Module 02: Numerical Methods

Unit 11: Finite Volume Method: IBVP

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Learning Objectives

- To derive the algebraic form for a IBVP using Finite Volume Method.

General Equation

A form of differential equation with a general variable ϕ:

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla.(\Upsilon_\phi \phi \mathbf{u}) = \nabla.(\mathbf{\Gamma}_\phi.\nabla\phi) + F_{\phi_o} + S_\phi \tag{1}$$

where

ϕ = general variable

Λ_ϕ, Υ_ϕ = problem dependent parameters

$\mathbf{\Gamma}_\phi$ = tensor

F_{ϕ_o} = other forces

S_ϕ = source/sink term

Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega : \quad \Lambda_\phi \frac{\partial \phi}{\partial t} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y)$$

$$\Omega : \quad \Lambda_\phi \frac{\partial \phi}{\partial t} = \nabla \cdot (\mathbf{\Gamma} \cdot \nabla \phi) + S_\phi(x, y)$$

Problem Definition

subject to

Initial Condition

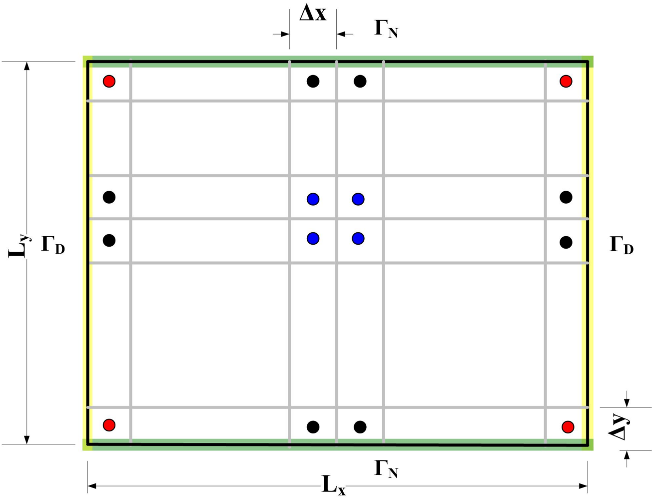
$$\phi(x, y, 0) = \phi_0(x, y)$$

and

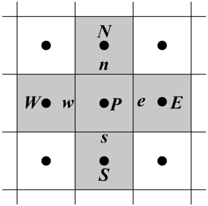
Boundary Condition

$$\begin{aligned}
\Gamma_D^1 &: \quad \phi(0, y, t) = \phi_1 \\
\Gamma_D^2 &: \quad \phi(L_x, y, t) = \phi_2 \\
\Gamma_N^3 &: \quad \left.\frac{\partial \phi}{\partial y}\right|_{(x,0,t)} = 0 \\
\Gamma_N^4 &: \quad \left.\frac{\partial \phi}{\partial y}\right|_{(x,L_y,t)} = 0
\end{aligned}$$

Domain Discretization

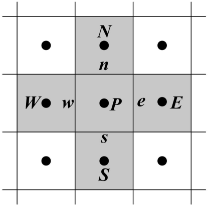


Discretization
Governing Equation



Discretization

Governing Equation



In Finite Volume Method, the governing equation is integrated over the element volume (in space) and time interval to form the discretized equation at node Point P (Versteeg and Malalasekra, 2008).

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P}\Lambda_\phi \frac{\partial \phi}{\partial t} d\Omega\right] dt = \int_{t}^{t+\Delta t}\left[\int_{\Omega_P}\nabla\cdot(\mathbf{\Gamma}\cdot\nabla\phi) d\Omega\right] dt + \int_{t}^{t+\Delta t}\left[\int_{\Omega_P} S_\phi(x,y) d\Omega\right] dt \tag{2}$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P} \Lambda_\phi \frac{\partial \phi}{\partial t} d\Omega\right] dt$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P}\Lambda_\phi\frac{\partial\phi}{\partial t}d\Omega\right]dt$$

$$=\Lambda_\phi\int_{t}^{t+\Delta t}\frac{\partial}{\partial t}\left(\int_{\Omega_P}\phi d\Omega\right)dt$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P}\Lambda_\phi\frac{\partial\phi}{\partial t}d\Omega\right]dt$$

$$=\Lambda_\phi\int_{t}^{t+\Delta t}\frac{\partial}{\partial t}\left(\int_{\Omega_P}\phi d\Omega\right)dt$$

$$=\Lambda_\phi\int_{t}^{t+\Delta t}\frac{\partial}{\partial t}\left(\phi_P\Delta\Omega_P\right)dt$$

