Journal of Hydraulic Research

ISSN: 0022-1686 (Print) 1814-2079 (Online) Journal homepage: https://www.tandfonline.com/loi/tjhr20

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Méthode de McCormack pour la simulation numérique unidimensionnelle des écoulements non permanents avec discontinuités dans des canaux à surface libre

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SUMMARY
This paper describes the use of the McCormack explicit finite difference scheme and the treatment of the boundary problem in the development of a one-dimensional simulation model that solves the St. Venant equations of the unsteady open channel flow. External and internal boundaries are considered. Various illustrative cases are presented to show the efficiency of this technique.

RÉSUMÉ
Dans cet article on décrit l'utilisation de la méthode explicite aux différences finies de McCormack et du traitement des conditions aux limites pour le développement d'une modélisation unidimensionnelle qui sert à résoudre les équations de St Venant pour des écoulements non permanents dans des canaux à surface libre. On a considéré des conditions aux limites intérieures et extérieures. On présente quelques cas illustratifs pour montrer le bon fonctionnement de cette méthode.

1 Introduction

It is generally accepted that one of the fundamental concerns in modern hydraulic engineering practice is the need for efficient and accurate numerical methods to solve the St. Venant equations. They should be able to numerically reproduce all the physically significant processes in the flow field.

Currently, finite difference techniques in the simulation of unsteady flow in open channels have become common as a predictive mathematical tool. In recent years a great variety of them have been applied and a comparison among them is extremely difficult to make [5, 6, 7]. A set of available criteria to evaluate the merits of a particular method would be useful. Unfortunately this is not realistic. Nevertheless, one may at least use some indicators to show the elements which are likely to influence the choice.

The purpose of this paper is to report the results of an investigation on the performance of a mathematical model based on the McCormack scheme. This explicit finite difference scheme

Revision received March 13, 1991. Open for discussion till August 31, 1992.

has been successfully applied to solve numerous problems in gas dynamics. It belongs to a class of fractional-step methods and is second order accurate in space and time. As a variant of the Lax-Wendroff scheme, it is a shock-capturing technique. Its main advantage is therefore its ability to simultaneously handle calculations of slowly varying flows as well as rapidly varying ones, containing shocks or discontinuities.

No solution of the differential equations will yield the correct results unless the boundary conditions are properly treated. Along the time evolution, the flow may vary its conditions (Froude number for instance) and the mathematical simulation should be able to automatically accommodate these changes. It is well known, that the method of characteristics is the only general technique for finding boundary conditions for explicit finite difference schemes.

It will be shown how this has been achieved and various examples of general one-dimensional practical cases will demonstrate the adaptability of the method to hydraulic studies.

2 Governing equations

The unsteady flow of water in a wide channel is governed by the one-dimensional shallow water equations. The fundamental hypothesis concerns the distribution of pressure in the vertical, which is assumed to be hydrostatic. Continuity of mass and momentum for a control volume under these conditions leads to a system of equations extensively described in the literature [1, 2]. For a prismatic channel with rectangular cross-section and smoothly varying bottom surface, they can be written in the conservative form:

$$\frac{\partial y}{\partial t}+\frac{1}{b}\frac{\partial Q}{\partial x}=0 \tag{2.1a}$$

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{by}+\frac{1}{2}gby^2\right)=gby(S_0-S_\mathrm{f}) \tag{2.1b}$$

Here S_0 is the bed slope and S_f corresponds to the friction slope. When approximated by the Manning formula, the latter takes the form:

$$S_\mathrm{f}=\frac{Q\,|\,Q\,|\,n^2}{A^2R^{4/3}} \tag{2.2}$$

n being the Manning roughness coefficient and R the hydraulic radius defined by $R=A/P$, with P representing the section wetted perimeter.

Equation (2.1b) expresses momentum conservation. Its right hand side acts as a momentum source or sink.

The system of partial differential equations (2.1) is hyperbolic and, as a consequence of its non-linearity, its solution may lead to spontaneous discontinuities which may have real physical meaning. They can be approached by weak solutions admitted by equations (2.1), since they are written in conservative form [3, 9]. Most of the conventional methods fail when employed to simulate such discontinuities. Therefore, the ability to numerically obtain these weak solutions is of considerable importance for certain applications. The technique to obtain them is described in the following sections.

