Module 04: Surface Water Hydraulics

Unit 03: Steady Channel Flow: Single/ Series

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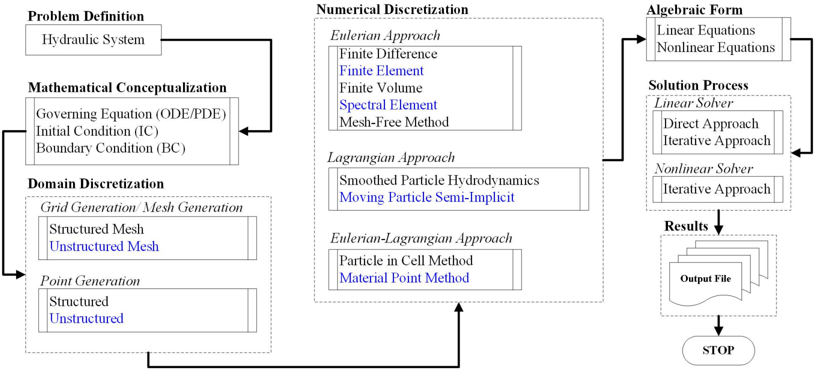
National Programme for Technology Enhanced Learning (NPTEL)

Learning Objective

- To solve steady channel flow problem (single or in series) using implicit method.

Problem Definition to Solution



Problem Definition

Governing Equation for Channel Flow can be written as,

Boundary Value Problem

Continuity Equation:

$$\frac{dQ}{dx} = 0$$

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Momentum Equation:

$$\frac{d\mathbb{E}}{dx} = -S_f$$

with

$$\mathbb{E} = y + z + \frac{\alpha Q^2}{2gA^2}$$

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with

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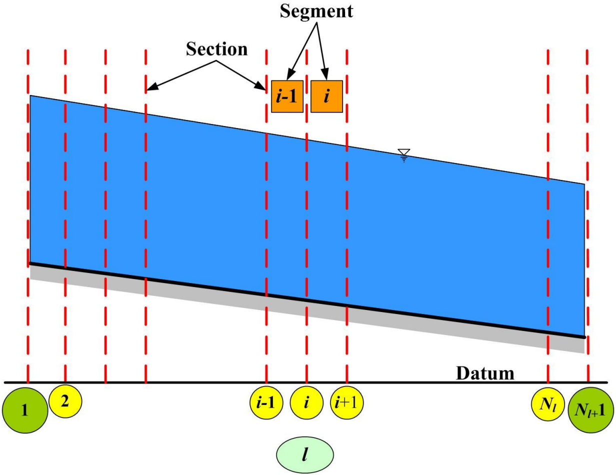
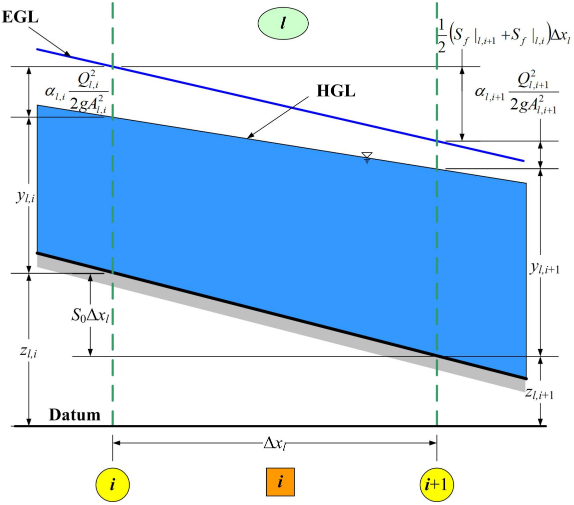
where

$y=$ depth of flow
$S_f=$ friction slope $\left(= \frac{n^2 Q^2}{R^{4/3} A^2}\right)$
$A=$ cross-sectional area
$R=$ hydraulic radius
$z=$ elevation of the channel bottom w.r.t. datum
$x=$ coordinate direction
$\alpha=$ momentum correction factor
$Q=$ discharge
$g=$ acceleration due to gravity

Channel Reach



Channel Flow



Discretization
Continuity Equation

Continuity equation for i^{th} segment of the l^{th} channel reach can be discretized as,

$$\frac{dQ}{dx} = 0$$
$$\frac{Q_{l,i+1} - Q_{l,i}}{\Delta x_l} = 0$$

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$l =$ index for channel number
$i =$ index for different sections within a channel reach.

