Mathematics II (BSM 102)

Manju Subedi

Gandaki University
Bachelor in Information Technology(BIT)
BSM 102

manjusubedi.061@gmail.com

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Power Series

A **power series** in powers of $z - z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

where z is a complex variable, $a_0, a_1, \cdots$ are complex (or real) constants, called the coeflicients of the series, and z_0 is a complex (or real) constant, called the center of the series.

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If $z_0 = 0$**,** we obtain as a particular case a power series in powers of z :

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

This generalizes real power series of calculus.

Convergence of a Power Series

Convergence in a Disk: The-geometric series

$$\sum_{n=1}^{\infty} z^n = 1 + z + z^2 + \cdots$$

converges absolutely if $|z| < 1$ and diverges if $|z| \geq 1$ as we have discussed earlier.

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Convergence for Every z: The-power series (which is known as the Maclaurin series of e^z)

$$\sum_{n=0}^{n} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

It absolutely convergent for every z. In fact, by the ratio test. for any fixed z

$$\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0 \quad \text{as } n \to \infty$$

Convergence Only at the Center: The following power series converges only at $z = 0$, but diverges for every $z \neq 0$, as we shall show.

$$\sum_{n=0}^{\infty} n! z^n = 1 + z + 2z^2 + 6z^2 + \cdots.$$

In fact, from the ratio test we have

$$\left| \frac{(n+1)! z^{n+1}}{n! z^n} \right| = (n+1)|z| \to \infty \quad \text{as} \quad n \to \infty \quad (z \text{ fixed and } \neq 0)$$

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(a) Every power series converges at the center z_0
(b) If a series converges at a point $z = z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, that is, $|z - z_0| < |z_1 - z_0|$.

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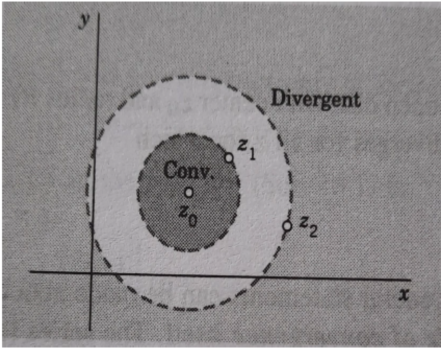

Radius of Convergence of a Power Series

Here, we consider the smallest circle with center z_0 that includes all the points at which a given power series converges. Let R denote its radius. The circle

$$|z - z_0| = R$$

is called the **circle of convergence** and its radius R **the radius of convergence** of the power series.

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Then convergence of power series implies convergence everywhere within that circle, that is, for all z for which

$$|z - z_0| < R$$

(the open disk with center z_0 and radius R). Also, since R is as small as possible, the power series diverges for all z for which

$$|z - z_0| > R$$

Radius of Convergence R

Suppose that the sequence $\left|\dfrac{a_{n+1}}{a_n}\right|, n = 1, 2, \cdots,$ converges with limit L^*, If $L^* = 0$, then $R = \infty$; that is, the power series converges for all z. If $L^* \neq 0$ (hence $L^* > 0$), then

$$R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

(Cauchy-Hadamard formula)
If $|a_{n+1}/a_n| \to \infty$, then $R = 0$ (convergence only at the center z_0).

Example

Find the center and the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$

Solution:

The radius of the convergence of the given power series is

$$R = \lim_{n\to\infty}\left[\frac{(2n!)}{(n!)^2}\Big/\frac{(2n+2)!}{((n+1)!)^2}\right] = \lim_{n\to\infty}\left[\frac{(2n!)}{(2n+2)!}\cdot\frac{((n+1)!)^2}{(n!)^2}\right]$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}.$$

The series converges in the open disk $|z-3i| < \frac{1}{4}$ of radius $\frac{1}{4}$ and center $3i$.

And the interval of the convergence is

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$|z-3i| < \frac{1}{4} \implies -1/4 < z-3i < 1/4 \implies 3i - 1/4 < z < 3i + 1/4$

Taylor's Series

Taylor Series: The Taylor series of $f(x)$ about $x = 0$ is the power series $\sum_{n=0}^{\infty} a_n x^n$, where

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Taylor Series about x = a**:** The Taylor series of $f(x)$ about $x = a$ is the power series

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Interval of the Convergence of Taylor's series

The corresponding Taylor series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Using the ratio test to determine the convergence set for this power series, we find that the limit of the ratio of consecutive terms is

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}(x-1)^{n+1}}{n+1}}{\frac{(-1)^{n+1}(x-1)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{n(x-1)^{n+1}}{(n+1)(x-1)^n} \right|$$

$$= |x-1|$$

so the series converges for

$$L = |x-1| < 1$$

or, equivalently, for

$$-1 < x-1 < 1$$

$$\implies 0 < x < 2 \implies \text{Thus, required interval is } (0,2)$$

Interval and Radius of Convergence

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Solution: Applying the ratio test, we find that the limit of the absolute value of the ratio of consecutive terms $n!x^n$ and $(n+1)!x^{n+1}$ is

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} |(n+1)x|$$
$$= \infty \quad \text{unless } x = 0$$

Therefore, the power series converges only for $x = 0$. The radius of convergence is $R = 0$, and the interval of absolute convergence is the single point $x = 0$.

For each of the following power series, find the radius of convergence and the interval of absolute convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$

c. $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

d. $\sum_{n=1}^{\infty} \frac{5^n x^n}{n!}$

e. $\sum_{n=0}^{\infty} \frac{n! x^n}{3^n}$

Thank You