3 The McCormack scheme

McCormack, in the process of studying the solution of the compressible Navier-Stokes equations, developed a non-centered, two-step finite difference scheme [14, 15]. Basically one-sided differences (forward or backward) are used to replace the spatial derivative. This presents three main advantages over the more common technique of using centered differences:

1. The program logic is simple because all dependent variables are calculated over the primary mesh points.
2. The inclusion of the free term is trivial.
3. The generalization to several space dimensions would be straightforward [13].

Using a compact expression of system (2.1) of the form:

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=H(U,x,t) \tag{3.1}$$

the solution at time $(n+1)\Delta t$ for the computational point i is obtained by application of the McCormack scheme written in a predictor-corrector sequence as follows:

$$U_i^{(1)}=U_i^n-\frac{\Delta t}{\Delta x}\left[(1-\varepsilon)F_{i+1}^n-(1-2\varepsilon)F_i^n-\varepsilon F_{i-1}^n\right]+\Delta t H_i^n \tag{3.2}$$

$$U_i^{n+1}=\tfrac{1}{2}(U_i^n+U_i^{(1)})-\frac{\Delta t}{2\Delta x}\left[\varepsilon F_{i+1}^{(1)}+(1-2\varepsilon)F_i^{(1)}+(\varepsilon-1)F_{i-1}^{(1)}\right]+\frac{\Delta t}{2}H_i^{(1)} \tag{3.3}$$

where superscript n indicates the time step and subscript i the computational mesh point. This allows two possibilities for replacing the space derivatives and gives rise to two versions of the scheme ($\varepsilon=0$ and $\varepsilon=1$). Some authors have suggested applying the two versions cyclically permuted to obtain the best results. The scheme has been termed preferential by Kutler [14] in the sense that the solution, when trying to reproduce shocks, will be more favorable depending on the version used.

The modified equation corresponding to McCormack's scheme (which reduces to Lax-Wendroff's for linear equations) shows that it is uniformly second-order accurate in both space and time [10]. It is consistent and is stable provided that the C.F.L. stability condition is fulfilled. According to Lax's theorem, the scheme is convergent [9]. The dissipation of the scheme is of order 4 and its dispersion of order 3, both of them being very sensitive to the Courant number. It is a shock-capturing technique [3] since it is capable of numerically predicting the location and intensity of predominant shock waves without explicit use of a shock-fitting procedure [4].

4 Boundary conditions

McCormack's scheme, like many other explicit time-marching schemes, can be used to advance the numerical solution one time step, Δt, from time $n\,\Delta t$ for all the computational points in a grid's row except for the first and last ones. If one of the flow variables is prescribed at one of these sections (boundary sections), a solution for the other dependent variable is still needed. It should be recalled that the only general technique available for the solution of this problem is the method of characteristics. Any other simplified way of dealing with it may lead to difficulties and errors.

The detailed description of the general principles of this method may be easily found in references [1, 8] and only its application to the boundary problem will be outlined here. It is useful to understand in an intuitive manner the boundary condition requirements.

The flow regime at the upstream or downstream end of a reach determines the boundary conditions required there. With subcritical flow, one physical condition ($y = y(t)$, $Q = Q(t)$ or $Q = Q(y)$) must be given. The other is then found by solving a difference equation based on the characteristic form of the flow equations:

$$\frac{\partial Q}{\partial t} + \left(\frac{Q}{A} \pm \sqrt{gy}\right)\frac{\partial Q}{\partial x} + b\left(-\frac{Q}{A} \pm \sqrt{gy}\right)\left[\frac{\partial y}{\partial t} + \left(\frac{Q}{A} \pm \sqrt{gy}\right)\frac{\partial y}{\partial x}\right] = gA(S_0 - S_f) \tag{4.1}$$

