Optimal Resource Planning for Abundance and Famine: A Lagrangian Approach

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Abstract

We present a simple model for optimal savings inspired by the narrative of Prophet Yusuf (peace be upon him): seven years of abundance followed by seven years of famine. Using a standard Lagrangian framework, we derive the optimal trajectory that balances smooth, gradual accumulation during abundance and consistent depletion during famine.

1 Introduction

The story of Prophet Yusuf (The Qur'an 12:43–49) describes the coming of a period of seven years of agricultural abundance, followed by seven years of drought and famine. The challenge is to plan the accumulation and subsequent use of stored grain to ensure food security throughout the entire period. We seek an optimal policy that maximizes sustainability and stability in storage and consumption rates. To the best of our knowledge, the application of the Lagrangian framework to model the abundance-famine resource allocation problem described in the story of Prophet Yusuf (peace be upon him) has not previously appeared in the literature. Although related problems in resource smoothing and optimal storage have been studied using various mathematical methods, a direct connection to the calculus of variations and the principle of stationary (least) action appears to be novel.

2 The Model

We propose applying a simplified Lagrangian framework to compute the optimal savings rate throughout the abundance and famine periods. This approach can be adapted to other resource management problems that involve accumulation and depletion under uncertainty or varying policy constraints.

2.1 Assumptions

- Let $s(t)$ denote the amount of grain in tonnes stored at time t.
- Let the savings rate be $\dot{s}(t) = ds(t)/dt$.
- The planning horizon is $t \in [0, 14]$.
- For $0 \leq t < 7$, there is agricultural abundance; grain is harvested and stored except for the amount necessary for consumption.
- For $7 \leq t \leq 14$, there is famine; stored grain is consumed.
- Storage in $t = 0$ is s_0; storage in $t = 14$ is s_{14}.
- We assume that, in general, maximum savings rate does not exceed maximum production rate: $\dot{s}_{max} < P_{max}$.

2.2 Objective

We seek a savings and depletion plan that:

- Accumulates grain during abundance and depletes it during famine,
- Avoids abrupt changes in storage/consumption rates,
- Satisfies $s(0) = s_0$ and $s(14) = s_{14}$.
- The savings rate, $\dot{s}(t)$, should be as smooth as possible.

We focus on "savings" rather than consumption because the utility of consumption is highly subjective and difficult to quantify. In contrast, savings is an objective and directly measurable quantity. Since consumption is simply the residual of production after accounting for savings, determining the optimal savings trajectory automatically yields the optimal consumption path by implication.

2.3 Lagrangian and Optimization Problem

The Lagrangian, $\mathcal{L}$, is generally defined as the difference between the kinetic and potential energies:

$$\mathcal{L} = T - V$$

In our context, we define the kinetic energy as $T = \frac{1}{2}m\dot{s}^2, m > 0$. The parameter m reflects resistance to abrupt changes in the savings rate, indicating a preference for a smooth and gradual change. Just as mass resists changes in velocity, a large m represents greater resistance to rapid changes in savings rate, reflecting the practical difficulty of abruptly adjusting savings strategies.

The potential energy is defined as $V = as(t), a > 0$. The parameter a reflects the tendency of the system to balance the accumulation and depletion rates over the planning horizon, shaping the smooth transition between storage during abundance and depletion during famine. A smoother path (one that avoids abrupt policy changes) requires a small a.

Accordingly, the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}m\dot{s}(t)^2 - as(t)$$

The objective is to find the stationary path $s(t)$ of the action:

$$J[s] = \int_0^{14} \left[\frac{1}{2}m\dot{s}(t)^2 - as(t)\right] dt$$

subject to

$$s(0) = s_0, \; s(14) = s_{14}$$

A stationary path means that small variations $\delta s(t)$ do not change the action $J[s]$ to first order. In many physical or economic problems, the stationary path is also a minimum (e.g., the least action).

