Mathematics II (BSM 102)

Manju Subedi

Gandaki University
Bachelor in Information Technology(BIT)
BSM 102

manjusubedi.061@gmail.com

July 4, 2023

Outlines: Total Derivatives

- Chain Rule for Functions of Two Variables
- Total Derivatives of First Order
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Total Derivatives

Chain Rule for Partial Derivatives:

Suppose z is a function of x and y, i.e. $z = f(x, y)$ each of which is a function of t i.e. $x = g(t)$ and $y = h(t)$.

Then z can be regarded as a function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = z_x\frac{dx}{dt} + z_y\frac{dy}{dt}$$

Observe that the expression for $\frac{dz}{dt}$ is the sum of two terms, each of which can be interpreted using the chain rule for a function of one variable.

In particular, $\frac{\partial z}{\partial x}\frac{dx}{dt} =$ rate of change of z with respect to t for fixed y and

$\frac{\partial z}{\partial y}\frac{dy}{dt} =$ rate of change of z with respect to t for fixed x

The chain rule for partial derivatives says that the total rate of change of z with respect to t is the sum of these two "partial" rates of change.

Total derivatives of second order

Similarly, second order derivative of z with respect to t is given by

$$\frac{d^2z}{dt^2} = z_{xx}\left(\frac{dx}{dt}\right)^2 + z_{yy}\left(\frac{dy}{dt}\right)^2 + 2z_{xy}\frac{dx}{dt}\frac{dy}{dt} + z_x\frac{d^2x}{dt^2} + z_y\frac{d^2y}{dt^2}$$

Example: Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution: Given that, $w = xy, x = \cos t, \quad y = \sin t$, then

$$\frac{\partial w}{\partial x} = y = \sin t, \quad \frac{\partial w}{\partial y} = x = \cos t, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t.$$

Example: Find the first and second order total derivatives of z with respect ot t, where $z = 2x^2 - xy + 3y^3$, $x = t + 1$ and $y = 2t - 1$

Chain Rule for Functions of Three Variables

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example: Changes in a function's values along a helix Find dw/dt if

$$w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t$$

What is the derivative's value at $t = 0$?

Solution:
$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t\end{aligned}$$

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

Example

A health store carries two kinds of vitamin water, brand A and brand B. Sales figures indicate that if brand A is sold for x dollars per bottle and brand B for y dollars per bottle, the demand for brand A will be

$$Q(x, y) = 300 - 20x^2 + 30y \text{ bottles per month}$$

It is estimated that t months from now the price of brand A will be

$$x = 2 + 0.05t \quad \text{dollars per bottle}$$

and the price of brand B will be

$$y = 2 + 0.1\sqrt{t} \text{ dollars per bottle}$$

At what rate will the demand for brand A be changing with respect to time 4 months from now?

Solution

Our goal is to find $\frac{dQ}{dt}$ when $t = 4$. Using the chain rule, we have

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial x}\frac{dx}{dt} + \frac{\partial Q}{\partial y}\frac{dy}{dt} = -40x(0.05) + 30\left(0.05t^{-1/2}\right)$$

When $t = 4$, $x = 2 + 0.05(4) = 2.2$ and hence,

$$\frac{dQ}{dt} = -40(2.2)(0.05) + 30(0.05)(0.5) = -3.65$$

That is, 4 months from now the monthly demand for brand A will be decreasing at the rate of 3.65 bottles per month.

If $z = f(x, y)$ and $y = g(x)$ then substituting $t = x$ in equation
$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$
we get,

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dt}$$

similarly,

$$\frac{d^2 z}{dx^2} = z_{xx} + z_{yy}\left(\frac{dy}{dx}\right)^2 + 2z_{xy}\frac{dy}{dx} + z_y \frac{d^2 y}{dx^2}$$

Example: If $z = x^2 + y^2$ and $y = 2x$ then find $\dfrac{dz}{dx}$ & $\dfrac{d^2 z}{dx^2}$

Exercise

Use the chain rule to find $\frac{dz}{dt}$. Express your answer in terms of y, and t.

1. $z = 2x + 3y; \quad x = t^2, \quad y = 5t$
2. $z = x^2 y; \quad x = 3t + 1, \quad y = t^2 - 1$
3. $z = \dfrac{3x}{y}; \quad x = t, \quad y = t^2$
4. $z = x^{1/2} y^{1/3}; \quad x = 2t, \quad y = 2t^2$
5. $z = xy; \quad x = e^{2t}, \quad y = e^{-3t}$
6. $z = \frac{x+y}{x-y}; \quad x = t^3 + 1, \quad y = 1 - t^2$
7. $z = x^2 y + xy^2, \quad y = x + 5$
8. $z = x^2 + 2xy + y^2, \quad y = 1/x$

Thank You