The first equation ($C+$, forward) is used at the end of the reach, the second ($C-$, backward), at the beginning. Since a fixed grid is being considered, the original method of characteristics is not used. Instead, a proper spatial interpolation is needed (Hartree method) [7]. In Fig. (1a) the grid points for a left-end (upstream) boundary are shown. An intermediate point R has to be added to make the solution progress to point M. The values of (y, Q) are known at points 1, 2. Those at point R, as well as its position, are determined using an iterative technique:

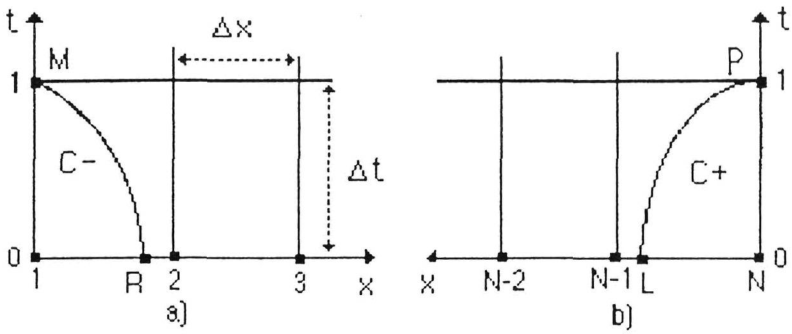

Fig. 1. Computational points involved to advance the solution one time step at the boundaries in the case of subcritical flow; a) upstream boundary; b) downstream boundary.

Points de calcul utilisés pour le calcul d'un pas de temps aux limites dans le cas d'un écoulement fluvial; a) limite amont; b) limite aval.

1st Approximation:

1. A first value of x_R is found by using the slope at point $(1, n)$:

$$x_R = x_M - \left(\frac{Q}{A} - \sqrt{gy}\right)_1^n \Delta t \tag{4.2}$$

2. Values for Q and y are found at R by linear interpolation after checking that inequality $x_1 < x_R < x_2$ holds. Otherwise, it would be necessary to restart again and change the interpolation to make use of point $(2, n)$.

$$Q_R = Q_2^n - (Q_2^n - Q_1^n)\frac{(x_2 - x_R)}{\Delta x} \tag{4.3}$$

$$y_R = y_2^n - (y_2^n - y_1^n)\frac{(x_2 - x_R)}{\Delta x} \tag{4.4}$$

3. Equation (4.1), in difference form, is used to obtain a condition between the variables at point M in the following way:

$$Q_M - Q_R + \left(-\frac{Qb}{A} - b\sqrt{gy}\right)_R (y_M - y_R) = \Delta t[gA(S_0 - S_f)]_R \tag{4.5}$$

4. The last expression can be written as:

$$Q_M = A_1 y_M + B_1 \tag{4.6}$$

where:

$$A_1 = \left(\frac{Qb}{A} + b\sqrt{gy}\right)_R \tag{4.7}$$

$$B_1 = Q_R + y_R\left(-\frac{Qb}{A} - b\sqrt{gy}\right)_R + g\,\Delta t[A(S_0 - S_f)]_R \tag{4.8}$$

are functions of the known quantities on row $n\,\Delta t$.

This provides the required relationship that completes the information given by the boundary condition. If the latter is of the type $y = y(t)$ or $Q = Q(t)$, the solution of the system of two equations is straightforward. If it consists of a rating curve $Q = Q(y)$, slightly more complex numerical techniques are necessary.

2nd and following approximations:

1′. Point R is more accurately located using

$$x_R = x_M - \frac{\Delta t}{2}\left[\left(\frac{Q}{A} - \sqrt{gy}\right)_M + \left(\frac{Q}{A} - \sqrt{gy}\right)_R\right] \tag{4.9}$$

2′. The interpolations (4.3) and (4.4) are repeated using the new value for x_R.

3′. Values of the dependent variables at point M are recalculated by averaging values of points R and the last approximation of M.

$$Q_M - Q_R + \frac{(y_M - y_R)}{2}\left[\left(-\frac{Qb}{A} - b\sqrt{gy}\right)_R + \left(-\frac{Qb}{A} - b\sqrt{gy}\right)_M\right] =$$

$$= \frac{\Delta t}{2}[(gA(S_0 - S_f))_R + (gA(S_0 - S_f))_M] \tag{4.10}$$

4′. The linear relationship (4.6) is now rewritten with improved coefficients.