Discretization
Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P}\Lambda_\phi\frac{\partial\phi}{\partial t}d\Omega\right]dt$$

$$=\Lambda_\phi\int_{t}^{t+\Delta t}\frac{\partial}{\partial t}\left(\int_{\Omega_P}\phi d\Omega\right)dt$$

$$=\Lambda_\phi\int_{t}^{t+\Delta t}\frac{\partial}{\partial t}\left(\phi_P\Delta\Omega_P\right)dt$$

$$=\Lambda_\phi(\phi_P^{l+1}-\phi_P^{l})\Delta\Omega_P$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P} \Lambda_\phi \frac{\partial \phi}{\partial t} d\Omega\right] dt$$

$$=\Lambda_\phi \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(\int_{\Omega_P} \phi d\Omega\right) dt$$

$$=\Lambda_\phi \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(\phi_P \Delta\Omega_P\right) dt$$

$$=\Lambda_\phi(\phi_P^{l+1} - \phi_P^{l})\Delta\Omega_P$$

$$=\Lambda_\phi(\phi_P^{l+1} - \phi_P^{l})\Delta x \Delta y$$

Discretization

Governing Equation: Spatial Term

$$\int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot(\mathbf{\Gamma}\cdot\nabla\phi)d\Omega\ dt = \int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot\left(\Gamma_x\frac{\partial\phi}{\partial x}\hat{i}+\Gamma_y\frac{\partial\phi}{\partial y}\hat{j}\right)d\Omega\ dt$$

Discretization

Governing Equation: Spatial Term

$$\int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot(\mathbf{\Gamma}\cdot\nabla\phi)d\Omega\ dt = \int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot\left(\Gamma_x\frac{\partial\phi}{\partial x}\hat{i}+\Gamma_y\frac{\partial\phi}{\partial y}\hat{j}\right)d\Omega\ dt$$

$$= \int_{t}^{t+\theta\Delta t}\int_{\Omega^P} \nabla\cdot\left(\Gamma_x\frac{\partial\phi}{\partial x}\hat{i}+\Gamma_y\frac{\partial\phi}{\partial y}\hat{j}\right)d\Omega\ dt + \int_{t+\theta\Delta t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot\left(\Gamma_x\frac{\partial\phi}{\partial x}\hat{i}+\Gamma_y\frac{\partial\phi}{\partial y}\hat{j}\right)d\Omega\ dt$$

Discretization

Governing Equation: Spatial Term

$$\int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot(\mathbf{\Gamma}\cdot\nabla\phi)d\Omega\ dt = \int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot\left(\Gamma_x\frac{\partial\phi}{\partial x}\hat{i}+\Gamma_y\frac{\partial\phi}{\partial y}\hat{j}\right)d\Omega\ dt$$

$$= \int_{t}^{t+\theta\Delta t}\int_{\Omega^P} \nabla\cdot\left(\Gamma_x\frac{\partial\phi}{\partial x}\hat{i}+\Gamma_y\frac{\partial\phi}{\partial y}\hat{j}\right)d\Omega\ dt + \int_{t+\theta\Delta t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot\left(\Gamma_x\frac{\partial\phi}{\partial x}\hat{i}+\Gamma_y\frac{\partial\phi}{\partial y}\hat{j}\right)d\Omega\ dt$$

$$= \left[\left(\Gamma_x\frac{\partial\phi}{\partial x}\right)_e^l A_{xe} - \left(\Gamma_x\frac{\partial\phi}{\partial x}\right)_w^l A_{xw} + \left(\Gamma_y\frac{\partial\phi}{\partial y}\right)_n^l A_{yn} - \left(\Gamma_y\frac{\partial\phi}{\partial y}\right)_s^l A_{ys}\right]\theta\Delta t +$$

$$\left[\left(\Gamma_x\frac{\partial\phi}{\partial x}\right)_e^{l+1} A_{xe} - \left(\Gamma_x\frac{\partial\phi}{\partial x}\right)_w^{l+1} A_{xw} + \left(\Gamma_y\frac{\partial\phi}{\partial y}\right)_n^{l+1} A_{yn} - \left(\Gamma_y\frac{\partial\phi}{\partial y}\right)_s^{l+1} A_{ys}\right](1-\theta)\Delta t$$

Discretization
Governing Equation

In a uniform grid system,
East Face:

$$\left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^l = \Gamma_{xe} \frac{\phi_E^l - \phi_P^l}{\Delta x} \quad \text{and} \quad \left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^{l+1} = \Gamma_{xe} \frac{\phi_E^{l+1} - \phi_P^{l+1}}{\Delta x} \tag{3}$$

Discretization
Governing Equation

In a uniform grid system,
East Face:

$$\left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^l = \Gamma_{xe} \frac{\phi_E^l - \phi_P^l}{\Delta x} \quad \text{and} \quad \left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^{l+1} = \Gamma_{xe} \frac{\phi_E^{l+1} - \phi_P^{l+1}}{\Delta x} \tag{3}$$

