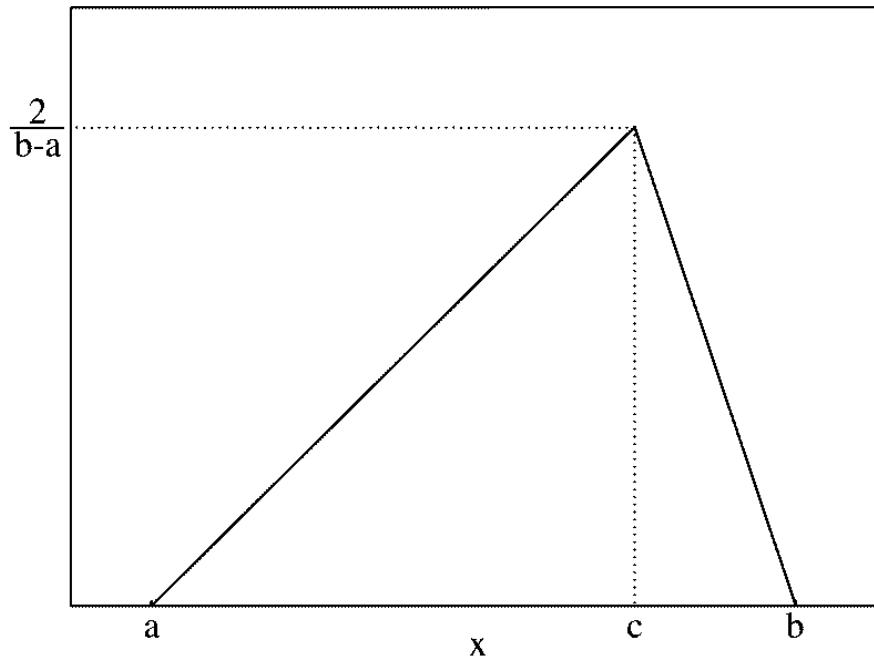


FIGURE 6.2. Triangular distribution function. $x_{\min}, x_{\text{mode}}, x_{\max} \in (-\infty, \infty)$, $x_{\max} > x_{\min}$, $x_{\min} \leq x_{\text{mode}} \leq x_{\max}$, $x_{\min} \leq x \leq x_{\max}$

$$\text{Triangular}(x|x_{\min}, x_{\max}, x_{\text{mode}}) = \begin{cases} \frac{2(x - x_{\min})}{(x_{\max} - x_{\min})(x_{\text{mode}} - x_{\min})} & \text{for } x_{\min} < x \leq x_{\text{mode}} \\ \frac{2(x_{\max} - x)}{(x_{\max} - x_{\min})(x_{\max} - x_{\text{mode}})} & \text{for } x_{\text{mode}} < x \leq x_{\max} \end{cases} \quad (6.2a)$$

6.4. Log-Normal distribution

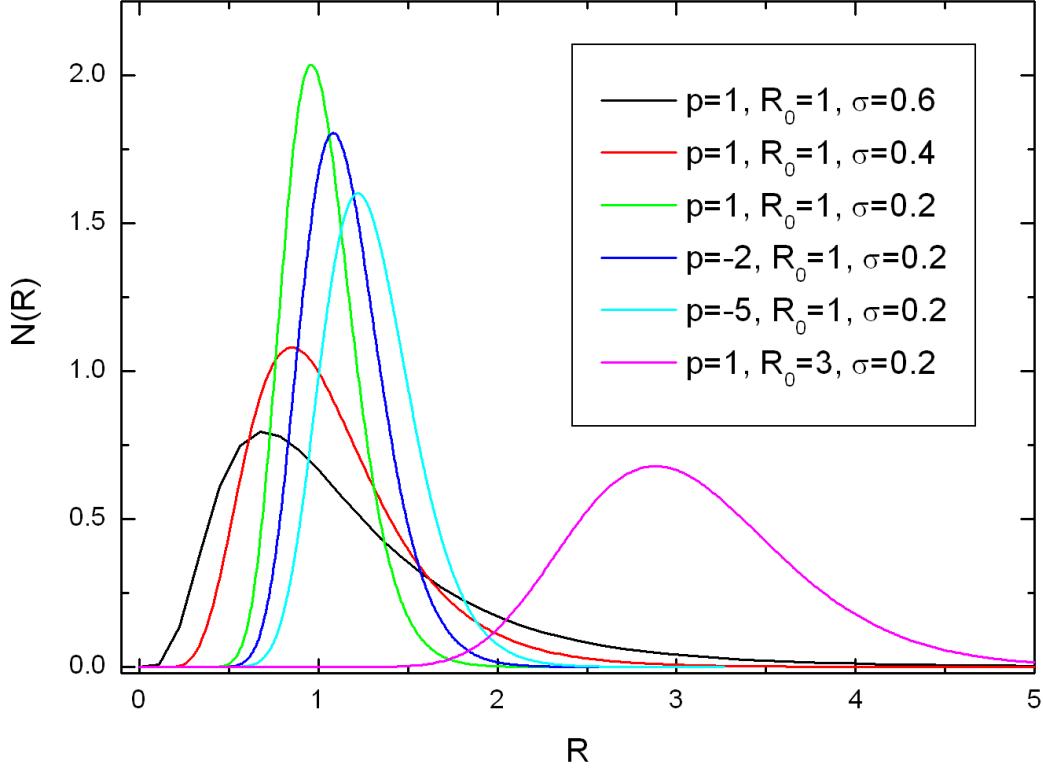


FIGURE 6.3. LogNormal distribution function ($R_0 = 1$ and $p = 1$ has been set both to one here). Valid parameter ranges: $R \in (0, \infty)$, $R_0 \in (0, \infty)$, $\sigma \geq 0$, $p \in (-\infty, \infty)$

The `LogNorm` distribution is a continuous distribution in which the logarithm of a variable has a normal distribution.

$$\text{LogNorm}(X, \mu, \sigma, p) = \frac{N}{c_{\text{LN}}} \frac{1}{X^p} \exp\left(-\frac{\ln(X/\mu)^2}{2\sigma^2}\right) \quad (6.3a)$$

$$c_{\text{LN}} = \sqrt{2\pi} \sigma \mu^{1-p} \exp\left((1-p)^2 \frac{\sigma^2}{2}\right) \quad (6.3b)$$

where σ is the width parameter, p a shape parameter, μ is the location parameter. c_{LN} is chosen so that $\int_0^\infty \text{LogNorm}(X, \mu, \sigma, p) dX = N$. The mode of the distribution X_{mode} and the variance $\text{Var}(X)$ are defined as

$$X_{\text{mode}} = \mu e^{-p\sigma^2} \quad (6.4)$$

$$X_{\text{mean}} = \mu e^{-\frac{1}{2}\sigma^2(2p-3)} \quad (6.5)$$

$$X_{\text{median}} = \mu e^{-\sigma^2(p-1)} \quad (6.6)$$

$$\text{Var}(X) = \mu^2 \left(e^{\sigma^2} - 1\right) e^{(3-2p)\sigma^2} \quad (6.7)$$

and the m^{th} moment $\langle X^m \rangle$ of the LogNorm distribution as

$$\langle X^m \rangle = \frac{\int X^m \text{LogNorm}(X) dX}{\int \text{LogNorm}(X) dX} = \mu^m e^{\frac{1}{2}\sigma^2 m(2-2p+m)}. \quad (6.8)$$

The cumulative distribution function is given by

$$F_{\text{LogNorm}} = 1/2\text{erf} \left[\frac{(p-1)\sigma^2 + \ln(x/\mu)}{\sqrt{2}\sigma} \right] \quad (6.9)$$

6.5. Schulz-Zimm (Flory) distribution

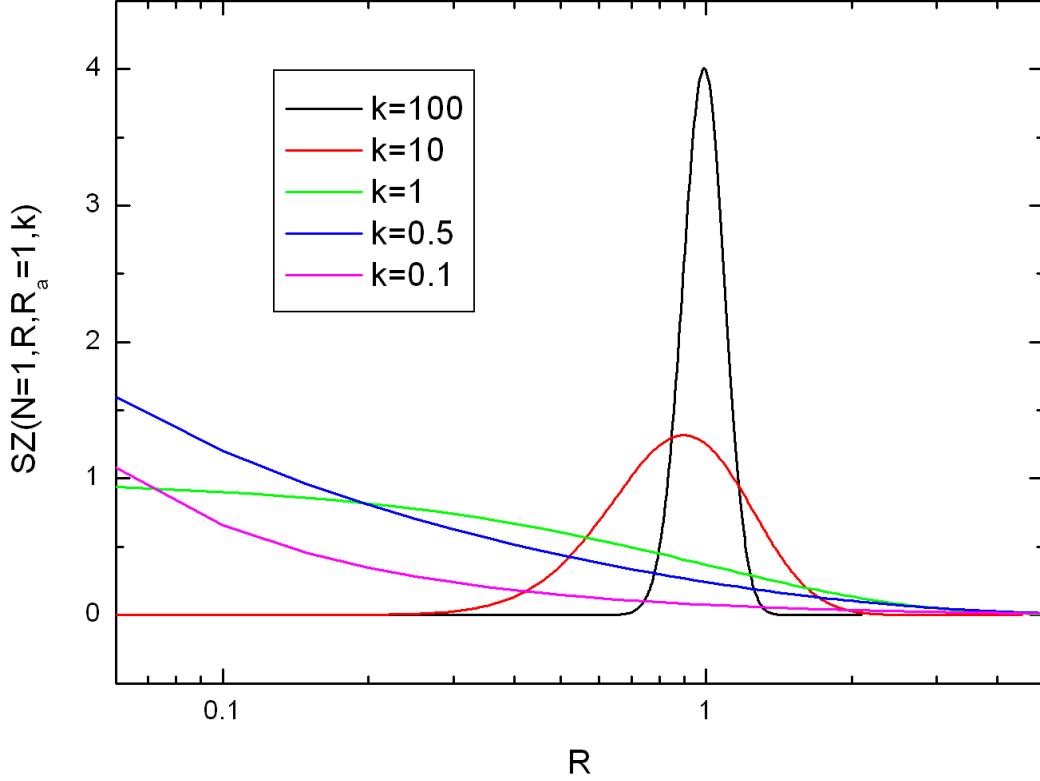


FIGURE 6.4. The $SZ(X, N, X_a, k)$ distribution function. Valid parameter ranges: $X \in [0, \infty)$, $X_a \in (0, \infty)$, $k = X_a^2/\sigma^2 > 0$

A function commonly used to present polymer molecular weight distributions is the Schulz-Zimm function

$$SZ_n(X, N, X_a, k) = \frac{N}{X_a} \left(\frac{X}{X_a} \right)^{k-1} \frac{k^k \exp(-kX/X_a)}{\Gamma(k)} \quad (6.10)$$

$SZ_n(X, N, X_a, k)$ is normalized so that $\int_0^\infty SZ_n(X, N, X_a, k) dX = N$. In polymer science X would be the molecular weight M , $\bar{M}_n = X_a$, and $\bar{M}_w = \bar{M}_n \frac{k+1}{k}$, and $\Gamma(k)$ is the gamma function. The above form (6.10) gives the number distribution. Its mode, mean, variance and m^{th} -moment are given by

$$X_{\text{mode}} = X_a \left(1 - \frac{1}{k} \right) \quad (6.11a)$$

$$X_{\text{mean}} = X_a \quad (6.11b)$$

$$\text{Var}(X) = \sigma^2 = \frac{X_a^2}{k} \quad (6.11c)$$

$$\langle X^m \rangle = \left(\frac{X_a}{k} \right)^m \frac{\Gamma(k+m)}{\Gamma(k)} \quad (6.11d)$$

The corresponding weight distribution is

$$\text{SZ}_w(X, N, X_a, k) = \frac{X}{X_a} \text{SZ}_n(X, N, X_a, k) = \frac{NX^k \left(\frac{k}{X_a}\right)^{k+1} e^{-\frac{kX}{X_a}}}{\Gamma(k+1)} \quad (6.12)$$

Also $\text{SZ}_w(X, N, X_a, k)$ is normalized so that $\int_0^\infty \text{SZ}_w(X, N, X_a, k) dX = N$. The mode, mean, variance and m^{th} -moment of the weight distribution are given by

$$X_{\text{mode}} = X_a \quad (6.13a)$$

$$X_{\text{mean}} = X_a \frac{1+k}{k} \quad (6.13b)$$

$$\text{Var}(X) = \sigma^2 = X_a^2 \frac{1+k}{k^2} \quad (6.13c)$$

$$\langle X^m \rangle = \left(\frac{X_a}{k}\right)^m \frac{\Gamma(k+m+1)}{\Gamma(k+1)} \quad (6.13d)$$

6.6. Gamma distribution

The Gamma distribution is a two parameter continuous distribution with a scale parameter θ and a shape parameter k .

$$\text{gammaSD}(x, N, x_{\text{mode}}, \sigma) = \frac{N}{\theta} \left(\frac{x}{\theta} \right)^{k-1} \frac{\exp(-x/\theta)}{\Gamma(k)} \quad (6.14)$$

The mean x_{mean} , mode x_{mode} and variance σ^2 of the distribution are given by

$$x_{\text{mean}} = k\theta \quad (6.15)$$

$$x_{\text{mode}} = (k - 1)\theta \text{ for } k \geq 1 \quad (6.16)$$

$$\sigma^2 = k\theta^2 \quad (6.17)$$

The gamma distribution is more flexible than the exponential or ξ^2 distribution function which are special cases of the gamma distribution function. When k is large, the gamma distribution closely approximates a normal distribution with the advantage that the gamma distribution has density only for positive real numbers. For small values of k the distribution becomes a right tailed distribution. The gamma distribution is equivalent to the Schulz-Zimm distribution (sec. 6.5), which is using the mean value instead of $\theta = x_{\text{mean}}/k$ for the parametrization. Also the Pearson III 6.7 is just another name for this distribution.

The m^{th} moment $\langle X^m \rangle$ of the size distribution is given by

$$\langle X^m \rangle = \theta^m \frac{\Gamma(k + m)}{\Gamma(k)}. \quad (6.18)$$

In the present version the Gamma distribution is parametrised as a function of the mode and variance, i.e. with

$$k = \frac{x_{\text{mode}} \sqrt{x_{\text{mode}}^2 + 4\sigma^2} + x_{\text{mode}}^2 + 2\sigma^2}{2\sigma^2} \quad (6.19)$$

and

$$\theta = \frac{1}{2} \left(\sqrt{x_{\text{mode}}^2 + 4\sigma^2} - x_{\text{mode}} \right) \quad (6.20)$$

$\text{gammaSD}(R, N, R_m, \sigma)$ is normalized so that $\int_0^\infty \text{gammaSD}(R, N, R_m, \sigma) dR = N$.

Input Parameters for model Sphere:

N: N

mode: mod of the distribution (maximum, most probable size) $x_{\text{mode}} > 0$

sigma: width parameter $\sigma > 0$. The variance of the distribution is σ^2 .

Note:

- The parameters **mode** and **sigma** needs to be positive.

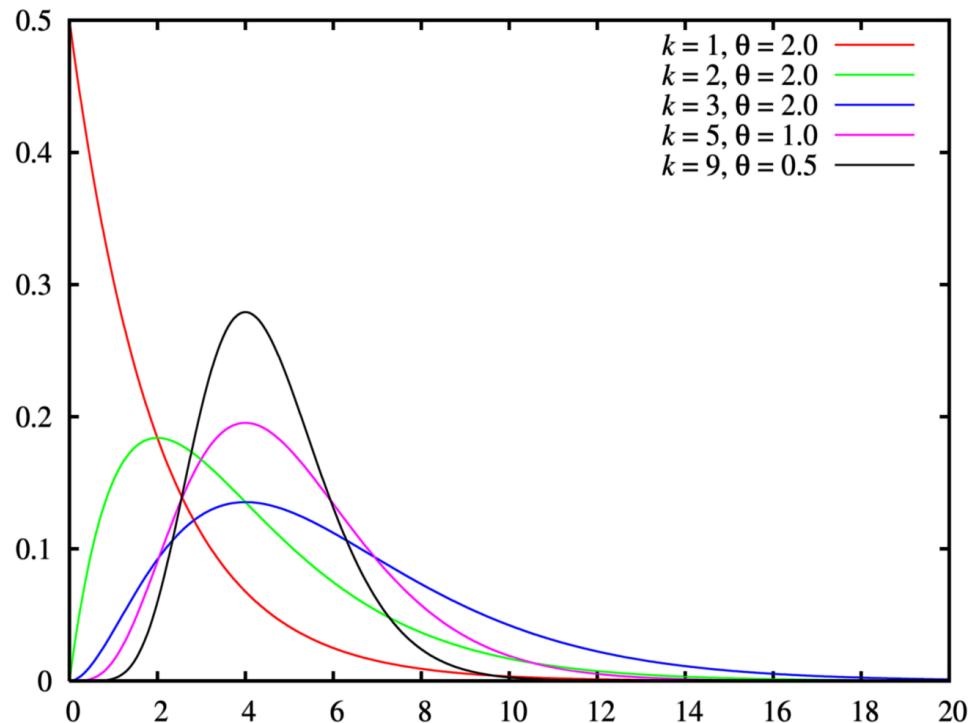


FIGURE 6.5. The $\text{gammaSD}(R, N, x_{\text{mode}}, \sigma)$ distribution function. Valid parameter ranges: $x \in [0, \infty)$, $x_{\text{mode}} \in (0, \infty)$, $\sigma > 0$

6.7. PearsonIII distribution

The Pearson distribution is a family of probability distributions that are a generalisation of the normal distribution. The Pearson Type III distribution is given by the probability density function

$$f(x) = \frac{1}{\beta \Gamma(p)} \left(\frac{x - \alpha}{\beta} \right)^{p-1} e^{-(x-\alpha)/\beta}, \quad (6.21)$$

where $x \in [\alpha, \infty)$ and α, β and p are parameters of the distribution with $\beta > 0$ and $p > 0$ (Abramowitz and Stegun 1954, p. 930). Here, $\Gamma()$ denotes the Gamma function.

- Mean:

$$\alpha + p\beta$$

- Variance:

$$p\beta^2$$

- Skewness:

$$\frac{2}{\sqrt{p}}$$

- Kurtosis:

$$\frac{6}{p}$$

The Pearson Type III distribution is identical to the Gamma distribution (6.14). When $\alpha = 0$, $\beta = 2$, and p is half-integer, the Pearson Type III distribution becomes the χ^2 distribution of $2p$ degrees of freedom.

6.8. Gauss distribution

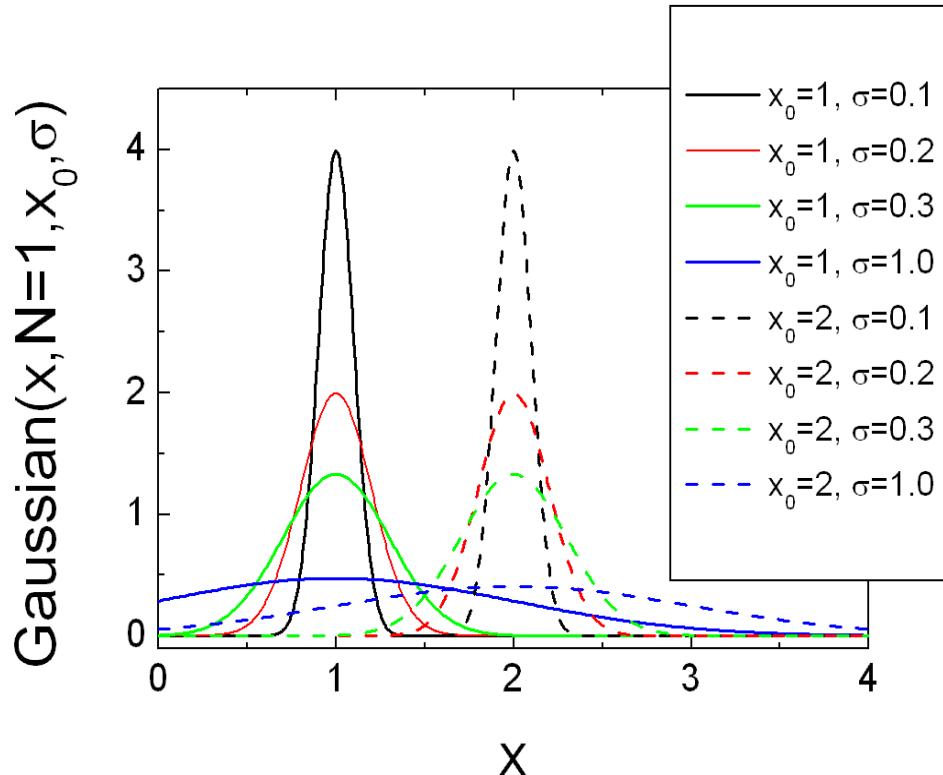


FIGURE 6.6. Normal or Gauss distribution function ($R_0 = \mu = 0$ has been chosen in the plot). Valid parameter ranges: $R \in (0, \infty)$, $R_0 \in (-\infty, \infty)$, $\sigma > 0$

$$\text{Gauss}(R, N, \sigma, R_0) = \frac{N}{c_{\text{Gauss}}} e^{-\frac{(R-R_0)^2}{2\sigma^2}} \quad (6.22a)$$

$$c_{\text{Gauss}} = \sqrt{\frac{\pi}{2}} \sigma \left(1 + \text{erf} \left(\frac{R_0}{\sqrt{2}\sigma} \right) \right) \quad (6.22b)$$

c_{Gauss} is chosen so that $\int_0^\infty \text{Gauss}(R, \sigma, R_0) dR = N$

6.9. Generalized exponential distribution (GEX)

The generalised exponential distribution is a three parameter distribution.

$$\text{GEX}(x, \beta, \lambda, \gamma) = N \frac{|\beta|}{\gamma} \left(\frac{x}{\gamma} \right)^{\lambda+1} \frac{e^{-(x/\gamma)^\beta}}{\Gamma\left(\frac{\lambda+2}{\beta}\right)} \quad (6.23)$$

$$\text{GEX}(x, \beta, p, \gamma) = N \frac{|\beta|}{\gamma} \left(\frac{x}{\gamma} \right)^{p-1} \frac{e^{-(x/\gamma)^\beta}}{\Gamma\left(\frac{p}{\beta}\right)} \quad (6.24)$$

For $\lambda + 2 = p = \beta$ it will transform in the Weibull distribution (eq. 6.33). For $\beta = 1$ one yields the Schulz-Zimm distribution (eq. 6.10) or the equivalent gamma distribution (eq. 7.31). In the limit $(\beta, \lambda) \rightarrow (0, \infty)$ whereas the product $\beta(\lambda + 2) \rightarrow \sigma$ is finite and the GEX distribution transforms into a Lognormal distribution.

$$\lim_{(\beta, \lambda) \rightarrow (0, \infty)} \text{GEX}(x, \beta, \lambda, \gamma) = \frac{1}{\sqrt{2\pi} \sigma x} \exp\left(-\frac{(\log x - \log \gamma)^2}{2\sigma^2}\right) \quad (6.25)$$

$$\text{with } \lim_{(\beta, \lambda) \rightarrow (0, \infty)} \beta(\lambda + 2) = \sigma \quad (6.26)$$

The mode x_{mode} , mean x_{mean} , and variance σ^2 of this distribution are given by

$$x_{\text{mode}} = \gamma \left(\frac{\lambda + 1}{\beta} \right)^{\frac{1}{\beta}} \quad (6.27)$$

$$x_{\text{mean}} = \begin{cases} \gamma \Gamma\left(\frac{\lambda+3}{\beta}\right) / \Gamma\left(\frac{\lambda+2}{\beta}\right) & \text{for } \beta > 0 \wedge \lambda > -3 \\ \text{undefined} & \text{otherwise} \end{cases} \quad (6.28)$$

$$\sigma^2 = \begin{cases} \gamma^2 \left(\frac{\Gamma\left(\frac{\lambda+4}{\beta}\right)}{\Gamma\left(\frac{\lambda+2}{\beta}\right)} - \left(\frac{\Gamma\left(\frac{\lambda+3}{\beta}\right)}{\Gamma\left(\frac{\lambda+2}{\beta}\right)} \right)^2 \right) & \text{for } \beta > 0 \wedge \lambda > -3 \\ \text{undefined} & \text{otherwise} \end{cases} \quad (6.29)$$

6.10. Generalized extreme value distribution (GEV)

FIGURE 6.7. The shape parameter governs the tail behaviour of the distribution, the sub-families defined by $\xi \rightarrow 0$, $\xi > 0$ and $\xi < 0$ correspond, respectively, to the Gumbel, Fréchet and Weibull families, whose cumulative distribution functions are reminded below. Gumbel or type I extreme value distribution

$$\text{GEV}(R, \mu, \sigma, \xi) = \frac{N}{c_1} \frac{e^{-(1+\frac{\xi(R-\mu)}{\sigma})^{-1/\xi}}}{\sigma \left(1 + \frac{\xi(R-\mu)}{\sigma}\right)^{1+1/\xi}} \quad (6.30)$$

with

$$c_1 = \begin{cases} 1 & \text{for } \xi > 0 \\ 1 - \exp\left(-\left(1 - \frac{\xi\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right) & \text{for } \xi < 0 \end{cases} \quad (6.31)$$

The shape parameter ξ governs the tail behaviour of the distribution, the sub-families defined by $\xi \rightarrow 0$, $\xi > 0$ and $\xi < 0$ correspond, respectively, to the Gumbel, Fréchet and Weibull families, whose cumulative distribution functions are reminded below.

- Gumbel or type I extreme value distribution

$$F(x; \mu, \sigma) = e^{-e^{-(x-\mu)/\sigma}} \quad \text{for } x \in \mathbb{R}$$

- Fréchet or type II extreme value distribution

$$F(x; \mu, \sigma, \alpha) = \begin{cases} 0 & x \leq \mu \\ e^{-((x-\mu)/\sigma)^{-\alpha}} & x > \mu \end{cases}$$

- Weibull or type III extreme value distribution

$$F(x; \mu, \sigma, \alpha) = \begin{cases} e^{-(-(x-\mu)/\sigma)^{-\alpha}} & x < \mu \\ 1 & x \geq \mu \end{cases}$$

where $\xi > 0$ and $\xi > 0$

Remark I: For reliability issues the Weibull distribution is used with the variable $t = \mu - x$, the time, which is strictly positive. Thus the support is positive - in contrast to the use in extreme value theory.

Remark II: Be aware of an important distinctive feature of the three extreme value distributions: The support is either unlimited, or it has an upper or lower limit.

- Parameters

$\mu \in [-\infty, \infty]$ location (real)

$\sigma \in (0, \infty]$ scale (real)

$\xi \in [-\infty, \infty]$ shape (real)

- Support

$$x > \mu - \sigma/\xi \ (\xi > 0)$$

$$x < \mu - \sigma/\xi \ (\xi < 0)$$

$$x \in [-\infty, \infty] \ (\xi = 0)$$

[1] http://en.wikipedia.org/wiki/Generalized_extreme_value_distribution

6.11. Maxwell distribution

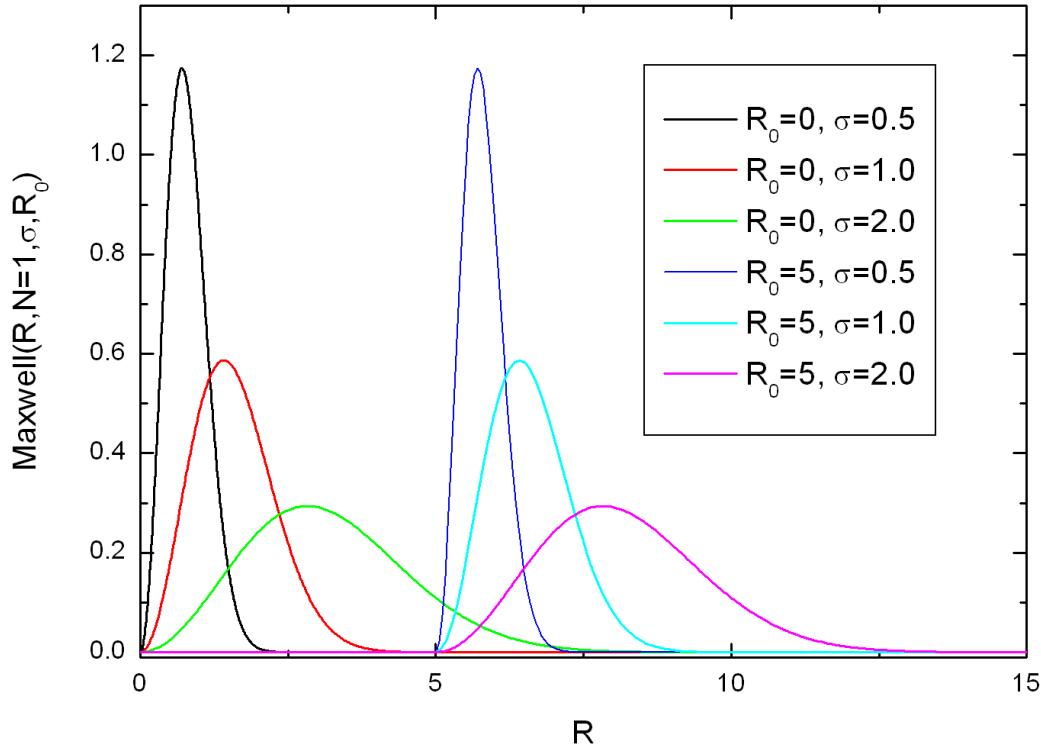


FIGURE 6.8. Maxwell distribution function. Valid parameter ranges: $R \in [0, \infty)$, $R_0 \in (-\infty, \infty)$, $\sigma > 0$

$$\text{Maxwell}(R, R_0, \sigma) = \begin{cases} \text{if } R \geq R_0: & N \frac{c}{c_{\text{mw}}} (R - R_0)^2 e^{-(R-R_0)^2/(2\sigma^2)} \\ \text{else:} & 0 \end{cases} \quad (6.32a)$$

$$c = \frac{4}{\sqrt{\pi}} (2\sigma^2)^{-3/2} \quad (6.32b)$$

$$c_{\text{mw}} = \begin{cases} \text{if } R_0 < 0: & 1 - \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \frac{R_0}{\sqrt{\exp(R_0^2/\sigma^2)}} + \text{erf}\left(\frac{R_0}{\sqrt{2}\sigma}\right) \\ \text{else:} & 1 \end{cases} \quad (6.32c)$$

6.12. Weibull distribution

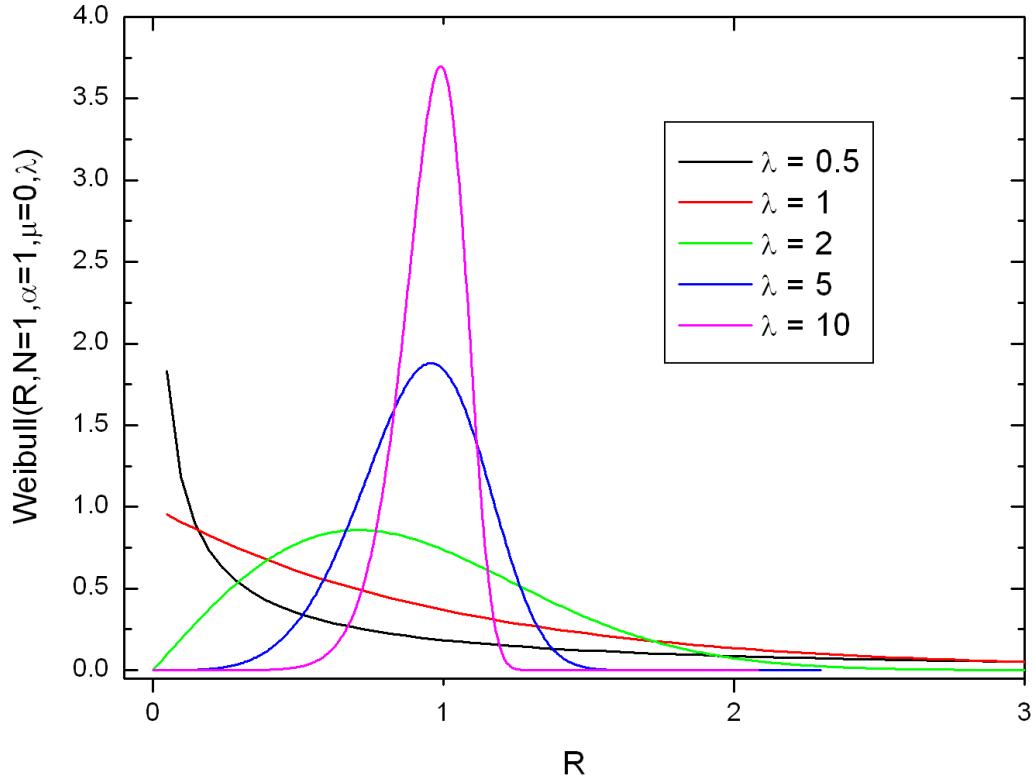


FIGURE 6.9. Weibull distribution function ($\mu = 0$, $\alpha = 1$ has been chosen in the plot). Valid parameter ranges: $R \in [0, \infty)$, $\mu \in [0, \infty)$, $\alpha > 0$, $\lambda > 0$

$$\text{Weibull}(R, \alpha, \lambda, \mu) = \frac{N\lambda}{\alpha} \left(\frac{R - \mu}{\alpha} \right)^{\lambda-1} e^{-\left(\frac{R-\mu}{\alpha}\right)^\lambda} e^{-\left(\frac{\mu}{\alpha}\right)^\lambda} \quad (6.33)$$

where λ is the shape parameter, μ is the location parameter and α is the scale parameter.

6.13. fractal size distribution

FIGURE 6.10. fractal size distribution function. Valid parameter ranges:
 $R \in [R_{\min}, R_{\max}]$, $f_D \in (-1, \infty)$, $R_{\max} > R_{\min} > 0$

$$\text{fractalsD}(R, N, R_{\min}, R_{\max}, f_D) = \frac{Nf_D}{R_{\min}^{-f_D} - R_{\max}^{-f_D}} R^{-(1+f_D)} \quad (6.34)$$

CHAPTER 7

Peak functions

7.1. Beta

The Beta distribution is a very versatile function which can be used to model several different shapes of probability density curves. In probability theory and statistics, the beta distribution is a family of continuous probability distributions defined on the interval $[0, 1]$ parameterized by two positive shape parameters, typically denoted by α and β .

$$p_{\text{Beta}}(x; \alpha, \beta) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (7.1)$$

The beta function, B , appears as a normalization constant to ensure that the total probability integrates to unity. α and β are positive numbers that define the shape parameters. The mode of the beta distribution for shape parameters $\alpha > 1$ and $\beta > 1$ is given by

$$\text{mode}(p_{\text{Beta}}) = \frac{\alpha - 1}{\alpha + \beta - 2} \quad (7.2)$$

7.1.1. Beta (Amplitude).

$$y_{\text{Beta(ampl)}}(x; A, x_{\min}, x_{\max}, \alpha, \beta, c_0) = A \frac{p_{\text{Beta}}\left(\frac{x-x_{\min}}{x_{\max}-x_{\min}}; \alpha, \beta\right)}{p_{\text{Beta}}\left(\frac{\alpha-1}{\alpha+\beta-2}; \alpha, \beta\right)} + c_0 \quad (7.3)$$

Required parameters:

- ampl.:** amplitude A of the Beta peak
- xmin:** continuous lower boundary parameters x_{\min}
- xmax:** continuous upper boundary parameters x_{\max}
- alpha:** first shape parameter $\alpha > 1$
- beta:** second shape parameter $\beta > 1$
- backgr:** offset c_0

Note

- Both shape parameter needs to be larger than one ($\alpha, \beta > 1$), as only than the distribution has a peak shape.
- where the Beta distribution is not defined the offset value is returned:
 $\forall x \notin (x_{\min}, x_{\max}) \quad y_{\text{Beta(ampl)}}(x) = c_0$
- Default (size) distribution: Monodisperse

7.1.2. Beta (Area).

$$y_{\text{Beta(area)}}(x; A, x_{\min}, x_{\max}, \alpha, \beta, c_0) = A \frac{p_{\text{Beta}}\left(\frac{x-x_{\min}}{x_{\max}-x_{\min}}; \alpha, \beta\right)}{x_{\max} - x_{\min}} + c_0 \quad (7.4)$$

Required parameters:

- area:** area A of the beta distribution

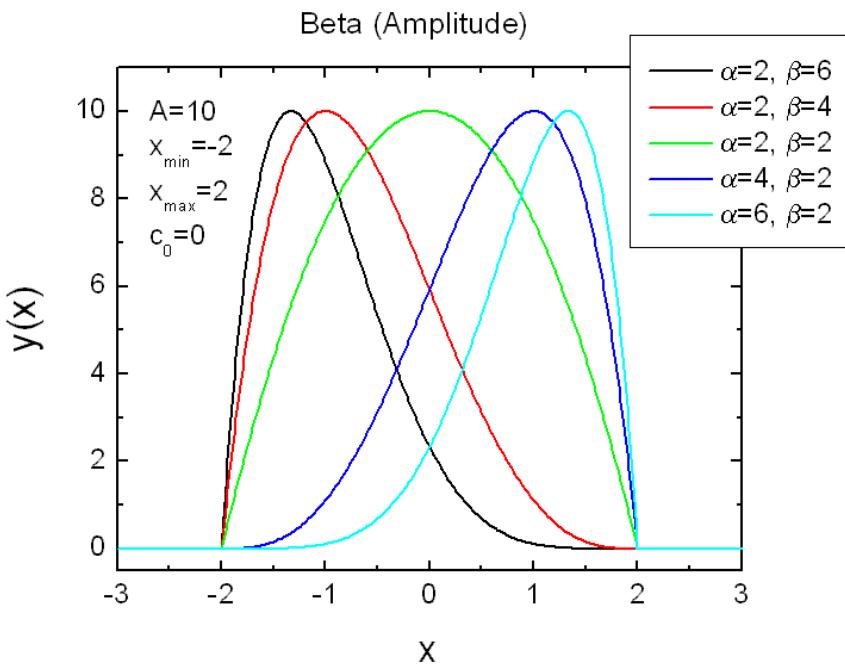


FIGURE 7.1. Plot of Beta (Amplitude) distribution.

xmin: continuous lower boundary parameters x_{\min}

xmax: continuous upper boundary parameters x_{\max}

alpha: first shape parameter $\alpha > 0$

beta: second shape parameter $\beta > 0$

backgr: offset c_0

Note

- Both shape parameter needs to be larger than zero ($\alpha, \beta > 0$)
- where the Beta distribution is not defined the offset value is returned:
 $\forall x \notin (x_{\min}, x_{\max}) \quad y_{\text{Beta(area)}}(x) = c_0$
- Default (size) distribution: Monodisperse

7.2. Chi-Squared

¹In probability theory and statistics, the chi-square distribution (also chi-squared or ξ^2 distribution) is one of the most widely used theoretical probability distributions in inferential statistics, e.g., in statistical significance tests. It is useful because, under reasonable assumptions, easily calculated quantities can be proven to have distributions that approximate to the chi-square distribution if the null hypothesis is true.

The best-known situations in which the chi-square distribution are used are the common chi-square tests for goodness of fit of an observed distribution to a theoretical one, and of the independence of two criteria of classification of qualitative data. Many other statistical tests also lead to a use of this distribution, like Friedman's analysis of variance by ranks.

A probability density function of the chi-square distribution is

$$f(x; k) = \begin{cases} \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \quad (7.5)$$

where Γ denotes the Gamma function, which has closed-form values at the half-integers. The mode of the distribution is

$$\text{mode} = k - 2 \text{ if } k \geq 2. \quad (7.6)$$

The χ^2 distribution is a special case of the gamma distribution 7.8 where $\theta = 2$ in the equation 7.31.

7.2.1. Chi-Squared (Amplitude).

$$\chi^2(x; A, x_c, \sigma, k, c_0) = \begin{cases} c_0 + A_0 (z + u)^v \exp\left(-\frac{z+u}{2}\right) & \text{for } z + u \geq 0 \\ c_0 & \text{otherwise} \end{cases} \quad (7.7)$$

with

$$z = \frac{x - x_c}{\sigma} \quad (7.8)$$

$$u = k - 2 \quad (7.9)$$

$$v = \frac{k}{2} - 1 \quad (7.10)$$

$$A_0 = \frac{A \exp(v)}{u^v} \quad (7.11)$$

The standard statistical form has been reparameterized. The parameter x_c has been added to enable variable x positioning, and σ to enable scaling. The mode is x_c . The function returns 0 for those x where it is undefined ($z + u < 0$).

Required parameters:

amplitude: amplitude a of the Gamma peak

center: location parameter (mode) x_c

¹Description taken partly from Wikipedia, the free encyclopedia

width: scaling parameter $\sigma > 0$

shape: shape parameter $k > 2$

backgr: offset c_0

Note

- The width parameter needs to be larger than zero ($\sigma > 0$).
- The shape parameter needs to be larger than two ($k > 2$)
- Default (size) distribution: Monodisperse

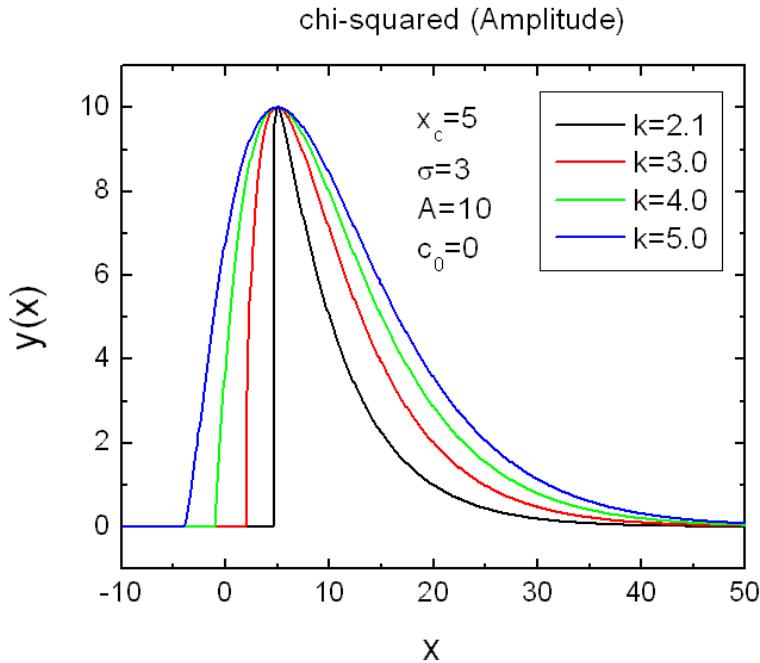


FIGURE 7.2. Plot of Chi-Squared (Amplitude) distribution.

7.2.2. Chi-Squared (Area).

$$\chi^2(x; A, x_c, \sigma, k, c_0) = \begin{cases} c_0 + A_0 (z + u)^v \exp\left(-\frac{z+u}{2}\right) & \text{for } z + u \geq 0 \\ c_0 & \text{otherwise} \end{cases} \quad (7.12)$$

with

$$z = \frac{x - x_c}{\sigma} \quad (7.13)$$

$$u = k - 2 \quad (7.14)$$

$$v = \frac{k}{2} - 1 \quad (7.15)$$

$$A_0 = \frac{A}{2^{\frac{k}{2}} \sigma \Gamma\left(\frac{k}{2}\right)} \quad (7.16)$$

The standard statistical form has been reparameterized. The parameter x_c has been added to enable variable x positioning, and σ to enable scaling. The mode is x_c . The function returns 0 for those x where it is undefined ($z + u < 0$).

Required parameters:

- area:** area a of the Gamma peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma > 0$
- shape:** shape parameter $k > 2$
- backgr:** offset c_0

Note

- The width parameter needs to be larger than zero ($\sigma > 0$).
- The shape parameter needs to be larger than two ($k > 2$)
- Default (size) distribution: Monodisperse

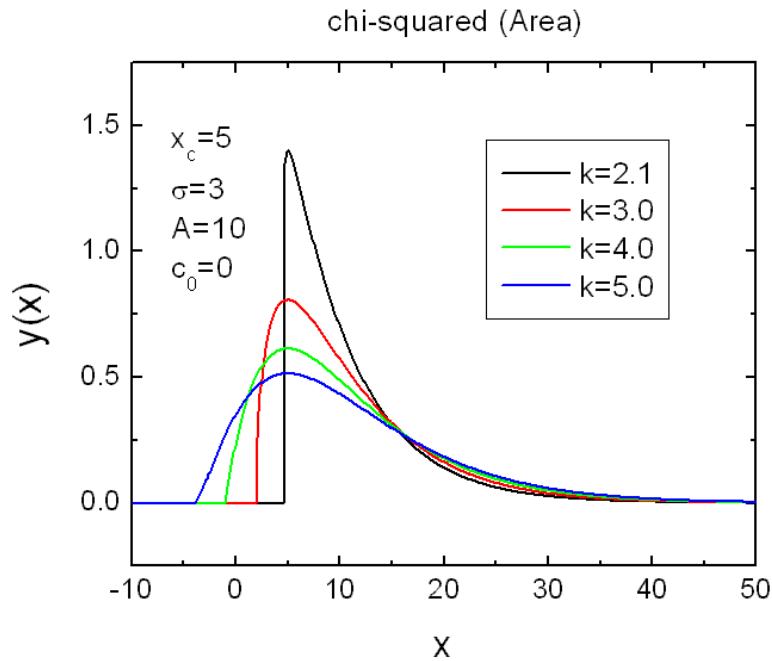


FIGURE 7.3. Plot of Chi-Squared (Area) distribution.

7.3. Erfc peak

7.3.1. Erfc (Amplitude).

$$y(x; a, x_c, \sigma, c_0) = a \operatorname{erfc} \left(\left(\frac{x - x_c}{\sigma} \right)^2 \right) + c_0 \quad (7.17)$$

Required parameters:

- ampl.:** amplitude a of the erfc peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma > 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- Default (size) distribution: Monodisperse

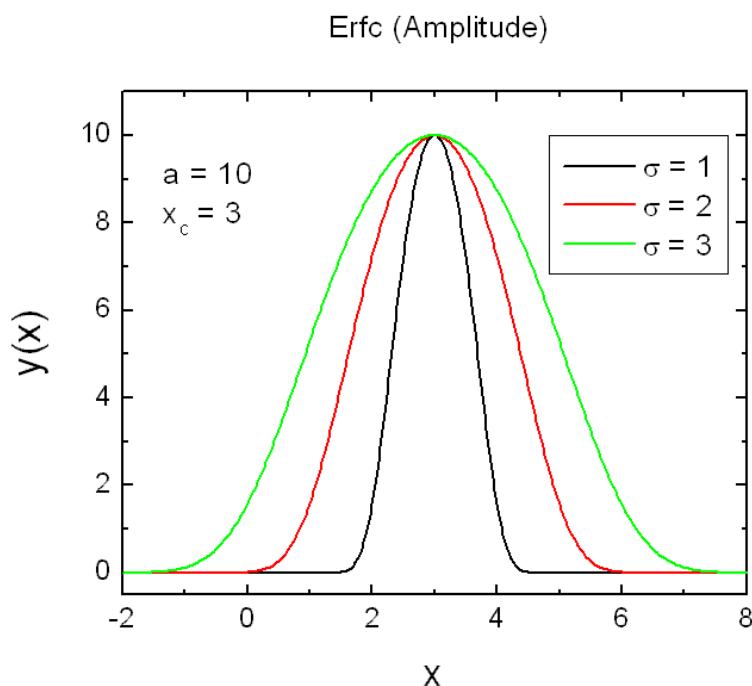


FIGURE 7.4. Plot of Erfc (Amplitude) distribution.

7.3.2. Erfc (Area).

$$y(x; a, x_c, \sigma, c_0) = a \frac{\operatorname{erfc}\left(\left(\frac{x-x_c}{\sigma}\right)^2\right)}{\int_{-\infty}^{\infty} \operatorname{erfc}\left(\left(\frac{x-x_c}{\sigma}\right)^2\right) dx} + c_0 \quad (7.18)$$

Required parameters:

- area:** area a below the erfc peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma > 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- Default (size) distribution: Monodisperse

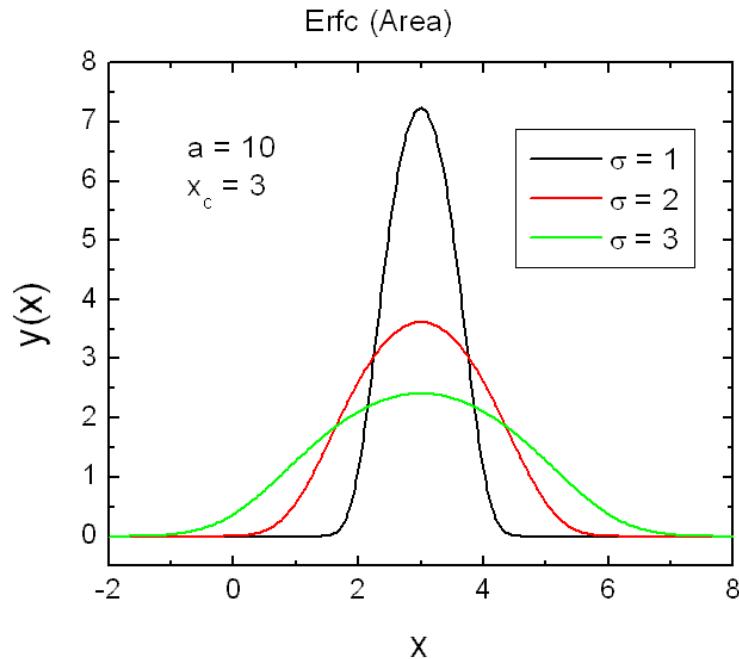


FIGURE 7.5. Plot of Erfc (Area) distribution.

7.4. Error peak

7.4.1. Error (Amplitude).

$$y(x; a, x_c, \sigma, k, c_0) = a \exp\left(-\frac{1}{2} \frac{|x - x_c|^{\frac{2}{k}}}{|\sigma|}\right) + c_0 \quad (7.19)$$

Required parameters:

ampl.: amplitude a of the error distribution

center: location parameter (mode) x_c

width: scaling parameter $\sigma \neq 0$

shape: shape parameter $k > 0$

backgr: offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- The shape parameter needs to be larger than zero ($k > 0$).
- Default (size) distribution: Monodisperse

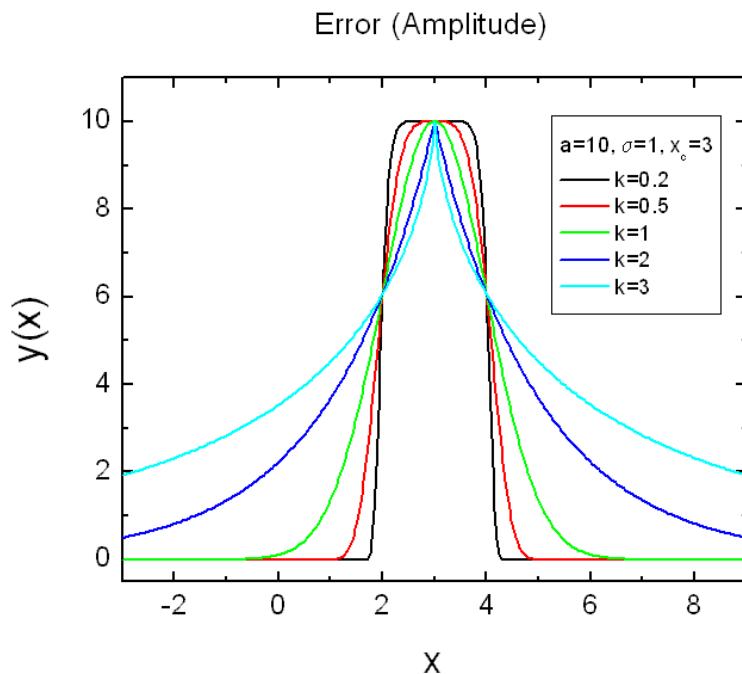


FIGURE 7.6. Plot of Error (Amplitude) distribution.

7.4.2. Error (Area).

$$y(x; a, x_c, \sigma, k, c_0) = \frac{a}{|\sigma|^{\frac{k}{2}} 2^{\frac{k}{2}+1} \Gamma\left(\frac{k}{2} + 1\right)} \exp\left(-\frac{1}{2} \frac{|x - x_c|^{\frac{2}{k}}}{|\sigma|}\right) + c_0 \quad (7.20)$$

Required parameters:

area: area a below the error distribution

center: location parameter (mode) x_c

width: scaling parameter $\sigma \neq 0$

shape: shape parameter $k > 0$

backgr: offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- The shape parameter needs to be larger than zero ($k > 0$).
- Default (size) distribution: Monodisperse

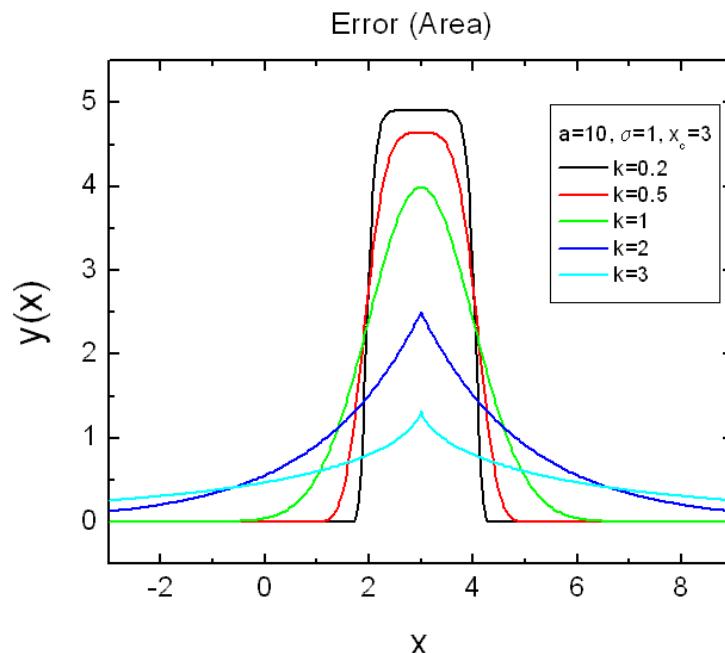


FIGURE 7.7. Plot of Error (Area) distribution.

7.5. Exponentially Modified Gaussian

7.5.1. Exponentially Modified Gaussian (Amplitude).

$$y(x; a, x_c, \sigma, \gamma, c_0) = \frac{a}{\text{const}} \exp \left(\frac{\sigma^2}{2\gamma^2} + \frac{x_c - x}{\gamma} \right) \left[\text{erf} \left(\frac{x - x_c}{\sqrt{2}\sigma} - \frac{\sigma}{\sqrt{2}\gamma} \right) + \frac{\gamma}{|\gamma|} \right] + c_0 \quad (7.21)$$

const is calculated numerically so that "a" represents the amplitude of the distribution.

Required parameters:

ampl.: amplitude a of the distribution

center: location parameter x_c

width: scaling parameter $\sigma > 0$

distortion: distortion parameter $\gamma \neq 0$

backgr: offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- The distortion parameter needs to be non-zero ($\gamma \neq 0$).
- Default (size) distribution: Monodisperse

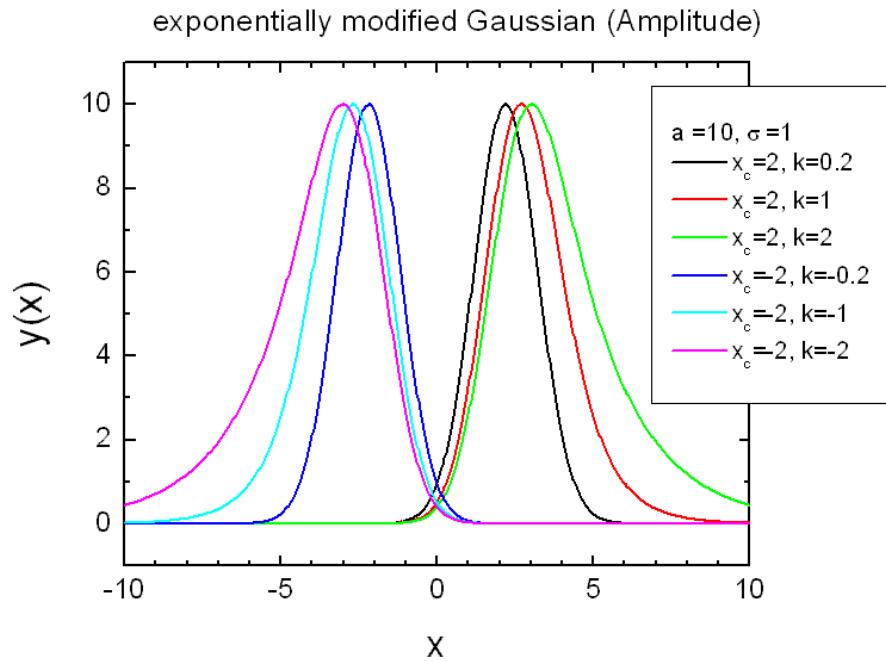


FIGURE 7.8. Plot of Exponentially Modified Gaussian (Amplitude) distribution.

7.5.2. Exponentially Modified Gaussian (Area).

$$\begin{aligned} y(x; a, x_c, \sigma, \gamma, c_0) &= \frac{a}{2\gamma} \exp\left(\frac{\sigma^2}{2\gamma^2} + \frac{x_c - x}{\gamma}\right) \\ &\quad \left[\operatorname{erf}\left(\frac{x - x_c}{\sqrt{2}\sigma} - \frac{\sigma}{\sqrt{2}\gamma}\right) + \frac{\gamma}{|\gamma|} \right] + c_0 \end{aligned} \quad (7.22)$$

Required parameters:

- area:** area a below the distribution
- center:** location parameter x_c
- width:** scaling parameter $\sigma > 0$
- distortion:** distortion parameter $\gamma \neq 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma > 0$).
- The distortion parameter needs to be non-zero ($\gamma \neq 0$).
- Default (size) distribution: Monodisperse

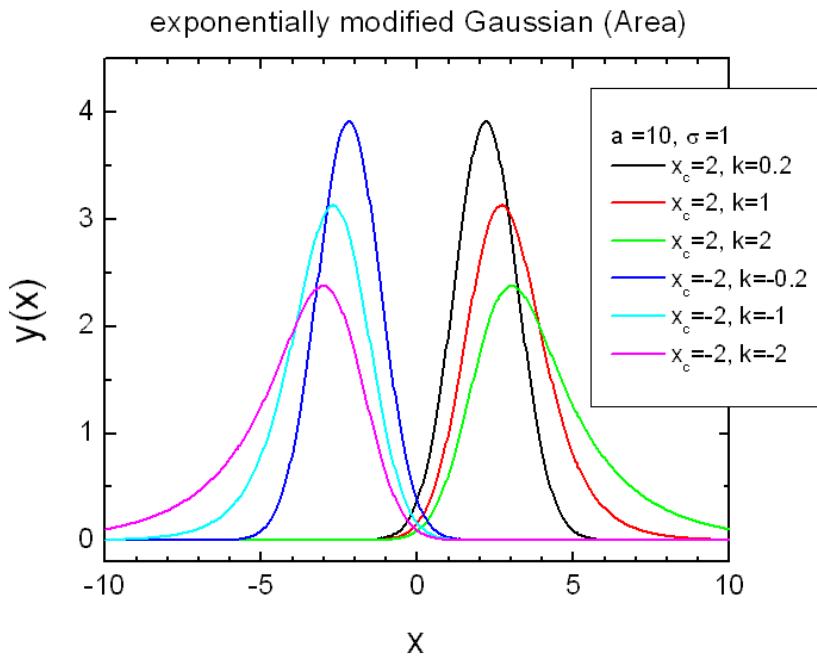


FIGURE 7.9. Plot of Exponentially Modified Gaussian (Area) distribution.

7.6. Extreme Value

7.6.1. Extreme Value (Amplitude).

$$y(x; a, x_c, \sigma, c_0) = a \exp \left[-\exp \left(-\frac{x - x_c}{|\sigma|} \right) - \frac{x - x_c}{|\sigma|} + 1 \right] + c_0 \quad (7.23)$$

Required parameters:

- ampl.:** amplitude a of the peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma \neq 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- Default (size) distribution: Monodisperse

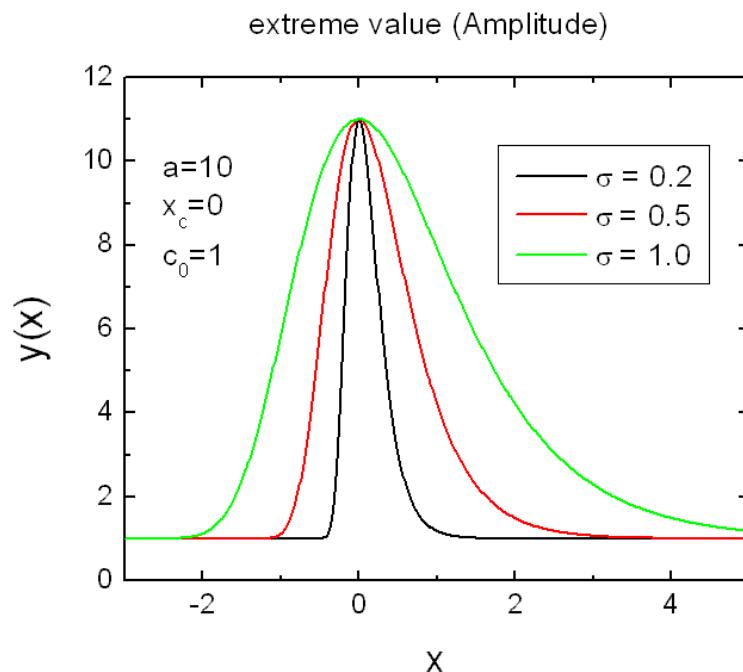


FIGURE 7.10. Plot of extreme value (Amplitude) distribution.

7.6.2. Extreme Value (Area).

$$y(x; a, x_c, \sigma, c_0) = \frac{a}{|\sigma|} \exp \left[-\exp \left(-\frac{x - x_c}{|\sigma|} \right) - \frac{x - x_c}{|\sigma|} \right] + c_0 \quad (7.24)$$

Required parameters:

- area:** area a below the peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma \neq 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- Default (size) distribution: Monodisperse

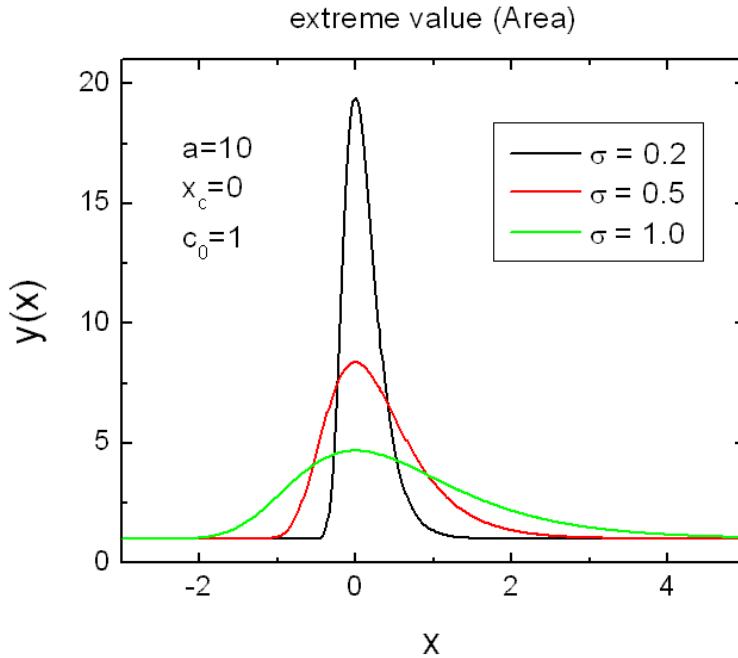


FIGURE 7.11. Plot of `extreme value (Area)` distribution.

7.7. F-Variance

In probability theory and statistics, the F-distribution is a continuous probability distribution. It is also known as Snedecor's F distribution or the Fisher-Snedecor distribution. The probability density is given by

$$p_{F-\text{var}}(x, \nu_1, \nu_2) = \frac{\sqrt{\frac{(\nu_1 x)^{\nu_1} \nu_2^{\nu_2}}{(\nu_1 x + \nu_2)^{\nu_1 + \nu_2}}}}{x B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \quad (7.25)$$

for real $x \geq 0$, where $\nu > 1$ and ν_2 are positive, and $B()$ is the beta function. For $\nu_1 > 2$ the mode of the distribution is defined by

$$\text{mode}_F(\nu_1, \nu_2) = \frac{\nu_1 - 2}{\nu_1} \frac{\nu_2}{\nu_2 + 2} \quad (7.26)$$

7.7.1. F-Variance (Amplitude).

The amplitude version represents a re-parametrization of the standard statistical form.

$$y(x; a, x_c, \sigma, \nu_1, \nu_2) = \begin{cases} c_0 + a \frac{p_{F-\text{var}}(z, \nu_1, \nu_2)}{p_{F-\text{var}}(\text{mode}_F(\nu_1, \nu_2), \nu_1, \nu_2)} & \text{for } z > 0 \\ c_0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} c_0 + a \frac{z^{\frac{\nu_1}{2}-1} \left(1 + \frac{\nu_1-2}{\nu_2+2}\right)^{\frac{\nu_1+\nu_2}{2}}}{\left(1 + \frac{\nu_1}{\nu_2} z\right)^{\frac{\nu_1+\nu_2}{2}} \left(\frac{\nu_2}{\nu_1} \frac{\nu_1-2}{\nu_2+2}\right)^{\frac{\nu_1-1}{2}}} & \text{for } z > 0 \\ c_0 & \text{otherwise} \end{cases} \quad (7.27)$$

with

$$z = \frac{x - x_c}{\sigma} + \frac{\nu_1 - 2}{\nu_1} \frac{\nu_2}{\nu_2 + 2} \quad (7.28)$$

The location parameter x_c has been added to enable variable x positioning, and σ to enable scaling. The mode of the distribution function is x_c due to the additional term $\frac{\nu_1-2}{\nu_1} \frac{\nu_2}{\nu_2+2}$ in the definition of z . The distribution function returns the offset c_0 for values $z \leq 0$.

Required parameters:

- ampl.:** amplitude a of the F-distribution
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma > 0$
- shape1:** shape parameter $\nu_1 > 2$
- shape2:** shape parameter $\nu_2 > 2$
- backgr:** offset c_0

Note

- The scale parameter needs to be larger than zero $\sigma > 0$
- The first shape parameter needs to be larger than zero $\nu_1 > 2$

- The second shape parameter needs to be larger than zero $\nu_2 > 2$
- Default (Size) distribution: Monodisperse

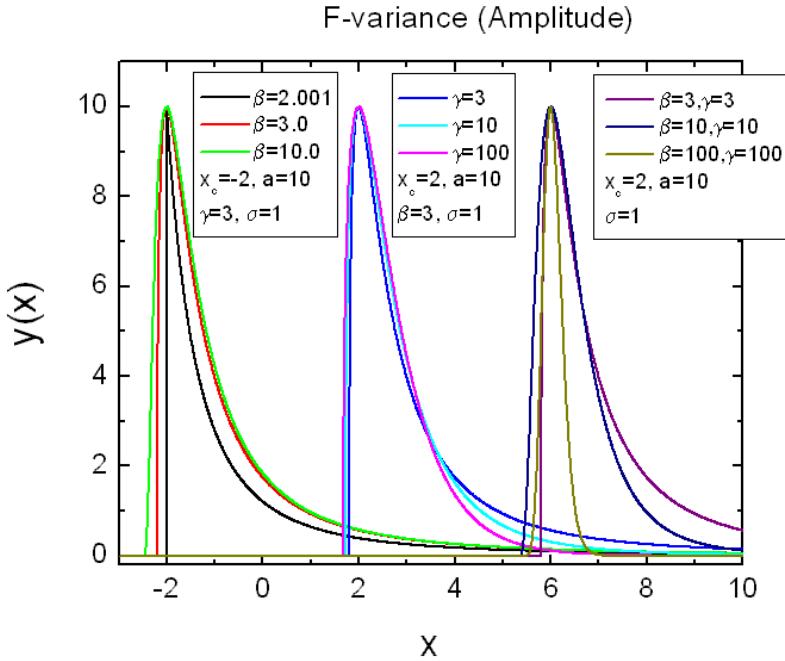


FIGURE 7.12. Plot of F-variance (Amplitude) distribution.

7.7.2. F-Variance (Area).

The area version represents a re-parametrization of the standard statistical form.

$$y(x; a, x_c, \sigma, \nu_1, \nu_2) = \begin{cases} c_0 + \frac{a}{\sigma} p_{F-\text{var}}(z, \nu_1, \nu_2) & \text{for } z > 0 \\ c_0 & \text{otherwise} \end{cases} \quad (7.29)$$

with

$$z = \frac{x - x_c}{\sigma} + \frac{\nu_1 - 2}{\nu_1} \frac{\nu_2}{\nu_2 + 2} \quad (7.30)$$

The location parameter x_c has been added to enable variable x positioning, and σ to enable scaling. The mode of the distribution function is x_c due to the additional term $\frac{\nu_1 - 2}{\nu_1} \frac{\nu_2}{\nu_2 + 2}$ in the definition of z . The distribution function returns the offset c_0 for values $z \leq 0$.

Required parameters:

- area:** area a of the F-distribution
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma > 0$
- shape1:** shape parameter $\nu_1 > 2$
- shape2:** shape parameter $\nu_2 > 2$

backgr: offset c_0

Note

- The scale parameter needs to be larger than zero $\sigma > 0$
- The first shape parameter needs to be larger than zero $\nu_1 > 2$
- The second shape parameter needs to be larger than zero $\nu_2 > 2$
- Default (Size) distribution: Monodisperse

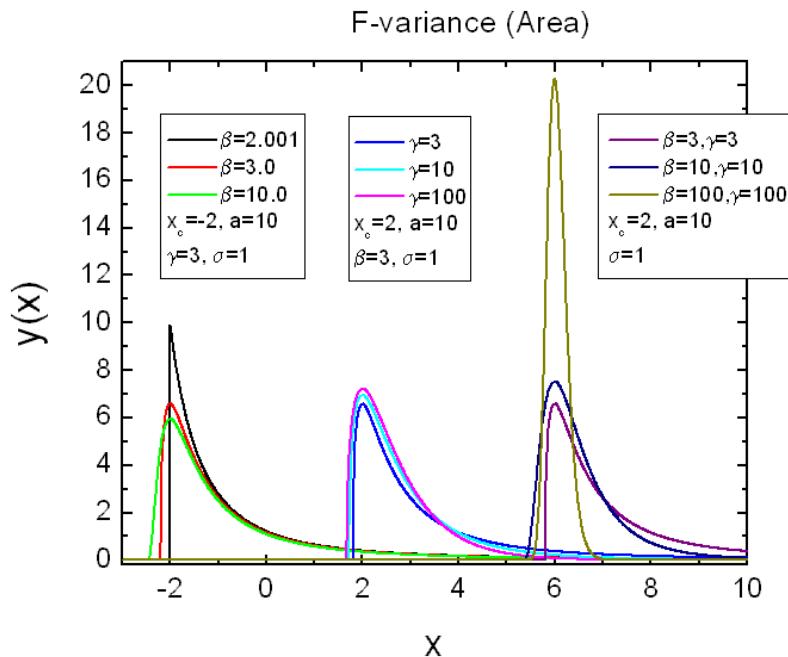


FIGURE 7.13. Plot of F-variance (Area) distribution.

7.8. Gamma

The gamma distribution models sums of exponentially distributed random variables.

The gamma distribution family is based on two parameters. The chi-square and exponential distributions, which are children of the gamma distribution, are one-parameter distributions that fix one of the two gamma parameters. The standard form is given by

$$p(x; k, \theta) = x^{k-1} \frac{e^{-x/\theta}}{\theta^k \Gamma(k)} \text{ for } x > 0 \text{ and } k, \theta > 0. \quad (7.31)$$

When k is large, the gamma distribution closely approximates a normal distribution with the advantage that the gamma distribution has density only for positive real numbers. In probability theory and statistics, the gamma distribution is a two-parameter family of continuous probability distributions. It has a scale parameter θ and a shape parameter k . If k is an integer then the distribution represents the sum of k independent exponentially distributed random variables, each of which has a mean of θ (which is equivalent to a rate parameter of θ^{-1}).

Alternatively, the gamma distribution can be parameterized in terms of a shape parameter $\alpha = k$ and an inverse scale parameter $\beta = 1/\theta$, called a rate parameter:

$$p(x; \alpha, \beta) = x^{\alpha-1} \frac{\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)} \text{ for } x > 0. \quad (7.32)$$

7.8.1. Gamma (Amplitude).

The parameter x_c has been added to enable variable x positioning, whereas the $+z\theta(k-1)$ adjusts x_c so that it represents the mode. c_0 is the offset value. The function returns the offset c_0 for those x where it is undefined

$$y(x) = \begin{cases} c_0 + a \exp(-z) \left(\frac{z+k-1}{k-1} \right)^{k-1} & \text{for } (z + k - 1) > 0 \\ c_0 & \text{otherwise} \end{cases} \quad (7.33)$$

with $z = \frac{x-x_c}{\theta}$

Required parameters:

- ampl.:** amplitude a of the Gamma peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\theta > 0$
- backgr:** offset c_0

Note

- The shape parameter needs to be larger than one $k > 1$.
- The scale parameter needs to be larger than zero $\theta > 0$
- Default (Size) distribution: Monodisperse

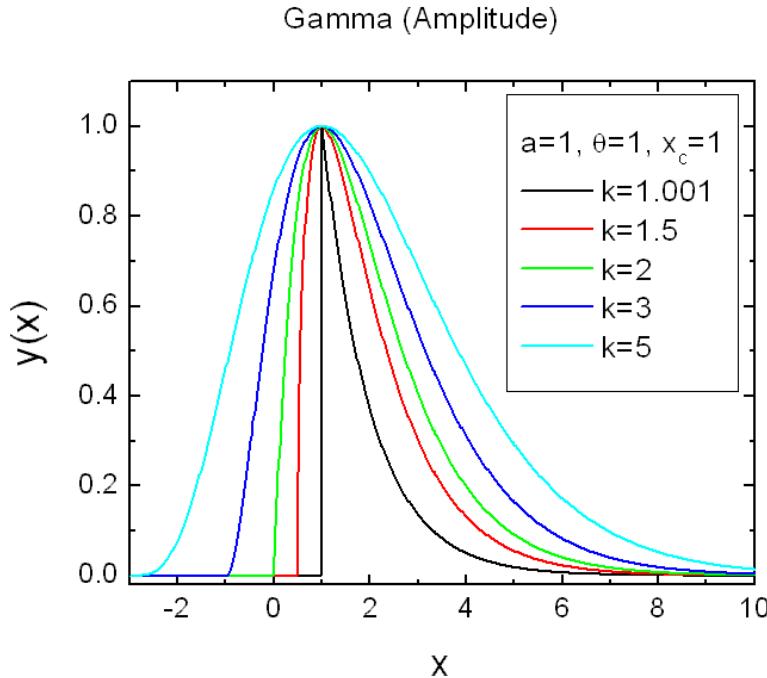


FIGURE 7.14. Plot of Gamma (Amplitude) distribution.

7.8.2. Gamma (Area).

The parameter x_c has been added to enable variable x positioning, whereas the $+\theta(k - 1)$ adjusts x_c so that it represents the mode. c_0 is the offset value. The function returns the offset c_0 for those x where it is undefined

$$y(x) = \begin{cases} c_0 + \frac{a}{\theta\Gamma(k)} \exp(-z) z^{k-1} & \text{for } z > 0 \\ c_0 & \text{otherwise} \end{cases} \quad (7.34)$$

with $z = \frac{x-x_c}{\theta} + k - 1$

Required parameters:

- area:** area a of the Gamma peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\theta > 0$
- backgr:** offset c_0

Note

- The shape parameter needs to be larger than one $k > 1$.
- The scale parameter needs to be larger than zero $\theta > 0$
- Default (Size) distribution: Monodisperse

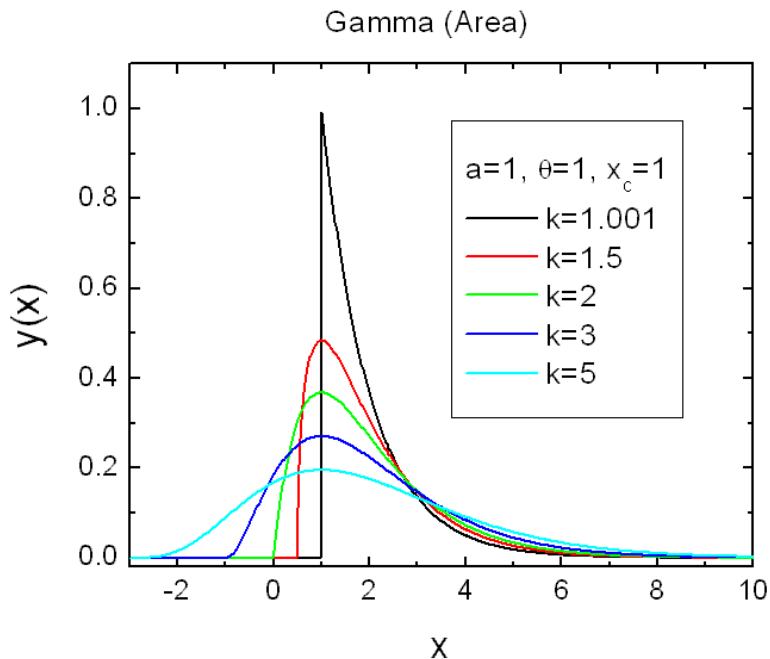


FIGURE 7.15. Plot of Gamma (Area) distribution.

7.9. Gaussian or Normal distribution

7.9.1. Gaussian (Amplitude).

$$y(x; a, x_c, \sigma, c_0) = a \exp \left[-\frac{1}{2} \left(\frac{x - x_c}{|\sigma|} \right)^2 \right] + c_0 \quad (7.35)$$

Required parameters:

- ampl.:** amplitude a of the peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma \neq 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- Default (size) distribution: Monodisperse

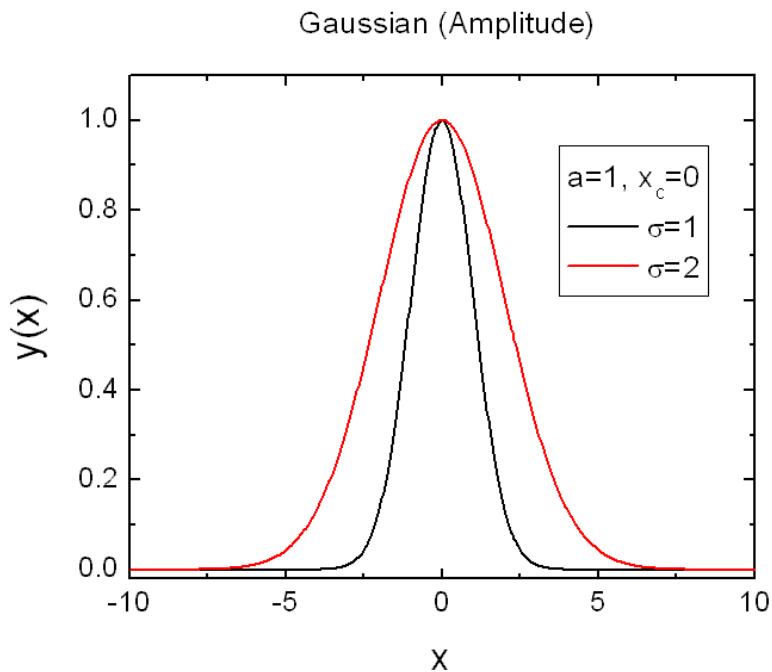


FIGURE 7.16. Plot of Gaussian (Amplitude) distribution.

7.9.2. Gaussian (Area).

$$y(x; a, x_c, \sigma, c_0) = \frac{a}{|\sigma| \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x - x_c}{|\sigma|} \right)^2 \right] + c_0 \quad (7.36)$$

Required parameters:

- area:** area a below the peak
- center:** location parameter (mode) x_c
- width:** scaling parameter $\sigma \neq 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- Default (size) distribution: Monodisperse

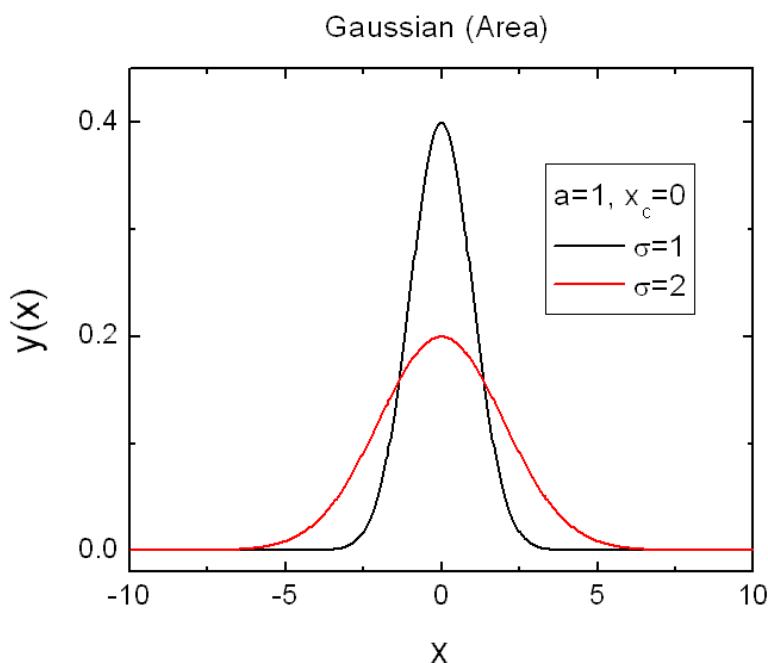


FIGURE 7.17. Plot of Gaussian (Area) distribution.

7.10. Gaussian-Lorentzian cross product

This distribution function is a Voigt approximation. It combines a Gaussian and Lorentzian in a multiplicative form. The shape parameter ν varies from 0 to 1. The pure Lorentzian occurs with $\nu = 1$ and the pure Gaussian with $\nu = 0$ but the transition from Lorentzian to Gaussian shape is not a linear function of ν .

7.10.1. Gaussian-Lorentzian cross product (Amplitude).

$$y(x, a, x_c, \sigma, \nu, c_0) = a \frac{\exp\left(-\frac{1-\nu}{2} \left(\frac{x-x_c}{|\sigma|}\right)^2\right)}{1 + \nu \left(\frac{x-x_c}{|\sigma|}\right)^2} + c_0 \quad (7.37)$$

Required parameters:

amplitude: amplitude a of the peak

center: location parameter (mode) x_c

shape: shape parameter $\nu \in [0, 1]$

width: scaling parameter $\sigma \neq 0$

backgr: offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- The shape parameter need to be between 0 and 1 ($\nu \in [0, 1]$)
- Default (size) distribution: Monodisperse

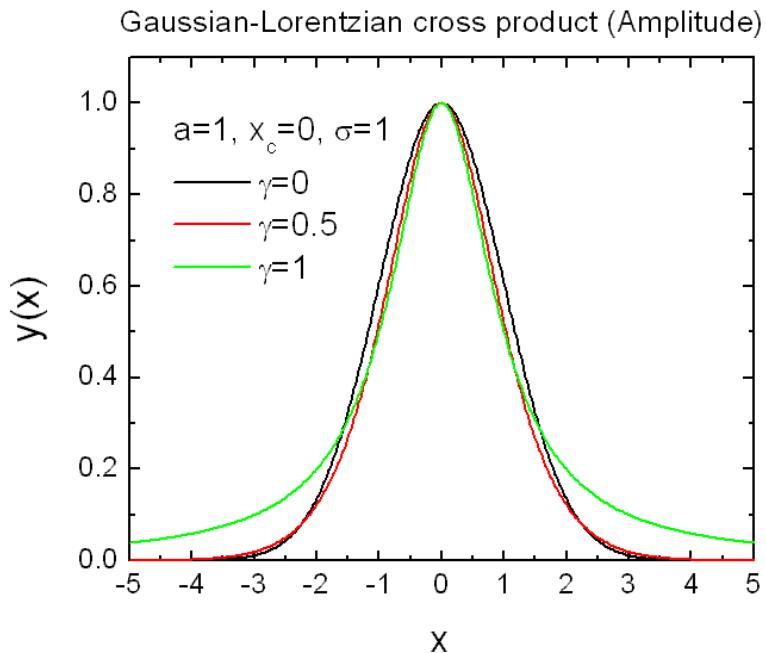


FIGURE 7.18. Plot of Gaussian-Lorentzian cross product (Amplitude) distribution.

7.10.2. Gaussian-Lorentzian cross product (Area).

$$y(x, a, x_c, \sigma, \nu, c_0) = a \frac{\sqrt{\nu}}{|\sigma|\pi} \frac{\exp\left(-\frac{1-\nu}{2\nu}\right)}{\operatorname{erfc}\left(\sqrt{\frac{1-\nu}{2\nu}}\right)} \frac{\exp\left(-\frac{1-\nu}{2}\left(\frac{x-x_c}{|\sigma|}\right)^2\right)}{1 + \nu\left(\frac{x-x_c}{|\sigma|}\right)^2} + c_0 \quad (7.38)$$

Required parameters:

- area:** area a below the peak
- center:** location parameter (mode) x_c
- shape:** shape parameter $\nu \in [0, 1]$
- width:** scaling parameter $\sigma \neq 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- The shape parameter need to be between 0 and 1 ($\nu \in [0, 1]$)
- Default (size) distribution: Monodisperse

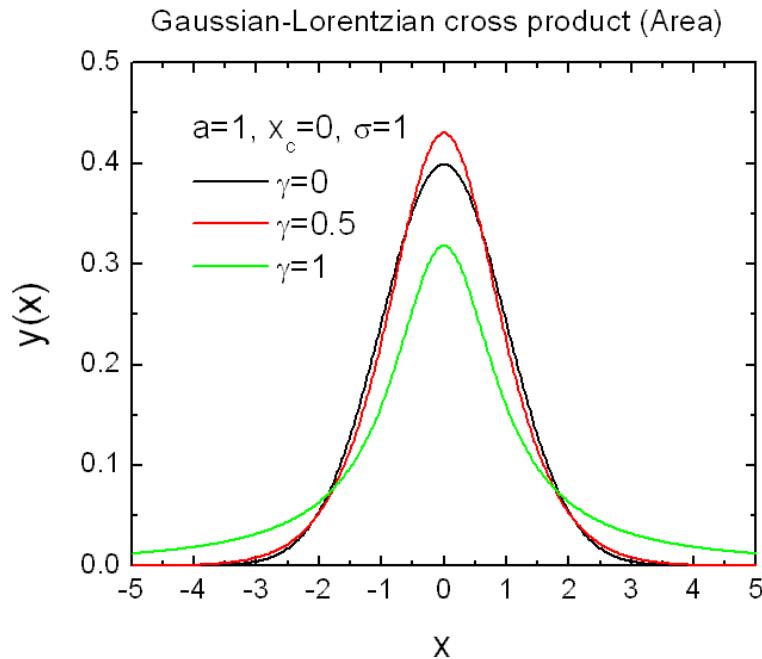


FIGURE 7.19. Plot of Gaussian-Lorentzian cross product (Area) distribution.

7.11. Gaussian-Lorentzian sum

This distribution function is another Voigt approximation, which is simply a sum of Lorentzian and Gaussian with equal FWHM. The shape parameter ν varies from 0 to 1. The pure Lorentzian occurs with $\nu = 1$ and the pure Gaussian with $\nu = 0$. The width parameter σ directly computes the full-width at half-maximum (FWHM).

7.11.1. Gaussian-Lorentzian sum (Amplitude).

$$y(x, a, x_c, \sigma, \nu, c_0) = a \frac{\frac{\nu}{|\sigma|} \sqrt{\frac{\ln 2}{\pi}} \exp \left(-4 \ln 2 \left(\frac{x-x_c}{|\sigma|} \right)^2 \right) + \frac{1-\nu}{\pi |\sigma| \left[1+4 \left(\frac{x-x_c}{|\sigma|} \right)^2 \right]} + c_0}{\frac{\nu}{|\sigma|} \sqrt{\frac{\ln 2}{\pi}} + \frac{1-\nu}{\pi |\sigma|}} \quad (7.39)$$

Required parameters:

amplitude: amplitude a of the peak

center: location parameter (mode) x_c

shape: shape parameter $\nu \in [0, 1]$

width: scaling parameter $\sigma \neq 0$

backgr: offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- The shape parameter need to be between 0 and 1 ($\nu \in [0, 1]$)
- Default (size) distribution: Monodisperse

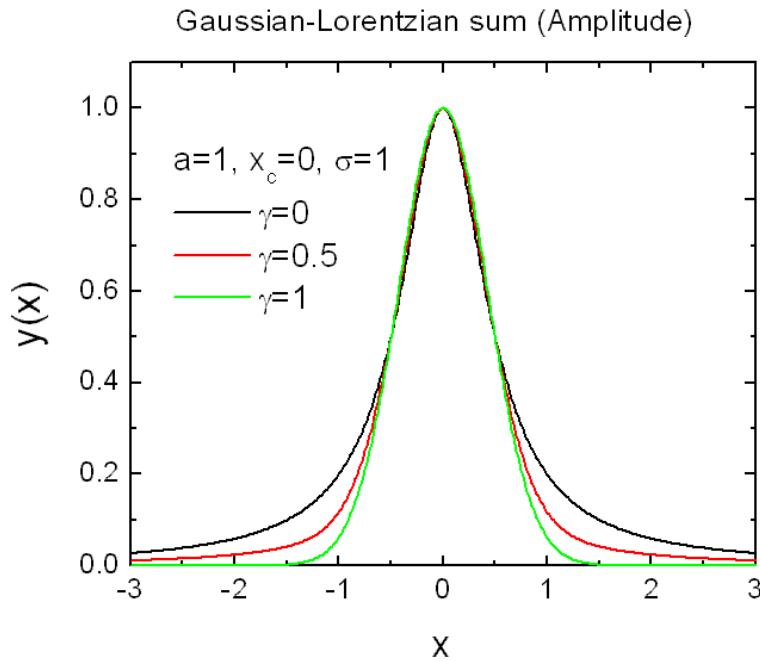


FIGURE 7.20. Plot of Gaussian-Lorentzian sum (Amplitude) distribution.

7.11.2. Gaussian-Lorentzian sum (Area).

$$y(x, a, x_c, \sigma, \nu, c_0) = 2a \left[\frac{\nu}{|\sigma|} \sqrt{\frac{\ln 2}{\pi}} \exp \left(-4 \ln 2 \left(\frac{x - x_c}{|\sigma|} \right)^2 \right) + \frac{1 - \nu}{\pi |\sigma| \left[1 + 4 \left(\frac{x - x_c}{|\sigma|} \right)^2 \right]} \right] + c_0 \quad (7.40)$$

Required parameters:

- area:** area a below the peak
- center:** location parameter (mode) x_c
- shape:** shape parameter $\nu \in [0, 1]$
- width:** scaling parameter $\sigma \neq 0$
- backgr:** offset c_0

Note

- The width parameter needs to be non-zero ($\sigma \neq 0$).
- The shape parameter need to be between 0 and 1 ($\nu \in [0, 1]$)
- Default (size) distribution: Monodisperse

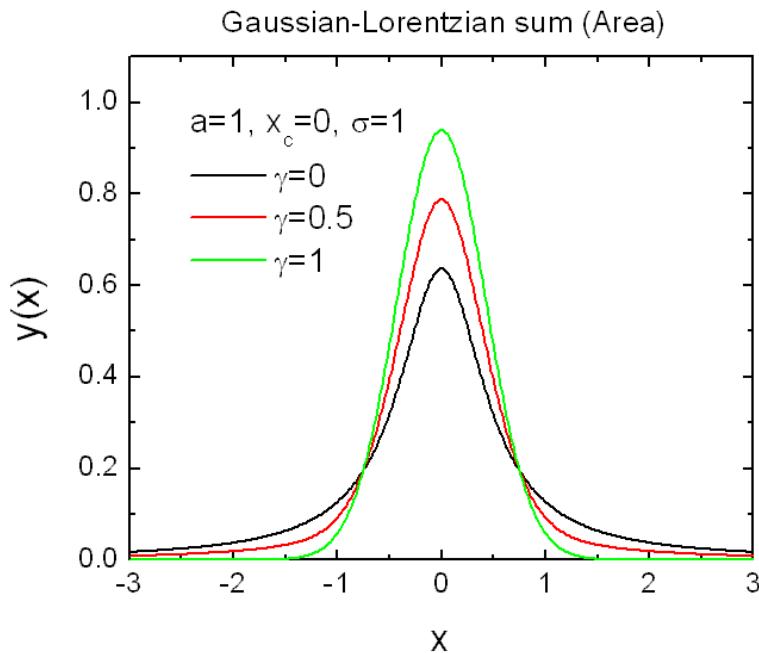


FIGURE 7.21. Plot of Gaussian-Lorentzian sum (Area) distribution.