Steps 1; to 4; are repeated until the desired accuracy is achieved.

Using the same procedure, the values of y and Q are determined for point P in the right-end boundary (Fig. 1b). It will lead to a similar linear relationship after every iteration step.

The preceeding algorithm must be changed if the flow in the neighborhood of the boundary is either supercritical or discontinuous [11, 12].

For supercritical flow, $x_R < x_1$ and the two characteristic curves have positive slope so that two boundary conditions must be given at the upstream end and none at the downstream end. The downstream boundary solution will be obtained using both equations (4.1) and the interpolation scheme, which should be adjusted accordingly.

In the case of discontinuous flow at the beginning or at the end of a reach, equations (4.1) cannot be used because they are valid only for continuous functions. The discontinuity or shock relationships should be used instead. It must be recalled that through a shock front which propagates with speed V the following equations for mass and momentum can be written [12].

$$(A_2 - A_1)V + Q_1 - Q_2 = 0 \tag{4.11a}$$

$$(Q_2 - Q_1)V + \left(\frac{Q^2}{A} + \frac{gby^2}{2}\right)_1 - \left(\frac{Q^2}{A} + \frac{gby^2}{2}\right)_2 = 0 \tag{4.11b}$$

Subscripts 1 and 2 denote values of the functions to the left and to the right of the front respectively. Eliminating the shock speed, they can be expressed:

$$\left(\frac{Q_2 - Q_1}{b(y_2 - y_1)}\right)^2 = (y_2 - y_1)^2 \frac{g(y_2 + y_1)}{2y_2 y_1} \tag{4.12}$$

If, for example, a discontinuous front exists starting at the beginning of a reach, where y_2, Q_2 are known (initial conditions), and Q_1 is given as boundary condition, then y_1 can be determined. If the flow behind the shock is subcritical, after a few steps the characteristic equations can be used again at the boundary. If the flow after the shock is supercritical upstream, equations (4.11) should be used to determine the values of (y, Q) at that point which will remain given there as two boundary conditions (if it is assumed that no other downstream information is available).
Analogous constraints are found for all the explicit schemes when an interior boundary is to be taken into account. It generally expresses compatibility conditions at interior points where the original St. Venant equations are not applicable and different laws must be introduced [1]. Examples of this are flow over weirs and channel junctions.
Internal boundaries are treated in the same way as it has been described for external ones. To illustrate how the equations would be formulated, the flow over an internal weir is considered. The end of the upstream reach (1) is treated as a downstream boundary whereas the beginning of reach (2) is treated as an upstream boundary (Fig. 2). In progressing from time level n to points L, R, four equations are necessary to solve for the two discharges and the two stages. When subcritical flow is considered on both sides, turning back to the preceeding discussion, we are allowed to make use of the linear relationships which hold along $C-$ and $C+$ characteristic curves passing through L and R respectively.
The two additional equations are provided by the mass continuity (the discharge must be the same on both sides of the weir) and a dynamic equation for the particular hydraulic device in the form of a free-flowing or flooded weir rating curve. When the flow becomes supercritical down-

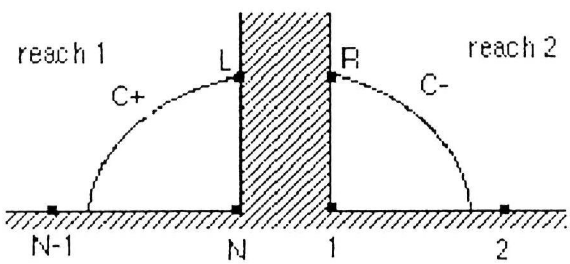

Fig. 2. Interior boundary condition. A weir is supposed to be placed between reaches 1 and 2 of the channel.

Condition à la limite intérieure. Un déversoir est supposé situé entre les biefs 1 et 2 du canal.

stream of the weir, the $C-$ curve leaving reach 2 through point R disappears. It must be replaced by an extra condition for the water stage at this location as, for instance, fixing the critical level there.
A continuous test of the flow regimes at the interior boundary points permits the automatic modification of the boundary conditions and the corresponding system of equations according to the flow variations.

