Mathematics II (BSM 102)

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First Order Partial Derivatives

If $z = f(x, y)$, we find the partial derivative of z with respect to x (denoted $\partial z/\partial x$) by treating the variable y as a constant and taking the derivative of $z = f(x, y)$ with respect to x.

Note that dz/dx represents the derivative of a function of one variable, x, and that $\partial z/\partial x$ represents the partial derivative of a function of two or more variables.

Notations used to represent the partial derivative of $z = f(x, y)$ with respect to x are

$$\frac{\partial z}{\partial x}, \quad \frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x} f(x, y), \quad f_x(x, y), \quad f_x, \quad \text{and } z_x$$

First Order Partial Derivatives

Similarly,
We can also take the derivative of z with respect to y by holding the variable x constant and taking the derivative of $z = f(x, y)$ with respect to y.

We denote this derivative as $\partial z/\partial y$.
and notations used to represent the partial derivative of $z = f(x, y)$ with respect to y are

$$\frac{\partial z}{\partial y}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial}{\partial y} f(x, y), \quad f_y(x, y), \quad f_y, \quad \text{and } z_y$$

Examples:

1. Find $\partial f/\partial y$ if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y) \\ &= (y\cos xy)\frac{\partial}{\partial y}(xy) + \sin xy \\ &= xy\cos xy + \sin xy.\end{aligned}$$

2. Find f_x if $f(x, y) = \dfrac{2y}{y + \cos x}$.

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned}f_x &= \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\dfrac{\partial}{\partial x}(2y) - 2y\dfrac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2} \\ &= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}.\end{aligned}$$

Higher Order Partial Derivatives

Just as we have taken derivatives of derivatives to obtain higher-order derivatives of functions of one variable, we may also take partial derivatives of partial derivatives to obtain higher-order partial derivatives of a function of more than one variable.

If $z = f(x, y)$, then the partial derivative functions z_x and z_y are called first partials. Partial derivatives of z_x and z_y are called second partials, so $z = f(x, y)$ has four second partial derivatives.

The notations for these second partial derivatives follow.

$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$: both derivatives taken with respect to x

$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$: both derivatives taken with respect to y

$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$: first derivative taken with respect to x, second with respect to y

$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$: first derivative taken with respect to y, second with respect to x

Examples

1. If $f(x,y) = x\cos y + ye^x$, then find second order partial derivatives.

Solution: Here,

$$\frac{\partial f}{\partial x} = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y.$$

Example: Find each of the second partial derivatives of $z = x^2y + e^{xy}$.
Solution Since, $z_x = 2xy + e^{xy} \cdot y = 2xy + ye^{xy}$

$$\begin{aligned} z_{xx} &= 2y + e^{xy} \cdot y^2 = 2y + y^2 e^{xy} \\ z_{xy} &= 2x + (e^{xy} \cdot 1 + ye^{xy} \cdot x) \\ &= 2x + e^{xy} + xye^{xy} \end{aligned}$$

Again, $z_y = x^2 + e^{xy} \cdot x = x^2 + xe^{xy}$

$$\begin{aligned} z_{yx} &= 2x + (e^{xy} \cdot 1 + xe^{xy} \cdot y) \\ &= 2x + e^{xy} + xye^{xy} \\ z_{yy} &= 0 + xe^{xy} \cdot x = x^2 e^{xy} \end{aligned}$$

The Mixed Derivative Theorem: If $z = f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

OR

$$z_{xy} = z_{yx}$$

OR

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Functions of More Than Two Variables

Function of the form $w = f(x, y, z)$ is the standard function in three variables.
The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example: If x, y and z are independent variables and then find f_x, f_y and f_z

$$f(x, y, z) = x\sin(y + 3z)$$

$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x\sin(y + 3z)] = x\frac{\partial}{\partial z}\sin(y + 3z)$$

$$= x\cos(y + 3z)\frac{\partial}{\partial z}(y + 3z) = 3x\cos(y + 3z).$$

Classwork

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

- $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
- $f(x, y) = (xy - 1)^2$
- $f(x, y) = (2x - 3y)^3$
- $f(x, y) = \sqrt{x^2 + y^2}$
- $f(x, y) = \left(x^3 + (y/2)\right)^{2/3}$

Find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$

1. $z = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
2. $z = (xy - 1)^2$

Find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$, from $f(x, y, z) = \sqrt{(x^2 + y^2 + z^2)}$

Thank You