7.12. generalized Gaussian 1

The generalized Gaussian distribution is one of two families of continuous probability distributions, which has an additional shape parameter to the normal distribution. Known also as the exponential power distribution, or the generalized error distribution, this is a parametric family of symmetric distributions. It includes all normal and Laplace distributions, and as limiting cases it includes all continuous uniform distributions on bounded intervals of the real line.

This family includes the normal distribution when $\beta = 2$ (with mean μ and variance $\frac{\alpha^2}{2}$) and it includes the Laplace distribution when $\beta = 1$. As $\beta \rightarrow \infty$, the density converges pointwise to a uniform density on $(\mu - \alpha, \mu + \alpha)$.

7.12.1. generalized Gaussian 1 (Amplitude).

$$y(x, a, \mu, \alpha, \beta) = ae^{-\left|\frac{x-\mu}{\alpha}\right|^\beta} \quad (7.41)$$

Required parameters:

amplitude: amplitude a of the peak

center: location parameter (mode) μ

width: scaling parameter $\alpha \neq 0$

shape: shape parameter β

backgr: offset c_0

Note

- The width parameter needs to be non-zero ($\alpha \neq 0$).
- The area parameter needs to be positive ($\beta > 0$).
- Default (size) distribution: Monodisperse

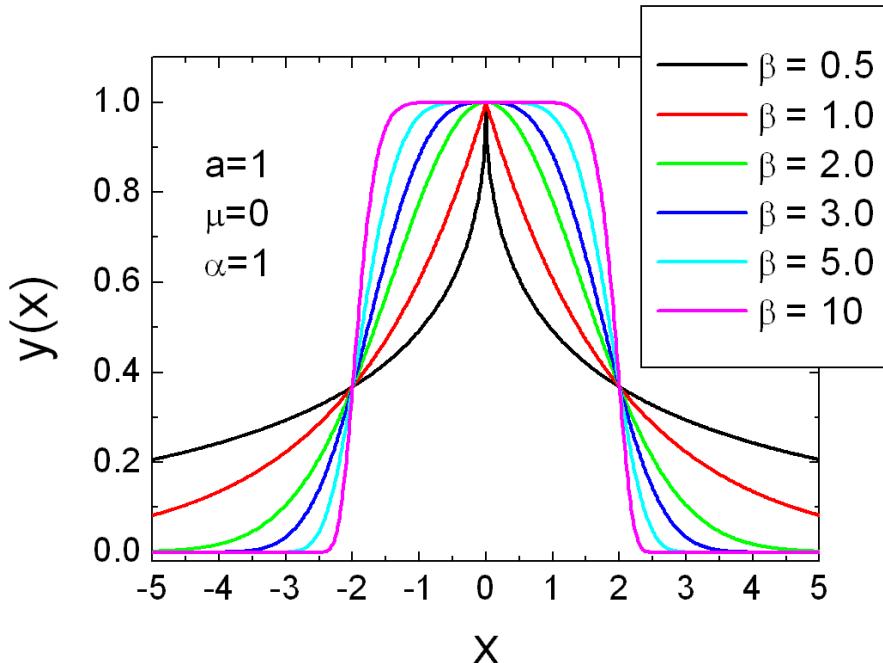


FIGURE 7.22. Plot of generalized Gaussian 1 (Amplitude) distribution.

7.12.2. generalized Gaussian 1 (Area).

$$y(x, a, \mu, \alpha, \beta) = a \frac{\beta}{2\alpha\Gamma(1/\beta)} e^{-|\frac{x-\mu}{\alpha}|^\beta} \quad (7.42)$$

Required parameters:

- area:** area a below the peak
- center:** location parameter (mode) μ
- width:** scaling parameter $\alpha \neq 0$
- shape:** shape parameter β
- backgr:** offset c_0

Note

- The scaling parameter needs to be positive ($\alpha > 0$).
- Default (size) distribution: Monodisperse

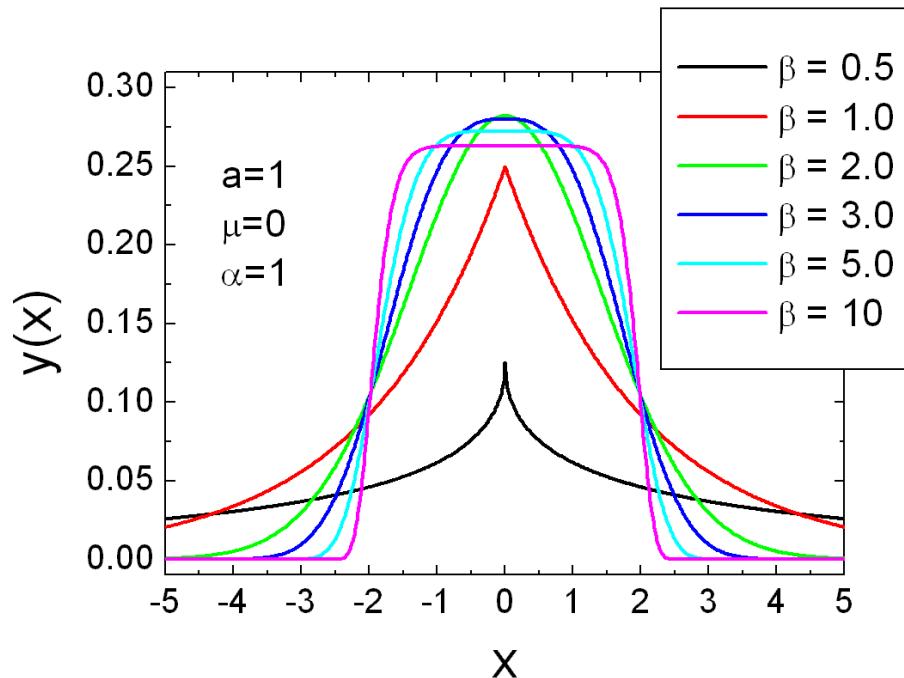


FIGURE 7.23. Plot of generalized Gaussian 1 (Area) distribution.

7.13. generalized Gaussian 2

$$x_{\text{mode}} = \begin{cases} \frac{\alpha}{\kappa} - \frac{\alpha}{\kappa} e^{-\kappa^2/2} + \xi & \text{if } \kappa \neq 0 \\ \xi & \text{if } \kappa = 0 \end{cases} \quad (7.43)$$

7.13.1. generalized Gaussian 2 (Amplitude).

$$y(x) = A \alpha \sqrt{2\pi} \exp\left(-\frac{\kappa^2}{2}\right) \frac{\phi(u)}{\alpha - \kappa(x - \xi)} + c_0 \quad (7.44)$$

where

$$u = \begin{cases} -\frac{1}{\kappa} \log \left[1 - \frac{\kappa(x-\xi)}{\alpha} \right] & \text{if } \kappa \neq 0 \\ \frac{x-\xi}{\alpha} & \text{if } \kappa = 0 \end{cases} \quad (7.45)$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (7.46)$$

Required parameters:

amplitude: amplitude A of the peak

location: location parameter ξ

width: scaling parameter $\alpha > 0$

shape: shape parameter κ

backgr: offset c_0

Note

- The scaling parameter needs to be positive ($\alpha > 0$).
- Default (size) distribution: Monodisperse

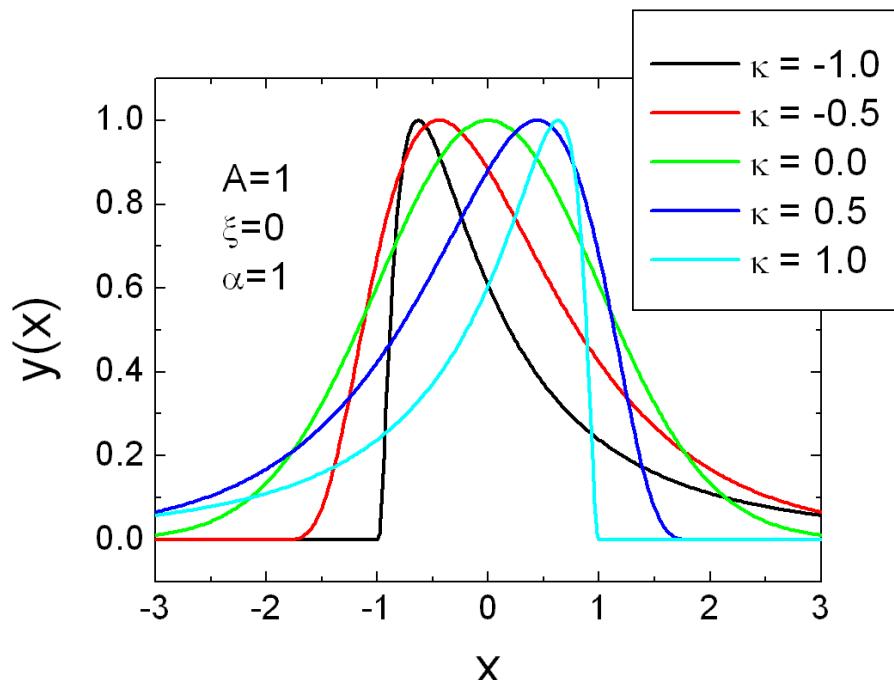


FIGURE 7.24. Plot of generalized Gaussian 2(Amplitude) distribution.

7.13.2. generalized Gaussian 2 (Area).

$$y(x) = A \frac{\phi(u)}{\alpha - \kappa(x - \xi)} + c_0 \quad (7.47a)$$

where

$$u = \begin{cases} -\frac{1}{\kappa} \log \left[1 - \frac{\kappa(x-\xi)}{\alpha} \right] & \text{if } \kappa \neq 0 \\ \frac{x-\xi}{\alpha} & \text{if } \kappa = 0 \end{cases} \quad (7.47b)$$

and

$$\phi(u) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{u^2}{2} \right) \quad (7.47c)$$

Required parameters:

area: area A below the peak

location: location parameter ξ

width: scaling parameter $\alpha > 0$

shape: shape parameter κ

backgr: offset c_0

Note

- The scaling parameter needs to be positive ($\alpha > 0$).
- Default (size) distribution: Monodisperse

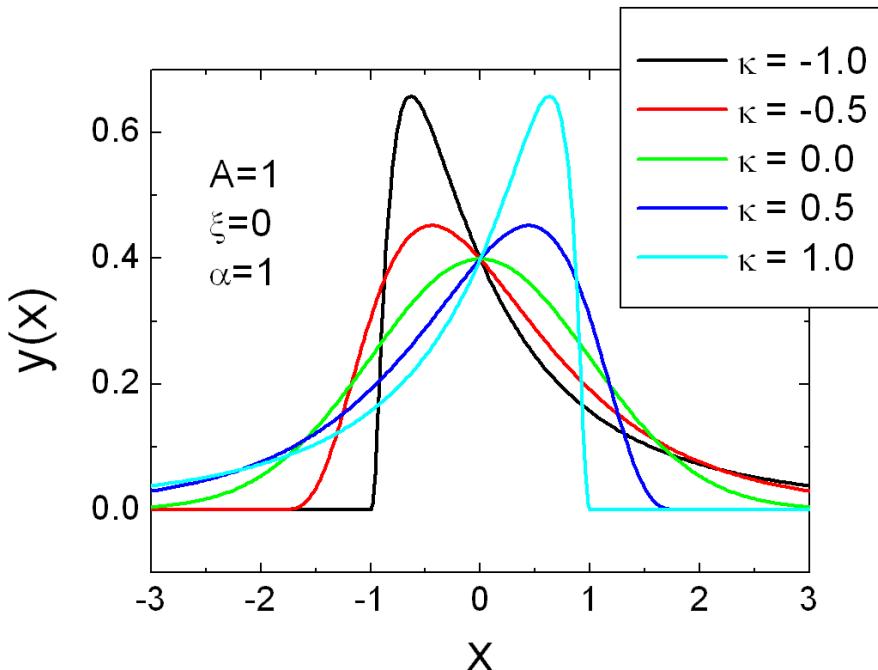


FIGURE 7.25. Plot of generalized Gaussian 2 (Area) distribution.

7.14. Giddings

The Giddings equation was derived by J. C. Giddings (*Dynamics of Chromatography, Part I*, Marcel Decker, New York, 1965). The equation provides a theoretical description for chromatographic peaks. The used formulae have been taken from the Manual of the Peakfit Software package (SeaSolve Software Inc.), which contains an expression for the for the **Giddings (Area)**. As the mode x_{mode} of the peak can not be calculated analytical the amplitude version of this peaks **Giddings (Amplitude)** calculates first numerically the mode of the peak and than normalizes the value at x_{mode} to be $y(x_{\text{mode}}) = A$.

7.14.1. Giddings (Amplitude).

As the mode x_{mode} of this peak can not be calculated analytical this version of the Giddings peak calculates first numerically the mode of the peak and than normalizes the value at x_{mode} to be $y(x_{\text{mode}}) = A$.

$$y(x) = \frac{A}{c} \sqrt{\frac{\beta}{x}} I_1 \left(\frac{2\sqrt{\beta x}}{\sigma} \right) \exp \left(-\frac{x + \beta}{\sigma} \right) \quad (7.48a)$$

with

$$c = \sqrt{\frac{\beta}{x_{\text{mode}}}} I_1 \left(\frac{2\sqrt{\beta x_{\text{mode}}}}{\sigma} \right) \exp \left(-\frac{x_{\text{mode}} + \beta}{\sigma} \right) \quad (7.48b)$$

Required parameters:

amplitude: amplitude A of the peak

location: location parameter β

width: scaling parameter $\sigma > 0$

backgr: offset c_0

Note

- The scaling parameter needs to be positive ($\sigma > 0$).
- Default (size) distribution: Monodisperse

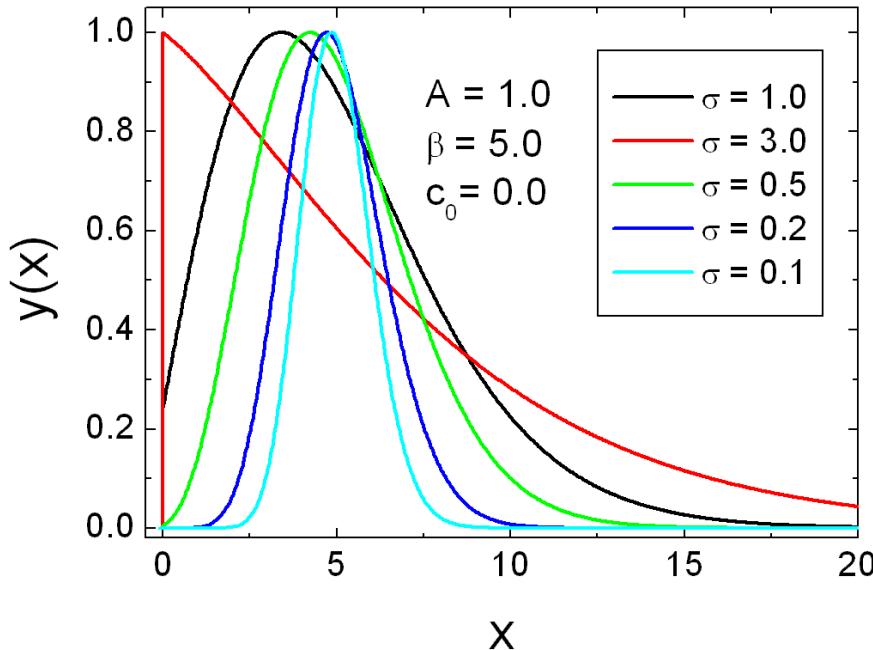


FIGURE 7.26. Plot of Giddings (Amplitude) distribution.

7.14.2. Giddings (Area).

$$y(x) = \frac{A}{1 - \exp(-\frac{\beta}{\sigma})} \frac{1}{\sigma} \sqrt{\frac{\beta}{x}} I_1 \left(\frac{2\sqrt{\beta x}}{\sigma} \right) \exp \left(-\frac{x + \beta}{\sigma} \right) \quad (7.49)$$

Required parameters:

area: area A below the peak

location: location parameter β

width: scaling parameter $\sigma > 0$

backgr: offset c_0

Note

- The scaling parameter needs to be positive ($\sigma > 0$).
- Default (size) distribution: Monodisperse

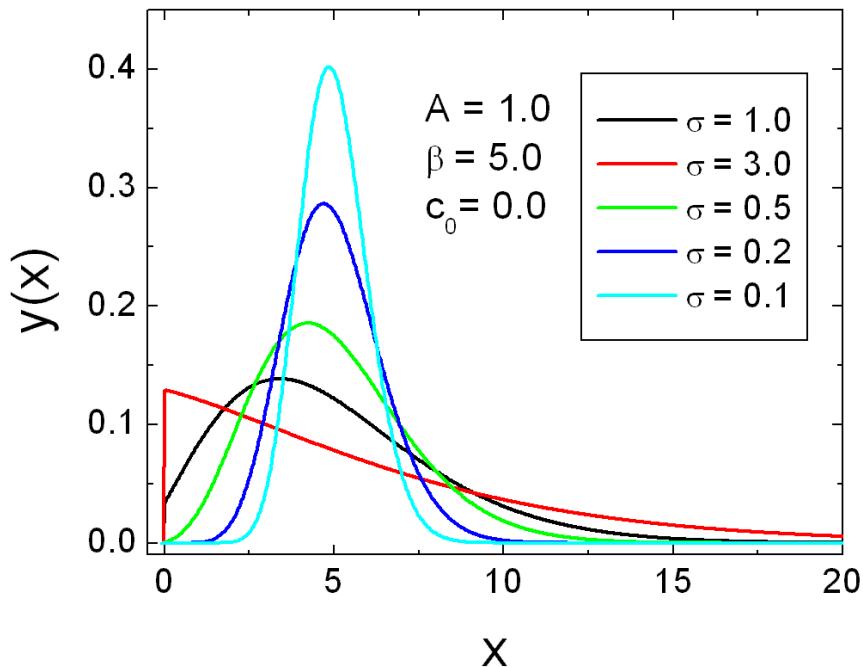


FIGURE 7.27. Plot of Giddings (Area) distribution.

7.15. Haarhoff - Van der Linde (Area)

$$\text{HVL}(x) = \frac{\frac{A\sigma}{\mu\delta\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]}{\left(\exp\left(\frac{\mu\delta}{\sigma^2}\right) - 1\right)^{-1} + \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]} \quad (7.50)$$

for $\delta \rightarrow 0$ one get the Gaussian distribution as a limiting case

$$\lim_{\delta \rightarrow 0} \text{HVL}(x) = \frac{A}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad (7.51)$$

as well as for $\mu \rightarrow 0$

$$\lim_{\mu \rightarrow 0} \text{HVL}(x) = \frac{A}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right] \quad (7.52)$$

Required parameters:

area: area A below the peak

width: scaling parameter $\sigma > 0$

backgr: offset c_0

Note

- The location parameter needs to be positive ($\mu > 0$).
- The scaling parameter needs to be positive ($\sigma > 0$).
- The distortion parameter needs to be nonzero ($\delta \neq 0$).
- Default (size) distribution: Monodisperse

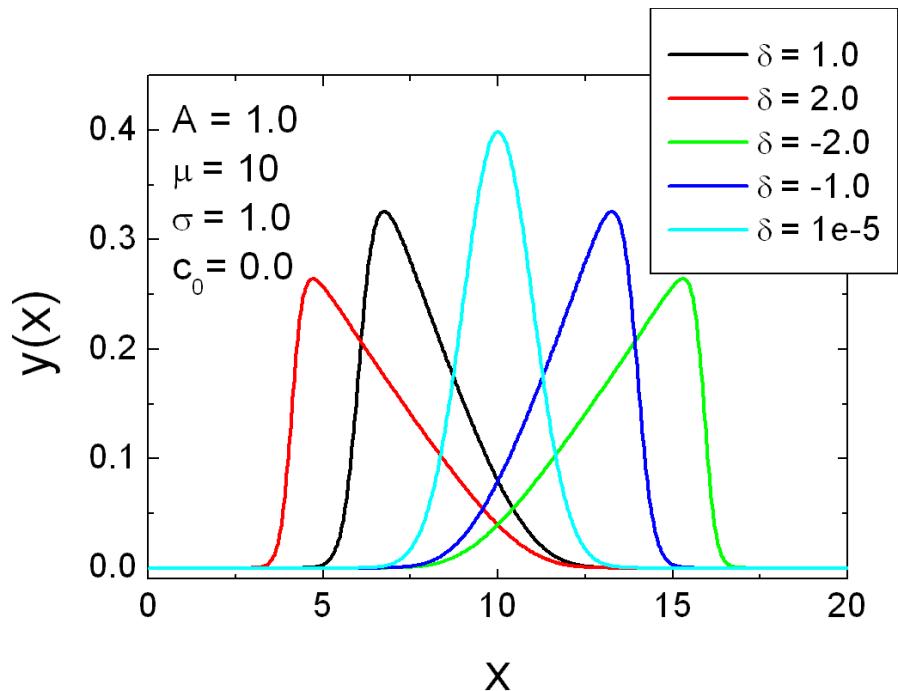


FIGURE 7.28. Plot of HaarhoffVanderLinde (Area) distribution.

7.16. Half Gaussian Modified Gaussian (Area)

The Half Gaussian Modified Gaussian (Area) is the mathematical convolution of a Gaussian with a half-Gaussian response function. There are only two components to this model, a primary Gaussian, and a response function which convolves or smears the Gaussian as in the profiles above. As the width of the half-Gaussian response increases, peaks become more asymmetric or tailed. This function directly fit both tailed and fronted peaks. The transition from tailed to smooth is continuous and occurs at $\delta = 0$. The formula has been taken from the Manual of the Peakfit Software package (SeaSolve Software Inc.).

$$y(x) = A \frac{\exp\left(-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2 + \delta^2}\right) \left[1 + \operatorname{erf}\left(\frac{\delta(x-\mu)}{\sqrt{2}\sigma\sqrt{\sigma^2 + \delta^2}}\right)\right]}{\sqrt{2\pi} \sqrt{\sigma^2 + \delta^2}} \quad (7.53)$$

Required parameters:

- area:** area A below the peak
- location:** location parameter μ
- width:** scaling parameter $\sigma > 0$
- distortion:** distortion parameter $\delta \neq 0$
- backgr:** offset c_0

Note

- The scaling parameter needs to be positive ($\sigma > 0$).
- The distortion parameter needs to be nonzero ($\delta \neq 0$).
- Default (size) distribution: Monodisperse

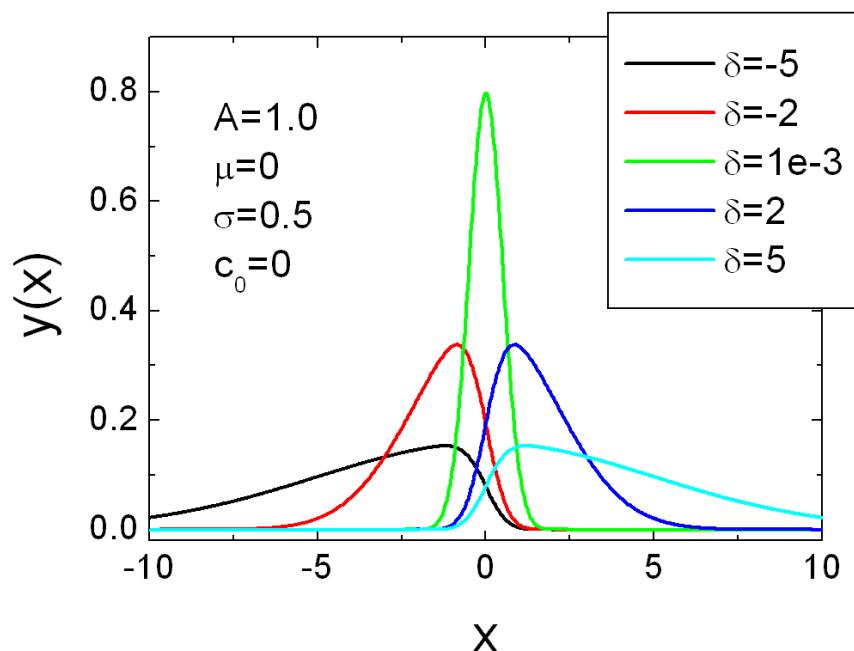


FIGURE 7.29. Plot of Half Gaussian Modified Gaussian (Area) distribution.

7.17. Inverted Gamma

The inverse gamma distribution is a two-parameter family of continuous probability distributions on the positive real line, which is the distribution of the reciprocal of a variable distributed according to the gamma distribution. The inverse gamma distribution's probability density function is defined over the support $x > 0$. The probability density function is given by

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(\frac{-\beta}{x}\right) \quad (7.54)$$

the mode x_{mode} of the probability function is given by $x_{\text{mode}} = \frac{\beta}{\alpha+1}$. The shape parameter α needs to be positive and non-zero as well as the scale parameter β ($\alpha > 0$, $\beta > 0$). The **SASfit** version represents a reparametrization of the standard statistical form. The parameter μ has been added to enable variable x positioning. Adjustment terms have been added so that μ is the mode x_{mode} . The function returns c_0 for those x where it is undefined. Note that the amplitude form is much faster.

7.17.1. Inverted Gamma (Amplitude).

$$y(x) = A \frac{\beta \exp\left(\frac{(x-\mu)(\alpha+1)^2}{x(\alpha+1)+\beta-\mu(\alpha+1)}\right) \left(\frac{x(\alpha+1)-\mu(\alpha+1)}{\beta} + 1\right)^{-\alpha}}{x(\alpha+1) + \beta - \mu(\alpha+1)} + c_0 \quad (7.55)$$

Required parameters:

amplitude: amplitude A of the peak

location: location parameter μ

width: scaling parameter $\beta > 0$

shape: shape parameter $\alpha > 0$

backgr: offset c_0

Note

- The scaling parameter needs to be positive ($\beta > 0$).
- The shape parameter needs to be positive ($\alpha > 0$).
- Default (size) distribution: Monodisperse

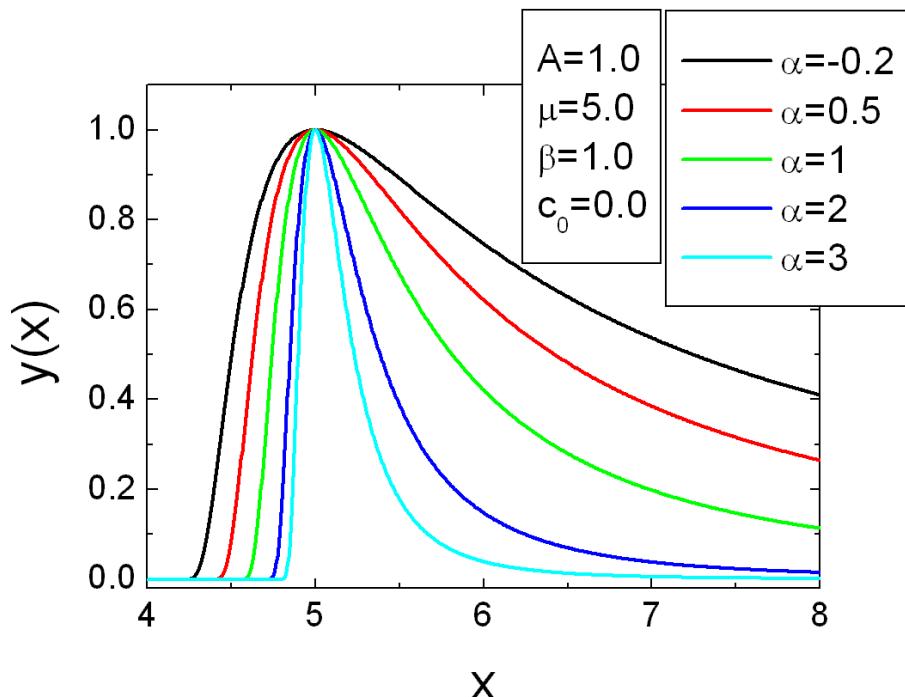


FIGURE 7.30. Plot of inverted Gamma (Amplitude) distribution.

7.17.2. Inverted Gamma (Area).

$$y(x) = A \frac{(\alpha + 1) \exp\left(\frac{\beta(\alpha+1)}{x(\alpha+1)+\beta-\mu(\alpha+1)}\right) \left(\frac{\beta(\alpha+1)}{x(\alpha+1)+\beta-\mu(\alpha+1)}\right)^\alpha}{(x(\alpha + 1) + \beta - \mu(\alpha + 1)) \Gamma(\alpha)} + c_0 \quad (7.56)$$

Required parameters:

area: area A below the peak

location: location parameter μ

width: scaling parameter $\beta > 0$

shape: shape parameter $\alpha > 0$

backgr: offset c_0

Note

- The scaling parameter needs to be positive ($\beta > 0$).
- The shape parameter needs to be positive ($\alpha > 0$).
- Default (size) distribution: Monodisperse

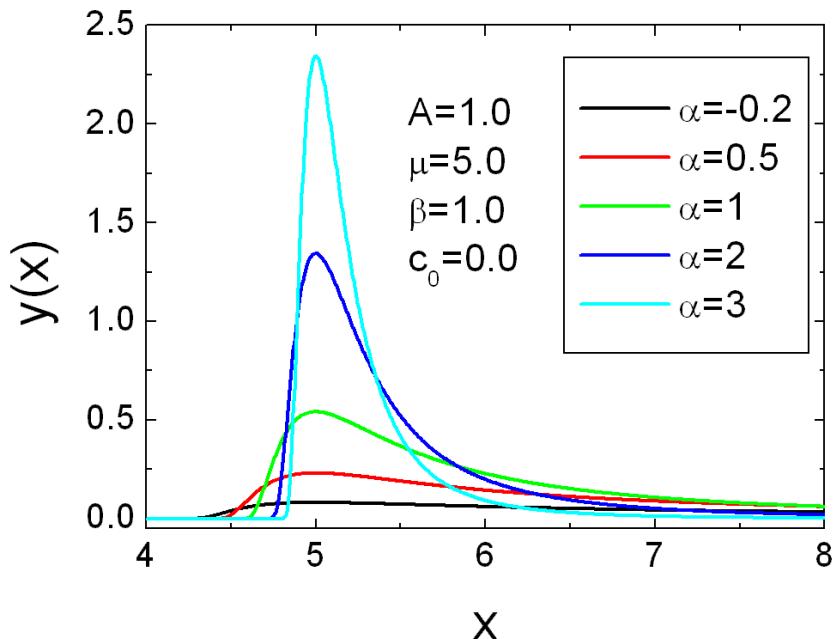


FIGURE 7.31. Plot of inverted Gamma (Area) distribution.

7.18. Kumaraswamy

The Kumaraswamy's double bounded distribution is a family of continuous probability distributions defined on the interval $[0, 1]$ differing in the values of their two non-negative shape parameters, a and b . It is similar to the Beta distribution, but much simpler to use especially in simulation studies due to the simple closed form of both its probability density function and cumulative distribution function. The probability density function of the Kumaraswamy distribution is

$$p(x; \alpha, \beta) = \alpha\beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1}. \quad (7.57)$$

For $\alpha > 1$ and $\beta > 1$ the mode of the distribution reads as

$$x_{\text{mode}} = \left(\frac{\alpha - 1}{\alpha\beta - 1} \right)^{1/\alpha} \quad (7.58)$$

7.18.1. Kumaraswamy (Amplitude).

$$y(x) = \begin{cases} \frac{A\alpha\beta \left(\frac{x+x_{\min}}{x_{\max}-x_{\min}} \right)^{\alpha-1} \left(1 - \left(\frac{x+x_{\min}}{x_{\max}-x_{\min}} \right)^\alpha \right)^{\beta-1}}{\alpha\beta x_{\text{mode}}^{\alpha-1} (1-x_{\text{mode}}^{\alpha})^{\beta-1}} + c_0 & \text{for } x \in [x_{\min}, x_{\max}] \\ c_0 & \text{for } x \notin [x_{\min}, x_{\max}] \end{cases} \quad (7.59)$$

Required parameters:

- ampl.:** amplitude A of the Kumaraswamy peak
- xmin:** continuous lower boundary parameters x_{\min}
- xmax:** continuous upper boundary parameters x_{\max}
- alpha:** first shape parameter $\alpha > 1$
- beta:** second shape parameter $\beta > 1$
- backgr:** offset c_0

Note

- Both shape parameter needs to be larger than one ($\alpha, \beta > 1$), as only than the distribution has a peak shape.
- where the Kumaraswamy distribution is not defined the offset value is returned:
 $\forall x \notin (x_{\min}, x_{\max}) \quad y_{\text{Beta(ampl)}}(x) = c_0$
- Default (size) distribution: Monodisperse

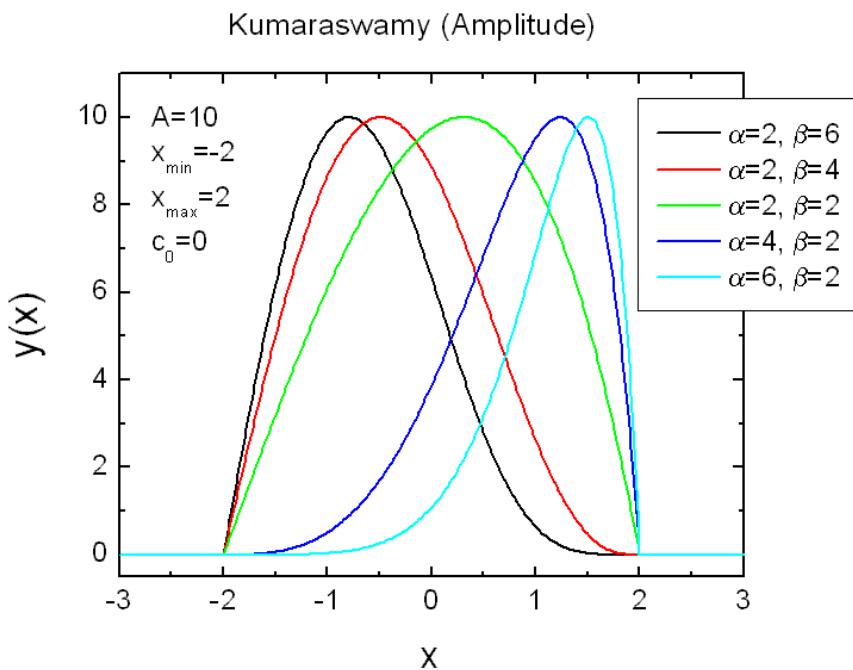


FIGURE 7.32. Plot of Kumaraswamy (Amplitude) distribution.

7.19. Kumaraswamy (Area)

$$y(x) = \begin{cases} \frac{A\alpha\beta\left(\frac{x+x_{\min}}{x_{\max}-x_{\min}}\right)^{\alpha-1}\left(1-\left(\frac{x+x_{\min}}{x_{\max}-x_{\min}}\right)^{\alpha}\right)^{\beta-1}}{x_{\max}-x_{\min}} + c_0 & \text{for } x \in [x_{\min}, x_{\max}] \\ c_0 & \text{for } x \notin [x_{\min}, x_{\max}] \end{cases} \quad (7.60)$$

Required parameters:

- area:** area A of the Kumaraswamy distribution
- xmin:** continuous lower boundary parameters x_{\min}
- xmax:** continuous upper boundary parameters x_{\max}
- alpha:** first shape parameter $\alpha > 0$
- beta:** second shape parameter $\beta > 0$
- backgr:** offset c_0

Note

- Both shape parameter needs to be larger than zero ($\alpha, \beta > 0$)
- where the Kumaraswamy distribution is not defined the offset value is returned:
 $\forall x \notin (x_{\min}, x_{\max}) \quad y_{\text{Beta}(area)}(x) = c_0$
- Default (size) distribution: Monodisperse

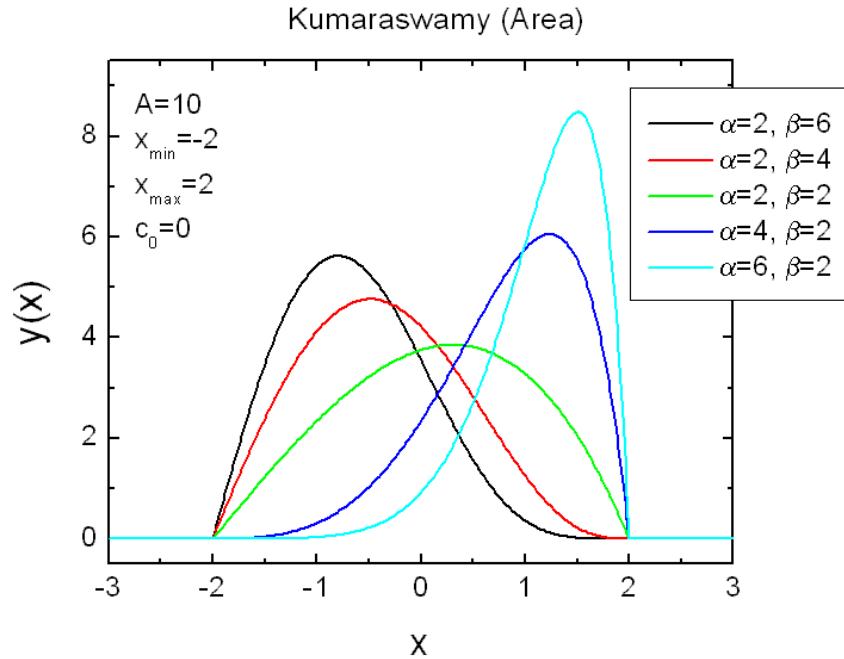


FIGURE 7.33. Plot of Kumaraswamy (Area) distribution.

7.20. Laplace

A random variable has a Laplace distribution if its probability density function is

$$p(x; x_0, \sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x - x_0|}{\sigma}\right) \quad (7.61)$$

Here, x_0 is a location parameter and $\sigma > 0$ is a scale parameter. The Laplace distribution is also sometimes called the double exponential distribution, because it can be thought of as two exponential distributions (with an additional location parameter) spliced together back-to-back, but the term double exponential distribution is also sometimes used to refer to the Gumbel distribution.

7.20.1. Laplace (Amplitude).

$$y(x; x_0, \sigma) = A \exp\left(-\frac{|x - x_0|}{\sigma}\right) + c_0 \quad (7.62)$$

Required parameters:

- amplitude:** amplitude A of the Laplace distribution
- center:** peak center (mode) x_0 of the Laplace distribution
- width:** width parameter $\sigma > 0$
- backgr:** offset c_0

Note

- Width parameter needs to be larger than zero ($\sigma > 0$)
- Default (size) distribution: Monodisperse

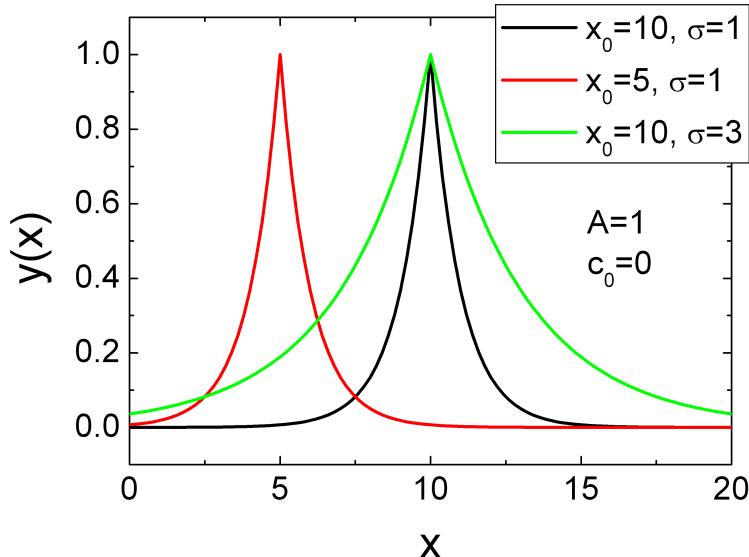


FIGURE 7.34. Plot of Laplace (Amplitude) distribution.

7.20.2. Laplace (Area).

$$y(x; x_0, \sigma) = \frac{A}{2\sigma} \exp\left(-\frac{|x - x_0|}{\sigma}\right) + c_0 \quad (7.63)$$

Required parameters:

area: area A of the Laplace distribution

center: peak center (mode) x_0 of the Laplace distribution

width: width parameter $\sigma > 0$

backgr: offset c_0

Note

- Width parameter needs to be larger than zero ($\sigma > 0$)
- Default (size) distribution: Monodisperse

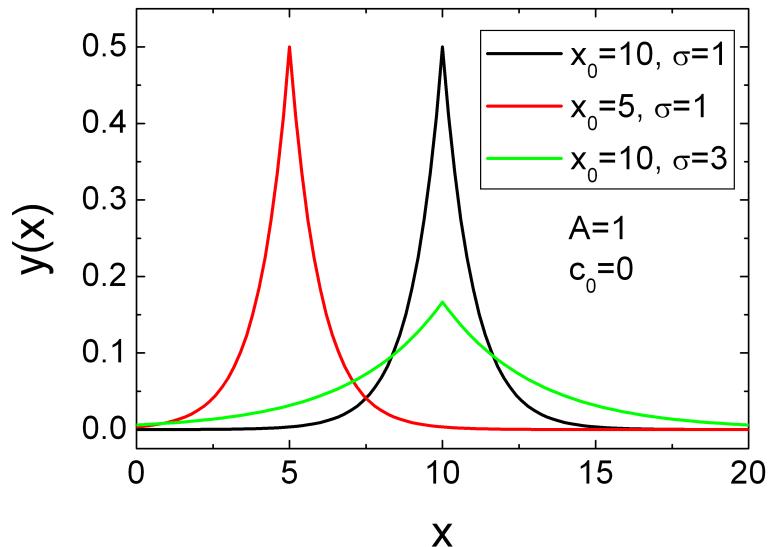


FIGURE 7.35. Plot of Laplace (Area) distribution.

7.21. Logistic

The logistic distribution is a continuous probability distribution. It resembles the normal distribution in shape but has heavier tails (higher kurtosis). The probability density function (pdf) of the logistic distribution is given by:

$$p(x; x_0, \sigma) = \frac{\exp\left(-\frac{x-x_0}{\sigma}\right)}{\sigma \left(1 + \exp\left(-\frac{x-x_0}{\sigma}\right)\right)^2} = \frac{1}{4\sigma} \operatorname{sech}^2\left(\frac{x-x_0}{2\sigma}\right). \quad (7.64)$$

Because the pdf can be expressed in terms of the square of the hyperbolic secant function sech, it is sometimes referred to as the sech-squared distribution. The mode, mean and median values are x_0 .

7.21.1. Logistic (Amplitude).

$$y(x; x_0, \sigma) = 4A \frac{\exp\left(-\frac{x-x_0}{\sigma}\right)}{\left(1 + \exp\left(-\frac{x-x_0}{\sigma}\right)\right)^2} \quad (7.65)$$

Required parameters:

amplitude: amplitude A of the Logistic distribution

x0: location parameter (mode) x_0

sigma: width parameters σ

backgr: offset c_0

Note

- the width parameter needs to be larger than zero $\sigma > 0$
- Default (size) distribution: Monodisperse

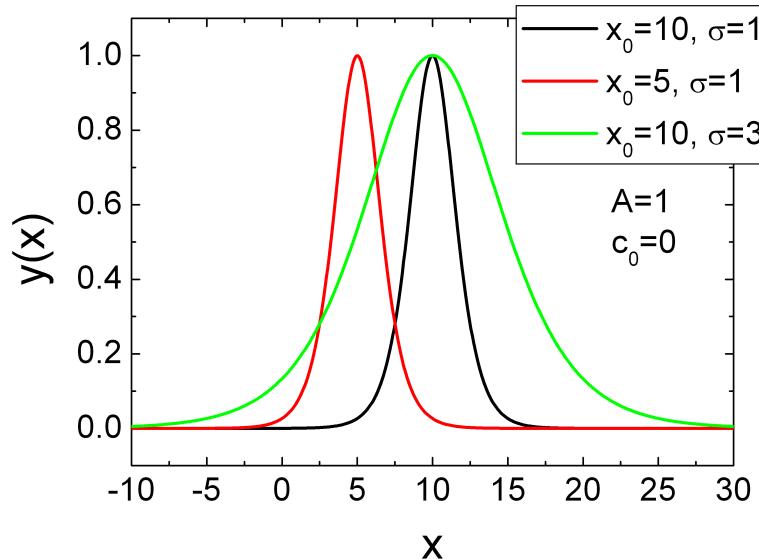


FIGURE 7.36. Plot of Logistic (Amplitude) distribution.

7.21.2. Logistic (Area).

$$y(x; x_0, \sigma) = A \frac{\exp\left(-\frac{x-x_0}{\sigma}\right)}{\sigma \left(1 + \exp\left(-\frac{x-x_0}{\sigma}\right)\right)^2} \quad (7.66)$$

Required parameters:

- area:** area A of the Logistic distribution
- x0:** location parameter (mode) x_0
- sigma:** width parameters σ
- backgr:** offset c_0

Note

- the width parameter needs to be larger than zero $\sigma > 0$
- Default (size) distribution: Monodisperse

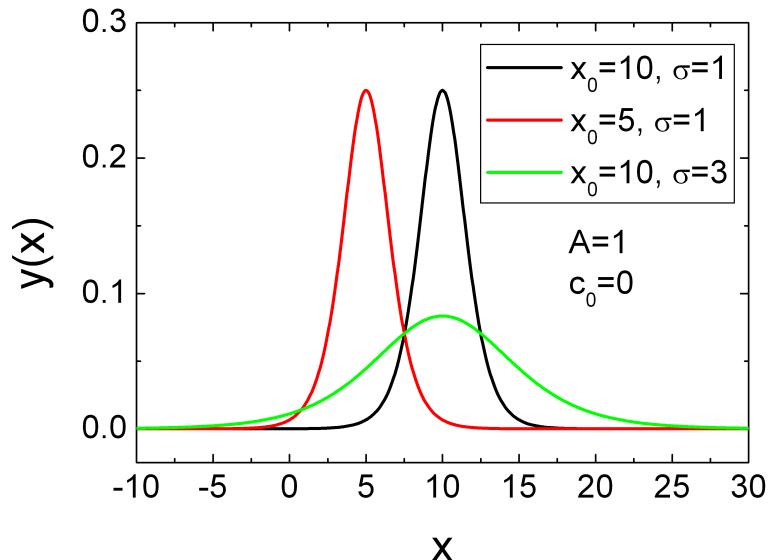


FIGURE 7.37. Plot of Logistic (Area) distribution

7.22. LogLogistic

As may be indicated by the name, the loglogistic (known as the Fisk distribution in economics) distribution has certain similarities to the logistic distribution. A random variable is loglogistically distributed if the logarithm of the random variable is logistically distributed. The **LogLogistic** distribution is a two-parameter distribution with parameters σ and x_0 . It is similar in shape to the log-normal distribution but has heavier tails.

The pdf for this distribution is given by:

$$p(x; \mu, \sigma) = \frac{\exp\left(-\frac{\log(x)-\log(\mu)}{\sigma}\right)}{\sigma(x)\left(1+\exp\left(-\frac{\log(x)-\log(\mu)}{\sigma}\right)\right)^2} = \frac{\left(\frac{x}{\mu}\right)^{-1/\sigma}}{x\sigma\left[1+\left(\frac{x}{\mu}\right)^{-1/\sigma}\right]^2}. \quad (7.67)$$

where $0 < x < \infty$, $-\infty < x_0 < \infty$ and $0 < \sigma < \infty$. The mode of the **LogLogistic** distribution, if $\sigma < 1$, is given by:

$$\text{mode} = \mu \left(\frac{1-\sigma}{1+\sigma}\right)^\sigma \quad (7.68)$$

7.22.1. LogLogistic (Amplitude).

$$y(x) = \begin{cases} A \frac{\left(\frac{x-x_0}{\mu}\right)^{-1/\sigma}}{\left(x-x_0\right)\sigma\left[1+\left(\frac{x-x_0}{\mu}\right)^{-1/\sigma}\right]^2} + c_0 & \text{for } x \geq x_0 \\ c_0 & \text{for } x < x_0 \end{cases} \quad (7.69)$$

Required parameters:

- amplitude:** amplitude A of the LogLogistic distribution
- x0:** location parameter x_0
- mu:** scale parameter μ
- sigma:** shape parameters σ
- backgr:** offset c_0

Note

- the width parameter needs to be larger than zero $0 < \sigma < 1$
- Default (size) distribution: Monodisperse

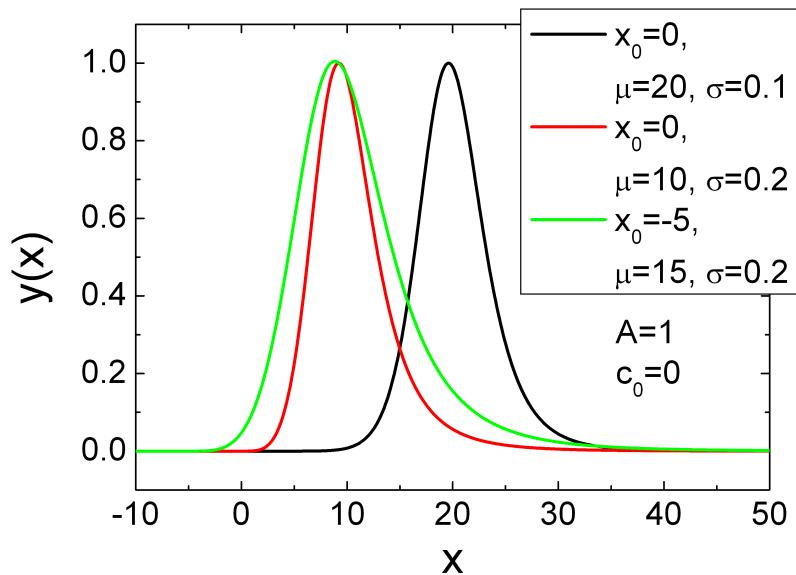


FIGURE 7.38. Plot of LogLogistic (Amplitude) distribution.

7.22.2. LogLogistic (Area).

$$y(x) = A \frac{\left(\frac{x}{x_0}\right)^{-1/\sigma}}{x\sigma \left[1 + \left(\frac{x}{x_0}\right)^{-1/\sigma}\right]^2}. \quad (7.70)$$

Required parameters:

area: area A of the LogLogistic distribution

x0: location parameter x_0

mu: scale parameter μ

sigma: shape parameters $0 < \sigma < 1$

backgr: offset c_0

Note

- the width parameter needs to be larger than zero $\sigma > 0$
- Default (size) distribution: Monodisperse

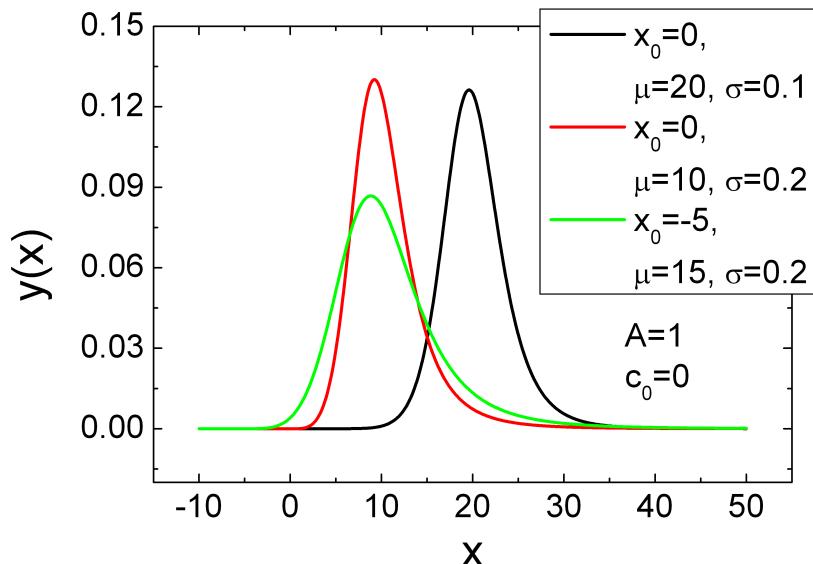


FIGURE 7.39. Plot of LogLogistic (Area) distribution.

7.23. Lognormal 4-Parameter

7.23.1. Lognormal 4-Parameter (Amplitude).

$$y(x) = \begin{cases} c_0 + A \exp \left[-\frac{\ln(2) \ln \left(\frac{(x-x_0)(\gamma^2-1)}{\sigma\gamma} + 1 \right)^2}{\ln(\gamma)} \right] & \text{for } \gamma \neq 1, \gamma > 0 \\ c_0 + A 2^{-4 \left(\frac{x-x_0}{\sigma} \right)^2} & \text{for } \gamma = 1 \end{cases} \quad (7.71)$$

For $\left(x \geq x_0 - \frac{\sigma\gamma}{\gamma^2-1} \wedge \gamma < 1 \right) \vee \left(x \leq x_0 - \frac{\sigma\gamma}{\gamma^2-1} \wedge \gamma > 1 \right)$ the function returns c_0 .

Required parameters:

amplitude: amplitude A of the LogLogistic distribution

x0: location parameter x_0

sigma: width parameter $\sigma > 0$

gamma: shape parameters $\gamma > 0$

backgr: offset c_0

Note

- the width parameter needs to be larger than zero $\sigma > 0$
- the shape parameter needs to be larger than zero $\gamma > 0$
- Default (size) distribution: Monodisperse

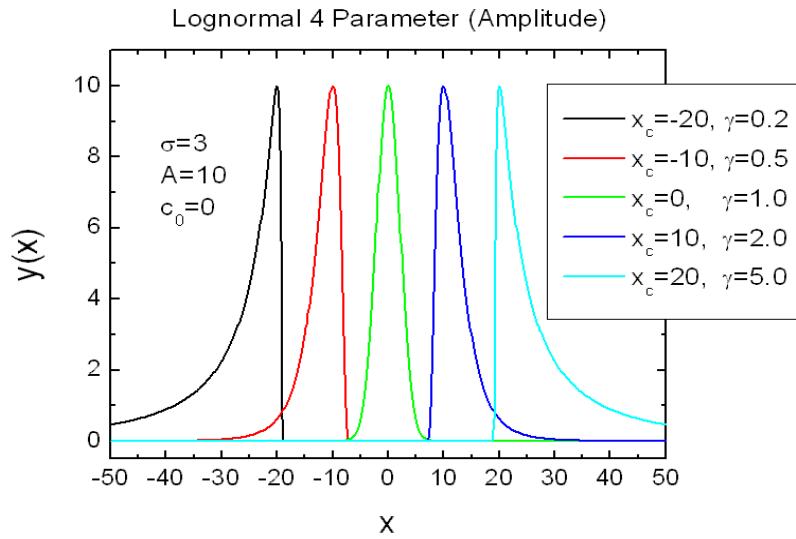


FIGURE 7.40. Plot of Lognormal 4-Parameter (Amplitude) distribution.

7.23.2. Lognormal 4-Parameter (Area).

$$y(x) = \begin{cases} c_0 + A \frac{\sqrt{\ln 2}(\gamma^2 - 1)}{\sigma\gamma \ln(\gamma)\sqrt{\pi} \exp\left(\frac{\ln(\gamma^2)}{4\ln 2}\right)} \exp\left[-\frac{\ln(2) \ln\left(\frac{(x-x_0)(\gamma^2-1)}{\sigma\gamma}+1\right)^2}{\ln(\gamma)}\right] & \text{for } \gamma \neq 1, \gamma > 0 \\ c_0 + A \frac{\sqrt{\ln 2}}{\sigma\sqrt{\pi}} 2^{-4\left(\frac{x-x_0}{\sigma}\right)^2} & \text{for } \gamma = 1 \end{cases} \quad (7.72)$$

For $\left(x \geq x_0 - \frac{\sigma\gamma}{\gamma^2-1} \wedge \gamma < 1\right) \vee \left(x \leq x_0 - \frac{\sigma\gamma}{\gamma^2-1} \wedge \gamma > 1\right)$ the function returns c_0 .

Required parameters:

area: area A of the LogLogistic distribution

x0: location parameter x_0

sigma: width parameter $\sigma > 0$

gamma: shape parameters $\gamma > 0$

backgr: offset c_0

Note

- the width parameter needs to be larger than zero $\sigma > 0$
- the shape parameter needs to be larger than zero $\gamma > 0$
- Default (size) distribution: Monodisperse

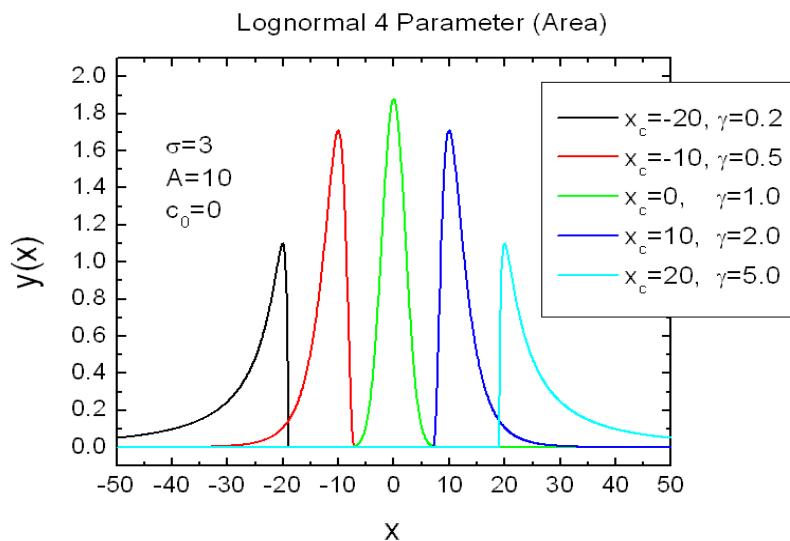


FIGURE 7.41. Plot of Lognormal 4-Parameter (Area) distribution.

7.24. LogNormal

The LogNormal distribution is defined with reference to the normal distribution. A random variable is Lognormally distributed if the logarithm of the random variable is normally distributed.

The LogNormal distribution is commonly used for general reliability analysis, cycles-to-failure in fatigue, material strengths and loading variables in probabilistic design. Another advantage of the LogNormal distribution is that it is positive-definite, so it is often useful for representing quantities that cannot have negative values. LogNormal distributions have proven useful as distributions for rainfall amounts, for the size distributions of aerosol particles or droplets, and for many other cases. The log-normal distribution has the probability density function

$$f(x') = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{x' - \mu'}{\sigma}\right)^2\right) \quad (7.73)$$

where $\mu' = \ln(\mu)$ and $x' = \ln(x)$. The lognormal pdf can be obtained, realizing that for equal probabilities under the normal and lognormal pdfs, incremental areas should also be equal, or:

$$f(x; \mu, \sigma) dx = f(x'; \mu, \sigma) dx' \quad (7.74)$$

Taking the derivative yields:

$$dx' = \frac{dx}{x} \quad (7.75)$$

Substitution yields:

$$f(x; \mu, \sigma) = \frac{f(x'; \mu, \sigma)}{x} \quad (7.76)$$

where:

$$f(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{\ln(x) - \ln(\mu)}{\sigma}\right)^2\right) \quad (7.77)$$

for $x \in (0, \infty]$, where $\mu > 0$ and $\sigma \neq 0$ are the location and scale parameter. The mode of the distribution is

$$\text{mode} = \mu \exp(-\sigma^2) \quad (7.78)$$

7.24.1. LogNormal (Amplitude).

$$y(x) = \begin{cases} \frac{A \exp(-\frac{1}{2}\sigma^2)\mu}{x-x_0} \exp\left(-\frac{1}{2}\left(\frac{\ln(x-x_0)-\ln(\mu)}{\sigma}\right)^2\right) + c_0 & \text{for } x > x_0 \\ c_0 & \text{for } x \leq x_0 \end{cases} \quad (7.79)$$

Required parameters:

amplitude: amplitude A of the LogNormal distribution

mu: location parameter μ

sigma: width parameter $\sigma > 0$

x0: shift parameters x_0

backgr: offset c_0

Note

- the width parameter needs to be larger than zero $\sigma > 0$
- the location parameter needs to be larger than the shift parameter $\mu > x_0$
- Default (size) distribution: Monodisperse

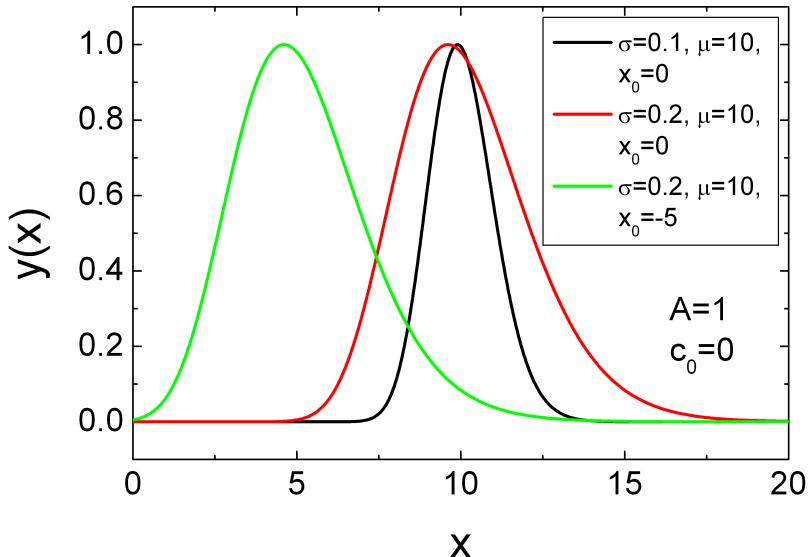


FIGURE 7.42. Plot of LogNormal (Amplitude) distribution.

7.24.2. LogNormal (Area).

$$y(x) = \begin{cases} \frac{A}{(x-x_0)\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x-x_0)-\ln(\mu)}{\sigma}\right)^2\right) + c_0 & \text{for } x > x_0 \\ c_0 & \text{for } x \leq x_0 \end{cases} \quad (7.80)$$

Required parameters:

area: area A of the LogNormal distribution

mu: location parameter μ

sigma: width parameter $\sigma > 0$

x0: shift parameters x_0

backgr: offset c_0

Note

- the width parameter needs to be larger than zero $\sigma > 0$
- the location parameter needs to be larger than the shift parameter $\mu > x_0$
- Default (size) distribution: Monodisperse

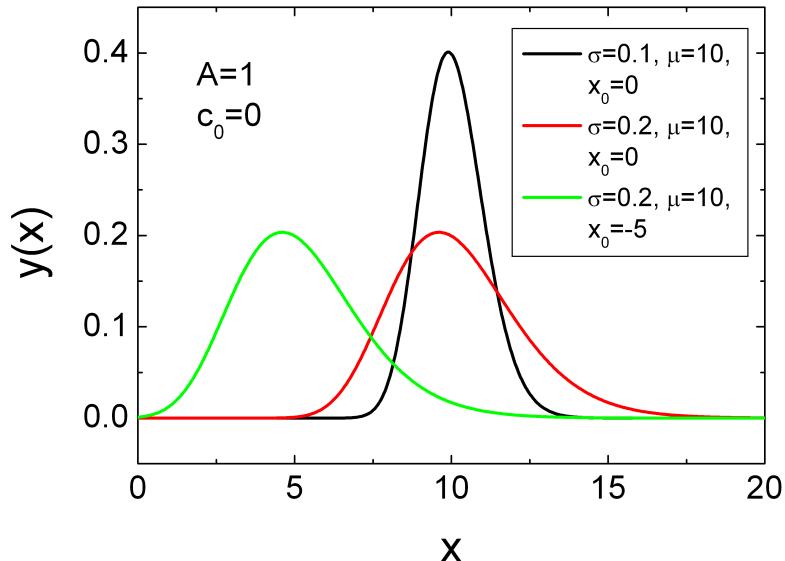


FIGURE 7.43. Plot of LogNormal (Area) distribution.

7.25. Lorentzian or Cauchy distribution

The CauchyLorentz Distribution, named after Augustin Cauchy and Hendrik Lorentz, is a continuous probability distribution. As a probability distribution, it is known as the Cauchy distribution, while among physicists, it is known as a Lorentz distribution, or a Lorentz(ian) function or the BreitWigner distribution. Its importance in physics is due to it being the solution to the differential equation describing forced resonance. The Lorntzian distribution has the probability density function

$$\begin{aligned} f(x; x_0, \sigma) &= \frac{1}{\pi\sigma \left[1 + \left(\frac{x-x_0}{\sigma} \right)^2 \right]} \\ &= \frac{1}{\pi} \left[\frac{\sigma}{(x - x_0)^2 + \sigma^2} \right] \end{aligned} \quad (7.81)$$

where x_0 is the location parameter, specifying the location of the peak of the distribution, and σ is the scale parameter which specifies the half-width at half-maximum (HWHM).

7.25.1. Lorentzian (Amplitude).

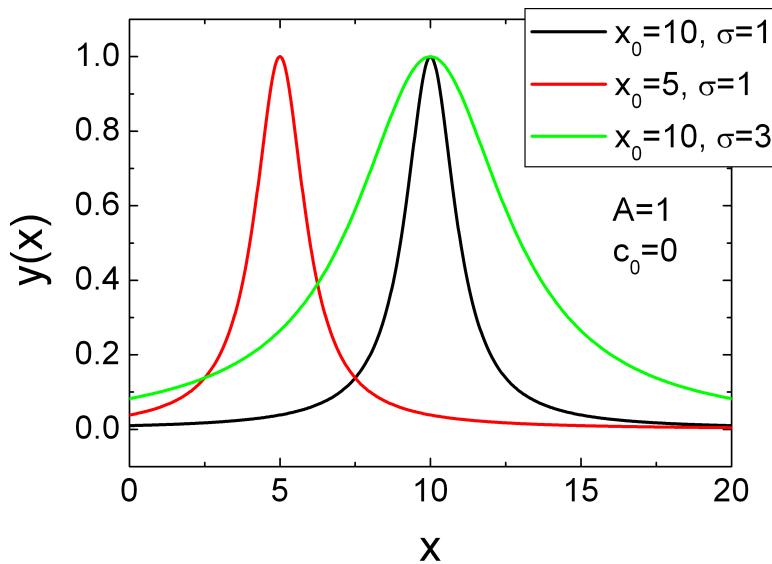


FIGURE 7.44. Plot of Lorentzian (Amplitude) distribution.

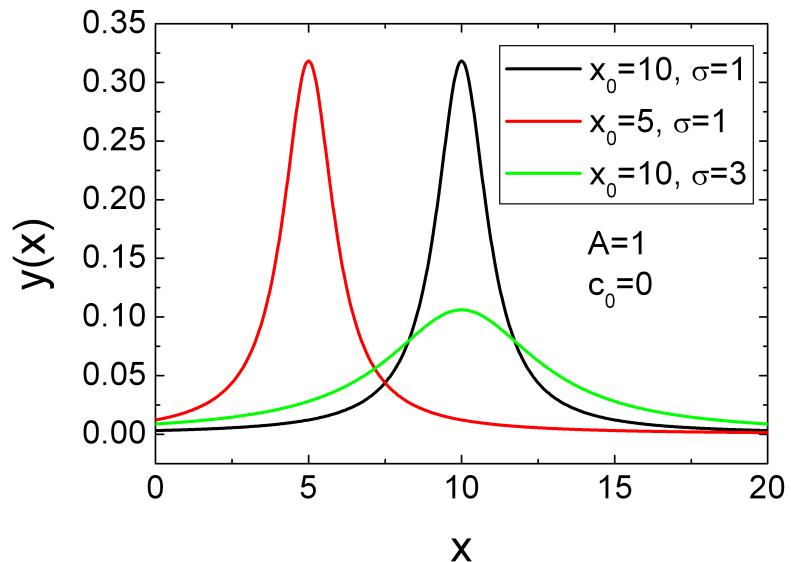
7.25.2. Lorentzian (Area).

FIGURE 7.45. Plot of Lorentzian (Area) distribution.

7.26. Maxwell-Boltzmann distribution

The Maxwell-Boltzmann distribution describes particle speeds in gases, where the particles do not constantly interact with each other but move freely between short collisions. It describes the probability of a particle's speed (the magnitude of its velocity vector) being near a given value as a function of the temperature of the system, the mass of the particle, and that speed value. This probability distribution is named after James Clerk Maxwell and Ludwig Boltzmann.

The Maxwell-Boltzmann distribution is usually thought of as the distribution for molecular speeds, but it can also refer to the distribution for velocities, momenta, and magnitude of the momenta of the molecules, each of which will have a different probability distribution function, all of which are related. Two Maxwell distributions have been implemented, one distribution for speed and a generalized Maxwell distribution, which includes also the energy distribution.

The generalized Maxwell distribution is here defined as

$$p(x; x_0, \sigma, n, m) = \begin{cases} 0 & \text{for } x < x_0 \\ \frac{(x-x_0)^m \exp\left(-\frac{1}{2}\left(\frac{x-x_0}{|\sigma|}\right)^n\right)}{2^{(1+m)/n} |\sigma|^{1+m} \frac{1}{|n|} \Gamma\left(\frac{1+m}{n}\right)} & \text{for } x \geq x_0. \end{cases} \quad (7.82)$$

where x_0 is the location parameter, specifying the location of the peak of the distribution, and σ is the scale parameter which specifies the width. The mode of the distribution is given by

$$x_{\text{mode}} = \left(\frac{2m}{n}\right)^{1/n} |\sigma| + x_0. \quad (7.83)$$

For the case $m = n = 2$ one gets the "Maxwell-Boltzmann distribution" to refers to the distribution of speed. To get the distribution for the energy one has to set $m = 1/2$ and $n = 1$. In case of the width parameter σ always the modulus is used in the calculation of the distribution function to avoid negative values for which the function is not always well defined. For $\sigma = 0$ the distribution function is not defined.

7.26.1. Maxwell (Amplitude).

$$y(x; A, \sigma, x_0, y_0) = \begin{cases} y_0 & \text{for } x < x_0 \\ y_0 + A \frac{(x-x_0)^2}{(x_{\text{mode}}-x_0)^2} \frac{\exp\left(-\frac{1}{2}\left(\frac{x-x_0}{\sigma}\right)^2\right)}{\exp\left(-\frac{1}{2}\left(\frac{x_{\text{mode}}-x_0}{\sigma}\right)^2\right)} & \text{for } x \geq x_0 \end{cases} \quad (7.84)$$

with $x_{\text{mode}} = \sqrt{2}|\sigma| + x_0$.

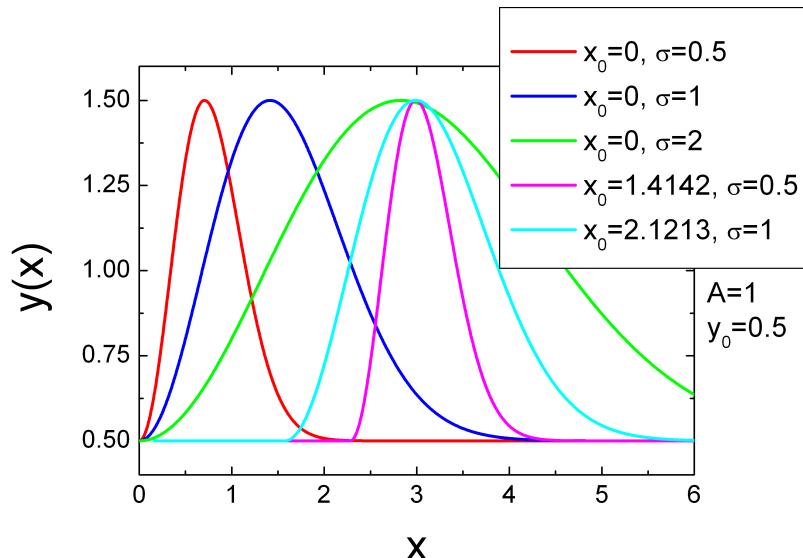


FIGURE 7.46. Plot of Maxwell (Amplitude) distribution.

7.26.2. Maxwell (Area).

$$y(x; A, \sigma, x_0, y_0) = \begin{cases} y_0 & \text{for } x < x_0 \\ y_0 + \sqrt{\frac{2}{\pi}} \frac{A(x-x_0)^2}{\sigma^3} \exp\left(-\frac{1}{2}\left(\frac{x-x_0}{\sigma}\right)^2\right) & \text{for } x \geq x_0 \end{cases} \quad (7.85)$$

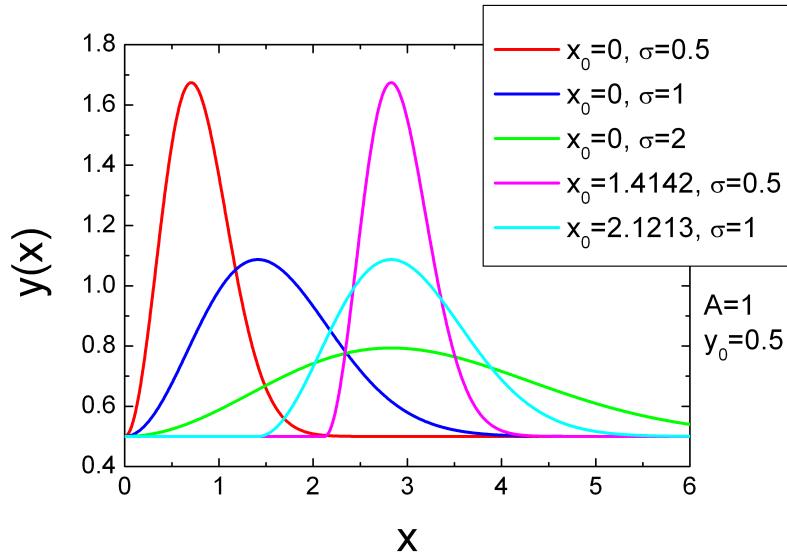


FIGURE 7.47. Plot of Maxwell (Area) distribution.

7.26.3. generalized Maxwell (Amplitude).

$$y(x; x_0, \sigma, n, m) = \begin{cases} y_0 & \text{for } x < x_0 \\ y_0 + A \frac{(x-x_0)^m \exp\left(-\frac{1}{2}\left(\frac{x-x_0}{|\sigma|}\right)^n\right)}{(x_{\text{mode}}-x_0)^m \exp\left(-\frac{1}{2}\left(\frac{x_{\text{mode}}-x_0}{|\sigma|}\right)^n\right)} & \text{for } x \geq x_0 \end{cases} \quad (7.86)$$

with $x_{\text{mode}} = \left(\frac{2m}{n}\right)^{1/n} |\sigma| + x_0$.

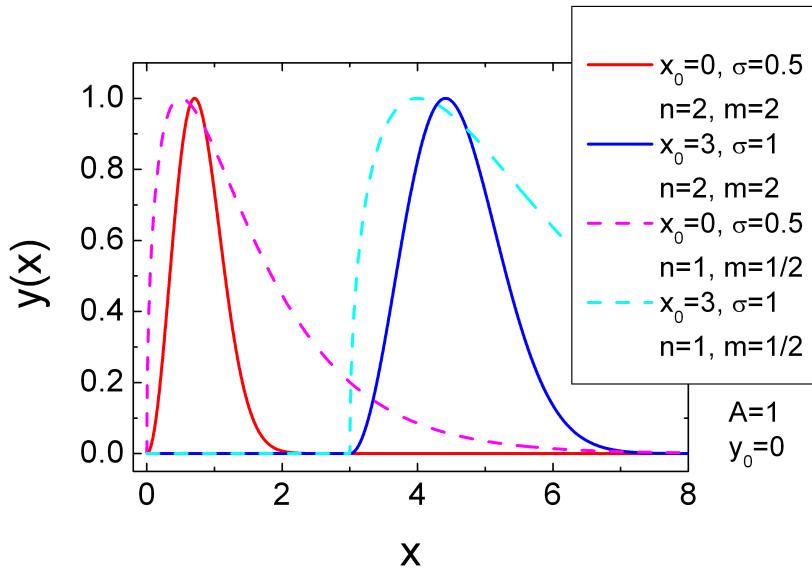


FIGURE 7.48. Plot of generalized Maxwell (Amplitude) distribution.

7.26.4. generalized Maxwell (Area).

$$y(x; x_0, \sigma, n, m) = \begin{cases} 0 & \text{for } x < x_0 \\ A \frac{(x-x_0)^m \exp\left(-\frac{1}{2}\left(\frac{x-x_0}{|\sigma|}\right)^n\right)}{2^{(1+m)/n} |\sigma|^{1+m} \frac{1}{|n|} \Gamma\left(\frac{1+m}{n}\right)} & \text{for } x \geq x_0. \end{cases} \quad (7.87)$$

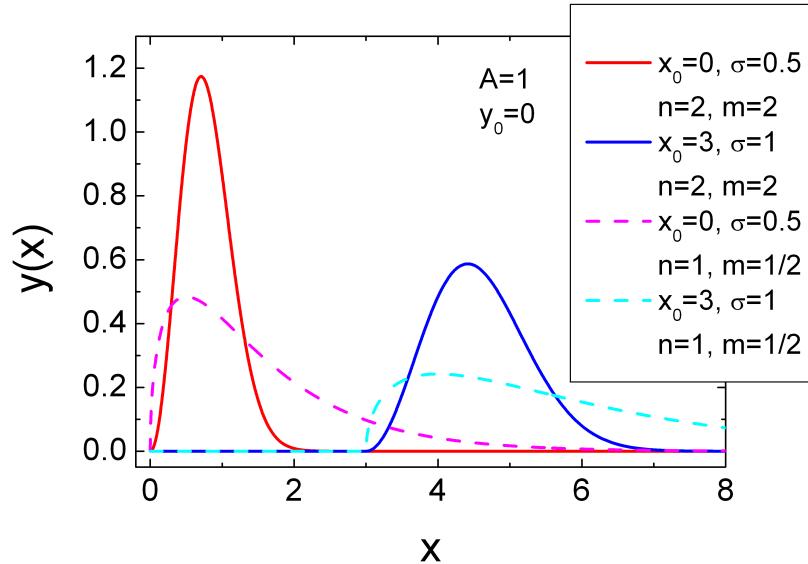


FIGURE 7.49. Plot of generalized Maxwell (Area) distribution.

7.27. Pearson-IV

Pearson type IV distribution:

$$p(x) = \frac{\left| \frac{\Gamma(m+\frac{\nu}{2}i)}{\Gamma(m)} \right|^2}{\alpha B(m - \frac{1}{2}, \frac{1}{2})} \left[1 + \left(\frac{x-\lambda}{\alpha} \right)^2 \right]^{-m} \exp \left[-\nu \arctan \left(\frac{x-\lambda}{\alpha} \right) \right]. \quad (7.88)$$

The normalizing constant involves the complex Gamma function (Γ) and the Beta function (B). For the distribution the mode, mean and variance are given by

$$\text{mode} = \mathcal{M} = \lambda - \frac{\alpha\nu}{2m} \quad (7.89)$$

$$\text{mean} = \langle x \rangle = \lambda - \frac{\alpha\nu}{2(m-1)} \quad ((m > 1)) \quad (7.90)$$

$$\text{variance} = \langle (x - \langle x \rangle)^2 \rangle = \frac{\alpha^2}{r^2(r-1)} (r^2 + \nu^2) \quad \text{for } m > \frac{3}{2} \quad (7.91)$$

$$\text{with } r = 2(m-1)$$

The shape parameter ν of the Pearson type IV distribution controls its skewness. If we fix its value at zero, we obtain a symmetric three-parameter family. This special case is known as the Pearson type VII distribution 7.28.

7.27.1. Pearson-IV (Amplitude).

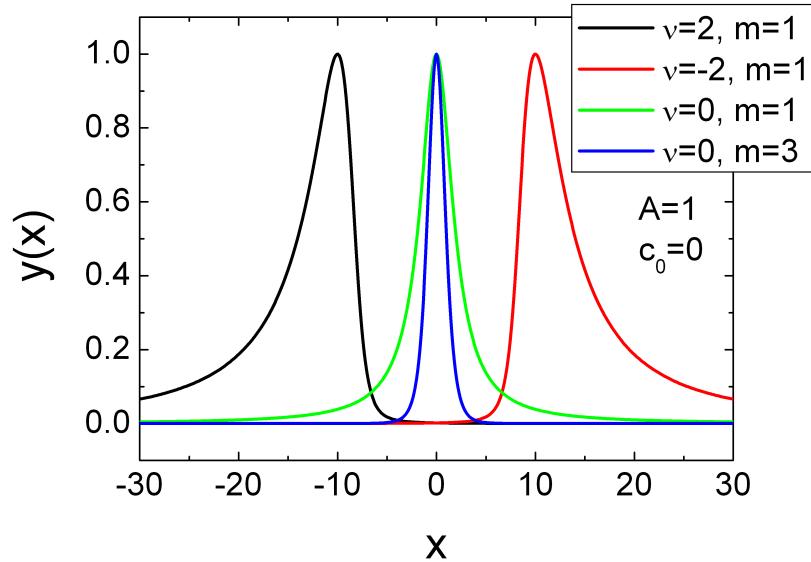


FIGURE 7.50. Plot of Pearson-IV (Amplitude) distribution.

7.27.2. Pearson-IV (Area).

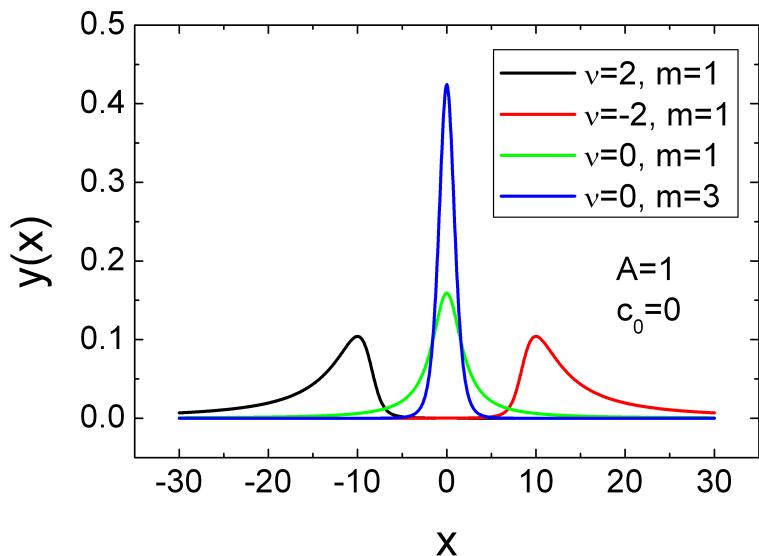


FIGURE 7.51. Plot of Pearson-IV (Area) distribution.

7.28. Pearson-VII

The Pearson-VII model has been used as an approximation for the Voigt function. The parameter σ is the FWHM (full-width at half-maxima). When m is 1.0, the function is an exact Lorentzian. As the m -power term increases, the function tends toward the Gaussian. For $m \sim 50$, the function is essentially Gaussian. The Pearson VII function is a different parametrization of the Student-t distribution function and reads as

$$p(x) = \frac{1}{\alpha B(m - \frac{1}{2}, \frac{1}{2})} \left[1 + \left(\frac{x - \lambda}{\alpha} \right)^2 \right]^{-m} \quad (7.92)$$

where B is the Beta function $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, with a and b not being negative integers. The mode \mathcal{M} of the distribution is equals $\mathcal{M} = \lambda$.

Sometimes distribution function are described by the mode \mathcal{M} and integral peak width or peak breadth β . The peak breadth is given by $\beta = p(\mathcal{M})/\int_{-\infty}^{\infty} p(x) dx = \alpha B(m - \frac{1}{2}, \frac{1}{2})$, so that

$$p(x) = \frac{1}{\beta} \left[1 + B^2 \left(m - \frac{1}{2}, \frac{1}{2} \right) \left(\frac{x - \mathcal{M}}{\beta} \right)^2 \right]^{-m} \quad (7.93)$$

An alternative parameterization (and slight specialization) of the type VII distribution is obtained by letting $\alpha = \sigma\sqrt{2m-3}$, which requires $m > 3/2$. By this constrain the variance of the distribution exists and is equal to σ^2 .

$$p(x) = \frac{1}{\sigma\sqrt{2m-3} B(m - \frac{1}{2}, \frac{1}{2})} \left[1 + \left(\frac{x - \lambda}{\sigma\sqrt{2m-3}} \right)^2 \right]^{-m} \quad (7.94)$$

Now the parameter m only controls the kurtosis of the distribution. If m approaches infinity as λ and σ are held constant, the normal distribution arises as a special case:

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{1}{\sigma\sqrt{2m-3} B(m - \frac{1}{2}, \frac{1}{2})} \left[1 + \left(\frac{x - \lambda}{\sigma\sqrt{2m-3}} \right)^2 \right]^{-m} \\ &= \frac{1}{\sigma\sqrt{2}\Gamma(\frac{1}{2})} \times \lim_{m \rightarrow \infty} \frac{\Gamma(m)}{\Gamma(m - \frac{1}{2}) \sqrt{m - \frac{3}{2}}} \times \lim_{m \rightarrow \infty} \left[1 + \frac{(\frac{x-\lambda}{\sigma})^2}{2m-3} \right]^{-m} \\ &= \frac{1}{\sigma\sqrt{2\pi}} \times 1 \times \exp \left[-\frac{1}{2} \left(\frac{x - \lambda}{\sigma} \right)^2 \right] \end{aligned}$$

For $m > \frac{5}{2}$ the distribution could also be expressed in terms of excess kurtosis γ_2 or kurtosis κ by letting

$$m = \frac{5}{2} + \frac{3}{\gamma_2} \quad (7.95)$$

$$\text{or } m = \frac{5}{2} + \frac{3}{\kappa - 3} \quad (7.96)$$

It is also possible to express the Pearson distribution in terms of the full width of half maximum ω_{FWHM} by substituting $\alpha = \omega_{\text{FWHM}} / (2\sqrt{2^{1/m} - 1})$. This leads to

$$p(x) = \frac{2\sqrt{2^{1/m} - 1}}{\omega_{\text{FWHM}} B(m - \frac{1}{2}, \frac{1}{2})} \left[1 + 4 \left(\sqrt{2^{1/m} - 1} \right) \left(\frac{x - \lambda}{\omega_{\text{FWHM}}} \right)^2 \right]^{-m} \quad (7.97)$$

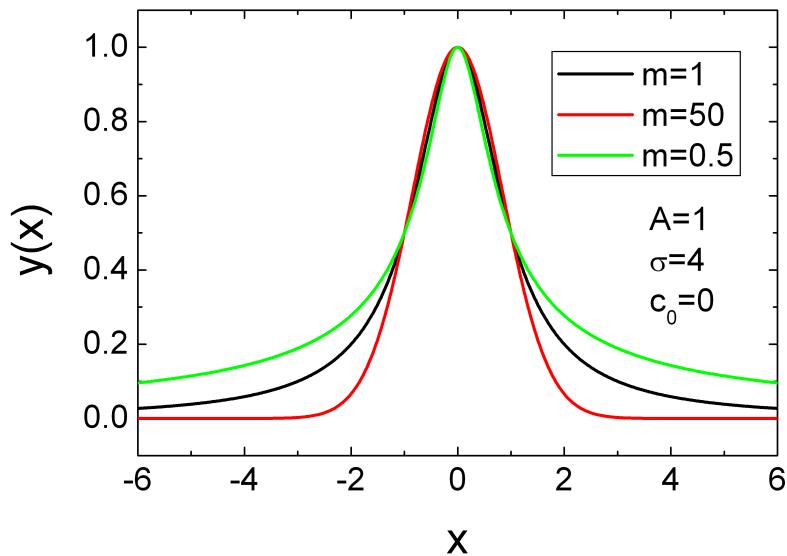
7.28.1. Pearson-VII (Amplitude).

FIGURE 7.52. Plot of Pearson-VII (Amplitude) distribution.

7.28.2. Pearson-VII (Area).

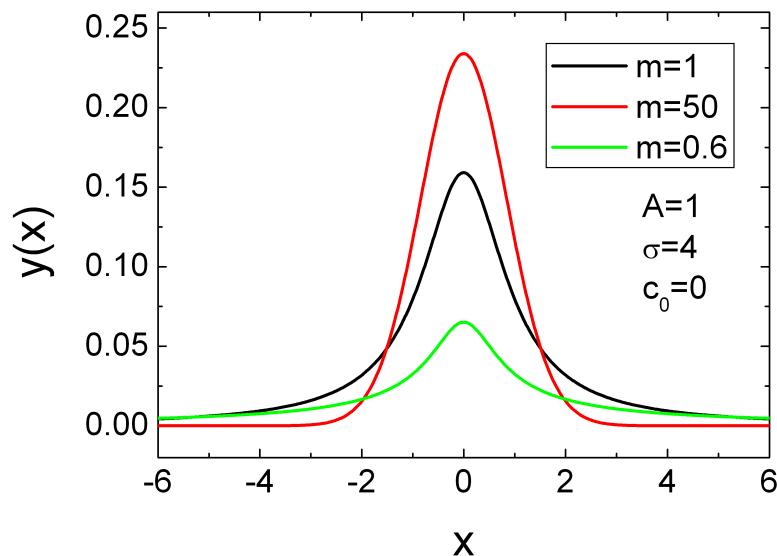


FIGURE 7.53. Plot of Pearson-VII (Area) distribution.

7.29. Pulse Peak

$$p(x) = \frac{2}{\sigma} \exp\left(\frac{x - x_0}{\sigma}\right) \left(1 - \exp\left(\frac{x - x_0}{\sigma}\right)\right) \quad (7.98)$$

$$\text{mode} = x_0 - \sigma \ln\left(\frac{1}{2}\right) \quad (7.99)$$

7.29.1. Pulse Peak (Amplitude).

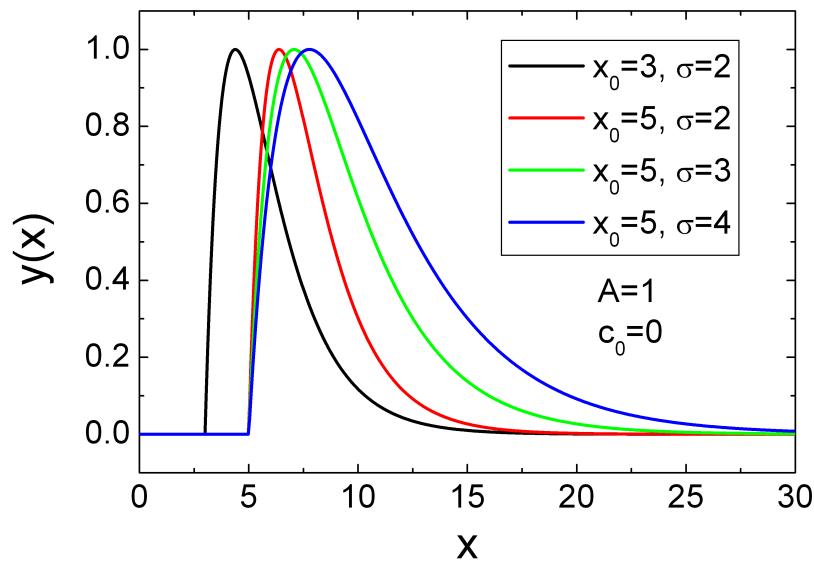


FIGURE 7.54. Plot of Pulse Peak (Amplitude) distribution.