5 Numerical results

Uniform motion of a shock
The preceeding technique was used to solve for the supercritical flow resulting from the sudden introduction of a subsequently constant discharge of 140 m^3/s into a rectangular, horizontal, frictionless channel 1 m wide in which, initially, water is at rest at a depth of 2 m.
Thus the upstream boundary condition during the first time step is the discontinuity ΔQ. The upstream depth which must be supplied as second boundary condition comes from the jump conditions equations (4.11). The new flow conditions at the first point are to be maintained so, since the flow is supercritical there, in the following time steps upstream boundary conditions will remain the same.
The results of the calculation for time $t = 40.5$s with $\Delta x = 10$ m and Δt given by the Courant criterion (variable for different steps of computation) are shown in Fig. 3. A comparison of version I (a) with version II (b) and with a combination of both of them (c) shows that the first one gives a more satisfactory solution. The intermediate procedure introduces less oscillations than version II but more than version I. All of them yield a spreading of the shock over a short distance (about three mesh intervals) reproducing a very accurate value of the depth just upstream of the front. A reduction on the Δx size would improve the result as a consequence of its 2nd order accuracy in space [4, 10]. On the other hand, how the solution wave properties can be worsened when using another way to obtain the value of the upstream water level that must be used as second boundary condition has been tested. Instead of using the jump relationships, the corresponding water depth has been calculated from the mass continuity equation discretized with McCormack's version I scheme. In Fig. 3d the profile of the water in the whole channel has been plotted.

Propagation and reflections of a shock wave
A similar shock wave is supposed to advance over still water in a wide, rectangular, horizontal and frictionless channel (t_1 in Fig. 4). The extreme downstream section is considered a dead-end, so zero discharge is imposed there.We are looking for the reflection of the advancing front which will travel upstream with increased height and reduced speed (leaving subcritical flow behind it) against the supercritical flow that is maintained by means of two suitable upstream boundary conditions (t_2 and t_3 in Fig. 4). When the reflected shock wave reaches the upstream end it is necessary to make use again of the discontinuity conditions. They will provide a new value for the height of the front according to the discharge that is supposed to enter continuously into the channel. This upstream second reflection of the wave originates also subcritical flow at the left end, so, after a few time steps, the characteristic equations should be applied again at both upstream and downstream ends (t_4 in Fig. 4). A control of the front position has been devised so that their reflections could be detected and the preferable version of the McCormack scheme accordingly changed. Parasitic oscillations have been eliminated as much as posible in this way.