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In simplified form for i^{th} segment of the l^{th} channel reach,

$$Q_{l,i+1} = Q_{l,i}$$

Discretization

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$l =$ index for channel number
$i =$ index for different sections within a channel reach.
In simplified form for i^{th} segment of the l^{th} channel reach,

$$Q_{l,i+1} = Q_{l,i}$$

For single channel,

$$Q_{l,1} = Q_{l,2} = \ldots = Q_{l,N_l+1} = Q_l$$

$N_l =$ number of segments for l^{th} channel reach.

Discretization

Momentum Equation

Momentum equation for i^{th} segment of the l^{th} channel reach can be discretized as,

$$\frac{d\mathbb{E}}{dx} = -S_f$$

$$\frac{\mathbb{E}_{l,i+1} - \mathbb{E}_{l,i}}{\Delta x_l} = -\frac{1}{2}\left(S_f\big|_{l,i+1} + S_f\big|_{l,i} \right)$$

Discretization
Momentum Equation

Momentum equation for i^{th} segment of the l^{th} channel reach can be discretized as,

$$\frac{d\mathbb{E}}{dx} = -S_f$$
$$\frac{\mathbb{E}_{l,i+1} - \mathbb{E}_{l,i}}{\Delta x_l} = -\frac{1}{2}\left(S_f\big|_{l,i+1} + S_f\big|_{l,i}\right)$$

In expanded form,

$$\frac{\left(y + z + \frac{\alpha Q^2}{2gA^2}\right)_{l,i+1} - \left(y + z + \frac{\alpha Q^2}{2gA^2}\right)_{l,i}}{\Delta x_l} = -\frac{1}{2}\left[\left(\frac{n^2 Q^2}{R^{4/3}A^2}\right)_{l,i+1} + \left(\frac{n^2 Q^2}{R^{4/3}A^2}\right)_{l,i}\right]$$

Discretization

Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q_{l,i+1}^2}{A_{l,i+1}^2} - \frac{Q_{l,i}^2}{A_{l,i}^2}\right) + \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q_{l,i+1}^2}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{Q_{l,i}^2}{R_{l,i}^{4/3} A_{l,i}^2}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

Discretization
Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q^2_{l,i+1}}{A^2_{l,i+1}} - \frac{Q^2_{l,i}}{A^2_{l,i}}\right) + \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q^2_{l,i+1}}{R^{4/3}_{l,i+1}A^2_{l,i+1}} + \frac{Q^2_{l,i}}{R^{4/3}_{l,i}A^2_{l,i}}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

In reduced form,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l Q_l^2}{2g}\left(\frac{1}{A^2_{l,i+1}} - \frac{1}{A^2_{l,i}}\right) + \frac{Q_l^2 n_l^2 \Delta x_l}{2}\left[\frac{1}{R^{4/3}_{l,i+1}A^2_{l,i+1}} + \frac{1}{R^{4/3}_{l,i}A^2_{l,i}}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

Discretization

Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q^2_{l,i+1}}{A^2_{l,i+1}} - \frac{Q^2_{l,i}}{A^2_{l,i}}\right)$$
$$+ \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q^2_{l,i+1}}{R^{4/3}_{l,i+1} A^2_{l,i+1}} + \frac{Q^2_{l,i}}{R^{4/3}_{l,i} A^2_{l,i}}\right], \quad \forall i \in \{1, \dots, N_l\}$$

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N_l non-linear equations with $N_l + 1$ unknown flow-depths

Discretization

Boundary Condition

For subcritical flows,

$$y_{l,N_l+1} = y_d$$
$$DB_{l,N_l+1} = y_{l,N_l+1} - y_d = 0$$

Discretization

Boundary Condition

For subcritical flows,

$$y_{l,N_l+1} = y_d$$
$$DB_{l,N_l+1} = y_{l,N_l+1} - y_d = 0$$

For supercritical flows,

$$y_{l,1} = y_u$$
$$UB_{l,N_l+1} = y_{l,1} - y_u = 0$$

Discretization

Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i}\left(y_{l,i+1}, y_{l,i}\right) = \left(y_{l,i+1} - y_{l,i}\right) + \left(z_{l,i+1} - z_{l,i}\right) + \frac{\alpha_l Q_l^2}{2g}\left(\frac{1}{A_{l,i+1}^2} - \frac{1}{A_{l,i}^2}\right)$$
$$+ \frac{Q_l^2 n_l^2 \Delta x_l}{2}\left[\frac{1}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{1}{R_{l,i}^{4/3} A_{l,i}^2}\right]$$