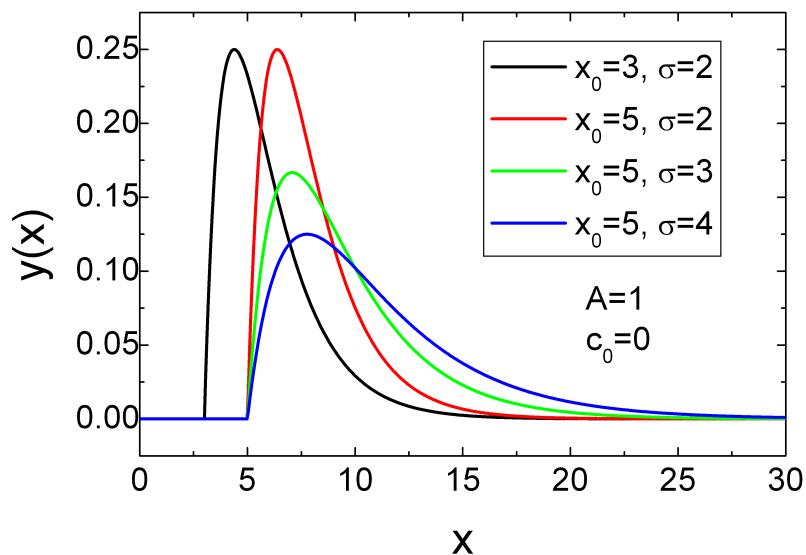
7.29.2. Pulse Peak (Area).

FIGURE 7.55. Plot of Pulse Peak (Area) distribution.

7.30. Pulse Peak with 2nd Width Term

$$p(x) = \frac{\sigma_1 + \sigma_2}{\sigma_2^2} \left(1 - \exp \left(\frac{x - x_0}{\sigma_1} \right) \right) \exp \left(\frac{x - x_0}{\sigma_2} \right) \quad (7.100)$$

$$\text{mode} = x_0 - \sigma_1 \ln \left(\frac{\sigma_1}{\sigma_2 + \sigma_1} \right) \quad (7.101)$$

7.30.1. Pulse Peak with 2nd Width Term (Amplitude).

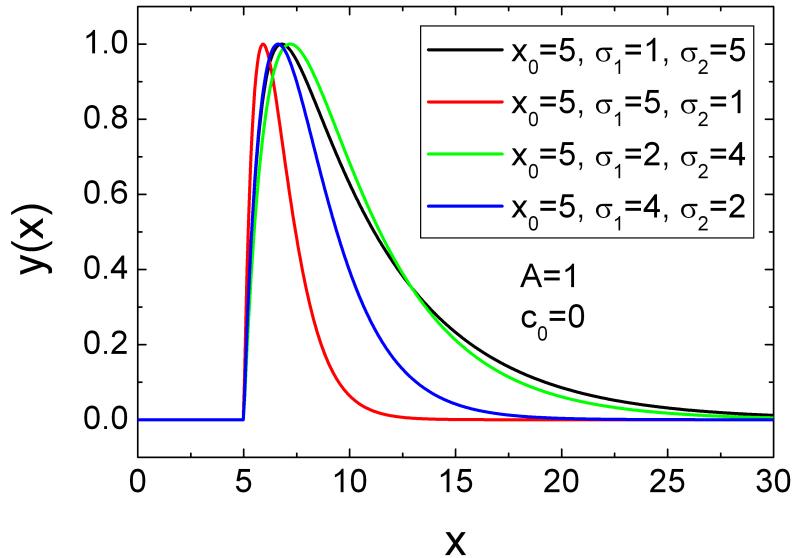


FIGURE 7.56. Plot of pulse with 2nd width (Amplitude) distribution.

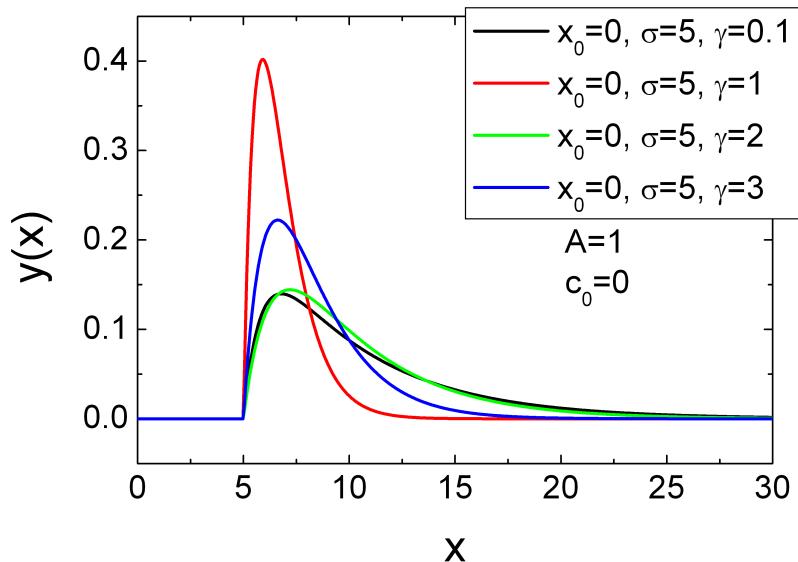
7.30.2. Pulse Peak with 2nd Width Term (Area).

FIGURE 7.57. Plot of pulse with 2nd width (Area) distribution.

7.31. Pulse Peak with Power Term

$$p(x) = \frac{\gamma + 1}{\sigma} \left(1 - \exp\left(\frac{x - x_0}{\sigma}\right)\right)^\gamma \exp\left(\frac{x - x_0}{\sigma}\right) \quad (7.102)$$

$$\text{mode} = x_0 - \sigma \ln\left(\frac{1}{\gamma + 1}\right) \quad (7.103)$$

7.31.1. Pulse Peak with Power Term (Amplitude).

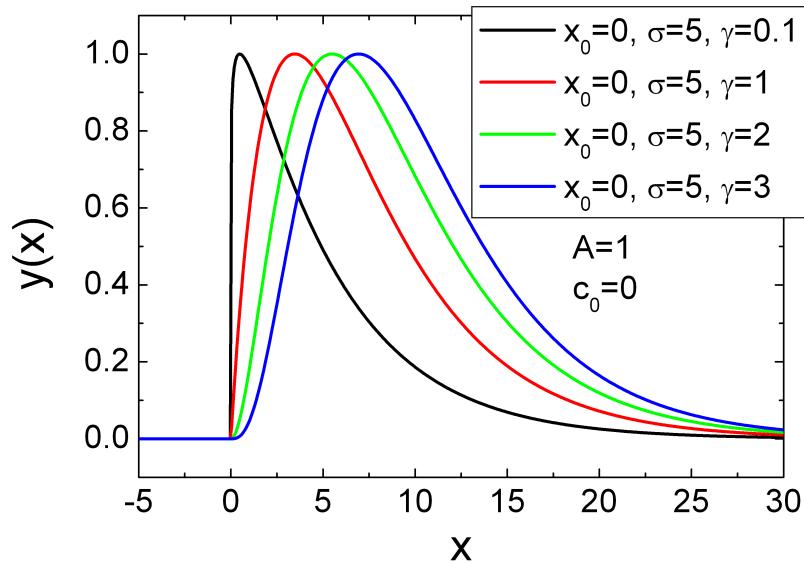


FIGURE 7.58. Plot of pulse with power term (Amplitude) distribution.

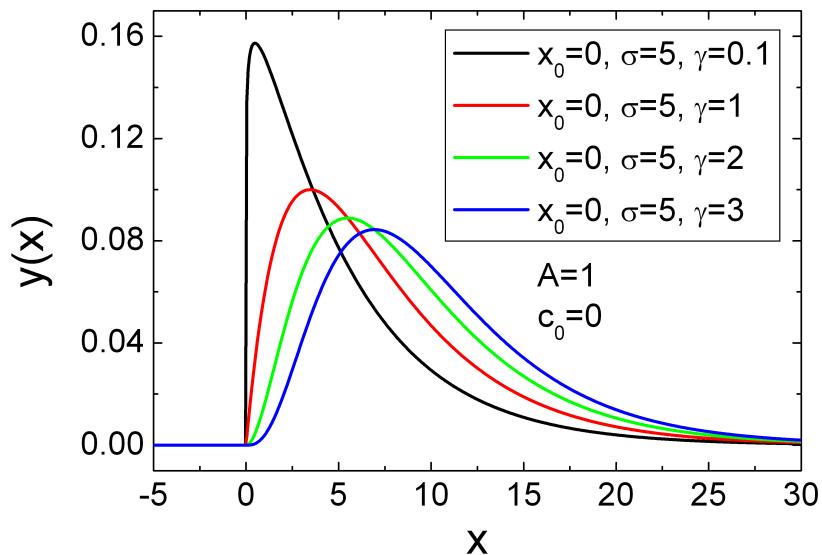
7.31.2. Pulse Peak with Power Term (Area).

FIGURE 7.59. Plot of pulse with power term (Area) distribution.

7.32. Student-t

$$p(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-(\frac{\nu+1}{2})} \quad (7.104)$$

7.32.1. Student-t (Amplitude).

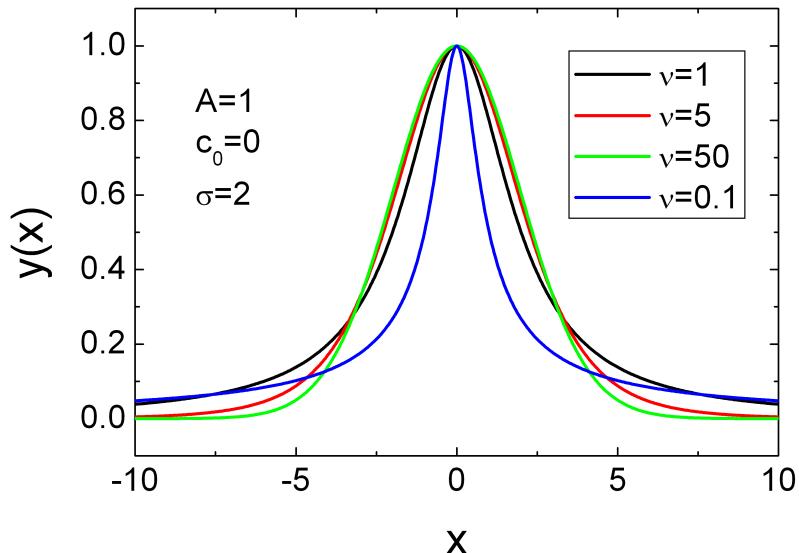


FIGURE 7.60. Plot of Student-t (Amplitude) distribution.

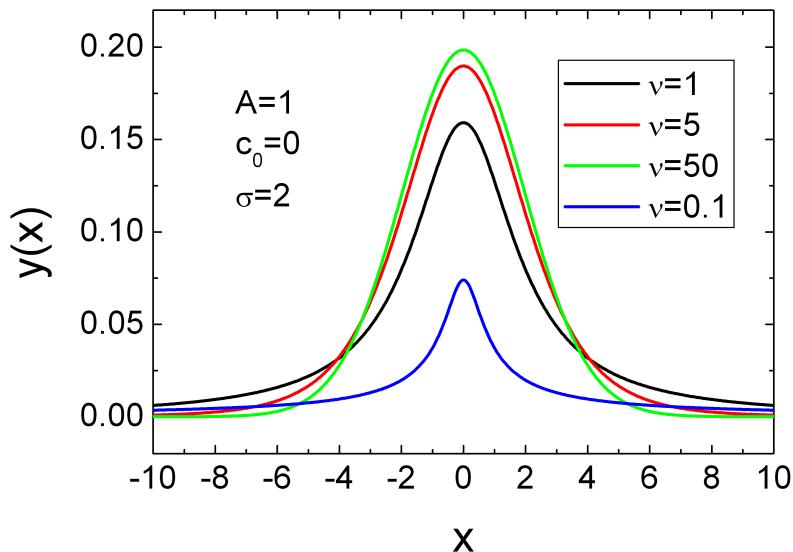
7.32.2. Student-t (Area).

FIGURE 7.61. Plot of Student-t (Area) distribution.

7.33. Voigt

The Voigt profile is a spectral line profile found in all branches of spectroscopy in which a spectral line is broadened by two types of mechanisms, one of which alone would produce a Gaussian profile (usually, as a result of the Doppler broadening), and the other would produce a Lorentzian profile. The Voigt profile is then a convolution of a Lorentz profile and a Gaussian profile:

$$V(x, x_c | \sigma, \gamma) = \int_{-\infty}^{\infty} D(x' | \sigma) L(x - x_c - x' | \gamma) dx' \quad (7.105a)$$

where $x - x_c$ is distance from line center x_c , $D(x | \sigma)$ is the centered Doppler profile:

$$D(x | \sigma) = \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \quad (7.105b)$$

and $L(x - x_c | \gamma)$ is the centered Lorentzian profile:

$$L(x - x_c | \gamma) = \frac{\gamma}{\pi((x - x_c)^2 + \gamma^2)}. \quad (7.105c)$$

The defining integral can be evaluated as [41, 99]:

$$V(x, x_c) = \frac{\Re[w(z)]}{\sigma\sqrt{2\pi}} \quad (7.105d)$$

where $\Re[w(z)]$ is the real part of the complex error function of z and

$$z = \frac{x - x_c + i\gamma}{\sigma\sqrt{2}} \quad (7.105e)$$

The full width at half maximum (FWHM) of the Voigt profile can be found from the widths of the associated Gaussian and Lorentzian widths. The FWHM of the Gaussian profile is $f_G = 2\sigma\sqrt{2\ln(2)}$. The FWHM of the Lorentzian profile is just $f_L = 2\gamma$. Define $\phi = f_L/f_G$. Then the FWHM of the Voigt profile (f_V) can be estimated as:

$$f_V \approx f_G \left(1 - c_0 c_1 + \sqrt{\phi^2 + 2c_1\phi + c_0^2 c_1^2} \right) \quad (7.106)$$

where $c_0 = 2.0056$ and $c_1 = 1.0593$. This estimate will have a standard deviation of error of about 2.4 percent for values of ϕ between 0 and 10. Note that the above equation will have the proper behavior in the limit of $\phi = 0$ and $\phi = \infty$. A different approximation was given by [158, 134]

$$f_V \approx 0.5346 f_L + \sqrt{0.2166 f_L^2 + f_G^2} \quad (7.107)$$

with an accuracy of 0.02

7.33.1. Voigt (Amplitude).

The amplitude version of the Voigt peak is parameterized as

$$V_{\text{Amplitude}}(x|A, \sigma, \gamma) = A \frac{\int_{-\infty}^{\infty} \frac{\exp(-u^2)}{\frac{\gamma^2}{2\sigma^2} + \left(\frac{x-x_c}{\sqrt{2}\sigma} - u\right)^2} du}{\int_{-\infty}^{\infty} \frac{\exp(-u^2)}{\frac{\gamma^2}{2\sigma^2} + u^2} du} = A \frac{V(x, x_c|\sigma, \gamma)}{V(x_c, x_c|\sigma, \gamma)} \quad (7.108)$$

Required parameters:

ampl.: amplitude A of the Voigt peak

center: location parameter (mode) x_c

sigma: width of Doppler (Gaussian) contribution $\sigma > 0$

gamma: width of Lorentzian contribution $\gamma > 0$

backgr: offset c_0

Note

- The Doppler (Gaussian) width parameter needs to be larger than 0 $\sigma > 0$.
- The Lorentzian width parameter needs to be larger than 0 $\gamma > 0$.
- Default (Size) distribution: Monodisperse

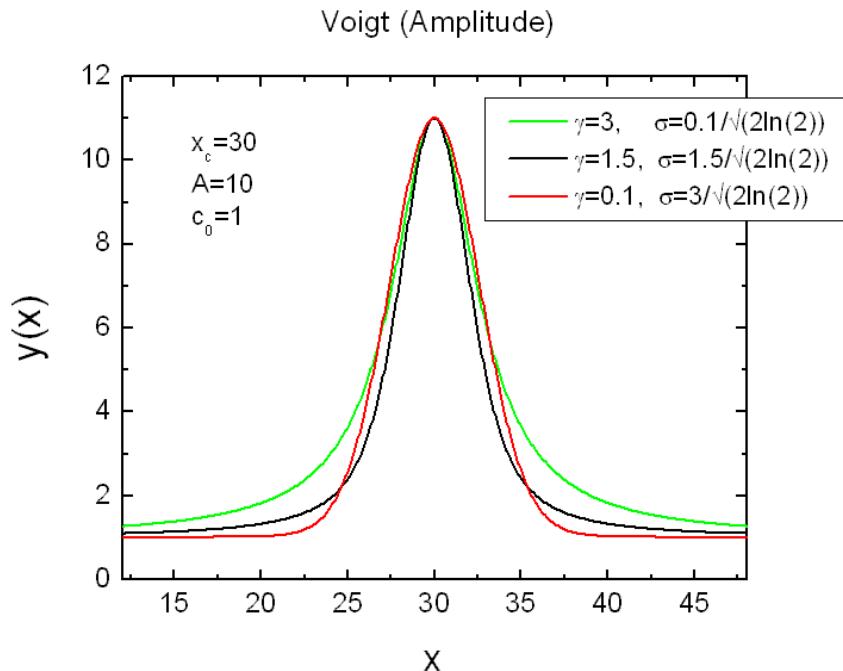


FIGURE 7.62. Plot of Voigt (Amplitude) distribution.

7.33.2. Voigt (Area).

The area version of the Voigt peak is parameterized as

$$V_{\text{Area}}(x|A, \sigma, \gamma) = A \frac{\gamma}{2\pi\sqrt{\pi}\sigma^2} \int_{-\infty}^{\infty} \frac{\exp(-u^2)}{\frac{\gamma^2}{2\sigma^2} + \left(\frac{x-x_c}{\sqrt{2}\sigma} - u\right)^2} du = A V(x, x_c|\sigma, \gamma) \quad (7.109)$$

Required parameters:

- area:** area A of the Voigt peak
- center:** location parameter (mode) x_c
- sigma:** width of Doppler (Gaussian) contribution $\sigma > 0$
- gamma:** width of Lorentzian contribution $\gamma > 0$
- backgr:** offset c_0

Note

- The Doppler (Gaussian) width parameter needs to be larger than 0 $\sigma > 0$.
- The Lorentzian width parameter needs to be larger than 0 $\gamma > 0$.
- Default (Size) distribution: Monodisperse

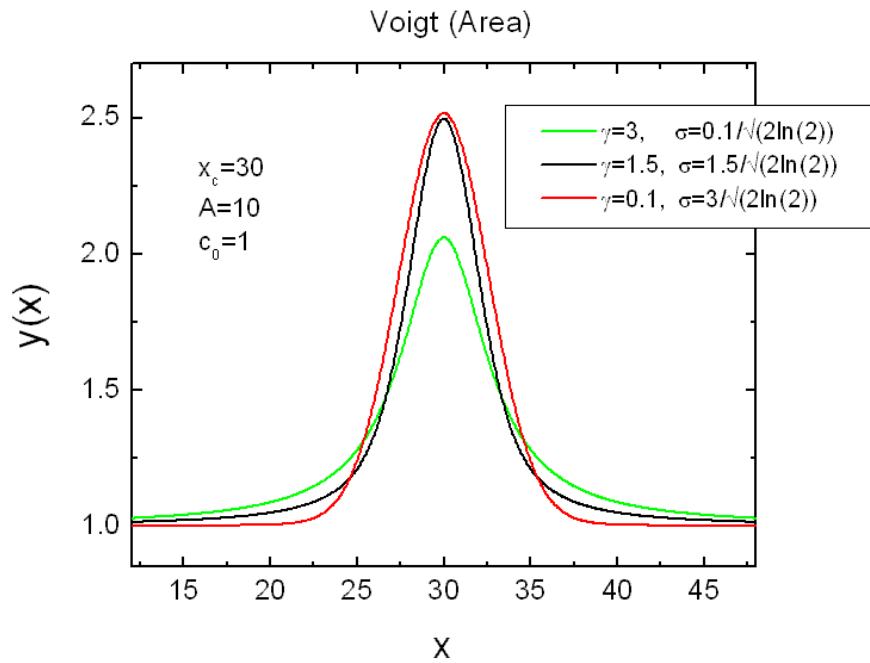


FIGURE 7.63. Plot of Voigt (Area) distribution.

7.33.3. Weibull.

The Weibull distribution is a continuous probability distribution. It is named after Waloddi Weibull who described it in detail in 1951, although it was first identified by Fréchet (1927) and first applied by Rosin & Rammler (1933) to describe the size distribution of particles. The probability density function of a Weibull random variable x is:

$$p(x; \lambda, k) = \begin{cases} \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (7.110)$$

where $k > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter of the distribution. For $k > 1$ the mode is given by

$$\text{mode} = \lambda \left(\frac{k-1}{k} \right)^{\frac{1}{k}} \text{ if } k > 1. \quad (7.111)$$

7.33.4. Weibull (Amplitude).

The amplitude version represents a reparametrization of the standard statistical form. The parameter x_0 has been added to enable variable x positioning. An additional adjustment term has been added so that x_0 represents the mode. The function returns c_0 for those x where it is undefined.

$$\begin{aligned} u &= \frac{k-1}{k} \\ z &= \frac{x - x_0}{\lambda} + u^{1/k} \\ y(x; x_0, k, \lambda, c_0, A) &= \begin{cases} c_0 + Au^{-u}z^{k-1} \exp(-z^k) & z \geq 0 \\ c_0 & z < 0 \end{cases} \end{aligned} \quad (7.112)$$

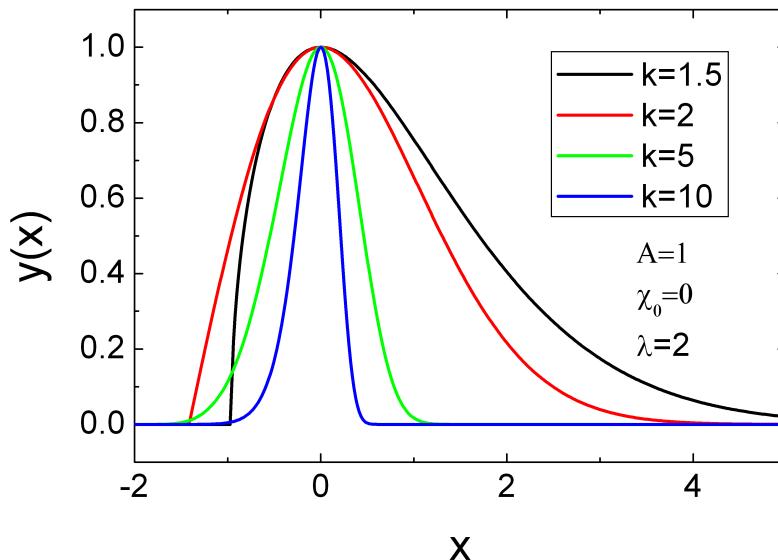


FIGURE 7.64. Plot of Weibull (Amplitude) distribution.

7.33.5. Weibull (Area).

The area version represents a reparametrization of the standard statistical form. The parameter x_0 has been added to enable variable x positioning. An additional adjustment term has been added so that x_0 represents the mode. The function returns c_0 for those x where it is undefined.

$$z = \frac{x - x_0}{\lambda} + \left(\frac{k - 1}{k} \right)^{1/k}$$

$$y(x; x_0, k, \lambda, c_0, A) = \begin{cases} c_0 + A \frac{k}{\lambda} z^{k-1} \exp(-z^k) & z \geq 0 \\ c_0 & z < 0 \end{cases} \quad (7.113)$$

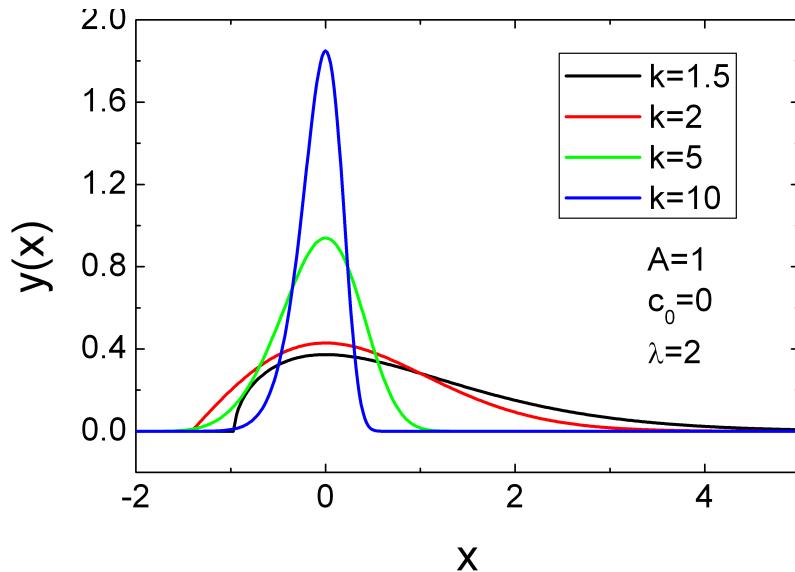


FIGURE 7.65. Plot of Weibull (Area) distribution.

CHAPTER 8

Plugin functions for form factors

8.1. Non spherical-symmetric randomly oriented particles

8.1.1. Rectangular parallelepiped. In a rectangular parallelepiped, all angles are right angles, and opposite faces of a parallelepiped are equal. Also the terms rectangular cuboid or orthogonal parallelepiped are used to designate this polyhedron. Four variants of rectangular parallelepipeds have been implemented, where either none, one, two, or all three axis have a common size distribution. The scattering intensity of a monodisperse

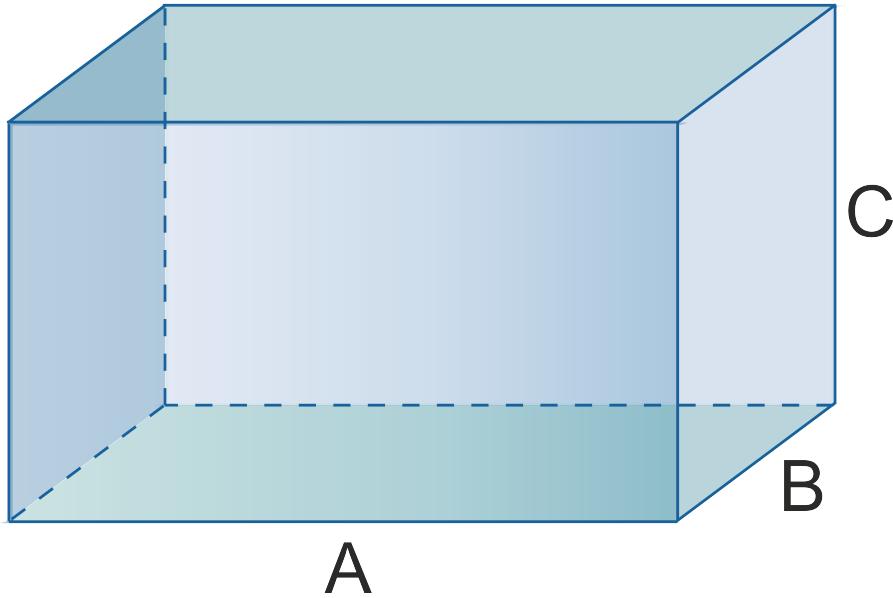


FIGURE 8.1. Rectangular parallelepiped with edge lengths A , B , and C .

randomly oriented rectangular parallelepiped with the edge lengths A , B , C , and a scattering length density contrast $\Delta\eta$ is given by

$$I_{\text{rPE}}(Q, A, B, C, \Delta\eta) = \int_0^{\pi} \int_0^{\pi/2} \frac{2}{\pi} \left[\text{sinc}\left(\frac{QA}{2} \sin(\alpha) \cos(\varphi)\right) \right. \\ \left. \text{sinc}\left(\frac{QB}{2} \sin(\alpha) \sin(\varphi)\right) \right. \\ \left. \text{sinc}\left(\frac{QC}{2} \cos(\alpha)\right) ABC \Delta\eta \right]^2 \sin(\alpha) d\alpha d\varphi \quad (8.1)$$

An additional size distribution is implemented in several ways either linearly, quadratically or cubically, depending if one two or all three axis scale with a single size distribution

$$I_{\text{rPE},1}(Q, \sigma, a, b, c, \Delta\eta) = \int_0^{\infty} \text{LogNorm}(\nu, \sigma) I_{\text{rPE}}(Q, \nu a, b, c, \Delta\eta) d\nu \quad (8.2)$$

$$I_{\text{rPE},2}(Q, \sigma, a, b, c, \Delta\eta) = \int_0^{\infty} \text{LogNorm}(\nu, \sigma) I_{\text{rPE}}(Q, \nu a, \nu b, c, \Delta\eta) d\nu \quad (8.3)$$

$$I_{\text{rPE},3}(Q, \sigma, a, b, c, \Delta\eta) = \int_0^{\infty} \text{LogNorm}(\nu, \sigma) I_{\text{rPE}}(Q, \nu a, \nu b, \nu c, \Delta\eta) d\nu \quad (8.4)$$

with

$$\text{LogNorm}(\nu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\ln^2(\nu)}{2\sigma^2}\right) \quad (8.5)$$

Input Parameters for model Parallelepiped_abc:

A: width A
B: depth B
C: hight C
dummy: not used
eta: scattering length density contrast $\Delta\eta$

Input Parameters for models Parallelepiped_abc1, Parallelepiped_abc2, and Parallelepiped_abc3:

A: width A
B: depth B
C: hight C
sigma: width parameter σ assuming a LogNorm-distribution
eta: scattering length density contrast $\Delta\eta$

Note:

- None

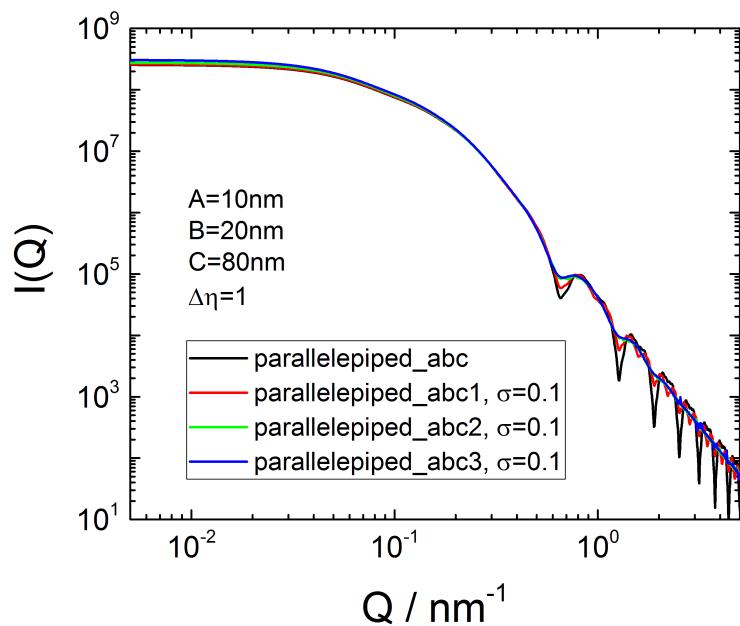


FIGURE 8.2. Scattering curves for parallelepiped with optional size distribution a single edge length (parallelepiped_abc1), of a surface area (parallelepiped_abc1), or volume parallelepiped_abc3.

8.2. Very anisotropic particles (local planar & local cylindrical objects)

For very anisotropic random orientated particles the form factor can be factorize according to Porod [173] in a cross section term $P_{\text{cs}}(Q)$ for the shorter dimension and a shape factor $P'(Q)$ for the long dimension.

$$I(Q) = P'(Q)P_{\text{cs}}(Q). \quad (8.6)$$

In this plugin the form factors of two types of anisotropic particles are collected, those with a local cylindrical and with a local planar geometry. In case of local planar objects the cross section term $P_{\text{cs}}(Q)$ can be homogeneous, a centro-symmetric bilayer, a gaussian bilayer, etc. . This cross section factor can than be combined with the overall shape factor $P'(Q)$ of for examples a thin spherical shell or elliptical shell, a thin cylindrical shell or a thin disc. As the total form factor is the product of the cross-section form factor and a shape form factor one can either programm all combination of cross-section and shape factors into individual form factor functions or one can programm the cross-section factors as form factor and the shape factor as a structure factors. Using the monodisperse approximation yields than the same result.

In this plugin the product of the cross-section and shape term have been implemented as form factor under "[by plugin|anisotropic obj.|local planar obj.]" and "[by plugin|anisotropic obj.|local cylindrical obj.]". The cross-section terms alone are also implemented as form factors under "[by plugin|anisotropic obj.|Pcs(Q) for planar obj.]" and "[by plugin|anisotropic obj.|Pcs(Q) for cylindrical obj.]". The shape factors are also available as structure factors under "[by plugin|anisotropic obj.|P'(Q): local planar obj.]" and "[by plugin|anisotropic obj.|P'(Q): local cylindrical obj.]".

The cross-section form factors can be easily calculated if the scattering length density contrast profile $\Delta\eta_{\text{cs}}(r)$ is known. For structures with a local planar geometry and a symmetric cross-section the form factor is given by

$$P_{\text{cs}}^{\text{planar}}(Q) = \left[2 \int_0^{\infty} \Delta\eta_{\text{cs}}(r) \cos(Qr) dr \right]^2 \quad (8.7)$$

In case of local cylindrical particles with a centro-symmetric scattering length density distribution the form factor is given by

$$P_{\text{cs}}^{\text{cylindrical}}(Q) = \left[2\pi \int_0^{\infty} \Delta\eta_{\text{cs}}(r) J_0(Qr) r dr \right]^2 \quad (8.8)$$

8.2.1. $P_{cs}(Q)$ for planar obj.

The cross-section form factors with local planar geometry are valid when the cross-section dimension is much smaller than the radius of curvature of the locally planar structure.

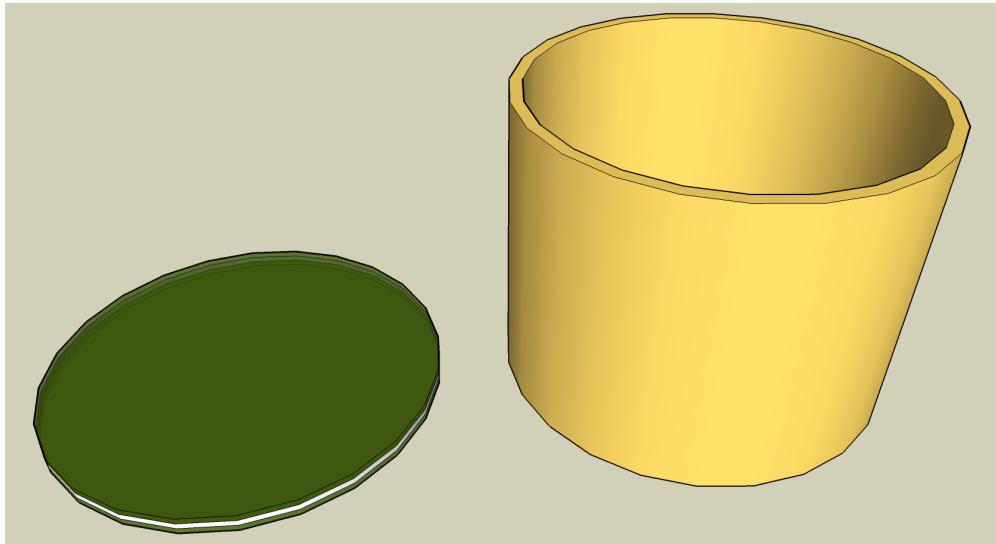


FIGURE 8.3. for local planar particles the cross section dimension is much smaller than the radius of curvature of the particle

Several cross-section profiles for local planar objects have been implemented, like a homogeneous cross-section, cross-section with two infinitely thin plates, layered centrosymmetric cross-section, bilayer with a Gaussian scattering length density profile, layer with Gaussian chains attached to the surface. These form factors are supposed to be combined with a shape factor for local planar objects which are implemented as structure plugins under "[by plugin|anisotropic obj.| $P'(Q)$: local planar obj.]".

8.2.1.1. $P_{cs}(Q)$ for a homogeneous cross-section.

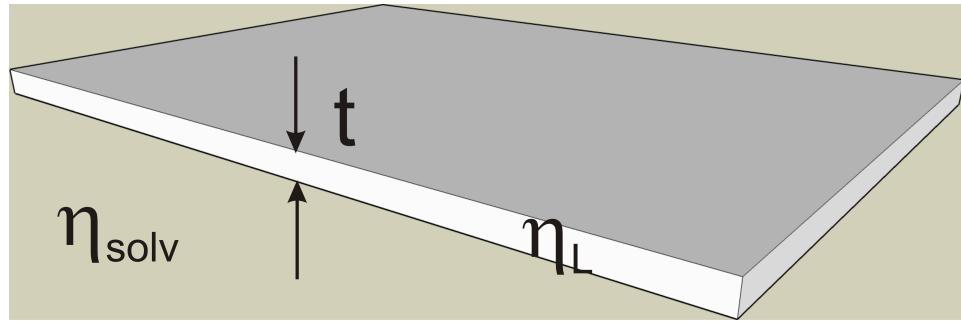


FIGURE 8.4. Plane with a homogeneous cross-section of thickness t .

This cross-section form factor describes the scattering of a layer with homogeneous scattering length density η_L in a matrix of a scattering length density η_{solv} . The thickness can have a distribution described by a log-normal distribution according to eq. 6.3b.

$$P_{\text{cs}}(Q, \sigma_t, t) = \int_0^{\infty} \text{LogNorm}(x, 1, \sigma_t, 1, t) \left[(\eta_L - \eta_{\text{solv}}) x \frac{\sin(Qx/2)}{Qx/2} \right]^2 dx \quad (8.9)$$

Input parameters for `Pcs:homogeneousPlate`:

- `t`: most probable layer thickness t
- `sigm_t`: width σ_t of thickness distribution (LogNorm)
- `dummy`: unused disabled parameter
- `dummy`: unused disabled parameter
- `eta_l`: scattering length density of layer η_L
- `eta_solv`: scattering length density of solvent η_{solv}

Note

- This form factor is supposed to be combined with a shape factor for local planar objects which are implemented as structure plugins under "[by plugin|anisotropic obj.|P'(Q): local planar obj.]".
- As the form factor already have the width distribution included one normally uses in SASfit as a size distribution the Delta-distribution.

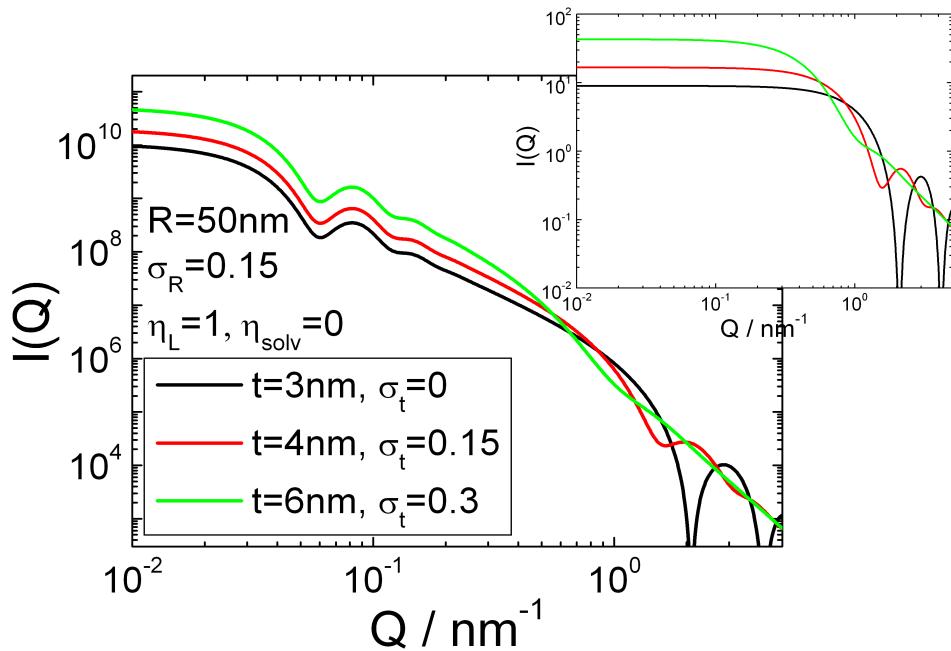


FIGURE 8.5. Scattering curve for the form factor "Pcs:homogeneousPlate" only (insert) and in combination with a structure factor "P"(Q): Thin Spherical Shell".

8.2.1.2. $P_{cs}(Q)$ for two infinitely thin parallel layers.

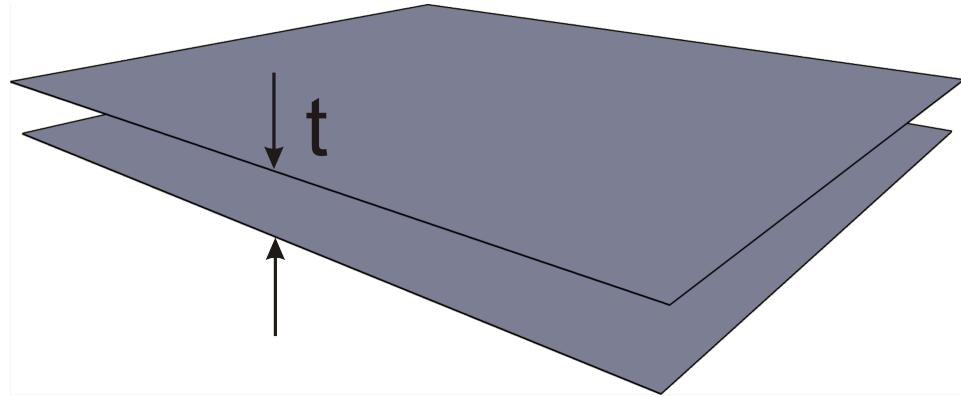


FIGURE 8.6. Two infinitely thin parallel layers separated by a distance t .

This cross-section form factor describes the scattering of two infinitely thin parallel layers. The separation distance can have a distribution described by a log-normal distribution according to eq. 6.3b.

$$P_{cs}(Q, \sigma_T, T) = \int_0^{\infty} \text{LogNorm}(x, 1, \sigma_T, 1, T) \cos^2(Qx/2) dx \quad (8.10)$$

Input parameters for `Pcs:TwoInfinitelyThinLayers`:

- t:** most probable layer separation t
- sigm_t:** width σ_t of separation distribution (LogNorm)

Note

- This form factor is supposed to be combined with a shape factor for local planar objects which are implemented as structure plugins under "[by plugin—anisotropic obj.— $P'(Q)$: local planar obj.]".
- As the form factor already have the width distribution included one normally uses in **SASfit** as a size distribution the **Delta**-distribution.

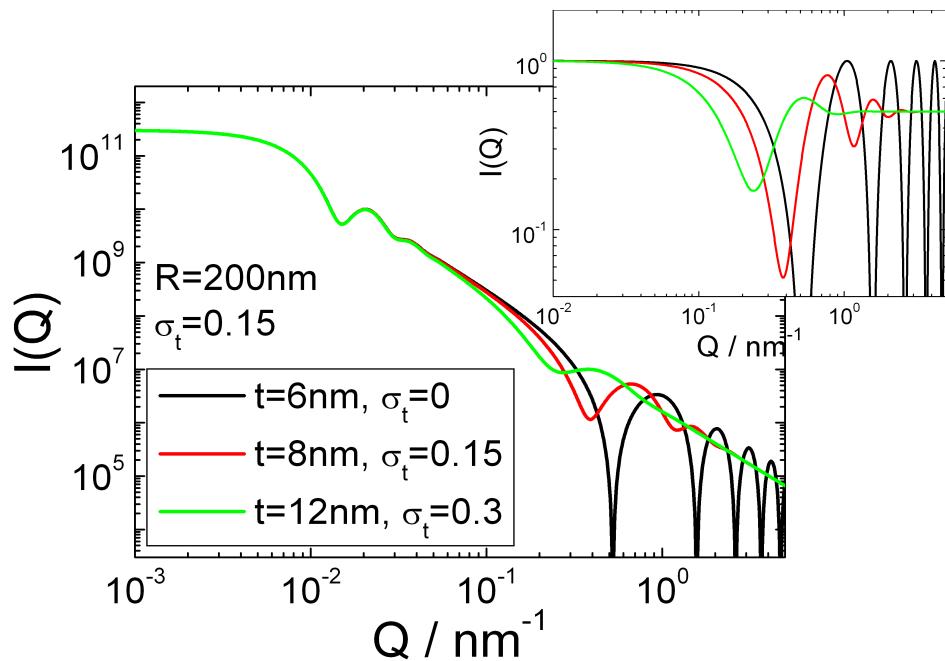


FIGURE 8.7. Scattering curve for the form factor "Pcs:TwoInfinitelyThinLayers" only (insert) and in combination with a structure factor "P'(Q): Thin Spherical Shell".

8.2.1.3. $P_{cs}(Q)$ for a layered centro symmetric cross-section structure.

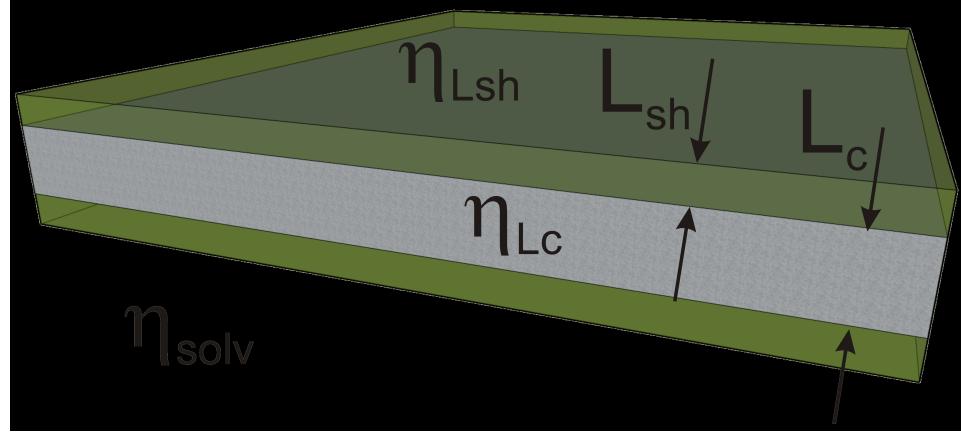


FIGURE 8.8. Two layered centro symmetric structure with a core thickness of L_c and an outer layer thickness $L_{L_{sh}}$. The corresponding scattering length densities of the core, the shell layer and the solvent are η_{L_c} , $\eta_{L_{sh}}$, and $\eta_{L_{solv}}$.

This cross-section form factor describes the scattering of a layered centro symmetric cross-section structure. Both the core thickness as well as the shell thickness can have a distribution described by a log-normal distribution as defined in eq. 6.3b.

$$\begin{aligned}
 P_{cs}(Q, \sigma_{L_c}, L_c, \sigma_{L_{sh}}, L_{sh}, \eta_{L_c}, \eta_{L_{sh}}, \eta_{solv}) = \\
 \int_0^{\infty} \text{LogNorm}(v, 1, \sigma_{L_c}, 1, L_c) \int_0^{\infty} \text{LogNorm}(u, 1, \sigma_{L_{sh}}, 1, L_{sh}) \\
 \left[\frac{(\eta_{L_{sh}} - \eta_{solv})(v + 2u) \sin(Q \frac{v+2u}{2})}{Q \frac{v+2u}{2}} - \frac{(\eta_{L_{sh}} - \eta_{L_c})v \sin(Q \frac{v}{2})}{Q \frac{v}{2}} \right]^2 du dv \quad (8.11)
 \end{aligned}$$

Input parameters for `Pcs:LayeredCentroSymmetricXS`:

- `L_c`: most probable layer separation L_c
- `sigm_Lc`: width σ_{L_c} of core thickness distribution (LogNorm)
- `L_sh`: most probable shell thickness L_{sh}
- `sigm_Lsh`: width $\sigma_{L_{sh}}$ of shell thickness distribution (LogNorm)
- `eta_Lc`: scattering length density of core layer η_{L_c}
- `eta_Lsh`: scattering length density of shell layer $\eta_{L_{sh}}$
- `eta_solv`: scattering length density of solvent η_{solv}

Note

- This form factor is supposed to be combined with a shape factor for local planar objects which are implemented as structure plugins under "[by plugin|anisotropic obj.|P'(Q): local planar obj.]".

- As the form factor already have the width distribution included one normally uses in SASfit as a size distribution the Delta-distribution.

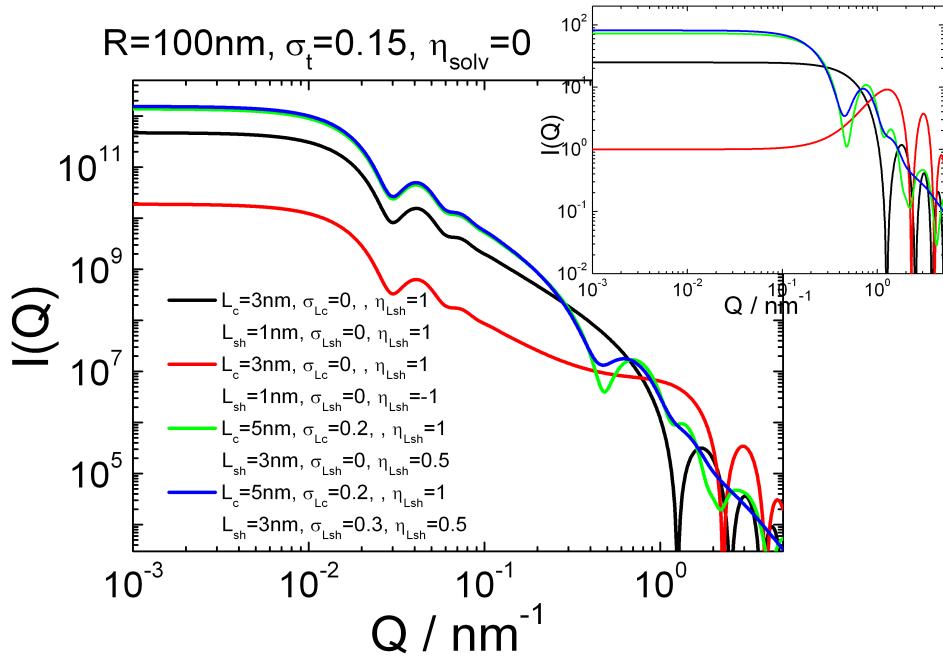


FIGURE 8.9. Scattering curve for the form factor "Pcs:LayeredCentroSymmetricXS" only (insert) and in combination with a structure factor "P'(Q): Thin Spherical Shell".

8.2.1.4. $P_{cs}(Q)$ for a bilayer with a Gaussian electron density profile [162, 161].

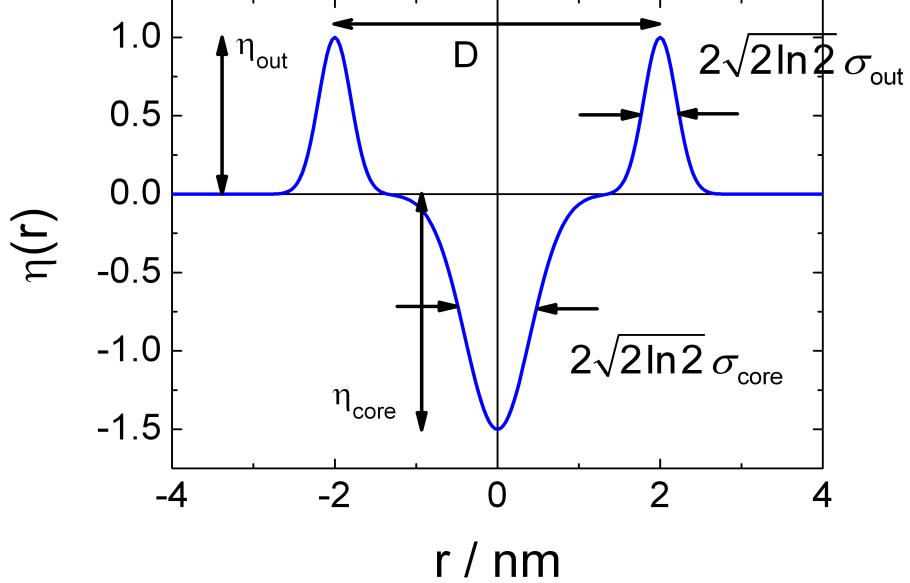


FIGURE 8.10. The plot shows a model for the Gaussian description of the bilayer electron density profile according to eq. 8.12. The origin of the profile is set to the bilayer centre. The model encountering a single Gaussian for the head group at $\pm t/2$. η_{out} is the amplitude of the head-group Gaussian and η_{core} that of the hydrocarbon chains with respect to the average electron density of water. The FWHM of the Gaussian profiles are $2\sqrt{2\ln 2}\sigma_{\text{out}}$ and $2\sqrt{2\ln 2}\sigma_{\text{core}}$.

This model for a bilayer is using a real-space representation of the electron density profile using a Gaussian description [162, 161]. In comparison to other models it is simpler and requiring the adjustment of only four parameters. The electron density profile (Fig. 8.10) is described by

$$\begin{aligned} \eta(r) = & \eta_{\text{out}} \left[\exp \left(-\frac{(r - \frac{t}{2})^2}{2\sigma_{\text{out}}^2} \right) + \exp \left(-\frac{(r + \frac{t}{2})^2}{2\sigma_{\text{out}}^2} \right) \right] \\ & + \eta_{\text{core}} \exp \left(-\frac{r^2}{2\sigma_{\text{core}}^2} \right) \end{aligned} \quad (8.12)$$

The scattering intensity of this cross section profile of a planar object can be calculated by eq. 8.7 and computes as

$$F_{\text{out}}(Q, D, \sigma_{\text{out}}, \eta_{\text{out}}) = \sqrt{2\pi} \sigma_{\text{out}} \eta_{\text{out}} \exp \left(-\frac{1}{2} (Q\sigma_{\text{out}})^2 \right) \cos \left(Q \frac{t}{2} \right) \quad (8.13)$$

$$F_{\text{core}}(Q, \sigma_{\text{core}}, \eta_{\text{core}}) = \sqrt{2\pi} \sigma_{\text{core}} \eta_{\text{core}} \exp \left(-\frac{1}{2} (Q\sigma_{\text{core}})^2 \right) \quad (8.14)$$

so that

$$P_{\text{cs}}(Q) = [F_{\text{core}}(Q, \sigma_{\text{core}}, \eta_{\text{core}}) + 2F_{\text{out}}(Q, D, \sigma_{\text{out}}, \eta_{\text{out}})]^2 \quad (8.15)$$

Input parameters for `Pcs:BilayerGauss`:

`sigma_core`: width σ_{out} of the central Gaussian profile

`eta_core`: scattering length density contrast of the central Gaussian profile

`sigma_out`: width σ_{out} of the two outer Gaussian profiles

`eta_out`: scattering length density contrast of the two outer Gaussian profiles

`t`: distance between the centers of the outer Gaussian profiles

Note

- This form factor is supposed to be combined with a shape factor for local planar objects which are implemented as structure plugins under "[by plugin|anisotropic obj.|P'(Q): local planar obj.]".

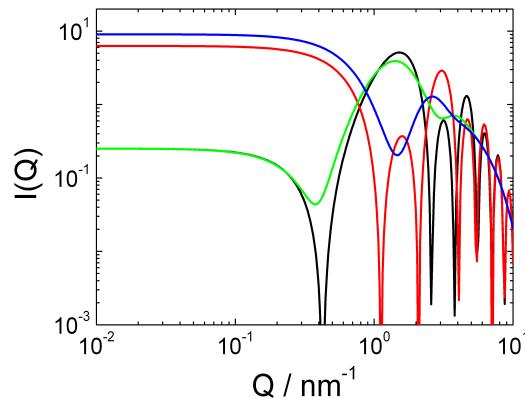
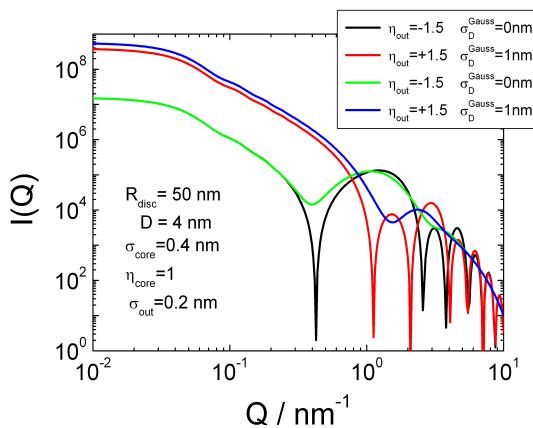


FIGURE 8.11. Scattering curve for the cross-section form factor "Pcs:BilayerGaussian". For some of the curves a distance distribution of the heads groups are assumed being Gaussian (see eq. 6.22a), i.e. calculating $\int_0^\infty \text{Gauss}(D, 1, \sigma_D^{\text{Gauss}}, D_0) P_{\text{cs}}(Q, D) dD$.

8.2.2. $P_{cs}(Q)$ for cylindrical obj.

The cross-section form factors with cylindrical geometry are valid when the cross-section dimension is much smaller than the segment length or Kuhn length of the local cylindrical structure.

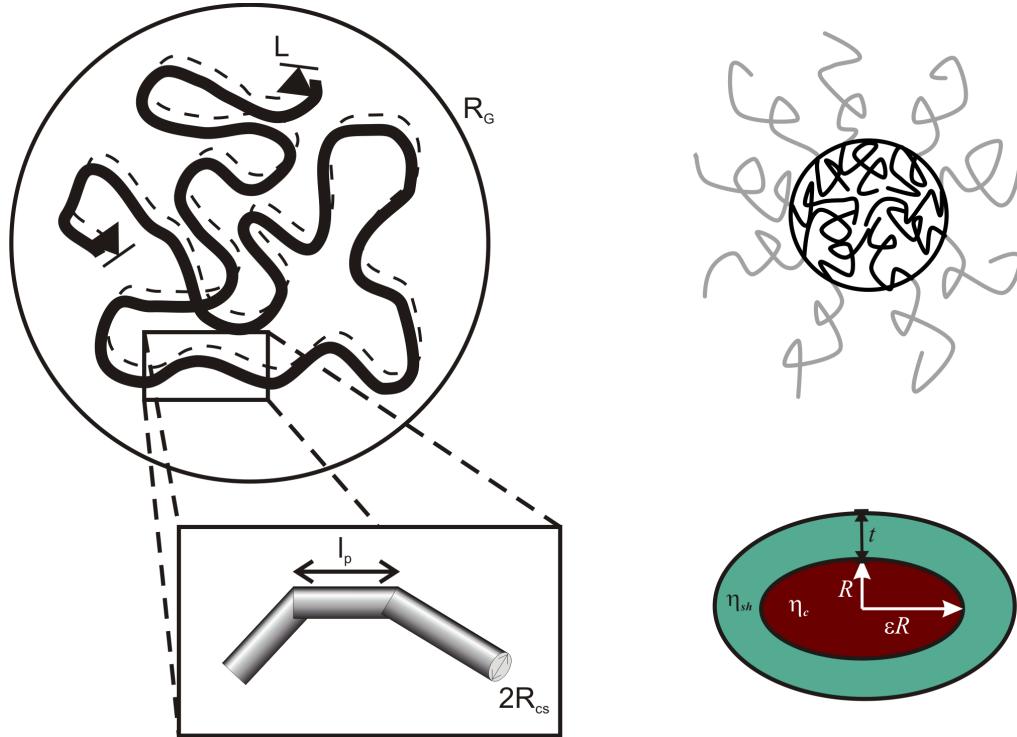


FIGURE 8.12. Sketch of wormlike structures which represent local cylindrical structures. The cross-section $2R_{cs}$ is much smaller than the Kuhn length l_p , which is a typical length scale where a freely jointed chain can randomly orient in any direction without the influence of any forces, independent of the directions taken by other segments. For the cross-section term several profiles have been implemented, like homogeneous round profile or elliptical shell profile

8.2.2.1. $P_{cs}(Q)$ for homogeneous cross-section of a cylinder.

This cross-section form factor describes the scattering of circular and homogeneous cross section. The cross-section radius R can have a distribution described by a log-normal distribution according to eq. 6.3b.

$$P_{cs}(Q, \sigma_R, R) = \int_0^{\infty} \text{LogNorm}(x, 1, \sigma_R, 1, R) \left((\eta_{\text{core}} - \eta_{\text{solv}}) \pi x^2 \frac{2J_1(Qx)}{Qx} \right)^2 dx \quad (8.16)$$

Input parameters for `Pcs:homogeneousCyl`:

`R`: most probable radius R
`sigm_R`: width σ_R of radius distribution (LogNorm)
`dummy`: not used
`eta_core`: scattering length density of the core η_{core}
`dummy`: not used
`eta_solv`: scattering length density of the solvent η_{solv}

Note

- This form factor is supposed to be combined with a shape factor for local cylindrical objects which are implemented as structure plugins under "[by plugin|anisotropic obj.|P'(Q): local cylindrical obj.]".
- As the form factor already have the width distribution included one normally uses in **SASfit** as a size distribution the Delta-distribution.

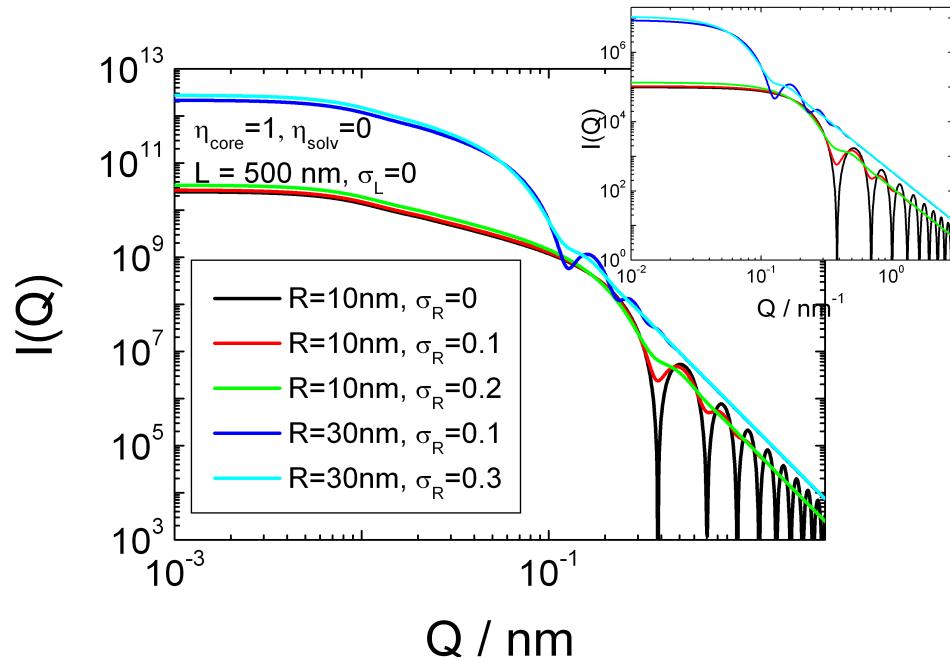


FIGURE 8.13. Scattering curve for the form factor "Pcs:homogeneousCyl" only (insert) and in combination with a structure factor "P'(Q): Thin Rod".

8.2.2.2. $P_{cs}(Q)$ for cross-section of a cylindrical shell with elliptical cross section.

This cross-section form factor describes the scattering of an elliptical core-shell cross-section. The cross-section radius R can have a distribution with a width of *sigma* as described by a log-normal distribution according to eq. 6.3b.

$$P_{cs}(Q) = \int_0^{\infty} \text{LogNorm}(x, 1, \sigma_R, 1, R) \times \\ \int_0^{\pi/2} \left[(\eta_{\text{shell}} - \eta_{\text{solv}}) F_{\text{cs,ell}}(Q, R + t, \epsilon, \phi) \right. \\ \left. + (\eta_{\text{core}} - \eta_{\text{shell}}) F_{\text{cs,ell}}(Q, R, \epsilon, \phi) \right]^2 d\phi \quad (8.17)$$

with

$$F_{\text{cs,ell}}(Q, R, \epsilon, \Delta\eta\phi) = \frac{2J_1(Qr(R, \epsilon, \phi))}{Qr(R, \epsilon, \phi)} \quad (8.18a)$$

$$r(R, \epsilon, \phi) = R\sqrt{\sin^2 \phi + \epsilon^2 \cos^2 \phi} \quad (8.18b)$$

Input parameters for `Pcs:homogeneousCyl`:

`R`: most probable radius R

`sigm_R`: width σ_R of radius distribution (LogNorm)

`epsilon`: eccentricity ϵ of elliptical cross-section

`t`: shell thickness t

`dummy`: not used

`dummy`: not used

`dummy`: not used

`eta_core`: scattering length density of the core η_{core}

`eta_shell`: scattering length density of the shell η_{shell}

`eta_solv`: scattering length density of the solvent η_{solv}

Note

- This form factor is supposed to be combined with a shape factor for local cylindrical objects which are implemented as structure plugins under "[by plugin|anisotropic obj.|P'(Q): local cylindrical obj.]".
- As the form factor already has the width distribution included one normally uses in SASfit as a size distribution the Delta-distribution.

8.2.3. $P'(Q)$ for local planar obj.

8.2.3.1. $P'(Q)$: thin discs.

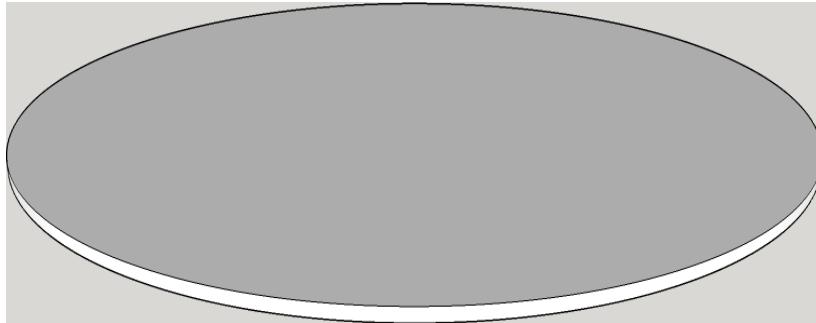


FIGURE 8.14. Sketch of a thin disc with a radius R . The thickness of the disc is assumed to be much smaller than its radius.

Input parameters for $P'(Q)$: Thin Disc:

R: most probable radius R

dummy: not used

sima: width σ of radius distribution (LogNorm)

Note

- This structure factor is supposed to be combined with a form factor of local planar objects which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local planar obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.

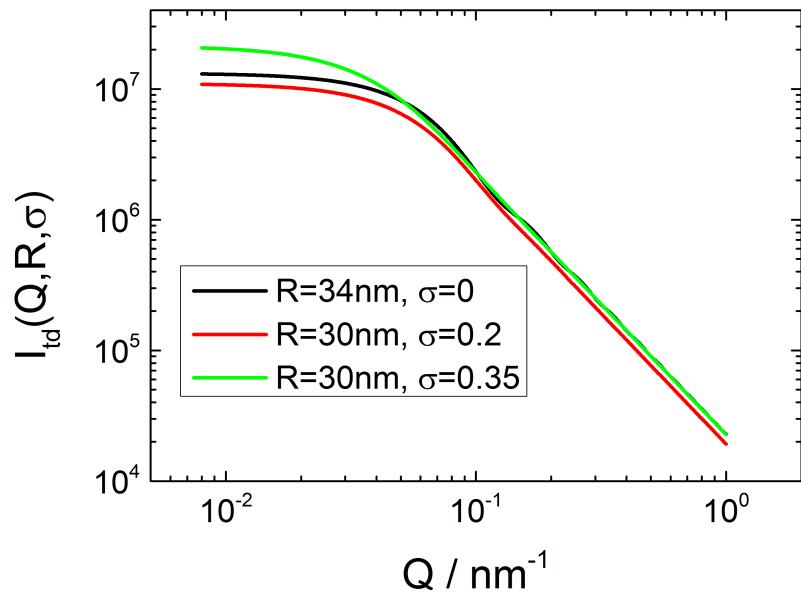


FIGURE 8.15. Scattering curve for the structure factor "P" (Q): Thin Disc" in combination with a constant background of 1".

8.2.3.2. $P'(Q)$: thin spherical shell.

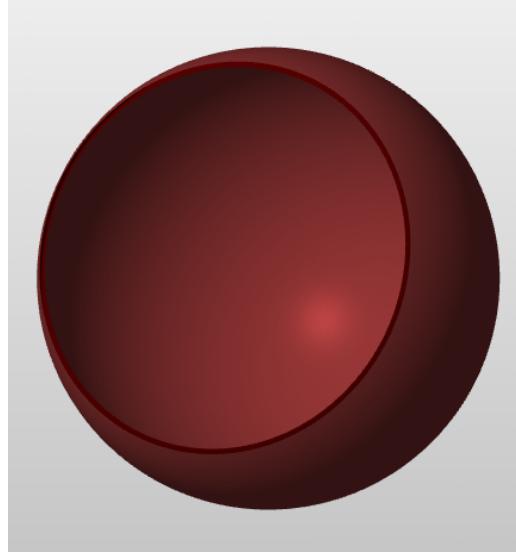


FIGURE 8.16. Sketch of a thin spherical shell with a radius R . The thickness of the shell is assumed to be much smaller than the radius of the sphere.

Input parameters for $P'(Q)$: Thin Spherical Shell:

R: most probable radius R
dummy: not used
sima: width σ of radius distribution (LogNorm)

Note

- This structure factor is supposed to be combined with a form factor of local planar objects which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local planar obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.

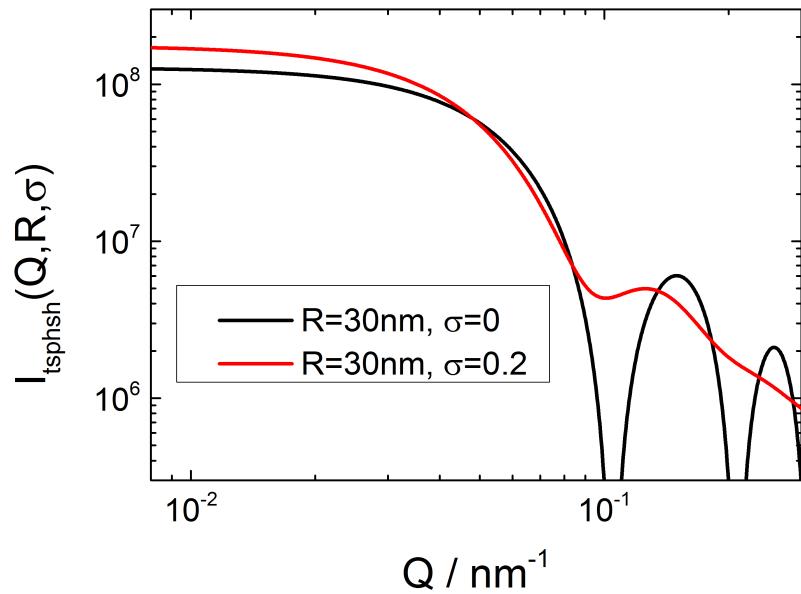


FIGURE 8.17. Scattering curve for the structure factor "P"(Q): Thin Spherical Shell" in combination with a constant background of 1".

8.2.3.3. $P'(Q)$: thin ellipsoidal shell.

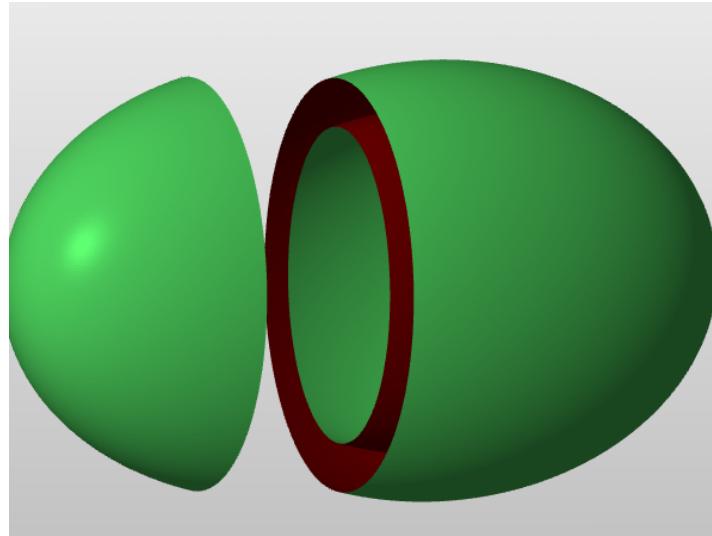


FIGURE 8.18. Sketch of a thin ellipsoidal shell with a radius R and eccentricity ϵ . The thickness of the shell is assumed to be much smaller than the two radii of the elliptical shell.

Input parameters for $P'(Q)$: Thin Ellipsoidal Shell:

`R`: most probable radius R

`epsilon`: eccentricity ϵ

`sima_R`: width σ_R of radius distribution (LogNorm)

Note

- This structure factor is supposed to be combined with a form factor of local planar objects which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local planar obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.

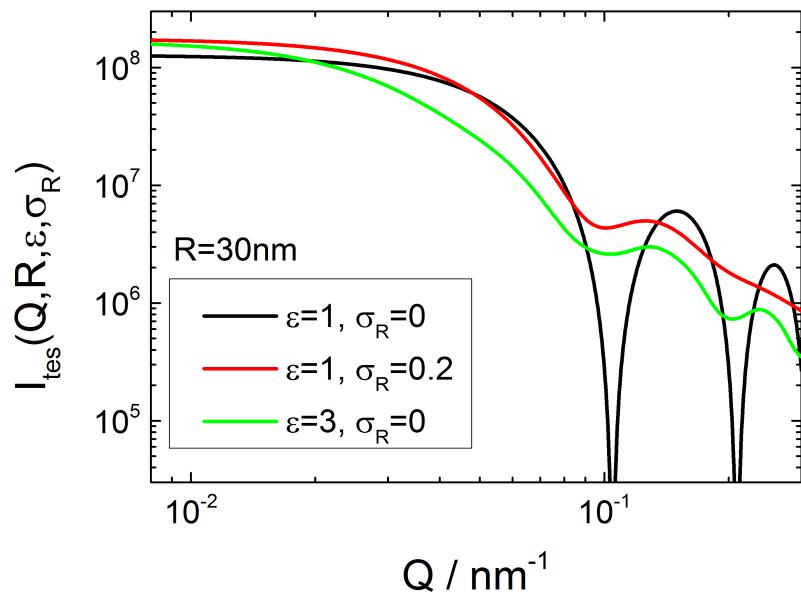


FIGURE 8.19. Scattering curve for the structure factor "P"(Q): Thin Ellipsoidal Shell" in combination with a constant background of 1".

8.2.3.4. $P'(Q)$: thin hollow cylinder.

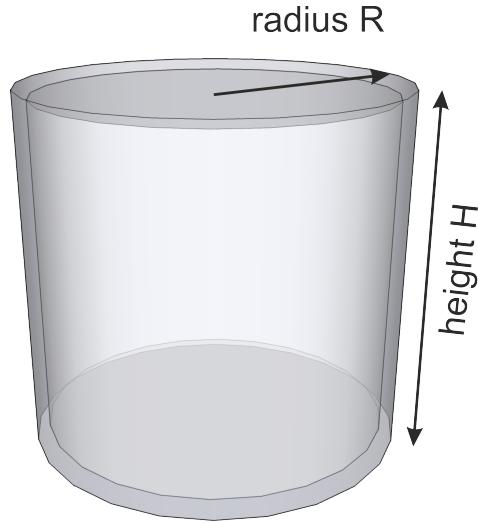


FIGURE 8.20. Sketch of a thin hollow cylinder of height H and radius R . The thickness of the wall is assumed to be much smaller than the outer dimensions of the cylinder.

$$\begin{aligned} \Xi(Q, R, H, \alpha) = 2\pi R & \left[R \frac{J_1(QR \sin \alpha)}{QR \sin \alpha} \right. \\ & \left. + H J_0\left(\frac{QH}{2} \cos \alpha\right) \frac{\sin\left(\frac{QH}{2} \cos \alpha\right)}{\frac{QH}{2} \cos \alpha} \right] \end{aligned} \quad (8.19)$$

$$P_{\text{thc}}(Q, R, H) = \int_0^{\pi/2} \Xi^2(Q, R, H, \alpha) \sin \alpha d\alpha \quad (8.20)$$

$$\begin{aligned} I_{\text{thc}}(Q, R, H) = \int_0^\infty & \int_0^\infty \text{LogNorm}(r, 1, \sigma_R, 1, R, 1) \\ & \text{LogNorm}(h, 1, \sigma_H, 1, H, 1) P_{\text{thc}}(Q, r, h) dh dr \end{aligned} \quad (8.21)$$

Input parameters for $P'(Q)$: Thin Hollow Cylinder:

- R: most probable cylinder radius R
- H: most probable cylinder height H
- sima_R: width σ_R of radius distribution (LogNorm)
- sima_H: width σ_H of height distribution (LogNorm)

Note

- This structure factor is supposed to be combined with a form factor of local planar objects which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local planar obj.]".

- The structure factor already has a log-normal width distribution for one parameter included.

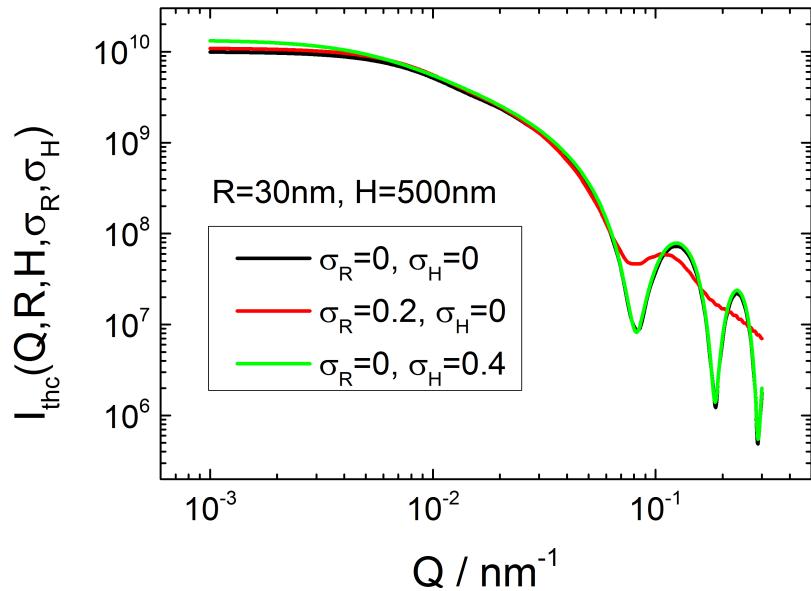


FIGURE 8.21. Scattering curve for the structure factor "P'(Q): Thin Hollow Cylinder" in combination with a constant background of 1".

8.2.4. $P'(Q)$ for local cylindrical obj.

8.2.4.1. $P'(Q)$: rods.

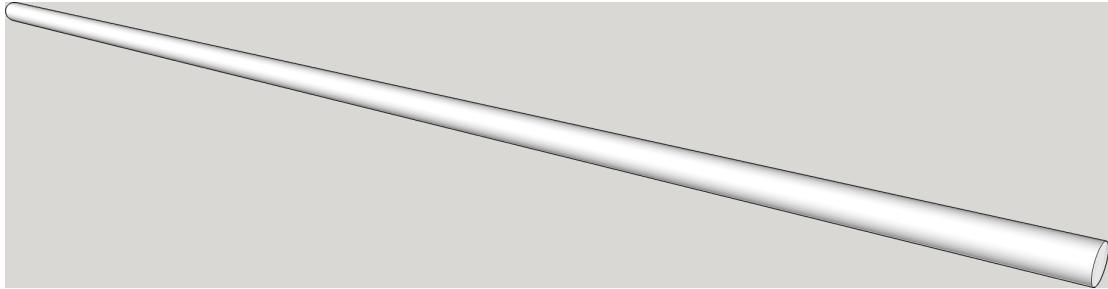


FIGURE 8.22. Sketch of a thin Rod of Length L . The diameter of the rod is assumed to be much smaller than its length.

Note

- This structure factor is supposed to be combined with a form factor with local cylindrical geometry which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local cylindrical obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.

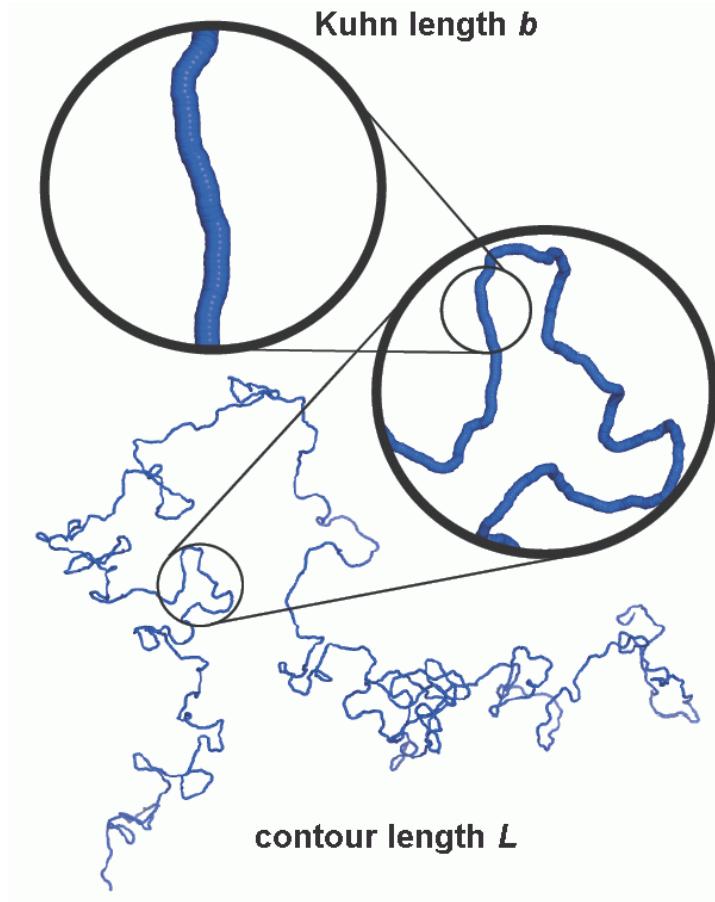
8.2.4.2. $P'(Q)$: Kholodenko's worm.

FIGURE 8.23.

By using the analogy between Dirac's fermions and semi-flexible polymers Kholodenko [112] could give a simple expression for the scattering behaviour of wormlike structures. The form factor $P_0(Q)$ resulting from Kholodenko's approach is designed to reproduce correctly the rigid-rod limit and the random-coil limit. Defining $x = 3L/l_b$ (L : contour length, l_b : Kuhn length), it is given by

$$P_0(Q, L, l) = \frac{2}{x} \left[I_{(1)} - \frac{1}{x} I_{(2)} \right] \quad (8.22)$$

where

$$I_{(n)}(x) = \int_0^x f(z) z^{n-1} dz \quad (8.23)$$

together with

$$f(z) = \begin{cases} \frac{1}{E} \frac{\sinh(Ez)}{\sinh(z)} & \text{for } Q \leq \frac{3}{l} \\ \frac{1}{F} \frac{\sin(Fz)}{\sinh(z)} & \text{for } Q > \frac{3}{l} \end{cases} \quad (8.24)$$

and

$$E = \sqrt{1 - \left(\frac{lQ}{3}\right)^2} \quad \text{and} \quad F = \sqrt{\left(\frac{lQ}{3}\right)^2 - 1} \quad (8.25)$$

$$P(Q, L, l_b, R) = P_0(Q, L, l_b) P_{cs}(Q, R) \quad (8.26)$$

Input Parameters for model P' (Q) Kholodenko Worm:

1b: Kuhn length¹ l of semi-flexible worm-like structure
L: contour length L of semi-flexible worm-like structure

Note

- This structure factor is supposed to be combined with a form factor with local cylindrical geometry which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local cylindrical obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.
- The equivalent solution of J.S. Pedersen [168] would be the expressions of worm-like structures without excluded volume effects.

¹The Kuhn length l_b is related to the length a of locally stiff segment simply via $l_b = 2a$

8.2.4.3. $P'(Q)$: wormlike PS1.

For worm-like micelles J.S. Pedersen [168] has developed three models for worm-like micelles, from which one is basing on a method developed by Yoshizaki et al. [232].

Without Excluded Volume Effects.

This first of the three solutions given by J.S. Pedersen [168] for worm-like micelles starts from the expression for the scattering function as follows:

$$S_{WC}(Q, L, l_B) = L^2 [(1 - \chi(Q, L, l_B)) S_{chain}(Q, L, l_B) + \chi(Q, L, l_B) S_{rod}(Q, L)] \Gamma(Q, L, l_B) \quad (8.27)$$

where $S_{chain}(Q, L, l_B)$ is the scattering function of a flexible chain without excluded volume effects and $S_{rod}(Q, L)$ is the scattering function of a rod. Furthermore, $\chi(Q, L, l_b)$ is a crossover function, and the function $\Gamma(q, L, b)$ corrects the crossover region. The function $S_{chain}(Q, L, l_B)$ is given by the Debye function:

$$S_{chain}(Q, L, l_B) = S_{Debye}(Q, L, l_B) = 2 [\exp(-u) + u - 1] / u^2 \quad (8.28)$$

with $u = \langle R_g^2 \rangle_0 Q^2$, where $\langle R_g^2 \rangle_0$ is the ensemble average of the square of the radius of gyration and given by

$$\langle R_g^2 \rangle_0 = \frac{Ll_B}{6} \left[1 - \frac{3}{2n_B} + \frac{3}{2n_B^2} - \frac{3}{4n_B^3} [1 - \exp(-2n_B)] \right] \quad (8.29)$$

where $n_B = L/l_B$ b is the number of statistical segments of the chain. The function $S_{rod}(Q, L)$ in 8.27 is the scattering function of an infinitely thin rod

$$S_{rod}(Q, L) = 2Si(Q, L)/(QL) - 4 \sin^2(QL/2)/(QL)^2 \quad (8.30)$$

where

$$Si(Q, L) = \int_0^x t^{-1} \sin t dt \quad (8.31)$$

Furthermore we have

$$\chi(Q, L, l_B) = \exp(-\xi^{-5}) \quad (8.32)$$

The parameter ξ is given by

$$\xi = Q \frac{\pi}{2L} \langle R_g^2 \rangle_0 \quad (8.33)$$

The function $\Gamma(Q, L, l_B)$ is given by

$$\Gamma(Q, L, l_B) = 1 + (1 - \chi) \sum_{i=2}^5 A_i \xi^i + \chi \sum_{i=0}^2 B_i \xi^{-i} \quad (8.34)$$

TABLE 1. Values for the parameters in the numerical expressions for the scattering function for worm-like chains **without** excluded volume effects [168]

$a1(2, 0)$	0.3054		$b_1(0, 0)$	-0.0162			
$a1(3, 0)$	0.05777		$b_1(1, 0)$	0.09046			
$a1(4, 0)$	-0.00604		$b_1(2, 0)$	0.1213			
$a1(5, 0)$	-0.03902		$b_1(0, 1)$	-0.3565	$b_2(0, 1)$	-0.3946	
$a1(2, 1)$	0.2316	$a2(2, 1)$	-0.4963	$b_1(1, 1)$	0.1909	$b_2(1, 1)$	-0.2231
$a1(3, 1)$	0.26531	$a2(3, 1)$	0.03688	$b_1(2, 1)$	0.15634	$b_2(2, 1)$	-0.2546
$a1(4, 1)$	0.3706	$a2(4, 1)$	0.30570	$b_1(0, 2)$	-0.3078	$b_2(0, 2)$	1.1361
$a1(5, 1)$	-1.0081	$a2(5, 1)$	0.39013	$b_1(1, 2)$	0.05176	$b_2(1, 2)$	-0.01615
$a1(2, 2)$	-22.779	$a2(2, 2)$	-0.4678	$b_1(2, 2)$	0.01568	$b_2(2, 2)$	-0.07606
$a1(3, 2)$	23.2457	$a2(3, 2)$	0.3365				
$a1(4, 2)$	8.1092	$a2(4, 2)$	0.4290				
$a1(5, 2)$	-3.3603	$a1(5, 2)$	0.3737				

where

$$A_i = \sum_{j=0}^2 a_1(i, j) \left(\frac{L}{l_B}\right)^{-j} \exp(-10l_B/L) + \sum_{j=1}^2 a_2(i, j) \left(\frac{L}{l_B}\right)^j \exp(-2L/l_B) \quad (8.35)$$

and

$$B_i = \sum_{j=0}^2 b_1(i, j) \left(\frac{L}{l_B}\right)^{-j} + \sum_{j=1}^2 b_2(i, j) \left(\frac{L}{l_B}\right)^j \exp(-2L/l_B) \quad (8.36)$$

The values of $a_1(i, j)$, $a_2(i, j)$, $b_1(i, j)$, and $b_2(i, j)$ are listed in table 1

With Excluded Volume Effects.

To include excluded volume effects in eq. 8.27 the function for a flexible chain $S_{\text{chain}}(Q, L, l_B)$ needs to be replaced by one with excluded volume effects included

$$S_{\text{WC}}(Q, L, l_B) = L^2 [(1 - \chi(Q, L, l_B)) S_{\text{exv}}(Q, L, l_B) + \chi(Q, L, l_B) S_{\text{rod}}(Q, L)] \Gamma(Q, L, l_B) \quad (8.37)$$

The scattering of a chain with excluded volume effects is given by

$$S_{\text{exv}}(Q, L, l_B) = w(Q, R_g) S_{\text{Debye}}(Q, L, l_B) + [1 - w(qR_g) [C_1 (QR_g)^{-1/\nu} + C_2 (QR_g)^{-2/\nu} + C_3 (QR_g)^{-3/\nu}]] \quad (8.38)$$

where $S_{\text{Debye}}(Q, L, l_B)$ is given by eq. 8.28 and

$$R_g^2 = \alpha(L/l_B)^2 \langle R_g^2 \rangle^2 \quad (8.39)$$

with $\alpha(L/l_B)$ being an expansion factor following the expression:

$$\alpha(x)^2 = [1 + (x/3.12)^2 + (x/8.67)^3]^{\epsilon/3} \quad (8.40)$$

and $\epsilon = 0.170$. The function $w(x)$ is another empirical cross-over function chosen as:

$$w(x) = \left[1 + \tanh \left(\frac{x - C_4}{C_5} \right) / C_5 \right] / 2 \quad (8.41)$$

The parameters C_i have been given in [168] as $C_1 = 1.220$, $C_2 = 0.4288$, $C_3 = -1.651$, $C_4 = 1.523$, and $C_5 = 0.1477$.

Furthermore the parameter ξ in the cross-over function $\chi(Q, L, l_B)$ needs to be chosen as

$$\xi = Ql_B \left(\frac{\pi l_B}{1.103L} \right)^{3/2} \left[\frac{R_g^2}{l_B^2} \right]^{1.282} \quad (8.42)$$

with R_g^2 defined in eq. 8.39. Last, but not least the values of $a_1(i, j)$, $a_2(i, j)$, $b_1(i, j)$, and $b_2(i, j)$ need to be taken from table 2.

Note

- This structure factor is supposed to be combined with a form factor with local cylindrical geometry which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local cylindrical obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.