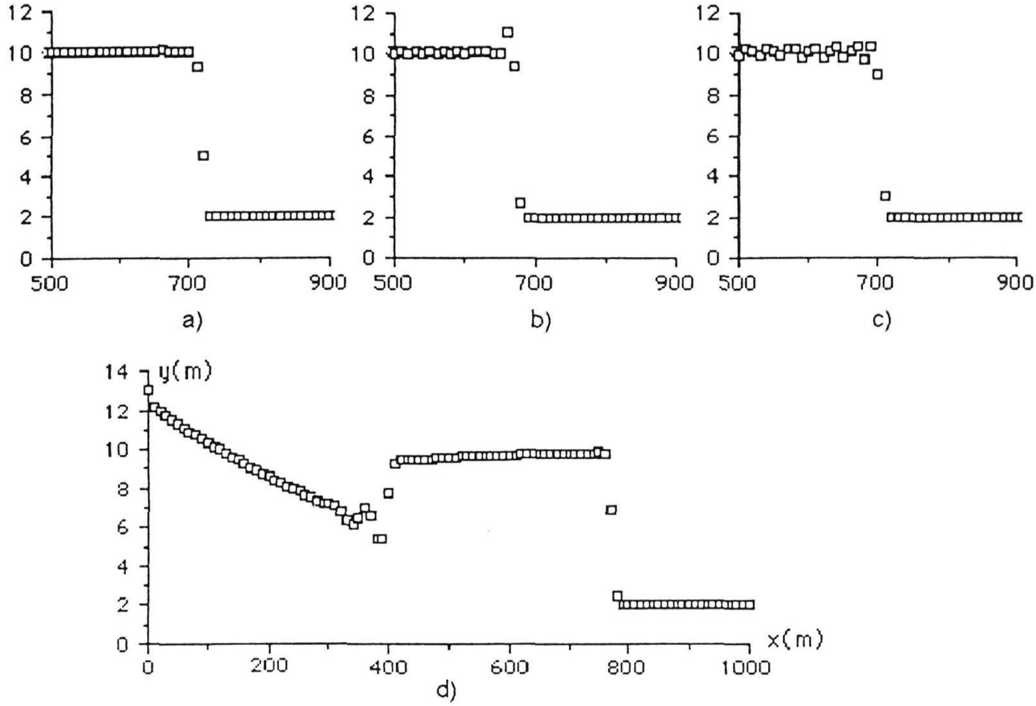

Fig. 3. a) Version I. b) Version II. c) Alternating versions. d) Calculation made using the mass continuity equation, to furnish the second upstream boundary condition, discretized with McCormack's version I scheme. In d) the profil of the water in the whole channel has been plotted.

a) Version I. b) Version II. c) Version mixte. d) Calcul fait à partir de l'équation de continuité en masse, pour fournir la seconde condition à la limite amont, discrétisée par le schéma de McCormack version I. d) Profil en long dans tout le canal.

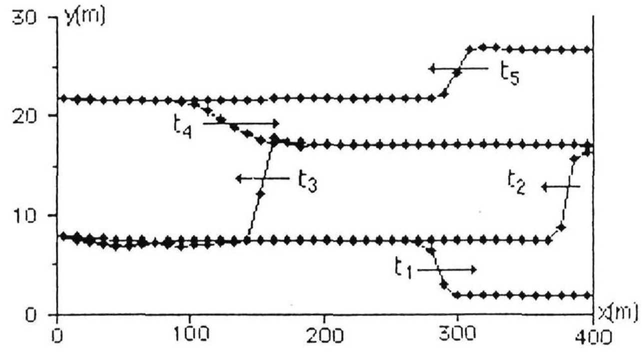

Fig. 4. Propagation and reflections of a shock wave in a rectangular frictionless channel.

Propagation et réflections d'une onde de choc dans un canal rectangulaire sans frottement.

Multiple-shock propagation

Fig. 5 shows the numerical solution obtained for a complex situation in which a shock front propagates over another existing previously. In a horizontal, unit-width, rectangular and frictionless channel of 1000 m, in which a water depth of 1 m is supposed to be initially at rest, a first discontinuous shock is originated by a rapid increase of the discharge from 0 to 11.9 m^3/s (the height of this front is $y = 2.7$ m from the jump conditions). It advances over the still water during 50 s ($t_1 = 30.4$ s in Fig. 5). A new modification of the discharge upstream from 11.9 to 47.62 m^3/s gives rise to a stronger, second shock 5.4 m high that travels faster than the first one ($t_2 = 56.8$ s and $t_3 = 87.7$ in Fig. 5) so that it catches up with it ($t_4 = 112.3$ s in Fig. 5). The Courant number has been chosen equal to 1 and the quality of the results is satisfactory. A comparison made by the authors between this solution and the one given by a third order explicit scheme showed that it is not worth going further for this kind of problems.