Discretization
Momentum Equation

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$$M_{l,i}\left(y_{l,i+1}, y_{l,i}\right) = \left(y_{l,i+1} - y_{l,i}\right) + \left(z_{l,i+1} - z_{l,i}\right) + \frac{\alpha_l Q_l^2}{2g}\left(\frac{1}{A_{l,i+1}^2} - \frac{1}{A_{l,i}^2}\right)$$
$$+ \frac{Q_l^2 n_l^2 \Delta x_l}{2}\left[\frac{1}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{1}{R_{l,i}^{4/3} A_{l,i}^2}\right]$$

Assuming

$$C_1 = \frac{\alpha_l Q_l^2}{2g} \quad \text{and} \quad C_2 = \frac{1}{2} Q_l^2 n_l^2 \Delta x_l$$

Discretization

Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i}\left(y_{l,i+1},y_{l,i}\right)=\left(y_{l,i+1}-y_{l,i}\right)+\left(z_{l,i+1}-z_{l,i}\right)+\frac{\alpha_l Q_l^2}{2g}\left(\frac{1}{A_{l,i+1}^2}-\frac{1}{A_{l,i}^2}\right)$$
$$+\frac{Q_l^2 n_l^2\Delta x_l}{2}\left[\frac{1}{R_{l,i+1}^{4/3}A_{l,i+1}^2}+\frac{1}{R_{l,i}^{4/3}A_{l,i}^2}\right]$$

Assuming

$$C_1=\frac{\alpha_l Q_l^2}{2g}\quad \text{and} \quad C_2=\frac{1}{2}Q_l^2 n_l^2\Delta x_l$$

$$M_{l,i}\left(y_{l,i+1},y_{l,i}\right)=\left(y_{l,i+1}-y_{l,i}\right)+\left(z_{l,i+1}-z_{l,i}\right)+C_1\left(\frac{1}{A_{l,i+1}^2}-\frac{1}{A_{l,i}^2}\right)$$
$$+C_2\left[\frac{1}{R_{l,i+1}^{4/3}A_{l,i+1}^2}+\frac{1}{R_{l,i}^{4/3}A_{l,i}^2}\right]$$

Algebraic Form

Momentum Equation

Elements of Jacobian Matrix can be calculated as,

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} = -1 + C_1 \frac{2}{A_{l,i}^3} \frac{dA}{dy}\Big|_{l,i} - C_2 \left[\frac{2}{A_{l,i}^3 R_{l,i}^{\frac{4}{3}}} \frac{dA}{dy}\Big|_{l,i} + \frac{4}{3A_{l,i}^2 R_{l,i}^{\frac{7}{3}}} \frac{dR}{dy}\Big|_{l,i} \right]$$

Algebraic Form

Momentum Equation

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$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}} = 1 - C_1 \frac{2}{A_{l,i+1}^3} \frac{dA}{dy}\Big|_{l,i+1} - C_2 \left[\frac{2}{A_{l,i+1}^3 R_{l,i+1}^{\frac{4}{3}}} \frac{dA}{dy}\Big|_{l,i+1} + \frac{4}{3A_{l,i+1}^2 R_{l,i+1}^{\frac{7}{3}}} \frac{dR}{dy}\Big|_{l,i+1} \right]$$

Algebraic Form

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Momentum Equation

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$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}} = 1 - C_1 \frac{2}{A_{l,i+1}^3} \frac{dA}{dy}\Big|_{l,i+1} - C_2 \left[\frac{2}{A_{l,i+1}^3 R_{l,i+1}^{\frac{4}{3}}} \frac{dA}{dy}\Big|_{l,i+1} + \frac{4}{3A_{l,i+1}^2 R_{l,i+1}^{\frac{7}{3}}} \frac{dR}{dy}\Big|_{l,i+1} \right]$$