TABLE 2. Values for the parameters in the numerical expressions for the scattering function for worm-like chains **with** excluded volume effects [168]

$a1(2, 0)$	-0.1222		$b_1(0, 0)$	-0.0699			
$a1(3, 0)$	0.3051		$b_1(1, 0)$	-0.0900			
$a1(4, 0)$	-0.0711		$b_1(2, 0)$	0.2677			
$a1(5, 0)$	0.0584		$b_1(0, 1)$	0.1342	$b_2(0, 1)$	-0.5171	
$a1(2, 1)$	1.761	$a2(2, 1)$	0.1212	$b_1(1, 1)$	0.0138	$b_2(1, 1)$	-0.2028
$a1(3, 1)$	2.252	$a2(3, 1)$	-0.4169	$b_1(2, 1)$	0.1898	$b_2(2, 1)$	-0.3112
$a1(4, 1)$	-1.291	$a2(4, 1)$	0.1988	$b_1(0, 2)$	-0.2020	$b_2(0, 2)$	0.6950
$a1(5, 1)$	0.6994	$a2(5, 1)$	0.3435	$b_1(1, 2)$	-0.0114	$b_2(1, 2)$	-0.3238
$a1(2, 2)$	-26.04	$a2(2, 2)$	0.0170	$b_1(2, 2)$	0.0123	$b_2(2, 2)$	-0.5403
$a1(3, 2)$	20.00	$a2(3, 2)$	-0.4731				
$a1(4, 2)$	4.382	$a2(4, 2)$	0.1869				
$a1(5, 2)$	1.594	$a1(5, 2)$	0.3350				

8.2.4.4. $P'(Q)$: wormlike PS2.

Note

- This structure factor is supposed to be combined with a form factor with local cylindrical geometry which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local cylindrical obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.

8.2.4.5. $P'(Q)$: wormlike PS3.

Note

- This structure factor is supposed to be combined with a form factor with local cylindrical geometry which are implemented as form factor plugins under "[by plugin|anisotropic obj.|Pcs(Q): local cylindrical obj.]".
- The structure factor already has a log-normal width distribution for one parameter included.

8.2.5. local planar obj.

8.2.6. local cylindrical obj.

8.3. Spherical core-shell structures with smooth or fuzzy interfaces

This plugin contains a collection of form factor for spherical core-shell structure with a smooth interface. The smooth interfaces are described by radial profiles of a form which are analytical integrable, i.e. for which the following integral for calculating the scattering amplitude $A_i(Q)$ of the i^{th} shell has an analytical solution.

$$A_i(Q) = \int_{R_i}^{R_i+t_i} \eta_i(r) 4\pi r^2 \frac{\sin(Qr)}{Qr} dr \quad (8.43)$$

Radial profiles for which the this integral can be solved are

$$\eta_{a,i}(r) = (\eta_{out,i} - \eta_{in,i}) \frac{r - R}{t} + \eta_{in,i} \quad (8.44a)$$

$$\eta_{b,i}(r) = (\eta_{out,i} - \eta_{in,i}) \left(\frac{r - R}{t} \right)^2 + \eta_{in,i} \quad (8.44b)$$

$$\eta_{c,i}(r) = (\eta_{out,i} - \eta_{in,i}) \exp \left(\frac{r - R}{t} \right) + \eta_{in,i} \quad (8.44c)$$

$$\eta_{d,i}(r) = \dots \quad (8.44d)$$

$$\eta_{e,i}(r) = \dots \quad (8.44e)$$

$$\eta_{f,i}(r) = \dots \quad (8.44f)$$

$$(8.44g)$$

8.3.1. JuelichCoreShell.

This model considers a dense core and original two shells [229]. Besides, it considers two different density profiles: a parabolic and a star-like profile for the second shell.

$$\eta_{\text{shell}}(r) \propto r^{-x} \quad \text{for starlike profile } x = 4/3 \quad (8.45)$$

$$\eta_{\text{shell}}(r) \propto 1 - \left(\frac{r}{L_p}\right)^2 \quad \text{for parabolic profile of thickness } L_p \quad (8.46)$$

Model parameters:

- b_{solv} : scattering length density of the solvent
- I_0 : forward scattering
- M_{core} : molecular weight of core (g/mol)
- M_{brush} : molecular weight brush (g/mol)
- ρ_{core} : mass density of core matter (g/cm³)
- ρ_{brush} : mass density of brush matter (g/cm³)
- b_{core} : scattering length density of core material (cm⁻²)
- b_{brush} : scattering length density of brush material (cm⁻²)
- N_{agg} : aggregation number (real number)
- d_c^+ : extra radius of core (compared to compact)
- p_{12} : relative distribution of shell amount in (1stshell:2ndshell) (0 . . . ∞)
- d_1^+ : extra radius of first shell (compared to compact)
- d_2^+ : extra radius of second shell (compared to compact)
- σ_c : core smearing
- σ_1 : smearing of 1st shell
- σ_2 : smearing of 2nd shell
- x_{star} : relative distribution of parabolic:starlike profile in 2nd shell, one has to put a very high value in order to consider only a star-like profile.
- γ : for star-like profile the exponent is 4/3 and for a constant profile chose 0
- L_p : thickness of parabolic brush (must fit in 2nd shell!)

$$I(Q) = [\Delta b_c F_c + \Delta b_b (F_1 + F_2)]^2 \quad (8.47)$$

$$\Delta b_c = b_{\text{core}} - b_{\text{solv}}(1 - f_{\text{core}}) \quad (8.48)$$

$$\Delta b_b = b_{\text{brush}} - b_{\text{solv}}(1 - f_{\text{brush}}) \quad (8.49)$$

V_c and V_b are the core and shell bulk volumes respectively.

Mass Conservation:

From the given values of the molecular weights of the two blocks and their densities, and an assumed aggregation number N_{agg} , the bulk volumes of the core and the shell, V_c and V_b , can be calculated.

Core:

$$\text{bulk core volume: } V_c = \frac{N_{\text{agg}} M_{\text{core}}}{\rho_{\text{core}} N_a} \quad (8.50)$$

$$\text{minimal radius of core: } R_c^0 = \left(\frac{3}{4\pi} V_c \right)^{1/3} \quad (8.51)$$

$$\text{effective core radius: } R_c = R_c^0 + d_c^+ \quad (8.52)$$

$$\text{swollen core volume: } V_{sc} = \frac{4}{3}\pi R_c^3 \quad (8.53)$$

$$\text{swelling factor: } s_c = \frac{V_{sc}}{V_c} \quad (8.54)$$

Shell:

$$\text{bulk shell volume: } V_b = \frac{N_{\text{agg}} M_{\text{shell}}}{\rho_{\text{shell}} N_a} \quad (8.55)$$

The relative amount of shell material in the first shell f_{shell1} is controlled by the parameter p_{12} , so that the portion of the second shell f_{shell2} can be obtained through:

$$f_{\text{shell1}} = \frac{p_{12}}{1 + p_{12}} \quad (8.56)$$

$$f_{\text{shell2}} = 1 - f_{\text{shell1}} \quad (8.57)$$

Shell 1:

$$\text{portion of the total shell volume in first shell: } V_{s1} = f_{\text{shell1}} V_b \quad (8.58)$$

$$\text{minimal radius of shell: } R_{c1} = \left(\frac{3}{4\pi} (V_{sc} + V_{s1}) \right)^{1/3} \quad (8.59)$$

$$\text{effective core radius: } R_1 = R_{c1} + d_1^+ \quad (8.60)$$

$$\text{swollen volume of first shell: } V_{s1s} = \frac{4}{3}\pi R_1^3 \quad (8.61)$$

$$\text{swelling factor: } s_{s1} = \frac{V_{s1s} - V_{sc}}{V_{s1}} \quad (8.62)$$

Shell 2:

$$\text{portion of the total shell volume in second shell: } V_{s2} = f_{\text{shell2}} V_b \quad (8.63)$$

$$\text{minimal radius of shell: } R_{c2} = \left(\frac{3}{4\pi} (V_{s1s} + V_{s2}) \right)^{1/3} \quad (8.64)$$

$$\text{effective core radius: } R_2 = R_{c2} + d_2^+ \quad (8.65)$$

$$\text{swollen volume of second shell: } V_{s2s} = \frac{4}{3}\pi R_2^3 \quad (8.66)$$

$$\text{swelling factor: } s_{s2} = \frac{V_{s2s} - V_{s1s}}{V_{s2}} \quad (8.67)$$

$$\text{fraction of star-like density profile in 2nd shell: } f_{\text{star}} = 2 \frac{\arctan(|p_{\text{star}}|)}{\pi} \quad (8.68)$$

Together with the profile functions $\Phi_c(r, R_c)$, $\Phi_1(r, R_1, R_2)$, $\Phi_2(r, R_1, R_2, f_{\text{star}})$ and

$$f_{\text{Fermi}}(x) = \frac{1}{1 + \exp(x)} \quad (8.69)$$

the volumes of the core and two shells and the corresponding form factor are determined by numerical integration.

Profiles:

$$\Phi_c(r, R_c) = f_{\text{Fermi}}(r - R_c) dr \quad (8.70)$$

$$\Phi_1(r, R_1, R_2) = (1 - f_{\text{Fermi}}(r - R_1)) f_{\text{Fermi}}(r - R_2) dr \quad (8.71)$$

for $r < R_1$

$$\begin{aligned} \Phi_2(r, R_1, R_2, f_{\text{star}}, \gamma) = & (1 - f_{\text{Fermi}}(r - R_1)) f_{\text{Fermi}}(r - R_2) \\ & \times \left[(1 - f_{\text{star}}) + \frac{f_{\text{star}}}{R_1^\gamma} \right] \end{aligned} \quad (8.72)$$

for $r > R_1$

$$\begin{aligned} \Phi_2(r, R_1, R_2, f_{\text{star}}, \gamma, L_p) = & (1 - f_{\text{Fermi}}(r - R_1)) f_{\text{Fermi}}(r - R_2) \\ & \times \left[(1 - f_{\text{star}}) \left(1 - \left(\frac{r - R_1}{L_p} \right)^2 \right) + \frac{f_{\text{star}}}{r^\gamma} \right] \end{aligned} \quad (8.73)$$

Input Parameters for model JuelichCoreShell:

- C: scaling constant C
- Mcore: molecular weight core (g/mol) M_{core}
- Mbrush: molecular weight brush (g/mol) M_{brush}
- rho_core: mass density core matter (g/cm³) ρ_{core}
- rho_brush: mass density brush matter (g/cm³) ρ_{brush}
- b_core: scattering length density of core material (cm⁻²) b_{core}
- b_brush: scattering length density of brush material (cm⁻²) b_{brush}
- Nagg: aggregation number N_{agg}
- d1_plus: extra radius of shell1=core (compared to compact) d_c^+
- part23: relative distribution of shell amount in (1stshell:2ndshell) (0 ... ∞) p_{12}
- d2_plus: extra radius of first shell2 (compared to compact) d_1^+
- d3_plus: extra radius of second shell3 (compared to compact) d_2^+
- sigma1: core smearing σ_c
- sigma2: smearing of 1st shell2 σ_1
- sigma3: smearing of 2nd shell3 σ_2
- partstar: relative distribution of parabolic:starlike profile in shell3 x_{star} ; one usually puts a very high value in order to consider only a star-like profile.
- gamma: for star-like profile the exponent is $\gamma = 4/3$ and for a constant profile $\gamma = 0$
- lparabol: thickness of parabolic brush L_p (must fit in shell3!)
- f_brush: scattering length density correction factor brush
- f_core: scattering length density correction factor core
- rhosolv: scattering length density of solvent b_{solv}

8.3.2. Fuzzy Sphere.

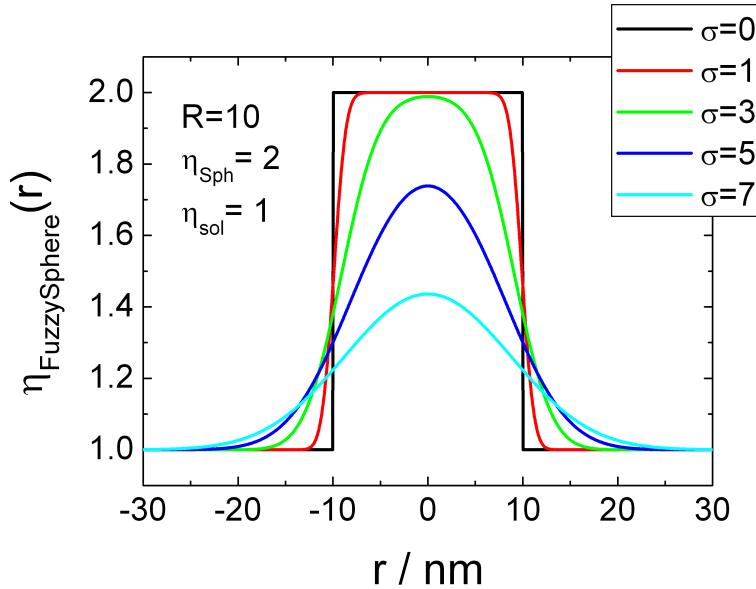


FIGURE 8.24. radial profile of a fuzzy sphere model

This model can be used to calculate the scattering from spherical particles with a "fuzzy" interface [201]. The fuzzy interface is obtained by convoluting the radial profile of a hard sphere with a Gaussian function.

$$\begin{aligned} \eta_{\text{FuzzySph}}(|\mathbf{r}|) &= (\eta_{\text{HS}} \star \eta_{\text{Gauss}})(\mathbf{r}) \\ &= \int_{\mathbb{R}^3} \eta_{\text{HS}}(\boldsymbol{\tau}) \eta_{\text{Gauss}}(\mathbf{r} - \boldsymbol{\tau}) d\boldsymbol{\tau} \end{aligned} \quad (8.74)$$

with

$$\eta_{\text{HS}}(|\mathbf{r}|) = \begin{cases} (\eta_{\text{sph}} - \eta_{\text{sol}}) & \text{for } |\mathbf{r}| \leq R \\ 0 & \text{for } |\mathbf{r}| > R \end{cases} \quad (8.75a)$$

$$\eta_{\text{Gauss}}(|\mathbf{r}|) = \frac{1}{2\sqrt{2}\pi^{3/2}|\sigma|^3} \exp\left[-\frac{|\mathbf{r}|^2}{2|\sigma|^2}\right] \quad (8.75b)$$

The convolution has to be done in \mathbb{R}^3 . As the hard sphere and Gaussian functions are radial symmetric also the profile of the fuzzy sphere only depends on $|\mathbf{r}|$. By defining the interface via a convolution the form factor can be easily calculated because the Fourier transform of a convolution is the pointwise product of the Fourier transforms according

to the convolution theorem, i.e.

$$\begin{aligned}
F(Q) &= \mathcal{F}[\eta_{\text{FuzzySph}}(r)] \\
&= \mathcal{F}[(\eta_{\text{HS}} * \eta_{\text{Gauss}})(r)] = \mathcal{F}[\eta_{\text{HS}}(r)] \mathcal{F}[\eta_{\text{Gauss}}(r)] \\
&= \int_0^\infty \eta_{\text{HS}}(r) 4\pi r^2 \frac{\sin(Qr)}{Qr} dr \int_0^\infty \eta_{\text{Gauss}}(r) 4\pi r^2 \frac{\sin(Qr)}{Qr} dr \\
&= (\eta_{\text{sph}} - \eta_{\text{sol}}) 4\pi R^3 \frac{\sin(QR) - QR \cos(QR)}{(QR)^3} e^{[-\frac{1}{2}\sigma^2 Q^2]} \tag{8.76}
\end{aligned}$$

Instead of calculating the convolution integral one also can get the radial profile of the fuzzy interface by the inverse Fourier transformation of the scattering amplitude

$$\eta_{\text{FuzzySph}}(r) = \int_0^\infty \frac{1}{(2\pi)^3} F(Q) 4\pi Q^2 \frac{\sin(Qr)}{Qr} dQ \tag{8.77}$$

$$\begin{aligned}
\eta_{\text{FuzzySph}}(r) &= (\eta_{\text{sph}} - \eta_{\text{sol}}) \\
&\left(\frac{\left(e^{-\frac{(r+R)^2}{2\sigma^2}} - e^{-\frac{(r-R)^2}{2\sigma^2}} \right) \sigma}{\sqrt{2\pi} r} + \frac{1}{2} \operatorname{erf} \left[\frac{r+R}{\sqrt{2}|\sigma|} \right] - \frac{1}{2} \operatorname{erf} \left[\frac{r-R}{\sqrt{2}|\sigma|} \right] \right) \tag{8.78}
\end{aligned}$$

Finally the scattering intensity is given by

$$\begin{aligned}
I_{\text{FuzzySph}}(Q) &= F^2(Q) = \\
&\left[(\eta_{\text{sph}} - \eta_{\text{sol}}) 4\pi R^3 \frac{\sin(QR) - QR \cos(QR)}{(QR)^3} e^{[-\frac{1}{2}\sigma^2 Q^2]} \right]^2 \tag{8.79}
\end{aligned}$$

The intensity $I_{\text{FuzzySph}}(Q)$ and also the scattering length density profile $\eta_{\text{FuzzySph}}(r)$ are normalized so that

$$\begin{aligned}
\lim_{Q \rightarrow 0} I_{\text{FuzzySph}}(Q) &= \left((\eta_{\text{sph}} - \eta_{\text{sol}}) \frac{4}{3}\pi R^3 \right)^2 \\
\int_0^\infty 4\pi r^2 \eta_{\text{FuzzySph}}(r) dr &= \frac{4}{3}\pi R^3
\end{aligned}$$

R = radius of the fuzzy sphere

σ = thickness of the fuzzy shell

η_{sph} : scattering length density of sphere

η_{sol} : scattering length density of the solvent

(8.80)

Input Parameters for model FuzzySphere and radial profile of FuzzySphere:

R: radius of the fuzzy sphere R

sigma: thickness of the fuzzy shell σ

eta_sph: scattering length density of sphere η_{sph}

eta_sol: scattering length density of solvent η_{sol}

Note:

- This form factor is only defined for positive radii $R > 0$.
- For $\sigma = 0$ the limiting case of a simple hard sphere form factor is used.
- In addition, scattering contributions arising from fluctuations of the microgel network are often included in this model expression as a Lorentzian function

$$I_{\text{fluct}}(Q) = \frac{I_{\text{fluct}}(0)}{1 + \xi^2 Q^2} \quad (8.81)$$

so that

$$I(Q) = I_{\text{FuzzySph}}(Q) + I_{\text{fluct}}(Q) \quad (8.82)$$

where $I_{\text{fluct}}(0)$ is the $Q = 0$ limiting intensity and ξ represents the correlation length of the fluctuations, which can be considered to be related to the blob or mesh size. It should be noted that the Lorentzian describes the ensemble average correlations in the polymer network.

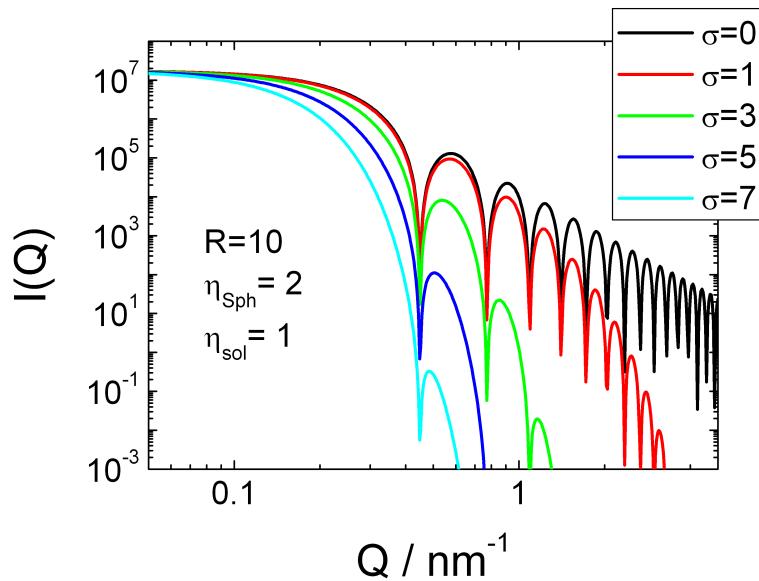


FIGURE 8.25. Scattering intensity of a fuzzy sphere. The scattering intensity has been calculated for a core radius $R = 10$, a scattering length density of the FuzzySphere of $\eta_{\text{sph}} = 1$, a scattering length density of the solvent $\eta_{\text{sol}} = 1$, and several widths of the "fuzzy" shell

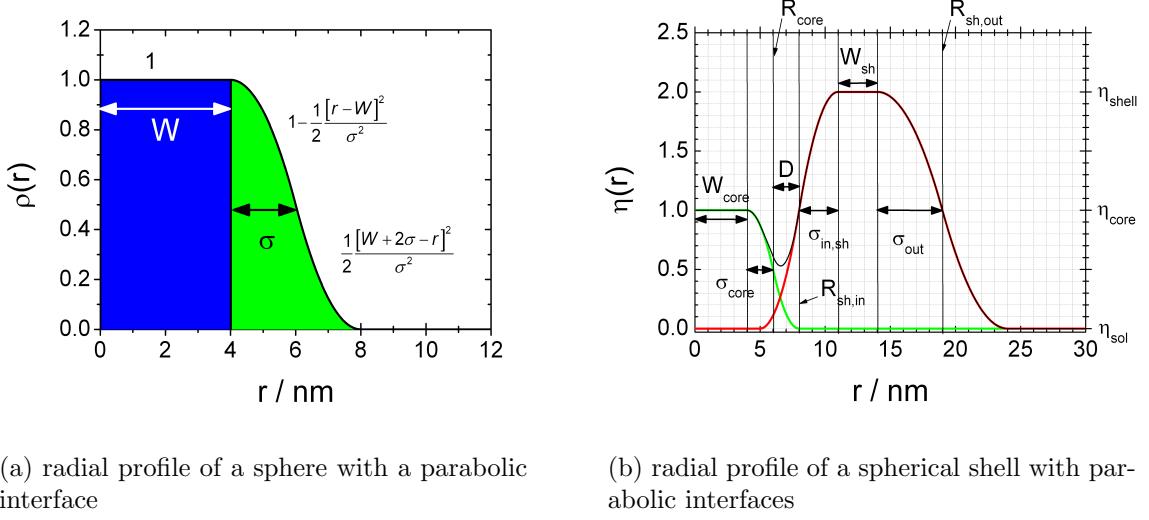


FIGURE 8.26.

8.3.3. CoreShellMicrogel. This model can be used to calculate the scattering from spherical particles with a parabolic "fuzzy" interface [15, 14, 16]. The radial profile is given by

$$\rho(r, R, \sigma) = \begin{cases} 1 & \text{for } r \leq R - \sigma \\ 1 - \frac{1}{2} \frac{((r-R)+\sigma)^2}{\sigma^2} & \text{for } R - \sigma < r \leq R \\ \frac{1}{2} \frac{((R-r)+\sigma)^2}{\sigma^2} & \text{for } R < r \leq R + \sigma \\ 0 & \text{for } r > R + \sigma \end{cases} \quad (8.83)$$

where $R = W + \sigma$. For such a radial profile the Fourier-transformation can be calculated analytically as

$$\begin{aligned} F(Q, R, \sigma) = \mathcal{F}[\rho(r, R, \sigma)] = \\ 4\pi \left(\left(\frac{R}{\sigma^2} + \frac{1}{\sigma} \right) \frac{\cos(q(R + \sigma))}{q^4} + \left(\frac{R}{\sigma^2} - \frac{1}{\sigma} \right) \frac{\cos(q(R - \sigma))}{q^4} \right. \\ \left. - 3 \frac{\sin(q(R + \sigma))}{q^5 \sigma^2} - 3 \frac{\sin(q(R - \sigma))}{q^5 \sigma^2} - 6 \frac{\sin(qR)}{q^5 \sigma^2} - 2R \frac{\cos(qR)}{q^4 \sigma^2} \right) \quad (8.84) \end{aligned}$$

The last term in the brackets needed to be corrected compared to the papers mentioned above due to a typo in the original papers. The radial scattering length density profile of a fuzzy core shell like in Fig. 8.26b can be obtained by

$$\begin{aligned} \eta_{core,sh}(r, W_{core}, \sigma_{core}, D, \sigma_{sh,in}, W_{sh}, \sigma_{sh,out}) = \eta_{sol} + (\eta_{shell} - \eta_{sol})\rho(r, R_{out}, \sigma_{out}) \\ + (\eta_{shell} - \eta_{sol})\rho(r, R_{sh,in}, \sigma_{sh,in}) + (\eta_{core} - \eta_{sol})\rho(r, R_{core}, \sigma_{core}) \quad (8.85) \end{aligned}$$

with

$$R_{\text{core}} = W_{\text{core}} + \sigma_{\text{core}} \quad (8.86\text{a})$$

$$R_{\text{sh,in}} = R_{\text{core}} + D \quad (8.86\text{b})$$

$$R_{\text{out}} = R_{\text{sh,in}} + \sigma_{\text{sh,in}} + W_{\text{sh}} + \sigma_{\text{sh,out}} \quad (8.86\text{c})$$

In the same way also the scattering amplitude $F_{\text{core,sh}}(Q, \dots)$ and the scattering intensity $I_{\text{core,sh}}(Q, \dots) = |F_{\text{core,sh}}(Q, \dots)|^2$ can be calculated

$$\begin{aligned} F_{\text{core,sh}}(Q, W_{\text{core}}, \sigma_{\text{core}}, D, \sigma_{\text{sh,in}}, W_{\text{sh}}, \sigma_{\text{sh,out}}) &= (\eta_{\text{shell}} - \eta_{\text{sol}})F(Q, R_{\text{out}}, \sigma_{\text{out}}) \\ &+ (\eta_{\text{shell}} - \eta_{\text{sol}})F(Q, R_{\text{sh,in}}, \sigma_{\text{sh,in}}) + (\eta_{\text{core}} - \eta_{\text{sol}})F(Q, R_{\text{core}}, \sigma_{\text{core}}) \end{aligned} \quad (8.87\text{a})$$

$$I_{\text{core,sh}}(Q, W_{\text{core}}, \sigma_{\text{core}}, D, \sigma_{\text{sh,in}}, W_{\text{sh}}, \sigma_{\text{sh,out}}) = |F_{\text{core,sh}}(Q, \dots)|^2 \quad (8.87\text{b})$$

Input parameters for "CoreShellMicrogel" and "radial profile of CoreShellMicrogel":

W_core: radius of center parts of core W_{core} with homogeneous scattering length density

sigma_core: interface half width of the core σ_{core}

W_shell: width of center parts of shell W_{sh} with homogeneous scattering length density

sigma_sh,in: half width of the inner interface of shell $\sigma_{\text{sh,in}}$

D: distance D between interface of core and in interface of shell

sigma_out: half width of the outer surface profile σ_{out}

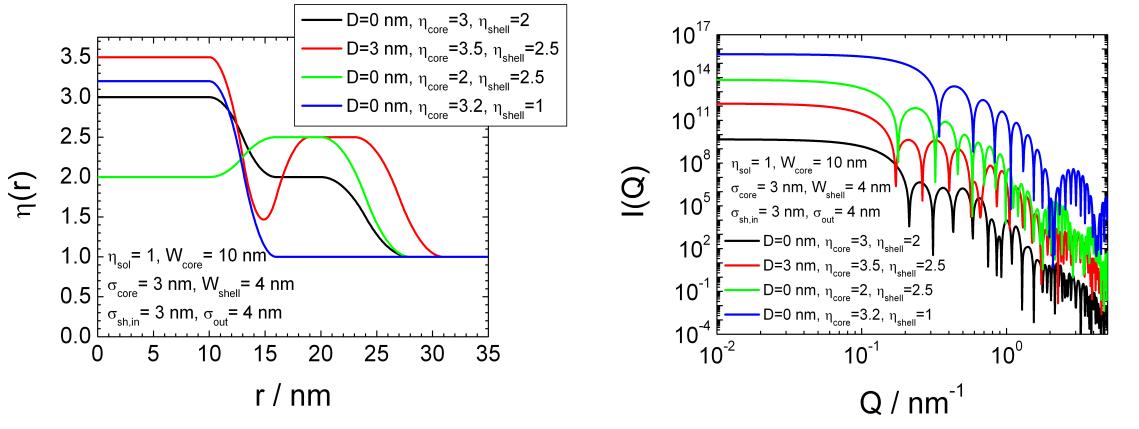
eta_core: scattering length density of homogeneous core part η_{core}

eta_shell: scattering length density of homogeneous shell part η_{shell}

eta_sol: scattering length density of solvent η_{sol}

Note

- If one like to simulate a simple step profile one should set $D = 0$ and $\sigma_{\text{core}} = \sigma_{\text{sh,in}}$. The last equality in case of fitting this parameter can be simply obtained by a global parameter under fitting multiple data sets.
- Instead of using the radii in eq. 8.86 as input parameters the thickness of the homogeneous parts of the core and the shell have been used to avoid a problems (negative dimensions) by applying an integration over a size distribution starting from 0 on the radii.



(a) Some radial profiles of spheres with a parabolic interfaces which have been used to calculate the scattering curve in Fig. (b).

(b) Scattering curves of the radial profiles shown in Fig. (a).

FIGURE 8.27. The profiles and scattering curves have been calculated with the plugin functions "CoreShellMicrogel" and "Radial Profile of CoreShellMicrogel".

8.3.4. Spherical shell with linear varying contrast profile (LinShell).

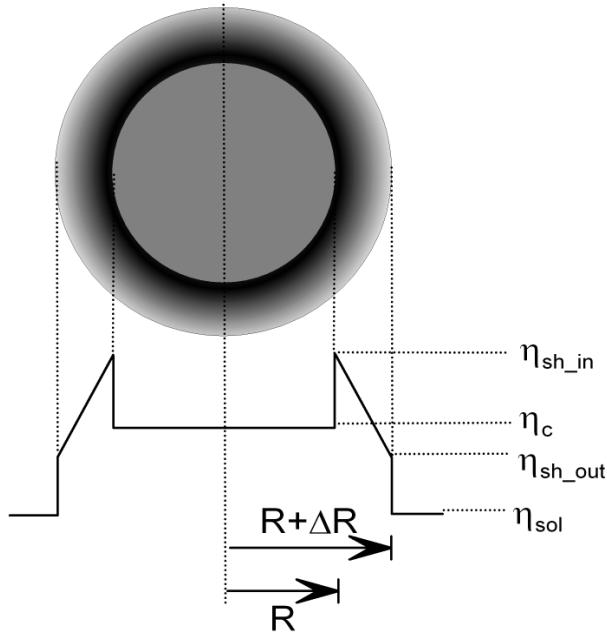


FIGURE 8.28. Radial profile for calculating the form factor of a spherical shell with a core radius R and a shell thickness of ΔR and a linear varying contrast profile.

Form factor of a spherical shell with a core radius R and a shell thickness of ΔR . Here a linear contrast profile within the shell has been assumed.

$$\Delta\eta(r) = \begin{cases} \eta_c & \text{for } r < R \\ mr + b & \text{for } r \in [R, R + \Delta R] \\ \eta_{\text{sol}} & \text{for } r > R + \Delta R \end{cases} \quad (8.88)$$

$$m = (\eta_{\text{sh,out}} - \eta_{\text{sh,in}})/\Delta R \quad (8.89)$$

$$b = -mR + \eta_{\text{sh,in}} \quad (8.90)$$

$$\eta_{\text{sh,in}} = (1 - x_{\text{in,sol}}) \eta_{\text{sh}} + x_{\text{in,sol}} \eta_{\text{sol}} - \eta_{\text{sol}} \quad (8.91)$$

: scattering length density at R

$$\eta_{\text{sh,out}} = (1 - x_{\text{out,sol}}) \eta_{\text{sh}} + x_{\text{out,sol}} \eta_{\text{sol}} - \eta_{\text{sol}} \quad (8.92)$$

: scattering length density at $R + \Delta R$

η_{sh} : scattering length density of pure shell material

η_c : scattering length density of core

$$F_{\text{sph}}(A, x) = \frac{4}{3}\pi x^3 3 \frac{\sin(A) - A \cos(A)}{A^3} \quad (8.93)$$

$$F_{\text{shlin}}(A, x) = 4\pi x^4 \frac{2 \cos(A) + 2A \sin(A) - A^2 \cos(A)}{A^4} \quad (8.94)$$

$$\begin{aligned} I_{\text{LinShell}} = & [(\eta_c - \eta_{\text{sol}} - b) F_{\text{sph}}(QR, R) \\ & - m F_{\text{shlin}}(QR, R) \\ & + m F_{\text{shlin}}(Q(R + \Delta R), R + \Delta R) \\ & + b F_{\text{sph}}(Q(R + \Delta R), R + \Delta R)]^2 \end{aligned} \quad (8.95)$$

Input Parameters for model LinShell and radial profile of LinShell:

R: radius of core R

dR: thickness of the shell ΔR

eta_c: scattering length density η_c

eta_sh: scattering length density of non-swollen shell η_{sh}

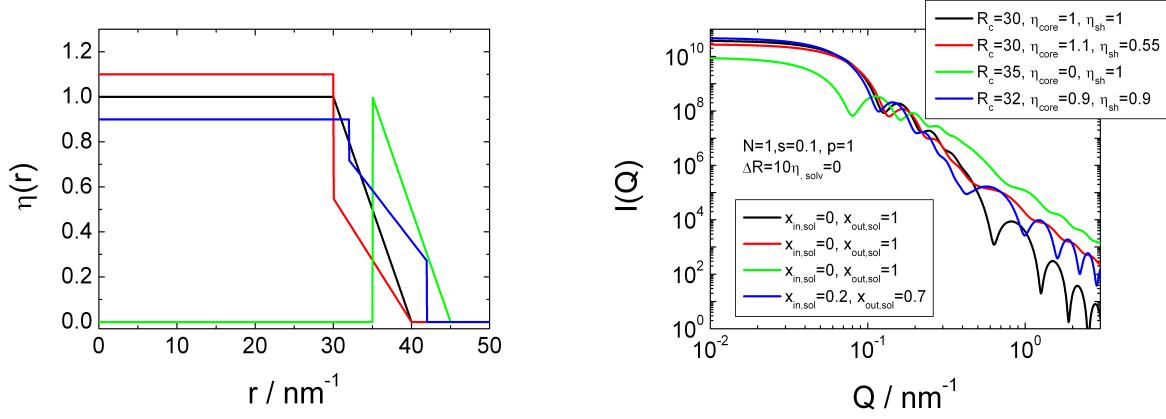
x_in: amount of solvent $x_{\text{in,sol}}$ on core-shell interface at R

x_out: amount of solvent $x_{\text{out,sol}}$ on shell-solvent interface at $R + \Delta R$

eta_sol: scattering length density of solvent η_{sol}

Note:

- $x_{\text{in,sol}}$ and $x_{\text{out,sol}}$ are only physical for values between 0 and 1.



(a) Some radial profiles of spheres with a linear interface profiles due to penetration of solvent into the shell which have been used to calculate the scattering curve in Fig. (b).

(b) Scattering curves of the radial profiles shown in Fig. (a).

FIGURE 8.29. Scattering intensity of a spherical shell with a linear shell profile. The scattering intensity has been calculated with a lognormal size distribution for the core radius R_c .

8.3.5. LinShell2.

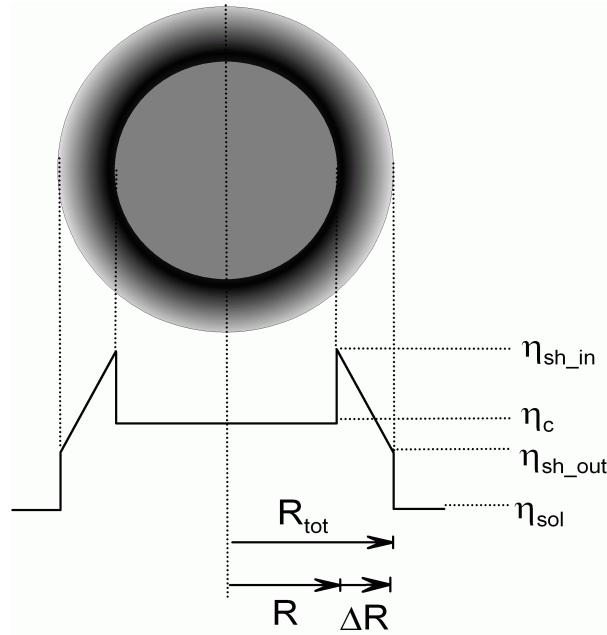


FIGURE 8.30. Radial profile for calculating the form factor of a spherical shell with a total radius R_{tot} , a shell thickness of ΔR , and a linear varying contrast profile.

Form factor of a spherical shell with a total radius R_{tot} and a shell thickness of ΔR . The definition are the same than for LinShell1 except that instead of the core radius R now the total radius R_{tot} is used to parameterize the form factor. The parameter definitions are the following:

$$R = \begin{cases} R_{\text{tot}} - \Delta R & \text{for } R_{\text{tot}} > \Delta R \\ 0 & \text{otherwise} \end{cases} \quad (8.96)$$

$$\Delta\eta(r) = \begin{cases} \eta_c & \text{for } r < R \\ mr + b & \text{for } r \in [R, R_{\text{tot}}] \\ \eta_{\text{sol}} & \text{for } r > R_{\text{tot}} \end{cases} \quad (8.97)$$

with

$$m = (\eta_{\text{sh_out}} - \eta_{\text{sh_in}})/\Delta R \quad (8.98)$$

$$b = -mR + \eta_{\text{sh_in}} \quad (8.99)$$

and

$$\eta_{sh,in} = (1 - x_{in,sol}) \eta_{sh} + x_{in,sol} \eta_{sol} \quad (8.100)$$

: scattering length density at R

$$\eta_{sh,out} = (1 - x_{out,sol}) \eta_{sh} + x_{out,sol} \eta_{sol} \quad (8.101)$$

: scattering length density at $R_{tot} = R + \Delta R$

η_{sh} : scattering length density of pure shell material

η_c : scattering length density of core

$x_{in,sol}$: amount of solvent at R

$x_{out,sol}$: amount of solvent at $R_{tot} = R + \Delta R$

$$F_{sph}(A, x) = \frac{4}{3}\pi x^3 \cdot 3 \frac{\sin(A) - A \cos(A)}{A^3} \quad (8.102)$$

$$F_{shlin}(A, x) = 4\pi x^4 \frac{2 \cos(A) + 2A \sin(A) - A^2 \cos(A)}{A^4} \quad (8.103)$$

$$I_{LinShell2} = [(\eta_c - \eta_{sol} - b) F_{sph}(QR, R) \\ - m F_{shlin}(QR, R) \\ + m F_{shlin}(QR_{tot}, R_{tot}) \\ + b F_{sph}(QR_{tot}, R_{tot})]^2 \quad (8.104)$$

Input Parameters for model LinShell2 and radial profile of LinShell2:

Rtot: total overall radius R_{tot}

dR: thickness of the shell ΔR

eta_c: scattering length density η_c

eta_sh: scattering length density of non-swollen shell η_{sh}

x_in: amount of solvent $x_{in,sol}$ on core-shell interface at R ($x_{in,sol} \in [0; 1]$)

x_out: amount of solvent $x_{out,sol}$ on shell-solvent interface at $R + \Delta R$ ($x_{out,sol} \in [0; 1]$).

eta_s: scattering length density of solvent η_{sol}

Note:

- $x_{in,sol}$ and $x_{out,sol}$ are only physical for values between 0 and 1.

8.3.6. ExpShell.

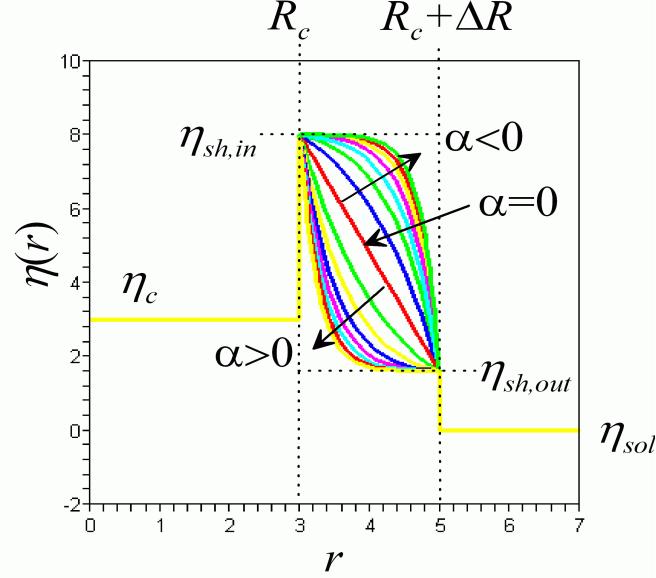


FIGURE 8.31. Radial profile for calculating the form factor of a spherical shell with a core radius R_c and a shell thickness of ΔR and a exponentially varying contrast profile. The profile shape can be varied by the parameter α describing the penetration of the solvent into the shell. A value of $\alpha = 0$ means linear profile and is equivalent to `LinShell` and `LinShell2`. For $\alpha > 0$ the solvent penetrates further into the shell and for $\alpha < 0$ less.

$$\eta_{\text{ExpShell}}(r, R_c, \Delta R, \alpha, \phi_{\text{in}}, \phi_{\text{out}}) = \begin{cases} \eta_c & r \leq R_c \\ \eta_{\text{exp}}\left(\frac{r-R_c}{\Delta R}\right) & R_c < r < R_c + \Delta R \\ \eta_{\text{sol}} & r > R_c + \Delta R \end{cases} \quad (8.105)$$

$$\eta_{\text{exp}}(x) = \begin{cases} \eta_{\text{sh,in}} + [\eta_{\text{sh,out}} - \eta_{\text{sh,in}}] x \exp([1-x]\alpha) & \alpha < 0 \\ [\eta_{\text{sh,in}} - \eta_{\text{sh,out}}] [1-x] \exp(-x\alpha) + \eta_{\text{sh,out}} & \alpha \geq 0 \end{cases} \quad (8.106)$$

$$\eta_{\text{sh,in}} = [\phi_{\text{in}} \eta_{\text{sol}} + (1 - \phi_{\text{in}}) \eta_{\text{sh}}] \quad (8.107)$$

$$\eta_{\text{sh,out}} = [\phi_{\text{out}} \eta_{\text{sol}} + (1 - \phi_{\text{out}}) \eta_{\text{sh}}] \quad (8.108)$$

The scattering intensity for the radial symmetric scattering length density profile $\eta_{\text{ExpShell}}(r)$ can be calculated analytical. The integral needed to be solved for that is

$$I_{\text{ExpShell}}(Q) = \int_0^\infty 4\pi r^2 \frac{\sin Qr}{Qr} \eta_{\text{ExpShell}}(r) dr \quad (8.109)$$

R_c = core radius

ΔR = shell thickness

$$\eta_{sh,in} = (1 - \phi_{in,sol}) \eta_{sh} + \phi_{in,sol} \eta_{sol} \quad (8.110)$$

: scattering length density at R_c

$$\eta_{sh,out} = (1 - \phi_{out,sol}) \eta_{sh} + \phi_{out,sol} \eta_{sol} \quad (8.111)$$

: scattering length density at $R_c + \Delta R$

η_{sh} : scattering length density of pure shell material

η_c : scattering length density of core

ϕ_{in} : amount of solvent at R_c

ϕ_{out} : amount of solvent at $R_c + \Delta R$

α : parameter for exponential diffuse profile of the shell (8.112)

Input Parameters for model ExpShell:

R_core: radius of core R_c

DR: thickness of the shell ΔR

eta_core: scattering length density η_c

eta_shell: scattering length density of non-swollen shell η_{sh}

x_in_sol: amount of solvent ϕ_{in} on core-shell interface at $r = R$

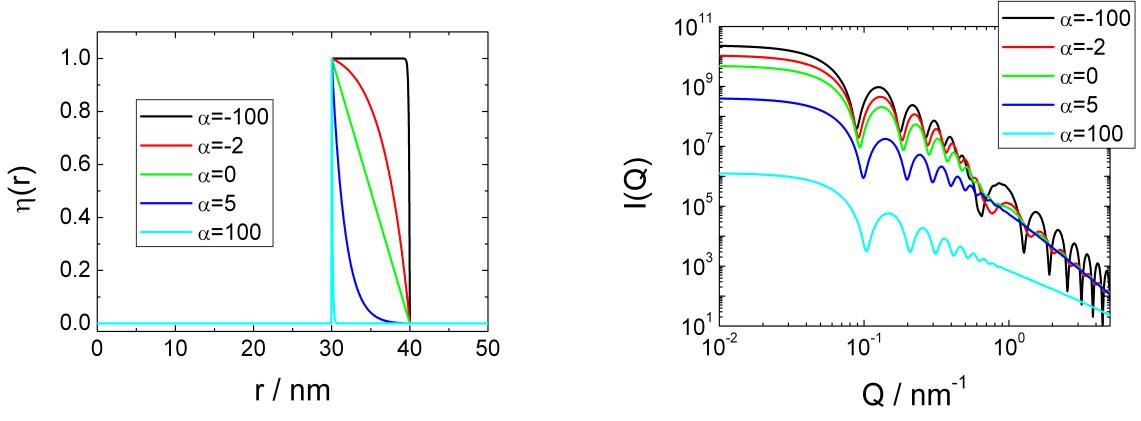
x_out_sol: amount of solvent ϕ_{out} on shell-solvent interface at $r = R + \Delta R$

alpha: a parameter (α) which describes the penetration profile of the solvent into the shell. A value of $\alpha = 0$ means linear profile and is equivalent to LinShell and LinShell2. For $\alpha > 0$ the solvent penetrates further into the shell and for $\alpha < 0$ less.

eta_solvent: scattering length density of solvent η_{sol}

Note:

- ϕ_{in} and ϕ_{out} are only physical for values between 0 and 1.



(a) Some radial profiles of spheres with exponential interfaces which have been used to calculate the scattering curve in Fig. (b).

(b) Scattering curves of the radial profiles shown in Fig. (a).

FIGURE 8.32. Scattering intensity of a spherical shell with an exponential shell profile. The scattering intensity has been calculated with a lognormal [$\text{LogNorm}(N = 1, \sigma = 0.05, p = 1, R = 30)$] size distribution for the core radius R_c . The scattering length density of the core η_c and the solvent η_{sol} are set to 0, $\eta_{\text{sh}} = 1$, $\phi_{\text{in}} = 0$, $\phi_{\text{out}} = 1$, and $\Delta R = 10$.

8.3.7. BoucherSphere, radial profile for a sphere resulting in a Porod law both below and above q^{-4} .

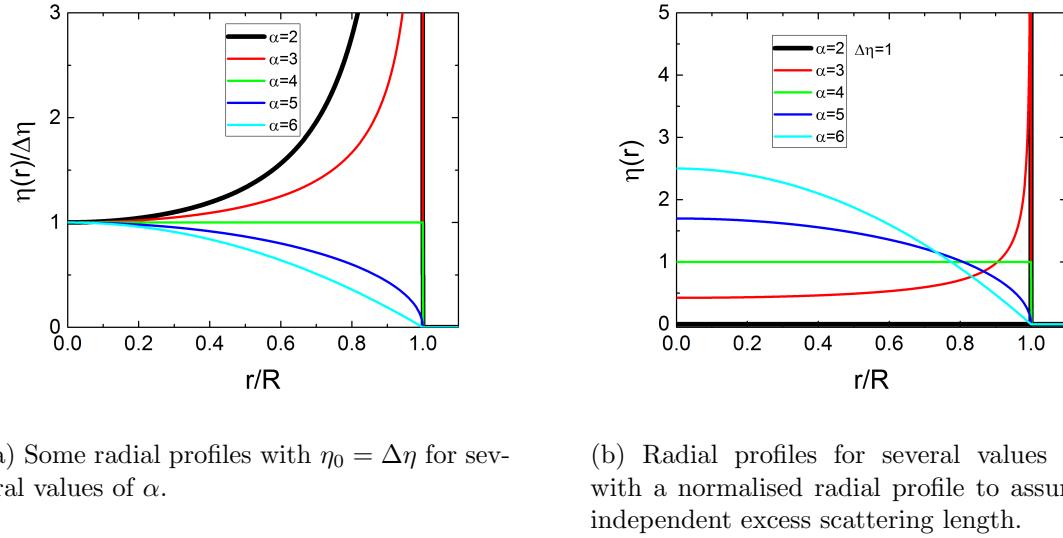


FIGURE 8.33. The two implemented case for the radial profiles according to Boucher [25]

Boucher has described in [25] a simple analytical radial profile for which also the form factor can be calculated analytically. The radial profile is parameterized in a way that any potential law in the Porod regime can be obtained. The radial profile is defined by

$$\eta(r) = \eta_0 \begin{cases} \left[1 - \left(\frac{r}{R}\right)^2\right]^{\frac{\alpha}{2}-2} & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \quad (8.113)$$

The excess scattering β of such a particle is

$$\beta = \int_0^R \eta(r) 4\pi r^2 dr = \eta_0 R^3 \pi^{\frac{3}{2}} \frac{\Gamma\left(\frac{\alpha}{2} - 1\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)} \quad (8.114)$$

The input value $\Delta\eta$ for this profile is the scatter contrast at $r = 0$, i.e.

$$\eta_0 = \Delta\eta \quad (\text{case BoucherSphere}) \quad (8.115)$$

The special case of $\alpha = 4$ corresponds to a homogeneous sphere with a scattering contrast $\Delta\eta$ for which the excess scattering length becomes $\beta_{sp} = \Delta\eta^{\frac{4}{3}} \pi R^3$.

Also a second variant of the form factor has been implemented. In the second case the excess scattering length is normalised to the one of a homogeneous sphere with

contrast $\Delta\eta$, so that the excess scattering length is for all α equal to $\beta_{\text{sp}} = \Delta\eta \frac{4}{3}\pi R^3$. This is done by setting the contrast at $r = 0$ in eq. 8.113 to

$$\eta_0 = \Delta\eta \frac{4}{3\sqrt{\pi}} \frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{\alpha}{2} - 1)} \quad (\text{case BoucherSphere2}) \quad (8.116)$$

By this normalisation the excess scattering length is for all α equal to the one of a homogeneous sphere with a scattering contrast $\Delta\eta$, i.e. $\beta_{\text{sp}} = \Delta\eta \frac{4}{3}\pi R^3$.

To get a feeling which part of the spherical scattering length density profile contributes most to the scattering it is convenient to define a fractional relative excess scattering length according to [25] which reads as

$$\begin{aligned} Z(x) &= \frac{\int_0^x \left(1 - \frac{r^2}{R^2}\right)^{\frac{\alpha}{2}-2} 4\pi r^2 dr}{\int_0^R \left(1 - \frac{r^2}{R^2}\right)^{\frac{\alpha}{2}-2} 4\pi r^2 dr} \\ &= \frac{4}{3} \frac{x^3 \Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi} \Gamma(\frac{\alpha}{2} - 1) R^3} {}_2F_1\left(\frac{3}{2}, -\frac{\alpha}{2} + 2; \frac{5}{2}; \frac{x^2}{R^2}\right) \end{aligned} \quad (8.117)$$

$Z(x)$ is the excess scattering of the sub-domain of the spherical particle ranging from $r = 0$ to $r = x$ relative to the excess scattering of the whole particle. The derivative $Z'(x)$ then defines the fractional contribution of a sub-shell with radius $r = x$ and thickness dx of the spherical particle compared to its total excess scattering length of the particle.

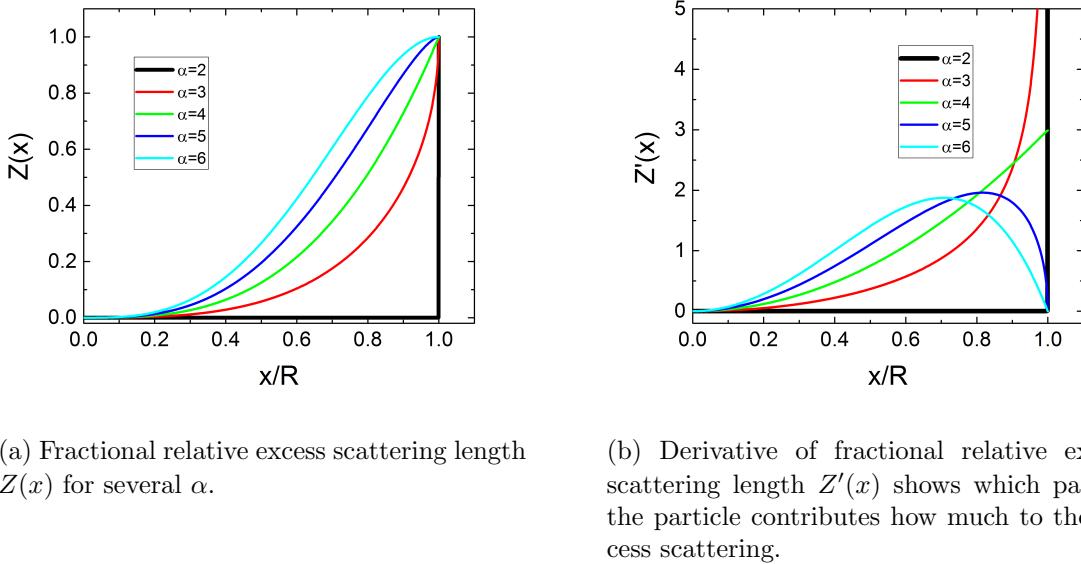


FIGURE 8.34. For $\alpha = 2$ all the scattering is caused by an infinitesimal thin outer shell of the particle where all the fractional excess scattering length is located.

The scattering amplitude for the radial profile in eq. 8.113 can be calculated as

$$\begin{aligned} F(Q) &= \int_0^R \eta(r) 4\pi r^2 \frac{\sin Qr}{Qr} dr \\ &= \beta \left(\frac{2}{QR} \right)^{\frac{\alpha-1}{2}} J_{\frac{\alpha-1}{2}}(QR) \Gamma \left(\frac{\alpha+1}{2} \right) \\ &= \beta {}_0F_1 \left(\frac{\alpha+1}{2}; -\frac{(QR)^2}{4} \right) \end{aligned} \quad (8.118)$$

making use of the fact that the bessel function can be expressed in terms of generalised hypergeometric functions $J_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+1)} {}_0F_1(\alpha+1; -\frac{x^2}{4})$. The scattering amplitude simplifies in case of $\alpha = 4$ exactly the one of a homogenous sphere and in case of $\alpha = 2$ to an infinitesimal thin spherical shell. For the scattering intensity $I(Q) = F^2(Q)$ we get in the Porod limit $Q \rightarrow \infty$

$$\lim_{Q \rightarrow \infty} I(Q) = \lim_{Q \rightarrow \infty} F^2(Q) = \beta^2 \frac{2^{\alpha-1}}{\pi} \Gamma^2 \left(\frac{\alpha+1}{2} \right) \frac{1}{(QR)^\alpha} \quad (8.119)$$

Input Parameters for model BoucherSphere and Boucher profile:

R: radius R

alpha: shape parameter of the profile α , which is also equal to the potential law at large Q -values

Delta_eta: scattering length density at $r = 0$, $\Delta\eta = \eta(r = 0)$

Note for model BoucherSphere and Boucher profile:

- For $\alpha < 4$ the scattering length density profile has a pole at $r = R$
- For $\alpha \leq 2$ the excess scattering length of the overall particle diverges and are therefore forbidden values.

Input Parameters for model BoucherSphere2 and Boucher profile2:

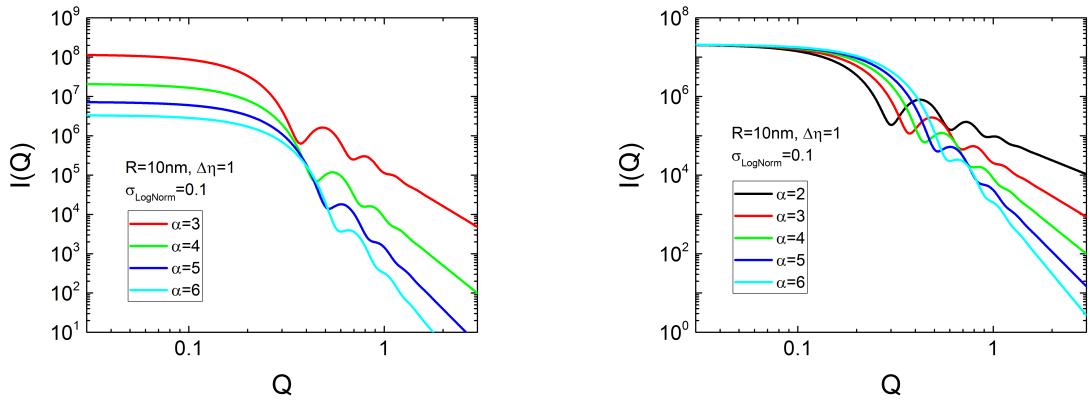
R: radius R

alpha: shape parameter of the profile α , which is also equal to the potential law at large Q -values

Delta_eta: is the average scattering length density so that the excess scattering length corresponds to the one of a homogeneous sphere with contrast $\Delta\eta$

Note for model BoucherSphere2 and Boucher profile:

- For $\alpha < 4$ the scattering length density profile has a pole at $r = R$
- There is no restriction for the input values of α in the form factor but in the profile, which only allows values for $\alpha > 2$. α -values below 2 do not have a standard physical interpretation.



(a) Scattering curves of **BoucherSpheres** for several α with a narrow LogNormal size distribution with $\sigma = 0.1$. As the excess scattering length diverges for $\alpha \leq 2$ these values are not allowed.

(b) Scattering curves of **BoucherSpheres2** for several α with a narrow LogNormal size distribution with $\sigma = 0.1$.

FIGURE 8.35. Scattering curves for the form factors **BoucherSpheres** and **BoucherSpheres2**.

8.4. Ellipsoidal Objects

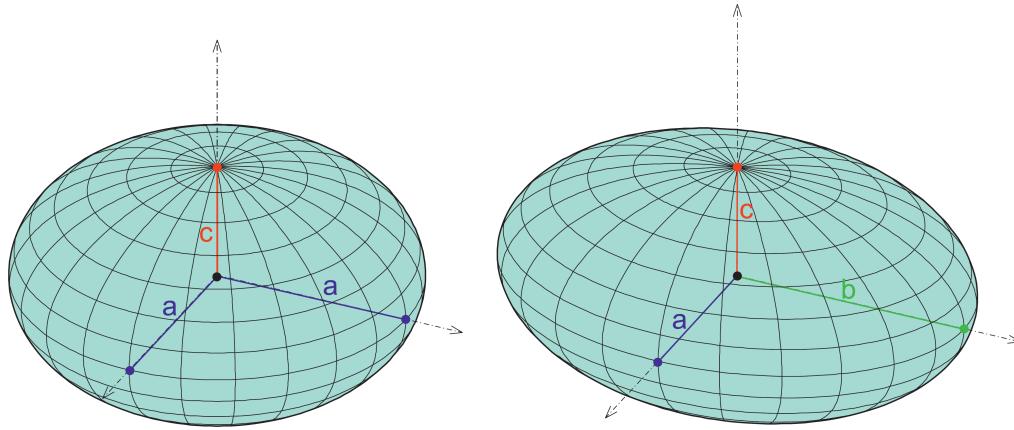


FIGURE 8.36. Ellipse of revolution (spheroid) and triaxial ellipse

The plugin "ellipsoidal obj." contains form factors of ellipsoidal objects with rotational symmetry, which are called ellipsoid of revolution or spheroid, as well as triaxial ellipsoidal objects. The three axes of the ellipsoids are assumed to be orthogonal. Because of the non-spherical symmetry of the objects an orientational average has to be performed for randomly oriented ellipsoids, which requires for spheroids a single integration and for triaxial ellipsoids a double integration. An optional integration over the size distribution would then slow down the numerical evaluation of the scattering intensity and care has to be taken, to use an appropriate integration routine. At the moment **SASfit** makes use of the **pcubature** algorithm from Steven G. Johnson [102] instead of nested 1-dimensional integration routines used so far. Due to **SASfit** internals an optional integration over the size distribution was realized via the form factor itself instead of including it via the GUI to profit from the speed-up of routines optimized for multi-dimensional integration.

8.4.1. Ellipsoid of revolution or spheroid.

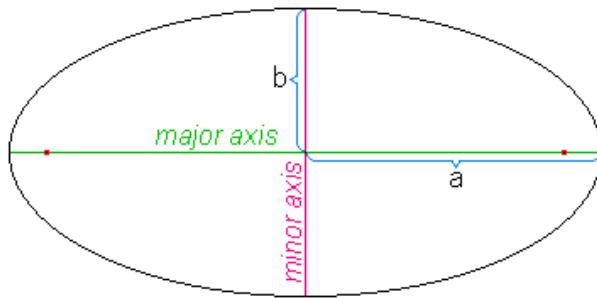


FIGURE 8.37. Ellipse, showing major and minor axes and parameters a and b

An ellipsoid is a quadric surface in three dimensions obtained by rotating an ellipse about one of its principal axes. Three particular cases of an ellipsoid are:

- If the ellipse is rotated about its major axis, the surface is a prolate spheroid.
- If the ellipse is rotated about its minor axis, the surface is an oblate spheroid.
- If the generating ellipse is a circle, the surface is a sphere.

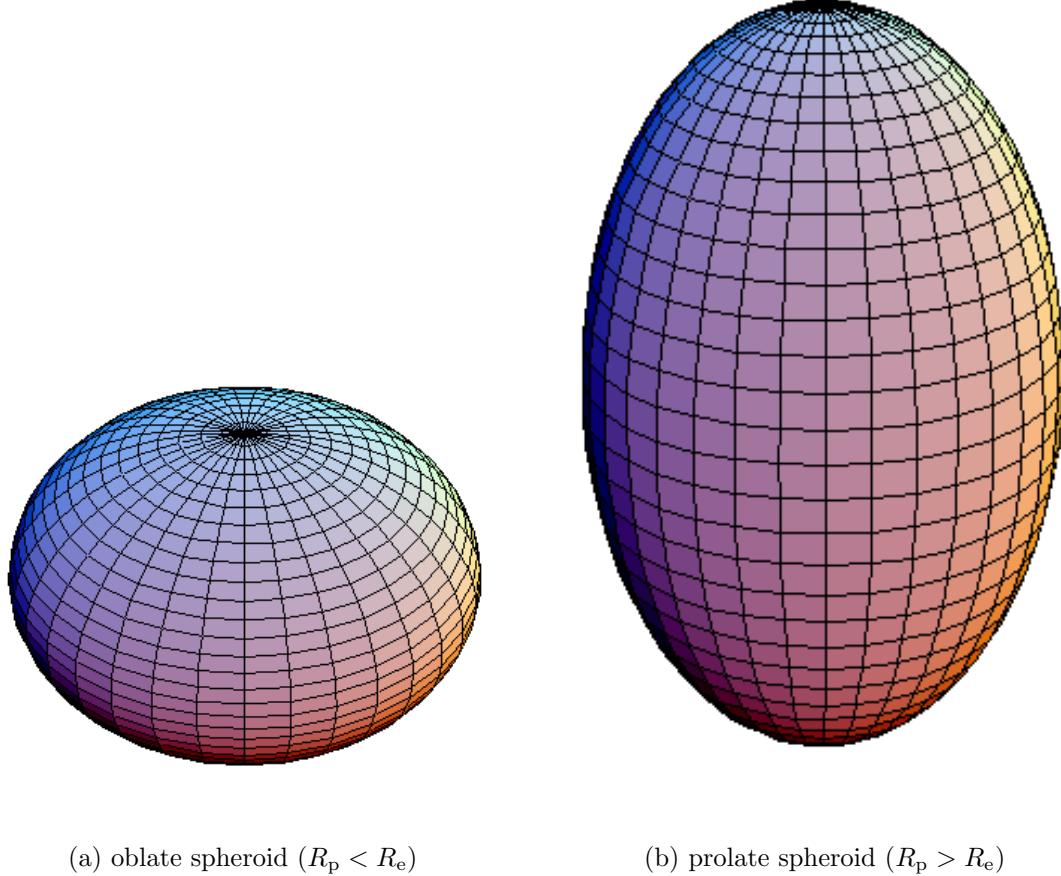


FIGURE 8.38. A spheroid is an ellipsoid having two equal equatorial semi-axes R_e . If the equatorial semi-axis are larger than the principal axis R_p the spheroid becomes oblate (a), if they are smaller it becomes prolate (b) and if they are equal the spheroid becomes a perfect sphere

The orientational averaged scattering amplitude and scattering intensity are given by [71]

$$\langle F_{\text{spheroid}}(Q, R_p, R_e, \Delta\eta) \rangle = V \Delta\eta \int_0^1 K \left(Q \sqrt{R_p^2 y^2 + R_e^2 (1-y)^2} \right) dy \quad (8.120)$$

$$I_{\text{spheroid}}(Q, R_p, R_e, \Delta\eta) = V^2 \Delta\eta^2 \int_0^1 K^2 \left(Q \sqrt{R_p^2 y^2 + R_e^2 (1-y)^2} \right) dy \quad (8.121)$$

with

$$V = \frac{4}{3}\pi R_e^2 R_p \quad (8.122)$$

$$K(x) = 3 \frac{\sin x - x \cos x}{x^3} \quad (8.123)$$

$$\Delta\eta = \eta_{\text{core}} - \eta_{\text{solv}} \quad (8.124)$$

The form factor of a spheroid has been implemented in three parametrisation variants. The variants **spheroid V** and **spheroid nu** replace the old variants **Ellipsoid i** and **Ellipsoid ii**, which are not anymore available. **spheroid R** is a new parametrisation variant. Their input parameters are

Input Parameters for model **spheroid nu**:

nu: ratio between radius of the polar axes and equatorial axis. Values of $\nu < 1$ describe oblate ellipsoids, a value of $\nu = 1$ a sphere, and $\nu > 1$ a prolate ellipsoids.

R_equatorial: length of the equatorial semi-axes R_e

dummy: not used

dummy: not used

dummy: not used

eta_core: scattering length density of spheroid

dummy: not used

eta_sol: scattering length density of solvent

Input Parameters for model **spheroid V**:

V: volume of spheroid

R_equatorial: length of equatorial semi-axes R_e

dummy: not used

dummy: not used

dummy: not used

eta_core: scattering length density of spheroid

dummy: not used

eta_sol: scattering length density of solvent

Input Parameters for model **spheroid R**:

R_polar: length of polar semi-axis R_p

R_equatorial: length of equatorial semi-axes R_e

dummy: not used

dummy: not used

dummy: not used

eta_core: scattering length density of spheroid

dummy: not used

eta_sol: scattering length density of solvent

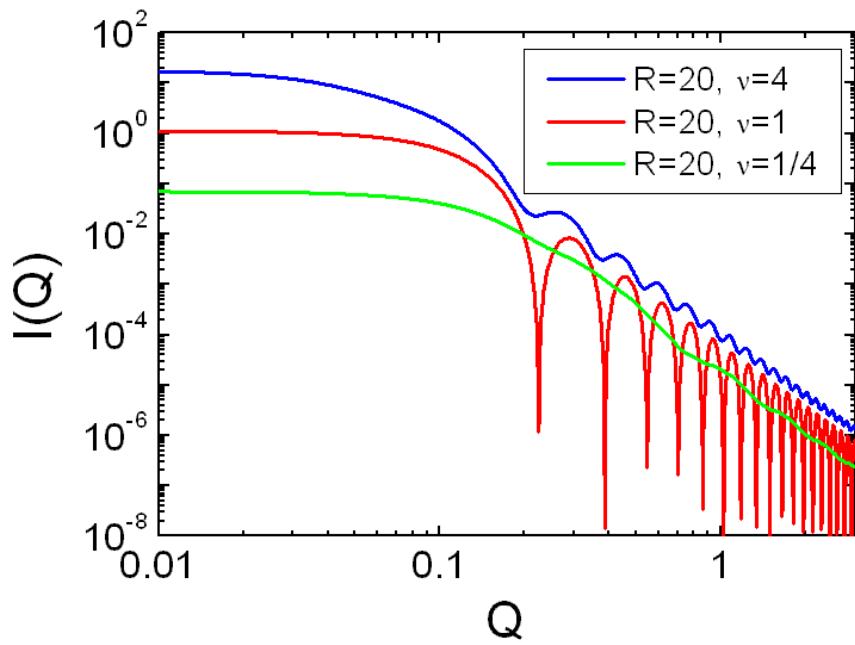


FIGURE 8.39. form factor of an ellipsoid with axis R , R and νR .

8.4.2. Ellipsoid with two equal equatorial semi-axis R and volume V .

$$I_i(Q, R, \nu) = (V \Delta \eta)^2 \int_0^{\frac{\pi}{2}} K^2 \left(Q, R \sqrt{\nu^2 \cos^2 \Theta + \sin^2 \Theta} \right) \sin \Theta d\Theta \quad (8.125)$$

with

$$\nu = \frac{V}{R^3} \frac{3}{4\pi} \quad \text{so that} \quad V = \frac{4}{3}\pi\nu R^3$$

and $\lim_{Q \rightarrow 0} I_i(Q, R, \nu) = (V \Delta \eta)^2$

Input Parameters for model Ellipsoid i:

R: radius of the rotational axes

V: total volume of the ellipsoid.

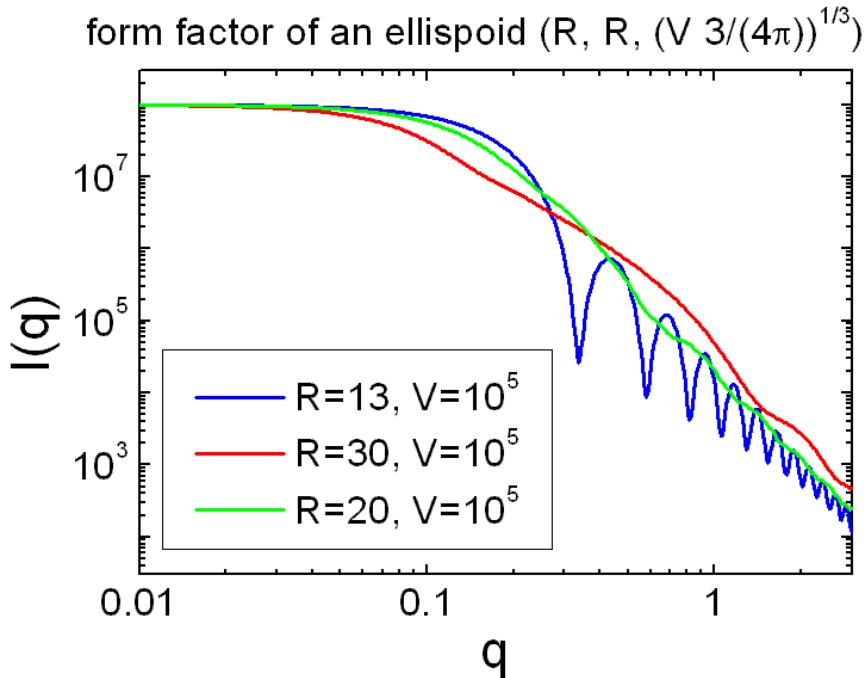


FIGURE 8.40. form factor of an ellipsoid with axis R , R and $\sqrt[3]{V \frac{3}{4\pi}}$.

8.4.3. Ellipsoidal core shell structure.

For these form factor plugins a size distribution parameter has been included directly in the form factor to avoid applying nested numerical integration routines. A routine for multi-dimensional integration (`pcubature` [102]) has been used instead.

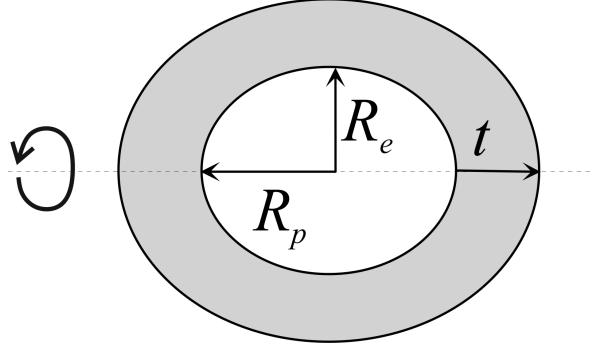


FIGURE 8.41. Ellipsoid of revolution with an outer shell of constant thickness t

$$I_{\text{ECSH}}(Q, R_p, R_e, t) = \langle F^2(Q, R_p, R_e, t) \rangle = \int_0^1 [F(Q, R_p, R_e, t, \mu)]^2 d\mu \quad (8.126)$$

$$\langle F(Q, R_p, R_e, t) \rangle = \int_0^1 F(Q, R_p, R_e, t, \mu) d\mu \quad (8.127)$$

with

$$\begin{aligned} F(Q, R_p, R_e, t, \mu) &= (\eta_{\text{core}} - \eta_{\text{shell}}) V_c \left[\frac{3j_1(x_c)}{x_c} \right] + (\eta_{\text{shell}} - \eta_{\text{sol}}) V_t \left[\frac{3j_1(x_t)}{x_t} \right] \\ j_1(x) &= \frac{\sin(x) - x \cos(x)}{x^2} \\ x_c &= Q \sqrt{R_p^2 \mu^2 + R_e^2 (1 - \mu^2)} \\ x_t &= Q \sqrt{(R_p + t)^2 \mu^2 + (R_e + t)^2 (1 - \mu^2)} \\ V_c &= \frac{4}{3} \pi R_p R_e^2 \\ V_t &= \frac{4}{3} \pi (R_p + t)(R_e + t)^2 \end{aligned}$$

- η_{core} : scattering length density of core
- η_{shell} : scattering length density of shell
- η_{sol} : scattering length density of solvent
- R_p : polar semi-axes of elliptical core
- R_e : equatorial semi-axis of elliptical core
- t : thickness of shell
- V_c : volume of core
- V_t : total volume of core along with shell

Several variants including a size distribution have been implemented which just differ if only none, one or both axis scale with the same size distribution and if additionally also scales with that distribution.

$$\begin{aligned}
 \text{Ellipsoidal Shell} &: \langle F^n(Q, R_p, R_e, t) \rangle \\
 \text{Ellipsoidal Shell (t)} &: \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, R_p, R_e, \nu t) \rangle d\nu \\
 \text{Ellipsoidal Shell (Rp)} &: \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu R_p, R_e, t) \rangle d\nu \\
 \text{Ellipsoidal Shell (Rp t)} &: \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu R_p, \nu R_e, \nu t) \rangle d\nu \\
 \text{Ellipsoidal Shell (Re)} &: \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, R_p, \nu R_e, t) \rangle d\nu \\
 \text{Ellipsoidal Shell (Re t)} &: \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, R_p, \nu R_e, \nu t) \rangle d\nu \\
 \text{Ellipsoidal Shell (Re Rp)} &: \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu R_p, \nu R_e, t) \rangle d\nu \\
 \text{Ellipsoidal Shell (Re Rp t)} &: \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu R_p, \nu R_e, \nu t) \rangle d\nu
 \end{aligned}$$

with n being either 1 or 2 depending if the size averaged amplitude or the size averaged intensity needs to be calculated. As a size distribution a lognormal distribution is assumed where all parameters are scaled simultaneously. The used lognormal distribution is defined as

$$\text{LogNorm}(x, \sigma) = \frac{1}{\sqrt{2\pi} x} \exp\left(-\frac{\ln^2(x)}{2\sigma^2}\right)$$

Input Parameters for model Ellipsoidal Shell:

R_p : semi-principal polar axes of elliptical core R_p
 R_e : equatorial semi-axis axes of elliptical core R_e
dummy: not used
 t : thickness of shell t
dummy: not used
 η_{core} : scattering length density of core η_c
 η_{shell} : scattering length density of shell η_{sh}
 η_{sol} : scattering length density of solvent η_{sol}

Input Parameters for model Ellipsoidal Shell (t), Ellipsoidal Shell (R_p),
Ellipsoidal Shell ($R_p t$), Ellipsoidal Shell (R_e), Ellipsoidal Shell ($R_e t$),
Ellipsoidal Shell ($R_e R_p$), Ellipsoidal Shell ($R_e R_p t$):

R_p : semi-principal polar axes of elliptical core R_p
 R_e : equatorial semi-axis axes of elliptical core R_e
dummy: not used
 t : thickness of shell t
 σ : width of the LogNorm size distribution
 η_{core} : scattering length density of core η_c
 η_{shell} : scattering length density of shell η_{sh}
 η_{sol} : scattering length density of solvent η_{sol}

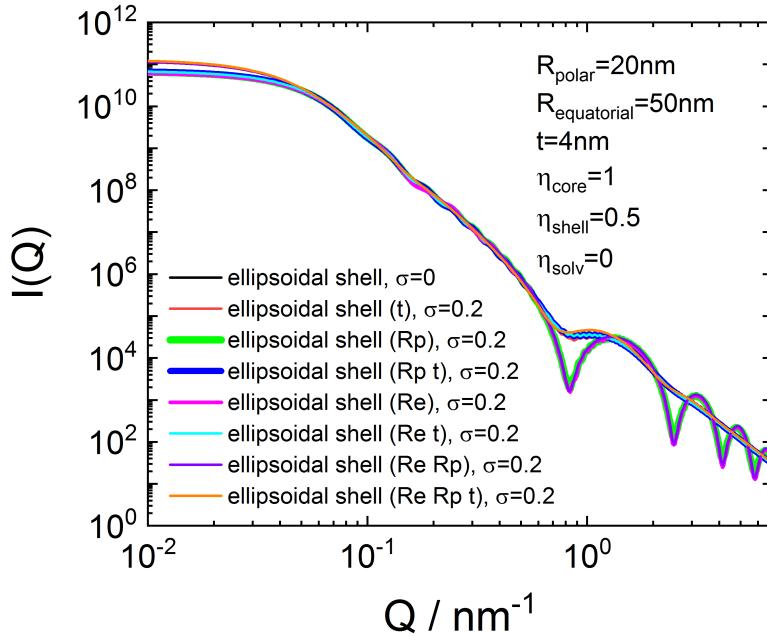


FIGURE 8.42. Form factor of an ellipsoidal core shell structure including a lognormal size distribution.

8.4.4. triaxial ellipsoidal core shell structure.

Also for these form factor plugins a size distribution parameter has been included directly in the form factor similar to the ellipsoidal shell of revolution to avoid applying nested numerical integration routines. As for the triaxial ellipsoid already a double integration for the orientational averaging is required it becomes even more important to have an optimized integration routine for including an additional size distribution, i.e an additional integration. The routine for multi-dimensional integration (`pcubature` [102]) has been used.

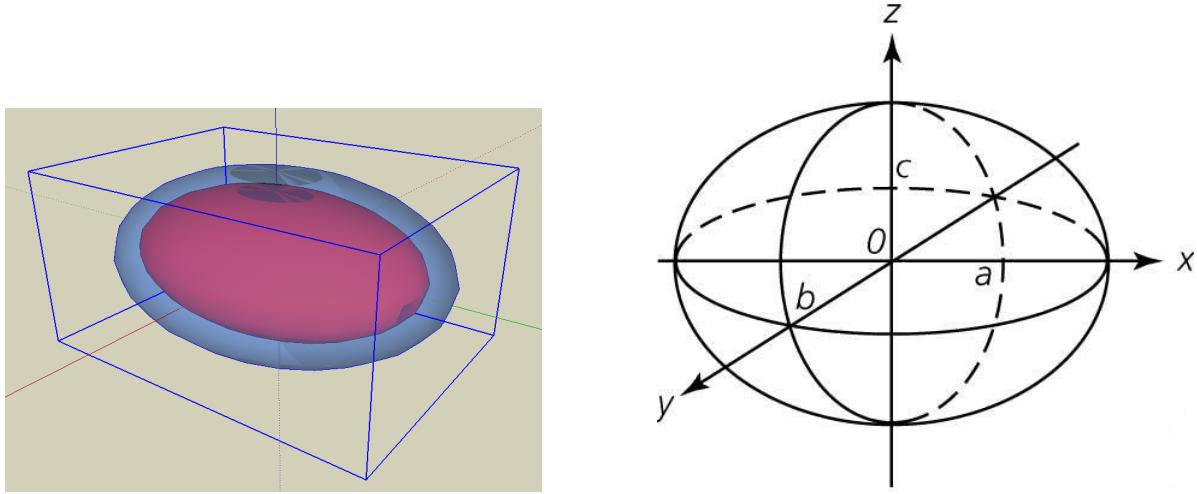


FIGURE 8.43. triaxial ellipsoidal core shell structure

$$I_{\text{triaxEllSh}}(Q) = \langle F^2(Q, a, b, c, t) \rangle = \int_0^1 \int_0^1 dx dy K_{\text{sh}}^2(Q, R, R_t) \quad (8.128)$$

$$\langle F(Q, a, b, c, t) \rangle = \int_0^1 \int_0^1 dx dy K_{\text{sh}}(Q, R, R_t) \quad (8.129)$$

with

$$K(QR) = 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \quad (8.130)$$

$$K_{\text{sh}}(Q, R, R_t) = (\eta_c - \eta_{\text{sh}}) K(QR) + (\eta_{\text{sh}} - \eta_{\text{sol}}) K(QR_t) \quad (8.131)$$

$$R^2 = [a^2 \cos^2(\pi x/2) + b^2 \sin^2(\pi x/2)] (1 - y^2) + c^2 y^2$$

$$R_t^2 = [(a + t)^2 \cos^2(\pi x/2) + (b + t)^2 \sin^2(\pi x/2)] (1 - y^2) + (c + t)^2 y^2$$

$$V_c = \frac{4}{3} \pi abc$$

$$V_t = \frac{4}{3} \pi (a + t)(b + t)(c + t)$$

η_c : scattering length density of core
 η_{sh} : scattering length density of shell
 η_{sol} : scattering length density of solvent
 a : semi-axes of elliptical core
 b : semi-axes of elliptical core
 c : semi-axes of elliptical core
 t : thickness of shell
 V_c : volume of core
 V_t : total volume of core along with shell

Several variants including a size distribution have been implemented which just differ if only none, one or both axis scale with the same size distribution and if additionally also scales with that distribution.

$$\begin{aligned}
 & \text{triax ellip shell} : \langle F^n(Q, a, b, c, t) \rangle \\
 & \text{triax ellip shell t} : \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, a, b, c, \nu t) \rangle d\nu \\
 & \text{triax ellip shell 1} : \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu a, b, c, t) \rangle d\nu \\
 & \text{triax ellip shell 1t} : \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu a, b, c, \nu t) \rangle d\nu \\
 & \text{triax ellip shell 2} : \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu a, \nu b, c, t) \rangle d\nu \\
 & \text{triax ellip shell 2t} : \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu a, \nu b, c, \nu t) \rangle d\nu \\
 & \text{triax ellip shell 3} : \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu a, \nu b, \nu c, t) \rangle d\nu \\
 & \text{triax ellip shell 3t} : \int_0^{\infty} \text{LogNorm}(\nu, \sigma) \langle F^n(Q, \nu a, \nu b, \nu c, \nu t) \rangle d\nu
 \end{aligned}$$

with n being either 1 or 2 depending if the size averaged amplitude or the size averaged intensity needs to be calculated. As a size distribution a lognormal distribution is assumed where all parameters are scaled simultaneously. The used lognormal distribution is defined as

$$\text{LogNorm}(x, \sigma) = \frac{1}{\sqrt{2\pi} x} \exp\left(-\frac{\ln^2(x)}{2\sigma^2}\right)$$

Input Parameters for model triax ellip shell:

- a: semi-axes of elliptical core a
- b: semi-axes of elliptical core b
- c: semi-axes of elliptical core c
- t: thickness of shell t
- dummy: not used
- eta_c: scattering length density of core η_c
- eta_sh: scattering length density of shell η_{sh}
- eta_sol: scattering length density of solvent η_{sol}

Input Parameters for model triax ellip shell t, triax ellip shell 1, triax ellip shell 1t, triax ellip shell 2, triax ellip shell 2t, triax ellip shell 3, triax ellip shell 3t:

- a: semi-axes of elliptical core a
- b: semi-axes of elliptical core b
- c: semi-axes of elliptical core c
- t: thickness of shell t
- sigma: width of the LogNorm size distribution σ
- eta_c: scattering length density of core η_c
- eta_sh: scattering length density of shell η_{sh}
- eta_sol: scattering length density of solvent η_{sol}

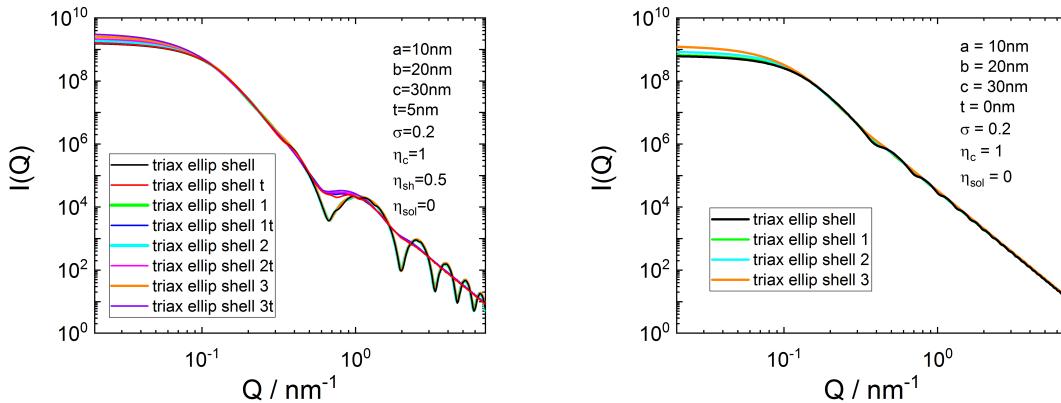


FIGURE 8.44. Form factor of a triaxial ellipsoidal core shell with semi axis a , b and c , with and without a shell thickness t plus an additional size distribution.

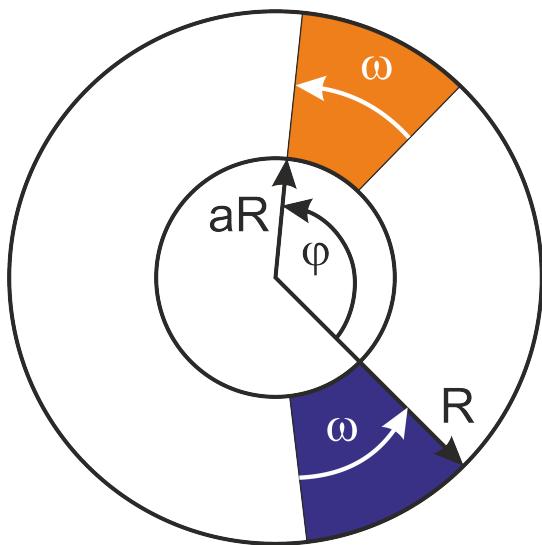
8.5. Cylindrical objects

8.5.1. Helical structures.

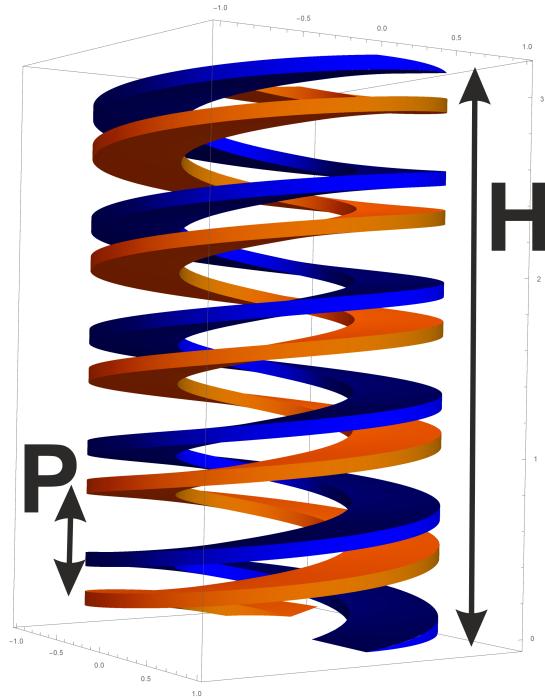
Several approaches for describing the small angle scattering signal from randomly oriented helical structures have been published. In [51, 176] a model of a double helix with a round cross section for each strand has been developed. For a fanlike cross section of a double helix a solution has been given by [191, 175]. Fukuda et al. [63] have extended the model to an arbitrary shaped cross section. In a model developed by Lebedev et al. [122] it is assumed, that beads are arranged on a helical path assuming a single strand. In all models the helix is straight and its length dimension is large compared to all other characteristic length of the helix. In [12] a solution of a single helical strand which can have a secondary coiling has been described. In this paper the cross section of the helical strand is assumed to be infinitesimal thin.

8.5.1.1. Fanlike helix.

Double helices with a fanlike cross section in the plane perpendicular to the helix axis have been described by Schmidt and Pringle [191, 175]. The model has been generalized by Fakuda et al. [63] for an arbitrarily shaped cross section and by Teixeira et al. [206] to a multi radial shell with fanlike cross section.



(a) top on view



(b) side view

FIGURE 8.45. Double helix with strands of round cross sections.

$$I(Q) = P_{\text{rod}}(Q, H) \left((\eta_h - \eta_{\text{solv}}) \omega R^2 (1 - a^2) \right)^2 \times \sum_{n=0}^{\infty} \epsilon_n \left(\cos(n\varphi/2) \frac{\sin(n\omega/2)}{n\omega/2} g_n(Q, R, a) \right)^2 \quad (8.132)$$

$$g_n(Q, R, a) = \frac{2}{R^2 (1 - a^2)} \int_{aR}^R r' J_n(Qr' (1 - q_n^2)) dr' \quad (8.133)$$

$$q_n = \begin{cases} \frac{2\pi n}{QP} & \text{for } Q \geq \frac{2\pi n}{P} \\ 1 & \text{for } Q < \frac{2\pi n}{P} \end{cases} \quad (8.134)$$

$$P_{\text{rod}}(Q, H) = H^2 \left(2 \frac{\text{Si}(QH)}{(QH)} - \left(\frac{\sin(QH/2)}{QH/2} \right)^2 \right) \quad (8.135)$$

with $\epsilon_n = 1$ for $n = 0$ and $\epsilon_n = 2$ for $n \geq 1$.

The sum converges very fast and for small Q -values the first few terms are already sufficient. However, **SASfit** is continuing the sum until the relative change of the sum is less than 10^{-10} .

The forward scattering of the model is normalized to the squared scattering length density contrast and squared total volume of the helix so that

$$I(Q = 0) = \left(H(\eta_h - \eta_{\text{solv}}) \omega R^2 (1 - a^2) \right)^2 \quad (8.136)$$

Input Parameters for model `fanlike helix`:

R: external helix radius R

a: inner helix radius aR , with $0 \leq a \leq 1$

omega: angular of the sector of material ω

phi: angle between the two sectors of material φ

dummy: not used

P: height of one helix period P

H: total length of helix H

eta_h: scattering length density of helix η_h

dummy: not used

eta_solv: scattering length density of solvent η_{solv}

Note:

- The helix is assumed to be stiff and long so that its scattering intensity can be factorized in a cross-section contribution and a shape contribution, whereas the shape contribution can be described by an infinitesimal thin rod of length H .
- R , P , and H are only physical for values larger than 0.
- The model is an approximation for the limit $H \gg P$ and $H \gg R$.
- $0 \leq a \leq 1$

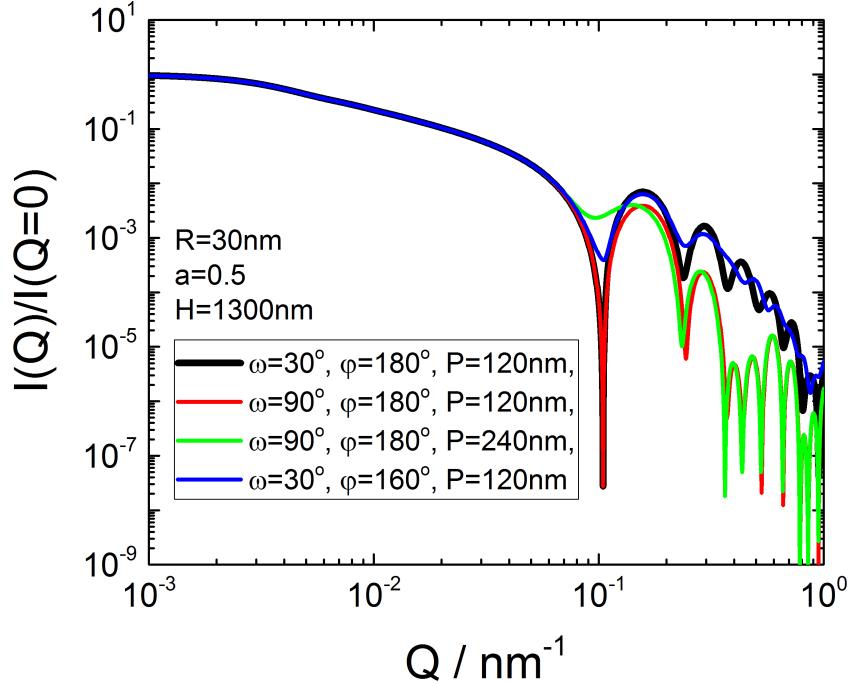


FIGURE 8.46. Normalized scattering curves of fanlike helices.

