Mathematics II (BSM 102)

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Outlines

- Introduction to sequence
- Convergent and Divergent sequence
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- Convergent and Divergent series

Introduction to Sequence

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the n^{th} term.
We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.
Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.
But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.
Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

Examples

Example 1: There are three standard ways to represent a sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1 .

(i) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ (ii) $a_n = \frac{n}{n+1}$ (iii) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

Example 2: The Fibonacci sequence $\{f_n\}$ is defined recursively by the-conditions

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2} \quad n \geqslant 3$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

Convergent and Divergent Sequence

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } \quad n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.
If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is convergent).
Otherwise, we say the sequence **diverges** (or is divergent).

Def^n: If $\{a_n\}$ is a sequence, then

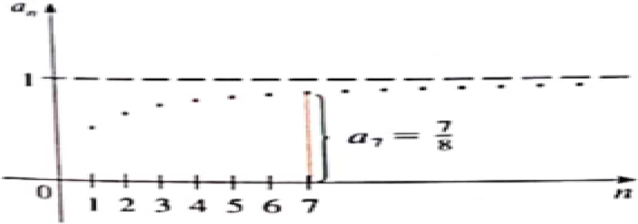
$$\lim_{n\to\infty} a_n = L$$

means that for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > N$$

Examples on Convergence/Divergence

The sequence $a_n = \frac{n}{n+1}$ is approaching 1 as n becomes large as shown in the figure below. Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$(show algebraically)



Examples:

1. The sequence $\left\{\frac{i^n}{n}\right\}$ is convergent with limit 0.
2. The sequence $\{i^n\}$ is divergent.
3. The sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is converges or diverges?
4. The sequence $\{a_n\} = \{\frac{n+1}{n}\}$ converges or diverges?
5. What about the sequence $\{a_n\} = \{n\}$ and $\{-1^n\}$?

Sequence of a Complex Numberes

Sequences of the Real and the Imaginary Parts: A sequence $z_1, z_2, \cdots, z_n, \cdots$ of complex numbers $z_n = x_n + iy_n$ (where $n = 1, 2, \cdots$) converges to $c = a + ib$ if and only if the sequence of the real parts $x_1, x_2, \cdots$ converges to a and the sequence of the imaginary parts $y_1, y_2, \ldots$ converges to b.

Introduction to Infinite Series

Given a sequence $a_1, a_2, \cdots, a_m, \cdots$, we may form the sequence of the sums

$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad \ldots$ and in general
$s_n = a_1 + a_2 + \cdots + a_n \quad (n = 1, 2, \cdots)$ s_n is called the nth partial sum of the **infinite series** or series

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots$$

$a_1, a_2, \cdots$ are called the terms of the series.
An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

Convergence/Divergence

A **convergent** series is one whose sequence of partial sums converges, say, $\lim_{n\to\infty} s_n = s$. Then we write

$$s = \sum_{m=1}^{\infty} a_m = a_1 + a_2 + \cdots$$

and call s the sum or value of the series. A series that is not convergent is called a **divergent series.**

Theorem: The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Complex Series

Real and Imaginary Parts: A complex series $\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$ with $z_m = x_m + iy_m$ converges and has the sum $s = u + iv$ if and only if $x_1 + x_2 + \cdots$ converges and has the sum u and $y_1 + y_2 + \cdots$ converges and has the sum v.

Therorem: If a series $z_1 + z_2 + \cdots$ converges, then $\lim\limits_{m\to\infty} z_m = 0$.

Proof: If $z_1 + z_2 + \cdots$ converges, with the sum s, then, since $z_m = s_m - s_{m-1}$

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} (s_m - s_{m-1}) = \lim_{m\to\infty} s_m - \lim_{m\to\infty} s_{m-1} = s - s = 0$$

But converse is not true in general.

Example: Show that the series $\lim\limits_{n\to\infty} \frac{1}{n(n+1)}$ is convergent, find its sum.

Test for Divergence: If $\lim\limits_{m\to\infty} z_m$ does not exist or $\lim\limits_{m\to\infty} z_m \neq 0$, then the series $\sum_{m=1}^{\infty} z_m$ is divergent.

Example: Show that the series $\sum_{n=1}^{\infty} \dfrac{n^2}{5n^2+4}$

Classwork Problem

1. Prove that the geometric series
$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$
is convergent if $|r| < 1$ and its sum is
$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$
If $|r| \geqslant 1$, the geometric series is divergent.
2. Is the series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ converges or diverges ? If the series converges, find its sum.
3. Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ converges or diverges ?
4. Show that the series $\lim_{n\to\infty} \frac{1}{n(n+1)}$ is convergent, find its sum.
5. Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges

1. let $a_n = \dfrac{2n}{3n+1}$

   a) Determine whether $\{a_n\}$ is convergent?

   b) Detrmine whether $\sum_{n=1}^{\infty} a_n$ is convergent?

2. Show that the harmonic series

$$\sum_{n=1}^{n} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

   is divergent.

Thank You