Mathematics II (BSM 102)

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BSM 102

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July 31, 2023

Outlines

- Modulus of Complex Numbers and its Properties
- Integer Powers of Z. De Moivre's Formula
- n^{th} Roots of Complex Numbers
- Square Roots of Complex Numbers

Modulus of Complex Numbers and its Properties

Modulus of complex number $z = x + iy$ is defined as

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Properties

1. Triangle Inequality:
   $|z_1 + z_2| \leq |z_1| + |z_2|$
   $|z_1 - z_2| \geq |z_1| - |z_2|$
   $|z_1 + z_2| \geq |z_1| - |z_2|$
2. Generalized triangle inequality:
   $|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|$
3. Product: $|z_1 z_2| = |z_1||z_2|$
4. Division: $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Example: If $z_1 = 3 + 4i$ and $z_2 = 2 + i$ then verify triangle inequality properties, product property and division property of modulus of complex numbers.

Integer Powers of z. De Moivre's Formula

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$
$z^n = [re^{i\theta}]^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$, Where $n = 0, 1, 2, \ldots$
De Moivre's formula: For any positive integer n,

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

n**th** **Roots of Complex Numbers:** If w be any non-zero complex number and n is positive integer such that $z^n = w$, then z is called n^{th} root of complex number w. It is denoted by

$$z = w^{1/n} \text{ or } \sqrt[n]{w}.$$

Let, $w = r(\cos\theta + i\sin\theta)$, then we may write
$w = r[\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k)]$, where k is a whole number.
Now, $w^{1/n} = z_k = [r(\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k))]^{\frac{1}{n}}$

$$= r^{1/n}\left[\cos\frac{(\theta + 2\pi \mathrm{k})}{n} + i\sin\left(\frac{\theta + 2\pi \mathrm{k}}{n}\right)\right]$$

Using De-moivre's theorem, where, $k = 0, 1, 2, \ldots\ldots, n-1$ Then z_k is n^{th} root of ω for $k = 0, 1, 2, \ldots\ldots..n-1$.

Example

Use De-Moivre's theorem to evaluate cube roots of unity.

Solution:

Here, $z = 1 = x + 0i \implies \theta = 0^\circ \, \text{and}, r = 1$

then, $z = \cos 0^\circ + i \sin 0^\circ = \cos\left(k \cdot 360^\circ + 0\right) + i \sin\left(k \cdot 360^\circ + 0\right)$

$= \cos \mathrm{k} \cdot 360^\circ + i \sin \mathrm{k} \cdot 360^\circ.$

Then, $z^{1/3} = z_k = \left[\cos k \cdot 360^\circ + i \sin k \cdot 360^\circ\right]^{1/3}$

So, when

$k = 0, \quad \mathrm{z}_0 = \cos 0 + i \sin 0 = 1 + i.0 = 1$

$k = 1, \quad z_1 = \cos \frac{360^\circ}{3} + i \sin \frac{360^\circ}{3} = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k = 2, \quad z_2 = \cos \frac{720}{3} + i \sin \frac{720^\circ}{3} = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\therefore \quad \sqrt[3]{1} = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

Exercise: Use De-Moivre's theorem to find the 5^{th} and 8^{th} roots of unity

Properties of Cube roots of Unity

Let ω an ω^2 are imaginary roots of unity, then we have

- Each imaginary cube root of unity is square of the other.
  i.e. $\omega^2 = \omega$
- The product of two imaginary cube roots of unity is 1.
  i.e. $\omega \cdot \omega^2 = \omega^3 = 1$
- Sum of all the cube roots of unity is zero.
  i.e. $1 + \omega + \omega^2 = 0$

If ω an ω^2 are imaginary roots of unity, then
(i) Verify all the properties of cube roots of unity.
(ii) Use above properties to show $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$

Square Roots of Complex Numbers

Consider a complex number $z = a + ib$
Let, $x + ib$ be the square root of $a + ib$
i.e. $x + iy = \sqrt{a + ib}$
Then, $(x + iy)^2 = a + ib$
or, $x^2 + 2ixy + i^2y^2 = a + ib$
or, $\left(x^2 - y^2\right) + i.2xy = a + ib$
Equating real and imaginary parts we get, $x^2 - y^2 = a \ldots\ldots$ (i) and $2xy = b \ldots\ldots (ii)$
Now, $\left(x^2 + y^2\right)^2 = \left(x^2 - y^2\right)^2 + 4x^2y^2 = a^2 + b^2$
$\therefore x^2 + y^2 = \sqrt{a^2 + b^2} \ldots\ldots$ (iii)
Adding (i) and (iii) we get,

$$2x^2 = a + \sqrt{a^2 + b^2} \implies x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

$$\therefore x = \pm \left(\frac{a + \sqrt{a^2 + b^2}}{2}\right)^{\frac{1}{2}}$$

Square Roots of Complex Numbers Continue

Again, subtracting (i) from (iii) we get,

$$2y^2 = \sqrt{a^2 + b^2 - a}$$
$$\text{or, } y^2 = \frac{\sqrt{a^2 + b^2 - a}}{2}$$
$$\text{or, } y = \pm \left(\frac{\sqrt{a^2 + b^2} - a}{2} \right)^{\frac{1}{2}}$$

$\therefore$ The required square roots of $a + ib$ are $x + iy$, where
$x = \pm \left(\frac{a + \sqrt{a^2 + b^2}}{2} \right)^{\frac{1}{2}}$ and $y = \pm \left(\frac{\sqrt{a^2 + b^2} - a}{2} \right)^{\frac{1}{2}}$

Example

Example: Find the square roots of $z = -8 + 6i$

Solution: let $a + ib$ be the square roots of $z = -8 + 6i = x + iy$ then,

$$a = \pm \left(\frac{x + \sqrt{x^2 + y^2}}{2} \right)^{1/2} = \pm \left(\frac{-8 - \sqrt{64 + 36}}{2} \right)^{1/2}$$

$$= \pm \left(\frac{-8 + 10}{2} \right)^{1/2} = \pm 1^{1/2} = \pm 1$$

$$b = \pm \left(\frac{\sqrt{x^2 + y^2} - x}{2} \right)^{1/2} = \pm \frac{\sqrt{64 + 36} + 8}{2} \Bigg)^{1/2}$$

$$= \pm \left(\frac{10 + 8}{2} \right)^{1/2} = \pm 3$$

$\therefore \pm(1 + 3i)$is the required square roots of given complex number

Exercise 1. Find the square roots of $5 - 12i$

2. Find the square roots of $7 - 24i$

Thank You