8.5.1.2. Helix with round cross-section.

Originally Franklin et al. [51] and Puigjaner et al. [176] have given the form factor of a cylinder with a groove of a double helix shape of round cross section. Fukada [63] is discussing the form factor of the random oriented double helices alone, but with arbitrary cross-sections.

$$I(Q) = P_{\text{rod}}(Q, H) \sum_{n=0}^{\infty} \epsilon_n (A_{1n}^2 + A_{2n}^2 + 2A_{1n}A_{2n} \cos n\alpha) \quad (8.137)$$

where

$$A_{in} = J_n(\delta_i Q_\perp) \pi R_i^2 (\eta_i - \eta_{\text{solv}}) \frac{2J_1(R_i Q_\perp)}{R_i Q_\perp} \quad (8.138)$$

$$Q_\perp^2 = Q^2 - \left[\frac{2\pi n}{P} \right]^2 \quad (8.139)$$

with $\epsilon_j = 1$ for $j = 0$ and $\epsilon_j = 2$ for $j \geq 1$. Furthermore R_1 and R_2 are the radii of the round helical strands. δ_1 and δ_2 are the distances of the strands to the helix axis. The pitch of the helix is denoted by P , whereas α is the angle between the two strands.

The sum converges very fast and for small Q -values the first few terms are already sufficient. However, SASfit is continuing the sum until the relative change of the sum is less than 10^{-10} .

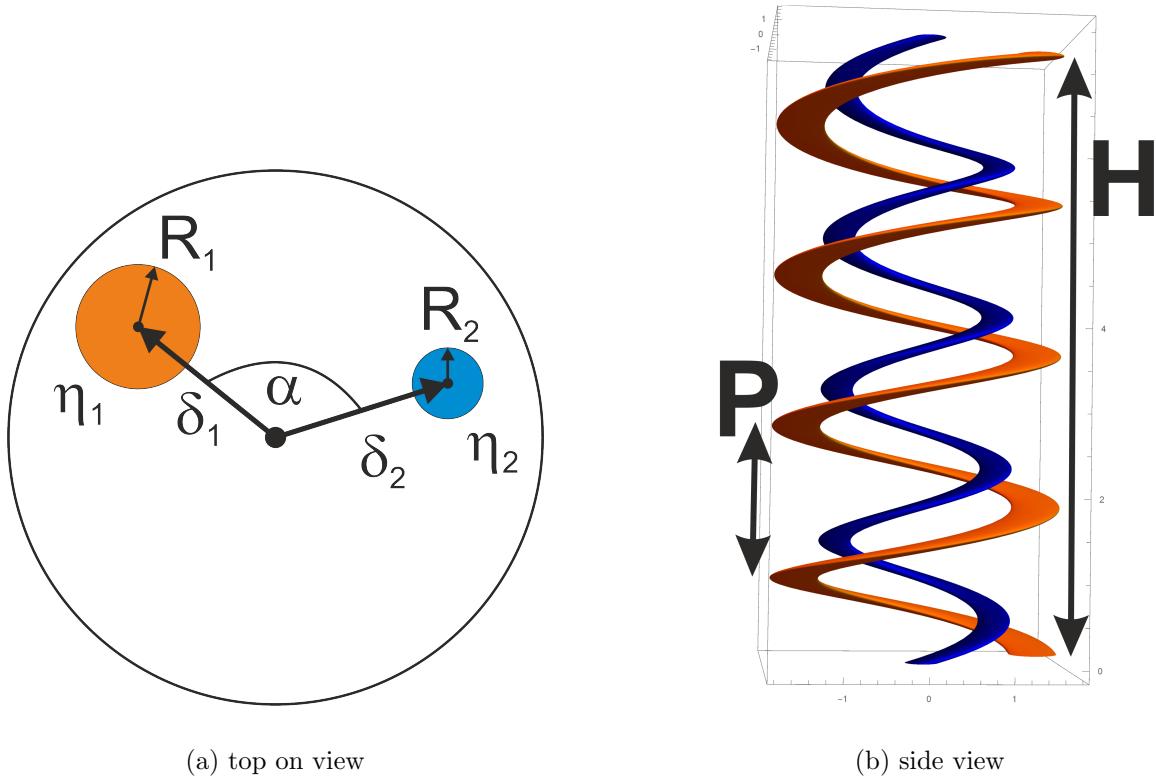


FIGURE 8.47. Double helix with strands of round cross sections. The cross-sections are round in the plane perpendicular to the helix axis.

The forward scattering of the model is normalized to the squared scattering length density contrast and squared total volume of the helix so that

$$I(Q=0) = H^2 \left((\eta_1 - \eta_{\text{solv}}) \pi R_1^2 + (\eta_2 - \eta_{\text{solv}}) \pi R_2^2 \right)^2 \quad (8.140)$$

Input Parameters for model helix with round XS:

R_1: radius of first strand thickness R_1

`delta_1`: distance of first strand to helix axis δ_1

R_2: radius of second strand thickness R_2

delta_2: distance of second strand to helix axis δ_2

alpha: angle α between the two strands

P: height of one helix period P

H : total length of helix H

eta_1: scattering length density of first strand η_1

eta_2: scattering length density of second strand η_2

eta_solv: scattering length density of solvent η_{solv}

Note:

- The helix is assumed to be stiff and long so that its scattering intensity can be factorized in a cross-section contribution and a shape contribution, whereas the shape contribution can be described by an infinitesimal thin rod of length H .
- $R_1, R_2, \delta_1, \delta_2, P$, and H are only physical for values larger than 0.
- The model is an approximation for the limit $H \gg P$ and $H \gg R$.

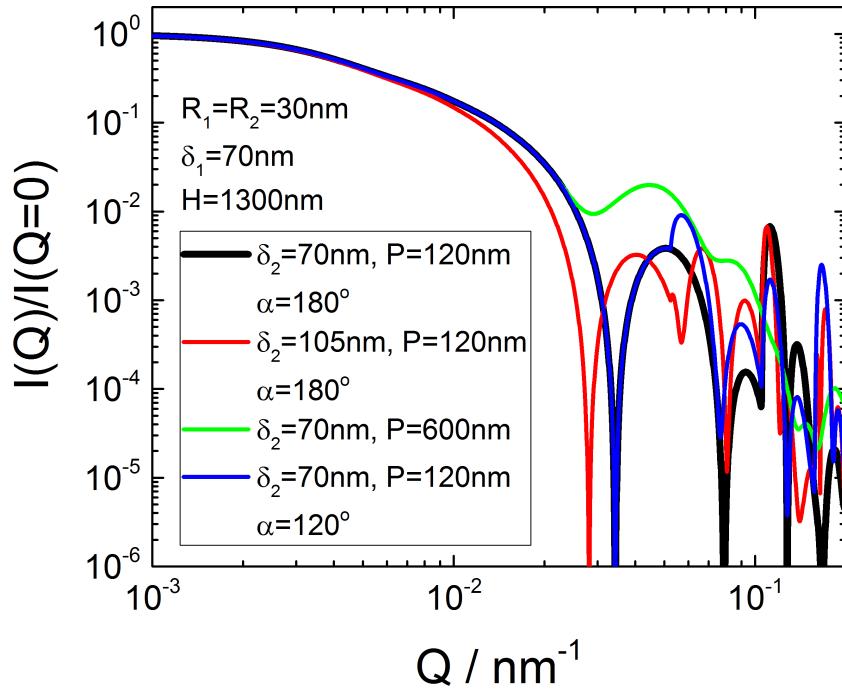


FIGURE 8.48. Normalized scattering curves of helices with round cross sections.

8.5.1.3. Beads model of a single helical strand.

In this model [122, 6] it is assumed that spherical beads are arranged along the path of a single helical strand. It is the product of the form factor of a sphere and the structure factor described by a helix with an infinitesimal thin single strand.

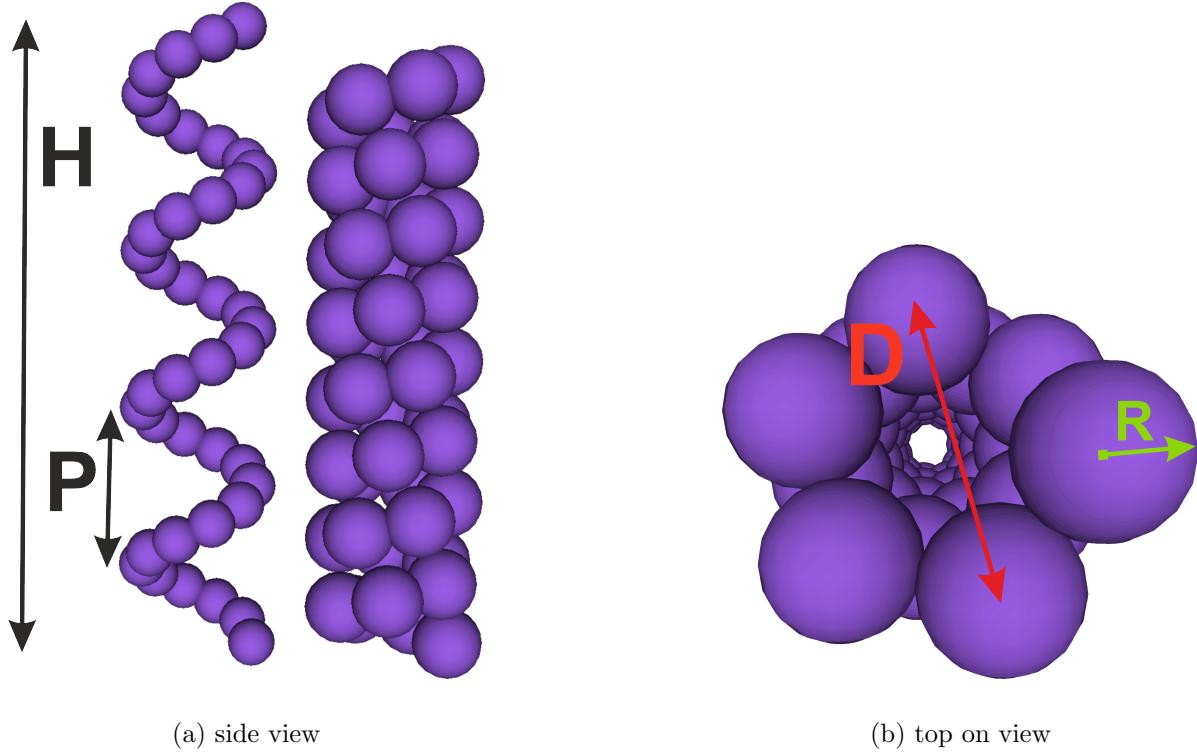


FIGURE 8.49. Bead model of a helix. In figure a) bead helices with different bead radius and different number of beads per turn are shown

$$I(Q) = P_{\text{rod}}(Q, H) \sum_{j=-\infty}^{\infty} \left| \frac{nH}{P} \Psi_j \left(QD, \frac{2\pi j}{PQ} \right) \Phi(Q, R) \right|^2 \quad (8.141)$$

with

$$\Psi_j \left(QD, \frac{2\pi j}{PQ} \right) = \begin{cases} J_j \left(\frac{QD}{2} \sqrt{1 - \left(\frac{2\pi j}{PQ} \right)^2} \right) & \text{for } Q \geq \frac{2\pi |j|}{P} \\ 0 & \text{for } Q < \frac{2\pi |j|}{P} \end{cases} \quad (8.142)$$

$$\Phi(Q, R) = 3 \frac{4\pi}{3} R^3 (\eta_b - \eta_{\text{solv}}) \frac{\sin(QR) - QR \cos QR}{(QR)^3} \quad (8.143)$$

$$P_{\text{rod}}(Q, H) = H^2 \left(2 \frac{\text{Si}(QH)}{(QH)} - \left(\frac{\sin(QH/2)}{QH/2} \right)^2 \right) \quad (8.144)$$

where $\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt$ is the sine integral. $P_{\text{rod}}(Q, H)$ is the form factor of an infinitesimal thin rod of length H . J_j is the regular cylindrical Bessel function, for which $J_j(x) = (-1)^{|j|} J_{-j}(x)$ if $j \in \mathbb{Z}$ (integer value) and $j \neq 0$. Therefore the sum in eq. 8.141 can be written as

$$I(Q) = P_{\text{rod}}(Q, H) \sum_{j=0}^{\infty} \epsilon_j \left| \frac{nH}{P} \Psi_j \left(QD, \frac{2\pi j}{PQ} \right) \Phi(Q, R) \right|^2 \quad (8.145)$$

with $\epsilon_j = 1$ for $j = 0$ and $\epsilon_j = 2$ for $j \geq 1$.

The sum converges very fast and for small Q -values the first two term are already sufficient. However, **SASfit** is continuing the sum until either the argument below the square root becomes negative or the relative change of the sum is less than 10^{-10} .

The forward scattering of the model is normalized so that

$$I(Q = 0) = \left(H \frac{nH}{P} \frac{4\pi}{3} R^3 (\eta_b - \eta_{\text{solv}}) \right)^2 \quad (8.146)$$

Input Parameters for model beads helix:

- R: radius of monomer units/beads R
- D: mean diameter of helix D
- n: number of monomer beads per turn n
- dummy: not used
- dummy: not used
- P: height of one helix period P
- H: total length of helix H
- eta_b: scattering length density of monomer beads η_b
- dummy: not used
- eta_solv: scattering length density of solvent η_{solv}

Note:

- The helix is assumed to be stiff and long so that it scattering intensity can be factorized in a cross-section contribution and a shape contribution, whereas the shape contribution can be described by an infinitesimal thin rod of length H .
- R , P , D , H , and n are only physical for values larger than 0.
- The model is an approximation for the limit $H \gg P$ and $H \gg D$.

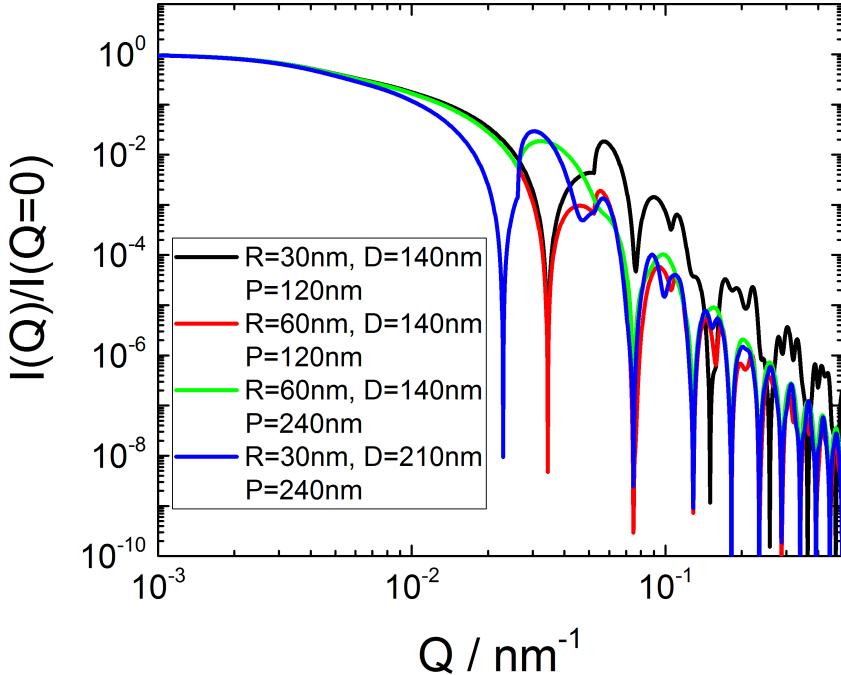


FIGURE 8.50. Normalized scattering curves of a single stranded helices formed by spherical beads.

8.5.1.4. straight superhelix.

Benham et al. [12] and Crick [37] describe single helices with a secondary helical curvature, which they call coiled superhelices or coiled coil. They assume in their analysis an infinitesimal thin helical strand. The straight superhelix is their starting geometry without a secondary coiling. The expression for a random oriented straight superhelix of finite length can be written exact in form of a single integral and is given by

$$I(Q) = 2 \int_0^{\mathcal{L}} \frac{(\mathcal{L} - w) \sin Q\psi}{Q\psi} dw \quad (8.147)$$

$$\psi = + \sqrt{2R_1^2 \left[1 - \cos \left(\frac{w}{\sqrt{R_1^2 + a^2}} \right) \right] + \frac{a^2 w^2}{R_1^2 + a^2}} \quad (8.148)$$

$$\mathcal{L} = 2\pi \frac{H}{P} \sqrt{R_1^2 + a^2} \quad (8.149)$$

$$a = \frac{P}{2\pi} \quad (8.150)$$

\mathcal{L} is the arclength of the helix, R_1 the distance to the helix axis and P the pitch of the helix. The intensity is normalized to the squared arclength of the helix. i.e.

$$I(Q = 0) = \mathcal{L}^2 \quad (8.151)$$

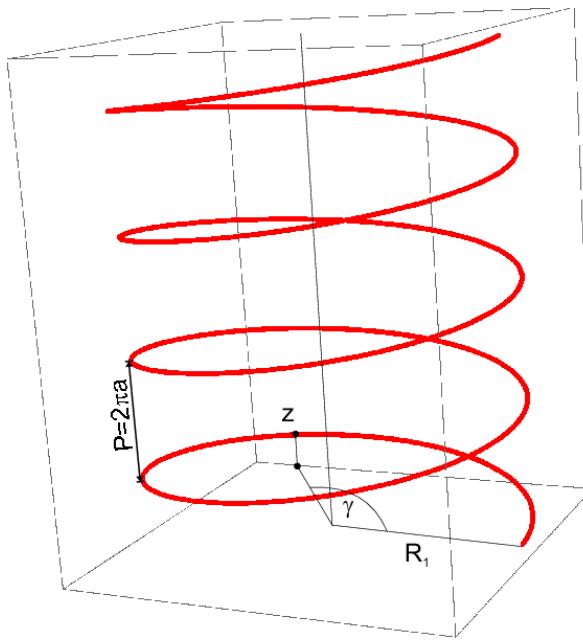


FIGURE 8.51. infinitesimal thin single stranded helix

Input Parameters for model `straight_superhelix`:

`R_1`: distance to the helix axis R_1

`dummy`: not used

`dummy`: not used

`dummy`: not used

`dummy`: not used

`P`: height of one helix period P

`H`: total length of helix H

`dummy`: not used

`dummy`: not used

`dummy`: not used

Note:

- the model assumes an infinitesimal thin single helical strand.
- R_1 , P , and H are only physical for values larger than 0.
- The model is exact.

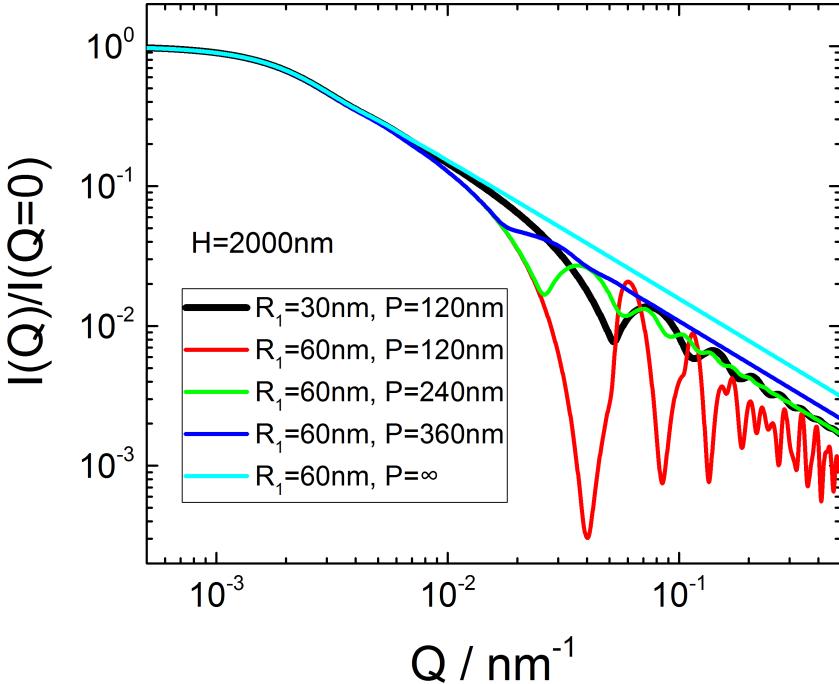


FIGURE 8.52. Normalized scattering curves of single stranded infinitesimal thin stranded helices.

8.5.1.5. coiled superhelix.

A coiled superhelix or coiled coil is a helix with a small repeat whose axis is slightly deformed and follows a larger more graded helical path [12, 37].

The points $p(\gamma) = [x(\gamma), y(\gamma), z(\gamma)]$ on the coiled superhelix are parametrized by

$$x(\gamma) = \cos(\gamma) (R_2 + R_1 \cos(N\gamma)) - R_1 \cos(\alpha_2) \sin(\gamma) \sin(N\gamma) \quad (8.152)$$

$$y(\gamma) = \sin(\gamma) (R_2 + R_1 \cos(N\gamma)) - R_1 \cos(\gamma) \sin(N\gamma) \cos(\alpha_2) \quad (8.153)$$

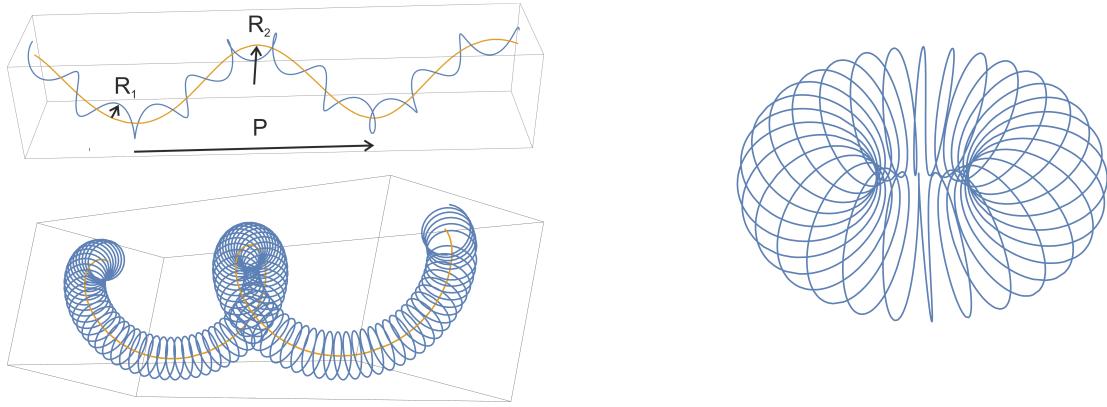
$$z(\gamma) = \frac{P}{2\pi}\gamma - R_1 \sin(\alpha_2) \sin(N\gamma) \quad (8.154)$$

R_1 and R_2 are the radii of the primary and secondary coiling of the superhelix. P is the pitch of the secondary coiling and $\alpha_2 = \arctan\left(\frac{2\pi R_2}{P}\right)$ its pitch angle. N is the number of first order turns in the primary coiling for each turn of the secondary coiling. The arc length of the helix \mathcal{L} can be calculated by

$$\mathcal{L} = \int_0^{\gamma_{\max}} \sqrt{f(\gamma)} d\gamma \quad (8.155)$$

with

$$f(\gamma) = R_2^2 + R_1^2 N^2 (\cos^2(N\gamma) + \sin^2(N\gamma) \cos^2(\alpha_2)) + \frac{P^2}{(2\pi)^2} + 2R_2 * R_1 \cos(N\gamma) + 2R_1^2 N \cos(\alpha_2) \quad (8.156)$$



(a) A superhelix with primary and secondary turns. The bottom one has ten times more primary turns per secondary turn than the top one

(b) A superhelix with zero pitch forms a toroidal helix

FIGURE 8.53. Combinations of two helical orders are referred to as coiled coils or coiled superhelix.

and

$$\gamma_{\max} = N_{2nd}/(2\pi) \quad (8.157)$$

where N_{2nd} is the number of secondary turns of the coiled superhelix. The scattering intensity is then given by

$$I(Q) = \int_0^{\gamma_{\max}} d\gamma_2 \int_0^{\gamma_{\max}} \sqrt{f(\gamma_2)f(\gamma_1)} \frac{\sin(Qr(\gamma_1, \gamma_2))}{Qr(\gamma_1, \gamma_2)} d\gamma_2 \quad (8.158)$$

with

$$r^2(\gamma_1, \gamma_2) = \left(\begin{pmatrix} x(\gamma_2) \\ y(\gamma_2) \\ z(\gamma_2) \end{pmatrix} - \begin{pmatrix} x(\gamma_1) \\ y(\gamma_1) \\ z(\gamma_1) \end{pmatrix} \right)^2 \quad (8.159)$$

The intensity is normalized to the squared arclength of the helix. i.e.

$$I(Q=0) = \mathcal{L}^2 \quad (8.160)$$

Input Parameters for model `coiled_superhelix`:

R_1: distance to the primary helix axis R_1

R_2: distance to the primary helix axis R_1

dummy: not used

N: number of primary turns N per secondary turn

dummy: not used

P: pitch length of secondary helix P
 turns: number of secondary turns N_{2nd}
 dummy: not used
 dummy: not used
 dummy: not used

Note:

- the model assumes an infinitesimal thin single helical strand with an additional secondary coiling.
- R_1, R_2, P, N_{2nd} and N are only physical for values larger than 0.
- The model is exact.
- The calculation of the scattering intensity requires a double integration, which is the reason for the longer computing time.

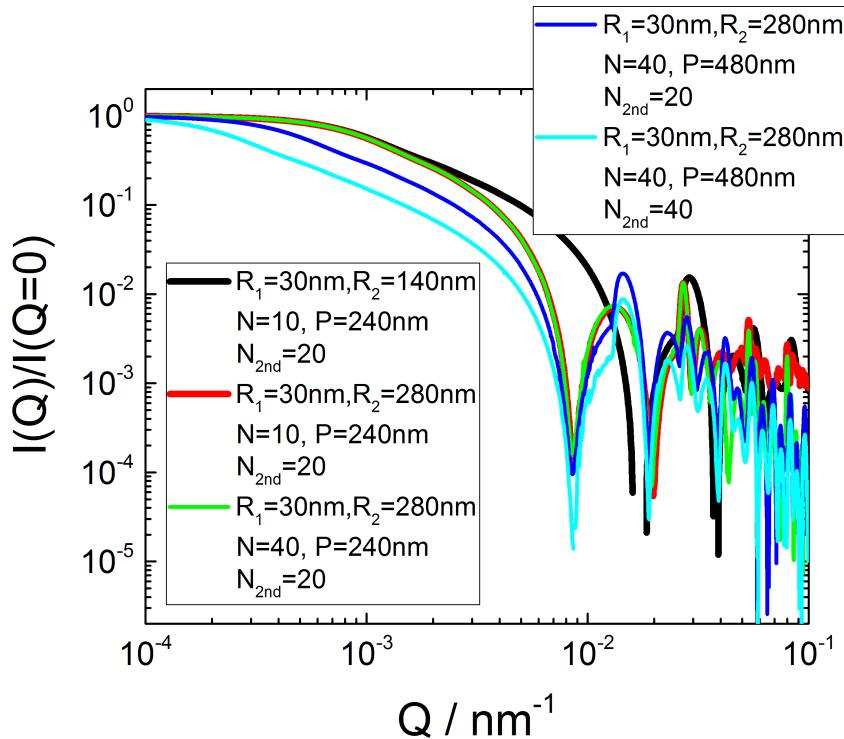


FIGURE 8.54. Normalized scattering curves of single stranded infinitesimal thin coiled helices.

8.6. Sheared and deformed objects

8.7. Non-equilibrium static form factor of a reptating chain

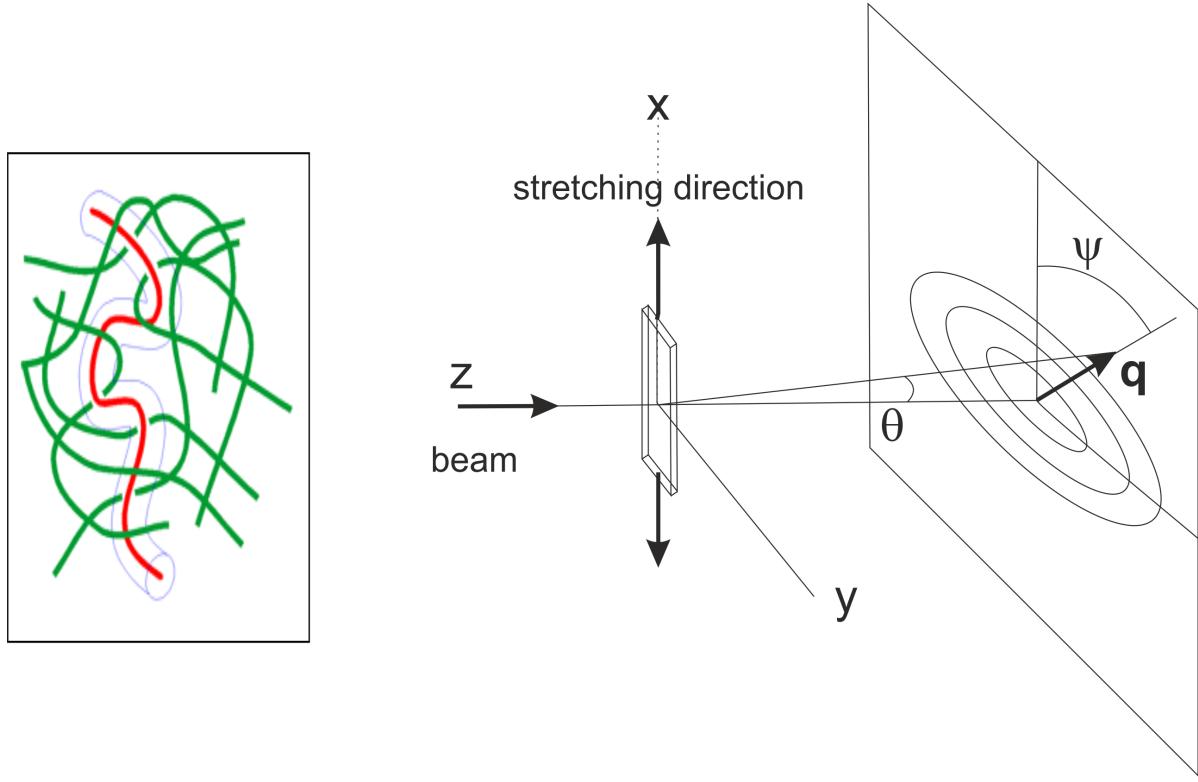


FIGURE 8.55. Sketch of experimental setup for stretching a polymer melt.

This model [94, 155] describes the scattering behaviour of an uniaxial deformed (stretched) Gaussian polymer chain as a function of the deformation ratio and Rouse relaxation time. The model might not include an up-to-date tube model but shows some main features of deformed polymer melts.

$$\begin{aligned} I\left(q, \frac{t}{\tau}\right) / I_0 = & D(\alpha) + \frac{1}{6}(\alpha - \beta)(1 + e^{-\alpha}) H\left(\alpha, \frac{t}{\tau}\right) \\ & + \frac{1}{6}(\alpha - \beta)^2 e^{-\beta} \int_0^1 dy y^3 e^{\beta y} (1 + e^{-\alpha y}) H\left(\alpha y, \frac{t}{\tau y^2}\right) \end{aligned} \quad (8.161)$$

with

$$H\left(x, \frac{t}{\tau}\right) = \frac{96}{\pi^2} \sum_{n=\text{odd}} \frac{1}{n^2(n^2\pi^2 + x^2)} e^{-n^2 \frac{t}{\tau}} \quad (8.162)$$

$D(x)$ is the Debye function $D(x) = 2(x - 1 + e^{-x})/x^2$, $\alpha = q^2 R_g^2$, R_g is the equilibrium radius of gyration of the chain and τ is the reptation time or tube disengagement time. Furthermore λ is the uniaxial elongation factor, q_{\parallel} and q_{\perp} are respectively the components of the wave vector q in directions parallel and perpendicular to the direction of

elongation which than define the parameter β as

$$\beta = (q_{\parallel}^2 \lambda^2 + q_{\perp}^2 / \lambda) R_g^2 / E \quad (8.163)$$

$$E = \frac{1}{2} \left\{ \lambda + \frac{\operatorname{asinh}(\sqrt{\lambda^3 - 1})}{\sqrt{\lambda(\lambda^3 - 1)}} \right\} \quad (8.164)$$

$$q_{\parallel} = q \cos(\psi - \theta_0) \quad (8.165)$$

$$q_{\perp} = q \sin(\psi - \theta_0) \quad (8.166)$$

Input Parameters for models CylShell1, CylShell2 and LongCylShell:

I0: forward scattering I_0

Rg: radius of gyration of unstretched polymer R_g

lambda: stretching factor λ

t/tau: relaxation time in units of Rouse time t/τ

theta_0: stretching direction in the detector plane θ_0 in degree

psi: q-direction ψ in the detector plane in degree

Note:

- The model only converges for $\lambda > 1$.

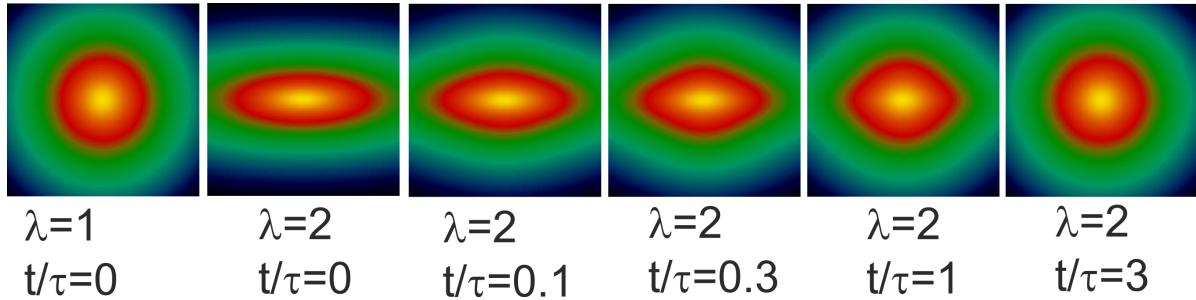


FIGURE 8.56. 2D scattering patterns for different relaxation times t/τ after a deformation step of $\lambda = 2$

8.7.1. ShearedCylinderHayterPenfold.

The orientation distribution of long cylinders in a shear cell has been studied by [190, 101, 87]. In contrast to the orientation distributions described in the next section this distribution also describes the tilting of the most probable orientation against the flow direction.

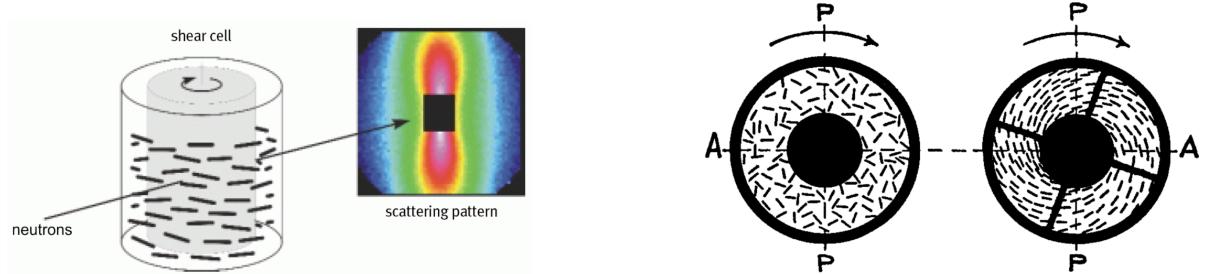


FIGURE 8.57. Shear orientation of micelles in a shear cell with the corresponding SANS-pattern. The orientation of rods at rest and under shear according to [190] are shown on the right as a top view on the cuvette cell.

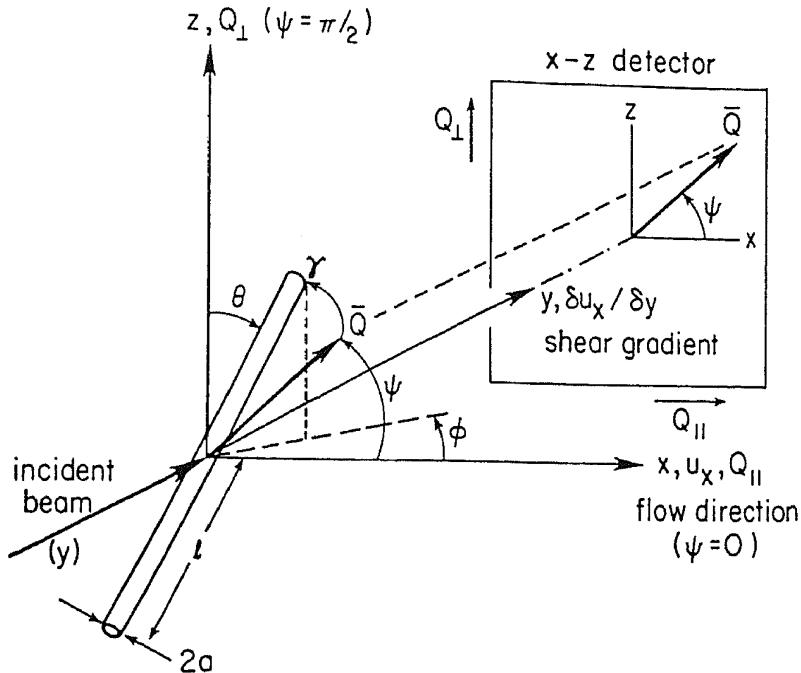


FIGURE 8.58. Cartesian and angular coordinates referred to the center of a cylindrical micelle at origin [87]. The relationship to the spectrometer geometry is shown schematically. The momentum transfer, Q , lies in the $z - x$ plane.

The scattering from monodisperse dilute (non-interacting) isotropic solution of anisotropic micelles is given by [87]

$$I(Q) = \langle |F(Q)|^2 \rangle_Q \quad (8.167)$$

where $F(Q)$ is the form factor for a micelle at a given orientation relative to the momentum transfer Q and $\langle \rangle_Q$ denotes an average over all such orientations.

For a uniform cylinder of length L and diameter $2R$ the form factor is given by:

$$F(Q) = F(Q, \gamma) = 2\Delta\eta V \frac{\sin(QL/2 \cos \gamma)}{QL/2 \cos \gamma} \frac{J_1(QR \sin \gamma)}{QR \sin \gamma} \quad (8.168)$$

where γ is the angle between Q and the cylinder axis, V is the volume, $\Delta\eta$ the scattering length density contrast relative to the solvent, $J_1(x)$ is the first order Bessel function of the first kind.

The scattering geometry for shear alignment is shown in Fig. 8.58. In general perfect alignment will not be achieved, and an orientation distribution must be employed such that the resultant scattering will be given by

$$I(Q, \psi) = \int_0^{2\pi} \int_0^\pi p_{HP}(\theta, \phi; \Gamma) (F^2(Q, \gamma^+) + F^2(Q, \gamma^-)) \sin \theta d\theta d\phi \quad (8.169)$$

with

$$p_{HP}(\theta, \phi; \Gamma) = \frac{(1 - \cos 2\Phi_0)(1 + \sin^2 \theta \cos 2\Phi_0)^{3/2}}{4\pi [1 - \sin^2 \theta \cos 2\Phi_0 \cos 2(\phi - \Phi_0)]^2} \quad (8.170)$$

and

$$2\Phi_0 = \arctan(8/\Gamma) \quad (8.171)$$

$(\cos \Phi_0, \sin \Phi_0, 0)^T$ is the direction of the most probable orientation ($\theta = \frac{\pi}{2}; \phi = \Phi_0$). Φ_0 varies between $\Phi_0 = \frac{\pi}{4}$ for the isotropic case ($\Gamma = 0$) to $\Phi_0 = 0$ for perfect alignment ($\Gamma \rightarrow \infty$). As the most probable direction is not the same than the flow direction, which

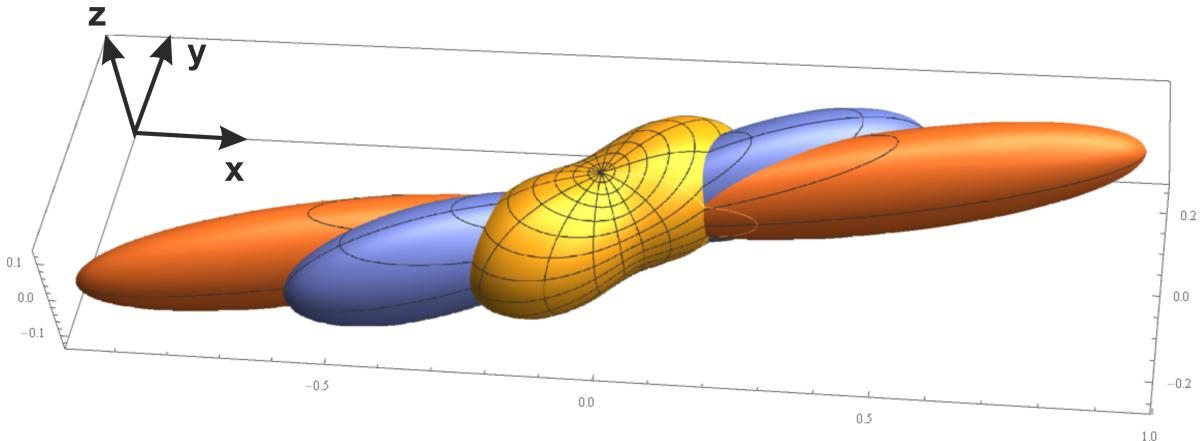


FIGURE 8.59. 3D plot of the orientation distributions $2p_{HP}(\theta, \phi; \Gamma = 2)$ (yellow), $p_{HP}(\theta, \phi; \Gamma = 8)$ (blue), $\frac{1}{2}p_{HP}(\theta, \phi; \Gamma = 16)$ (orange).

is assumed to be the x -direction, and secondly that the beam is passing twice the cuette cell with flow direction \mathbf{x} and $-\mathbf{x}$ we have to take the sum over two values for γ^\pm in eq. 8.169. The angle $\gamma^\pm = \angle(\mathbf{Q}, \mathbf{n}^\pm)$ between the scattering vector \mathbf{Q} and the director of the cylinder axis \mathbf{n}^\pm is

$$\frac{\mathbf{Q}}{|\mathbf{Q}|} = \begin{pmatrix} \cos \psi \\ 0 \\ \sin \psi \end{pmatrix} \quad \frac{\mathbf{n}^\pm}{|\mathbf{n}^\pm|} = \begin{pmatrix} \pm \sin \theta \cos \phi \\ \pm \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} \quad (8.172)$$

$$\cos \angle(\mathbf{Q}, \mathbf{n}^\pm) = \cos \gamma^\pm = \frac{\mathbf{Q} \cdot \mathbf{n}^\pm}{|\mathbf{Q}| |\mathbf{n}^\pm|} = \pm \cos \psi \sin \theta \cos \phi + \sin \psi \cos \theta \quad (8.173)$$

For the reversed direction of the flow the x -component of the vector \mathbf{n}^- changes its sign. In the original paper [87] one finds $\cos \gamma_{\text{HP}}^- = \cos \psi \sin \theta \cos \phi - \sin \psi \cos \theta$, which, however, results to the same scattering intensity as the scattering intensity of a cylinder is invariant by a rotation of π , i.e. as $\cos(\pi + \gamma^-) = -\cos \gamma^- = \cos \gamma_{\text{HP}}^-$ the resulting scattering patterns are the same for both value γ^- and γ_{HP}^- .

8.7.2. partly aligned cylindrical shell.

In contrast to the model of Hayter and Penfold [87] in section 8.7.1 here we use an orientation distribution with the most probable orientation along the x-axis and independent on the polar angle ϕ . Therefore the coordination system is slightly differently defined than in the Hayter-Penfold model of sheared cylinders.

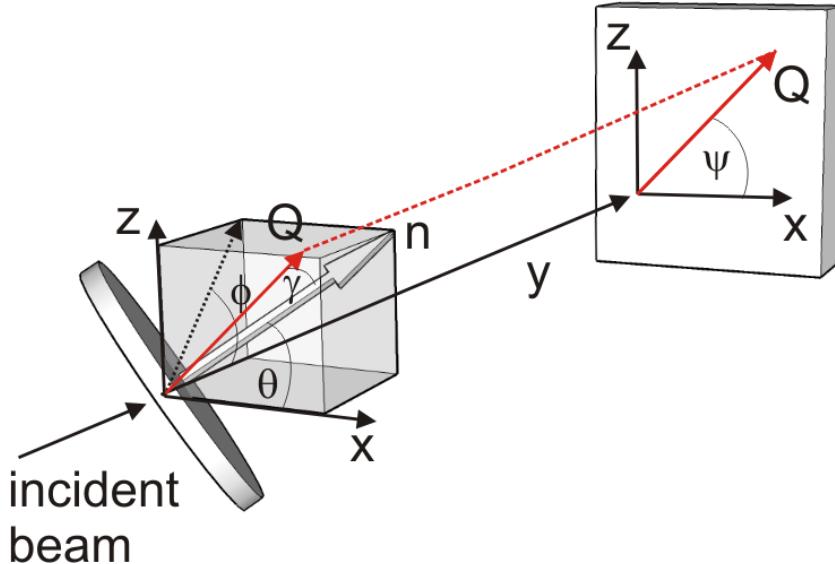


FIGURE 8.60. Sketch of relative orientation \mathbf{n} of partly aligned cylinders or discs to the scattering vector \mathbf{Q} .

The scattering amplitude of a cylindrical shell is given by

$$K_{\text{CylShell}}(Q, \dots, \gamma) = K_{\text{Cyl}}(Q, \eta_{\text{core}} - \eta_{\text{shell}}, R, L, \gamma) \quad (8.174)$$

$$+ K_{\text{Cyl}}(Q, \eta_{\text{shell}} - \eta_{\text{solv}}, R + \Delta R, L, \gamma) \quad (8.175)$$

with

$$K_{\text{Cyl}}(Q, \Delta\eta, R, L, \gamma) = 2\pi R^2 L \Delta\eta \frac{J_1(QR \sin \gamma)}{QR \sin \gamma} \frac{\sin(\frac{QL}{2} \cos \gamma)}{\frac{QL}{2} \cos \gamma} \quad (8.176)$$

where γ is the angle between \mathbf{Q} and the cylinder axis \mathbf{n} . L is the length of the cylinder, R its radius, $\Delta\eta$ the scattering length density contrast relative to the solvent and $J_1(x)$ is the first order Bessel function of the first kind. γ can be calculated from the orientation (θ, ϕ) of the cylinder and the direction of the scattering vector ψ in the plane of the detector by

$$\frac{\mathbf{Q}}{|\mathbf{Q}|} = \begin{pmatrix} \cos \psi \\ 0 \\ \sin \psi \end{pmatrix} \quad \frac{\mathbf{n}}{|\mathbf{n}|} = \begin{pmatrix} \cos \theta \\ \sin \theta \cos \phi \\ \sin \theta \sin \phi \end{pmatrix} \quad (8.177)$$

$$\cos \angle(\mathbf{Q}, \mathbf{n}) = \cos \gamma = \frac{\mathbf{Q} \cdot \mathbf{n}}{|\mathbf{Q}| |\mathbf{n}|} = \cos \psi \cos \theta + \sin \psi \sin \theta \sin \phi \quad (8.178)$$

If the orientation distribution of the orientation vector \mathbf{n} is described by $p(\theta, \phi)$ so that the scattering intensity is given by

$$I_{\text{p.a.CylShell}}(Q) = \int_0^\pi d\theta \int_0^{2\pi} d\phi K_{\text{CylShell}}(Q, \dots, \gamma) p(\theta, \phi; \kappa) \sin(\theta) \quad (8.179)$$

For this form factor it is assumed that the orientation distribution is independent of ϕ , i.e. $p(\theta, \phi; \kappa) = p(\theta; \kappa)$ and that $p(\theta; \kappa) = p(\pi - \theta; \kappa)$, which means that turning the cylinder by 180° results in the same scattering intensity. Several orientation distributions have been implemented in a way that their resulting order parameter S_2 can have values between -0.5 and 1, which correspond to perfect alignment perpendicular to the x axis and perfect alignment parallel to it. all probability distributions have been normalized so that

$$\int_0^\pi \int_0^{2\pi} p(\theta, \phi; \kappa) \sin \theta d\theta d\phi = 1 \quad (8.180)$$

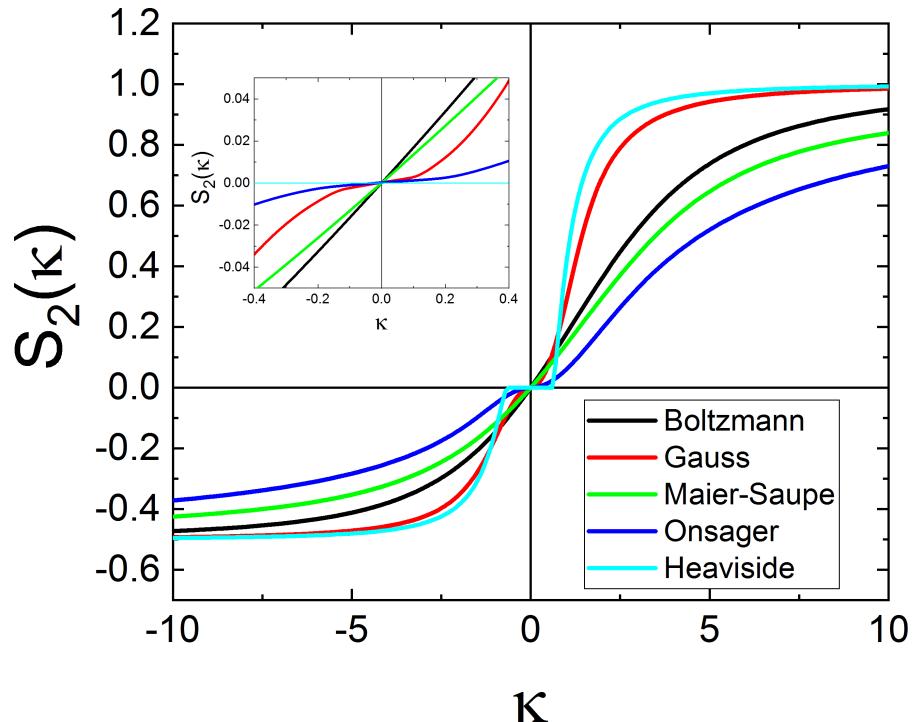
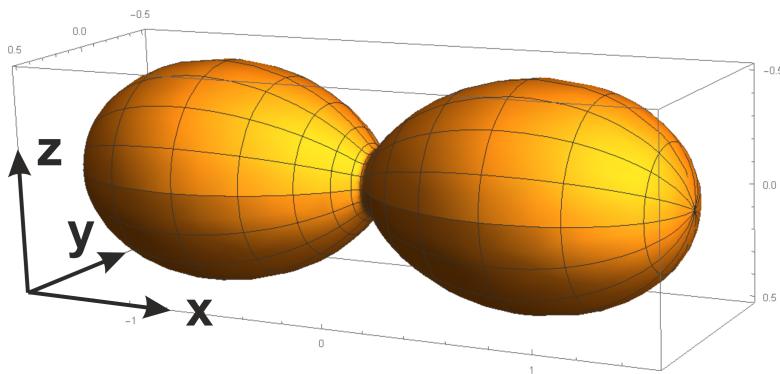


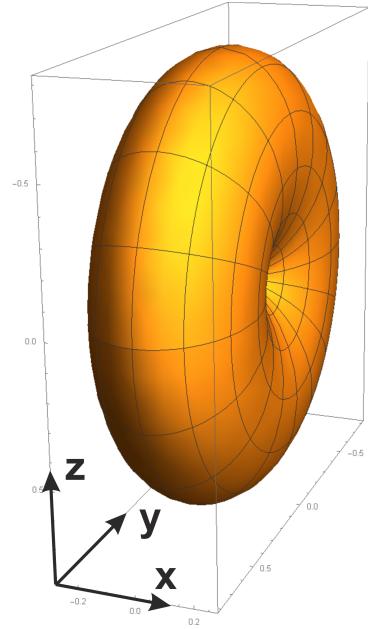
FIGURE 8.61. Order parameter $S_2(\kappa)$ for the orientation distributions described in the next subsections.

The order parameters S_2 can be calculated by

$$\begin{aligned} S_2(\kappa) &= \int_0^{\pi} \int_0^{2\pi} p(\theta, \phi; \kappa) \frac{1}{2} (3 \cos^2 \theta - 1) \sin \theta d\theta d\phi \\ &= \int_0^{\pi} 2\pi p(\theta; \kappa) \frac{1}{2} (3 \cos^2 \theta - 1) \sin \theta d\theta \end{aligned} \quad (8.181)$$



(a) Onsager orientation distribution for $\kappa = 3$.



(b) Onsager orientation distribution for $\kappa = -3$.

FIGURE 8.62. Orientation distributions are all independent on ϕ and have for $\kappa > 0$ a positive order parameter, i.e. the most probable orientation is in the x -direction and for $\kappa < 0$ a negative order parameter, i.e. the most probable orientation lies in the zy -plane

TABLE 3. Values for κ to obtain certain order parameters $S_2(\kappa)$ for the different orientation distributions $p(\theta, \phi, \kappa)$.

$p(\theta, \phi, \kappa)$	$S_2(\kappa)$	-0.45	-0.4	-0.2	0	0.2	0.4	0.6	0.8	0.9	0.95
Gauss		-3.739	-1.62	-1.128	0	0.83	1.220	1.686	2.576	3.762	5.4
Boltzmann		-7.141	-4.59	-1.431	0	1.114	2.218	3.581	5.989	9	13.08
Maier-Saupe		-15	-7.49	-1.874	0	1.367	2.709	4.444	8.241	15.59	30.54
Onsager		-28.43	-13.35	-2.918	0	2.042	3.629	6.313	13.92	28.96	58.99
Heavyside		-3.108	-2.157	-1.129	0	0.794	0.982	1.267	1.868	2.692	3.840
Hayter-Penfold		\emptyset	\emptyset	\emptyset	0	3.008	6.269	10.93	20.8	34.91	55.64

8.7.2.1. Maier-Saupe orientation distribution.

$$p(\theta, \phi; \kappa) = \frac{1}{c_{\text{MS}}} \exp(\kappa \cos^2 \theta) \quad (8.182)$$

with

$$c_{\text{MS}} = \begin{cases} 4\pi \exp(\kappa) \frac{D(\sqrt{\kappa})}{\sqrt{\kappa}} & \text{for } \kappa > 0 \\ 2\pi\sqrt{\pi} \frac{\text{erf}(\sqrt{|\kappa|})}{\sqrt{|\kappa|}} & \text{for } \kappa < 0 \\ 4\pi & \text{for } \kappa = 0 \end{cases} \quad (8.183)$$

with $D(x)$ being Dawson's integral $D(x) = e^{-x^2} \int_0^x e^{y^2} dy = \frac{1}{2}\sqrt{\pi} e^{-x^2} \text{erfi}(x)$

8.7.2.2. *Onsager orientation distribution.*

$$p(\theta, \phi; \kappa) = \begin{cases} \frac{\kappa \cosh(\kappa \cos(\theta))}{4\pi \sinh(\kappa)} & \text{for } \kappa \geq 0 \\ \frac{\cosh(|\kappa| \sin(\theta))}{2\pi^2 L_1(|\kappa|)} & \text{for } \kappa < 0 \end{cases} \quad (8.184)$$

where $L_\nu(x)$ is the modified Struve function.

8.7.2.3. *Boltzmann orientation distribution.*

$$p(\theta, \phi; \kappa) = \begin{cases} \frac{1}{2\pi c_B} \exp(-\theta\kappa) & \text{for } \theta \leq \frac{\pi}{2} \\ \frac{1}{2\pi c_B} \exp(-(\pi - \theta)\kappa) & \text{for } \theta > \frac{\pi}{2} \end{cases} \quad (8.185)$$

with

$$c_B = \frac{2(1 - \kappa \exp(-\frac{\pi}{2}\kappa))}{\kappa^2 + 1} \quad (8.186)$$

8.7.2.4. Gaussian orientation distribution.

$$p(\theta, \phi; \kappa) = \begin{cases} \frac{1}{2\pi c_G} \exp(-\theta^2 \kappa^2) & \text{for } \kappa \geq 0 \text{ and } \theta \leq \frac{\pi}{2} \\ \frac{1}{2\pi c_G} \exp(-(\pi - \theta)^2 \kappa^2) & \text{for } \kappa \geq 0 \text{ and } \theta > \frac{\pi}{2} \\ \frac{1}{2\pi c_G} \exp\left(-\left(\frac{\pi}{2} - \theta\right)^2 |\kappa|^2\right) & \text{for } \kappa < 0 \end{cases} \quad (8.187)$$

with

$$c_G = \begin{cases} \frac{D(\kappa/2)}{\kappa} + \frac{\sqrt{\pi}}{2\kappa} e^{-\frac{1}{4\kappa^2}} \Re \left(\operatorname{cerfi} \left(-\frac{1}{2\kappa} + i\frac{\pi}{2}\kappa \right) \right) & \text{for } \kappa \geq 0 \\ \frac{\sqrt{\pi}}{|\kappa|} e^{-\frac{1}{4\kappa^2}} \Im \left(\operatorname{cerfi} \left(-\frac{1}{2|\kappa|} + i\frac{\pi}{2}|\kappa| \right) \right) & \text{for } \kappa < 0 \end{cases} \quad (8.188)$$

8.7.2.5. *ShearedCylinderHeaviside.*

$$p(\theta, \phi; \bar{\theta}) = \Theta[\theta - \bar{\theta}] \quad (8.189)$$

$$\cos \gamma^\pm = \sin \theta \cos \phi \cos \psi \pm \cos \theta \sin \psi \quad (8.190)$$

8.8. Form factors of magnetic objects

8.8.1. Magnetic spin misalignment.

In [146] the autocorrelation function $C(r)$ of the spin misalignment has been published. The approach assumes the material to be close to magnetic saturation and considers uniform values for the exchange interaction and the saturation magnetization, i.e. it's a micro magnetic theory of single phase magnets. The correlation function is defined by

$$C(r) = \frac{KR^4}{H_i^2} \int_0^\infty \frac{j_0(qr)j_1^2(qR)}{(1 + l_H^2 q^2)^2} dq \quad (8.191)$$

where j_1 and j_2 are the spherical Bessel functions² and $K = 8H_P^2/V$. H_i is the internal magnetic field, $l_H = \sqrt{2A/(\mu_0 M_s H_i)}$ the exchange length of the magnetic field, H_P the mean magnetic anisotropy field, A the exchange-stiffness constant, and μ_0 the permeability of free space. The internal magnetic field $H_i = H_0 - NM_s$ is the sum of the external applied magnetic field H_0 and the demagnetisation field $-NM_s$ (N : demagnetisation factor). V is the sample volume and M_s the saturation magnetisation.

It is assumed in this model that the magnetic anisotropy field, which perturbs the magnetization from the homogeneously magnetized state, is uniform (constant) within a spherical volume; the parameter R represents the corresponding anisotropy-field radius, and it is emphasized that R is not necessarily identical to the crystallite size.

The correlation is related to the scattering intensity by

$$C(r) = \frac{1}{2\pi^2} \int_0^\infty I(q)q^2 \frac{\sin(qr)}{qr} dq = \frac{1}{2\pi^2} \int_0^\infty I(q)q^2 j_0(qr) dq \quad (8.192)$$

Therefore the intensity $I(q)$ can be computed as

$$I(q) = \frac{KR^6}{H_i^2} \frac{2\pi^2}{(qR)^2} \frac{j_1^2(qR)}{(1 + l_H^2 q^2)^2} \quad (8.193)$$

Input Parameters for the models $C(r)$ for spin misalignment and $I(q)$ for spin misalignment:

K: $K = 8H_P^2/V$

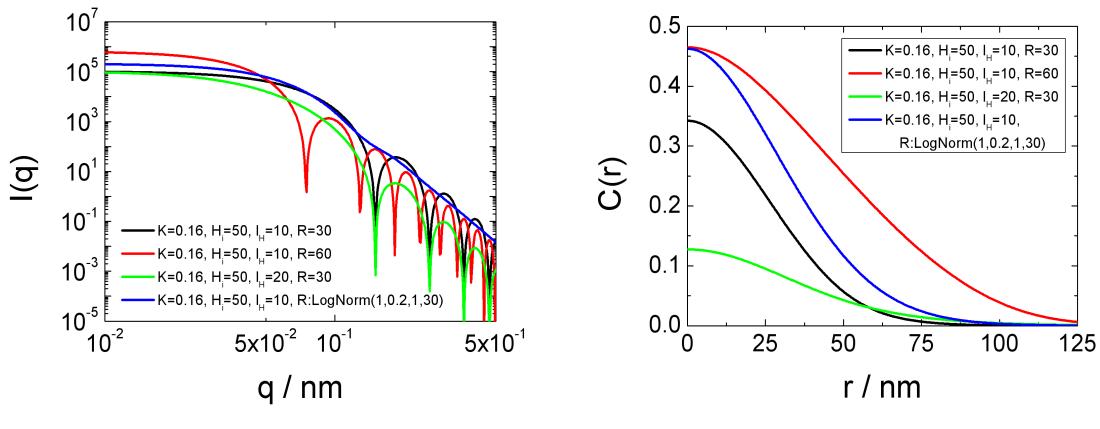
H_i: internal magnetic field H_i

l_H: exchange length of magnetic field l_H

R: anisotropy field radius R

Note: none

²the spherical Bessel functions j_n and y_n , and are related to the ordinary Bessel functions J_n and Y_n by: $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$ and $y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+\frac{1}{2}}(x) = (-1)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-\frac{1}{2}}(x)$. The first two spherical Bessel function are computed by $j_0(x) = \frac{\sin(x)}{x}$ and $j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}$



(a) The scattering curves for the correlation functions on to right

(b) correlation functions for the scattering curves to the left

FIGURE 8.63. Scattering curves and corresponding correlation functions. In one case the radius of the anisotropy field is assumed to have a lognormal size distribution.

8.8.2. Ferrofluids.

”Ferrofluids” and ”Magnetic Liquids” are two common expressions for suspensions in which nanometer sized particles are dispersed in a carrier liquid. Figure 8.64 gives a schematic illustration. The particle material can be ferri- or ferromagnetic, and is often coated with a stabilizing dispersing agent (surfactant). The particle size is smaller than the size of magnetic domains of the corresponding bulk material. This means that every particle is single-domain ferri- or ferromagnetic. Additionally we assume in this model, that the metallic core has a shell structure to be able to model a thin oxidation layer with a reduced magnetisation on the surface. The surfactant molecules consist of a head and a polar tail. The head can coalesce with the magnetic particle, and the tail protrudes into the carrier liquid and can dissolve in it. This polar tails protruding into the liquid lead to repulsion between the colloids. Different substances like organic acids or polymers usually serve as surfactants. The carrier liquid is selected to meet the requirements for a particular applications: It can be hydrocarbon, ester, water or others. As a result of the small size of the particles they are very mobile due to Brownian motion. In zero field the magnetic moment is random oriented. Applying a magnetic field the orientation distribution of the magnetic moments of the ferrofluid particles follow a Langevin statistic.

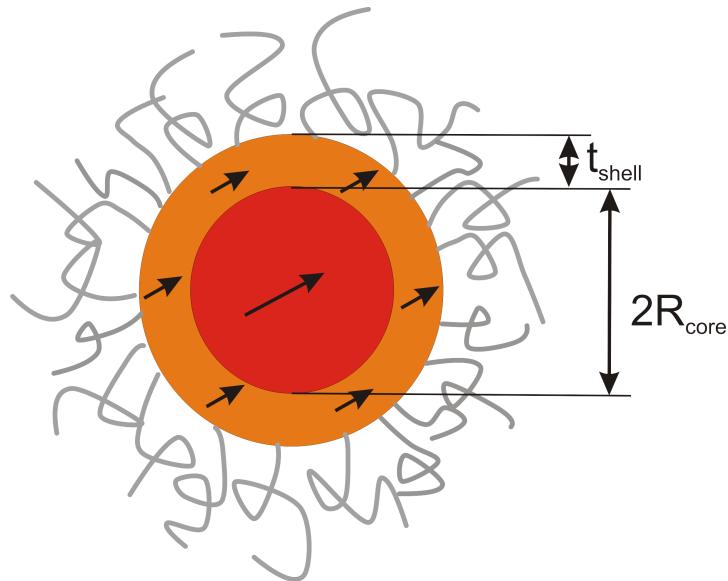


FIGURE 8.64. Sketch of a ferrofluid particle stabilised with surfactant molecules on its surface. The magnetic core can have a shell with lower magnetisation. The direction of the shell magnetisation is assumed to be parallel or antiparallel to the core magnetisation, depending on the signs of the magnetic scattering length densities.

In the present implementation of scattering function of superparamagnetic particles geometrical parameters are calculated like magnetic scattering length densities are calculated from the magnetization of the shell and the core, whereas the nuclear scattering length densities are an explicit input parameter. In such a case one has to be careful using the right units.

The relevant parameters to describe the ferrofluid particle in this model are:

- Radius of the R_{core} in reciprocal units of scattering vector q .
- Thickness of the metallic shell t_{shell} in reciprocal units of scattering vector q .
- Specific aggregation number of stabilizing surfactant molecules per surface area in units of cm^{-2} .
- Molecular volume V_{brush} of a single surfactant molecule in units of nm^3 . The scattering length density calculator of **SASfit** might be useful to supply this value.
- Nuclear scattering length density η_{core} of the solid ferromagnetic core.
- Nuclear scattering length density η_{sh} of an optional demagnetised shell of the solid ferromagnetic core.
- Nuclear scattering length density of the solvent η_{solv} .
- Magnetisation of the metallic core M_{core} . The magnetic scattering length density is calculated via eq. 8.194.
- Magnetisation of the metallic shell M_{sh} . Also here the magnetic scattering length density of the shell is calculated via eq. 8.194.
- Temperature T in Kelvin.
- External applied magnetic field B in Tesla or milliTesla depending of the units of q .
- Lagrange parameter α for describing superparamagnetic properties is calculated by ratio of the potential energy of the total magnetic moment of the superparamagnetic particle in the applied magnetic field to the thermal energy via eq. 8.197
- As aligned magnetic structures scatter anisotropically the angle ψ between scattering vector and the applied magnetic field needs to be supplied.
- using polarised neutrons the incident polarisation p needs to be known. p can have values $p \in [-1, 1]$.
- in case of full polarisation analysis also the transmission of the analyser needs to be known

The magnetic scattering length density η_{mag} in units of cm^{-2} is calculated from the magnetisation M given in units of A/m by

$$\eta_{\text{mag}} = D_M \mu_0 M \times 10^{-4} \quad (8.194)$$

$$D_M = 2.3161 \frac{1}{\text{Vs}} \quad (8.195)$$

$$\mu_0 = 4\pi \times 10^{-7} \text{Hm}^{-1} = 4\pi \times 10^{-7} \frac{\text{Vs}}{\text{Am}} \quad (8.196)$$

To get the magnetic scattering length density in cm^{-2} a multiplication factor of 10^{-4} has to be applied. By choosing the unit cm^{-2} for the magnetic scattering length density we have to use the corresponding units for the nuclear scattering length densities.

The other unit which has to be chosen with care is the molecular volume V_{brush} of a single surfactant molecule. From the specific aggregation and the molecular volume the excess scattering of the surfactant shell is calculated. In case that the q -values are given in units of nm^{-1} the molecular volume has to be in units of nm^3 and in case of \AA^{-1} for the q -values one also need to use \AA^3 units for the molecular volume.

The next parameter, where we need to use proper units is the Langevin parameter α , describing the ratio between potential energy of the magnetic moment of the particle to the thermal energy:

$$\alpha = \mu_{\text{FF}} B / (k_B T) \quad (8.197)$$

$$\mu_{\text{FF}} = \frac{4}{3} \pi [R_{\text{core}}^3 M_{\text{core}} + \{(R_{\text{core}} + t_{\text{shell}})^3 - R_{\text{core}}^3\} M_{\text{shell}}] \times \frac{10^{-27} \text{m}^3}{\text{nm}^3} \quad (8.198)$$

$$k_B = 1.3806488 \times 10^{-23} \text{m}^2 \text{kg s}^{-2} \text{K}^{-1} \quad (8.199)$$

The magnetic moment of a particle μ_{FF} as given in eq. 8.198 is calculated from the magnetisation of the core and the shell and their corresponding dimensions. The magnetisation has units of A/m whereas the radius and shell thickness have reciprocal dimensions of q . A magnetic moment is measured in amperesquare meters (Am^2) or in joules per tesla (JT^{-1}) which is equivalent: $1\text{Am}^2 = 1\text{JT}^{-1}$. In case that q values are given in \AA^{-1} the applied magnetic field should be given in units of mT instead of T.

The implemented form factor for ferrofluids is done for neutrons with or without incident polarisation as well as with polarisation analysis. In case of full polarisation analysis the measured intensity $I_m(\mathbf{Q})$ would depend on the incident polarisation, efficiency of the spin-flipper and in case of an opaque spin filter from its transmission values for the two spin states according to eq. 3.268. However here we just have implemented the intensities for a given incident polarisation without polarisation analysis (SANSPOL) and the intensities for POLARIS data are assumed to be corrected so that the resulting intensities correspond to perfect incident polarisation and perfect polarisation analysis. For calculating the scattering cross-section for the 4 scattering processes involved the differential cross-sections in the decoupling approach are given by

$$\frac{d\sigma_{\pm\pm}}{d\Omega}(\mathbf{Q}) = N \left\{ \left\langle \left| F_{\pm\pm}(\mathbf{Q}) \right|^2 \right\rangle + \left| \left\langle F_{\pm\pm}(\mathbf{Q}) \right\rangle \right|^2 (S(\mathbf{Q}) - 1) \right\} \quad (8.200)$$

The averages $\langle \rangle$ needs to be performed over the orientation distribution of the superparamagnetic moments of the ferrofluids and later on also about their size distribution. The averaging over the orientation distribution of the magnetic moments can be done analytically in terms of order parameters according to [227, 222, 115].

The spin dependent scattering amplitudes $F_{\pm\pm}(\mathbf{Q})$ can be calculated from the nuclear and magnetic amplitudes

$$F_{\pm\pm}(\mathbf{Q}) = F_N(\mathbf{Q}) \mp F_{M_{\perp}x}(\mathbf{Q}) \quad (8.201)$$

$$F_{\pm\mp}(\mathbf{Q}) = -(F_{M_{\perp}y}(\mathbf{Q}) \pm iF_{M_{\perp}z}(\mathbf{Q})) \quad (8.202)$$

The nuclear scattering amplitude is proportional to the nuclear excess scattering $\beta_N = F_N(Q = 0)$ and the nuclear form factor $f_N(\mathbf{Q})$

$$F_N(\mathbf{Q}) = \beta_N f_N(\mathbf{Q}) \quad (8.203)$$

The magnetic scattering amplitude $\mathbf{F}_{M_{\perp}}(\mathbf{Q})$ can be described as a vector, with

$$\mathbf{F}_{M_{\perp}}(\mathbf{Q}) = \hat{\boldsymbol{\mu}}_{\perp} D_M \mu_{\text{FF}} f_M(\mathbf{Q}) = \hat{\boldsymbol{\mu}}_{\perp} F_M(\mathbf{Q}) \quad (8.204)$$

where $f_M(Q)$ is the magnetic form factor, μ_{FF} the magnetic moment of the particle, $D_M = \frac{\gamma e}{2\pi\hbar}$, and $\hat{\mu}_\perp$ the Halpern-Johnson vector defined as

$$\hat{\mu}_\perp = \frac{\mathbf{Q}}{Q} \times \left(\frac{\mu_{\text{FF}}}{\mu_{\text{FF}}} \times \frac{\mathbf{Q}}{Q} \right) \quad (8.205)$$

It is assumed here that the neutron spin polarization is parallel or antiparallel to the axes e_x which is the direction perpendicular to the incoming neutron beam.

In the following we derive the average over the squared scattering amplitude and the averaged scattering amplitude for superparamagnetic particles. The description of the nuclear scattering amplitude is based on the form factor described in [166, 164] for block copolymer micelles from Pedersen et al.

The different versions of form factor implemented in the plugin differ on one side on the model for describing the scattering of the molecules on the surface and on the other side how the data are pre-treated. At the moment three different model for the polymer chains on the surface are implemented, namely that the chains are noninteracting and follow random walk behaviour (**Chain,RW**), secondly they are interacting due to the high grafting density and a model for self avoiding walk (**Chain,SAW**) is applied, thirdly a parabolic scattering length density profile for the outer shell is used (**Chain,parabolic**). For each models a version for a sector cut in the direction ψ (**psi**), a radial averaged intensity (**rad. avg.**), the anisotropic scattering contribution, which follows a \sin^2 -law (**magnetic**), the angle dependent difference signal between two spin states (**cross-term**), as well as the radial averaged cross term (**cross-term (rad. avg.)**) has bee implemented. Also the for individual angle dependent cross-sections with full polarisation analysis are available. Finally we end up with the following possible combinations:

$$\left. \begin{array}{c} \text{FF+}(\text{Chain,RW}) \\ \text{FF+}(\text{Chain,SAW}) \\ \text{FF+}(\text{Chain,parabolic}) \end{array} \right\} - \left\{ \begin{array}{l} \text{psi} \\ (\text{rad. avg.}) \\ \text{magnetic} \\ \text{cross-term} \\ \text{cross-term (rad. avg.)} \\ (++) \\ (--) \\ (+-) \\ (-+) \end{array} \right\} \quad (8.206)$$

8.8.3. Langevin statistics for averages of the form factor and averages of the squared form factor.

It is assumed here that the neutron spin polarization is parallel or antiparallel to the axes e_x which is the direction perpendicular to the incoming neutron beam. If only the Halpern-Johnson vector $\hat{\mu}_\perp$ depends on the orientation distribution $f(\theta)$ of the magnetic moments μ of the particles but not the form factor $f_M(\mathbf{Q})$, which is valid for spherical symmetric particles or anisotropic shaped particles where the particle shape is not correlated to the direction of the magnetic moment, the averages in 8.200 can be

written in terms of the order parameters S_1 and S_2

$$\langle F_{\pm\pm}(\mathbf{Q}) \rangle = F_N(\mathbf{Q}) \mp F_M(\mathbf{Q}) S_1 \sin^2 \psi \quad (8.207a)$$

$$\langle F_{\pm\mp}(\mathbf{Q}) \rangle = F_M(\mathbf{Q}) S_1 \sin \psi \cos \psi \quad (8.207b)$$

$$\begin{aligned} \langle |F_{\pm\pm}(\mathbf{Q})|^2 \rangle &= |F_N(\mathbf{Q})|^2 + |F_M(\mathbf{Q})|^2 \left[S_2 \sin^4 \psi + \frac{1 - S_2}{3} \sin^2 \psi \right] \\ &\mp [F_M(\mathbf{Q}) F_N^*(\mathbf{Q}) + F_M^*(\mathbf{Q}) F_N(\mathbf{Q})] S_1 \sin^2 \psi \end{aligned} \quad (8.207c)$$

$$\langle |F_{\pm\mp}(\mathbf{Q})|^2 \rangle = |F_M(\mathbf{Q})|^2 \left[2 \frac{1 - S_2}{3} - S_2 \sin^4 \psi + \frac{4S_2 - 1}{3} \sin^2 \psi \right] \quad (8.207d)$$

The above equations for the spin dependent scattering intensities of magnetic particles with an orientation distribution $f(\theta, \phi)$ of its magnetic moment can be calculated in terms of order parameters S_l if one assumes that the particle are spherical symmetric or the orientation of the magnetic moment of a particle is not correlated to the particle orientation. Furthermore one has to assume that an external magnetic field is applied perpendicular to the incident neutron beam and that all orientations ϕ for a given angle θ , which is defined as the angle between the magnetisation vector of the particle and the direction of the external field \mathbf{B} have the same probability, i.e. the orientation distribution only depends on θ so that $f(\theta, \phi) = f(\theta)$. The orientation probability distribution function can be expanded in terms of a complete set of orthogonal functions. Expanding it in terms of Legendre polynomials $P_l(\cos \theta)$ gives

$$f(\theta) = \sum_l c_l P_l(\cos \theta) = \sum_l \frac{2l+1}{2} S_l P_l(\cos \theta) \quad (8.208)$$

The expansion coefficients can be calculated by

$$c_l = \frac{2l+1}{2} \int_0^\pi f(\theta) P_l(\cos \theta) \sin \theta \, d\theta \quad (8.209)$$

or

$$S_l = \int_0^\pi f(\theta) P_l(\cos \theta) \sin \theta \, d\theta$$

The first three Legendre polynomials are defined by

$$P_0(\cos \theta) = 1 \quad (8.210a)$$

$$P_1(\cos \theta) = \cos \theta \quad (8.210b)$$

$$P_2(\cos \theta) = \frac{1}{2} (3 \cos^2 \theta - 1) \quad (8.210c)$$

For superparamagnetic particle the orientation probability distribution is given by

$$f(\theta) = \frac{\alpha}{4\pi \sinh \alpha} \exp(\alpha \cos \theta) \quad (8.211)$$

with $\alpha = BM_p V_p / (k_B T)$ being the Langevin parameter. For this orientation probability distribution the first order parameters can be calculated as

$$S_0 = 1 \quad (8.212a)$$

$$S_1 = L(\alpha) = \coth \alpha - \frac{1}{\alpha} \quad (8.212b)$$

$$S_2 = 1 - 3 \frac{L(\alpha)}{\alpha} \quad (8.212c)$$

The spin-flip and spin-nonflip cross-section $\frac{d\sigma_{\pm\pm}}{d\Omega}(\mathbf{Q})$ can be obtained by combining 8.200 and 8.207. The cross-sections without polarization analysis $I_{\pm}(\mathbf{Q})$ and for unpolarized neutrons $I_{unp}(\mathbf{Q})$ are given by

$$\begin{aligned} I_{\pm}(\mathbf{Q}) &= I_{\pm\pm}(\mathbf{Q}) + I_{\pm\mp}(\mathbf{Q}) \\ &= \left[|F_N(\mathbf{Q})|^2 + |F_M(\mathbf{Q})|^2 S_1^2 \sin^2 \psi \right. \\ &\quad \left. \mp [F_M(\mathbf{Q}) F_N^*(\mathbf{Q}) + F_N^*(\mathbf{Q}) F_M(\mathbf{Q})] S_1 \sin^2 \psi \right] S(\mathbf{Q}) \\ &\quad |F_M(\mathbf{Q})|^2 \left(\frac{2}{3} (1 - S_2) + (S_2 - S_1^2) \sin^2 \psi \right) \end{aligned} \quad (8.213a)$$

$$\begin{aligned} I_{unp}(\mathbf{Q}) &= \frac{1}{2} (I_+(\mathbf{Q}) + I_-(\mathbf{Q})) \\ &= \left(|F_N(\mathbf{Q})|^2 + |F_M(\mathbf{Q})|^2 S_1^2 \sin^2 \psi \right) S(\mathbf{Q}) \\ &\quad + |F_M(\mathbf{Q})|^2 \left(\frac{2}{3} (1 - S_2) + (S_2 - S_1^2) \sin^2 \psi \right) \end{aligned} \quad (8.213b)$$

In the case of a Boltzmann orientation distribution $f(\theta) = \exp\left(\frac{B\mu}{k_B T}\right) = \exp\left(\frac{B\mu \cos \theta}{k_B T}\right)$ the order parameter S_l already have been given in eq. 8.212 and the spin dependent intensities can be written as

$$\begin{aligned} \frac{I_{\pm\pm}(\mathbf{Q})}{N} &= |F_M(\mathbf{Q}) L(\alpha) \sin^2 \psi \pm F_N(\mathbf{Q})|^2 S(\mathbf{Q}) \\ &\quad + |F_M(\mathbf{Q})|^2 \left(\frac{L(\alpha)}{\alpha} \sin^2 \psi - \left(L^2(\alpha) - 1 + 3 \frac{L(\alpha)}{\alpha} \right) \sin^4 \psi \right) \end{aligned} \quad (8.214a)$$

$$\begin{aligned} \frac{I_{\mp\pm}(\mathbf{Q})}{N} &= (\sin^2 \psi - \sin^4 \psi) L^2(\alpha) |F_M(\mathbf{Q})|^2 S(\mathbf{Q}) \\ &\quad + |F_M(\mathbf{Q})|^2 \left((\sin^4 \psi - \sin^2 \psi) \left(L^2(\alpha) - 1 + 3 \frac{L(\alpha)}{\alpha} \right) + (2 - \sin^2 \psi) \frac{L(\alpha)}{\alpha} \right) \end{aligned} \quad (8.214b)$$

ψ is the angle between \mathbf{Q} and the horizontal axis in the plane of the detector. $L(\alpha) = \coth(\alpha) - \frac{1}{\alpha}$ is the Langevin function. In the case of a static field the superparamagnetic

particle are thermodynamic equilibrium and α is given by

$$\alpha = \frac{BM_P V_P}{k_B T}, \quad (8.215)$$

with M_P being the particle magnetization, V_P the particle volume, T the temperature in Kelvin and k_B the Boltzmann constant.

8.9. Bi-continuous and non-particulate structures

8.9.1. Teubner-Strey.

The Teubner and Strey [207, 192, 199] phenomenological model often accurately describes scattering from bi-continuous micro-emulsions. The scattered intensity for this model is

$$I(q) = \frac{8\pi\langle\eta^2\rangle/\xi}{a^2 - 2bq^2 + q^4} \quad (8.216)$$

where $a^2 = (k^2 + 1/\xi^2)^2$ is a positive quantity, and $b = k^2 - 1/\xi^2$ can be a positive or negative depending on the relative magnitude of $d = 2\pi/k$ and ξ . A positive b , i.e. $\xi > d/2\pi$, leads to a peak at $q_{\max} = \sqrt{b}$ whereby for $\xi < d/2\pi$, hence negative b , no distinct peak appears. The length scale d represents a quasi-periodic repeat distance between water and oil regions within the solution, while the correlation length, ξ , corresponds to a characteristic length for positional correlation. k is defined as $2\pi/d$. The corresponding isotropic real space correlation function, $\gamma(r)$, that incorporates alternating regions of the two phases in the bi-continuous system³ (e.g. water and oil), is given by

$$\gamma(r) = \frac{\sin(kr)}{kr} \exp\left(-\frac{r}{\xi}\right) \quad (8.217)$$

The Fourier transformation of this function $\langle\eta^2\rangle \int_0^\infty 4\pi r^2 \frac{\sin qr}{qr} \gamma(r) dr$ leads to the scattering intensity in eq. 8.216. The correlation length defined by eq. 13.73

$$l_c = \frac{2}{\gamma(0)} \int \gamma(r) dr \quad (8.218)$$

leads to

$$l_c = \frac{d}{\pi} \arctan\left(\frac{2\pi\xi}{d}\right) \quad (8.219)$$

For large q -values the intensity decays as

$$I(q) \xrightarrow{q \rightarrow \infty} \frac{8\pi\langle\eta^2\rangle}{\xi} q^{-4} \quad (8.220)$$

The scattering invariant is given by (see also eq. 13.49)

$$\tilde{Q} = \int_0^\infty I(q) q^2 dq = 2\pi^2 \langle\eta^2\rangle \quad (8.221)$$

In case of a two phase model with sharp interphases, which have a contrast of $\Delta\rho$ one can write $\langle\eta^2\rangle = \Delta\rho^2 f_p(1 - f_p)$ (see also eq. 13.48). Next to the invariant also the first

³In case of a correlation function $\gamma(r) = \exp\left(-\frac{r}{\xi}\right)$ the correlation length is $l_c = 2\xi$ and the expression of the scattering intensity is given by the form factor of Debye-Anderson-Brumberger in section 8.9.2 by $I(q) = I_0/(1 + (q\xi)^2)^2$

moment of the scattering curve is given analytically.

$$\int_0^\infty I(q) q dq = d\langle\eta^2\rangle \left(\frac{\pi}{2} - \arctan \left(\frac{d}{4\pi\xi} - \frac{\pi\xi}{d} \right) \right) \quad (8.222)$$

Therefore also a correlation length l_c (eq. 13.73 and 13.74) can be determined via

$$l_c = \pi \frac{\int_0^\infty I(q) q dq}{\int_0^\infty I(q) q^2 dq} = d \left(\frac{1}{4} - \frac{1}{2\pi} \arctan \left(\frac{d}{4\pi\xi} - \frac{\pi\xi}{d} \right) \right) \quad (8.223)$$

By using the identity $\frac{\pi}{4} - \frac{1}{2} \arctan \left(\frac{1}{4x} - x \right) = \arctan(2x)$ we get $l_c = \frac{d}{\pi} \arctan \left(\frac{2\pi\xi}{d} \right)$ which is consistent with eq. 8.219.

At $q_{\max} = \sqrt{b}$ the intensity is given by $I(q_{\max}) = \frac{d^2 \langle\eta^2\rangle \xi}{2\pi}$ and for $q = 0$ one gets $I(0) = \frac{8\pi d^4 \langle\eta^2\rangle \xi^3}{(d^2 + 4\pi^2 \xi^2)^2}$. In the limit of large values of d the form factor converges to the one from Debye-Anderson-Brumberger in section 8.9.2 which is $I(q) = I_0 / (1 + (q\xi)^2)^2$

Sometimes one finds the following parametrization

$$I(q) = \frac{I_0}{1 + c_1 q^2 + c_2 q^4} \quad (8.224)$$

As the intensity needs to be a positive number the conditions $c_2 > 0$ and $c_1 > -2\sqrt{c_2}$ have to hold. In this case the parameters are related to the above one via

$$I_0 = \frac{8\pi d^4 \langle\eta^2\rangle \xi^3}{(d^2 + 4\pi^2 \xi^2)^2} \quad (8.225)$$

$$c_1 = \frac{2d^2 \xi^2 (d^2 - 4\pi^2 \xi^2)}{(d^2 + 4\pi^2 \xi^2)^2} \quad (8.226)$$

$$c_2 = \frac{d^4 \xi^4}{(d^2 + 4\pi^2 \xi^2)^2} \quad (8.227)$$

or

$$\xi^2 = 4 \frac{c_2}{c_1 + 2\sqrt{c_2}} \quad (8.228)$$

$$\langle\eta^2\rangle = \frac{I_0 \xi / 2}{4\pi c_2} \quad (8.229)$$

$$d^2 = \frac{16\pi^2 c_2}{2\sqrt{c_2} - c_1} \quad (8.230)$$

Input Parameters for model Teubner-Strey:

xi: characteristic length for positional correlation ξ

d: characteristic domain size d

eta: $\langle\eta^2\rangle$ is the mean square scattering length density fluctuation. In case of a two phase system with sharp interphases $\langle\eta^2\rangle = \phi(1-\phi)\Delta\rho^2$ where $\Delta\rho$ is the scattering contrast. In [202] a discussion about a smooth interface can be found.

