Mathematics II (BSM 102)

Manju Subedi

Gandaki University
Bachelor in Information Technology(BIT)
BSM 102

manjusubedi.061@gmail.com

June 14, 2023

Cauchy's Convergence Principle for Series:
A series $z_1 + z_2 + \cdots$ is convergent if and only if for every given $\epsilon > 0$ (no matter how small) we can find an N (which depends on ϵ, in general) such that $|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon$ for every $n > N$ and $p = 1, 2, \cdots$
Absolute Convergence: A series $z_1 + z_2 + \cdots$ is called absolutely convergent if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \cdots$$

is convergent.
If $z_1 + z_2 + \cdots$ converges but $|z_1| + |z_2| + \cdots$ diverges, then the series $z_1 + z_2 + \cdots$ is called, more precisely, **conditionally convergent.**
Example:
The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges, but only conditionally since the harmonic series diverges, as proved earlier.

P-series/ Comparison Test

P-series Test: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

Comparison Test:

i. Suppose there exists an integer N such that $0 \leq a_n \leq b_n$ for all $n \geq N$,
If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \geq b_n \geq 0$ for all $n \geq N$,
If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

(Compare either with p-series: $\sum_{i=1}^{\infty} \frac{1}{n^p}$ or with geometric series: $\sum ar^{n-1}$)

Examples:

$\sum_{n=1} \frac{1}{n^3 + 3n + 1}$

$\sum_{n=1} \frac{1}{2^n + 1}$

Limit Comparison Test:

Let $a_n, b_n \geq 0$ for all $n \geq 1$.

- If $\lim_{n\to\infty} a_n/b_n = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} a_n/b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Examples: Consider the two series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ and $\sum_{n=1}^{\infty} 1/n^2$.

These series are both p-series with $p = 1/2$ and $p = 2$, respectively. Since $p = 1/2 < 1$, the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges.

On the other hand, since $p = 2 > 1$, the series $\sum_{n=1}^{\infty} 1/n^2$ converges.

Ratio Test:

Ratio Test If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0 (n = 1, 2, \cdots)$ has the property that for every n greater than some N,

$$\left| \frac{z_{n+1}}{z_n} \right| \leqq q < 1$$

(where $q < 1$ is fixed), this series converges absolutely. If for every $n > N$,

$$\left| \frac{z_{n+1}}{z_n} \right| \geq 1$$

$$(n > N)$$

the series diverges.

Ratio Test

Ratio Test: If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0 (n = 1, 2, \cdots)$ is such that $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$, then:
a) If $L < 1$, the series converges absolutely.
b) If $L > 1$, the series diverges.
c) If $L = 1$, the series may converge or diverge, so that the test fails and permits no conclusion.

Cauchy's Root Test

Root Test: If a series $z_1 + z_2 + \cdots$ is such that for every n greater than some N,

$$\sqrt[n]{|z_n|} \leqq q < 1$$

$(n > N)$ (where $q < 1$ is fixed), this series converges absolutely. If for infinitely many n,

$$\sqrt[n]{|z_n|} \geq 1$$

the series diverges.

Root Test: If a series $z_1 + z_2 + \cdots$ is such that

$$\lim_{n \to \infty} \sqrt[n]{|z_n|} = L$$

then:

(a) The series converges absolutely if $L < 1$.

(b) The series diverges if $L > 1$.

(c) If $L = 1$, the test fails; that is, no conclusion is possible.

Properties of Convergence

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

i. The series

$$\sum_{n=1}^{\infty} (a_n + b_n)$$

converges and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(Sum Rule)

ii. The series $\sum_{n=1}^{\infty} (a_n - b_n)$ converges and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Exercise:

1.Test the absolute convergence of the following series:

(a) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$

(b) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+2}}{n^2}$

2. Test for convergence or divergence by ratio test:

(a) $\sum_{n=0}^{\infty} \dfrac{n!}{5^n}$

(b) $a_n = \dfrac{(-10)^n}{4^{2n+1}(n+1)}$

(c) $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$

Thank You