3 Solution

Finding the stationary path for the action requires solving the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) - \frac{\partial \mathcal{L}}{\partial s} = 0$$

Assuming for simplicity that $m = 1$, we find that:

$$\frac{\partial \mathcal{L}}{\partial \dot{s}} = \dot{s}, \quad \frac{d}{dt}(\dot{s}) = \ddot{s}, \quad \frac{\partial \mathcal{L}}{\partial s} = -a$$

Thus

$$\ddot{s} + a = 0 \quad \Longrightarrow \quad \ddot{s} = -a$$

The general solution for $s(t)$ therefore will be:

$$s(t) = -\frac{a}{2}t^2 + bt + c$$

where b and c are determined by the boundary conditions:

$$\begin{aligned} s(0) = s_0 &\implies c = s_0 \\ s(14) = s_{14} &\implies s_{14} = -\frac{a}{2}(14)^2 + 14b + s_0 \end{aligned}$$

Solving for b:

$$b = \frac{s_{14} - s_0}{14} + \frac{a}{2} \cdot 14$$

4 Optimal Trajectory

The optimal storage trajectory over 14 years is:

$$s(t) = -\frac{a}{2}t^2 + \left[\frac{s_{14} - s_0}{14} + 7a\right] t + s_0$$

If $s_0 = s_{14} = 0$, this simplifies to:

$$s(t) = -\frac{a}{2}t^2 + 7at$$

which is an **inverted parabola** peaking at $t = 7$, which corresponds to the end of abundance, the start of famine.

The *optimal policy* is therefore:

- **Years 0-7 (Abundance):** Store grain at a decreasing rate, reaching peak storage at $t = 7$.
- **Years 7-14 (Famine):** Deplete storage at an increasing rate, reaching s_{14} at $t = 14$.

The accumulation rate is:

$$\dot{s}(t) = -at + b$$

which is maximum at $t = 0$ (highest accumulation), zero at $t = 7$ (the transition from accumulation to depletion), and minimum at $t = 14$ (highest depletion rate). To normalize the peak so that $s(7) = 100$, we set $a = 100/(0.125(14^2))$. See Figures 1 and 2 below.

5 Discussion

Figure 1 shows the cumulative grain savings starting from year 0. The level of savings increases steadily throughout the period of abundance, reaching its maximum at year 7, when abundance ends. From that point onward, the savings level diminishes as grain is depleted during the famine years, reaching zero by year 14.

The Quran states that the durations of abundance and famine are equal (see Appendix A.1). The symmetry between the is reflected in the parabola that reaches the peak at the end of the abundance period. The symmetry of the parabola implies the linearity of the savings rate.

Figure 2 shows the rate of change of savings, corresponding to the slope of the inverted parabola in Figure 1. The accumulation rate is highest at year 0, then decreases steadily, becoming zero at year 7—the transition from accumulation to depletion. After year 7, the rate becomes negative, indicating grain withdrawal, and continues to decrease until year 14.

It is important to note the smoothness of the savings rate. This smoothness is the result of the optimal allocation of resources throughout the period of abundance and famine. The optimal savings trajectory is the stationary path of the action integral, determined by the Euler–Lagrange equation. In this context, the stationary path also corresponds to a minimum, ensuring the smoothest possible transition between accumulation and depletion.

The continuous-time formulation can be implemented in discrete time with an Euler update ($\Delta t = 0.25$ year, say) without altering the linear saving-rate profile or the inverted-parabola shape of the stock path (see Appendix 3).