Note:

- None

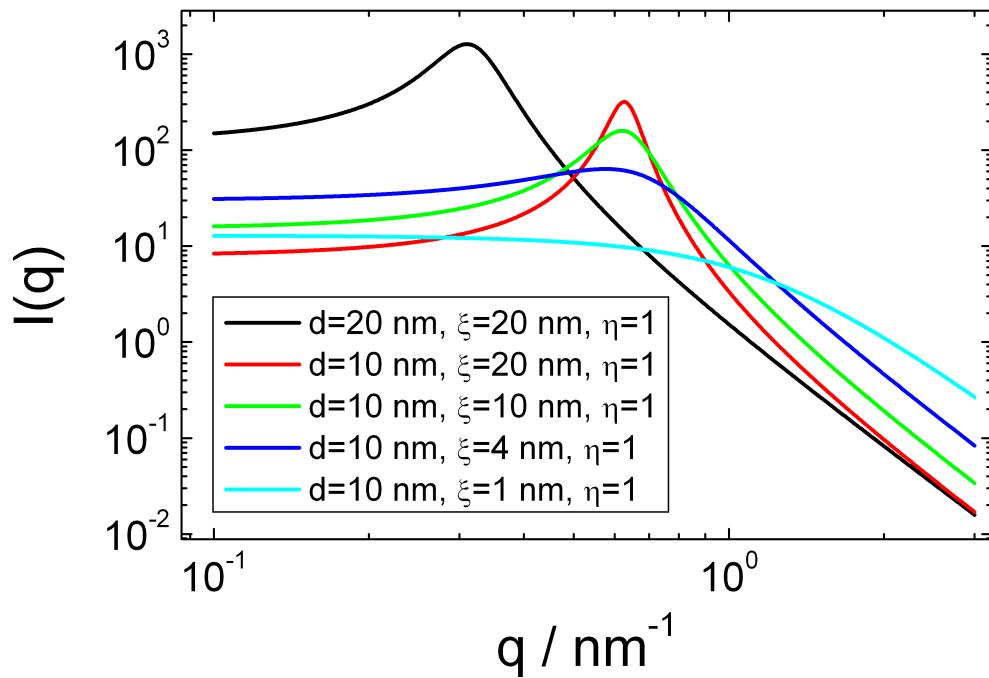


FIGURE 8.65. Some typical curves for the Teubner-Strey model function

8.9.2. Debye-Anderson-Brumberger(DAB).

This form factor calculates the scattering from a randomly distributed (i.e. non-particulate), two-phase system based on the Debye-Anderson-Brumberger (DAB) [44, 45] model for such systems. The two-phase system is characterized by a single length scale, the correlation length ξ , which is a measure of the average spacing between regions of phase 1 and phase 2. The model also assumes smooth interfaces between the phases and hence exhibits Porod behavior ($I \propto q^{-4}$) at large q ($q\xi \gg 1$). The pair correlation function is given by [45]

$$\gamma(r) = \exp(-r/\xi) \quad (8.231)$$

The macroscopic scattering cross-section in the DBA model is given by

$$I(q) = I_0 \frac{1}{[1 + (q\xi)^2]^2} \quad (8.232)$$

Input Parameters for model Debye-Anderson-Brumberger:

xi: correlation length ξ

eta: scattering length density contrast $\Delta\eta$

Note:

- None

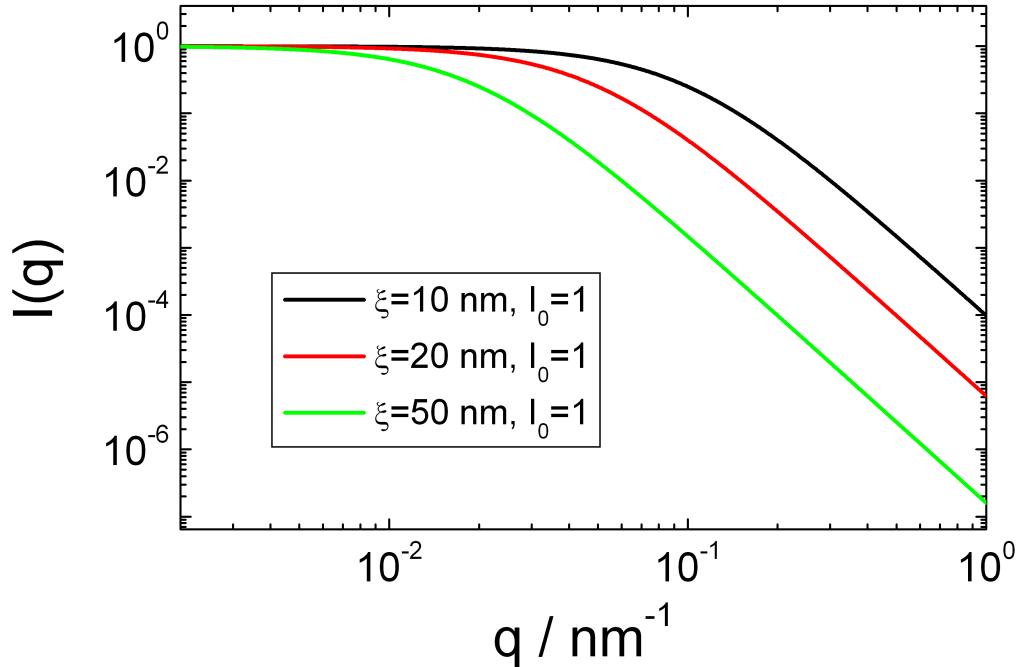


FIGURE 8.66. Some typical curves for the Debye-Anderson-Brumberger model function

8.9.3. Generalization of the Debye-Anderson-Brumberger model (gDAB).

The Debye-Anderson-Brumberger(DAB) model from the previous section 8.9.2 can be considered as a special case from a more general correlation function describing self-affine random density distributions [113, 96, 4]

$$\gamma(r) = \frac{2}{\Gamma(H)} \left(\frac{r}{2a} \right)^H K_H \left(\frac{r}{a} \right) \quad (8.233)$$

where $K_n(x)$ is the modified Bessel function of the second kind and Γ is the Gamma function. H is the so-called Hurst exponent with typical value around $\frac{1}{2}$. For $H = \frac{1}{2}$ the gDAB-model results into the DAB model described in section 8.9.2. The scattering intensity is implemented as

$$I(q) = \frac{(\Delta\eta 4\pi(1+2H)\xi^3)^2}{[1+(q\xi)^2]^{\frac{3}{2}+H}} \quad (8.234)$$

Input Parameters for model gDAB:

xi: correlation length ξ

H: Hurst exponent H

eta: scattering length density contrast $\Delta\eta$

Note:

- None

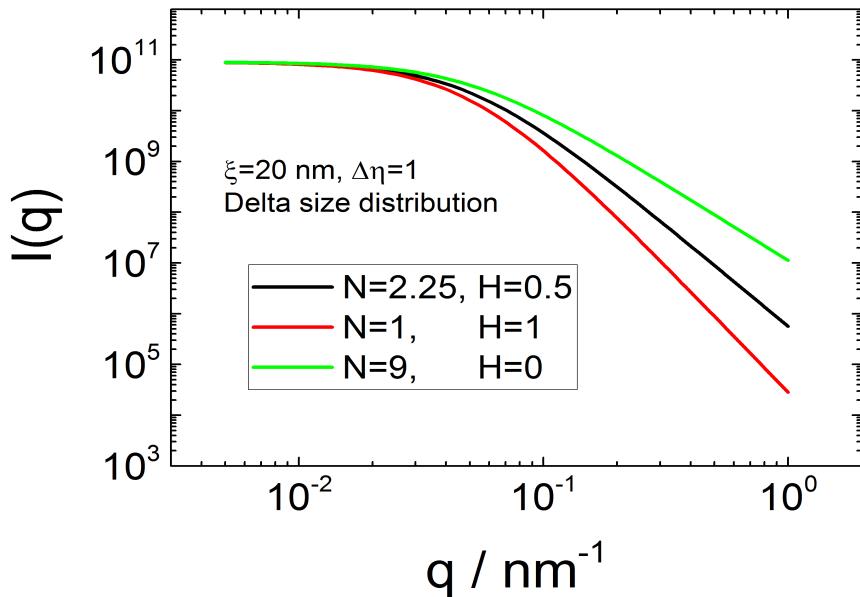


FIGURE 8.67. Some typical curves for the generalized Debye-Anderson-Brumberger model function

8.9.4. Spinodal.

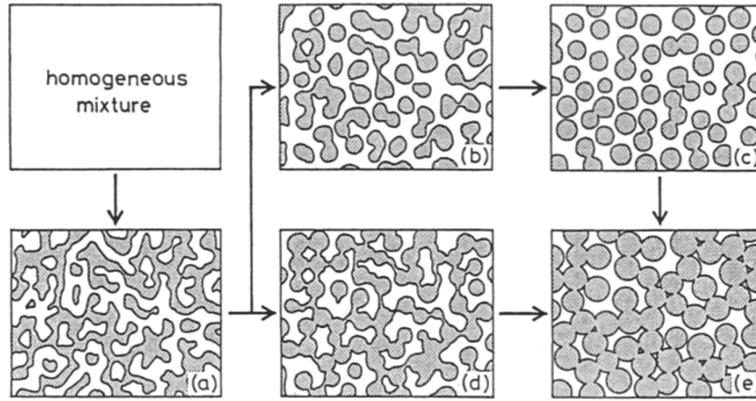


FIGURE 8.68. Schematic representation of a phase separation scheme resulting in a connected globule structure.

Spinodal decomposing systems show a characteristic small angle scattering signal with a correlation peak at some scattering value q_{\max} . The scattering curve $I(q)$ can be approximated by

$$I(q) = I_{\max} \frac{(1 + \gamma/2)x^2}{\gamma/2 + x^{2+\gamma}} \quad (8.235)$$

according to Furukawa [64], where $x = q/q_{\max}$. The position of the correlation peak at q_{\max} contain information about the size of the structures, which scatter. The exponent γ is equal to $\gamma = D + 1$ for off-critical mixtures and $\gamma = 2D$ for critical concentration mixtures, whereby D is the dimensionality of the system.

Input Parameters for model Spinodal:

I_{max}: scattering intensity at peak position I_{\max}

Q_{max}: peak maximum q_{\max}

gamma: exponent γ

Note:

- None

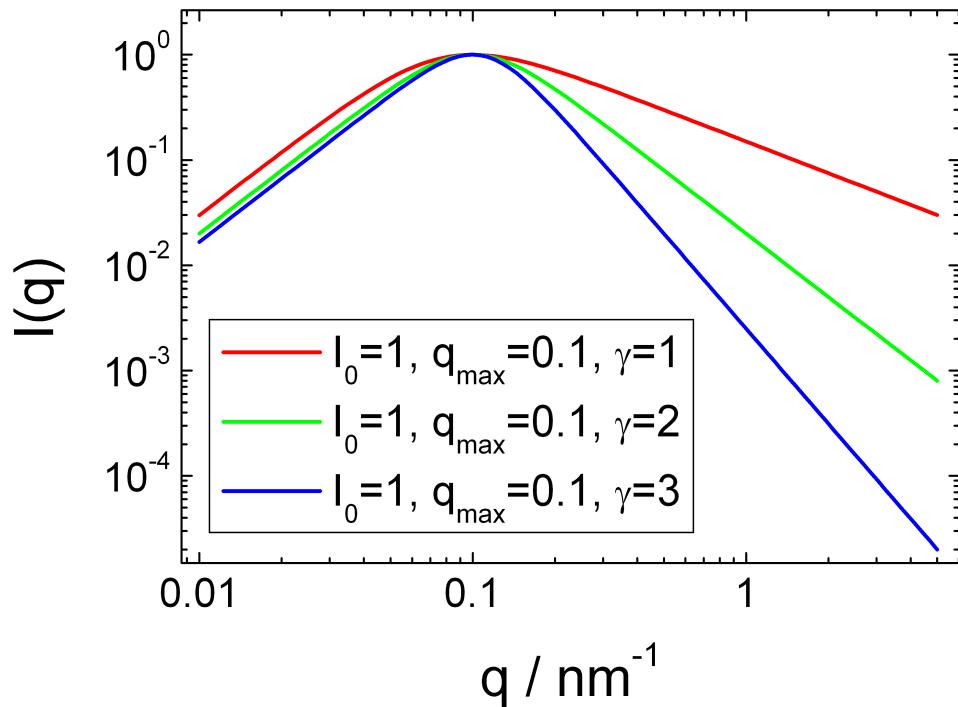


FIGURE 8.69.

8.9.5. OrnsteinZernike.

The low-angle scattering of thermal composition fluctuations can be described according to the Ornstein-Zernike formulation by a Lorentzian profile

$$I(q) = \frac{I_0}{1 + q^2 \xi^2} \quad (8.236)$$

characterizing the exponential decay of the composition fluctuations correlation function, with correlation length ξ . The Fourier transform of a Lorentzian function corresponds to correlations dying out as $\gamma(r) \simeq \frac{1}{r} \exp(-r/\xi)$. Note that the low- Q limit of this empirical form reproduces the Guinier law.

Input Parameters for model OrnsteinZernike:

I0: forward scattering I_0 at $q = 0$.

xi: correlation length ξ

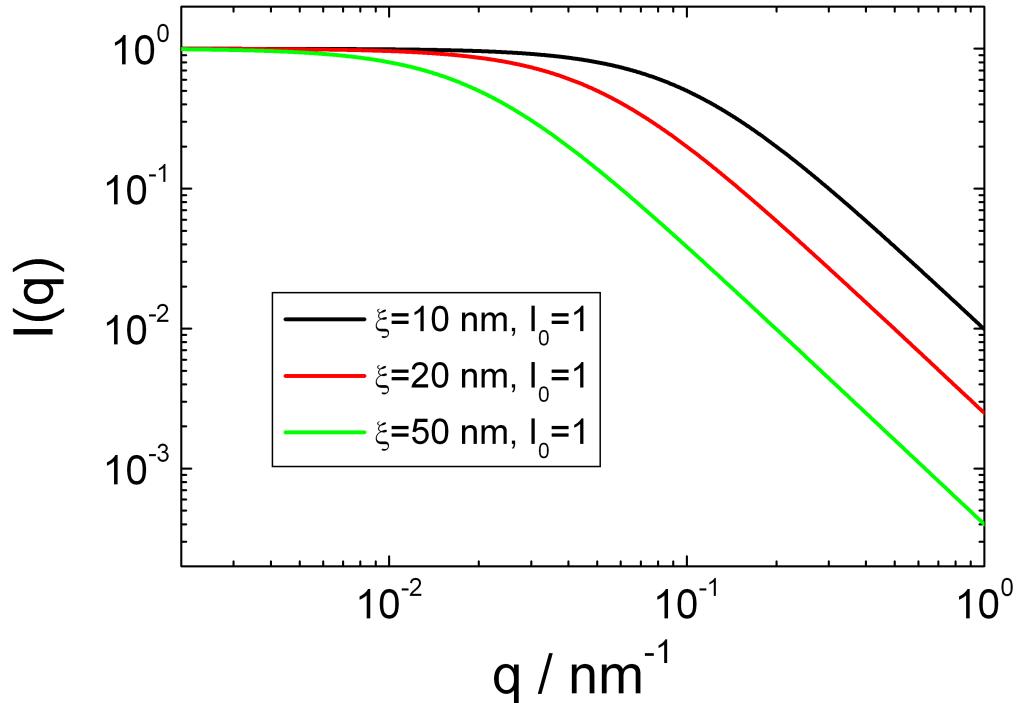


FIGURE 8.70. Ornstein-Zernike Scattering intensity $I(q)$ for different correlation lengths ξ

8.9.6. Broad-Peak.

Many SANS spectra are characterized by a broad peak even though they are from amorphous soft materials. The d -spacing corresponding to the broad peak is a characteristic distance between the scattering inhomogeneities (such as in lamellar, cylindrical, or spherical morphologies or for bicontinuous structures). The following simple functional form reproduces the broad peak feature:

$$I(q) = \frac{I_0}{(1 + (|q - q_0|\xi)^m)^p} \quad (8.237)$$

Here the peak position is related to the d -spacing as $q_0 = 2\pi/d$. Soft systems that show a SANS peak include copolymers, polyelectrolytes, multiphase systems, layered structures, etc.

Input Parameters for model Broad-Peak:

I0: scattering intensity $I(q)$ at $q = q_0$.

xi: correlation length ξ

m: exponent m

p: exponent p

Note:

- For $q_0 = 0$, $m = 2$ and $p = 1$ one gets the Ornstein-Zernike model.
- For $q_0 = 0$, $m = 2$ and $p = 2$ The Broad-Peak model is identical to the Debye-Anderson-Brumberger model.

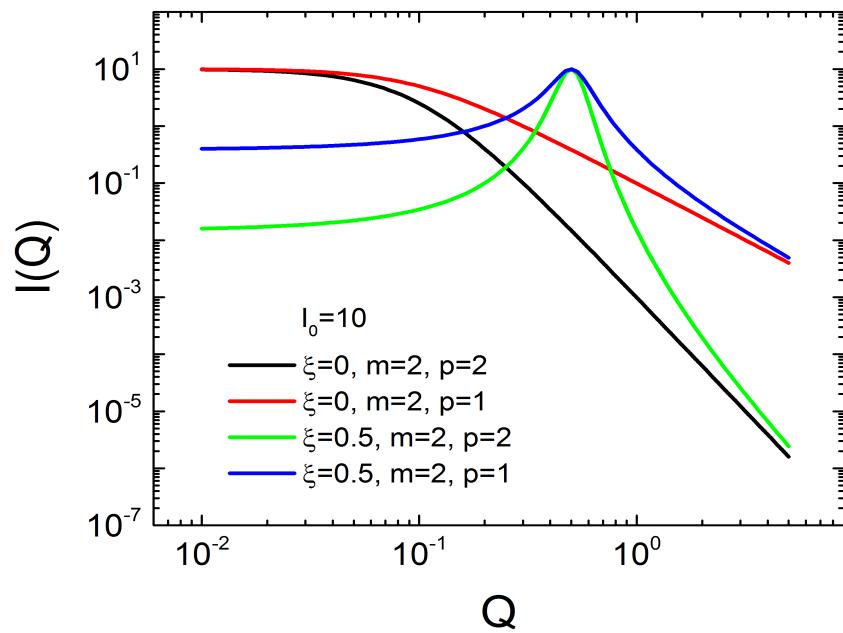


FIGURE 8.71. Empirical form factor of Broad-Peak

8.9.7. generalized Guinier approximation [52, 92, 90, 91].

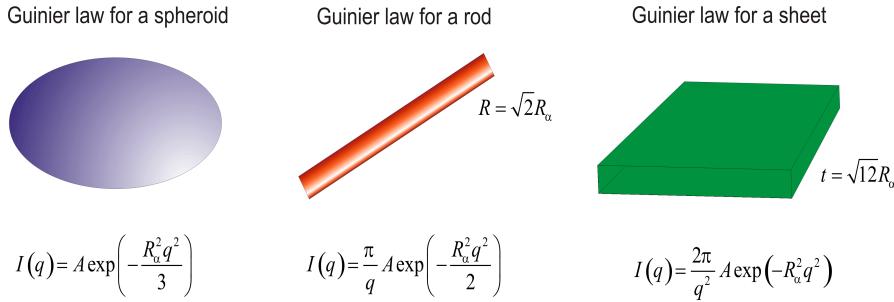


FIGURE 8.72. generalized Guinier approximation

Quantitative analysis of particle size and shape starts with the Guinier approximations. For three-dimensional objects the Guinier approximation is given by $I(q) = \exp(-Rg^2q^2/3)$. This approximation can be extended also to rod-like and plane objects by

$$I(q) = \begin{cases} 1 & \text{for } \alpha = 0 \\ \alpha\pi q^{-\alpha} & \text{for } \alpha = 1, 2 \end{cases} A \exp\left(-\frac{R_\alpha^2 q^2}{3-\alpha}\right) \quad (8.238)$$

$\alpha = 0$: spheroid

$\alpha = 1$: rod-like

$\alpha = 2$: plane

The apparent particle shape (also called the dimensionality) is represented in eq. 8.238 by α , which has integer values of 0, 1, and 2 for a point, a line, and a plane, respectively. Equation 8.238 states that there are q ranges, corresponding to length scales as q^{-1} , from which the particle dimension or shape, α , the radius of gyration, R_α , and the pre-factor, A , characteristic of α can be inferred. α has a value of 0 for a q range such that $qR_g < 1 - 1.3$ (the larger applies when the particle is known to be a spheroid), where R_g is the particle radius of gyration (computed about the particle centroid). In this case, the pre-factor A describes the excess differential cross-section per unit mass ($\text{cm}^2 \text{ g}^{-1}$) of a particle. If the particle has one dimension of length L , that is, much larger than the others (i.e., elongated, rod-like, or worm-like), then there is a q range such that $qR_c < 1 \ll qL$, where $\alpha = 1$. Here, R_c is the radius of gyration (computed about a line centered along L) of the cross-section perpendicular to L . If these conditions apply, the pre-factor A describes the excess differential cross section per unit length per unit mass ($\text{cm}^2 \text{ \AA}^{-1} \text{ g}^{-1}$). Finally, for planar shapes, including single bilayer vesicles, with two locally large dimensions, D , and planar cross-sectional radius of gyration (computed about a central plain), R_d , there may be a region of q such that $qR_d < 1 \ll qD$, where $\alpha = 2$. For such planar structures, the pre-factor is the excess differential cross-section per unit area per unit mass ($\text{cm}^2 \text{ \AA}^{-2} \text{ g}^{-1}$) of a sheet.

Input Parameters for model **generalized Guinier law**:

I0: scaling factor I_0 .

a: dimensionality parameter α for potential law at small Q -values; $\alpha = 0$: sphere,
 $\alpha = 1$: rod, $\alpha = 2$: lamellar

Ra: radius of gyration R_a

Note:

- parameter constrain: $3 > \alpha \geq 0$

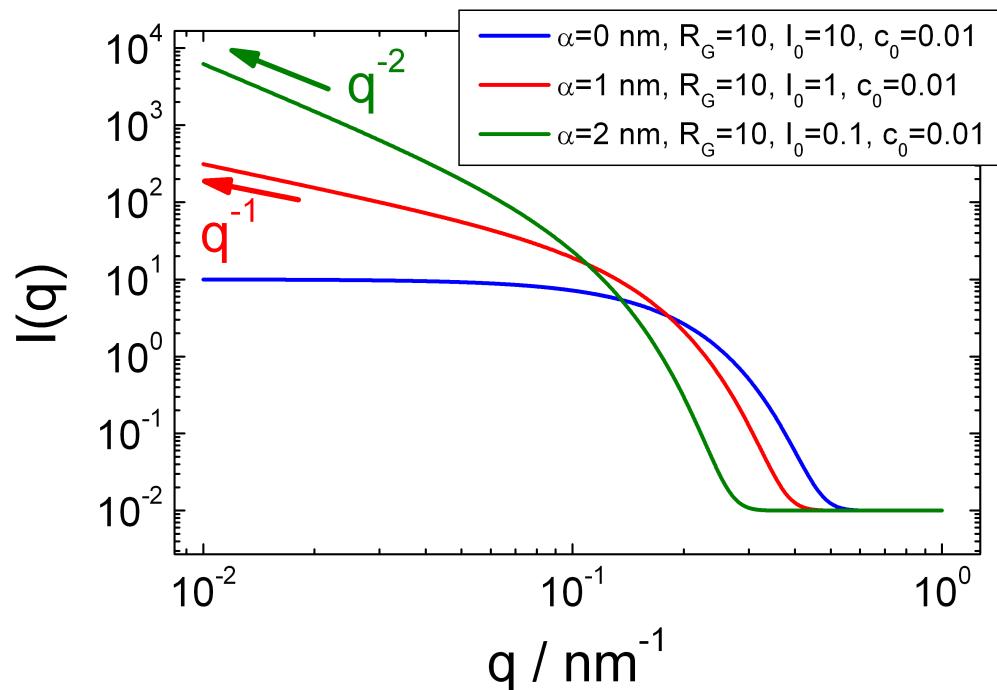


FIGURE 8.73. generalized Guinier law

8.9.8. generalized Guinier-Porod law.

The generalised Guinier-Porod approach from [77] combined the generalized Guinier approximation 8.9.7 with a potential law.

$$I(q) = \begin{cases} \frac{G_2}{Q^{s_2}} \exp\left(-\frac{Q^2 R_{g2}^2}{3-s_2}\right) & \text{for } Q \leq Q_2 \\ \frac{G_1}{Q^{s_1}} \exp\left(-\frac{Q^2 R_{g1}^2}{3-s_1}\right) & \text{for } Q_2 < Q \leq Q_1 \\ \frac{D}{Q^d} & \text{for } Q > Q_1 \end{cases} \quad (8.239)$$

To get a continuous function in $I(Q)$ and its derivative yield constrains for two parameters, which have been chosen to be Q_2 and G_2 , for which one get

$$Q_2 = \sqrt{\frac{s_1 - s_2}{\frac{2}{3-s_2} R_{g2}^2 - \frac{2}{3-s_1} R_{g1}^2}} \quad (8.240)$$

$$G_1 = G_2 \exp\left[-Q_2^2 \left(\frac{R_{g1}^2}{3-s_1} - \frac{R_{g2}^2}{3-s_2}\right)\right] Q_2^{(s_2-s_1)} \quad (8.241)$$

$$Q_1 = \frac{1}{R_{g1}} \sqrt{(m - s_1) \frac{3 - s_1}{2}} \quad (8.242)$$

$$D = G_1 \exp\left(-\frac{Q_1^2 R_{g1}^2}{3 - s_1}\right) Q_1^{m - s_1} \quad (8.243)$$

The constrains for the parameters are $R_{g2} > R_{g1}$, $3 > s_1 > s_2$, and $m > s_1$. The reason for that is, that the functions and their derivatives used in 8.239 are strongly monotonic decaying and the constrains are required to get a continuous function and derivative at Q_1 and Q_2 .

Input Parameters for model generalized Guinier-Porod law:

G2: scaling factor G_2 .

s2: potential law for the small Q -region, s_2 is a dimensionality parameter

RG2: radius of gyration R_{g2} (larger dimension)

s1: potential law in intermediate Q -range, s_1 is a dimensionality parameter

RG1: radius of gyration R_{g1} (smaller dimension)

m: potential law at large Q -value: Q^{-m}

Note:

- parameter constrain: $R_{g2} > R_{g1}$
- parameter constrain: $3 > s_1 > s_2$
- parameter constrain: $m > s_1$

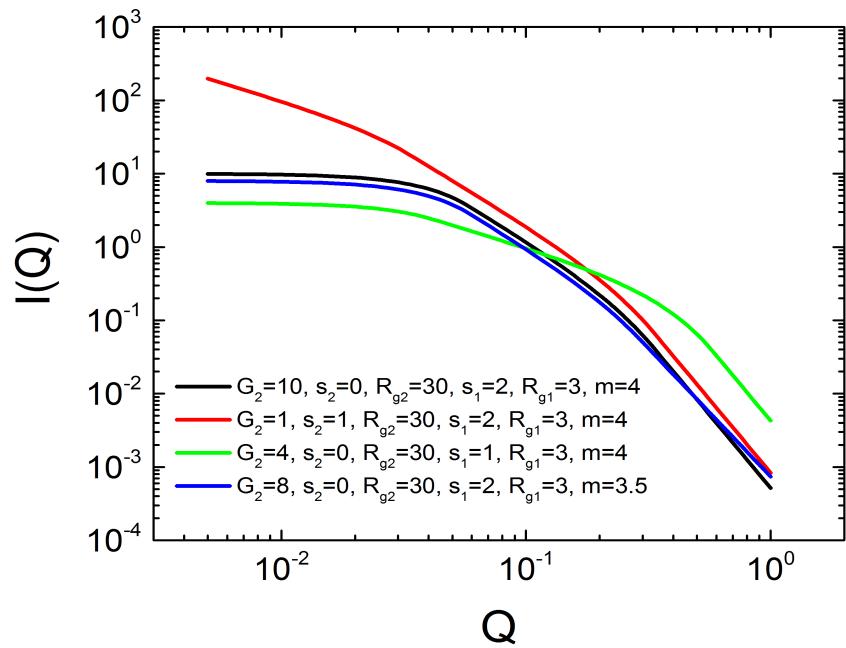


FIGURE 8.74. some examples showing the behaviour of the generalized Guinier-Porod law

8.10. Functions for analysing azimuthal averaged data**8.10.1. \sin^2 - \sin^4 azimuthal analysis.****8.10.2. Maier-Saupe azimuthal analysis.****8.10.3. Ellipsoidal azimuthal analysis.****[203, 148, 182, 82, 81, 189, 205, 69, 70]**

8.11. fractal size distribution of particles

Scattering curves of samples having a fractal structure show on a loglog scale a linear behaviour. One way to define the fractal dimension of an object is to look how the mass m of an object scales with its size x . For an object with a fractal dimension of f_D the mass scales m with $m \propto x^{f_D}$. A similar behaviour also can be found if the sample contains particles with a size distribution $n(x)$ proportional to $\propto x^{-(1+f_D)}$.

$$n(x, N, f_D, x_{\min}, x_{\max}) = \begin{cases} x < x_{\min} & 0 \\ x_{\min} \leq x \leq x_{\max} & N \frac{x^{-(1+f_D)}}{(x_{\min}^{-f_D} - x_{\max}^{-f_D})/f_D} \\ x > x_{\max} & 0 \end{cases} \quad (8.244)$$

The integral over size distribution $\int n(x, N, f_D, x_{\min}, x_{\max}) dx$ is normalized to N . In case the distribution follows such a potential law over a large enough size range a slope of $Q^{-(6-f_D)}$ in the scattering curve will be observed similar to the scattering behaviour of a surface fractal. Deviations of it can occur if the size range $[x_{\min}, x_{\max}]$ of such a size distribution is too narrow. The integration of a spherical form factor $I(Q) = \int_{\xi_{\min}}^{\xi_{\max}} N \frac{r^{-(1+f_D)}}{(R_{\min}^{-f_D} - R_{\max}^{-f_D})/f_D} \left(\frac{4}{3} \pi r^3 \frac{\sin Qr - Qr \cos Qr}{(Qr)^3} \right)^2 dr$ has not been found yet. However, using the Debye-Anderson-Brumberger (DAB) model from 8.9.2 instead of the form factor for spheres allows to express the integral in terms of hypergeometric functions:

$$I_{f_D}(Q; N, f_D, \xi_{\min}, \xi_{\max}) = \int_0^\infty n(\xi, N, f_D, \xi_{\min}, \xi_{\max}) \xi^6 I_{\text{DAB}}(Q, \xi) d\xi \quad (8.245)$$

$$= \int_{\xi_{\min}}^{\xi_{\max}} N \frac{\xi^{-(1+f_D)}}{(\xi_{\min}^{-f_D} - \xi_{\max}^{-f_D})/f_D} \frac{\xi^6}{(1 + \xi^2 Q^2)^2} d\xi \quad (8.246)$$

$$= N \frac{(f_D - 4)f_D}{(f_D - 2)2Q^4 (\xi_{\max}^{f_D} - \xi_{\min}^{f_D})} \left[\begin{aligned} & \xi_{\max}^2 \xi_{\min}^{f_D} {}_2F_1 \left(1, \frac{f_D - 2}{2}; \frac{f_D}{2}; -\frac{1}{Q^2 \xi_{\max}^2} \right) \\ & - \xi_{\min}^2 \xi_{\max}^{f_D} {}_2F_1 \left(1, \frac{f_D - 2}{2}; \frac{f_D}{2}; -\frac{1}{Q^2 \xi_{\min}^2} \right) \end{aligned} \right] \quad (8.247)$$

$$+ N \frac{f_D Q^2 \left(\frac{\xi_{\min}^4 \xi_{\max}^{f_D}}{Q^2 \xi_{\min}^2 + 1} - \frac{\xi_{\max}^4 \xi_{\min}^{f_D}}{Q^2 \xi_{\max}^2 + 1} \right)}{2Q^4 (\xi_{\max}^{f_D} - \xi_{\min}^{f_D})}$$

The DAB Model follows a Guinier law at small Q -values and the Porod law (Q^{-4}) at large Q values like a sphere. The term ξ^6 is added to account for the fact, that intensities scale with the squared volume of a particle. The correlation length ξ in the DAB model

is related to the radius of gyration R_G or the radius of a sphere R via $\xi^2 = \frac{1}{6}R_G^2 = \frac{1}{10}R^2$.⁴ This plugin has three variants of the above fractal size distribution implemented, namely a series of 1 to 3 of such distribution. The distributions are continuously continued when the fractal dimension is changing at a certain correlation length. For the form factor **fractal series 1** only one potential law with a fractal dimension $f_{D,1}$ is assumed. For the other cases we just assume a summation of k such distribution.

$$N(x) = \sum_{i=1}^k n(x, N_i, f_{D,i}, x_i, x_{i+1}) \quad (8.248)$$

We assume, that N is the overall number density of particles and normalize the size distribution accordingly

$$\int N(x)dx = N \quad (8.249)$$

To have a continuous sized distribution we also need to fulfill the conditions

$$n(x_{i+1}, N_i, f_{D,i}, x_i, x_{i+1}) = n(x_{i+1}, N_{i+1}, f_{D,i+1}, x_{i+1}, x_{i+2}) \forall i = 1, k-1 \quad (8.250)$$

These conditions define the parameters N_i . Therefore we get

- (1) For **fractal series 1** we assume a single potential law, i.e. $k = 1$ and therefore only have $N_1 = N$
- (2) For **fractal series 2** we assume two potential laws. The first one ranges from x_{\min} to $x_{1,2}$ and the second one from $x_{1,2}$ to x_{\max} . Therefore N_1 and N_2 are calculated by

$$N_1 = N \frac{f_{D,2}x_{\max}^{f_{D,2}} \left(x_{1,2}^{f_{D,1}} - x_{\min}^{f_{D,1}} \right)}{f_{D,2}x_{1,2}^{f_{D,1}}x_{\max}^{f_{D,2}} - x_{\min}^{f_{D,1}} \left(f_{D,1}x_{1,2}^{f_{D,2}} + (f_{D,2} - f_{D,1})x_{\max}^{f_{D,2}} \right)} \quad (8.251)$$

$$N_2 = N \frac{f_{D,1}x_{\min}^{f_{D,1}} \left(x_{1,2}^{f_{D,2}} - x_{\max}^{f_{D,2}} \right)}{x_{\min}^{f_{D,1}} \left(f_{D,1}x_{1,2}^{f_{D,2}} + (f_{D,2} - f_{D,1})x_{\max}^{f_{D,2}} \right) - f_{D,2}x_{1,2}^{f_{D,1}}x_{\max}^{f_{D,2}}} \quad (8.252)$$

- (3) For **fractal series 3** we assume three potential laws. The first one ranges from x_{\min} to $x_{1,2}$, the second one from $x_{1,2}$ to $x_{2,3}$ and the third from $x_{2,3}$ to

⁴The Taylor series of the DAB model at small Q-values gives $I_{\text{DAB}}(Q \rightarrow 0, \xi) \sim 1 - 2\xi^2 Q^2 + 3\xi^4 Q^4$ whereas the Taylor series of a form factor in terms of the Guinier radius R_G or the Guinier law at small Q-values is $I(Q \rightarrow 0) \sim I_0 \exp(-R_G^2 Q^2/3) \sim I_0 (1 - R_G^2 Q^2/3)$

x_{\max} . Therefore N_1 , N_2 and N_3 are calculated by

$$\begin{aligned} N_1 &= N \left(f_{D,2} f_{D,3} x_{2,3}^{f_{D,2}} x_{\max}^{f_{D,3}} \left(x_{1,2}^{f_{D,1}} - x_{\min}^{f_{D,1}} \right) \right) \\ &\quad / \left\{ f_{D,2} f_{D,3} x_{1,2}^{f_{D,1}} x_{2,3}^{f_{D,2}} x_{\max}^{f_{D,3}} - x_{\min}^{f_{D,1}} \left(f_{D,1} f_{D,2} x_{1,2}^{f_{D,2}} x_{2,3}^{f_{D,3}} \right. \right. \\ &\quad \left. \left. + x_{\max}^{f_{D,3}} \left(f_{D,3} (f_{D,2} - f_{D,1}) x_{2,3}^{f_{D,2}} - f_{D,1} (f_{D,2} - f_{D,3}) x_{1,2}^{f_{D,2}} \right) \right) \right\} \end{aligned} \quad (8.253)$$

$$\begin{aligned} N_2 &= N \left(f_{D,1} f_{D,3} x_{\min}^{f_{D,1}} x_{\max}^{f_{D,3}} \left(x_{1,2}^{f_{D,2}} - x_{2,3}^{f_{D,2}} \right) \right) \\ &\quad / \left\{ x_{\min}^{f_{D,1}} \left(f_{D,1} f_{D,2} x_{1,2}^{f_{D,2}} x_{2,3}^{f_{D,3}} + x_{\max}^{f_{D,3}} \left(f_{D,3} (f_{D,2} - f_{D,1}) x_{2,3}^{f_{D,2}} \right. \right. \right. \\ &\quad \left. \left. \left. - f_{D,1} (f_{D,2} - f_{D,3}) x_{1,2}^{f_{D,2}} \right) \right) - f_{D,2} f_{D,3} x_{1,2}^{f_{D,1}} x_{2,3}^{f_{D,2}} x_{\max}^{f_{D,3}} \right\} \end{aligned} \quad (8.254)$$

$$\begin{aligned} N_3 &= N \left(f_{D,1} f_{D,2} x_{\min}^{f_{D,1}} x_{1,2}^{f_{D,2}} \left(x_{2,3}^{f_{D,3}} - x_{\max}^{f_{D,3}} \right) \right) \\ &\quad / \left\{ x_{\min}^{f_{D,1}} \left(f_{D,1} f_{D,2} x_{1,2}^{f_{D,2}} x_{2,3}^{f_{D,3}} + x_{\max}^{f_{D,3}} \left(f_{D,3} (f_{D,2} - f_{D,1}) x_{2,3}^{f_{D,2}} \right. \right. \right. \\ &\quad \left. \left. \left. - f_{D,1} (f_{D,2} - f_{D,3}) x_{1,2}^{f_{D,2}} \right) \right) - f_{D,2} f_{D,3} x_{1,2}^{f_{D,1}} x_{2,3}^{f_{D,2}} x_{\max}^{f_{D,3}} \right\} \end{aligned} \quad (8.255)$$

Input Parameters for the models `fractal series1`:

N: scaling factor, total number density of particles N

xi_min: minimum correlation length for DAB ξ_{\min}

fD1: first fractal dimension $f_{D,1}$

xi_max: maximum correlation length for DAB ξ_{\max}

Input Parameters for the models `fractal series2`:

N: scaling factor, total number density of particles N

xi_min: minimum correlation length for DAB ξ_{\min}

fD1: first fractal dimension $f_{D,1}$

xi_12: correlation length of DAB model where fractal dimension changes $\xi_{1,2}$

fD2: second fractal dimension $f_{D,2}$

xi_max: maximum correlation length for DAB ξ_{\max}

Input Parameters for the models `fractal series1`:

N: scaling factor, total number density of particles N

xi_min: minimum correlation length for DAB ξ_{\min}

fD1: first fractal dimension $f_{D,1}$

xi_12: correlation length of DAB model where fractal dimension changes $\xi_{1,2}$

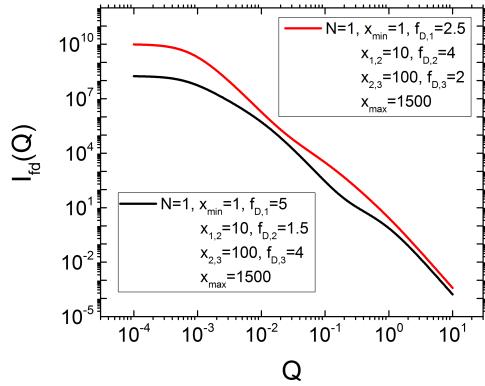
fD2: second fractal dimension $f_{D,2}$

xi_23: correlation length of DAB model where fractal dimension changes $\xi_{2,3}$

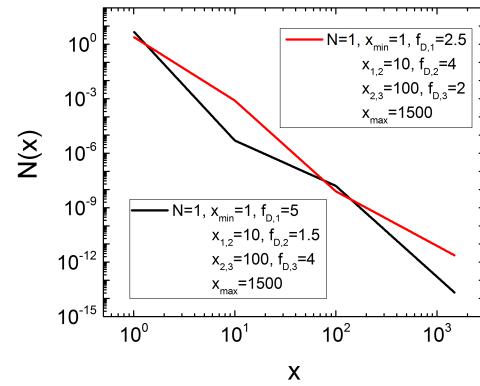
fD3: third fractal dimension $f_{D,3}$

`xi_max`: maximum correlation length for DAB ξ_{\max}

Note: The fractal dimensions need to be larger than 0 as well as ξ_{\min} . For the numerical evaluation it is required that $0 > \xi_{\min} > \xi_{1,2} > \xi_{2,3} > \xi_{\max}$.



(a) scattering curves of a fractal size distribution of particles where the scattering of the individual particle is described by the DAB form factor



(b) fractal size distribution, the size distribution is zero for $x < x_{\min}$ and $x > x_{\max}$

FIGURE 8.75. Scattering curve and corresponding size distribution for the plugin "fractal series 3".

8.12. Plugin functions for SESANS

An overview about the transformation cycle between the spherically symmetric correlation function, projected correlation function and differential cross section, which is relevant for the SESANS plugin functions, is described in detail in [114].

The depolarization of the neutron beam as a function of the spin-echo length z is given by

$$\frac{P_n(\delta)}{P_{n0}(\delta)} = \exp(\mathcal{G}(\delta) - \mathcal{G}(0)) \quad (8.256)$$

where the unnormalized SESANS correlation function reads as

$$\mathcal{G}(\delta) = \frac{1}{k_0^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dQ_y dQ_z \frac{d\Sigma(\mathbf{Q})}{d\Omega} \frac{\cos(Q_z \delta)}{S} \quad (8.257)$$

$$= \frac{1}{k_0^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dQ_y dQ_z \frac{d\sigma(\mathbf{Q})}{d\Omega} t \cos(Q_z \delta) \quad (8.258)$$

with $\mathbf{Q} = (0, Q_y, Q_z)^T$ and $\frac{d\Sigma(\mathbf{Q})}{d\Omega}$ being the macroscopic differential scattering cross-section in units of cm^2 and $\frac{d\sigma(\mathbf{Q})}{d\Omega} = \frac{1}{V} \frac{d\Sigma(\mathbf{Q})}{d\Omega}$ the differential scattering cross-section normalized on the sample volume $V = tS$ in units of cm^{-1} . t is the sample thickness and S the aperture cross-section defining the illuminated sample area. For isotropic scattering $\frac{d\sigma(\mathbf{Q})}{d\Omega} = \frac{d\sigma(Q)}{d\Omega}$, where the scattering only depends on the modulus of the scattering vector $Q = |\mathbf{Q}|$ the unnormalized SESANS correlation function simplifies to a Hankel transform of the scattering intensity multiplied by $\frac{\lambda^2 t}{2\pi}$. To see this, one has to change from cartesian coordinates $(0, Q_y, Q_z)$ to polar coordinates $Q(0, \sin \phi, \cos \phi)$ with $dQ_y dQ_z = Q dQ d\phi$. The integration over ϕ yield the cylindrical zeroth-order Bessel function

$$J_0(Qz) = \frac{1}{2\pi} \int_0^{2\pi} \cos(Q \cos(\phi) \delta) d\phi \quad (8.259)$$

so that one gets

$$\mathcal{G}(\delta) = \frac{2\pi t}{k_0^2} \int_0^{\infty} Q J_0(Q\delta) \frac{d\sigma(Q)}{d\Omega} dQ \quad (8.260)$$

$$= \frac{\lambda^2 t}{4\pi^2} 2\pi \underbrace{\int_0^{\infty} Q J_0(Q\delta) \frac{d\sigma(Q)}{d\Omega} dQ}_{\mathcal{H}_0\left[\frac{d\sigma(Q)}{d\Omega}\right](\delta)} = \frac{\lambda^2 t}{4\pi^2} 2\pi \mathcal{H}_0\left[\frac{d\sigma(Q)}{d\Omega}\right](\delta) = \lambda^2 t \tilde{G}(\delta) \quad (8.261)$$

where $\tilde{G}(\delta) = \frac{1}{2\pi} \mathcal{H}_0 \left[\frac{d\sigma(Q)}{dQ} \right] (\delta)$. $\mathcal{H}_0 [\dots]$ denotes the Hankel transform of zeroth order.⁵

The instrument independent projected sample correlation function $\tilde{G}(\delta)$ can be calculated via the Abel transform \mathcal{A} of the scattering length density autocorrelation function $\tilde{\gamma}(\mathbf{r}) = \int \rho(\mathbf{r}')\rho(\mathbf{r}' - \mathbf{r})d\mathbf{r}'$.

$$\tilde{G}(\delta) = 2 \int_z^\infty \frac{\tilde{\gamma}(r)r}{\sqrt{r^2 - z^2}} dr = \mathcal{A}[\tilde{\gamma}(r)](\delta) \quad (8.262)$$

The scattering intensity $I(Q)$ and $\tilde{\gamma}(r)$ are related by Fourier transform⁶ $\mathcal{F}_{I,3D}[\dots]$

$$I(Q) = \int_0^\infty \tilde{\gamma}(r) \frac{\sin Qr}{Qr} 4\pi r^2 dr = \mathcal{F}_{I,3D}[\tilde{\gamma}(r)](Q) \quad (8.263)$$

In literature the expressions for the scattering length density autocorrelation function are often normalized to 1 for $r = 0$, i.e. the expressions $\gamma_0(r) = \tilde{\gamma}(r)/\tilde{\gamma}(0)$ are given. Also one finds that

$$\tilde{\gamma}(0) = \Delta\eta^2 V_P \quad (8.264)$$

and

$$\int_0^\infty \tilde{\gamma}(r) 4\pi r^2 dr = \Delta\eta^2 V_P^2 \quad \text{or} \quad \int_0^\infty \gamma_0(r) 4\pi r^2 dr = V_P \quad (8.265)$$

where in case of spherical particle $V_P = \frac{4}{3}\pi R^3$. In Fig. 8.76 it is shown how the functions $\tilde{\gamma}(r)$, $\tilde{G}(\delta)$ and $I(Q)$ are related to each other. However, the cycle is mathematically only exact for a one-dimensional Fourier transformation on a radial symmetric function. Such a transformation cycle is called the projection-slice theorem which is a standard analysis in imaging [27, 26]. The projection-slice theorem makes use of the fact that

$$\mathcal{H} \circ \mathcal{F}_{1D} \circ \mathcal{A} = \mathcal{F}_{1D} \circ \mathcal{A} \circ \mathcal{H} = \mathcal{A} \circ \mathcal{H} \circ \mathcal{F}_{1D} = \mathcal{I} \quad (8.266)$$

⁵ There are several possible definitions of a Hankel transform $\mathcal{H}_\nu[f(r)]$ of order ν of a function $f(r)$ and its inverse $\mathcal{H}_\nu^{-1}[F_\nu(k)]$

- (1) $F_\nu(k) = \int_0^\infty f(r) J_\nu(kr) r dr$ and its inverse $f(r) = \int_0^\infty F_\nu(k) J_\nu(kr) k dk$
- (2) $F_\nu(k) = \int_0^\infty f(r) J_\nu(kr) \sqrt{kr} dr$ and its inverse $f(r) = \int_0^\infty F_\nu(k) J_\nu(kr) \sqrt{kr} dk$
- (3) $F_\nu(k) = 2\pi \int_0^\infty f(r) J_\nu(2\pi kr) r dr$ and its inverse $f(r) = 2\pi \int_0^\infty F_\nu(q) J_\nu(2\pi kr) k dk$

⁶ Similar to the Hankel transform also for the Fourier transform several conventions (spherical symmetric functions are assumed here) are in use which are

- (1) $\mathcal{F}_{I,3D}[f(r)](Q) = F(Q) = \int_0^\infty f(r) \frac{\sin 2\pi Qr}{2\pi Qr} 4\pi r^2 dr$ and
 $\mathcal{F}_{I,3D}^{-1}[F(Q)](r) = f(r) = \int_0^\infty F(Q) \frac{\sin 2\pi Qr}{2\pi Qr} 4\pi r^2 dQ$
- (2) $\mathcal{F}_{I,3D}[f(r)](Q) = F(Q) = \int_0^\infty f(r) \frac{\sin Qr}{Qr} 4\pi r^2 dr$ and
 $\mathcal{F}_{I,3D}^{-1}[F(Q)](r) = f(r) = \frac{1}{(2\pi)^3} \int_0^\infty F(Q) \frac{\sin Qr}{Qr} 4\pi r^2 dQ$
- (3) $\mathcal{F}_{I,3D}[f(r)](Q) = F(Q) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty f(r) \frac{\sin Qr}{Qr} 4\pi r^2 dr$ and
 $\mathcal{F}_{I,3D}^{-1}[F(Q)](r) = f(r) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty F(Q) \frac{\sin Qr}{Qr} 4\pi r^2 dQ$

In small angle scattering convention 2 is normally used.

with $\mathcal{F}_{1D}(s) = \int_{-\infty}^{\infty} \exp(-i2\pi sr) f(r) dr$, i.e. using the pair (1) of Fourier as well pair (1) of Hankel transform conventions as given in footnote⁶ and footnote⁵. That the \mathcal{FAH} cycle is still valid using the isotropic 3 dimensional Fourier transform $\mathcal{F}_{I,3D}$ as shown in fig. 8.76 has been shown in [114].

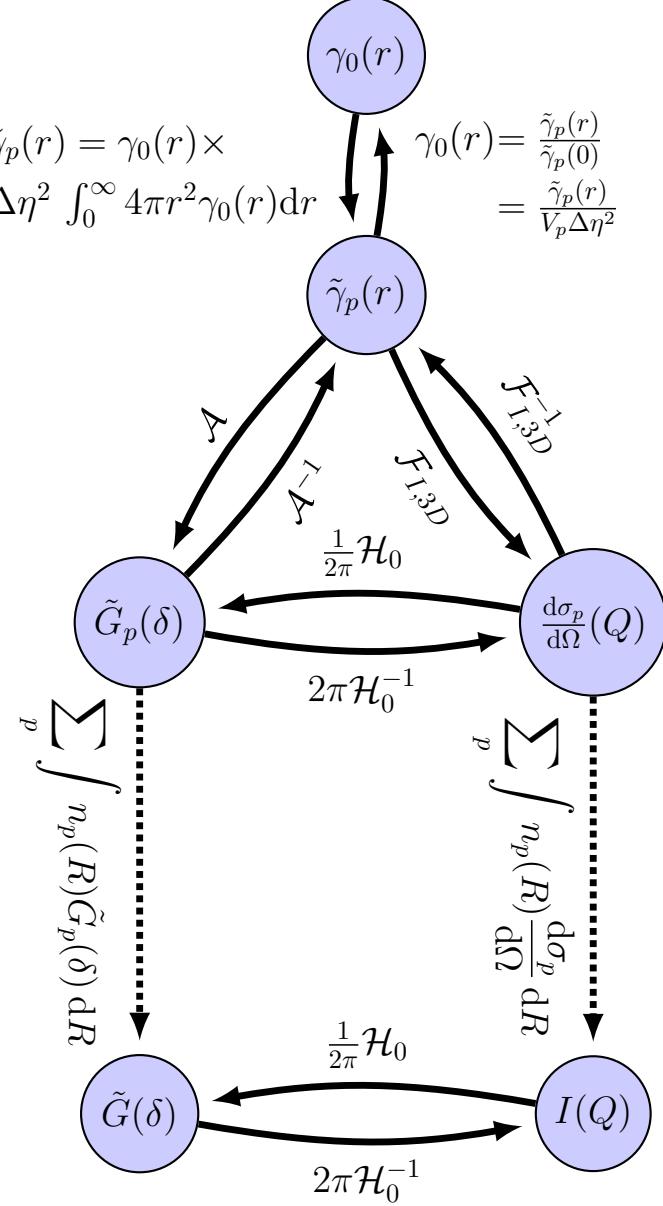


FIGURE 8.76. Transformation cycle between $\tilde{\gamma}(r)$, $I(q)$, and $\tilde{G}(\delta)$.

8.12.1. $G(z)$ of a sphere. The scattering intensity for a single sphere is given by

$$I(Q) = \left(\frac{4}{3} \pi R^3 \Delta\eta \right)^2 \left(3 \frac{\sin QR - QR \cos QR}{Q^3 R^3} \right)^2 \quad (8.267)$$

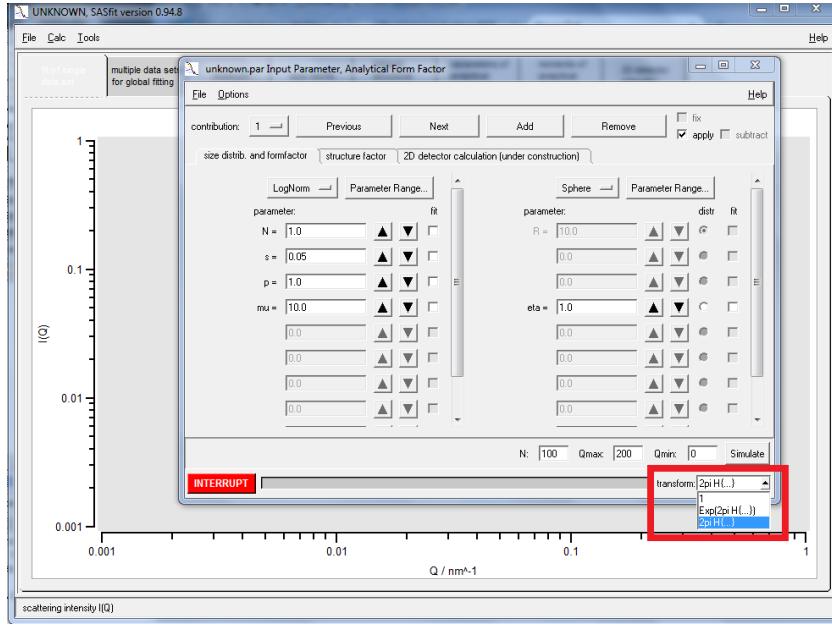


FIGURE 8.77. SASfit allows to perform a final transformation on the model function to convert a SANS signal into a SESANS signal

The unnormalized autocorrelation function for this sphere is given by

$$\tilde{\gamma}(r) = \begin{cases} \Delta\eta^2 \frac{4}{3}\pi R^3 \left(1 - \frac{3}{4}\frac{r}{R} + \frac{1}{16}\left(\frac{r}{R}\right)^3\right) & \text{for } r \leq 2R \\ 0 & \text{for } r > 2R \end{cases} \quad (8.268)$$

and the unnormalized SESANS correlation function

$$\tilde{G}_{\text{sphere}}(z) = \Delta\eta^2 \pi R^4 \left(\sqrt{1 - \xi^2} (2 + \xi^2) - \xi^2 (\xi^2 - 4) \ln \left(\frac{\xi}{1 + \sqrt{1 - \xi^2}} \right) \right) \quad (8.269)$$

with $\xi = \frac{z}{2R}$.

8.13. $G(z)$ of a randomly distributed, two-phase system (DAB)

The scattering intensity for a randomly distributed, two-phase system (DAB) is given by [45, 44, 4]

$$I(Q) = \frac{(8\pi xi^3 \Delta\eta)^2}{(1 + Q^2 \xi^2)^2} \quad (8.270)$$

The unnormalized autocorrelation function for this sphere is given by

$$\tilde{\gamma}(r) = \Delta\eta^2 8\pi \xi^3 \exp(-r/\xi) \quad (8.271)$$

and the unnormalized SESANS correlation function

$$\tilde{G}_{\text{DAB}}(z) = 8\pi \xi^4 2 \frac{z}{\xi} K_1 \left(\frac{z}{\xi} \right) \quad (8.272)$$

8.14. $G(z)$ of a self-affine random density distribution (gDAB)

The scattering intensity for a randomly distributed, two-phase system (DAB) is given by [113, 96, 4]

$$I(Q) = \frac{(4\pi\xi^3(1+2H)\Delta\eta)^2}{(1+Q^2\xi^2)^{3/2+H}} \quad (8.273)$$

The unnormalized autocorrelation function for this sphere is given by

$$\tilde{\gamma}(r) = \Delta\eta^2 8a^3 \pi \sqrt{\pi} \frac{\Gamma(\frac{3}{2}+H)}{\Gamma(H)} \frac{2}{\Gamma(H)} \left(\frac{r}{2a}\right)^H K_H\left(\frac{r}{a}\right) \quad (8.274)$$

and the unnormalized SESANS correlation function

$$\tilde{G}_{\text{gDAB}}(z) = 8a^3 \pi \frac{\Gamma(\frac{3}{2}+H)}{\Gamma^2(H)} 2\sqrt{az} \left(\frac{z}{2a}\right)^{H+\frac{1}{2}} K_{\frac{1}{2}+H}\left(\frac{z}{a}\right) \quad (8.275)$$

8.15. $G(z)$ for a generalized Gaussian coil (gGc)

The scattering intensity for a a generalized Gaussian coil (gGc) has been given by Hammouda [80, 78, 76, 79] (see also section 3.2.1.5) as

$$I_{\text{gGc}}(q) = I_0 \left(\frac{1}{\nu U^{\frac{1}{2\nu}}} \gamma\left(\frac{1}{2\nu}, U\right) - \frac{1}{\nu U^{\frac{1}{\nu}}} \gamma\left(\frac{1}{\nu}, U\right) \right) \quad (8.276)$$

with the modified variable

$$U = (2\nu+1)(2\nu+2) \frac{q^2 R_g^2}{6} \quad (8.277)$$

and the lower incomplete Gamma Function $\gamma(a, x) = \int_0^x dt t^{a-1} \exp(-t)$. ν is the excluded volume parameter from the Flory mean field theory and typical values for them are

- $\nu = 1/3$: partially precipitate in poor solvents
- $\nu = 1/2$: thermally relaxed in "theta"-solvents
- $\nu = 3/5$: swollen in good solvents

To be able to perform the Hankel transform we start from eq. 3.28c which leads to eq. 8.276

$$P(q) = 2 \int_0^1 dx (1-x) e^{-q^2 R_g^2 (2\nu+1)(2\nu+2)x^{2\nu}} \quad (8.278)$$

The unnormalized autocorrelation function the order of integration over x and over q for the Hankel transform can be changed and one gets

$$\tilde{G}_{\text{gGc}}(z) = \frac{I_0 4\pi}{\nu z^2} \left(w^{\frac{1}{2\nu}} \Gamma\left(1 - \frac{1}{2\nu}, w\right) - w^{\frac{1}{\nu}} \Gamma\left(1 - \frac{1}{\nu}, w\right) \right) \quad (8.279)$$

$$w = \frac{3}{4} \frac{z^2}{(2\nu^2 + 3\nu + 1) R_g^2} \quad (8.280)$$

where $\Gamma(a, x) = \int_x^\infty dt t^{a-1} \exp(-t)$ is the upper incomplete Gamma Function. The limit $\tilde{G}_{\text{gGc}}(0)$ is only finite for $\nu \in (0, \frac{1}{2})$

$$\tilde{G}_{\text{gGc}}(0) = I_0 \frac{3\pi}{(4\nu^4 - 5\nu^2 + 1) R_g^2} \text{ for } \nu \in \left(0, \frac{1}{2}\right) \quad (8.281)$$

CHAPTER 9

Plugin functions for size distributions

9.1. LogNorm_fp

The `LogNorm` distribution is a continuous distribution in which the logarithm of a variable has a normal distribution.

$$\text{LogNorm}(x, N, \sigma, p, \mu) = \frac{N}{c_{\text{LN}}} \frac{1}{x^p} \exp\left(-\frac{\ln(x/\mu)^2}{2\sigma^2}\right) \quad (9.1a)$$

$$c_{\text{LN}} = \sqrt{2\pi} \sigma \mu^{1-p} \exp\left((1-p)^2 \frac{\sigma^2}{2}\right) \quad (9.1b)$$

where σ is the width parameter, p a shape parameter, μ is the location parameter. c_{LN} is chosen so that $\int_0^\infty \text{LogNorm}(x, \mu, \sigma, p) dR = N$. The mode of the distribution is defined as

$$x_{\text{mode}} = \mu e^{-p\sigma^2} \quad (9.2)$$

and the n^{th} moment $\langle X^n \rangle$ of the `LogNorm` distribution as

$$\langle X^n \rangle = \frac{\int X^n \text{LogNorm}(X) dX}{\int \text{LogNorm}(X) dX} = \mu^n e^{\frac{1}{2}\sigma^2 n(2-2p+n)}. \quad (9.3)$$

Instead of using the parameter N (particle number density) another Log-Normal size distribution namely `LogNorm_fp` with the volume fraction f_p as a parameter has been implemented. Using the volume fraction as a scaling parameter requires that the intensity is given in units of cm^{-1} and the scattering vector in nm^{-1} . Furthermore the scattering contrast needs to be supplied in units of cm^{-2} . More details about absolute intensity can be found in chapter 11. The volume fraction f_p can be obtained from the `LogNorm`-distribution (eq. 9.1b) by integrating over the particle volume V_P . In case of spheres we get

$$f_p = 10^{21} \int_0^\infty \text{LogNorm}(R, N, \sigma, p, \mu) V_P(R) dR \quad (9.4)$$

$$= 10^{21} \int_0^\infty \text{LogNorm}(R, N, \sigma, p, \mu) \frac{4}{3} \pi R^3 dR = 10^{21} N \frac{4}{3} \pi \langle X^3 \rangle. \quad (9.5)$$

The scaling factor 10^{21} depends on the actual units. More details are given in section 11.4.

For other shapes than spheres the corresponding volume of the object has to be used in eq. 9.4. In case of cylinders the volume is given by $V_{\text{cyl}} = \pi R^2 L$. Depending whether

the radius R or the cylinder length L has a size distribution the volume fraction f_p is calculated differently namely in case for a radius distribution by

$$f_p = 10^{21} \int_0^{\infty} \text{LogNorm}(R) V_{\text{cyl}}(R, L) dR \quad (9.6)$$

$$= 10^{21} \int_0^{\infty} \text{LogNorm}(R) \pi R^2 L dR = 10^{21} N \pi L \langle X^2 \rangle \quad (9.7)$$

and in case of a length distribution by

$$f_p = 10^{21} \int_0^{\infty} \text{LogNorm}(L) \pi R^2 L dL = 10^{21} N \pi R^2 \langle X \rangle. \quad (9.8)$$

As the cylinder volume depends on R^2 and L either the second or the first moment of the distribution function is involved in calculating the volume fraction depending which parameter has a distribution. For a spherical shell a sum of different moments has to be used as listed in table 1.

shape	form factor	distrib. param.	length2	length3	volume	V	$N(f_p)$
1	Sphere	R	not used	not used	whole sph.	$\frac{4}{3}\pi R^3$	$\frac{f_p}{10^{21}} \frac{3}{4\pi} \frac{1}{\langle X^3 \rangle}$
2	Cylinder	R	L	not used	whole cyl.	$\pi R^2 L$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{\langle X^2 \rangle L}$
3	Cylinder	L	R	not used	whole cyl.	$\pi R^2 L$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{\langle X^2 \rangle L}$
4	Sph. Sh. iii	R	ΔR	not used	core+shell	$4\pi (R^2 \Delta R + R \Delta R^2 + \frac{1}{3} \Delta R^3 + \frac{1}{3} R^3)$	$\frac{f_p}{10^{21}} \frac{1}{4\pi} \frac{1}{3} \langle X^3 \rangle + \langle X^2 \rangle \Delta R + \langle X^1 \rangle \Delta R^2 + \langle X^0 \rangle \frac{\Delta R^3}{3}$
5	Sph. Sh. iiii	ΔR	R	not used	core+shell	$4\pi (R^2 \Delta R + R \Delta R^2 + \frac{1}{3} \Delta R^3 + \frac{1}{3} R^3)$	$\frac{f_p}{10^{21}} \frac{1}{4\pi} \frac{1}{3} \langle X^3 \rangle + \langle X^2 \rangle \Delta R + \langle X^1 \rangle \Delta R^2 + \langle X^0 \rangle \frac{\Delta R^3}{3}$
6	Sph. Sh. iii	R	ΔR	not used	core	$\frac{4}{3}\pi R^3$	$\frac{f_p}{10^{21}} \frac{1}{4\pi} \frac{3}{\pi} \langle X^3 \rangle$
7	Sph. Sh. iii	ΔR	R	not used	core	$\frac{4}{3}\pi R^3$	$\frac{f_p}{10^{21}} \frac{1}{4\pi} \frac{3}{\pi} \langle X^3 \rangle$
8	Sph. Sh. iii	R	ΔR	not used	shell	$4\pi (R^2 \Delta R + R \Delta R^2 + \frac{1}{3} \Delta R^3)$	$\frac{f_p}{10^{21}} \frac{1}{4\pi} \frac{1}{\langle X^2 \rangle \Delta R + \langle X^1 \rangle \Delta R^2 + \langle X^0 \rangle \frac{\Delta R^3}{3}}$
9	Sph. Sh. iii	ΔR	R	not used	shell	$4\pi (R^2 \Delta R + R \Delta R^2 + \frac{1}{3} \Delta R^3)$	$\frac{f_p}{10^{21}} \frac{1}{4\pi} \frac{1}{\langle X^2 \rangle \Delta R + \langle X^1 \rangle \Delta R^2 + \langle X^0 \rangle \frac{\Delta R^3}{3}}$
10	CylShell1	R	ΔR	L	core+shell	$\pi L (\Delta R^2 + 2R \Delta R + R^2)$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{L (\Delta R^2 \langle X^0 \rangle + 2R \langle X^1 \rangle \Delta R + \langle X^2 \rangle)}$
11	CylShell1	ΔR	R	L	core+shell	$\pi L (\Delta R^2 + 2R \Delta R + R^2)$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{L (\langle X^2 \rangle + 2R \langle X^1 \rangle + R^2 \langle X^0 \rangle)}$
12	CylShell1	L	R	ΔR	core+shell	$\pi L (\Delta R^2 + 2R \Delta R + R^2)$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{(X^1) (\Delta R^2 + 2R \Delta R + R^2)}$
13	CylShell1	R	ΔR	L	core	$\pi L R^2$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{\langle X^2 \rangle L}$
14	CylShell1	ΔR	R	L	core	$\pi L R^2$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{R^2 L \langle X^0 \rangle}$
15	CylShell1	L	R	ΔR	core	$\pi L R^2$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{R^2 L \langle X^0 \rangle}$
16	CylShell1	R	ΔR	L	shell	$\pi L (\Delta R^2 + 2R \Delta R)$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{L (\Delta R^2 \langle X^0 \rangle + 2R \langle X^1 \rangle \Delta R)}$
17	CylShell1	ΔR	R	L	shell	$\pi L (\Delta R^2 + 2R \Delta R)$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{L (\langle X^2 \rangle + 2R \langle X^1 \rangle)}$
18	CylShell1	L	R	ΔR	shell	$\pi L (\Delta R^2 + 2R \Delta R)$	$\frac{f_p}{10^{21}} \frac{1}{\pi} \frac{1}{(X^1) (\Delta R^2 + 2R \Delta R)}$

TABLE 1. The number density N expressed in terms of volume fraction f_p and moments $\langle X^n \rangle$ of the distribution function for some particle shapes and different parameters having a distribution. The factor 10^{21} is needed due to unit conversion. It is assumed that the radius is given in nm, the intensity in cm^{-1} and the scattering length densities in cm^{-2} .

CHAPTER 10

Plugin functions for structure factors

10.1. Structure factor of a random flight model

The random flight model describes a discrete chain, where the positions of the N units forming the discrete chains follow a 3D random walk. The distance between neighbouring units is constant. The structure factor $S_N(QD)$ describing such an ar-

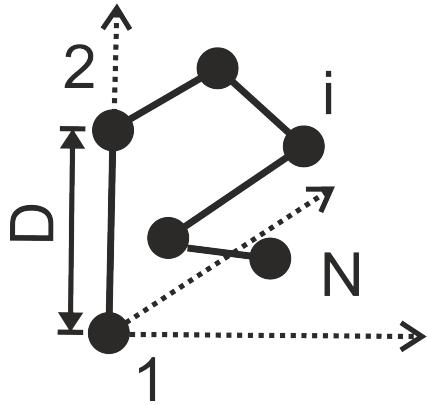


FIGURE 10.1. Random flight of N particles with constant distances D .

angement of N particles on a random flight with a constant step size D is given by [29] as

$$S_N(QD) = \frac{2}{1 - \frac{\sin(QD)}{QD}} - 1 - \frac{2 \left[1 - \left[\frac{\sin(QD)}{QD} \right]^N \right]}{N \left[1 - \frac{\sin(QD)}{QD} \right]^2} \frac{\sin(QD)}{QD} \quad (10.1)$$

The formula above is only real for all values of QD for integer values of N . For non integer values n a linear interpolation between $[n]$ and $[n] + 1$ is taken [66], where $[n]$ is the largest integer smaller or equal to n . With $w = n - [n]$ we get

$$S_n(QD) = (1 - w)S_{[n]}(QD) + wS_{[n]+1}(QD) \quad (10.2)$$

Input Parameters for model `random flight`:

D: step length D

n: number of steps n

Note:

- N needs to be larger or equal to 1. For non-integer values of N the curve is linear interpolated between the structure factor for N and $N + 1$.

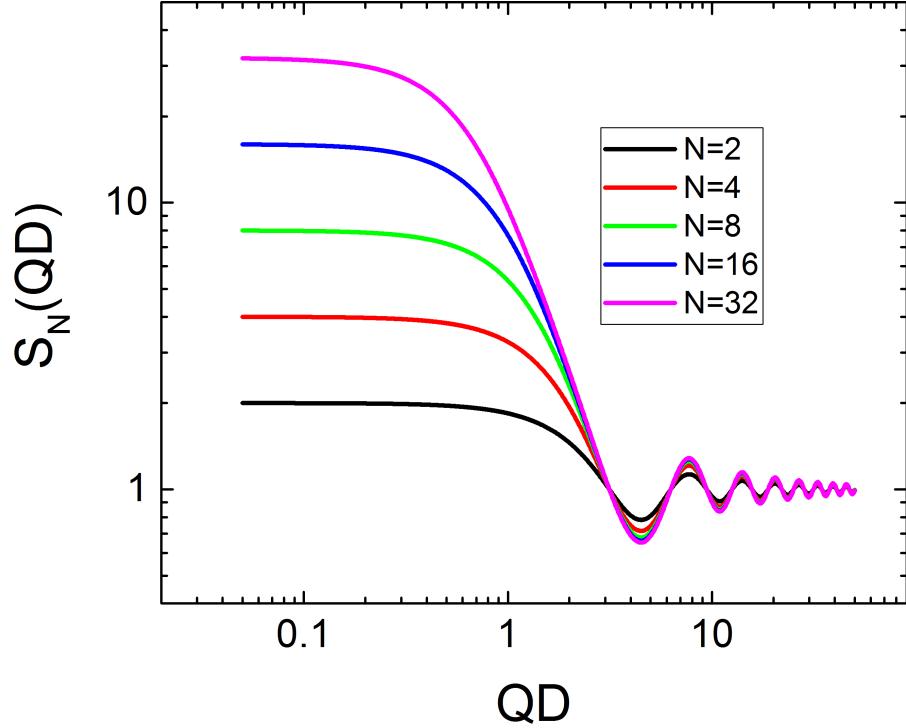


FIGURE 10.2. Random flight structure factor with N steps of length D .

10.2. ordered particle systems

This plugin contains the structure factor of ordered mesoscopic materials in case the domains are random orientated like in powder diffraction as described in [62]. For oriented domains the scattering pattern is not anymore radial symmetric and depends both on the direction and modulus of the scattering vector. For this case the structure factor are taken from [59]. This plugin tries to re-implement the functions which are originally supplied by the software package `scatter` described in [58]. The software package `scatter` is specialised on calculating and fitting ordered structures and has much more options and better GUI for this kind of studies. In both case the structure factor is approximated in the decoupling approach ([117]) as defined in 4.1.2 or in eq. 13.56 of section 13.2.4

In case of random oriented domains of three dimensional ordered particle systems the decoupling approach is implemented as

$$I(Q) = \overline{\langle F^2(Q) \rangle}_{or} + \langle F(Q) \rangle_{or}^2 (S(Q) - 1) \quad (10.3)$$

The overline symbol denotes the average over a particle size distribution and the brackets $\langle \rangle_{or}$ for the orientational average. In the above case it is assumed that the position of the scatterer is independent of their size and orientation. For small size distributions and only small deviations from spherical symmetry of the scatterer the decoupling approximation works quite well. For two and one dimensional ordered particle systems the particles can be very anisotropic, i.e. very long cylindrical in case of 2D ordering and thin planar objects in case of 1D ordering. For this very anisotropic shaped particles the

scattering amplitude and scattering intensity can be written as a product 8.2 in terms of a cross section term for the short dimension L_{short} and a shape factor for the long dimension L_{long} as well as there averages

$$F(Q, L_{\text{short}}, L_{\text{long}}) = F_{\text{cs}}(Q, L_{\text{short}})F'(Q, L_{\text{long}}) \quad (10.4a)$$

$$\langle \overline{F^2(Q, L_{\text{short}}, L_{\text{long}})} \rangle_{\text{or}} = \langle \overline{F_{\text{cs}}^2(Q, L_{\text{short}})} \rangle_{\text{or}} \langle \overline{F'^2(Q, L_{\text{long}})} \rangle_{\text{or}} \quad (10.4b)$$

$$\langle \overline{F(Q, L_{\text{short}}, L_{\text{long}})} \rangle_{\text{or}}^2 = \langle \overline{F_{\text{cs}}(Q, L_{\text{short}})} \rangle_{\text{or}}^2 \langle \overline{F'(Q, L_{\text{long}})} \rangle_{\text{or}}^2 \quad (10.4c)$$

In case of one (multi-lamellar structures) as well as of two dimensional (ordering of structures on a planar surface) ordered particles strongly anisotropic particles are often ordered along their short dimension, i.e. cylindrical pillar have their cylindrical axis often perpendicular to the ordering plane or in case of lamellar structures ordering happens always in the direction of the normal of the planar scattering objects. The averages, which needs to be performed in the decoupling approximation are

$$I(\mathbf{Q}) = \langle \langle \overline{F^2(\mathbf{Q})} \rangle_i \rangle_d + \langle \langle \overline{F(\mathbf{Q})} \rangle_i^2 (S(\mathbf{Q}) - 1) \rangle_d \quad (10.5a)$$

$$= \langle \overline{F^2(\mathbf{Q})} \rangle_i \rangle_d + \langle \langle \overline{F(\mathbf{Q})} \rangle_i^2 (Z(\mathbf{Q}) - 1) G(\mathbf{Q}) \rangle_d \quad (10.5b)$$

with

$$S(\mathbf{Q}) = (Z(\mathbf{Q}) - 1) G(\mathbf{Q}) + 1 \quad (10.6)$$

In the last equation the structure factor $S(\mathbf{Q})$ has been expressed in terms of the lattice factor function $Z(\mathbf{Q})$ and the Debye-Waller factor $G((Q))$. The term $Z(\mathbf{Q})G(\mathbf{Q})$ describes the decay of the Bragg peaks due to displacement and $1 - G(\mathbf{Q})$ the concomitant increase of diffuse scattering. For a perfect lattice $G((Q)) = 1$. The orientation averages is done here slightly different than in the paper from [59]. The orientational averaging of the scatterer within the domain are denoted by $\langle \dots \rangle_i$ and the orientation averaging of the whole domains by $\langle \dots \rangle_d$. In the appendix of [59] it is discussed under which conditions the orientation distribution over the structure factor and form factor can be factorized. If the orientation distribution of the scatterer in a domain is significant larger than the orientation distribution of the domains the averages can be factorized $\langle (\langle \dots \rangle_i)^2 \dots \rangle_d \simeq (\langle \dots \rangle_i)^2 \langle \dots \rangle_d$ and one obtains

$$I(\mathbf{Q}) = \langle \overline{F^2(\mathbf{Q})} \rangle_i + \langle \langle \overline{F(\mathbf{Q})} \rangle_i^2 (S(\mathbf{Q}) - 1) \rangle_d \quad (10.7a)$$

$$\simeq \langle \overline{F^2(\mathbf{Q})} \rangle_i + \langle \overline{F(\mathbf{Q})} \rangle_i^2 \langle (Z(\mathbf{Q}) - 1) G(\mathbf{Q}) \rangle_d \quad (10.7b)$$

The formula above is different to the one given in [59], where the orientational averaging is done after squaring the size average of the scattering amplitude $\langle \overline{F(\mathbf{Q})}^2 \rangle_{\text{or}}$ instead of doing the orientation averaging first $\langle \overline{F(\mathbf{Q})} \rangle_{\text{or}}^2$.

However, in case of a powder signal, where the orientation distribution within a domain is small and the domains are random oriented. Also the structures can be very anisotropic in two or one dimensional ordered structures and the periodicity is in most cases perpendicular to the long dimension of the scattering object.

10.2.1. Domains of ordered particle systems isotropically oriented.

For random oriented domains of ordered particle systems the structure factor can be written as

$$S(Q) = (Z_0(Q) - 1) G(Q) + 1 \quad (10.8)$$

$Z_0(Q)$ is the lattice factor got an ideal undistorted lattice and $G(Q)$ the Debye-Waller factor. The lattice factor expressed with Miller indices reads as

$$Z_0(Q) = \frac{(2\pi)^{d-1}}{nv_d\Omega_d Q^{d-1}} \sum_{\{hkl\}} m_{hkl} f_{hkl}^2 L_{hkl}(Q - Q_{hkl}) \quad (10.9)$$

where n is the number of particles per unit cell, f_{hkl} is the symmetry factor taking into account extinction rules, v_d is the volume ($d = 3$), surface ($d = 2$), or long-period ($d = 1$) of the d -dimensional unit cell, Ω_d is the d -dimensional solid angle, $L_{hkl}(Q - Q_{hkl})$ is a normalised peak-shape function, and m_{hkl} is the multiplicity. If the sum is done over all reflections $\{hkl\}$, i.e. $\sum_{\{hkl\}} = \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty}$, one automatically accounts for multiplicity but one the costs for summing over all combinations of $\{hkl\}$. For the normalized peak shape function $L_{hkl}(x)$ the user can choose between Lorentzian, Gaussian, and Pearson VII peak shape

$$L_{hkl}(x) = \begin{cases} \frac{2}{\pi\delta} \exp\left(-4\frac{x^2}{\pi\delta}\right) & \text{for Gaussian} \\ \frac{\delta}{2\pi} \frac{1}{x^2 + (\frac{\delta}{2})^2} & \text{for Lorentzian} \\ \frac{\left(1+B^2(\nu-\frac{1}{2}, \frac{1}{2})\left(\frac{x}{\delta}\right)^2\right)^{-\nu}}{\delta} & \text{for Pearson VII} \end{cases} \quad (10.10)$$

To describe the diffraction pattern one has to define the unit cell of the ordered structure and the position of the particles with in the unit cell. The unit cell can be specified totally by six scalar quantities, which are called the unit cell dimensions or lattice parameters. These are (see also Fig. 10.3):

$$a, b, c, \alpha, \beta, \gamma$$

The first three parameters (a, b and c) represent the lengths of the unit cell edges, and the last three (α, β and γ) represent the angles between them. By convention, α is the angle between b and c , β is the angle between a and c , and γ is the angle between a and b .

If we assume that the vectors \mathbf{a} and \mathbf{b} are in the x - y plane and $\mathbf{a} \parallel \mathbf{e}_x$ we can write the vector of the unit cell as

$$\mathbf{a} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{b} = b \begin{pmatrix} \cos \gamma \\ \sin \gamma \\ 0 \end{pmatrix} \quad \mathbf{c} = c \begin{pmatrix} \frac{\cos \beta}{\frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \gamma}} \\ \sqrt{1 - \cos^2 \beta - \left(\frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \gamma}\right)^2} \\ \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \gamma} \end{pmatrix} \quad (10.11)$$

Next to the direct lattice with \mathbf{a} , \mathbf{b} , and \mathbf{c} be the elementary translations in a three-dimensional lattice a second lattice, reciprocal to the direct lattice, is defined by three elementary translations \mathbf{a}^* , \mathbf{b}^* and \mathbf{c}^*

$$\mathbf{a}^* = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} \quad \mathbf{b}^* = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} \quad \mathbf{c}^* = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} \quad (10.12)$$

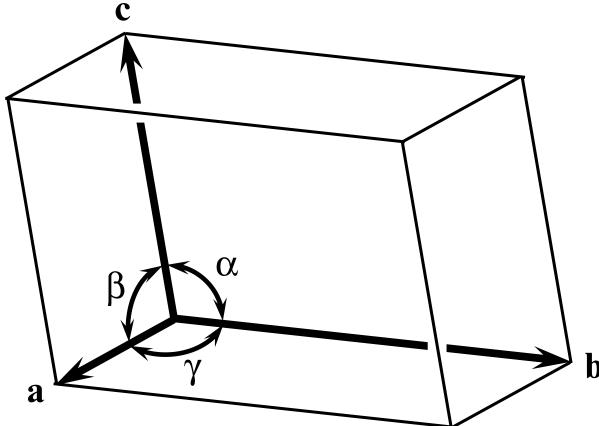


FIGURE 10.3. Unit cell in three dimensions.

For the scalar product between the direct and reciprocal lattice the following conditions holds:

$$\mathbf{a} \cdot \mathbf{a}^* = 1 \quad \mathbf{a} \cdot \mathbf{b}^* = 0 \quad \mathbf{a} \cdot \mathbf{c}^* = 0 \quad (10.13a)$$

$$\mathbf{b} \cdot \mathbf{a}^* = 0 \quad \mathbf{b} \cdot \mathbf{b}^* = 1 \quad \mathbf{b} \cdot \mathbf{c}^* = 0 \quad (10.13b)$$

$$\mathbf{c} \cdot \mathbf{a}^* = 0 \quad \mathbf{c} \cdot \mathbf{b}^* = 0 \quad \mathbf{c} \cdot \mathbf{c}^* = 1 \quad (10.13c)$$

For a two dimensional periodic lattice with the direct lattice vectors \mathbf{a}, \mathbf{b} and reciprocal lattice vector $\mathbf{a}^*, \mathbf{b}^*$ the orthogonal relations above also hold and by using eq. 10.12 with $\mathbf{c} = (0, 0, 1)^T$ ($c = 1, \alpha = \beta = \pi/2$) one can calculate them in the same way.

$$\mathbf{a} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{b} = b \begin{pmatrix} \cos \gamma \\ \sin \gamma \\ 0 \end{pmatrix} \quad (10.14a)$$

$$\mathbf{a}^* = \frac{1}{a} \begin{pmatrix} 1 \\ -\frac{\cos \gamma}{\sin \gamma} \\ 0 \end{pmatrix} \quad \mathbf{b}^* = \frac{1}{b} \begin{pmatrix} 0 \\ \frac{1}{\sin \gamma} \\ 0 \end{pmatrix} \quad (10.14b)$$

Last but not least, for a one dimensional lattice $\mathbf{a} = (a, 0, 0)^T$ the reciprocal lattice vector is simply $\mathbf{a}^* = \frac{1}{a} (1, 0, 0)^T$.

Diffraction peaks can occur at integer multiples, called Miller indices (hkl) , of the reciprocal lattice

$$\mathbf{Q}_{hkl}^{3D} = 2\pi (h\mathbf{a}^* + k\mathbf{b}^* + l\mathbf{c}^*) \quad (10.15a)$$

$$\mathbf{Q}_{hk}^{2D} = 2\pi (h\mathbf{a}^* + k\mathbf{b}^*) \quad (10.15b)$$

$$\mathbf{Q}_h^{1D} = 2\pi h\mathbf{a}^* \quad (10.15c)$$

Due to the symmetry of the unit cell certain (hkl) reflections might be forbidden. This is described by the structure factor of the unit cell f_{hkl} . The structure factor depends next to the Miller indices also from the type and position of the scattering objects within the unit cell. The position $\mathbf{R}_i(u, v, w)$ of the i^{th} scatterer with the scattering amplitude

$F_i(\mathbf{Q})$ in the unit cell is normally given in terms of the direct lattice vectors \mathbf{a} , \mathbf{b} , and \mathbf{c}

$$\mathbf{R}_i(u_i, v_i, w_i) = u_i \mathbf{a} + v_i \mathbf{b} + w_i \mathbf{c} \quad (10.16)$$

The scattering amplitude of the unit cell f_{hkl} is than given by

$$f_{hkl}(\mathbf{Q}_{hkl}) = \sum_{i=1}^N F_i(\mathbf{Q}_{hkl}) \exp(-i\mathbf{Q}_{hkl}\mathbf{R}_i) \quad (10.17a)$$

$$= \sum_{i=1}^N F_i(\mathbf{Q}_{hkl}) \exp(-i2\pi(hu_i + kv_i + lw_i)) \quad (10.17b)$$

$$= \sum_{i=1}^N F_i(\mathbf{Q}_{hkl}) [\cos(2\pi(hu_i + kv_i + lw_i)) - i \sin(2\pi(hu_i + kv_i + lw_i))] \quad (10.17c)$$

$$= \sqrt{A^2 + B^2} \exp(-i \arctan(A/B)) \quad (10.17d)$$

with

$$A = \sum_{i=1}^N F_i(\mathbf{Q}_{hkl}) \cos(2\pi(hu_i + kv_i + lw_i)) \quad (10.18a)$$

$$B = \sum_{i=1}^N F_i(\mathbf{Q}_{hkl}) \sin(2\pi(hu_i + kv_i + lw_i)) \quad (10.18b)$$

For a 2D and 1D lattice the scattering amplitude of the unit cell f_{hk} and f_h are calculated accordingly.

10.2.2. Domains of ordered particle systems with preferred orientation.

$$\mathbf{k}_i = \begin{pmatrix} k_{i,x} \\ k_{i,y} \\ k_{i,z} \end{pmatrix} = \frac{2\pi}{\lambda} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad (10.19)$$

$$\mathbf{k}_s = \begin{pmatrix} k_{i,x} \\ k_{i,y} \\ k_{i,z} \end{pmatrix} = \frac{2\pi}{\lambda} \begin{pmatrix} \cos(\psi) \sin(\theta) \\ \sin(\psi) \sin(\theta) \\ -\cos(\theta) \end{pmatrix} \quad (10.20)$$

$$\mathbf{Q} = \mathbf{k}_s - \mathbf{k}_i = \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \frac{2\pi}{\lambda} \begin{pmatrix} \cos(\psi) \sin(\theta) \\ \sin(\psi) \sin(\theta) \\ 1 - \cos(\theta) \end{pmatrix} \quad (10.21)$$

lattice	LAM	SQ (P4/mm)	HEX (P6/mm)	CREC (cmm)
n	1	1	1	2
v_d	a	a^2	$\sqrt{3}a^2/2$	ab
d	1	2	2	2
f_{hkl}	$f_h = 1$	$f_{hk} = 1$	$f_{hk} = 1$	$f_{hk} = 1$
m_{hkl}				
Ω_d	1	2π	2π	
\bar{a}	a	a	a	$\min \{a, b, \frac{1}{2}\sqrt{a^2 + b^2}\}$
Q_{hkl}	$\frac{2\pi h}{a}$	$\frac{2\pi\sqrt{h^2+k^2}}{a}$	$\frac{4\pi\sqrt{h^2+hk+k^2}}{\sqrt{3}a}$	$2\pi\sqrt{\frac{h^2}{a^2} + \frac{k^2}{b^2}}$

lattice	BCT (I4/mmm)	FCC (Fm3m)	BCC (Im3m)	HCP (P6/mmc)	SC (Pm3m)
n	2	4	2	2	1
$\mathbf{R}_i = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
v_d	$a^2 c$	a^3	a^3	$\sqrt{2}a^3$	a^3
d	3	3	3	3	3
$ f_{hkl} $	$ 1+\cos(\pi(h+k+l)) $	$\left \begin{array}{l} 1+\cos(\pi(h+k)) \\ +\cos(\pi(h+l)) \\ +\cos(\pi(k+l)) \end{array} \right $	$ 1+\cos(\pi(h+k+l)) $	$ 2\cos(\pi(\frac{h+2k}{3} + \frac{l}{2})) $	$f_{hkl}=1$
m_{hkl}					
Ω_d	4π	4π	4π	4π	4π
\bar{a}	$\sqrt{2}a/2$	$\sqrt{2}a/2$	$\sqrt{3}a/2$	a	a
Q_{hkl}	$\frac{2\pi\sqrt{h^2+k^2+l^2}}{a^2 + c^2}$	$\frac{2\pi\sqrt{h^2+k^2+l^2}}{a}$	$\frac{2\pi\sqrt{h^2+k^2+l^2}}{a}$	$\frac{2\pi\sqrt{\frac{4}{3}(h^2+hk+k^2) + \frac{3}{8}l^2}}{a}$	$\frac{2\pi\sqrt{h^2+k^2+l^2}}{a}$

TABLE 1.

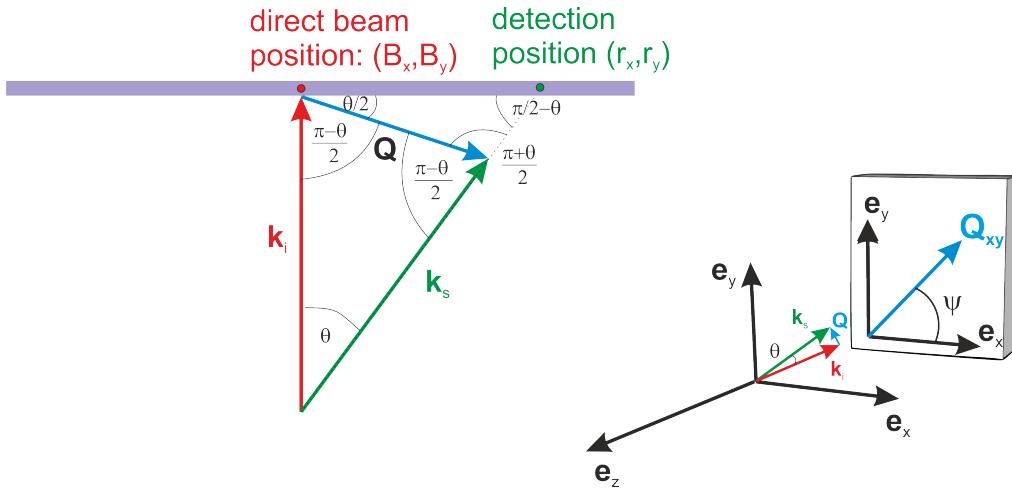


FIGURE 10.4. The scattering vector in polar coordinates coordination system with respect to a laboratory-fixed coordinate system based on the three orthogonal unit vectors (\mathbf{e}_x , \mathbf{e}_y , \mathbf{e}_z). They are arranged such that the x-direction coincides with the x-direction of the detector, and the y-direction coincides with the y-direction of the detector. The direction of the incoming neutron beam is chosen to be $-\mathbf{e}_z$.

CHAPTER 11

Absolute intensities, moments and volume fractions

11.1. Fitting absolute intensities

Absolute intensities in the simulation can be obtained by using proper units for the scattering vector \mathbf{Q} , the size dimensions of the scatterer, the scattering length densities etc. In the following a few example are discussed for absolute calibrated data sets. One question which is asked quite frequently is "What is the meaning of N in the size distribution and what are its units?". The answer is normally "That depends on the units of your data you are fitting and the units of your scattering length densities". In the following a few explanations will be given to clarify this in some more detail.

Let us consider first the scattering intensity of a single sphere. The form factor of a sphere is given by eq. 3.1a as

$$I_{\text{Sphere}}(Q, R, \Delta\eta) = \left[\frac{4}{3} \pi R^3 \Delta\eta 3 \frac{\sin QR - QR \cos QR}{(QR)^3} \right]^2 \quad (11.1)$$

The radius R and the scattering vector Q have reciprocal units, i.e. if Q is given in $1/\text{nm}$ the radius R has a unit of nm. The other variable in the form factor is the scattering length density contrast $\Delta\eta$ between sphere and surrounding matrix or solvent. The unit of the scattering length density is length/volume and has therefore a unit $1/\text{cm}^2$ or some other sites are using units of $1/\text{\AA}^2$. The difference is only a constant factor of

$$\Delta\eta \frac{1}{\text{cm}^2} = 10^{16} \Delta\eta \frac{1}{\text{\AA}^2}. \quad (11.2)$$

The overall unit of the scattering intensity (differential cross-section) of a single sphere is therefore

$$[I_{\text{Sphere}}(Q, R, \Delta\eta)] = [R]^6 [\Delta\eta]^2 = \frac{\text{nm}^6}{\text{cm}^4} = 10^{-42} \text{cm}^2 \quad (11.3)$$

for the case that $[R] = \text{nm}$ and $[\Delta\eta] = \text{cm}^{-2}$. The unit for the scattering cross-section of a single sphere with $[R] = \text{\AA}$ and $[\Delta\eta] = \text{\AA}^{-2}$ is than

$$[I_{\text{Sphere}}(Q, R, \Delta\eta)] = [R]^6 [\Delta\eta]^2 = \frac{\text{\AA}^6}{\text{\AA}^4} = \text{\AA}^2 = 10^{-16} \text{cm}^2, \quad (11.4)$$

respectively. The scattering cross-section of a single scatterer is calculated by **SASfit** if one chooses in the tab for distribution functions the probability functions **Monodisperse**.

Differential cross-section have a unit of an area

$$\left[\frac{d\Sigma}{d\Omega}(Q) \right] = \text{cm}^2. \quad (11.5)$$

Many instruments deliver with their data reduction software a cross-section normalized by the sample volume so that the unit is in reciprocal length:

$$\left[\frac{d\sigma}{d\Omega}(Q) \right] = \frac{1}{[V]} \left[\frac{d\Sigma}{d\Omega}(Q) \right] = \frac{1}{\text{cm}}. \quad (11.6)$$

For fitting a form factor to experimental data one needs next to the size parameter also a scaling parameter. For the simplest case this is done by choosing as a distribution function **Delta**. **Delta** simply multiplies a constant value N to the form factor. The meaning and the unit of N now depends on the unit of the cross-section, whether it is normalized or not normalized on the sample volume. **SASfit** calculates in the case of a form factor of **Sphere** with **Delta** as a distribution function

$$I_{\text{SASfit}} = N \times I_{\text{Sphere}}(Q, R, \Delta\eta). \quad (11.7)$$

Fitting N to a data set, which is given in units of $1/\text{cm}$ and where $[Q] = \text{nm}^{-1}$, $[R] = \text{nm}$ and $[\Delta\eta] = \text{cm}^{-2}$ would mean that N has the unit

$$[N] = \frac{\left[\frac{d\sigma}{d\Omega}(Q) \right]}{[I_{\text{Sphere}}(Q, R, \Delta\eta)(Q)]} = \frac{\frac{1}{\text{cm}}}{10^{-42}\text{cm}^2} = 10^{42}\text{cm}^{-3}. \quad (11.8)$$

One therefore needs to multiply the value N obtained by **SASfit** with 10^{42} to get the number density of scatterers in units of cm^{-3} .

Let us now consider the simplest case of spheres with a size distribution and no structure factor, which are fitted to experimental data. All the size distribution have a scaling parameter N . The units of the parameter N in the size distribution is the same than for **Delta**. The size distribution $n(x)$ are implemented as distribution function $n(x) = Np(x)$ with $p(x)$ being a probability function. In case of polydisperse spheres **SASfit** calculates the integral

$$I_{\text{SASfit}}(Q) = \int_0^\infty n(R) I_{\text{Sphere}}(Q, R, \Delta\eta) dR \quad (11.9)$$

$$= N \int_0^\infty p(R) I_{\text{Sphere}}(Q, R, \Delta\eta) dR \quad (11.10)$$

The probability function $p(x)$ is normalized to

$$\int_0^\infty p(x) dx = 1, \quad (11.11)$$

so that the parameter N has like for the **Delta**-distribution the unit $[N] = 10^{42}\text{cm}^{-3}$ if the data set is given in units of $1/\text{cm}$ and $[Q] = \text{nm}^{-1}$, $[R] = \text{nm}$ and $[\Delta\eta] = \text{cm}^{-2}$.