For general channel cross-section,

$$\frac{dR}{dy} = \frac{T}{P} - \frac{R}{P}\frac{dP}{dy}$$

Algebraic Form
Boundary Conditions

For subcritical flows,

$$\frac{\partial DB_{l,N_l+1}}{\partial y_{l,N_l}} = 0$$
$$\frac{\partial DB_{l,N_l+1}}{\partial y_{l,N_l+1}} = 1$$

Algebraic Form
Boundary Conditions

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$$\frac{\partial DB_{l,N_l+1}}{\partial y_{l,N_l}} = 0$$

$$\frac{\partial DB_{l,N_l+1}}{\partial y_{l,N_l+1}} = 1$$

For supercritical flows,

$$\frac{\partial UB_{l,N_l+1}}{\partial y_{l,1}} = 1$$

$$\frac{\partial UB_{l,N_l+1}}{\partial y_{l,2}} = 0$$

Algebraic Form

Subcritical flow

In general form, governing equation including boundary condition can be written as,

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} \Delta y_{l,i} + \frac{\partial M_{l,i}}{\partial y_{l,i+1}} \Delta y_{l,i+1} = -M_{l,i}, \quad \forall i \in \{1, \ldots, N_l\}$$

For subcritical flow,

$$\Delta y_{l,N_l+1} = -DB_{l,N_l+1}$$

Jacobian Matrix Structure

Subcritical flow

$$
\begin{bmatrix}
\times & \times & & & & & & & & \\
 & \times & \times & & & & & & & \\
 & & \times & \times & & & & & & \\
 & & & \ddots & \ddots & & & & & \\
 & & & & \ddots & \ddots & & 0 & & \\
 & 0 & & & & \ddots & \ddots & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & \times & \times & \\
 & & & & & & & & \times & \times \\
 & & & & & & & & & \bullet
\end{bmatrix}
$$

Algebraic Form

Supercritical flow

In general form, governing equation including boundary condition can be written as,

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} \Delta y_{l,i} + \frac{\partial M_{l,i}}{\partial y_{l,i+1}} \Delta y_{l,i+1} = -M_{l,i}, \quad \forall i \in \{1, \ldots, N_l\}$$

For supercritical flow,

$$\Delta y_{l,1} = -UB_{l,N_l+1}$$

Jacobian Matrix Structure

Supercritical flow

$$
\begin{bmatrix}
\bullet & & & & & & & \\
\times & \times & & & & & & \\
 & \times & \times & & & & 0 & \\
 & & \times & \times & & & & \\
 & & & \ddots & \ddots & & & \\
 & 0 & & & \ddots & \ddots & & \\
 & & & & & \times & \times & \\
 & & & & & & \times & \times
\end{bmatrix}
$$

Channel Series

Junction Condition

Continuity

$$Q_{l,N_l+1} = Q_{l+1,1}$$

Energy

Neglecting losses,

$$y_{l,N_l+1} + z_{l,N_l+1} = y_{l+1,1} + z_{l+1,1}$$

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

$n = 0.015$

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

$n = 0.015$

$L_x = 200m$

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

$n = 0.015$

$L_x = 200m$

$Q = 20m^3/s$

Problem Statement

Single Channel

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

$n = 0.015$

$L_x = 200m$

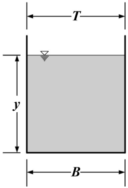
$Q = 20m^3/s$

$y_d = 0.60m$

Required

Estimate the flow depth across the channel reach.

Rectangular Cross-section



$$A = By$$

$$P = B + 2y$$

$$R = \frac{A}{P}$$

$$T = B$$

$$\frac{dR}{dy} = \frac{B^2}{(B+2y)^2}$$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

Problem Statement

Channels in Series

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Channel Cross-Section Type: Rectangular

$B = 15m$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

$L_{x2} = 100m$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

$L_{x2} = 100m$

$Q = 20m^3/s$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

$L_{x2} = 100m$

$Q = 20m^3/s$

$y_d = 0.60m$

Required

Estimate the flow depth across the channels in series.

List of Source Codes

Gradually Varied Flow-Implicit Approach

- Single Channel
  - steady_1D_channel_single.sci
- Channels in Series
  - steady_1D_channel_series.sci

Thank You