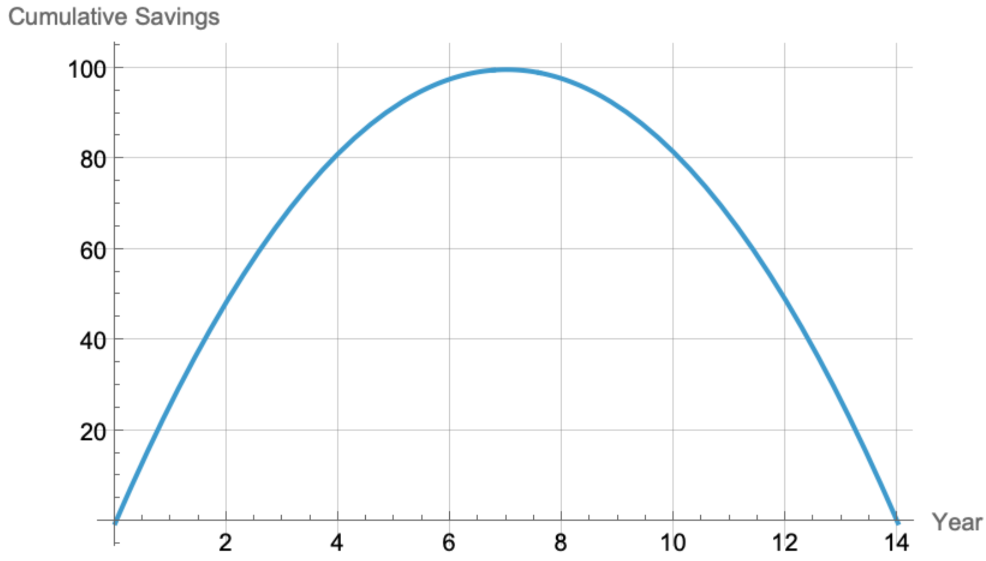

Figure 1: Cumulative savings during the abundance and famine years

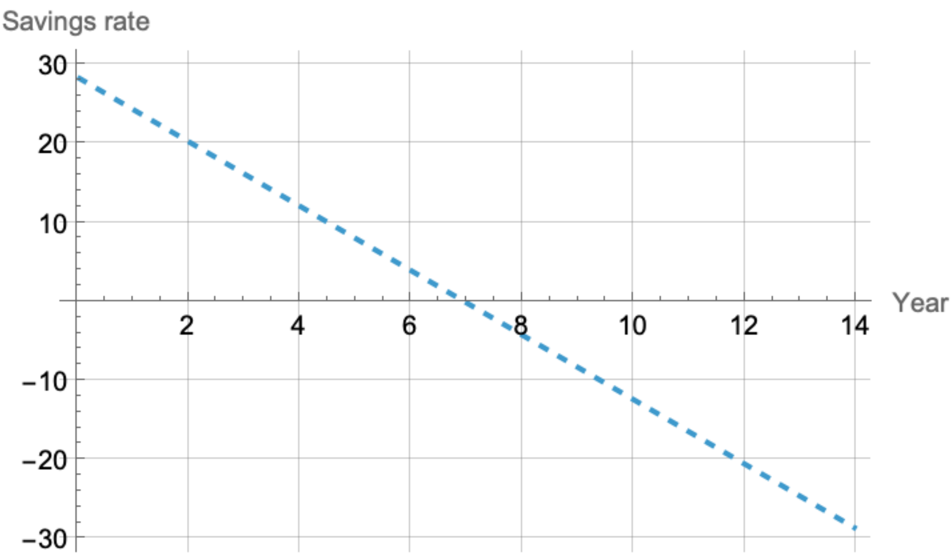

Figure 2: The rate of change in savings

6 Conclusion

The Lagrangian framework was originally developed to describe how natural systems, such as winds, rivers, and planets, manage their energy efficiently and elegantly. It is remarkable that this same mathematical approach can be applied to the optimal allocation of resources during periods of abundance and famine, as recounted in the time of Prophet Yusuf (peace be upon him).

The solution to the Lagrangian model yields an inverted parabola for savings, which naturally balances prudent accumulation with stable depletion. This optimal trajectory is robust to parameter choices and boundary conditions, reflecting both mathematical elegance and practical fairness in resource management.

References

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[2] Kamien, M., & Schwartz, N. (2012). *Dynamic optimization: The calculus of variations and optimal control in economics and management*. Dover Publications. (Original work published 1991 by Elsevier Science.)

[3] Lanczos, C. (1970). *The variational principle of mechanics*. Dover Books.

Appendix

A.1 The Story in the Quran

The story of abundance and famine in the Quran (12:43-49)

> And the king said, 'I saw in a dream seven fat kine, and seven lean ones devouring them; likewise seven green ears of corn, and seven withered. My counsellors, pronounce to me upon my dream, if you are expounders of dreams.' (43) 'A hotchpotch of nightmares!' they said. 'We know nothing of the interpretation of nightmares.' (44) Then said the one who had been delivered, remembering after a time, 'I will myself tell you its interpretation; so send me forth.' (45) 'Joseph, thou true man, pronounce to us regarding seven fat kine, that seven lean ones were devouring, seven green ears of corn, and seven withered; haply I shall return to the men, haply they will know.' (46) He said, 'You shall sow seven years after your wont; what you have harvested leave in the ear, excepting a little whereof you eat. (47) Then thereafter there shall come upon you seven hard years, that shall devour what you have laid up for them, all but a little you keep in store. (48) Then thereafter there shall come a year wherein the people will be succored and press in season. (49).