Most of the form factor are implemented in a way that they return the scattering cross-section of a single object like the example of a sphere above, but a few are not, like for example the standard form of a gaussian chain **Gauss**. In this particular case two other versions **Gauss2** and **Gauss3** with different parameterizations of the forward scattering of a single gaussian chain are available. However, there are some form factors, which have been implemented according to the literature but which are normalized

differently. This has to be checked before the parameter N in the size distribution is interpreted in terms of number density of scatterers.

11.2. Contrast - Concentration - Forward Scattering - Particle Volume - Absolute Scale

A frequently asked question is if the scattering intensity is consistent with the concentration of material in the sample. Especially people working with micellar solution, star polymers, but also proteins, etc. want to cross-check the absolute intensity with the known concentration. In the dilute case the differential cross-section is simply N times the cross-section of an individual scatterer

$$\frac{d\sigma}{d\Omega}(Q) = \frac{N}{V_{\text{tot}}} P(Q) \quad (11.12)$$

N is number of particles/molecules/proteins in the illuminated volume, V_{tot} the illuminated sample volume, $n = N/V_{\text{tot}}$ the particle number density, and $P(Q)$ the scattering cross-section of a single particle. $P(Q)$ has the dimension cm^2 , N/V_{tot} the dimension cm^{-3} , and $\frac{d\sigma}{d\Omega}(Q)$ the dimension cm^{-1} . Eq. 11.12 can also be expressed in terms of concentration c in units of g/cm^3

$$c = n m_{\text{mol}} = n M_r u = n \frac{M_r M_u}{N_A} = \frac{N}{V_{\text{tot}}} \frac{M_r M_u}{N_A} \quad (11.13)$$

so that

$$\frac{d\sigma}{d\Omega}(Q) = c \frac{N_A}{M_r M_u} P(Q) \quad (11.14)$$

M_r is the relative molar mass of the particle¹ (Molecular weight (M.W.) and formula weight (F.W.) are older terms) which is a dimensionless quantity (i.e., a pure number, without units). To get units in g/mol the relative molar mass needs to be multiplied by the molar mass constant M_u . The value of the molar mass constant M_u is defined to be 1 g/mol in SI units. The molar mass constant is important in writing dimensionally correct equations. It is common to see phrases such as "*The molar mass of an element is the atomic weight in grams per mole.*" However molecular or atomic weight are dimensionless quantities, and cannot take the units of grams per mole. Formally, the operation is the multiplication by a constant which has the value 1 g/mol , that is the molar mass constant². The molecule mass m_m in units of g is $m_m = M_r M_u / N_A$.

¹Definition of relative atomic mass and relative molecular mass can be found on the url-address <http://physics.nist.gov/Pubs/SP811/sec08.html>

Relative atomic mass (formerly atomic weight): ratio of the average mass per atom of an element to 1/12 of the mass of the atom of the nuclide ^{12}C .

Relative molecular mass (formerly molecular weight): ratio of the average mass per molecule or specified entity of a substance to 1/12 of the mass of an atom of the nuclide ^{12}C .

²Definition of unified atomic mass unit: $1\text{u} = m_u = m(^{12}\text{C})/12 = 1M_u/N_A = 1(\text{g/mol})/(6.02214129 \times 10^{23}\text{mol}^{-1}) = 1.660538921 \times 10^{-24}\text{g}$

Now we need to look on the forward scattering $P(Q=0)$ of a single particle/protein/polymer chain. For a simple particle like a sphere, the forward scattering is given by

$$P(Q=0) = (\eta_{\text{sol}} - \eta_{\text{sp}})^2 V_{\text{sp}}^2 \quad (11.15)$$

where $(\eta_{\text{sol}} - \eta_{\text{sp}})$ is the scattering contrast between spherical particle and solvent and V_{sp} the volume of a single sphere. In case of a spherical particle the boundary between particle and solvent is well defined and therefore also the volume of the particle as it has a sharp interface. The scattering contrast of a spherical particle can also be written in terms of the overall scattering length of the sphere b_{sp} , i.e. the sum of the scattering length of all atoms forming the sphere, the volume of the sphere and the scattering length density of the solvent. The volume of the sphere can be calculated from its mass m_{sp} or relative molar mass $M_{r,\text{sp}}$ and its density ρ_{sp} .

$$(\eta_{\text{sol}} - \eta_{\text{sp}}) = \left(\eta_{\text{sol}} - \frac{b_{\text{sp}}}{V_{\text{sp}}} \right) = \left(\eta_{\text{sol}} - \frac{b_{\text{sp}}\rho_{\text{sp}}}{m_{\text{sp}}} \right) = \left(\eta_{\text{sol}} - \frac{b_{\text{sp}}\rho_{\text{sp}}N_A}{M_{r,\text{sp}}M_u} \right) \quad (11.16)$$

But what about the forward scattering of a gaussian polymer coil? A polymer does not have a sharp boundary to the solvent. Polymer and solvent can penetrate each other. To determine the polymer volume one would need a detailed model for the polymer molecule and its interaction with solvent molecules. As a first approximation the volume of a polymer molecule can be obtained by $V_{\text{polym}} = \frac{\rho_{\text{polym}}}{m_{\text{polym}}}$. For a polymer coil with a relative molar mass $M_{r,\text{polym}}$ the forward scattering in a solvent is given by

$$P(Q=0) = \left(\frac{M_{r,\text{polym}}M_u}{\rho_{\text{polym}}N_A} \right)^2 \left(\eta_{\text{sol}} - \frac{b_{\text{polym}}\rho_{\text{polym}}N_A}{M_{r,\text{polym}}M_u} \right)^2 \quad (11.17)$$

The volume of a polymer molecule

$$V_{\text{polym}} = \frac{M_{r,\text{polym}}M_u}{\rho_{\text{polym}}N_A}$$

is the volume occupied by single polymer chain in the solvent or in other word the amount of solvent volume displaced by one polymer chain. For the forward scattering it does not matter, if the coil is collapsed or swollen. As long as the scattering length density of the solvent inside the swollen polymer coil is the same than in the bulk and the molecular volume of the polymer chain does not change with the solvent quality the forward scattering does not depend on the conformation of the polymer. Sometimes the relative molar mass of a polymer is given in degree of polymerization p and relative molar mass of the monomer $M_{r,\text{m}}$. In this case and assuming that the mass densities of polymer and monomer are the same the forward scattering is given by

$$\begin{aligned} P(Q=0) &= \left(\frac{p M_{r,\text{m}}M_u}{\rho_{\text{m}}N_A} \right)^2 \left(\eta_{\text{sol}} - \frac{b_{\text{m}}\rho_{\text{m}}N_A}{M_{r,\text{m}}M_u} \right)^2 \\ &= \left(\frac{p M_{r,\text{m}}M_u}{\rho_{\text{m}}N_A} \right)^2 \left(\eta_{\text{sol}} - \frac{b_{\text{m}}\rho_{\text{m}}N_A}{M_{r,\text{m}}M_u} \right)^2 \end{aligned} \quad (11.18)$$

Let us now come back to the measured differential cross-section $\frac{d\sigma}{d\Omega}(Q)$. For $Q = 0$ we get

$$\frac{d\sigma}{d\Omega}(Q=0) = \frac{N}{V_{\text{tot}}} P(Q=0) = c \frac{N_A}{M_{r,\text{Gauss}} M_u} P(Q=0) \quad (11.19)$$

$$= c \frac{N_A}{p M_{r,\text{m}} M_u} \left(\frac{p M_{r,\text{m}} M_u}{\rho_{\text{m}} N_A} \right)^2 \left(\eta_{\text{sol}} - \frac{b_{\text{m}} \rho_{\text{m}} N_A}{M_{r,\text{m}} M_u} \right)^2 \quad (11.20)$$

$$= c \frac{p M_{r,\text{m}} M_u}{N_A \rho_{\text{m}}^2} \left(\eta_{\text{sol}} - \frac{b_{\text{m}} \rho_{\text{m}} N_A}{M_{r,\text{m}} M_u} \right)^2 \quad (11.21)$$

The last equation says, that the forward scattering of a solution of dilute non-interacting polymer molecules consisting of p monomer units depends linearly on the number of monomer units p in the polymer. Even though the scattering of a single polymer molecule depends quadratically on the number of monomer units. The reason is simply that the larger the degree of polymerization the lower the number density of molecules in the solution as we assume a constant concentration, i.e. $c \propto p n$ and therefore $\frac{d\sigma}{d\Omega}(Q=0) \propto p^2 n$.

11.3. Moments of scattering curves and size distribution

The relevance of moments both for scattering curves as well as size distributions has been discussed in several publications [39, 195, 38, 224, 150, 73].

Moments $\langle x^m \rangle$ of any order m of a function $f(x)$ are defined by integrating $f(x)$ with a suitable power of x over its domain $[a, b]$

$$\langle x^m \rangle = \int_a^b f(x) x^m dx \quad (11.22)$$

The different moments of the scattering curve together with the forward scattering $I(0)$ and the Porod constant can be used to calculate easily several structural parameters

of the scatterers.

$$\tilde{Q}_{\text{inv}} = \int_0^{\infty} Q^2 I(Q) dQ \text{ (scattering invariant)} \quad (11.23a)$$

$$\frac{S}{V} = \frac{\pi}{\tilde{Q}_{\text{inv}}} \lim_{Q \rightarrow \infty} \{Q^4 I(Q)\} \text{ (specific surface)} \quad (11.23b)$$

$$\langle R_G \rangle^2 = 3 \left(- \lim_{Q \rightarrow 0} \left\{ \frac{d[\ln I(Q)]}{d(Q^2)} \right\} \right) \text{ (squared Guinier radius)} \quad (11.23c)$$

$$l_i = \langle d \rangle = \frac{4}{\pi} \frac{\int_0^{\infty} Q^2 I(Q) dQ}{\lim_{Q \rightarrow \infty} \{Q^4 I(Q)\}} \text{ (average intersection length)} \quad (11.23d)$$

$$l_c = \langle l \rangle = \frac{\pi}{\tilde{Q}_{\text{inv}}} \int_0^{\infty} Q I(Q) dQ \text{ (correlation length)} \quad (11.23e)$$

$$A_c = \langle A \rangle = \frac{2\pi}{\tilde{Q}_{\text{inv}}} \int_0^{\infty} I(Q) dQ \text{ (correlation surface)} \quad (11.23f)$$

$$V_P = \langle V \rangle = \frac{2\pi^2}{\tilde{Q}_{\text{inv}}} I(0) \text{ (correlation volume, Porod volume)} \quad (11.23g)$$

These structural parameters are calculated by **SASfit** via the menu [**Calc|integral parameters...**].

On the other side the structural parameters from above can depend on specific moments of the size distribution in the case the scattering objects are spheres. The m -th moment $\langle R^m \rangle$ of a size distribution $n(R)$ is given by

$$\langle R^m \rangle = \frac{\int_0^{\infty} n(R) R^m dR}{\int_0^{\infty} n(R) dR} \quad (11.24)$$

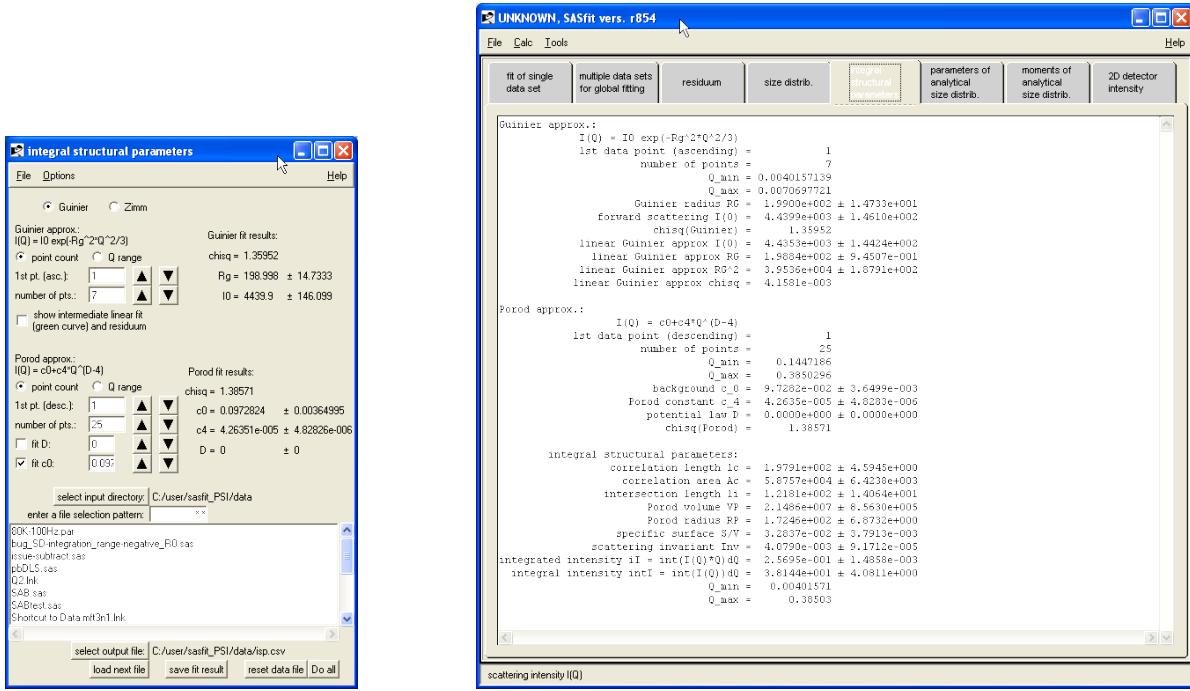
From these moments the following integral structural parameters in case of polydisperse spheres can be calculated and are listed together with a hypothetical radius of monodisperse spheres having the same structural parameter.

intersection length l_i :

$$l_i = \frac{\langle R^3 \rangle}{\langle R^2 \rangle} \text{ and } R_{l_i} = \frac{3}{4} l_i$$

correlation length l_c :

$$l_c = \frac{\langle R^4 \rangle}{\langle R^3 \rangle} \text{ and } R_{l_c} = \frac{2}{3} l_c$$



(a) GUI for defining fit ranges for Porod and Guinier approximations

(b) Tabbed menu displaying the integral structural parameters calculated via the different moments of the scattering curve the Porod and Guinier extrapolations to $Q \rightarrow 0$ and $Q \rightarrow \infty$.

FIGURE 11.1. Menu and tabbed window for integral structural parameters. SASfit also supports analysis of series of data, whereby the structural parameters are stored in CSV format readable by many software packages in a separate file for further analysis.

Guinier radius R_G :

$$R_G = \sqrt{\frac{\langle R^8 \rangle}{\langle R^6 \rangle}} \text{ and } R_{R_G} = \sqrt{\frac{5}{3}} R_G$$

correlation cross section A_c :

$$A_c = \frac{4\pi}{5} \frac{\langle R^5 \rangle}{\langle R^3 \rangle} \text{ and } R_{A_c} = \sqrt{\frac{5}{4\pi} A_c}$$

Porod Radius R_{V_P} :

$$V_P = \frac{4\pi}{3} \frac{\langle R^6 \rangle}{\langle R^3 \rangle} \text{ and } R_{V_P} = \sqrt[3]{\frac{3}{4\pi} V_P}$$

Fig. 11.2 shows the SASfit menu displaying these values for each scattering contribution having a size distribution and also for the sum of all scattering contributions. Next to the integral structural parameters also the different moments of the size distribution up the 8th moment are supplied.

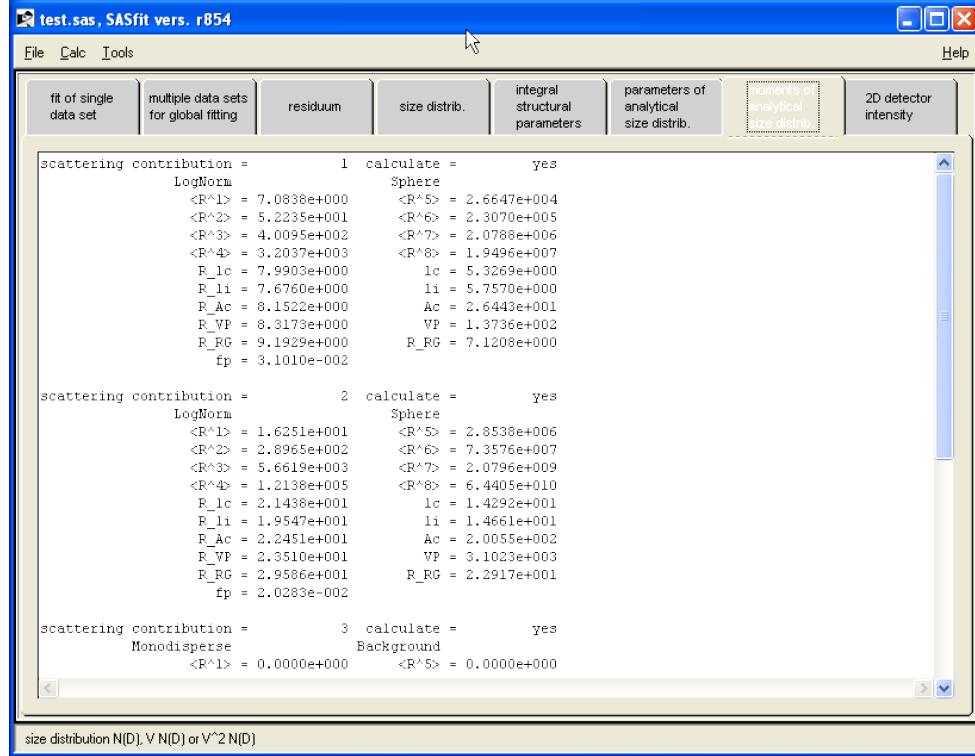


FIGURE 11.2. Menu displaying the different moments of a size distribution. At the moment these values are only calculated for single data sets but not yet for multiple data sets

11.4. Volume fractions

Having measured SAS-data versus $[Q] = nm^{-1}$ in absolute scale ($1/cm$) and knowing the scattering contrast also in absolute scale ($1/cm^2$) one can get the number density of particles. However, in general the volume fraction is known by other means but not the number density. The volume fraction can be calculated from the size distribution for some simple geometric shapes of the particles.

Let us first consider the case of simple spheres (**Sphere**) with a size distribution over their radii. The size distribution can be interpreted as a number density distribution function. The volume fraction f_p of the spheres can be easily calculated by

$$f_p = \int_0^\infty n(R) \frac{4}{3} \pi R^3 dR = \int_0^\infty N p(R) \frac{4}{3} \pi R^3 dR = N \frac{4}{3} \pi \langle R^3 \rangle \quad (11.25)$$

where $\langle R^3 \rangle$ is the third moment of the size distribution. The different moments of a size distribution can be calculated analytically for some special cases like the log-normal distribution. However, SASfit calculates the moments and displays them on the menu [calc|single data single]. Up to the 8th-moment of a distribution function is displayed in the menu tab **moments of analytical size distrib.** like in Fig. 11.2 together with some other parameters defined in section 11.3. To compute the volume fraction f_p , which is a dimensionless parameter, one has to use the proper

units for N and $\langle R^3 \rangle$. $\langle R^3 \rangle$ has here units of $[\langle R^3 \rangle] = \text{nm}^3 = 10^{-21}\text{cm}^3$ and N is in units of $[N] = 10^{42}\text{cm}^{-3}$. The volume fraction f_p can as an example be computed for $N = 8.55241 \times 10^{-28}$ and $\langle R^3 \rangle = 5.6619 \times 10^3$ as

$$f_p = 10^{42} \times 8.55241 \times 10^{-28} \frac{4}{3}\pi 5.6619 \times 10^3 10^{-21} = 0.020283. \quad (11.26)$$

The numbers for N and $\langle R^3 \rangle$ can be directly taken from the **SASfit** gui.

Let us now consider the case of cylinders with a circular cross-section with radius R and length L . We will have a look on the two cases of having either a distribution in the radius R or a distribution in the length L . The volume of a cylinder V_{cyl} is given by

$$V_{\text{cyl}}(R, L) = \pi R^2 L \quad (11.27)$$

To calculate the volume fraction from the size distribution we need to integrate over the particle volume. The integration is done either over the radius dR

$$f_p = \int_0^\infty n(R)V_{\text{cyl}}(R, L) dR = \int_0^\infty Np(R)\pi R^2 L dR = NL\pi\langle R^2 \rangle \quad (11.28)$$

or over the cylinder length dL

$$f_p = \int_0^\infty n(L)V_{\text{cyl}}(R, L) dL = \int_0^\infty Np(L)\pi R^2 L dL = NR^2\pi\langle L \rangle \quad (11.29)$$

depending if we have a distribution over the radius R or the length L . In both cases the volume fraction can be expressed in terms of moments of the size distribution supplied by **SASfit**. In the first case it can be expressed by the second moment $\langle R^2 \rangle$ of the cylinder radius and in the second case by the first moment $\langle L \rangle$ of the cylinder length, i.e. the mean cylinder length. The required moments are displayed in **SASfit** in the menu shown in Fig. 11.2. Also here one has to take care using proper units, but this is done equivalently to the first example of a sphere in eq. 11.26.

The three examples above show that the volume fraction f_p of scatterers can be calculated in many cases via the moments of the size distribution and for simple cases all necessary parameters are supplied in the **SASfit** menu interface. The volume fraction **fp** in Fig. 11.2 is numerically calculated from the size distribution. For some specific other form factor and the special case of a **LogNorm** distribution a plugin size distribution named **LogNorm_fp** described in section 9.1 has been implemented. Calculating volume fractions for any size distribution and for any form factor is not easy to implement. It would require quite some knowledge about the form factor and how exactly the volume fraction is defined. The plugin **LogNorm_fp** distinguishes between volume fraction of a core only, a volume fraction of a core together with a shell and a volume fraction of a shell only. For the general case one also needs to know which size parameter of the form factor has a distribution. This already shows that the user has to supply additional information. For the calculation a volume function has to be associated to each form factor. If this is not the case the **SASfit** routine returns 0. For those functions a volume function is associated to the form factor **SASfit** calculates numerically the volume fraction for any size distribution by integration. The plugin function **LogNorm_fp** on the other side has a lognormal distribution implemented and the information about the form factor and the

size parameter of the form factor having a distribution has to be given by the user via an input value called **shape**. Only a very limited number of form factor can be selected by this parameter. For other form factors the plugin needs to be extended or an routine calculating the volume for the specific form factor needs to be implemented.

CHAPTER 12

Basic Analysis of Dynamic Light Scattering Data

In a typical dynamic light scattering (DLS) or photo correlation scattering (PCS) experiment, the autocorrelation function $G^{(2)}(\tau)$ of the intensity scattered by dispersed particles is determined as a function of the delay τ . $G^{(2)}(\tau)$ is related to the modulus of the normalized field autocorrelation function $g_1(\tau)$ by a Siegert relationship

$$G^{(2)}(\tau) = Ag_1^2(\tau) + B. \quad (12.1)$$

Here B is a background term often designated as the baseline and A can be considered as another instrumental factor. The time dependence of $g_1(\tau)$ is related to the dynamics of the dispersed particles. For particles in Brownian motion, the time decay of $g_1(\tau)$ is determined by the diffusion coefficient of the dispersed particles. In particular, for monodisperse samples $g_1(\tau)$ is an exponentially decaying function:

$$g_1(\tau) = \exp(-\Gamma\tau) \quad (12.2)$$

or

$$G^{(2)}(\tau) = A \exp(-2\Gamma\tau) + B \quad (12.3)$$

where the decay rate Γ is linked to the particles' diffusion coefficient D by $\Gamma = DQ^2$, where Q is the modulus of the scattering vector

$$Q = \frac{4\pi m_1}{\lambda_0} \sin(\theta/2) \quad (12.4)$$

m_1 is the refraction index of the solution, λ_0 the wavelength in vacuo of the incident light and θ the scattering angle. At the end the Stokes-Einstein expression for the diffusion coefficient is used to get an average particle radius R_{DLS}

$$D = \frac{kT}{6\pi\eta R_{\text{DLS}}} \quad (12.5)$$

where k is Boltzmann's constant, T the absolute temperature, η the viscosity of the dispersion medium and R_{DLS} the particle radius (only valid for noninteracting particles).

12.1. Cumulant Analysis

The formulas in eq. 12.2 to 12.5 are valid for monodisperse dispersions only. For polydisperse dispersions the cumulants method of Koppel (1972) [116] is widely used, which assumes a multi-exponential behaviour so that $g_1(\tau)$ and $G^{(2)}(\tau)$ can be written in a series expansion as:

$$g_1(\tau) = \exp \left(-\Gamma_1 \tau + \frac{\Gamma_2 \tau^2}{2} - \frac{\Gamma_3 \tau^3}{6} + \dots \right) \quad (12.6)$$

$$G^{(2)}(\tau) = A \exp \left(-2\Gamma_1 \tau + \Gamma_2 \tau^2 - \frac{\Gamma_3 \tau^3}{3} + \dots \right) + B \quad (12.7)$$

SASfit assumes for the cumulant fit-routine that the function $G^{(2)}(\tau)$ is supplied. As normally no error bar is available from the correlator a robust least square procedure is implemented.

The formula given by Koppel has been obtained by a Taylor expansion of $\ln g_1(\tau)$ around $\tau = 0$. A more stable expansion can be obtained by a series expansion of $g_1(\tau)$ around the average relaxation time Γ_1 [177, 46, 57, 139].

$$g_1(\tau) = \exp(-\Gamma_1 \tau) \left(1 + \frac{\Gamma_2 \tau^2}{2} - \frac{\Gamma_3 \tau^3}{6} + \dots \right) \quad (12.8)$$

$$G^{(2)}(\tau) = A \exp(-2\Gamma_1 \tau) \left(1 + \frac{\Gamma_2 \tau^2}{2} - \frac{\Gamma_3 \tau^3}{6} + \dots \right)^2 + B \quad (12.9)$$

12.2. Double Decay Cumulant Analysis

$$G^{(2)}(\tau) = A \left[p e^{-2\Gamma_{a,1}\tau + \Gamma_{a,2}\tau^2} + (1-p) e^{-2\Gamma_{b,1}\tau + \Gamma_{b,2}\tau^2} \right] + B \quad (12.10)$$

12.3. Fit of Double Stretched Exponentials

$$G^{(2)}(t) = A \left\{ p \exp \left(\left[\frac{t}{\tau_1} \right]^{\gamma_1} \right) + (1-p) \exp \left(\left[\frac{t}{\tau_2} \right]^{\gamma_2} \right) \right\} + B \quad (12.11)$$

12.3.1. The least squares minimiser and the robust least squares procedure.

The function to be minimised is

$$\chi^2 = \sum_i \left(\frac{r_i}{\Delta I_i} \right)^2$$

where the residual is defined as

$$r_i = I_i - I_{i,th}.$$

Here I_i is the intensity correlation function $G^{(2)}(t_i)$ at time t_i with already subtracted baseline as received from a correlator, $I_{i,th}$ is the value of the cumulant fit according to eq. 12.7 or a double exponential decay according to eq. 12.11. Normally no error values ΔI_i are supplied from the correlator so that all data points are weighted the same. A robust fitting with bisquare weights is implemented which uses an iteratively reweighted least squares algorithm, and follows the procedure:

- (1) Fit the model by an unweighted least squares (that is, χ).
- (2) Standardize the residuals via $u_i = r_i/(Ks)$. Here K is a tuning constant equal to 4.685, and s is the robust variance given by $MAD/0.6745$, where MAD is the median absolute deviation of the residuals

$$MAD = \sum_{i=1}^N \frac{1}{N} |I_{i,th} - I_i|$$

- (3) Compute the robust weights w_i as a function of the standardized residuals u_i . The bisquare weights are given by

$$w_i = \begin{cases} (1 - u_i^2)^2 & |u_i| < 1 \\ 0 & |u_i| \geq 1 \end{cases}$$

- (4) Re-do the fit using the weighted minimiser:

$$\chi^2 = \sum_i w_i \left(\frac{r_i}{\Delta I_i} \right)^2$$

- (5) The fit converges when the MAD changes by no more than the fraction set by `residual_tolerance` (which has been chosen to be 10^{-8}). Otherwise, perform the next iteration of the fitting procedure by returning to the first step.

CHAPTER 13

Scattering Theory

13.1. Scattering Cross-Section

In a scattering experiment one is interested in a detailed analysis of the scattering pattern as a function of the characteristics of the incident beam. Monochromator and collimator specify direction and energy of the incident radiation. The radiation interacts with the sample and receives thereby a momentum transfer $\hbar\mathbf{Q}$. By this process the radiation receives beside a direction change also an energy change. The result is described with the help of a cross-section.

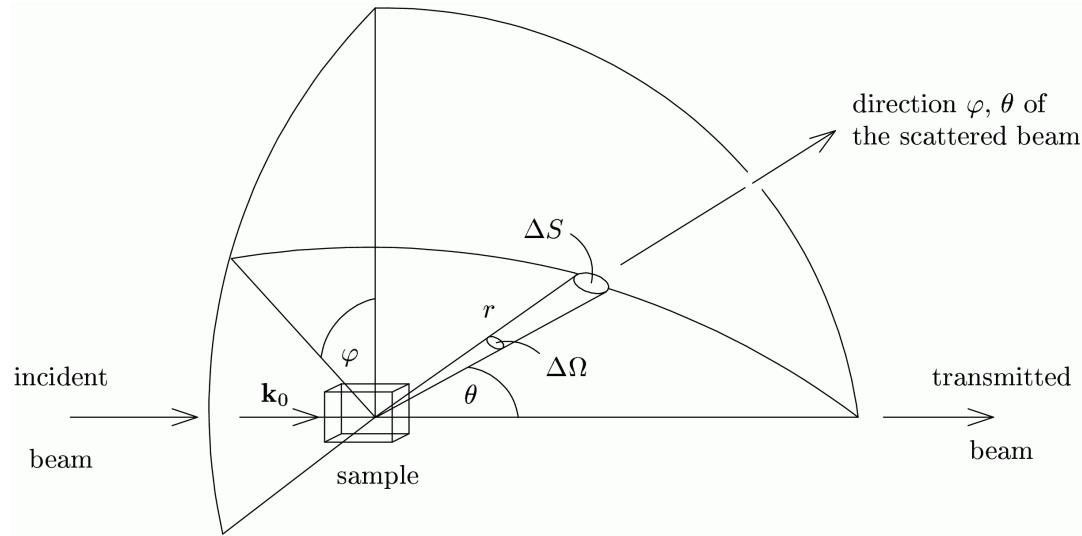


FIGURE 13.1. Schematic representation of a scattering process

A detector with an efficiency ϵ measures for this the number of the scattered neutrons or photons in a given direction $\mathbf{k}_0 + \mathbf{Q}$. The distance between detector and sample should be large in comparison to the linear dimension of the detector, so that the solid angle included by the detector element $\Delta\Omega$ is small.

If the incident beam has a homogeneous, continuous flow density Φ_0 ($[\Phi_0] = \text{neutron (photons -) per cm}^2 \text{ and second}$) and if the beam contain N identical particles, then the counting rate C of the detector is proportional to all these quantities. The proportionality constant is called differential scattering cross-section $\frac{d\sigma}{d\Omega} = \frac{C}{\Phi_0 \epsilon N \Delta\Omega}$. In the case of inelastic scattering the counting rate is in a certain interval δE of transferred energy in addition proportionally to δE . The appropriate proportionality constant is the partial

(or double) differential scattering cross-section

$$\frac{d^2\sigma}{d\Omega dE} = \frac{C}{\Phi_0 \epsilon N \Delta\Omega \Delta E} = \frac{\text{Number of neutrons (photons), which are scattered per second into the solid angle } d\Omega \text{ toward } \phi, \theta \text{ with an energy between } E \text{ and } E + dE}{N\Phi_0 d\Omega dE}$$

The total differential cross-section is defined as

$$\sigma_t = (\text{total number of scattered neutrons (photons) per sec})/\Phi_0. \quad (13.1)$$

The three different cross-section are related to each other by

$$\sigma_t = \int d\Omega \frac{d\sigma}{d\Omega} \quad \text{and} \quad \frac{d\sigma}{d\Omega} = \int_0^\infty dE \frac{d^2\sigma}{d\Omega dE}. \quad (13.2)$$

13.1.1. Scattering of neutrons on atoms. The scattering of neutrons can be explained by two types of interactions between neutrons and matter; once the strong spin dependent nuclear forces between nuclei and neutron (interaction range $\sim 10^{-15}\text{m}$) and secondly the dipole-dipole interaction between the magnetic moment of the neutron and that of an unpaired electron, or nuclei accordingly. Both interactions have in common that their interaction potential $V(\mathbf{r})$ for $r \rightarrow \infty$ decays faster than $1/r$. For this kind of interaction potential the quantum mechanical scattering theory [89] yields as an asymptotic approximation for the Schrödinger equation of the wave function

$$\psi_{\mathbf{k}_0}(\mathbf{r}) \xrightarrow[r \rightarrow \infty]{} \frac{1}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}_0 \cdot \mathbf{r}} + f(\theta, \phi) \frac{e^{ik_0 r}}{r} \right]. \quad (13.3)$$

For low energy particles and short range potential the partial waves methods yields for the scattering amplitude $f(\theta, \phi) = -b$, whereby the so called scattering length b can be determined experimentally. Via another Ansatz, the Born approximation, one gets a series expansion for $f(\theta, \phi)$. The first term of this series expansion is give by

$$f^{(0)}(\theta, \phi) = -\frac{m_N}{2\pi\hbar^2} \int d\mathbf{r} e^{i\mathbf{Q} \cdot \mathbf{r}} V(\mathbf{r}), \quad (13.4)$$

with $\mathbf{Q} = \mathbf{k} - \mathbf{k}_0$. A comparison of this result with the partial wave method ($f(\theta, \phi) = -b$), shows that the pseudo potential $V(\mathbf{r}) = \frac{2\pi\hbar^2}{m_N} b \delta(\mathbf{r})$ (Fermi's pseudo potential) has an equivalent solution. For an ensemble of N atoms, e.g. a crystal, the total potential is in kinematic approximation the sum of the individual potentials

$$V(\mathbf{r}) = \frac{2\pi\hbar^2}{m_N} \sum_{j=1}^N b_j \delta(\mathbf{r} - \mathbf{r}_j), \quad (13.5)$$

whereby \mathbf{r}_j describes the position of nuclei j with scattering the length b_j and N the number of scattering atoms. The scattering length b_j depends on the element or isotope which is the scattering center. Furthermore it can depend on the spin state of the neutron and the nuclei and on unpaired electrons in non fully occupied atomic electron shells. In the kinematic approximation it is assumed that the intensity of the incoming beam is identical at each scattering center, i.e. that the scattered intensity does not attenuate the incoming beam. Furthermore it is assumed that the incoming beam is only scattered once and multiple scattered can be neglected.

In the static approximation, i.e. for a scattering process which does not change the state of the scattering center and therefore is an elastic scattering process, the differential scattering cross-section can be described in terms of a scattering amplitude $f^{(0)}(\theta, \phi)$ by

$$\frac{d\sigma}{d\Omega}(\mathbf{Q}) = \frac{1}{N} |f^{(0)}(\theta, \phi)|^2 = \frac{1}{N} \left(\frac{m_N}{2\pi\hbar^2} \right)^2 \left| \int d\mathbf{r} e^{i\mathbf{Q}\mathbf{r}} V(\mathbf{r}) \right|^2 . \quad (13.6)$$

Using Fermi's pseudo potentials (eq. 13.5) leads to the expression

$$\frac{d\sigma}{d\Omega}(\mathbf{Q}) = \frac{1}{N} \left| \sum_{j=1}^N b_j e^{i\mathbf{Q}\mathbf{r}_j} \right|^2 = \frac{1}{N} \sum_{i,j} b_i b_j e^{i\mathbf{Q}\mathbf{r}_i} e^{-i\mathbf{Q}\mathbf{r}_j} \quad (13.7)$$

Let us now consider a new system of scatterers, which are only different to those from eq. 13.7 that the scattering length of the nuclei are exchanged. Hereby both the position and fraction of the scattering length b_i are kept the same. For a large number of scattering centers the average over the cross-sections of all possible systems of scatterers which are identical to the one in 13.7 can be described by

$$\frac{d\sigma}{d\Omega}(\mathbf{Q}) = \frac{1}{N} \sum_{i,j} \overline{b_i b_j} e^{i\mathbf{Q}\mathbf{r}_i} e^{-i\mathbf{Q}\mathbf{r}_j} . \quad (13.8)$$

If the scattering length b_i occur in the same fraction x_i , whereby $\sum_i x_i = 1$, so that the averages \bar{b} and \bar{b}^2 can be written as

$$\bar{b} = \sum_i x_i b_i \text{ and } \bar{b}^2 = \sum_i x_i b_i^2 . \quad (13.9)$$

Under the condition that there are no correlations between the scattering lengths of the individual nuclei one can write

$$\overline{b_i b_j} = \bar{b}^2 \text{ for } i \neq j \quad \text{and} \quad \overline{b_i b_j} = \bar{b}^2 \text{ for } i = j \quad (13.10)$$

From this it follows for the differential cross-section

$$\frac{d\sigma}{d\Omega}(\mathbf{Q}) = \frac{1}{N} \left(\bar{b}^2 + \bar{b}^2 \sum_{i,j}^{i \neq j} e^{i\mathbf{Q}\mathbf{r}_i} e^{-i\mathbf{Q}\mathbf{r}_j} \right) = \underbrace{\frac{1}{N} (\bar{b}^2 - \bar{b}^2)}_{\frac{d\sigma_{\text{inc}}}{d\Omega}} + \underbrace{\frac{1}{N} \bar{b}^2 \left| \sum_i e^{i\mathbf{Q}\mathbf{r}_i} \right|^2}_{\frac{d\sigma_{\text{coh}}}{d\Omega}} \quad (13.11)$$

13.1.1.1. *Nuclear scattering.* The simplest system of scatterers consist of only one isotope with nuclear spin I . As the spin of the neutron is $s = \pm 1/2$ only two orientations are possible: parallel spins of neutron and nuclei, i.e. a total spin of $J_{(+)} = I + 1/2$ or antiparallel spins, i.e $J_{(-)} = I - 1/2$. The corresponding scattering length are named $b_{(+)}$ and $b_{(-)}$.¹

The number of possible states for the total spin $J_{(\pm)}$ are $2J_{(+)} + 1 = 2I + 2$ and $2J_{(-)} + 1 = 2I$. In case of unpolarized neutrons and/or random oriented nuclear spins

¹For coherent scattering the spin of the neutrons keep constant in contrast to incoherent scattering where a part come along with a spin-flip. This can e.g. be used to change the ration between coherent and incoherent scattering.

parallel and antiparallel spin states have the same probability. The fraction $x_{(\pm)}$ of the scattering lengths $b_{(\pm)}$ is therefore proportional to the corresponding number of states

$$x_{(+)} = \frac{2J_{(+)}}{(2J_{(+)} + 1) + (2J_{(-)} + 1)} = \frac{I + 1}{2I + 1} \text{ and } x_{(-)} = \frac{I}{2I + 1} . \quad (13.12)$$

For the average scattering length \bar{b} we therefore get

$$\bar{b} = \sum_{i=(+),(-)} x_i b_i = \frac{1}{2I + 1} [(I + 1) b_{(+)} + I b_{(-)}] . \quad (13.13)$$

For a mixture of different elements and isotopes of type l with nuclear spin I_l and the fraction x_l (with $\sum_l x_l = 1$) the averages can be written as

$$\bar{b} = \sum_l \frac{x_l}{2I_l + 1} [(I_l + 1) b_{l(+)} + I_l b_{l(-)}] \quad (13.14)$$

$$\bar{b^2} = \sum_l \frac{x_l}{2I_l + 1} [(I_l + 1) (b_{l(+)}^2) + I_l (b_{l(-)}^2)] . \quad (13.15)$$

13.1.1.2. Magnetic Scattering. In magnetic materials the contribution of the interaction between neutrons and atomic magnetic dipole moments to the scattering length has the same order of magnitude than the nuclear scattering length. The magnetic scattering is based on the interaction of the magnetic moment of the neutron μ_n with the magnetic moment of the scattering atom μ_A . The magnetic interaction potential $V(\mathbf{r})$ is described by

$$V(\mathbf{r}) = -\mu_n \cdot \mathbf{B}(\mathbf{r}), \quad (13.16)$$

whereby $\mu_n = \gamma \frac{e\hbar}{2m_p} \boldsymbol{\sigma} = \gamma \mu_N$ is the magnetic dipole moment² of the neutron, $\boldsymbol{\sigma}$ Pauli's spin operator, γ the neutron magnetic moment to nuclear magneton ratio³ and $\mathbf{B}(\mathbf{r})$ the magnetic field of an atom at the position of the neutron. An atom generates a magnetic field due to the magnetic dipole moment μ_S of its electrons $\mathbf{B}_S(\mathbf{r})$

$$\mathbf{B}_S(\mathbf{r}) = \nabla \times \mathbf{A} \text{ with } \mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mu_S \times \mathbf{r}}{r^3} \quad (13.17)$$

and due to the orbital angular momentum of the electrons $\mathbf{l} = -\mathbf{p} \times \mathbf{r}$ which generates a field of $\mathbf{B}_L(\mathbf{r})$

$$\mathbf{B}_L(\mathbf{r}) = -\frac{\mu_0}{4\pi} \frac{2\mu_B}{\hbar} \frac{\mathbf{p} \times \mathbf{r}}{r^3} . \quad (13.18)$$

The magnetic interaction potential $V(\mathbf{r}) = -\mu_n \cdot (\mathbf{B}_S(\mathbf{r}) + \mathbf{B}_L(\mathbf{r}))$ is a weak long range potential which also can be treated with Born's approximation. Compared to the nuclear

²neutron magnetic moment: $\mu_n = -0.96623645 \times 10^{-26} \text{ JT}^{-1}$, neutron magnetic moment to Bohr magneton ratio: $\mu_n/\mu_B = -1.04187563 \times 10^{-3}$, nuclear magneton: $\mu_n = 5.05078343 \times 10^{-27} \text{ JT}^{-1}$, Bohr magneton: $\mu_B = \frac{e\hbar}{2m_e} = 927.400949 \times 10^{-26} \text{ JT}^{-1}$, proton mass: $m_p = 1.67262171 \times 10^{-27} \text{ kg}$, neutron mass: $m_n = 1.67492728 \times 10^{-27} \text{ kg}$, electron mass: $m_e = 9.1093826 \times 10^{-31} \text{ kg}$, elementary charge: $e = 1.60217653 \times 10^{-19} \text{ C}$, Planck constant over 2π : $\hbar = h/2\pi = 1.05457168 \times 10^{-34} \text{ J s}$

³neutron magnetic moment to nuclear magneton ratio $\gamma = \mu_n/\mu_N = 1.91304273$

scattering amplitude the corresponding magnetic scattering amplitude b_M is given by the Fourier transformation of the magnetic interaction potential $\mathcal{F}[V(\mathbf{r})]$

$$b_M = -\frac{m_n}{2\pi\hbar^2} \boldsymbol{\mu}_n \cdot \int d^3r e^{i\mathbf{Q}\mathbf{r}} (\mathbf{B}_S(\mathbf{r}) + \mathbf{B}_L(\mathbf{r})). \quad (13.19)$$

The Fourier transformation of the magnetic field is related in case of a static magnetic field to the Fourier transformation of the local magnetization $\mathbf{M}(\mathbf{Q}) = \mathcal{F}[\mathbf{M}(\mathbf{r})]$ [200] by

$$\mathbf{B}(\mathbf{Q}) = \mu_0 \frac{\mathbf{Q} \times [\mathbf{M}(\mathbf{Q}) \times \mathbf{Q}]}{Q^2} = \mu_0 \mathbf{M}_\perp(\mathbf{Q}), \quad (13.20)$$

whereby $\mathbf{M}_\perp(\mathbf{Q})$ is the component of $\mathbf{M}(\mathbf{Q})$ perpendicular to \mathbf{Q} and $\mu_0 = 4\pi 10^{-7}$ Vs/Am the magnetic constant. For the magnetic scattering amplitude⁴ b_M we find than

$$b_M = D_M \mu_0 \boldsymbol{\sigma} \cdot \mathbf{M}_\perp(\mathbf{Q}) \text{ with } D_M = -\gamma \frac{m_n}{2\pi\hbar^2} \mu_N = 2.31605 \times 10^{14} \frac{1}{\text{m}^2 \text{Tesla}}. \quad (13.21)$$

For scattering on magnetic structures always two interactions have to be considered, nuclear scattering which is caused by fluctuations in the number density and composition and magnetic scattering caused by fluctuations in amplitude and/or orientation of the local magnetization. In case of a preferred orientation, e.g. the direction of an external applied magnetic field \mathbf{H} , the magnetic scattering depends on the spin state $\boldsymbol{\sigma}$ of the neutron. If \mathbf{e}_x describes the direction of the preferred axis and (+) and (-) the neutron spin polarisation antiparallel and parallel to \mathbf{e}_x than the scattering can be described by four scattering processes; these are two spin non-flip $(++,--)$ and two spin flip $(+-,-+)$ processes. Moon, Riste und Koehler [151] have shown that for coherent scattering the four scattering length are given by

$$b_{\pm\pm} = b_N \mp D_M \mu_0 M_{\perp x} \quad (13.22)$$

$$b_{\pm\mp} = -D_M \mu_0 (M_{\perp z} \pm i M_{\perp y}). \quad (13.23)$$

whereby b_N is the nuclear scattering length. In case of unpolarized neutrons the differential scattering cross-section can be written as

$$\frac{d\sigma_{\text{unp}}}{d\Omega}(\mathbf{Q}) = \frac{d\sigma_{\text{nuc}}}{d\Omega}(\mathbf{Q}) + \frac{d\sigma_{\text{mag}}}{d\Omega}(\mathbf{Q}), \quad (13.24)$$

because $(b_{++}^2 + b_{--}^2 + b_{+-}^2 + b_{-+}^2)/2 = b_N^2 + D_M^2 \mu_0^2 M_\perp^2$. For unpolarized neutrons the interference contribution only has an influence on the degree of polarization of the scattered neutrons but not on the scattering intensity.

⁴Frequently the magnetization is given in units of Bohr magnetons ($\mu_B = \frac{e\hbar}{2m_e} = 927.400949 \times 10^{-26}$ J/T, $1[\text{J/T}] = 1[\text{Am}^2]$) per atomic volume Ω so that the magnetic scattering length density can be written as $b_M = D_\mu \sum_i c_i M_i / \Omega_i$ with $D_\mu = 2.69914 \times 10^{-15} \text{ m}^2$. The two constants are related via $D_\mu = D_M \mu_0 \mu_B$.

13.1.2. Scattering of x-ray at atoms. The scattering of x-rays on matter practically exclusively depends on the interaction of the incoming radiation with electrons. The contribution on the nuclei is negligible small because the mass of the nuclei is more than 10^3 times larger than the mass of an electron and the energy of the nuclear scattering more than 10^6 smaller than the energy of the scattering on electrons.

The frequency $\nu_0 = c/\lambda$ of the incoming x-ray beam is in general large against the resonance frequency of the electrons. In this case the electrons can be considered to be free and the special properties of the chemical binding are of no importance. These are the conditions for Thomson-scattering. J.J. Thomson developed a simple classical model for this type of scattering. Under the influence of an electric field of x-rays electrons start to oscillate. For an incoming plane and monochromatic wave with an electric field $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}_0 \mathbf{r} - \omega t)}$ the amplitude E_s of a wave scattered on a free electron is

$$E_s = -E_0 \frac{e^2}{m_e c^2} \frac{1}{r} \sin \psi. \quad (13.25)$$

Hereby e and m_e are the charge⁵ and mass⁶ of the electron, respectively. c is the speed of light⁷, r the distance between sample and detector and ψ the angle between the direction of the accelerated electrons by the incoming wave and the direction of the scattered wave.

Analogously to the neutron scattering length for an electron the x-ray scattering length (far field of a Hertz dipole) is defined as

$$b_{\text{x-ray}} = \frac{e^2}{m_e c^2} \sin \psi = r_0 \sin \psi \quad (13.26)$$

whereby $r_0 = e^2 / (m_e c^2) = 2.82 \times 10^{-13}$ cm is the classical electron radius. For small angle scattering $\psi \simeq \pi/2$ whereby the angle dependent polarization factor is approximately 1.

For calculating the scattering amplitude of an atom with Z electrons one has to sum up the scattered waves from the different electrons with the correct phase. To do this an electron density distribution $\rho_e(\mathbf{r})$ can be introduced which describes the time average probability distribution of the electron in the atom. The scattering amplitude of an atom is than

$$f_a(\mathbf{Q}) = r_0 \underbrace{L(Q)}_{\substack{\text{polarization} \\ \text{factor} \sim 1}} \int d\mathbf{r} \rho_e(\mathbf{r}) e^{i\mathbf{Q}\mathbf{r}} \stackrel{Q \rightarrow 0}{=} r_0 Z. \quad (13.27)$$

The charge distribution in an atom can be described in good approximation by a radial symmetric function so that

$$f_a(Q) = r_0 \int_0^\infty \frac{\sin Qr}{Qr} \rho_e(r) 4\pi r^2 dr \stackrel{Q \rightarrow 0}{=} r_0 \int_0^\infty \rho_e(r) 4\pi r^2 dr = r_0 Z. \quad (13.28)$$

In small angle scattering the scattering length of an atom is therefore $f_a = r_0 Z$ or in units of "electron units [e.u.]" $f = f_a/r_0 = Z$.

⁵ $e = 1.60217653 \times 10^{-19}$ C

⁶ $m_e = 9.1093826 \times 10^{-31}$ kg

⁷ $c = 299792458$ m s⁻¹

13.1.2.1. *Anomalous scattering of x-rays.* The relation $f = Z$ for atomic scattering length is only valid as long as the energy of the incoming radiation is much larger than the energy of the K-, L-, etc. shells. In case that the absorption edge of an atom is close to the energy of the incoming beam the scattering length has to be corrected by a dispersion term. In general the scattering length of an atom depends on the energy of the x-rays and is a complex number

$$f(E) = Z + f'(E) + i f''(E) . \quad (13.29)$$

The correction terms f' and f'' change the scattering length f near a K_α absorption edge typically up to 30%. Figure 13.2 shows the energy dependency of the scattering length of iron f'_{Fe} und f''_{Fe} . The dispersion terms are related via the Karamer-Kronig relation

$$f'(E) = \frac{2}{\pi} \int_0^\infty dE' \frac{E' f''(E')}{E'^2 - E^2}. \quad (13.30)$$

In general the imaginary part f'' can be determined experimentally by the mass ab-

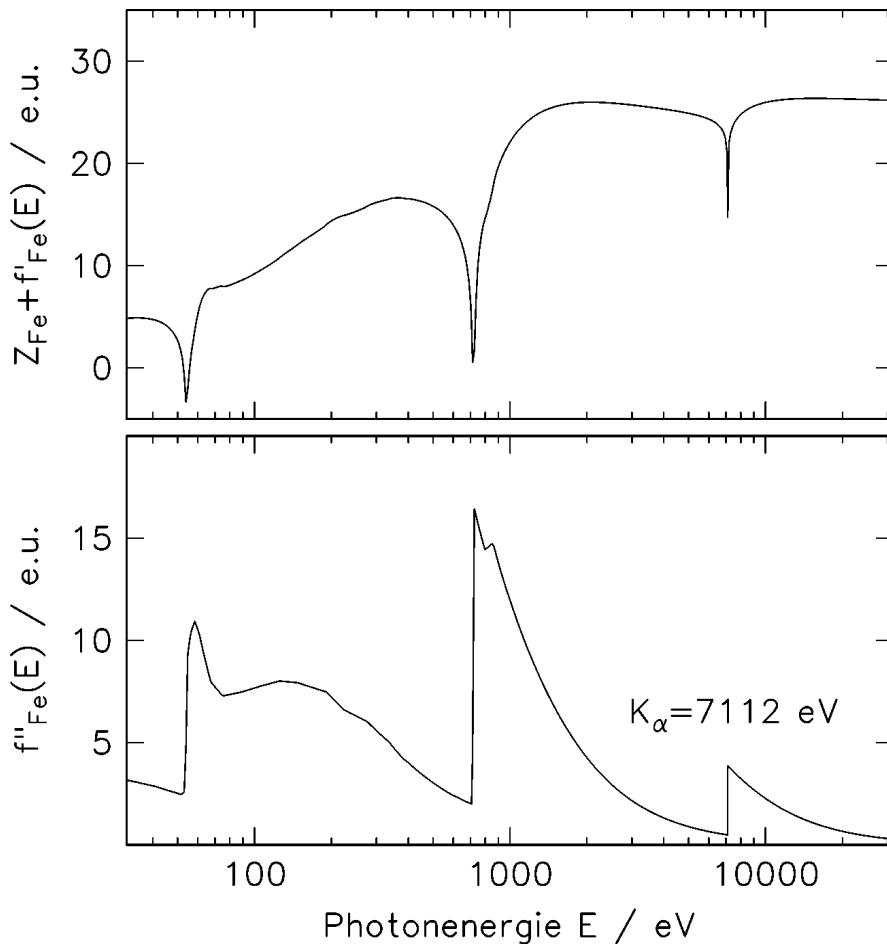


FIGURE 13.2. Energy dependence of the real and imaginary part of the scattering length of iron: $Z_{\text{Fe}} + f'_{\text{Fe}}(E)$ and $f''_{\text{Fe}}(E)$ ($Z_{\text{Fe}} = 26$)

sorption coefficient $\mu_m(E) = \frac{2N_A}{A} \lambda r_0 f''(E)$ (N_A Avogadro number, A mass of atom in resonance) to calculate than with equation 13.30 the real part of the scattering length $f'(E)$.

13.2. Small angle scattering

In small angle scattering geometry the structural unit down to single atoms can not be resolved, only structures larger than several atomic layers can be seen by that method. Small angle scattering techniques measure the beam scattered close to forward direction whereby the beam divergence is in the order $\lambda/L \ll 1$. For a wavelength of $\lambda = 1$ nm and a characteristic dimension $L = V^{1/3}$ of the scatterer of about 10 nm the divergency of the beam is about $\lambda/L = 0.1$ rad $\simeq 6.3^\circ$.

In principle with small angle scattering both information on size as well as shape of the scatterer and information on the relative arrangement of the scatterer can be obtained. In the following an overview of the theoretical basics to analyze small angle scattering data will be given.

In section 13.1 the scattering of an ensemble of atoms has been discussed. There interference terms between the waves scattered by the individual atoms were important for the differential scattering cross-section (eq. 13.7 ,13.28).

$$\frac{d\sigma}{d\Omega}(\mathbf{Q}) = \frac{1}{N} \left| \sum_{j=1}^N b_j e^{i\mathbf{Q}\mathbf{r}_j} \right|^2 \quad \text{or} \quad \frac{d\sigma}{d\Omega}(\mathbf{Q}) = \frac{1}{N} \left| \sum_{j=1}^N f_{a,j}(\mathbf{Q}) e^{i\mathbf{Q}\mathbf{r}} \right|^2. \quad (13.31)$$

Small angle scattering normally does not resolve dimension down to atomic dimensions. Therefore the summation over the individual atoms can be replaced by an integration over the illuminated volume V :

$$\frac{d\sigma}{d\Omega}(\mathbf{Q}) = \frac{1}{N} \left| \int_V d\mathbf{r} \rho(\mathbf{r}) e^{i\mathbf{Q}\mathbf{r}} \right|^2 = \frac{1}{N} I(\mathbf{Q}), \quad (13.32)$$

whereby $\rho(\mathbf{r})$ is the local scattering length density and $I(\mathbf{Q})$ the scattering amplitude. The differential scattering cross-section is mathematically the square of the modulus of the Fourier transformation of the scattering length density. The scattering length density $\rho(\mathbf{r})$ is proportional to the locally averaged scattering potential $\bar{V}(\mathbf{r})$. Thats why equation 13.32 is except a constant identical to equation 13.6. The scattering intensity $I(\mathbf{Q}) = |z|^2 = z z^*$ can therefore be written as

$$I(\mathbf{Q}) = \iint d\mathbf{r}_1 d\mathbf{r}_2 \rho(\mathbf{r}_1) \rho^*(\mathbf{r}_2) e^{i\mathbf{Q}(\mathbf{r}_1 - \mathbf{r}_2)} . \quad (13.33)$$

By using the substitution $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ one get

$$I(\mathbf{Q}) = \int d\mathbf{r} e^{i\mathbf{Q}\mathbf{r}} \underbrace{\int d\mathbf{r}_1 \rho(\mathbf{r}_1) \rho^*(\mathbf{r}_1 - \mathbf{r})}_{\Gamma(\mathbf{r}) = \rho(\mathbf{r}) \circledast \rho(\mathbf{r})} = \int d\mathbf{r} \Gamma(\mathbf{r}) e^{i\mathbf{Q}\mathbf{r}} . \quad (13.34)$$

$\Gamma(\mathbf{r})$ is the autocorrelation function⁸ In case of a real scattering length density eq. 13.34 is the convolution integral $\Gamma(\mathbf{r}) = \rho(\mathbf{r}) * \rho(\mathbf{r}) = \int d\mathbf{r}_1 \rho(\mathbf{r}_1) \rho(\mathbf{r}_1 - \mathbf{r})$ of the scattering length density and is called Patterson function⁹.

13.2.1. Autocorrelation function $\Gamma(\mathbf{r})$ and $\gamma(\mathbf{r})$. In the following we will make use of two simplifications. Firstly the scattering system is assumed to be isotropic. Hereby the isotropy can have its origin both in the shape of the scatterer or being a consequence of the temporal change of the particle orientation. The consequence is that $\Gamma(\mathbf{r})$ only depends on the modulus of r and $e^{i\mathbf{Q}\mathbf{r}}$ can be averaged over all orientations of \mathbf{r} . (The second simplification will follow further below)

13.2.1.1. *Isotropic averages.* If α is assigned to the angle between \mathbf{Q} and \mathbf{r} and if all orientations α are equal probable than the probability $p_\alpha d\alpha$ that \mathbf{Q} and \mathbf{r} include the angle α is equal to $\frac{1}{2} \sin \alpha d\alpha$.

$$p_\alpha d\alpha = \frac{2\pi R \sin \alpha}{4\pi R^2} R d\alpha = \frac{\sin \alpha}{2} d\alpha \quad (13.35)$$

The average of $e^{i\mathbf{Q}\mathbf{r}}$ over all orientations of \mathbf{r} is

$$\begin{aligned} \overline{e^{i\mathbf{Q}\mathbf{r}}}^{\mathbf{r}} &= \int_0^\pi d\alpha \frac{\sin \alpha}{2} e^{iQr \cos \alpha} = \frac{\sin Qr}{Qr} \\ &= \underbrace{\int_0^\pi d\alpha \frac{\sin \alpha}{2} \cos(Qr \cos \alpha)}_{\text{symmetric to } \alpha=\pi/2} + i \underbrace{\int_0^\pi d\alpha \frac{\sin \alpha}{2} \sin(Qr \cos \alpha)}_{\text{antisymmetric to } \alpha=\pi/2 \Rightarrow =0} \\ &= 2 \int_0^{\frac{\pi}{2}} d\alpha \frac{\sin \alpha}{2} \cos(Qr \cos \alpha) = \frac{1}{Qr} \int_0^{Qr} du \cos u = \frac{\sin Qr}{Qr} \end{aligned} \quad (13.36)$$

and equation 13.34 is simplified to

$$I(Q) = \int dr 4\pi r^2 \Gamma(r) \frac{\sin Qr}{Qr} . \quad (13.37)$$

⁸Normally the autocorrelation function is defined as $\rho(\mathbf{r}) \otimes \rho(\mathbf{r}) = \int d\mathbf{r}_1 \rho(\mathbf{r}_1) \rho^*(\mathbf{r}_1 + \mathbf{r})$, so that in fact $\Gamma(\mathbf{r})$ should be $\Gamma(\mathbf{r}) = \rho(-\mathbf{r}) \otimes \rho(-\mathbf{r})$.

⁹Here the Patterson function is defined as the autocorrelation of the scattering length density $\rho(\mathbf{r})$. Instead of defining the cross-section via a scattering length density as in 13.34 in neutron scattering the differential cross-section is often defined as $d\sigma_{coh}(\mathbf{Q})/d\Omega = (\sigma_{coh}/4\pi N) \int d\mathbf{r} \rho(\mathbf{r}) \otimes \rho(\mathbf{r}) \exp(i\mathbf{Q}\mathbf{r})$, whereby $\sigma_{coh} = 4\pi \bar{b}^2$ is the coherent cross-section and $\rho(\mathbf{r})$ the particle number density. The Patterson function is than the autocorrelation of the particle number density. This function is independent of the scattering lengths and only dependent on the geometric arrangement of the scattering centers. The Fourier transformation of the Patterson function is therefore sometimes called Structure factor $S(\mathbf{Q})$ and is related in case of a static approximation to the differential cross-section by the equation $d\sigma_{coh}(\mathbf{Q})/d\Omega = (\sigma_{coh}/4\pi N) S(\mathbf{Q})$.

13.2.1.2. *Absence of long range order.* As a second simplification it will be assumed that long range order is absent. The consequence is that the autocorrelation $\Gamma(\mathbf{r})$ becomes constant for large values of r . It converges to $\bar{\Gamma} = \bar{\rho}^2 V$. $\bar{\rho}$ is the average scattering length density defined by

$$\int d\mathbf{r} (\rho(\mathbf{r}) - \bar{\rho}) = \int d\mathbf{r} \eta(\mathbf{r}) = 0 . \quad (13.38)$$

Structural information is therefore only contained in a finite range of $\Gamma(\mathbf{r})$ where it deviates from the average value $\bar{\Gamma}$. This is due to the fact that only a deviation of $\eta(\mathbf{r})$ from the average $\bar{\rho}$ leads to a scattering contribution for $Q \neq 0$. An additional constant value $\bar{\rho}$ only contributes to the scattering at $Q = 0$ and is therefore not accessible. Consequently the average scattering length density can be subtracted without loss of generality and only deviations $\eta(\mathbf{r}) = \rho(\mathbf{r}) - \bar{\rho}$ have to be considered. The autocorrelation function is therefore defined as

$$\gamma(r) = \frac{1}{V} \overline{\eta(\mathbf{r}) \circledast \eta(\mathbf{r})}^{\mathbf{r}} = \frac{1}{V} \overline{(\rho(\mathbf{r}) - \bar{\rho}) \circledast (\rho(\mathbf{r}) - \bar{\rho})}^{\mathbf{r}} \quad (13.39)$$

$$\Rightarrow I(Q) = V \int_0^\infty dr 4\pi r^2 \gamma(r) \frac{\sin Qr}{Qr}. \quad (13.40)$$

$V\gamma(\mathbf{r})$ is different from $\Gamma(\mathbf{r})$ due to the definition of $\bar{\rho}$ (eq. 13.38) only by a constant term $\bar{\rho}^2 V$, i.e. $V\gamma(\mathbf{r}) = \Gamma(\mathbf{r}) - \bar{\rho}^2 V$ because

$$\Gamma(\mathbf{r}) = \int d\mathbf{r}_1 \rho(\mathbf{r}_1) \rho^*(\mathbf{r}_1 - \mathbf{r}) = \int d\mathbf{r}_1 (\bar{\rho} + \eta(\mathbf{r}_1)) (\bar{\rho} + \eta(\mathbf{r}_1 - \mathbf{r}))^* \quad (13.41)$$

$$= \bar{\rho}^2 V + \underbrace{\int d\mathbf{r}_1 (\bar{\rho} \eta(\mathbf{r}_1) + \bar{\rho}^* \eta^*(\mathbf{r}_1 - \mathbf{r}))}_{= 0 \text{ due to def. of } \eta(\mathbf{r}) \text{ in eq. 13.38}} + \int d\mathbf{r}_1 \eta(\mathbf{r}_1) \eta^*(\mathbf{r}_1 - \mathbf{r}) \quad (13.42)$$

$$= \bar{\rho}^2 V + \gamma(\mathbf{r}) V. \quad (13.43)$$

13.2.1.3. *Limits $r = 0$ and $r = \infty$.* The limits $r = 0$ and $r \rightarrow \infty$ for the autocorrelation $\gamma(\mathbf{r})$ are $\gamma(0) = \bar{\eta}^2$ and $\gamma(\infty) = 0$. The limit $r \rightarrow \infty$ is zero because $\eta(\mathbf{r})$ is defined as the deviation of the average scattering length density $\bar{\rho}$. As long range correlation is assumed not to be present $\eta(\mathbf{r}) \circledast \eta(\mathbf{r}) \xrightarrow{r \rightarrow \infty} \bar{\eta}^2 = 0$. $\gamma(r)$ can be calculated by the inverse Fourier transformation of the scattering intensity $I(Q)$ (eq. 13.40)

$$V\gamma(r) = \frac{1}{2\pi^2} \int_0^\infty dQ Q^2 I(Q) \frac{\sin Qr}{Qr} . \quad (13.44)$$

An important special case is $r = 0$ for which equation 13.44 can be simplified to

$$V\gamma(0) = V\bar{\eta}^2 = \frac{1}{2\pi^2} \int_0^\infty dQ Q^2 I(Q) = \frac{\tilde{Q}}{2\pi^2} . \quad (13.45)$$

The integration of the intensity $I(Q)$ in reciprocal space is therefore directly related to the average quadratic deviation of the scattering length density $\bar{\eta}^2$ but independent to the shape of the scatterers. If for example the scattering particle undergoes deformation

the scattering pattern may change drastically but the integral \tilde{Q} in eq. 13.45 keeps invariant against such a deformation.

13.2.2. Volume fraction. The average quadratic deviation of the scattering length density is directly related in a two-phase system to the scattering contrast $\Delta\eta$ and volume fraction f_p of one or the other ($1 - f_p$) phase. The average quadratic deviation of the scattering length density is defined as

$$\overline{\eta^2} = \eta_1^2 f_p + \eta_2^2 (1 - f_p), \quad (13.46)$$

whereby η_1 and η_2 are the scattering length density differences of the two phases from the average value $\bar{\rho}$ and the scattering contrast is than defined as $\Delta\eta = \eta_1 + \eta_2$. The average value $\bar{\rho}$ is given by

$$\begin{aligned} \bar{\rho} = (\bar{\rho} - \eta_1)f_p + (\bar{\rho} + \eta_2)(1 - f_p) &\Leftrightarrow f_p(\eta_1 + \eta_2) = f_p \Delta\eta = \eta_2 \\ \text{or } (1 - f_p) \Delta\eta &= \eta_1 . \end{aligned} \quad (13.47)$$

Replacing η_1 and η_2 in eq. 13.46 leads to

$$\overline{\eta^2} = \Delta\eta^2 f_p(1 - f_p). \quad (13.48)$$

The second moment of the scattering intensity, the so-called scattering invariant \tilde{Q} from eq. 13.45, can therefore be related to the volume fractions of the two phases f_p and $(1 - f_p)$ by

$$\tilde{Q} = \int dQ Q^2 I(Q) = 2\pi^2 V \Delta\eta^2 f_p(1 - f_p) . \quad (13.49)$$

Due to the Babinet principle in a scattering experiment the volume fractions can not be uniquely assigned to one or the other phase. A system with exchanged phases would have the same invariant A unique solution of eq. 13.49 for the volume fraction can only be obtained if either one already knows from somewhere else that the volume fraction of one phase is much smaller than the volume fraction of the other (dilute case) or if time resolved experiments are performed during the formation of the structure and when it is known that one phase is growing on the cost of the other one (Ostwald ripening).

13.2.3. Interparticle interferences. The square of the Fourier transformation of the scattering length density of a sample with a volume V is equal to the scattering intensity $I(\mathbf{Q})$

$$I(\mathbf{Q}) = \left| \int_V d\mathbf{r} \rho(\mathbf{r}) e^{i\mathbf{Q}\mathbf{r}} \right|^2 . \quad (13.50)$$

The integration has to be carried out over the whole illuminated sample volume V . If the sample volume contains N particles embedded in a matrix with a constant scattering length density ρ_M and if \mathbf{R}_i defines the center of particle i with a constant scattering

length density $\rho_{P,i} = \Delta\eta_i + \rho_M$ the scattering intensity can be written as

$$I(\mathbf{Q}) = \left| \sum_{i=1}^N F_i(\mathbf{Q}) e^{i\mathbf{Q}\cdot\mathbf{R}_i} \right|^2 \quad (13.51)$$

$$\text{with } F_i(\mathbf{Q}) = \int_{V_i(\mathbf{R}_i)} d\mathbf{r} \Delta\eta_i e^{i\mathbf{Q}(\mathbf{r}-\mathbf{R}_i)} = \Delta\eta_i \int_{V_i(\mathbf{0})} d\mathbf{r} e^{i\mathbf{Q}\cdot\mathbf{r}} = \Delta\eta_i V_i f_i(\mathbf{Q}).$$

$V_i(\mathbf{R}_i)$ describes the integration volume of scatterer i located at \mathbf{R}_i and $V_i(\mathbf{0})$ describes the integration volume of the scatterer i moved to the origin of the coordinate system. $\Delta\eta_i$ and V_i are the scattering contrast and particle volume, respectively. The square of the modulus in eq. 13.51 can be rewritten to

$$I(\mathbf{Q}) = \sum_{i=1}^N \sum_{j=1}^N F_i(\mathbf{Q}) F_j^*(\mathbf{Q}) e^{i\mathbf{Q}\cdot\mathbf{R}_{ij}}$$

$$= \sum_{i=1}^N |F_i(\mathbf{Q})|^2 + \underbrace{2 \sum_{i=1}^N \sum_{j>i}^N [\Re(F_i(\mathbf{Q}) F_j^*(\mathbf{Q})) \cos \mathbf{Q}\cdot\mathbf{R}_{ij} - \Im(F_i(\mathbf{Q}) F_j^*(\mathbf{Q})) \sin \mathbf{Q}\cdot\mathbf{R}_{ij}]}_{\Psi(\mathbf{Q})} \quad (13.52)$$

The first term for which $i = j$ the phase factor is identical to 1 and describes the sum of the scattering intensity of individual particles. The double sum in the second term describes interference effects of the scattering amplitudes scattered from different particles which depends on their relative arrangement $\mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j$. The scattering amplitude $F_i(\mathbf{Q})$ is among other things dependent on the scattering contrast $\Delta\eta_i$ and on the normalized form factor $F_i(\mathbf{Q})/\Delta\eta_i V_i = f_i(\mathbf{Q})$, whereby $f_i(\mathbf{Q})$ is a real-valued function with $f_i(Q = 0) = 1$. Analytical expressions for scattering intensities $i_0(Q) = |f(Q)|^2$ of simple geometric bodies are listed in the chapter 3. In case $\Delta\eta_i$ is complex valued, i.e. the scatterer has an absorption contrast $\Delta\eta''_i$, the scattering contrast can be written as $\Delta\eta_i = \Delta\eta'_i + i\Delta\eta''_i$ and for the interference term we get

$$\Psi(\mathbf{Q}) = 2 \sum_{i=1}^N \sum_{j>i}^N f_i(\mathbf{Q}) f_j^*(\mathbf{Q}) V_i V_j$$

$$\times [(\Delta\eta'_i \Delta\eta'_j + \Delta\eta''_i \Delta\eta''_j) \cos \mathbf{Q}\cdot\mathbf{R}_{ij} - (\Delta\eta'_i \Delta\eta''_j - \Delta\eta''_i \Delta\eta'_j) \sin \mathbf{Q}\cdot\mathbf{R}_{ij}] \quad . \quad (13.53)$$

If now all particle have an identical scattering contrast $\Delta\eta = \Delta\eta_i = \Delta\eta_j$ we get for the scattering intensity (eq. 13.51) the expression

$$I(\mathbf{Q}) = \sum_{i=1}^N V_i^2 (\Delta\eta'^2 + \Delta\eta''^2) |f_i(\mathbf{Q})|^2$$

$$+ 2 \sum_{i=1}^N \sum_{j>i}^N f_i(\mathbf{Q}) f_j^*(\mathbf{Q}) V_i V_j (\Delta\eta'^2 + \Delta\eta''^2) \cos \mathbf{Q}\cdot\mathbf{R}_{ij} \quad . \quad (13.54)$$

For identical scattering contrasts there is consequently no interference between the scattering amplitude of waves scattered at the real and at the imaginary (absorption) part of the scattering contrast.

13.2.3.1. Isotropic ensemble of particles. In the following we assume as another simplification an isotropic ensemble of particles. Such a system of scatterer is defined as follows: If \mathbf{R}_i defines the position of any particle i and \mathbf{R}_{ij} is the difference vector between the position of particle i and j than a system of particles is called isotropic if all vectors \mathbf{R}_{ij} of the same length will take with equal probability any direction. Under this simplification the interference term $\Psi(\mathbf{Q})$ in eq. 13.53 can be averaged over all directions \mathbf{R}_{ij} . This average yield for $\overline{\sin Q\mathbf{R}_{ij}}^{\mathbf{R}_{ij}} = 0$ and for $\overline{\cos Q\mathbf{R}_{ij}}^{\mathbf{R}_{ij}} = \frac{\sin QR_{ij}}{QR_{ij}}$ (compare with eq. 13.37). The interference term can then be written as

$$\Psi(\mathbf{Q}) = 2 \sum_{i=1}^N \sum_{j>i}^N f_i(\mathbf{Q}) f_j(\mathbf{Q}) V_i V_j (\Delta\eta'_i \Delta\eta'_j + \Delta\eta''_i \Delta\eta''_j) \frac{\sin QR_{ij}}{QR_{ij}}. \quad (13.55)$$

Therefore also for an isotropic ensemble of scatterers no interferences between waves scatterer at the real and imaginary part of the scattering contrast disappears as already shown for systems of particles with identical scattering contrast.

13.2.4. Influence of the relative arrangement of scatterers on interparticle interferences. The expression for $\Psi(\mathbf{Q})$ can be further simplified if the scattering system consist of identical particles which fulfill the condition that for all of them each orientation be likewise probable and furthermore the relative position of two particles do not have an influence on their orientation. The second part of the assumption is for radial symmetric particle automatically fulfilled. However, in case of a system of close packed ellipsoidal particles with half axis R , R and νR with $\nu > 1$ distance of $2R$ between the centers of the ellipsoids are possible, but for such an arrangement not all orientations of the ellipsoids are allowed anymore. That means that the relative distance has an influence on allowed orientations of the particles. The general case without the restrictions made in this paragraph are discussed by Guinier and Fournet in [73]. For only slightly anisotropic and not to closely packed systems the assumptions made here are at least fulfilled in first approximation. Under the made assumptions the averaging over the particle orientations can be separated from the averaging of the particle positions. As a result from the average one gets for the scattering intensity

$$\overline{I(Q)} = N \overline{|F(Q)|^2} + 2 \left| \overline{F(Q)} \right|^2 \sum_{i=1}^N \sum_{j>i}^N \frac{\sin QR_{ij}}{QR_{ij}} \quad (13.56)$$

For low concentrations this averaging is in first approximation also valid for particles with a size distribution. The probability to find a particle at the position \mathbf{R}_i is in average $\frac{N}{V} d\mathbf{R}_i$ and the probability to find at the same time another particle at position \mathbf{R}_j is $\frac{N}{V} d\mathbf{R}_i \frac{N}{V} d\mathbf{R}_j$. A deviation from this is considered by $P(R_{ij})$ so that the double sum in eq. 13.56 can be written as

$$\sum_{i=1}^N \sum_{j>i}^N \frac{\sin QR_{ij}}{QR_{ij}} = \iint_V \frac{\sin QR_{ij}}{QR_{ij}} P(R_{ij}) \frac{N^2}{V^2} d\mathbf{R}_i d\mathbf{R}_j. \quad (13.57)$$

For isotropic media the function $P(R_{ij})$ is independent of the indices i and j and only a function of the distance R . $P(R)$ has the property to converge for large distance against one. By the substitution $P(R) = 1 - (1 - P(R))$ one gets for the interference term

$$\Psi(Q) = \left| \overline{F(Q)} \right|^2 \frac{N^2}{V^2} \iint_V \frac{\sin QR_{ij}}{QR_{ij}} d\mathbf{R}_i d\mathbf{R}_j \quad (13.58)$$

$$- \left| \overline{F(Q)} \right|^2 \frac{N^2}{V^2} \iint_V \frac{\sin QR_{ij}}{QR_{ij}} (1 - P(R_{ij})) d\mathbf{R}_i d\mathbf{R}_j. \quad (13.59)$$

The first term can be interpreted as the scattering of a particle with the volume V and the average scattering contrast $\overline{F(Q)} \frac{N}{V}$. As the illuminated sample volume V is relatively large this contribution is practically zero for all experimental accessible scattering angles. As for isotropic media the integration over $d\mathbf{R}_i$ is independent from the integration over $d\mathbf{R}_j$ and $(1 - P(R))$ quickly converges against zero the first integration in the second term (neglecting side effects) can be written as

$$\int_0^\infty dR \frac{\sin QR}{QR} (1 - P(R)) \frac{N^2}{V^2} 4\pi R^2. \quad (13.60)$$

The second integration only yields an additional multiplication factor V . Finally one gets for the scattering intensity the expression

$$\overline{I(Q)} = N \left\{ \overline{|F(Q)|^2} - \left| \overline{F(Q)} \right|^2 \underbrace{\frac{N}{V} \int dR 4\pi R^2 (1 - P(R)) \frac{\sin QR}{QR}}_{\Upsilon(Q)} \right\} . \quad (13.61)$$

13.2.4.1. Formula from Prins and Zernicke. For radial symmetric identical scatterer the square of the average form factor $\left| \overline{F(Q)} \right|^2$ and the average of the squared form factor $\overline{|F(Q)|^2}$ are the same so that one gets for the expression from Prins and Zernicke [234] and from Debye und Mencke [42]

$$\overline{I(Q)} = N F^2(Q) \underbrace{\left\{ 1 - \frac{N}{V} \int dR 4\pi R^2 (1 - P(R)) \frac{\sin QR}{QR} \right\}}_{S(Q)=1-\Upsilon(Q)}. \quad (13.62)$$

The problem of applying eq. 13.61 or 13.62 is the evaluation of $P(R)$. $P(R)$ depends on the geometric arrangement of the scatterer. For liquid emulsions the geometric arrangement can be related to thermodynamic quantities like concentration, temperature and interaction potential between the particles. In a theory of Raman [178] the geometric arrangement is associated to the interaction potential $U(r)$ in a simple way. According to his theory the potential is the sum of two-body interactions and he finds for $P(r)$ the relation $P(r) = \exp(-U(r)/kT)$. For larger concentrations many-body interactions have to be taken into account. In a more general theory of Born and Green [23] three-body interactions are at least considered in first approximation and they find for the

scattering intensity the expression

$$\begin{aligned}\overline{I(Q)} &= N \left\{ \overline{|F(Q)|^2} + \left| \overline{F(Q)} \right|^2 \frac{\chi(Q)}{\frac{V}{N}(2\pi)^{-3/2} - \chi(Q)} \right\} \quad (13.63) \\ \text{with } \chi(Q) &= \sqrt{\frac{2}{\pi}} \int_0^\infty dr [e^{-U(r)/kT} - 1] r^2 \frac{\sin Qr}{Qr}.\end{aligned}$$

For a simple hard sphere model with the interaction potential $U(r) = \begin{cases} 0 & \text{for } r > 2R \\ \infty & \text{for } r \leq 2R \end{cases}$ the scattering intensity can be calculated analytically. According to Raman's theory Debye calculates for the scattering intensity of spheres with a radius R and a volume V_P the expression

$$\overline{I(Q)} = N K^2(QR) \left\{ 1 - 8N \frac{V_P}{V} K(2QR) \right\}, \quad (13.64)$$

whereby $K(x)$ is the scattering function of a sphere given in section 3.1.1. Following the theory of Born and Green one gets a similar expression

$$\overline{I(Q)} = N K^2(QR) \left\{ 1 + 8N \frac{V_P}{V} K(2QR) \right\}^{-1}. \quad (13.65)$$

Nonetheless both theories are only valid for monodisperse systems of scatterer. In practice, however, most scattering systems have a more or less pronounced size distribution and frequently also an additional variety of shapes. For such systems the interaction potential can not be expressed in a closed form. One has to introduce for each pair of particle type a separate potential [5] which complicates the analytical treatment a lot.

13.2.4.2. Isolate particles. A system of isolate particles is characterized by its property that the positions of a particle is not influenced by the positions of any other particle. For this case the interaction potential $U(r)$ and consequently also $\chi(Q)$ is identical zero. Therefore the expression for the scattering intensity simplifies to

$$I(Q) = N \overline{|F(Q)|^2} \quad \text{or} \quad I(\mathbf{Q}) = \sum_{i=1}^N |F_i(\mathbf{Q})|^2. \quad (13.66)$$

The total scattering intensity is simply the sum of the intensity scattered by the individual particles. A system of non-interacting particles can be realized by diluting the system. Thereby the average distance between the particles is increased and the interaction potential becomes negligible small and does not influence anymore the arrangement of the scatterer.

13.2.4.3. Polydisperse System of isolated particles. The scattering function of a polydisperse system of isolated particles is determined by the shape and size of the particles. In a scattering experiment it is not possible to separate both quantities at the same time. In the data analysis one has to assume one of the quantities either the shape or the size distribution to get then the other quantity. Normally the shape is assumed whereby the size distribution is than obtained from the scattering curve [67, 220]. To get an information about the shape one normally need a system of identical particles, like for example a system of identical proteins. An additional but known size distribution would smear out the scattering too much to still allow to extract confidential information about

the particle shape. The scattering intensity of a system of isolated particles of different shape and a size distribution is given by

$$I(\mathbf{Q}) = \sum_{\mu=1}^M \int N_{\mu}(R) \overline{|F_{\mu}(Q, R)|^2} dR, \quad (13.67)$$

whereby $N_{\mu}(R)$ is the size distribution of particles of type μ and $F_{\mu}(Q, R)$ its form factor. Form factor of simple objects are listed in chapter 3 and frequently used size distributions in chapter 6.

13.2.5. Influence of $N(R)$ and $F(Q, R)$ on interparticle interferences. Interparticle interferences in small angle scattering signals have been described in eq. 13.61 and 13.62 by

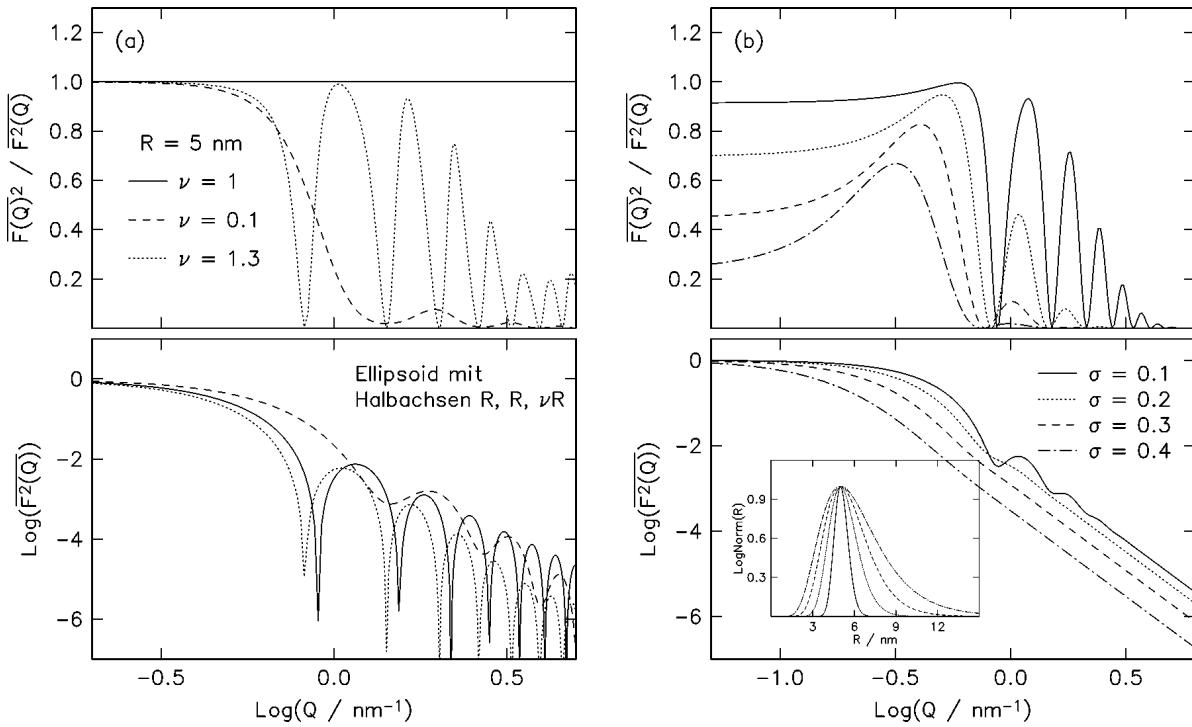
$$\overline{I(Q)} = N \overline{|F(Q)|^2} \left\{ 1 + \frac{\overline{|F(Q)|^2}}{\overline{|F(Q)|^2}} (S(Q) - 1) \right\} \quad (13.68)$$

Hereby $S(Q)$ describes the influence of the relative arrangement of the scatterers on $\overline{I(Q)}$ and can be calculated by the interaction potential between the scattering particles (eq. 13.63). Interparticle interferences depends next to the relative arrangement also on the square of the average form factor $\overline{|F(Q)|^2}$. Only for the case of identical and spherical symmetric scatterer the relation $\overline{|F(Q)|^2} \equiv |\overline{F(Q)}|^2$ is valid. In this case the interparticle interferences only depends on $S(Q)$. For particles with an irregular shape, however, the two averages are different and depends both on the size distribution as well as on the particle shape and their orientation distribution. For such systems the condition for the derivation the relation from Prins and Zernicke 13.61 namely the independency of orientation and size for higher concentrations is not anymore fulfilled. This condition allowed in eq. 13.56 to separate the averaging of the particle positions from the averaging of the form factor. Nevertheless the principle influence of a irregular particle shape and a size distribution can be made clear by the means of eq. 13.68. For judging their influence the ratio $\overline{|F(Q)|^2}/|\overline{F(Q)}|^2$ is considered.

Taking for example a system of identical but random oriented rotational ellipsoids with the half axis R , R , νR the ratio of the two averages is shown in figure 13.3 together with the scattering intensity $|\overline{F(Q)}|^2$ of the corresponding ideal dilutes system. The figure shows that the ratio decreases for larger scattering vectors Q the more the shaper of the rotational ellipsoid is different from the ideal spherical case ($\nu = 1$). An irregular particle shape already is sufficient to reduce the influence interparticle interferences.

Accordingly also the particle size distribution give rise to a difference in the averages $|\overline{F(Q)}|^2$ and $\overline{|F(Q)|^2}$. Figure 13.3b shows the ratio of the two averages of spherical scatterers with a lognormal size distribution of different width as a function of Q . For $Q = 0$ the ratio is given by $|\overline{F(Q)}|^2/\overline{|F(Q)|^2} = \exp(-9\sigma^2)$. One can see that the width of the size distribution strongly reduces the interference effect [213].

In general one can say that each kind of disorder reduces interferences. The disorder can have its origin in a random arrangement of particles (dilute systems), in an orientation distribution of irregular shaped particles, or also in a size distribution.



(a) The influence of irregular particle shapes on $\overline{|F(Q)|^2}/\overline{|F(Q)|^2}$ is shown for an example of random orientated rotational ellipsoids with the half axis $R, R, \nu R$ for $\nu = 1, 0.1$, and 1.3.

(b) Influence of a particle size distribution $\text{LogNorm}(R) = \exp(-(ln \mu - ln R)^2/(2\sigma^2))$ of the widths $\sigma = 0.1, 0.2, 0.3, 0.4$ for $\mu = 5 \text{ nm}$ on $\overline{|F(Q)|^2}/\overline{|F(Q)|^2}$.

FIGURE 13.3. For identical and spherical symmetric scatterers the ratio $\overline{|F(Q)|^2}/\overline{|F(Q)|^2} \equiv 1$. Interparticle interferences depend then only on the relative positions of the scatterers. Size distribution and irregular shapes reduce these interference effects.

13.2.6. Scattering laws and structural parameter. In this section a series of useful scattering laws are presented, which are useful for a simple analysis of experimental data and which allow an easy determination of structural parameters. [73].

13.2.6.1. Porod volume. The ratio of the scattering intensity in forward direction $I(Q = 0) = N V_P^2 \Delta \eta^2$ and the scattering invariant $\tilde{Q} = 2\pi^2 \Delta \eta^2 V f_p(1 - f_p)$ can be used to determine the particle volume V_P (V describes the illuminated sample volume). $f_p V = N V_P$ corresponds to the total volume of all scatterers whereby the particle volume V_P can be calculated by

$$\frac{V_P}{1 - f_p} \stackrel{f_p \ll 1}{\approx} V_P = 2\pi^2 \frac{I(Q = 0)}{\tilde{Q}}. \quad (13.69)$$

The measurement of the scattering curve in relative units is therefore sufficient to determine the volume of a homogeneous scatterer. Sources of errors for this way of determination of particle sizes are the extrapolation into forward direction und more important

the extrapolation to large scattering angles (Q^{-4} -extrapolation). Furthermore for large volume fractions f_p the particle volume has to be eventually corrected by a prefactor $1/(1 - f_p)$ which is not always known.

13.2.6.2. Radius of gyration and Guinier approximation. The scattering intensity for small angles can be approximated in a series expansion by replacing the expression $\frac{\sin Qr}{Qr}$ in eq. 13.40 by a McLaurin series which leads to

$$I(Q) = V \int 4\pi r^2 \gamma(r) \left[1 - \frac{Q^2 r^2}{3!} + \frac{Q^4 r^4}{5!} - \dots \right] dr , \quad (13.70)$$

i.e. $I(Q)$ is expanded by moments \bar{r}^n of $\gamma(r)$. The first term corresponds to the scattering intensity of $Q = 0$. For the second term Guinier and Fournet [73] have shown, that it can be related to the gyration radius of the scattering length density R_G by

$$V \int 4\pi r^2 \frac{Q^2 r^2}{3!} \gamma(r) dr = I(0) \frac{Q^2 R_G^2}{3} \Rightarrow I(Q) = I(0) \left(1 - \frac{Q^2 R_G^2}{3} \right) \quad (13.71)$$

with $R_G^2 = \int \eta(r) r^2 dr / \int \eta(r) dr$. Up to the term of Q^4 this series expansion at the beginning of the scattering curve is identical to the series expansion of an exponential function which then leads to the well known Guinier approximation

$$I(Q) = I(0) e^{-Q^2 R_G^2 / 3} . \quad (13.72)$$

The Guinier law is valid for any particle shape which is roughly isodiametric. For flat or elongated structures the Guinier law has to be corrected slightly [50, 73]. The radius of gyration for a sphere with radius R is given by $R_G = \sqrt{3/5} R$. The Guinier law is valid in the scattering vector interval $0 < Q < 1/R_G$.

13.2.6.3. Correlation length. Another characteristic quantity, which can be easily extracted from the scattering curve $I(Q)$ is the correlation length l_c . The correlation length is defined as the average width of the correlation function $\gamma(r)$:

$$l_c = \frac{2}{\gamma(0)} \int \gamma(r) dr . \quad (13.73)$$

Together with the definition of the scattering invariant \tilde{Q} in eq. 13.49 and the relation between $I(Q)$ and $\gamma(r)$ in eq. 13.44 this results to

$$\begin{aligned} l_c &= \frac{2}{\frac{1}{2\pi^2 V} \int_0^\infty Q^2 I(Q) dQ} \int \frac{1}{2\pi^2 V} \int_0^\infty Q^2 I(Q) \frac{\sin Qr}{Qr} dQ dr \\ &= \frac{2}{\int_0^\infty Q^2 I(Q) dQ} \int_0^\infty Q^2 I(Q) \int_0^\infty \frac{\sin Qr}{Qr} dr dQ \\ &= \frac{2}{\int_0^\infty Q^2 I(Q) dQ} \int_0^\infty Q^2 I(Q) \frac{\pi}{2Q} dQ \\ &= \pi \frac{\int Q I(Q) dQ}{\int Q^2 I(Q) dQ} . \end{aligned} \quad (13.74)$$

A sphere with radius R has therefore a correlation length of $l_c = \frac{3}{2} R$.