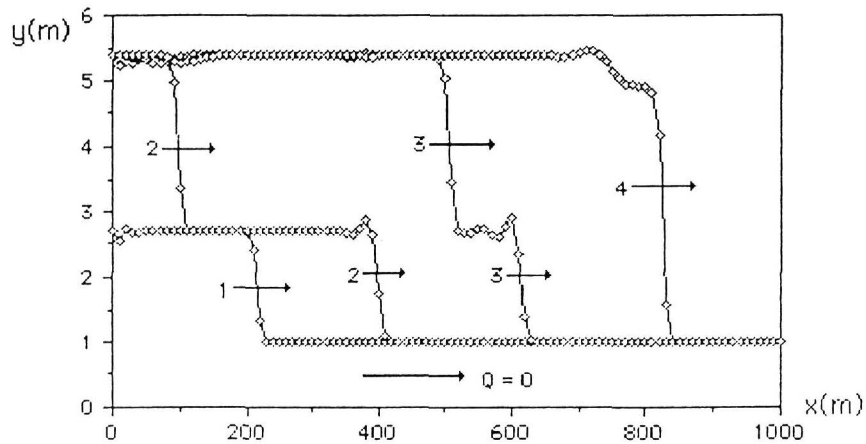

Fig. 5. Several profils of the water in a channel in which two different shocks propagate.
Plusieurs profils en long de l'eau dans un canal où se propagent deux chocs différents.

Ladder of cascades

As an example of application of the treatment of the internal boundary problem, the discontinuous stationary flow in a channel ($S_0 = 0.003$, $n = 0.009$, $b = 6$ m) that contains three identical weirs ($H_v = 0.25$ m) is shown on Fig. 6. Starting with the non stationary initial conditions $Q(x, 0) = 20$ m^3/s and $y(x, 0) = 2$ m, the numerical scheme locates the discontinuities of the corresponding stationary solution. Following the procedure described in §4, we have used the characteristic equations together with mass continuity and flow over a weir conditions while flow is in a subcritical state on both sides of the weir. Once flow becomes critical at the beginning of the right side of the weir, the condition of keeping fixed the critical depth at that point (instead of using the backward characteristic) is introduced as second boundary condition.

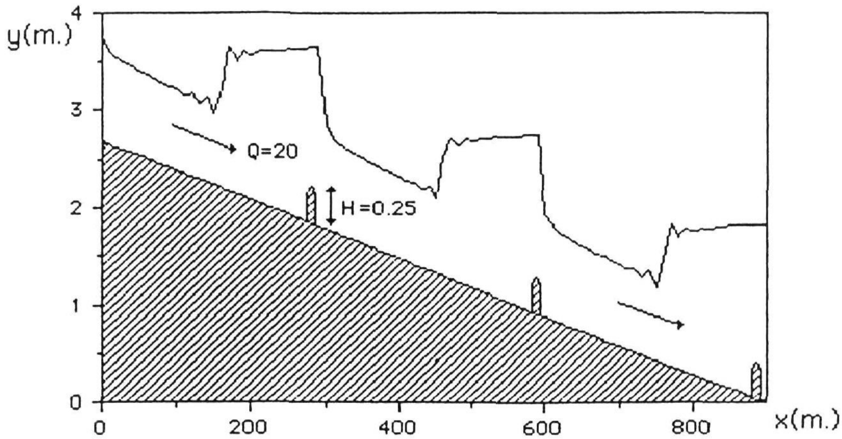

Fig. 6. Ladder of cascades. Final steady state of the flow in a slopping channel with three weirs which must be treated as interior boundary conditions.

Succession de cascades. Etat final permanent de l'écoulement dans un canal en pente avec trois déversoirs qui doivent être traités comme des conditions aux limites intérieures.

6 Conclusions

This paper has described the development of a mathematical model which solves the one-dimensional unsteady water flow equations. It is based on the McCormack second order explicit scheme, that has become an increasingly popular method for numerically integrating problems involving shocks.

The treatment of the external and internal boundaries has been made by means of the method of characteristics. This technique is the only correct one if we want to incorporate the true information in every problem.

The method is able to treat rapidly varying flow situations and relatively complex phenomena. It appears to introduce numerical oscillations which can be reduced by modifying the method of approximating the spatial derivative. The overall performance of the model can be considered good.

7 Acknowledgments

The work described in this paper forms part of a research activity developped by the author [16] with the funds of a *F.P.I.* grant from the Spanish *Ministerio de Educación y Ciencia.* The author is also grateful to Dr. J. Cunge of L. H. F. (Grenoble) for helpful suggestions and a critical reading of the manuscript.

Symbols

Q	flow discharge
y	flow depth
x	distance along channel
t	time
g	gravity acceleration
b	cross section width
A	cross section surface
P	wetted perimeter
R	hydraulic radius
n	Manning roughness coefficient
S_0	bottom slope
S_f	friction slope
V	shock speed

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