The story is in the Bible, Book of Genesis 41:1–40.

A.2: Wolfram Mathematica Code

```
ClearAll["Global`*"]
(*Parameters*)
T = 14.0;
s0 = 0.0;
sT = 0.0;
(*Positive for inverted U*)
A = 100/(0.125*T^2); (*Adjusted to normalize peak to 100*)
(*Coefficients*)
c0 = s0;
b = (sT - s0 + 0.5*A*T^2)/T;
(*Storage path as an explicit function*)
s[t_] := -0.5*A*t^2 + b*t + c0;
(*Derivative as an explicit function*)
dsdt[t_] := -A*t + b;
(*Plot storage (inverted U)*)
Plot[
s[t],
{t, 0, T},
PlotRange -> All,
AxesLabel -> {"Year", "Cumulative Savings"}, PlotStyle -> Thick,
GridLines -> Automatic
]
(*Plot rate of change (line)*)
Plot[
dsdt[t],
{t, 0, T},
PlotRange -> All,
AxesLabel -> {"Year", "Savings rate"}, PlotStyle -> Dashed,
GridLines -> Automatic
]
```

A.3: A Discrete Model

Continuous benchmark

The optimal storage stock derived in Section 2 satisfies

$$\ddot{s}(t) \;=\; -a, \qquad \Longrightarrow \qquad s(t) \;=\; -\tfrac{1}{2}at^2 + b\,t + c, \tag{1}$$

with the associated saving/withdrawal rate

$$v(t) \;=\; \dot{s}(t) \;=\; -at + b. \tag{2}$$

Forward Euler discretisation

Introduce a uniform step size Δt and grid points $t_k = k\Delta t$, $k = 0, 1, \ldots, K$ where $T = K\Delta t = 14$ years. Define the discrete approximations

$$s_k \approx s(t_k), \qquad v_k \approx v(t_k).$$

The first–order (explicit) Euler scheme for (2)–(1) is

$$v_{k+1} = v_k - a\,\Delta t, \tag{3a}$$

$$s_{k+1} = s_k + v_k\,\Delta t. \tag{3b}$$

Starting from the boundary value $v_0 = b$ one obtains the closed-form recursion

$$v_k = b - ak\Delta t, \tag{4}$$

$$s_k = s_0 + bk\Delta t - \tfrac{1}{2}a(k-1)k\,\Delta t^2. \tag{5}$$

Yearly versus quarterly grids

(a) Yearly updates. With $\Delta t = 1$ yr ($K = 14$) the maximum deviation between the discrete and continuous stocks is

$$|s_k - s(t_k)| \;=\; \tfrac{1}{2}ak\Delta t^2 \;\le\; \tfrac{1}{2}aT\Delta t \;=\; 7a.$$

(b) Quarterly updates. Refining to $\Delta t = \frac{1}{4}$ yr ($K = 56$) yields

$$|s_k - s(t_k)| \;\le\; \tfrac{1}{2}aT\Delta t \;=\; 1.75a,$$

a four-fold reduction in the worst-case deterministic error while preserving the linear rate profile (4) and the inverted parabolic stock path (5).

Convergence order

For a fixed horizon T the Euler error obeys

$$|s_k - s(t_k)| \;=\; \mathcal{O}(\Delta t), \qquad |v_k - v(t_k)| \;=\; \mathcal{O}(\Delta t), \tag{6}$$

so halving Δt halves the approximation error. In practical terms, quarterly (or finer) grids reproduce the continuous optimum to within observational noise while remaining trivial to implement in a spreadsheet or script.

Take-away. Discretising the model with a quarter-year step retains all qualitative features—linear savings rate and inverted-parabola stocks—while cutting deterministic error by 75 % relative to yearly updates. Planners can therefore work comfortably in discrete time without compromising the optimal trajectory.