13.2.6.4. Porod law and specific surfaces. The Porod law describes the scattering behavior at large Q values. As the scattering intensity $I(Q)$ and the correlation function $\gamma(r)$ are related via the Fourier transformation the intensity $I(Q)$ is determined at large values of Q mainly by $\gamma(r)$ at small r . For small r the correlation function $\gamma(r)$ can be expanded in a Taylor series and one gets according to Guinier and Fournet [73]

$$\frac{\gamma(r)}{\gamma(0)} = 1 - \frac{1}{4} \frac{S}{V} + \dots , \quad (13.75)$$

whereby S is the total surface of all scatterer in the illuminated sample volume V . Eq. 13.75 together with 13.40 result for large Q values into the Porod law

$$I(Q) \longrightarrow \Delta\eta^2 \frac{2\pi S}{Q^4}. \quad (13.76)$$

In case of a system of N identical spheres with radii R a scattering contrast $\Delta\eta$ this law can be easily verified. The scattering intensity of N spherical particle is given by

$$I(Q) = N \left[\Delta\eta^2 \frac{4}{3} \pi R^3 \frac{\sin QR - QR \cos QR}{(QR)^3} \right]^2 \quad (13.77)$$

in the limit $Q \rightarrow \infty$ we get

$$\lim_{Q \rightarrow \infty} I(Q)(QR)^4 = N \left[\Delta\eta^2 \frac{4}{3} \pi R^3 \frac{\sin QR - QR \cos QR}{(QR)^3} \right]^2 (QR)^4 \quad (13.78)$$

$$\rightarrow N [\Delta\eta^2 4\pi R^3]^2 \cos^2 QR \quad (13.79)$$

Due to resolution limits the oscillations normally smear out in average so that

$$I(Q) \rightarrow N \frac{[\Delta\eta^2 4\pi R^3]^2}{2(QR)^4} = N \Delta\eta^2 \frac{2\pi A_{sp}}{Q^4} = \Delta\eta^2 \frac{2\pi S}{Q^4} \quad (13.80)$$

where $A_{sp} = 4\pi R^2$ is the area of the sphere's surface and $NA_{sp} = S$ the total surface of all spheres in the beam. By scaling the scattering intensity on \tilde{Q} one obtains an expression for the specific surface S/V of

$$\lim_{Q \rightarrow \infty} \pi \frac{I(Q)}{\tilde{Q}} Q^4 f_p (1 - f_p) = \frac{S}{V}. \quad (13.81)$$

The Porod law can be applied to all systems having sharp interfaces.

CHAPTER 14

Experimental Setup of a Small Angle Scattering Instrument

14.1. SANS-Camera

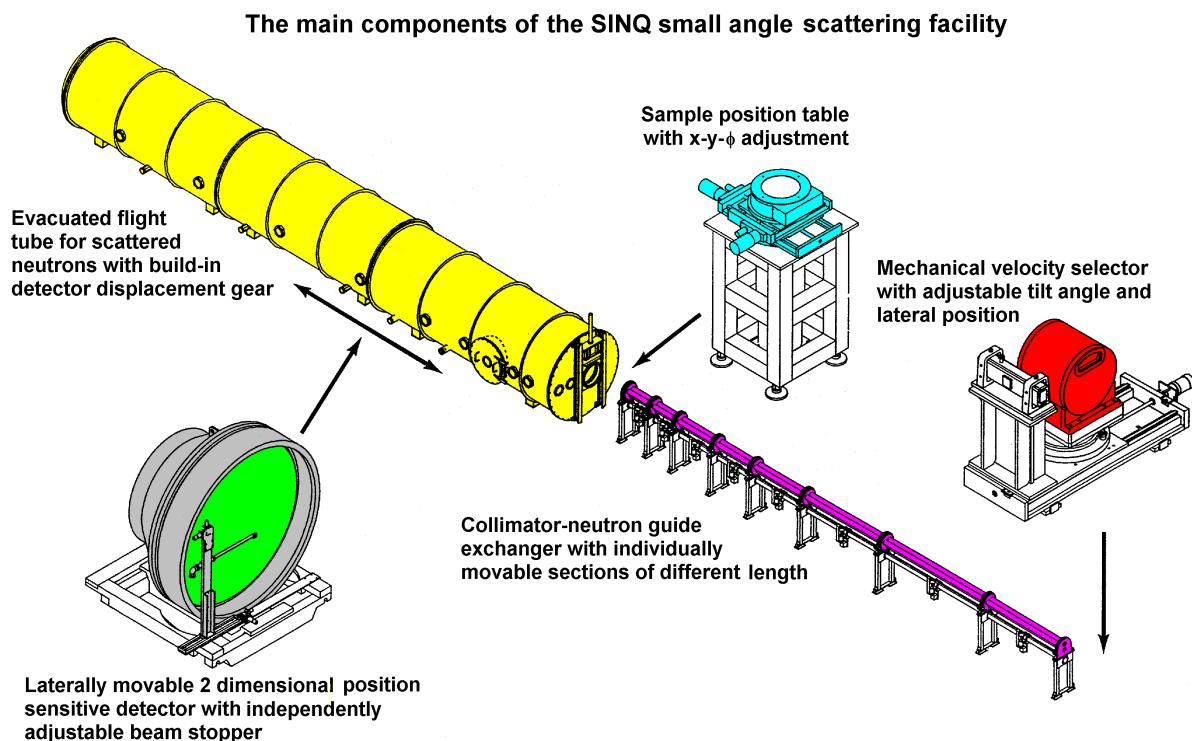


FIGURE 14.1. SANS-1 instrument at PSI, Switzerland

14.2. SAXS-Camera

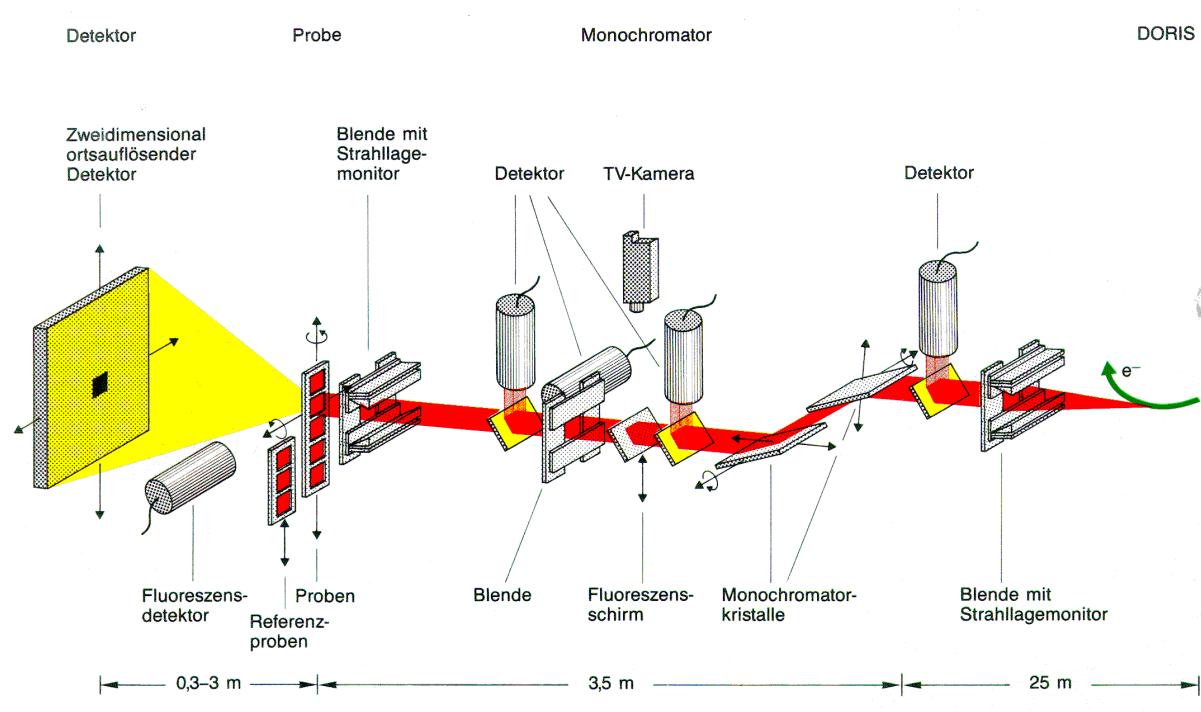


FIGURE 14.2. Small angle x-ray scattering instrument Jusifa at the synchrotron light source HASYLAB in Hamburg

CHAPTER 15

Data Reduction in SAS

15.1. Correction and Normalization of SANS-Raw data

A detector of a small angle scattering camera measures the superposition of intensities of different origin:

- (1) Background noise I_B
- (2) Scattering of the empty sample holder I_H
- (3) scattering of the isolated sample I_S

Furthermore the detector elements can have different efficiencies ϵ_i . To determine the differential cross-section of the sample all the different contribution to the total scattering intensity have to be considered and determined separately by different measurements. The quantity to be known is the scattering contribution of the isolated sample I_S , which in general can not be measured directly. Experimental accessible scattering contributions are the scattering of the sample in the sample holder I_{S+H} , the contribution of the empty sample holder I_H and the background noise I_B . From this experimental accessible data the wanted scattering contribution of the isolated sample has to be determined.

15.1.1. Contribution of the isolated sample. The intensity of the incident beam will be attenuated by absorption and scattering effects within the sample. Also the scattered neutrons will be further attenuated on their residual path through the sample. The measured intensity in a detector element i is given by

$$I_{S,i}^0 = \int_0^d dx \Phi_0 \Delta\Omega_i \epsilon_i e^{-\alpha x} A \rho e^{-\frac{\alpha(d-x)}{\cos\theta}} \left\{ \frac{d\sigma_{coh}}{d\Omega} + \frac{d\sigma_{inc}}{d\Omega} \right\}. \quad (15.1)$$

Hereby d describes the sample thickness in cm, Φ_0 the incident neutron flux in neutrons per $\text{cm}^2 \times \text{sec}$, $d\Omega_i$ is the solid angle of the detector element i in steradian, ϵ_i the detection efficiency of detector element i and α the extinction coefficient of the sample in cm^{-1} . A the illuminated sample area in cm^2 . θ describes the angle between the wave vector \mathbf{k}_0 of the incident neutrons and the wave vector \mathbf{k} of the scattered neutrons. $\frac{d\sigma_{coh}}{d\Omega}$ and $\frac{d\sigma_{inc}}{d\Omega}$ are the coherent and incoherent differential cross-sections, respectively. For small angle scattering $\cos\theta \simeq 1$ and one yields after integration

$$I_{S,i}^0 = \Phi_0 \Delta\Omega_i \epsilon_i \underbrace{A \rho d}_{N_S} \underbrace{e^{-\alpha d}}_{T_S} \left\{ \frac{d\sigma_{coh}}{d\Omega} + \frac{d\sigma_{inc}}{d\Omega} \right\}. \quad (15.2)$$

The quantity $N_S = A \rho d$ corresponds to the number of scattering atoms and $T_S = e^{-\alpha d} = \frac{I_{trans}}{I_0}$ to the transmissions coefficient, which can be determined from the ration of the intensity of the transmitted primary beam I_{trans} and the intensity of the incident

beam I_0 . The incoherent scattering is isotropically and equally distributed over the whole solid angle of 4π . Therefore one gets for $I_{S,i}^0$,

$$I_{S,i}^0 = \Phi_0 \Delta\Omega_i \epsilon_i \left\{ N_S T_S \frac{d\sigma_{coh}}{d\Omega} + A \frac{1 - T_S}{4\pi} \right\}. \quad (15.3)$$

15.1.2. Correction for sample holder and background noise. The scattering intensity of an isolated sample can practically never measured directly. The scattering of the sample is always superposed by scattering from the sample holder and by background noise. Background noise is meant to be electronic noise, cosmic radiation, and detection of neutrons, which have not passed through the sample like scattered neutrons from neighboring experiments. Because of this reasons additional measurements have to be carried out, which are a measurement of the empty sample holder I_H and a measurement with a strong absorber like Cadmium in front of the sample to measure the background noise I_B . Together with the measurement of the sample in the sample holder $I_{S+H,i}$ the scattering of the isolated sample $I_{S,i}^0$ on the detector element i can be calculated by

$$I_{S,i}^0 = \left(\frac{I_{S+H,i}}{M_{S+H}} - \frac{I_{B,i}}{M_B} \right) - \frac{T_{S+H}}{T_H} \left(\frac{I_{H,i}}{M_H} - \frac{I_{B,i}}{M_B} \right). \quad (15.4)$$

The index B stands for background, H for the empty sample holder, $S+H$ for the sample in the sample holder and S for the scattering of the isolated sample. All intensities have to be normalized on the number of incident neutrons. This can be done for example by division of the measured intensity my a monitor count rate M . The factor $\frac{T_{S+H}}{T_H}$ takes account for the attenuation of the beam by the sample.

The differential cross-section in eq. 15.3 can now be calculated from the measurable intensities in eq. 15.4. Nevertheless the quantities Φ_0 , $\Delta\Omega_i$ and ϵ_i still have to be known. Furthermore I_S^0 is not given in physical standard units but in units per monitor count. All this can be overcome by a comparison with a standard substance St . Common standard materials are in general materials with a small coherent cross-section and a large incoherent cross-section like vanadium or water. For both of these materials the coherent cross-section is negligible small. The scattering intensities of the standard materials have to be corrected in the same way than the sample itself according to eq. 15.4. The ratio of both intensities $I_{S,i}^0/I_{St,i}^0$ leads to

$$\frac{I_{S,i}^0}{I_{St,i}^0} = \frac{T_S N_S \left(\frac{d\sigma_{coh}}{d\Omega} + \frac{d\sigma_{inc}}{d\Omega} \right)}{T_{St} N_{St} \frac{d\sigma_{inc}^{St}}{d\Omega}} \quad (15.5)$$

$$\Leftrightarrow \frac{d\sigma_{coh}}{d\Omega} + \frac{d\sigma_{inc}}{d\Omega} = \frac{I_{S,i}^0}{I_{St,i}^0} \frac{T_{St} N_{St}}{T_S N_S} \frac{d\sigma_{inc}^{St}}{d\Omega} \quad (15.6)$$

$$\text{or } \frac{d\sigma_{coh}}{d\Omega} + \frac{d\sigma_{inc}}{d\Omega} = \frac{I_{S,i}^0}{I_{St,i}^0} \frac{(1 - T_{St}) \frac{A_{St}}{4\pi}}{T_S N_S}. \quad (15.7)$$

If water is used as a standard the last formula has to be multiplied on the right side with an empirical factor $f(\lambda, \sigma_t, T) \simeq 1$. This factor corrects for the different efficiencies

of the detector for different neutron energies. This correction can become important in case of water because of its inelastic scattering behavior.

15.2. Correction and normalization of SAXS raw data

History

2018-xx-xx: SASfit 0.94.10

- The form factors "EllipsoidalCoreShell", "Ellipsoid i", "Ellipsoid ii", and "triaxEllShell1" have been disabled and replaced by a series of plug-in functions. Those replacements make use of a faster routine for multidimensional integrals (pcubature). Because of this an optional size distribution became part of the supplied plug-in form factors.
- bug fix of unit conversion via clipboard
- added a plug-in for a random flight structure factor
- bug fix in the algorithm for calculating the resolution parameter in case of averaging neighboring data points
- plug-in implementation of several variants of helices
- plug-in for rectangular parallelepipeds have been extended so that it can have a size distribution of one, two or all three axis. The multiple integration is done by using the pcubature code from Steven G. Johnson
- multi dimensional integration package "cubature" from Steven G. Johnson becomes generally available also in plug-ins (cubature)

2017-08-16: SASfit 0.94.9

- changed scaling of SESANS correlation functions by factor $1/(2\pi)^2$
- unit conversion for reading ASCII files corrected and extended
- update to FFTW 3.3.6-pl2 did not work, went back to 3.3.5
- update to gsl 2.4
- bug fix: resolution bar was not plotted properly
- Uncertainties on x and y axis are now plotted symmetrically from $x \pm \Delta x$ and $y \pm \Delta y$. Before they were plotted from $x \pm 0.5\Delta x$ and $y \pm 0.5\Delta y$.

2016-12-16: SASfit 0.94.8

- Moved all peak functions into the plugin area.
- Added the possibility to apply after the summation of all scattering contributions a final operator. At the moment between three operator can be chosen:
 - (1) Unit operator which does not do anything with the data
 - (2) Taking the Hankel transform $2\pi H[I(Q)]$ for converting SAS model into a projected correlation function $G(\delta)$, which defines the SESANS signal
 - (3) Applying the operator from 2) and taking the exponential function of it $\exp(2\pi H[I(Q)])$ to get SESANS signal
- For the extension to SESANS also an option for reading SESANS data has been supplied. The format has been defined by TU Delft and has the default extension *.ses.
- added a model for of a self-fine random density distribution (gDAB) both under non-particular structures as well as under SESANS as a correlation function.
- added a SESANS correlation function for a generalized Gaussian chain for Flory exponents between nu in (0,1/2)
- updated to FFTW 3.3.5

- updated to sundials 2.7.0
- updated to gsl 2.3
- supplying some alternative integration routines for integrating over the size distribution
- bug fix of sasfit_integrate routine, which caused a crash when called from plugin functions
- first minimal routine for reading simple ascii data from ALV-5000 with single correlation function for DLS-analysis
- adding the PLHNC and RMSA closure to the OZ solver
- fixed a bug in the scaling parameter of the Teubner-Strey model
- bugfix in routine for reading data in BerSANS format

2016-04-25: SASfit 0.94.7

- implementation of another cumulant formula for DLS
- bug-fix in the unit conversion routine
- implementation of GMRES, Bi-CGStab, TFQMR and Andersen Acceleration for solving efficiently the Ornstein Zernicke fixpoint problem
- bug fix: since version 0.94.4 the new interrupt option suppressed a proper error reporting to the GUI for undefined input values
- removed some old structure factors, which were rarely used and theoretically not up to date
- a first version of a plugin for ordered mesoscopic and nano structures. The plugin is providing a part of the structure factors available in the software package "scatter" from S. Frster.
- plugin of a radial profile for a sphere resulting in a Porod law both below and above Q^{-4} (Boucher Sphere)

2014-12-14: SASfit 0.94.6

- improved analysis of confidence intervals of fit parameters:
- clickable correlation coefficients in the matrix highlight associated pair of fit parameters
- covariance matrix elements are highlighted according to their correlation coefficient
- highlighting and selection of correlated params improved

2014-10-03: SASfit 0.94.5

- storing intensity and size distribution in batch processing routine
- changed the width of the error bar, assuming that the supplied error is FWHM
- changed the internal procedure for plotting error bars
- resolution parameter can now also be plotted
- changed the order of plotting the fit results in "integral structural parameters"
- interruption of batch processing or series analysis implemented

2014-09-03: SASfit 0.94.4

- a new interrupt button can stop now the intensity evaluation of fitting procedure after each q-value. Before the whole scattering curve needed to be calculated.
- progress bar has been added to the GUI

- bug fix in plotting error bars of multiple data sets.
- implementation of another method for thinning out oversampled data sets.
The new methods is performing an averaging of data points depending on a user-defined maximum allowed q-smearing and a user-defined maximum distance in intensity in units of the error bar of the data points, i.e. an averaging is only performed, if the intensities look similar with n-times the intensity error bars.
- bug fix in GUI if one wants to forget old data and load a new data sets.

2014-07-02: SASfit 0.94.3

- bug fix in the plugin for parallel epiped
- spelling errors in the menu interface
- in case of slow convergence the OZ solver can be interrupted now

2014-02-06: SASfit 0.94.2

- new binding for zoom option, which works better for a mouse with one button (plot zoom by ctrl+left-mouse btn)
- removed <delete> key binding for scattering contributions
- Added several iteration schemes for finding the fixpoint of the OZ equation.
- Added a tab for the total correlation function $h(r)$
- Added also some root finding algorithms for solving OZ equation.
- resolution parameter from file was not scaled during the change of units for Q
- bug fix in calculation of χ_i for sq for wormlike structures (PS1 model)
- wrong label for contrast in Teubner-Strey model
- bug fix in SPHERE+R $\hat{=}$ a_Nagg form factor in assignement of core volume
- data export format set to scientific notation in the form x.yyyE?zz only

2014-02-05: SASfit 0.94.1

- added missing parameter label for Hamaker constant in DLVO potential
- Penetrable Sphere Model was not properly assigned.
- added all available plugins into the distribution binaries

2014-01-20: SASfit 0.94.0

- A new interface for solving the Ornstein Zernike equation for different closure relations and potentials (OZ-solver) has been added. The solutions of the OZ-solver can be used as a structure factor as a spline function without the possibility to fit a parameter of the potential at the moment.
- manual has been extended for the ferrofluid plugin scattering functions
- orientation of the 2D simulation was rotated by 90deg.
- correction of the manual for Porod's approximation of cylinders.
- minor bug fix for Porod's approximation of long cylinders.

2013-06-09: SASfit 0.93.5

- implementation of a plugin for a form factor and correlation function for spin misalignment
- bug fix for the scattering contribution of the individual chains in the form factors WORM*, ROD*, and DISC*
- bug fix in the form factor MagneticShellCrossTerm
- added some subfolders for form factor plugins
- bug fix in the calculation of the scattering length density SLD(E) of x-rays

2013-04-30: SASfit 0.93.4

- In the menu for confidence interval the non-diagonal elements of correlation matrix are shown together with the confidence interval for the fitted parameters as diagonal elements.
- added some additional parameters into the fit menu useful to evaluate the goodness of a fit
- new plugin for `Parallelepiped_abc` of dimension a^*b^*c to be found under [by `plugins|anisotropic obj`]
- new plugin for `generalized Guinier law` to be found under [by `plugins|non-particular structures`]
- Renamed HMI format into `BerSANS` format and did some debugging. Now also all masked data points (negative errors) are ignored.
- added key bindings `Home`, `End`, `PgUp`, `PgDn`, `Insert`, `Delete` for going to first-last-next-previous entry or to add and remove an entry
- adding plugin with a series of form factor for strongly anisotropic structures with local planar and local cylindrical shapes
- adding a new plugin form factor for spheres with fuzzy interfaces. The existing form factors `ExpShell`, `LinShell` and `LinShell2` have been moved to this plugin.
- New plugin of `FuzzySphere` and `CoreShellMicrogel`. Also the related functions for calculating the corresponding radial profiles have been added.
- new section in the manual about absolute scale, molecular weight, etc.
- for the form factors `flat cylinder` `long cylinder`, and `Porod cylinder` the limiting case $q \equiv 0$ is now treated properly
- small bug fix in the peak function `Gamma (Area)` for checking validity of parameter
- bug fix of the resolution parameter handling in case it will be read in from a data file
- included under peak function the `Maxwell` distribution and the `generalized Maxwell` distribution
- batch fitting

2011-05-04: SASfit 0.93.3

- bug fix in the model `Stacked Discs`. The structure factor describing the stacking order contained a bug.
- extended plugin for stroboscopic measurements, especially for `TISANE`
- plotting: ignore negative y values on `log()` and `sqrt()` scales
- included `Pcs_homogeneousCyl` form factor
- bug fix of `ferrofluid` plugin
- added radial averaged form factor in `ferrofluid` plugin
- extended the spline plugin to be used also as form factors. In case somebody wants to fit a spline function to e.g. a TEM size distribution, this function need to be available as a form factor and not only as a size distribution.
- added configuration file `config.ini` as replacement for deprecated `sasfit_init_public.tcl`
- added switch for disabling `About` popup at start time via config file

- added checkbox (ascii options) for ignoring zero(0) intensity at the beginning of data
- removed , as data column separator
- added substitution of , \Rightarrow . for data columns (german decimal format to english format)
- added interface function for covariance matrix output
- covar matrix visualisation with parameter highlighting
- added a chapter about particle number densities, volume fraction and absolute intensities in the manual.
- LogNorm_fp size distribution is now plugin function. The new plugin is not backwards compatible. The manual explains a bit the difficulties in describing the size distribution in terms of a volume fraction.
- added scrollbars for fit parameter window

2010-07-08: SASfit 0.93.2

- bug fix in ferrofluid plugin
- added radial averaged form factor. Included radial averaged form factors also for SAW model
- Extended the spline plugin to be used also as form factors. In case somebody wants to fit a spline function to a size distribution, this function needs to be available as a form factor and not only as a size distribution.
- forwarding intermediate linear Guinier approximation results to the plot window (green curve)
- display of linear Guinier approximation results in ISP text output window
- residuum window updated with linear Guinier approximation residuum
- by default disabled, see checkbox in ISP window
- fixed SLDcalculator in source package (missing data files)
- added KNOWN_BUGS.txt (not complete)
- optical (layout) GUI improvements:
 - removed thick margin around text boxes for ISP/analyt results
 - added resizable file list in ISP window
 - added resizable 'merge files' list when loading data files
- added menu->tools->toggle console to show the console, it is hidden by default now
- added OPTIM parameter to `src/CMakeLists.txt` for optimized binary generation on the underlying hardware, use: '`cmake -DOPTIM=TRUE`'
- added configuration file `config.ini` as (working) replacement for deprecated `sasfit_init_public.tcl`
- added switch for disabling 'about' popup at start time via config file
- added switch to set the default data directory
- added checkbox (ascii options) for ignoring zero(0) intensity at the beginning of data

2010-05-13: SASfit 0.93.1

removed obsolete print menu entries and fix of textual output bug

2010-05-05: SASfit 0.93.0

- copy&paste-able text output, as well as `csv` export (semicolon separated) for

- integral structural parameters (ISP) data
- parameters of contributions
- moments of size distribution
- improved/rewritten file selection GUI for ISP series fitting
- new and improved plugins: **Kratky Sphere**, **JuelichCoreShell** (rewritten), **Langevin**
- data is always plotted first, below the calculated lines
- error bars are drawn behind data points
- fixed wrong plotting of very large error bars
- for log-plotting on the y-axis, negative data is ignored (not plotted, was `abs()` before)
- fix to prevent the user from loading a **SASfit** project file as data
- fix in **Form-Factor Background** (improved numerical stability)
- bug fix for saving parameter files on windows
- bug fix in gui when selecting form factor **TwoAttachedSpheres**
- bug fix for "Singular Matrix" error

2010-01-02: SASfit 0.92.3

- implemented three different versions for worm like chains as described in Macromolecules 1996, 29, 7602-7612. They have been implemented as structure factors [`anisotropic obj|P'(Q):local cylindrical geometry`], so that it can be combined with different cross-section form factors of local cylindrical objects [`anisotropic obj.|Pcs(Q) for cylindrical obj.`].
- new form factor plugin for a sphere with 3 shells
- new structure factor for a regular cluster up to maximal 5 particle (tetrahedron like)
- correction of menu entry order for magnetic shell and superparamagnetic shell
- new plugin for ferrofluid particles with a scheme similar to the one of J.S. Pedersen for Gaussian Chains attached to a spherical particle
- bug fix in `mMemberedTwistedRing`
- two more default plot: **Guinier (rods)** and **Guinier (sheets)**
- bug fix in `loglogistic` peak
- implementation of asymptotic limits for fractals, which require a numerical integration. The integration often fails for large q-values for which an asymptotic solution is available or has been constructed.
- reprogrammed `SquareWell11` structure factor for a squared well potential
- replaced `sasfit_qromb` function by `sasfit_integrate` function in the form factor for the torus. The `sasfit_qromb` routine did not work for some unknown reason. This needs to be checked.
- new structure factor for a thin square well potential
- bug fix in `BeaucageExpPowLaw2`
- Bug fix for setting plot option `Holtzer` in multiple data set tab
- public initialization file `sasfit_init_public.tcl` for setting the default working directory by the user to any path

2009-10-08: SASfit 0.92.2

- data reduction without data loss, after loading a project file the data reduction can be reversed
- info message about a guessed error bar is displayed only once when the ascii options are changed (not for every file)
- added tooltip for complete filenames in merge window
- color for selected fit region stays at dark grey after loading an old project file
- removed unused help buttons in file open dialogs
- fixed error loop when adding new data to previously loaded project file
- fix for a homedir being e.g. 'U:át startup on Windows
- minor correction in the routine to guess the error bar (normalization) when only two column are supplied
- fix of rare error "form factor param out of range: -1"

13th of September: SASfit 0.92.1

- fix for GUI problem with two plugin form factors (contribution updates). If there are two plugin model functions with a different number of parameters, you can't switch/cycle through the contributions anymore (Next, Previous).
- fix for saving a parameter file (file creation was disabled by accident in previous version)

8.7.2009: SASfit 0.91.1: Since the previous version of SASfit (0.90.1, January 2009) there were a lot of changes to primarily improve the quality and portability of the code. Here is only a short summary of the larger changes done:

- added detailed documentation on setup and installation of SASfit , as well as plugin development (how to add own model functions)
- fixed some bugs in plugin framework
- added automatic determination of available plugins at build time
- enabled static building for plugins
- increasing maximum number of model parameter in GUI
- verified build compatibility for MacOS
- Extended and improved Scattering Length Density (SLD) Calculator. Now the scattering length density for x-ray energies between 1keV and 24.9 keV can be calculated.
- a few new form factor have been included: generalized Gaussian coil, generalized Gaussian coil 1, generalized Gaussian coil 2, ellCylShell1, ellCylShell2

12.01.2009: SASfit 0.90.1 : bug-fix in plugin-GUI

5.01.2009: SASfit 0.90: new release including full source code and binaries for windows and linux. Since the previous version of v (0.87, March 2008) there were a lot of changes to primarily improve the quality and portability of the code. Here only a short summary of the 'big' changes done (as of Dec. 17th, 2008):

- (1) Structured the source code into the modules `sasfit_common`, `sasfit_sd`, `sasfit_sq`, `sasfit_ff`, `sasfit_core`.
- (2) Switched to CMake build environment for platform independence. Build and tested SASfit on Linux and Windows, 32bit as well as 64bit.

- (3) Replaced intensive string comparisons for model function selection in each iteration step by more reasonable selection of functions pointers at initialization time and direct call of the according function at iteration time.
- (4) Fixed a lot of bugs and typos in the GUI (but there are still some)
- (5) Added flexible plugin system for external model functions. This way, all model functions can be provided as plugins and though move out of the core algorithms. Also enables easy customization. In the future the modules `sasfit_sd`, `sasfit_sq`, `sasfit_ff` and `sasfit_peaks` will be converted to external plugins.
- (6) Added `sasfit_peaks`, a new class of model functions containing peaks.
- (7) Added capability to ship **SASfit** as standalone executable, allows running on system without the need of external libraries (e.g. Tcl, BLT, ...)

4.03.2008: (**SASfit** version 0.87) The last modification in the menu navigation still had bugs. Hopefully they are removed in this version.

28.02.2008: (**SASfit** version 0.86) The menu navigation has been debugged and optimized. Corrected a bug in calculating the polydispersity index (PDI) in DLS cumulant analysis (PDI=Gamma2/Gamma1^2)

25.01.2008: (**SASfit** version 0.85) A bug for the form factors `ROD+R^-a*` has been corrected. The implementation of the scaling approximation, partial structure factors and local monodisperse approach has been improved.

9.01.2008: (**SASfit** version 0.84) Form factor for worm-like micelles (`WORM+Chains(RW)`, `WORM+Chains(RW)_Rc`, `WORM+Chains(RW)_Nagg`) and for cluster aggregates (`Mass Fractal (Exp(-x) Cut-Off)`, `Fisher-Burford`, `MassFractExp`, `MassFractGauss`, `Mass Fractal (Exp(-x^a) Cut-Off)`, `DLCAggregation`, `RLCAggregation`, `MassFractOverlappingSph` have been implemented. Furthermore a simple scheme for importing data from the clipboard has been implemented, which e.g. allows to copy/paste data from spread-sheets directly into **SASfit**. Corrected a bug for the form factors `ROD+R^-a*`

4.10.2007: (`SPHERE+Chains(RW)`, `SPHERE+Chains(RW)_Rc`, `SPHERE+Chains(RW)_Nagg`) (**SASfit** version 0.80) Next to the correction of some bugs a simulation option for multiple data sets has been implemented. Furthermore an option has been implemented to subtract a theoretical scattering contribution from the experimental data set, like e.g. a constant background signal. The format of the project files have up to now never been tested for compatibility. An attempt has been started to change this for the future versions.

20.8.2007: A couple of form factors for spherical, elliptical, cylindrical and very long rod-like micelles consisting of a homogeneous core and which are either grafted with Gaussian chains `*(RW)*`, or grafted with semi-flexible self-avoiding and interacting chains `*(SAW)*` or a corona with a power-law decaying profile $r^{-\alpha} * (R^-a)*$ have been implemented: `SPHERE+Chains(RW)`, `SPHERE+Chains(RW)_Rc`, `SPHERE+Chains(RW)_Nagg`, `SPHERE+Chains(SAW)`, `SPHERE+Chains(SAW)_Rc`, `SPHERE+Chains(SAW)_Nagg`, `SPHERE+R^-a`, `SPHERE+R^-a_Rc`, `SPHERE+R^-a_Nagg`, `ELL+Chains(RW)`, `ELL+Chains(RW)_Rc`,

`ELL+Chains(RW)_Nagg`, `CYL+Chains(RW)`, `CYL+Chains(RW)_Rc`,
`CYL+Chains(RW)_Nagg`, `ROD+Chains(RW)`, `ROD+Chains(RW)_Rc`,
`ROD+Chains(RW)_nagg`, `ROD+R^-a`, `ROD+R^-a_Rc`, `ROD+R^-a_nagg`

30.6.2007: Rudimental copy algorithm to copy plots or parameters into window-clipboard. Everything is copied in wmf-format and the option only works fine for information in non-scrolled widgets. Copy-option can be activated with right mouse button or double click of left mouse button. A triaxial ellipsoidal (`triaxEllShell`) shell with semiaxis a , b , c and shell thickness t is available.

4.6.2007: Implementation of form factors for cylindrical shells with circular cross-section and capped ends (`CylShell2`) and without capped ends (`CylShell1`) together with an approximation for very long cylindrical shells (`LongCylShell`)

27.3.2007: Implementation of form factors for bi-continuous systems (`TeubnerStrey` and `DAB`)

23.2.2007: Implementation of the structure factor for a system of charged, spheroidal objects in a dielectric medium according to the RMSA model of Hayter and Penfold

11.11.2006: Implementation of a semiflexible polymer according to Kholodenko, some form factors with plane geometry, which are intended to be used with lamellar structure factors (`homogenousXS`, `TwoInfinitelyThinPlates`, `LayeredCentroSymmetricXS`, `BiLayerGauss`), a sphere with Gaussian chains attached `SphereWithGaussChains` and a slightly different parametrised form factor named `BlockCopolymerMicelle`. An additional option for reading ASCII data files is now available, which allows to convert values for the scattering vector from nm^{-1} into \AA^{-1} and vice versa.

23.10.2006: Implementation of a polydisperse star `PolydisperseStar` and of flexible ring polymers `FlexibleRingPolymer` and `mMemberedTwistedRing`.

22.10.2006: Implementation of the form factor of a flexible polymer with Gaussian statistics (in different parameterisations `Gauss`, `Gauss2`, `Gauss3`) for a polydisperse flexible polymer with Gaussian statistics (`GaussPoly`) and a flexible ring of polymer with Gaussian statistics (`FlexiblePolymerRing`)

12.10.2006: Under the menu option `Calc/DLS...` next to a cumulant fit also a double stretched exponential decay can be selected to fit dynamic light scattering data.

5.10.2006: Implementation of a spherical shell with a diffuse (exponential) scattering length density profile inside the shell caused by solvent penetration into the shell

13.9.2006: Implementation of a form factor for spheres with gaussian chains attached.

19.7.2006: First release (current version: 0.71) at the moment only a windows version is available. The installation files for tcl/tk and blt are included in the distribution file.

Bibliography

- [1] R. P. Agarwal, D. O'Regan, and D. R. Sahu. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *Journal of Nonlinear and Convex Analysis*, 8(1):61–79, 2007. URL <http://www.ybook.co.jp/online2/jncav8-1.html>.
- [2] S. Alexander, P. M. Chaikin, P. Grant, G. J. Morales, P. Pincus, and D. Hone. Charge renormalization, osmotic pressure, and bulk modulus of colloidal crystals: Theory. *The Journal of Chemical Physics*, 80(11):5776–5781, jun 1984. doi: 10.1063/1.446600.
- [3] D. G. Anderson. Iterative procedures for nonlinear integral equations. *Journal of the ACM*, 12(4):547–560, oct 1965. doi: 10.1145/321296.321305.
- [4] R. Andersson, L. F. van Heijkamp, I. M. de Schepper, and W. G. Bouwman. Analysis of spin-echo small-angle neutron scattering measurements. *Journal of Applied Crystallography*, 41(5):868–885, sep 2008. doi: 10.1107/s0021889808026770.
- [5] N. W. Ashcroft and J. Lekner. Structure and resistivity of liquid metals. *Physical Review*, 145(1):83–90, may 1966. doi: 10.1103/physrev.145.83.
- [6] M. V. Avdeev, V. L. Aksenov, Z. Gazová, L. Almásy, V. I. Petrenko, H. Gojzewski, A. V. Feoktystov, K. Siposova, A. Antosova, M. Timko, and P. Kopcansky. On the determination of the helical structure parameters of amyloid protofilaments by small-angle neutron scattering and atomic force microscopy. *Journal of Applied Crystallography*, 46(1):224–233, jan 2013. doi: 10.1107/s0021889812050042.
- [7] P. Ballone, G. Pastore, G. Galli, and D. Gazzillo. Additive and non-additive hard sphere mixtures. *Molecular Physics*, 59(2):275–290, oct 1986. doi: 10.1080/00268978600102071.
- [8] B. Barboy. On a representation of the equation of state of fluids in terms of the adhesive hard-spheres model. *The Journal of Chemical Physics*, 61(8):3194–3196, oct 1974. doi: 10.1063/1.1682475.
- [9] R. Baxter. Percus-yevick equation for hard spheres with surface adhesion. *J. Chem. Phys.*, 49(6):2770–2774, 1968. doi: 10.1063/1.1673684.
- [10] G. Beaucage. Approximations leading to a unified exponential/power-law approach to small-angle scattering. *Journal of Applied Crystallography*, 28(6):717–728, dec 1995. doi: 10.1107/s0021889895005292.
- [11] G. Beaucage. Small-angle scattering from polymeric mass fractals of arbitrary mass-fractal dimension. *Journal of Applied Crystallography*, 29(2):134–146, apr 1996. doi: 10.1107/s0021889895011605.
- [12] C. Benham, G. Brady, and D. Fein. X-ray scattering from randomly oriented superhelices. circular superhelical DNA. *Biophysical Journal*, 29(3):351–366, mar 1980. doi: 10.1016/S0006-3495(80)85139-3.
- [13] H. Benoit. On the effect of branching and polydispersity on the angular distribution of the light scattered by gaussian coils. *Journal of Polymer Science*, 11(5):507–510, nov 1953. doi: 10.1002/pol.1953.120110512.
- [14] I. Berndt, J. S. Pedersen, P. Lindner, and W. Richtering. Influence of shell thickness and cross-link density on the structure of temperature-sensitive poly-n-isopropylacrylamide-poly-n-isopropylmethacrylamide core-shell microgels investigated by small-angle neutron scattering. *Langmuir*, 22(1):459–468, jan 2006. doi: 10.1021/la052463u.
- [15] I. Berndt, J. S. Pedersen, and W. Richtering. Structure of multiresponsive “intelligent” core-shell microgels. *Journal of the American Chemical Society*, 127(26):9372–9373, jul 2005. doi: 10.1021/ja051825h. PMID: 15984856.

- [16] I. Berndt, J. S. Pedersen, and W. Richtering. Temperature-sensitive core-shell micro-gel particles with dense shell. *Angewandte Chemie*, 118(11):1769–1773, mar 2006. doi: 10.1002/ange.200503888.
- [17] P. Bevington and D. K. Robinson. *Data Reduction and Error Analysis for the Physical Sciences*. McGraw-Hill, 3rd edition, 23 July 2002. URL <http://catalogs.mhhe.com/mhhe/viewProductDetails.do?isbn=0072472278>. ISBN-10: 0072472278, ISBN-13: 978-0072472271.
- [18] A. E. Blaurock. Evidence of bilayer structure and of membrane interactions from x-ray diffraction analysis. *Biochimica et Biophysica Acta (BBA) - Reviews on Biomembranes*, 650(4):167–207, may 1982. doi: 10.1016/0304-4157(82)90016-8.
- [19] C. Bohren and D. Huffman. *Absorption and Scattering of Light by Small Particles*. Wiley-VCH Verlag GmbH, apr 1998. doi: 10.1002/9783527618156.
- [20] J.-M. Bomont. *Recent Advances in the Field of Integral Equation Theories: Bridge Functions and Applications to Classical Fluids*, chapter Chapter 1, pages 1–84. John Wiley & Sons, Inc., apr 2008. doi: 10.1002/9780470259498.ch1.
- [21] J. M. Bomont and J. L. Bretonnet. A self-consistent integral equation: Bridge function and thermodynamic properties for the lennard-jones fluid. *The Journal of Chemical Physics*, 119(4):2188–2191, jul 2003. doi: 10.1063/1.1583675.
- [22] J.-M. Bomont and J.-L. Bretonnet. A consistent integral equation theory for hard spheres. *The Journal of Chemical Physics*, 121(3):1548–1552, jul 2004. doi: 10.1063/1.1764772.
- [23] M. Born and H. S. Green. A general kinetic theory of liquids. i. the molecular distribution functions. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 188(1012):10–18, dec 1946. doi: 10.1098/rspa.1946.0093.
- [24] M. Borowko, S. Sokolowski, and D. Henderson. Integral equations in the theory of simple fluids. In *Surfactant Science*, chapter 3, pages 135–165. CRC Press, feb 2000. doi: 10.1201/9781420030037.ch3.
- [25] B. Boucher, P. Chieux, P. Convert, and M. Tournarie. Small-angle neutron scattering determination of medium and long range order in the amorphous metallic alloy TbCu_{3.54}. *Journal of Physics F: Metal Physics*, 13(7):1339–1357, jul 1983. doi: 10.1088/0305-4608/13/7/006.
- [26] R. Bracewell. Strip integration in radio astronomy. *Australian Journal of Physics*, 9(2):198, 1956. doi: 10.1071/ph560198.
- [27] R. Bracewell. *The Projection-Slice Theorem*, pages 493–504. Springer US, Boston, MA, 2003. doi: 10.1007/978-1-4419-8963-5_14.
- [28] W. Burchard. Statistics of star-shaped molecules. i. stars with polydisperse side chains. *Macromolecules*, 7(6):835–841, nov 1974. doi: 10.1021/ma60042a027.
- [29] W. Burchard and K. Kajiwara. The statistics of stiff chain molecules. i. the particle scattering factor. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 316(1525):185–199, apr 1970. doi: 10.1098/rspa.1970.0074.
- [30] W. Burchard, E. Michel, and V. Trappe. Conformational properties of multiply twisted ring systems and daisy-like structures. *Macromolecules*, 29(18):5934–5939, jan 1996. doi: 10.1021/ma9603286.
- [31] C. Caccamo. Integral equation theory description of phase equilibria in classical fluids. *Physics Reports*, 274(1-2):1–105, sep 1996. doi: 10.1016/0370-1573(96)00011-7.
- [32] A. Caille. X-ray scattering in smectic a. *Comptes Rendus des Seances de l'Academie des Sciences, Serie B: Sciences Physiques*, 274:891–893, 1972.
- [33] I. Charpentier and N. Jakse. Exact numerical derivatives of the pair-correlation function of simple liquids using the tangent linear method. *The Journal of Chemical Physics*, 114(5):2284–2292, feb 2001. doi: 10.1063/1.1332808.
- [34] J. Chihara. Integral euqations for fluids with long-range and short-range potentials. *Progress of Theoretical Physics*, 50(2):409–423, aug 1973. doi: 10.1143/ptp.50.409.
- [35] N. Choudhury and S. K. Ghosh. Integral equation theory of lennard-jones fluids: A modified verlet bridge function approach. *The Journal of Chemical Physics*, 116(19):8517, 2002. doi: 10.1063/1.1467894.

- [36] R. Chugh, V. Kumar, and S. Kumar. Strong convergence of a new three step iterative scheme in banach spaces. *American Journal of Computational Mathematics*, 02(04):345–357, 2012. doi: 10.4236/ajcm.2012.24048.
- [37] F. H. C. Crick. The fourier transform of a coiled-coil. *Acta Crystallographica*, 6(8):685–689, sep 1953. doi: 10.1107/s0365110x53001952.
- [38] G. Damaschun and H.-V. Prschel. Rntgen-kleinwinkelstreuung von isotropen proben ohne fernordnung. i. allgemeine theorie. *Acta Crystallographica Section A*, 27(3):193–197, may 1971. doi: 10.1107/s0567739471000457.
- [39] G. Damaschun, H.-V. Prschel, and G. Sommer. Bestimmung der Gestalt gelöster Makromoleküle aus Röntgen-Kleinwinkeldiagrammen mit Hilfe von Gleichungssystemen. *Acta Crystallographica Section A*, 25(6):708, nov 1969. doi: 10.1107/s0567739469001586.
- [40] M. Daoud and J. Cotton. Star shaped polymers : a model for the conformation and its concentration dependence. *Journal de Physique*, 43(3):531–538, 1982. doi: 10.1051/jphys:01982004303053100.
- [41] W. I. F. David. Powder diffraction peak shapes. parameterization of the pseudo-voigt as a voigt function. *Journal of Applied Crystallography*, 19(1):63–64, feb 1986. doi: 10.1107/s0021889886089999.
- [42] P. Debye. Über die Zerstreuung von Röntgenstrahlen an amorphen Körpern. *Physik. Z.*, 28:135–141, 1927.
- [43] P. Debye. Mollecular-weight determination by light scattering. *Journal of Physical and Colloid Chemistry*, 51(1):18–32, jan 1947. doi: 10.1021/j150451a002.
- [44] P. Debye, H. R. Anderson, and H. Brumberger. Scattering by an inhomogeneous solid. ii. the correlation function and its application. *Journal of Applied Physics*, 28(6):679–683, jun 1957. doi: 10.1063/1.1722830.
- [45] P. Debye and A. M. Bueche. Scattering by an inhomogeneous solid. *Journal of Applied Physics*, 20(6):518–525, jun 1949. doi: 10.1063/1.1698419.
- [46] J. Dhont. *An Introduction to Dynamics of Colloids*. Studies in Interface Science. Elsevier Science, 1996. URL <https://books.google.ch/books?id=mmArTF5SJ9oC>.
- [47] W. D. Dozier, J. S. Huang, and L. J. Fetters. Colloidal nature of star polymer dilute and semidilute solutions. *Macromolecules*, 24(10):2810–2814, may 1991. doi: 10.1021/ma00010a026.
- [48] D.-M. Duh and A. D. J. Haymet. Integral equation theory for uncharged liquids: The lennard-jones fluid and the bridge function. *The Journal of Chemical Physics*, 103(7):2625–2633, aug 1995. doi: 10.1063/1.470724.
- [49] J. Dzubiella, A. Jusufi, C. N. Likos, C. von Ferber, H. Lwen, J. Stellbrink, J. Allgaier, D. Richter, A. B. Schofield, P. A. Smith, W. C. K. Poon, and P. N. Pusey. Phase separation in star-polymer-colloid mixtures. *Physical Review E*, 64(1):010401, jun 2001. doi: 10.1103/physreve.64.010401.
- [50] L. A. Feigin and D. I. Svergun. *Structure Analysis by Small-Angle X-ray and Neutron Scattering*. Plenum Press, 1987.
- [51] R. E. Franklin and A. Klug. The nature of the helical groove on the tobacco mosaic virus particle x-ray diffraction studies. *Biochimica et Biophysica Acta*, 19(Supplement C):403–416, jan 1956. doi: 10.1016/0006-3002(56)90463-2.
- [52] P. Fratzl. Statistical model of the habit and arrangement of mineral crystals in the collagen of bone. *Journal of Statistical Physics*, 77(1-2):125–143, 1994. doi: 10.1007/BF02186835.
- [53] R. W. Freund. A transpose-free quasi-minimal residual algorithm for non-hermitian linear systems. *SIAM Journal on Scientific Computing*, 14(2):470–482, mar 1993. doi: 10.1137/0914029.
- [54] T. Frhwirth, G. Fritz, N. Freiberger, and O. Glatter. Structure and order in lamellar phases determined by small-angle scattering. *Journal of Applied Crystallography*, 37(5):703–710, sep 2004. doi: 10.1107/s0021889804012956.
- [55] M. Frigo and S. G. Johnson. The fastest Fourier transform in the west. Technical Report MIT-LCS-TR-728, Massachusetts Institute of Technology, September 1997. URL <http://www.fftw.org/fftw-paper.pdf>.

- [56] M. Frigo and S. G. Johnson. The design and implementation of FFTW3. *Proceedings of the IEEE*, 93(2):216–231, 2005. doi: 10.1109/JPROC.2004.840301. Special issue on “Program Generation, Optimization, and Platform Adaptation”.
- [57] B. J. Friskin. Revisiting the method of cumulants for the analysis of dynamic light-scattering data. *Applied Optics*, 40(24):4087, aug 2001. doi: 10.1364/ao.40.004087.
- [58] S. Frster, L. Apostol, and W. Bras. Scatter: software for the analysis of nano- and mesoscale small-angle scattering. *Journal of Applied Crystallography*, 43(3):639–646, apr 2010. doi: 10.1107/s0021889810008289.
- [59] S. Frster, S. Fischer, K. Zielske, C. Schellbach, M. Sztucki, P. Lindner, and J. Perlich. Calculation of scattering-patterns of ordered nano- and mesoscale materials. *Advances in Colloid and Interface Science*, 163(1):53–83, mar 2011. doi: 10.1016/j.cis.2010.12.003.
- [60] S. Frster, N. Hermsdorf, C. Bttcher, and P. Lindner. Structure of polyelectrolyte block copolymer micelles. *Macromolecules*, 35(10):4096–4105, may 2002. doi: 10.1021/ma011565y.
- [61] S. Frster, N. Hermsdorf, W. Leube, H. Schnablegger, M. Regenbrecht, S. Akari, P. Lindner, and C. Bttcher. Fusion of charged block copolymer micelles into toroid networks. *The Journal of Physical Chemistry B*, 103(32):6657–6668, aug 1999. doi: 10.1021/jp990076l.
- [62] S. Frster, A. Timmann, M. Konrad, C. Schellbach, A. Meyer, S. S. Funari, P. Mulvaney, and R. Knott. Scattering curves of ordered mesoscopic materials. *The Journal of Physical Chemistry B*, 109(4):1347–1360, feb 2005. doi: 10.1021/jp0467494.
- [63] H. Fukuda, A. Goto, and T. Imae. Structure determination of helical fibers by numerical simulation for small-angle neutron scattering†. *Langmuir*, 18(19):7107–7114, sep 2002. doi: 10.1021/la025528r.
- [64] H. Furukawa. Dynamics-scaling theory for phase-separating unmixing mixtures: Growth rates of droplets and scaling properties of autocorrelation functions. *Physica A*, 123:497, 1984. doi: 10.1016/0378-4371(84)90168-7.
- [65] D. Gazzillo, A. Giacometti, R. G. D. Valle, E. Venuti, and F. Carsughi. A scaling approximation for structure factors in the integral equation theory of polydisperse nonionic colloidal fluids. *The Journal of Chemical Physics*, 111(16):7636–7645, oct 1999. doi: 10.1063/1.480089.
- [66] L. Giehm, C. L. P. Oliveira, G. Christiansen, J. S. Pedersen, and D. E. Otzen. SDS-induced fibrillation of α -synuclein: An alternative fibrillation pathway. *Journal of Molecular Biology*, 401(1):115–133, aug 2010. doi: 10.1016/j.jmb.2010.05.060.
- [67] O. Glatter. Data evaluation in small angle scattering: Calculation of the radial electron density distribution by means of indirect Fourier transformation. *Acta Physica Austriaca*, 47:83–102, 1977.
- [68] G. S. Greschner. Mathematical properties of polydisperse systems, 2. application to light scattering. *Die Makromolekulare Chemie*, 170(1):203–229, 1973. doi: 10.1002/macp.1973.021700117.
- [69] X. Gu and D. F. R. Mildner. Ultra-small-angle neutron scattering with azimuthal asymmetry. *Journal of Applied Crystallography*, 49(3):934–943, may 2016. doi: 10.1107/S1600576716005586.
- [70] X. Gu and D. F. R. Mildner. Determination of porosity in anisotropic fractal systems by neutron scattering. *Journal of Applied Crystallography*, 51(1):175–184, feb 2018. doi: 10.1107/S1600576718000080.
- [71] A. Guinier. La diffraction des rayons x aux très petits angles : application à l'étude de phénomènes ultramicroscopiques. *Annales de Physique*, 11(12):161–237, 1939. doi: 10.1051/annphys/19391120161.
- [72] A. Guinier. *X-ray Diffraction*. San Francisco: Freeman, 1963.
- [73] A. Guinier and G. Fournet. *Small-Angle Scattering of X-Rays*. John Wiley & Sons, Inc., 1955.
- [74] F. Gürsoy and V. Karakaya. A Picard-S hybrid type iteration method for solving a differential equation with retarded argument. *ArXiv e-prints*, Mar. 2014, 1403.2546. URL <https://arxiv.org/abs/1403.2546>.
- [75] W. C. Hamilton. Significance tests on the crystallographic r factor. *Acta Crystallographica*, 18(3):502–510, may 1965. doi: 10.1107/s0365110x65001081.
- [76] B. Hammouda. *SANS from homogeneous polymer mixtures: A unified overview*, pages 87–133. Springer-Verlag, Berlin, Heidelberg, 1993. doi: 10.1007/bfb0025862.

- [77] B. Hammouda. A new guinier–porod model. *Journal of Applied Crystallography*, 43(4):716–719, may 2010. doi: 10.1107/s0021889810015773.
- [78] B. Hammouda. Small-angle scattering from branched polymers. *Macromolecular Theory and Simulations*, 21(6):372–381, mar 2012. doi: 10.1002/mats.201100111.
- [79] B. Hammouda. Form factors for branched polymers with excluded volume. *Journal of Research of the National Institute of Standards and Technology*, 121:139, apr 2016. doi: 10.6028/jres.121.006.
- [80] B. Hammouda. the sans toolbox. /the_SANS_toolbox.pdf, 2016. URL http://www.ncnr.nist.gov/staff/hammouda/the_SANS_toolbox.pdf.
- [81] B. Hammouda, R. A. Bubeck, and D. F. Mildner. Macromolecular orientation in hot stretched and injection moulded polystyrene. *Polymer*, 27(3):393–397, mar 1986. doi: 10.1016/0032-3861(86)90155-2.
- [82] B. Hammouda, D. F. R. Mildner, R. A. Bubeck, and M. T. Malanga. The inclination angle of nonisotropic inhomogeneities determined by SANS. *Journal of Applied Crystallography*, 19(5):320–323, oct 1986. doi: 10.1107/S002188988608929X.
- [83] H. J. M. Hanley, C. D. Muzny, D. L. Ho, and C. J. Glinka. A small-angle neutron scattering study of a commercial organoclay dispersion. *Langmuir*, 19(14):5575–5580, jul 2003. doi: 10.1021/la026342v.
- [84] J.-P. Hansen and J. B. Hayter. A rescaled MSA structure factor for dilute charged colloidal dispersions. *Molecular Physics*, 46(3):651–656, jun 1982. doi: 10.1080/00268978200101471.
- [85] J.-P. Hansen and I. R. McDonald. *Theory of Simple Liquids with Applications to Soft Matter*. Academic Press, fourth edition, 2013. doi: 10.1016/B978-0-12-387032-2.00013-1.
- [86] J. B. Hayter and J. Penfold. An analytic structure factor for macroion solutions. *Molecular Physics*, 42(1):109–118, jan 1981. doi: 10.1080/00268978100100091.
- [87] J. B. Hayter and J. Penfold. Use of viscous shear alignment to study anisotropic micellar structure by small-angle neutron scattering. *The Journal of Physical Chemistry*, 88(20):4589–4593, sep 1984. doi: 10.1021/j150664a030.
- [88] A. C. Hindmarsh, P. N. Brown, K. E. Grant, S. L. Lee, R. Serban, D. E. Shumaker, and C. S. Woodward. SUNDIALS. *ACM Transactions on Mathematical Software*, 31(3):363–396, sep 2005. doi: 10.1145/1089014.1089020.
- [89] O. Hittmair. *Lehrbuch der Quantentheorie*. Verlag Karl Thiemicg, 1972.
- [90] R. P. Hjelm, C. Schteingart, A. F. Hofmann, and D. S. Sivia. Form and structure of self-assembling particles in monoolein-bile salt mixtures. *The Journal of Physical Chemistry*, 99(44):16395–16406, nov 1995. doi: 10.1021/j100044a030.
- [91] R. P. Hjelm, C. D. Schteingart, A. F. Hofmann, and P. Thiyagarajan. Structure of conjugated bile salt-fatty acid-monoglyceride mixed colloids: Studies by small-angle neutron scattering. *The Journal of Physical Chemistry B*, 104:197–211, 2000. doi: 10.1021/jp992157n.
- [92] R. P. Hjelm, P. Thiyagarajan, and H. Alkan-Onyuksel. Organization of phosphatidylcholine and bile salt in rodlike mixed micelles. *The Journal of Physical Chemistry*, 96(21):8653–8661, oct 1992. doi: 10.1021/j100200a080.
- [93] H. H. Homeier, S. Rast, and H. Krienke. Iterative solution of the ornstein-zernike equation with various closures using vector extrapolation. *Computer Physics Communications*, 92(2-3):188 – 202, dec 1995. doi: 10.1016/0010-4655(95)00116-0.
- [94] K. M. Hong and J. Noolandi. Non-equilibrium static form factor of a reptating chain. *Die Makromolekulare Chemie, Rapid Communications*, 4(9):617–621, sep 1983. doi: 10.1002/marc.1983.030040907.
- [95] R. Hosemann and S. Bagchi. *Direct Analysis of Diffraction by Matter*. Amsterdam: North-Holland., 1962.
- [96] M. Hunter, V. Backman, G. Popescu, M. Kalashnikov, C. W. Boone, A. Wax, V. Gopal, K. Badizadegan, G. D. Stoner, and M. S. Feld. Tissue self-affinity and polarized light scattering in the born approximation: A new model for precancer detection. *Physical Review Letters*, 97(13):138102, sep 2006. doi: 10.1103/physrevlett.97.138102.

- [97] A. J. Hurd and W. L. Flower. In situ growth and structure of fractal silica aggregates in a flame. *Journal of Colloid and Interface Science*, 122(1):178–192, mar 1988. doi: 10.1016/0021-9797(88)90301-3.
- [98] M. Huš, M. Zalar, and T. Urbic. Correctness of certain integral equation theories for core-softened fluids. *The Journal of Chemical Physics*, 138(22):224508, jun 2013. doi: 10.1063/1.4809744.
- [99] T. Ida, M. Ando, and H. Toraya. Extended pseudo-voigt function for approximating the voigt profile. *Journal of Applied Crystallography*, 33(6):1311–1316, dec 2000. doi: 10.1107/s0021889800010219.
- [100] S. Ishikawa. Fixed points by a new iteration method. *Proceedings of the American Mathematical Society*, 44(1):147–150, jan 1974. doi: 10.1090/s0002-9939-1974-0336469-5.
- [101] H. G. Jerrard. Theories of streaming double refraction. *Chemical Reviews*, 59(3):345–428, jun 1959. doi: 10.1021/cr50027a001.
- [102] S. G. Johnson. multi-dimensional adaptive integration (cubature) in C, 2017. URL <https://github.com/stevengj/cubature>.
- [103] J. E. Jones. On the determination of molecular fields. II. from the equation of state of a gas. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 106(738):463–477, oct 1924. doi: 10.1098/rspa.1924.0082.
- [104] A. Jusufi, J. Dzubiella, C. N. Likos, C. von Ferber, and H. Lwen. Effective interactions between star polymers and colloidal particles. *Journal of Physics: Condensed Matter*, 13(28):6177, jun 2001. doi: 10.1088/0953-8984/13/28/303.
- [105] I. Karahan and M. Ozdemir. Fixed point problems of the picard-mann hybrid iterative process for continuous functions on an arbitrary interval. *Fixed Point Theory and Applications*, 2013(1):244, 2013. doi: 10.1186/1687-1812-2013-244.
- [106] I. Karahan and M. Ozdemir. A general iterative method for approximation of fixed points and their applications. *Advances in Fixed Point Theory*, 3(3):510–526, 2013. URL <http://scik.org/index.php/afpt/article/view/1100/527>.
- [107] T. Kawaguchi. Radii of gyration and scattering functions of a torus and its derivatives. *Journal of Applied Crystallography*, 34(5):580–584, sep 2001. doi: 10.1107/s0021889801009517.
- [108] H. Kaya. Scattering from cylinders with globular end-caps. *Journal of Applied Crystallography*, 37(2):223–230, mar 2004. doi: 10.1107/s0021889804000020.
- [109] H. Kaya and N.-R. de Souza. Scattering from capped cylinders. addendum. *Journal of Applied Crystallography*, 37(3):508–509, may 2004. doi: 10.1107/s0021889804005709.
- [110] C. T. Kelley. *Solving Nonlinear Equations with Newton's Method*. Society for Industrial and Applied Mathematics, jan 2003. doi: 10.1137/1.9780898718898.
- [111] S. Khan. A picard-mann hybrid iterative process. *Fixed Point Theory and Applications*, 2013(1):69, 2013. doi: 10.1186/1687-1812-2013-69.
- [112] A. L. Kholodenko. Analytical calculation of the scattering function for polymers of arbitrary flexibility using the dirac propagator. *Macromolecules*, 26(16):4179–4183, aug 1993. doi: 10.1021/ma00068a017.
- [113] L. Klimeš. Correlation functions of random media. *Pure and Applied Geophysics*, 159(7-8):1811–1831, jul 2002. doi: 10.1007/s00024-002-8710-2.
- [114] J. Kohlbrecher and A. Studer. Transformation cycle between the spherically symmetric correlation function, projected correlation function and differential cross section as implemented in SASfit. *Journal of Applied Crystallography*, 50(5):1395–1403, sep 2017. doi: 10.1107/s1600576717011979.
- [115] J. Kohlbrecher, A. Wiedenmann, and H. Wollenberger. Magnetic coupling between the different phases in nanocrystalline fe-si-b studied by small angle neutron scattering. *Zeitschrift fr Physik B Condensed Matter*, 104(1):1–4, jan 1997. doi: 10.1007/s002570050411.
- [116] D. E. Koppel. Analysis of macromolecular polydispersity in intensity correlation spectroscopy: The method of cumulants. *The Journal of Chemical Physics*, 57(11):4814–4820, 1972. doi: 10.1063/1.1678153.
- [117] M. Kotlarchyk and S.-H. Chen. Analysis of small angle neutron scattering spectra from polydisperse interacting colloids. *The Journal of Chemical Physics*, 79(5):2461–2469, sep 1983. doi: 10.1063/1.446055.

- [118] A. Kovalenko and F. Hirata. Self-consistent description of a metal–water interface by the kohn–sham density functional theory and the three-dimensional reference interaction site model. *The Journal of Chemical Physics*, 110(20):10095–10112, may 1999. doi: 10.1063/1.478883.
- [119] M. A. Krasnoselskij. Two remarks on the method of successive approximations, (russian). *Uspehi Mat. Nauk.* 10, 63(1):123–127, 1955. URL <http://mi.mathnet.ru/eng/umn7954>.
- [120] O. Kratky and G. Porod. Diffuse small-angle scattering of x-rays in colloid systems. *Journal of Colloid Science*, 4(1):35–70, feb 1949. doi: 10.1016/0095-8522(49)90032-x.
- [121] C. G. D. Kruif, P. W. Rouw, W. J. Briels, M. H. G. Duits, A. Vrij, and R. P. May. Adhesive hard-sphere colloidal dispersions. a small-angle neutron-scattering study of stickiness and the structure factor. *Langmuir*, 5(2):422–428, mar 1989. doi: 10.1021/la00086a023.
- [122] D. Lebedev, D. Baitin, A. Islamov, A. Kuklin, V. Shalguev, V. Lanzov, and V. Isaev-Ivanov. Analytical model for determination of parameters of helical structures in solution by small angle scattering: comparison of RecA structures by SANS. *FEBS Letters*, 537(1-3):182–186, feb 2003. doi: 10.1016/s0014-5793(03)00107-8.
- [123] J. L. Lebowitz and J. K. Percus. Mean spherical model for lattice gases with extended hard cores and continuum fluids. *Physical Review*, 144(1):251–258, apr 1966. doi: 10.1103/physrev.144.251.
- [124] L. L. Lee. An accurate integral equation theory for hard spheres: Role of the zero-separation theorems in the closure relation. *The Journal of Chemical Physics*, 103(21):9388–9396, dec 1995. doi: 10.1063/1.469998.
- [125] H. N. Lekkerkerker and R. Tuinier. *Colloids and the Depletion Interaction*, volume 833 of *Lecture Notes in Physics*. Springer Netherlands, 2011. doi: 10.1007/978-94-007-1223-2.
- [126] C. N. Likos. Effective interactions in soft condensed matter physics. *Physics Reports*, 348(4-5):267–439, jul 2001. doi: 10.1016/s0370-1573(00)00141-1.
- [127] C. N. Likos. Soft matter with soft particles. *Soft Matter*, 2(6):478, 2006. doi: 10.1039/B601916C.
- [128] C. N. Likos. Structure and thermodynamics of ionic microgels. In A. F.-N. H. M. Wyss, J. Mattsson, and D. A. Weitz, editors, *Microgel Suspensions*, chapter Chapter 7., pages 163–193. Wiley-VCH Verlag GmbH & Co. KGaA, feb 2011. doi: 10.1002/9783527632992.ch7.
- [129] C. N. Likos, H. Lwen, M. Watzlawek, B. Abbas, O. Jucknischke, J. Allgaier, and D. Richter. Star polymers viewed as ultrasoft colloidal particles. *Physical Review Letters*, 80(20):4450–4453, may 1998. doi: 10.1103/physrevlett.80.4450.
- [130] M. Lin, H. Lindsay, D. Weitz, R. Ball, R. Klein, and P. Meakin. Universality of fractal aggregates as probed by light scattering. *Proc. R. Soc. Lond. A*, 423:71–87, 1989. doi: 10.1098/rspa.1989.0042.
- [131] M. Y. Lin, H. M. Lindsay, D. A. Weitz, R. C. Ball, R. Klein, and P. Meakin. Universal reaction-limited colloid aggregation. *Physical Review A*, 41(4):2005–2020, feb 1990. doi: 10.1103/physreva.41.2005.
- [132] M. Y. Lin, H. M. Lindsay, D. A. Weitz, R. Klein, R. C. Ball, and P. Meakin. Universal diffusion-limited colloid aggregation. *Journal of Physics: Condensed Matter*, 2(13):3093–3113, apr 1990. doi: 10.1088/0953-8984/2/13/019.
- [133] M. Y. Lin, H. M. Lindsay, D. A. Weitz, R. Klein, R. C. Ball, and P. Meakin. Universal diffusion-limited colloid aggregation. *Journal of Physics: Condensed Matter*, 2(23):5283–5283, jun 1990. doi: 10.1088/0953-8984/2/23/521.
- [134] Y. Liu, J. Lin, G. Huang, Y. Guo, and C. Duan. Simple empirical analytical approximation to the voigt profile. *Journal of the Optical Society of America B*, 18(5):666, may 2001. doi: 10.1364/josab.18.000666.
- [135] E. Lomba and L. L. Lee. Consistency conditions for the integral equations of liquid structures. *International Journal of Thermophysics*, 17(3):663–672, may 1996. doi: 10.1007/bf01441512.
- [136] A. A. Louis, P. G. Bolhuis, and J. P. Hansen. Mean-field fluid behavior of the gaussian core model. *Physical Review E*, 62(6):7961–7972, dec 2000. doi: 10.1103/physreve.62.7961.
- [137] Y. Lyatskaya, Y. Liu, S. Tristram-Nagle, J. Katsaras, and J. F. Nagle. Method for obtaining structure and interactions from oriented lipid bilayers. *Phys. Rev. E*, 63(1):011907, Dec 2000. doi: 10.1103/PhysRevE.63.011907.
- [138] W. G. Madden and S. A. Rice. The mean spherical approximation and effective pair potentials in liquids. *The Journal of Chemical Physics*, 72(7):4208–4215, apr 1980. doi: 10.1063/1.439651.

- [139] A. G. Mailer, P. S. Clegg, and P. N. Pusey. Particle sizing by dynamic light scattering: non-linear cumulant analysis. *Journal of Physics: Condensed Matter*, 27(14):145102, mar 2015. doi: 10.1088/0953-8984/27/14/145102.
- [140] W. R. Mann. Mean value methods in iteration. *Proceedings of the American Mathematical Society*, 4(3):506–510, mar 1953. doi: 10.1090/s0002-9939-1953-0054846-3.
- [141] C. Marquest and T. Witten. Simple cubic structure in copolymer mesophases. *Journal de Physique*, 50(10):1267–1277, 1989. doi: 10.1051/jphys:0198900500100126700.
- [142] G. Martynov and G. Sarkisov. Exact equations and the theory of liquids. v. *Molecular Physics*, 49(6):1495–1504, aug 1983. doi: 10.1080/00268978300102111.
- [143] G. A. Martynov and A. G. Vompe. Differential condition of thermodynamic consistency as a closure for the ornstein-zernike equation. *Physical Review E*, 47(2):1012–1017, feb 1993. doi: 10.1103/physreve.47.1012.
- [144] S. V. G. Menon, V. K. Kelkar, and C. Manohar. Application of baxter’s model to the theory of cloud points of nonionic surfactant solutions. *Physical Review A*, 43(2):1130–1133, jan 1991. doi: 10.1103/physreva.43.1130.
- [145] S. V. G. Menon, C. Manohar, and K. S. Rao. A new interpretation of the sticky hard sphere model. *J. Chem. Phys.*, 95(12):9186–9190, dec 1991. doi: 10.1063/1.461199.
- [146] A. Michels and J.-P. Bick. Autocorrelation function of the spin misalignment in magnetic small-angle neutron scattering: application to nanocrystalline metals. *Journal of Applied Crystallography*, 46(3):788–790, apr 2013. doi: 10.1107/s0021889813005402.
- [147] G. Mie. Beitrge zur optik trber medien, speziell kolloidaler metallsungen. *Annalen der Physik*, 330(3):377–445, 1908. doi: 10.1002/andp.19083300302.
- [148] D. F. R. Mildner. Efficient use of two-dimensional detectors for small-angle neutron scattering contours from polymers under external constraint. *Macromolecules*, 16(11):1760–1763, nov 1983. doi: 10.1021/ma00245a014.
- [149] B. Mladek, M. Fernaud, G. Kahl, and M. Neumann. On the thermodynamic properties of the generalized gaussian core model. *Condensed Matter Physics*, 8(1):135–148, 2005. doi: 10.5488/cmp.8.1.135.
- [150] J. Mller, R. Kranold, J. Schmelzer, and U. Lembke. Small-angle x-ray scattering size parameters and higher moments of the particle-size distribution function in the asymptotic stage of ostwald ripening. *Journal of Applied Crystallography*, 28(5):553–560, oct 1995. doi: 10.1107/s0021889895003827.
- [151] R. M. Moon, T. Riste, and W. C. Koehler. Polarization analysis of thermal-neutron scattering. *Physical Review*, 181(2):920–931, may 1969. doi: 10.1103/physrev.181.920.
- [152] F. Muller, M. Delsanti, L. Auvray, J. Yang, Y. Chen, J. Mays, B. Demé, M. Tirrell, and P. Guenoun. Ordering of urchin-like charged copolymer micelles: Electrostatic, packing and polyelectrolyte correlations. *The European Physical Journal E*, 3(1):45–53, sep 2000. doi: 10.1007/s101890070040.
- [153] G. Ngele. On the dynamics and structure of charge-stabilized suspensions. *Physics Reports*, 272(5–6):215–372, jul 1996. doi: 10.1016/0370-1573(95)00078-x.
- [154] G. Ngele. The physics of colloidal soft matter, 2004.
- [155] J. Noolandi and K. Hong. Non-equilibrium correlation function of a reptating chain. *Journal de Physique Lettres*, 45(3):149–157, 1984. doi: 10.1051/jphyslet:01984004503014900.
- [156] M. A. Noor. New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications*, 251(1):217–229, nov 2000. doi: 10.1006/jmaa.2000.7042.
- [157] O. D. of crystallography. R factor. http://reference.iucr.org/dictionary/R_factor. URL http://reference.iucr.org/dictionary/R_factor.
- [158] J. Olivero and R. Longbothum. Empirical fits to the voigt line width: A brief review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 17(2):233–236, feb 1977. doi: 10.1016/0022-4073(77)90161-3.
- [159] L. Ornstein and F. Zernike. Accidental deviations of density and opalescence at the critical point of a single substance. *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings*, 17:793–806, 1914. URL <http://www.dwc.knaw.nl/DL/publications/PU00012727.pdf>.

- [160] S. Oversteegen and H. Lekkerkerker. On the accuracy of the derjaguin approximation for depletion potentials. *Physica A: Statistical Mechanics and its Applications*, 341(1-4):23–39, oct 2004. doi: 10.1016/j.physa.2004.04.103.
- [161] G. Pabst, R. Koschuch, B. Pozo-Navas, M. Rappolt, K. Lohner, and P. Laggner. Structural analysis of weakly ordered membrane stacks. *Journal of Applied Crystallography*, 36(6):1378–1388, nov 2003. doi: 10.1107/s0021889803017527.
- [162] G. Pabst, M. Rappolt, H. Amenitsch, and P. Laggner. Structural information from multilamellar liposomes at full hydration: Fullq-range fitting with high quality x-ray data. *Physical Review E*, 62(3):4000–4009, sep 2000. doi: 10.1103/physreve.62.4000.
- [163] J. S. Pedersen. Determination of size distribution from small-angle scattering data for systems with effective hard-sphere interactions. *Journal of Applied Crystallography*, 27(4):595–608, aug 1994. doi: 10.1107/s0021889893013810.
- [164] J. S. Pedersen. Form factors of block copolymer micelles with spherical, ellipsoidal and cylindrical cores. *Journal of Applied Crystallography*, 33(3):637–640, jun 2000. doi: 10.1107/s0021889899012248.
- [165] J. S. Pedersen. *Modelling of Small-Angle Scattering Data from Colloids and Polymer Systems*, chapter 6, pages 391–420. Elsevier Science, 2002.
- [166] J. S. Pedersen and M. C. Gerstenberg. Scattering form factor of block copolymer micelles. *Macromolecules*, 29(4):1363–1365, jan 1996. doi: 10.1021/ma9512115.
- [167] J. S. Pedersen, D. Posselt, and K. Mortensen. Analytical treatment of the resolution function for small-angle scattering. *Journal of Applied Crystallography*, 23(4):321–333, aug 1990. doi: 10.1107/s0021889890003946.
- [168] J. S. Pedersen and P. Schurtenberger. Scattering functions of semiflexible polymers with and without excluded volume effects. *Macromolecules*, 29(23):7602–7612, jan 1996. doi: 10.1021/ma9607630.
- [169] J. K. Percus and G. J. Yevick. Analysis of classical statistical mechanics by means of collective coordinates. *Physical Review*, 110(1):1–13, apr 1958. doi: 10.1103/physrev.110.1.
- [170] H. I. Petracche, N. Gouliaev, S. Tristram-Nagle, R. Zhang, R. M. Suter, and J. F. Nagle. Interbilayer interactions from high-resolution x-ray scattering. *Phys. Rev. E*, 57(6):7014–7024, Jun 1998. doi: 10.1103/PhysRevE.57.7014.
- [171] W. Phuengrattana and S. Suantai. On the rate of convergence of mann, ishikawa, noor and SP-iterations for continuous functions on an arbitrary interval. *Journal of Computational and Applied Mathematics*, 235(9):3006–3014, mar 2011. doi: 10.1016/j.cam.2010.12.022.
- [172] E. Picard. Mmoire sur la thorie des quations aux drives partielles et la mthode des approximations successives. *Journal de mathmatiques pures et appliquées*, 6(4):145–210, 1890. URL http://portail.mathdoc.fr/JMPA/PDF/JMPA_1890_4_6_A3_0.pdf.
- [173] G. Porod. Die Abhngigkeit der Rntgenkleinwinkelstreuung von Form und Grsse der Kolloiden Teilchen in verdunnten Systemen, IV. *Acta Physica Austriaca*, 2:255–292, 1948.
- [174] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes. The Art of Scientific Computing*. Cambridge University Press, 3rd edition, 2007.
- [175] O. A. Pringle and P. W. Schmidt. Small-angle x-ray scattering from helical macromolecules. *Journal of Applied Crystallography*, 4(4):290–293, aug 1971. doi: 10.1107/s002188987100699x.
- [176] L. C. Puigjaner and J. A. Subirana. Low-angle x-ray scattering by disordered and partially ordered helical systems. *Journal of Applied Crystallography*, 7(2):169–173, apr 1974. doi: 10.1107/s002188987400906x.
- [177] P. N. Pusey, D. E. Koppel, D. E. Schaefer, R. D. Camerini-Otero, and S. H. Koenig. Intensity fluctuation spectroscopy of laser light scattered by solutions of spherical viruses. r17, q.beta., BSV, PM2, and t7. i. light-scattering technique. *Biochemistry*, 13(5):952–960, feb 1974. doi: 10.1021/bi00702a020.
- [178] C. Raman. LXI. on the mean distance between neighbouring molecules in a fluid. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 47(280):671–679, may 1924. doi: 10.1080/14786442408634406.

- [179] L. Rayleigh. On the theory of long waves and bores. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 90(619):324–328, jul 1914. doi: 10.1098/rspa.1914.0055.
- [180] C. Regnaut and J. C. Ravey. Application of the adhesive sphere model to the structure of colloidal suspensions. *The Journal of Chemical Physics*, 91(2):1211–1221, jul 1989. doi: 10.1063/1.457194.
- [181] C. Regnaut and J. C. Ravey. Erratum: Application of the adhesive sphere model to the structure of colloidal suspensions [j. chem. phys. 91, 1211 (1989)]. *The Journal of Chemical Physics*, 92(5):3250–3250, mar 1990. doi: 10.1063/1.458616.
- [182] L. E. Reynolds and D. F. R. Mildner. The elliptical average of small-angle scattering data. *Journal of Applied Crystallography*, 17(6):411–416, dec 1984. doi: 10.1107/S0021889884011857.
- [183] C. Robertus, W. H. Philipse, J. G. H. Joosten, and Y. K. Levine. Solution of the percolation approximation of the multicomponent adhesive sphere system to the small angle x-ray scattering from microemulsions. *The Journal of Chemical Physics*, 90(8):4482–4490, apr 1989. doi: 10.1063/1.456635.
- [184] F. J. Rogers and D. A. Young. New, thermodynamically consistent, integral equation for simple fluids. *Physical Review A*, 30(2):999–1007, Aug 1984. doi: 10.1103/physreva.30.999.
- [185] Y. Rosenfeld and N. W. Ashcroft. Theory of simple classical fluids: Universality in the short-range structure. *Physical Review A*, 20(3):1208–1235, sep 1979. doi: 10.1103/physreva.20.1208.
- [186] D. Roux and C. Safinya. A synchrotron x-ray study of competing undulation and electrostatic interlayer interactions in fluid multimembrane lyotropic phases. *Journal de Physique*, 49(2):307–318, 1988. doi: 10.1051/jphys:01988004902030700.
- [187] Y. Saad. A flexible inner-outer preconditioned GMRES algorithm. *SIAM Journal on Scientific Computing*, 14(2):461–469, mar 1993. doi: 10.1137/0914028.
- [188] Y. Saad and M. H. Schultz. GMRES: A generalized minimal residual algorithm for solving non-symmetric linear systems. *SIAM Journal on Scientific and Statistical Computing*, 7(3):856–869, jul 1986. doi: 10.1137/0907058.
- [189] R. F. Saraf. Small-angle scattering from anisotropic systems in the guinier region. *Macromolecules*, 22(2):675–681, mar 1989. doi: 10.1021/ma00192a028.
- [190] H. A. Scheraga, J. T. Edsall, and J. O. Gadd. Double refraction of flow: Numerical evaluation of extinction angle and birefringence as a function of velocity gradient. *The Journal of Chemical Physics*, 19(9):1101–1108, sep 1951. doi: 10.1063/1.1748483.
- [191] P. W. Schmidt. Small angle x-ray scattering from helical filaments. *Journal of Applied Crystallography*, 3(4):257–264, aug 1970. doi: 10.1107/s0021889870006155.
- [192] K.-V. Schubert, R. Strey, S. R. Kline, and E. W. Kaler. Small angle neutron scattering near Lifshitz lines: Transition from weakly structured mixtures to microemulsions. *J. Chem. Phys.*, 101:5343, 1994. doi: 10.1063/1.467387.
- [193] S. Schwartz, J. Cain, E. Dratz, and J. Blasie. An analysis of lamellar x-ray diffraction from disordered membrane multilayers with application to data from retinal rod outer segments. *Biophysical Journal*, 15(12):1201–1233, dec 1975. doi: 10.1016/s0006-3495(75)85895-4.
- [194] R. V. Sharma and K. C. Sharma. The structure factor and the transport properties of dense fluids having molecules with square well potential, a possible generalization. *Physica A: Statistical Mechanics and its Applications*, 89(1):213 – 218, oct 1977. doi: 10.1016/0378-4371(77)90151-0.
- [195] B. Sjberg. Small-angle x-ray investigation of the equilibria between copper(II) and glycyl-l-histidylglycine in water solution. a method for analysing polydispersed systems. *Journal of Applied Crystallography*, 7(2):192–199, apr 1974. doi: 10.1107/s0021889874009241.
- [196] W. R. Smith, D. Henderson, and Y. Tago. Mean spherical approximation and optimized cluster theory for the square-well fluid. *The Journal of Chemical Physics*, 67(11):5308–5316, dec 1977. doi: 10.1063/1.434709.
- [197] C. M. Sorensen, J. Cai, and N. Lu. Test of static structure factors for describing light scattering from fractal soot aggregates. *Langmuir*, 8(8):2064–2069, aug 1992. doi: 10.1021/la00044a029.
- [198] C. M. Sorensen and G. M. Wang. Size distribution effect on the power law regime of the structure factor of fractal aggregates. *Physical Review E*, 60(6):7143–7148, dec 1999. doi: 10.1103/physreve.60.7143.

- [199] T. Sottmann, R. Strey, and S.-H. Chen. A small-angle neutron scattering study of nonionic surfactant molecules at the water–oil interface: Area per molecule, microemulsion domain size, and rigidity. *The Journal of Chemical Physics*, 106(15):6483–6491, apr 1997. doi: 10.1063/1.473638.
- [200] G. L. Squires. *Thermal Neutron Scattering*. Cambridge University Press, 1978. doi: 10.1017/CBO9781139107808.
- [201] M. Stieger, J. S. Pedersen, P. Lindner, and W. Richtering. Are thermoresponsive microgels model systems for concentrated colloidal suspensions? a rheology and small-angle neutron scattering study. *Langmuir*, 20(17):7283–7292, aug 2004. doi: 10.1021/la049518x. PMID: 15301516.
- [202] R. Strey, J. Winkler, and L. Magid. Small-angle neutron scattering from diffuse interfaces. 1. mono- and bilayers in the water-octane-pentaoxyethylene monododecyl ether (c12es) system. *The Journal of Physical Chemistry*, 95(19):7502–7507, sep 1991. doi: 10.1021/j100172a070.
- [203] G. C. Summerfield and D. F. R. Mildner. Small-angle scattering with azimuthal asymmetry. *Journal of Applied Crystallography*, 16(4):384–389, aug 1983. doi: 10.1107/S0021889883010663.
- [204] C. Svaneborg and J. S. Pedersen. Form factors of block copolymer micelles with excluded-volume interactions of the corona chains determined by monte carlo simulations. *Macromolecules*, 35(3):1028–1037, jan 2002. doi: 10.1021/ma011046v.
- [205] D. A. Svetogorsky. The properties of anisotropic small-angle scattering. *Journal of Applied Crystallography*, 23(2):79–81, apr 1990. doi: 10.1107/S0021889889008253.
- [206] C. V. Teixeira, H. Amenitsch, T. Fukushima, J. P. Hill, W. Jin, T. Aida, M. Hotokka, and M. Lindén. Form factor of anN-layered helical tape and its application to nanotube formation of hexa-peri-hexabenzocoronene-based molecules. *Journal of Applied Crystallography*, 43(4):850–857, jun 2010. doi: 10.1107/s0021889810015736.
- [207] M. Teubner and R. Strey. Origin of the scattering peak in microemulsions. *J. Chem. Phys.*, 87:3195, 1987. doi: 10.1063/1.453006.
- [208] O. B. Toon and T. P. Ackerman. Algorithms for the calculation of scattering by stratified spheres. *Applied Optics*, 20(20):3657, oct 1981. doi: 10.1364/ao.20.003657.
- [209] A. Toth and C. T. Kelley. Convergence analysis for anderson acceleration. *SIAM Journal on Numerical Analysis*, 53(2):805–819, jan 2015. doi: 10.1137/130919398.
- [210] A. Trokhymchuk, I. Nezbeda, J. Jirsák, and D. Henderson. Hard-sphere radial distribution function again. *The Journal of Chemical Physics*, 123(2):024501, jul 2005. doi: 10.1063/1.1979488.
- [211] A. Trokhymchuk, I. Nezbeda, J. Jirsák, and D. Henderson. Erratum: “hard sphere radial distribution function again” [j. chem. phys. 123, 024501 (2005)]. *The Journal of Chemical Physics*, 124(14):149902, apr 2006. doi: 10.1063/1.2188941.
- [212] J. Ulander, H. Greberg, and R. Kjellander. Primary and secondary effective charges for electrical double layer systems with asymmetric electrolytes. *The Journal of Chemical Physics*, 115(15):7144–7160, oct 2001. doi: 10.1063/1.1398587.
- [213] P. van Beurten and A. Vrij. Polydispersity effects in the small-angle scattering of concentrated solutions of colloidal spheres. *The Journal of Chemical Physics*, 74(5):2744–2748, mar 1981. doi: 10.1063/1.441443.
- [214] H. van de Hulst. *Light Scattering by Small Particles*. Wiley, New York, 1957.
- [215] H. A. van der Vorst. Bi-CGSTAB: A fast and smoothly converging variant of bi-CG for the solution of nonsymmetric linear systems. *SIAM Journal on Scientific and Statistical Computing*, 13(2):631–644, mar 1992. doi: 10.1137/0913035.
- [216] L. Verlet. Integral equations for classical fluids. *Molecular Physics*, 41(1):183–190, sep 1980. doi: 10.1080/00268978000102671.
- [217] L. Verlet. Integral equations for classical fluids. *Molecular Physics*, 42(6):1291–1302, apr 1981. doi: 10.1080/00268978100100971.
- [218] L. Viererblová, J. Kolafa, S. Labík, and A. Malijevský. Virial coefficients and equation of state of the penetrable sphere model. *Phys. Chem. Chem. Phys.*, 12(1):254–262, 2010. doi: 10.1039/b917204a.
- [219] A. G. Vompe and G. A. Martynov. The bridge function expansion and the self-consistency problem of the ornstein–zernike equation solution. *The Journal of Chemical Physics*, 100(7):5249–5258, apr 1994. doi: 10.1063/1.467189.

- [220] C. G. Vonk. On two methods of determination of particle size distribution functions by means of small-angle x-ray scattering. *Journal of Applied Crystallography*, 9(6):433–440, dec 1976. doi: 10.1107/s0021889876011874.
- [221] A. Vrij. Mixtures of hard spheres in the percus-yevick approximation. light scattering at finite angles. *J. Chem. Phys.*, 71(8):3267–3270, oct 1979. doi: 10.1063/1.438756.
- [222] W. Wagner and J. Kohlbrecher. Small-angle neutron scattering. In Y. Zhu, editor, *Modern Techniques for Characterizing Magnetic Materials*, chapter Small-angle neutron scattering, pages 65–105. Springer-Verlag, 2005. doi: 10.1007/0-387-23395-4_2.
- [223] H. F. Walker and P. Ni. Anderson acceleration for fixed-point iterations. *SIAM Journal on Numerical Analysis*, 49(4):1715–1735, jan 2011. doi: 10.1137/10078356X.
- [224] G. Walter, R. Kranold, T. Gerber, J. Baldrian, and M. Steinhart. Particle size distribution from small-angle x-ray scattering data. *Journal of Applied Crystallography*, 18(4):205–213, aug 1985. doi: 10.1107/s0021889885010160.
- [225] J. D. Weeks, D. Chandler, and H. C. Andersen. Role of repulsive forces in determining the equilibrium structure of simple liquids. *The Journal of Chemical Physics*, 54(12):5237–5247, jun 1971. doi: 10.1063/1.1674820.
- [226] M. S. Wertheim. Exact solution of the percus-yevick integral equation for hard spheres. *Physical Review Letters*, 10(8):321–323, apr 1963. doi: 10.1103/physrevlett.10.321.
- [227] A. Wiedenmann, R. Ghler, C. D. Dewhurst, U. Keiderling, S. Prévost, and J. Kohlbrecher. Relaxation mechanisms in magnetic colloids studied by stroboscopic spin-polarized small-angle neutron scattering. *Physical Review B*, 84(21):214303, dec 2011. doi: 10.1103/physrevb.84.214303.
- [228] C. M. Wijmans and E. B. Zhulina. Polymer brushes at curved surfaces. *Macromolecules*, 26(26):7214–7224, dec 1993. doi: 10.1021/ma00078a016.
- [229] L. Willner, A. Poppe, J. Allgaier, M. Monkenbusch, P. Lindner, and D. Richter. Micellization of amphiphilic diblock copolymers: Corona shape and mean-field to scaling crossover. *Europhysics Letters (EPL)*, 51(6):628–634, sep 2000. doi: 10.1209/epl/i2000-00384-1.
- [230] W. Wiscombe. Mie scattering calculations: Advances in technique and fast, vector-speed computer codes. Technical report, National Center For Atmospheric Research, Boulder, Colorado, 1979. doi: 10.5065/d6zp4414.
- [231] W. J. Wiscombe. Improved mie scattering algorithms. *Applied Optics*, 19(9):1505–1509, may 1980. doi: 10.1364/AO.19.001505.
- [232] T. Yoshizaki and H. Yamakawa. Scattering functions of wormlike and helical wormlike chains. *Macromolecules*, 13(6):1518–1525, nov 1980. doi: 10.1021/ma60078a030.
- [233] G. Zerah and J.-P. Hansen. Self-consistent integral equations for fluid pair distribution functions: Another attempt. *The Journal of Chemical Physics*, 84(4):2336–2343, feb 1986. doi: 10.1063/1.450397.
- [234] F. Zernike and J. A. Prins. Die beugung von rntgenstrahlen in flssigkeiten als effekt der moleklanordnung. *Zeitschrift fr Physik A Hadrons and nuclei*, 41(2-3):184–194, jun 1927. doi: 10.1007/bf01391926.
- [235] R. Zhang, R. M. Suter, and J. F. Nagle. Theory of the structure factor of lipid bilayers. *Phys. Rev. E*, 50(6):5047–5060, Dec 1994. doi: 10.1103/PhysRevE.50.5047.
- [236] B. H. Zimm. The scattering of light and the radial distribution function of high polymer solutions. *The Journal of Chemical Physics*, 16(12):1093–1099, dec 1948. doi: 10.1063/1.1746738.