Mathematics II (BSM 102)

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BSM 102

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July 28, 2023

Outlines

- Introduction to Complex Number
- Operation on Complex Numbers (addition, multiplication)
- Quotient of Complex Numbers
- Complex Conjugate Numbers
- Properties of Complex Conjugates
- Polar Form of Complex Numbers

Introduction to Complex Number [i]

While solving equation like $x^2 + 1 = 0$, early in history that led to the introduction of complex numbers.
By definition, a complex number z is an ordered pair (x, y) of real numbers x and y, written $z = (x, y)$ x is called the real part and y the imaginary part of z, written

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z$$

By definition, two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. $(0, 1)$ is called the imaginary unit and is denoted by i, $i = (0, 1)$.

First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501-1576), who found the formula for solving cubic equations. The term"complex number" was introduced by CARL FRIEDRICH GAUSS who also paved the way for a general use of complex numbers.

Operation on Complex Number

In practice, complex number $z = (x, y)$ are written as $z = x + iy$
Addition of two complex numbers: $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \quad y_1 + y_2).$$

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Multiplication of two complex number:

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) \\ &= (x_1 x_2 - y_1 y_2, \quad x_1 y_2 + x_2 y_1) \\ (x_1 + iy_1)(x_2 + iy_2) &= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Quotient of Complex Number

The quotient $z = z_1/z_2\,(z_2 \neq 0)$ is the complex number z for which $z_1 = zz_2$. If we equate the real and the imaginary parts on both sides of this equation, setting $z = x + iy$, we obtain
$x_1 = x_2x - y_2y, \quad y_1 = y_2x + x_2y.$
The solution is

$$z = \frac{z_1}{z_2} = a + ib \quad \text{where,} \quad a = \frac{x_1x_2 + y_1y_2}{{x_2}^2 + {y_2}^2}, \quad b = \frac{x_2y_1 - x_1y_2}{{x_2}^2 + {y_2}^2}$$

The practical rule used to get this is by multiplying numerator and denominator of z_1/z_2 by $x_2 - iy_2$ and simplifiying:

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{{x_2}^2 + {y_2}^2} + i\frac{x_2y_1 - x_1y_2}{{x_2}^2 + {y_2}^2}$$

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point z in the real axis. Figure below shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

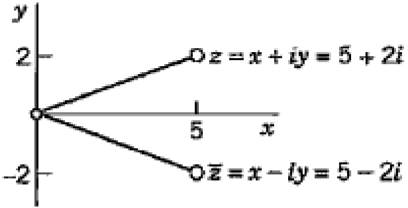

Fig. 319. Complex conjugate numbers

Properties of Complex Conjugate

The complex conjugate is important because it permits us to switch from complex to real.
Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!).
By addition and subtraction, $z + \bar{z} = 2x, z - \bar{z} = 2iy$.
We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

$$\operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z})$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely.
Properties:

$$\begin{aligned}
1.\quad & \overline{(z_1 + z_2)} && = \bar{z}_1 + \bar{z}_2 \\
2.\quad & \overline{(z_1 - z_2)} && = \bar{z}_1 - \bar{z}_2, \\
3.\quad & \overline{(z_1 z_2)} && = \bar{z}_1 \bar{z}_2, \\
4.\quad & \overline{\left(\frac{z_1}{z_2}\right)} && = \frac{\bar{z}_1}{\bar{z}_2}.
\end{aligned}$$

Examples

1. Find the conjugate of the complex number $z = \frac{1+2i}{1-2i}$.
2. If $z_1 = 3 + 4i$ and $z_2 = 2 + i$ then verify that all four properties of complex conjugate.

Polar Form of Complex Numbers

Polar form of complex number is the another representation of complex number. $z = x + iy$ is the rectangular form, with (x, y) as rectangular coordinate.

Polar form of $z = x + iy$ is

$$z = r(\cos\theta + i\sin\theta)$$

$$x = r\cos\theta, \quad y = r\sin\theta.$$

$$\theta = \arg z = \arctan\frac{y}{x} = tan^{-1}\left(\frac{y}{x}\right)$$

r is called the absolute value or modulus or magnitude of z and is denoted by $|z|$. Hence

$$r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Geometrically, $|z|$ is the distance of the point z from the origin. $-\pi < \operatorname{Arg} z \leqq \pi$

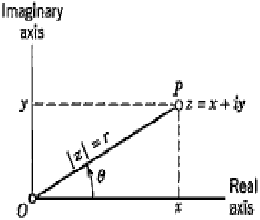

Fig. 320. Complex plane, polar form of a complex number

Examples

1. Write the polar representation of $z = 1 + i$
2. Prove that polar form of $z = 3 + 3\sqrt{3}i$ is $6\left(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi\right)$

Classwork

1. Express the complex number $z = 4i$ into polar form.
2. Express $z = \sqrt{3} + i$ in polar form.
3. Convert the polar form of the given complex number to rectangular form: $z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

Exercise:

A. Represent in polar form and graph in the complex plane

1. $3 - 3i$ 2. $2i, -2i$ 3. -5 4. $\frac{1}{2} + \frac{1}{4}\pi i$

5. $\frac{1+i}{1-i}$ 6. $\frac{3\sqrt{2}+2i}{-\sqrt{2}-(2/3)i}$ 7. $\frac{-6+5i}{3i}$ 8. $\frac{2+3i}{5+4i}$

B. Represent in the form $x + iy$ and graph it in the complex plane.

1. $\cos\frac{1}{2}\pi + i\sin\left(\pm\frac{1}{2}\pi\right)$ 2. $3(\cos 0.2 + i\sin 0.2)$
3. $4\left(\cos\frac{1}{3}\pi \pm i\sin\frac{1}{3}\pi\right)$ 4. $\cos(-1) + i\sin(-1)$
5. $12\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right)$

$$\textit{\textbf{Thank You}}$$