West Face:

$$\left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_w^l = \Gamma_{xw} \frac{\phi_P^l - \phi_W^l}{\Delta x} \quad \text{and} \quad \left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_w^{l+1} = \Gamma_{xw} \frac{\phi_P^{l+1} - \phi_W^{l+1}}{\Delta x} \tag{4}$$

Discretization
Governing Equation

In a uniform grid system,

East Face:

$$\left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^l = \Gamma_{xe} \frac{\phi_E^l - \phi_P^l}{\Delta x} \quad \text{and} \quad \left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^{l+1} = \Gamma_{xe} \frac{\phi_E^{l+1} - \phi_P^{l+1}}{\Delta x} \tag{3}$$

West Face:

$$\left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_w^l = \Gamma_{xw} \frac{\phi_P^l - \phi_W^l}{\Delta x} \quad \text{and} \quad \left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_w^{l+1} = \Gamma_{xw} \frac{\phi_P^{l+1} - \phi_W^{l+1}}{\Delta x} \tag{4}$$

North Face:

$$\left(\Gamma_y \frac{\partial \phi}{\partial y}\right)_n^l = \Gamma_{yn} \frac{\phi_N^l - \phi_P^l}{\Delta y} \quad \text{and} \quad \left(\Gamma_y \frac{\partial \phi}{\partial y}\right)_n^{l+1} = \Gamma_{yn} \frac{\phi_N^{l+1} - \phi_P^{l+1}}{\Delta y} \tag{5}$$

Discretization

Governing Equation

In a uniform grid system,

East Face:

$$\left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^l = \Gamma_{xe} \frac{\phi_E^l - \phi_P^l}{\Delta x} \quad \text{and} \quad \left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_e^{l+1} = \Gamma_{xe} \frac{\phi_E^{l+1} - \phi_P^{l+1}}{\Delta x} \tag{3}$$

West Face:

$$\left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_w^l = \Gamma_{xw} \frac{\phi_P^l - \phi_W^l}{\Delta x} \quad \text{and} \quad \left(\Gamma_x \frac{\partial \phi}{\partial x}\right)_w^{l+1} = \Gamma_{xw} \frac{\phi_P^{l+1} - \phi_W^{l+1}}{\Delta x} \tag{4}$$

North Face:

$$\left(\Gamma_y \frac{\partial \phi}{\partial y}\right)_n^l = \Gamma_{yn} \frac{\phi_N^l - \phi_P^l}{\Delta y} \quad \text{and} \quad \left(\Gamma_y \frac{\partial \phi}{\partial y}\right)_n^{l+1} = \Gamma_{yn} \frac{\phi_N^{l+1} - \phi_P^{l+1}}{\Delta y} \tag{5}$$

South Face:

$$\left(\Gamma_y \frac{\partial \phi}{\partial y}\right)_s^l = \Gamma_{ys} \frac{\phi_P^l - \phi_S^l}{\Delta y} \quad \text{and} \quad \left(\Gamma_y \frac{\partial \phi}{\partial y}\right)_s^{l+1} = \Gamma_{ys} \frac{\phi_P^{l+1} - \phi_S^{l+1}}{\Delta y} \tag{6}$$

Discretization

Governing Equation

In a uniform grid system,

$$
\begin{aligned}
A_{xe} &= A_{xw} = \Delta y \\
A_{yn} &= A_{ys} = \Delta x
\end{aligned}
\tag{7}
$$

Discretization

Governing Equation

In a uniform grid system,

$$\begin{aligned} A_{xe} &= A_{xw} = \Delta y \\ A_{yn} &= A_{ys} = \Delta x \end{aligned} \tag{7}$$

Source Term:

$$\int_{t}^{t+\Delta t} \int_{\Omega^P} S_\phi(x,y) d\Omega \, dt = \left[\theta S_\phi^{l}(x_P, y_P) + (1-\theta) S_\phi^{l+1}(x_P, y_P)\right] \Delta x \Delta y \Delta t \tag{8}$$

Discretization
Governing Equation

Compact Form of the equation can be written as,

$$
\begin{aligned}
&\Lambda_\phi(\phi_P^{l+1} - \phi_P^l)\Delta x \Delta y \\
&= \left[\left(\Gamma_x \frac{\partial \phi}{\partial x} \right)_e^l A_{xe} - \left(\Gamma_x \frac{\partial \phi}{\partial x} \right)_w^l A_{xw} \right] \theta \Delta t \\
&+ \left[\left(\Gamma_y \frac{\partial \phi}{\partial y} \right)_n^l A_{yn} - \left(\Gamma_y \frac{\partial \phi}{\partial y} \right)_s^l A_{ys} \right] \theta \Delta t \\
&+ \left[\left(\Gamma_x \frac{\partial \phi}{\partial x} \right)_e^{l+1} A_{xe} - \left(\Gamma_x \frac{\partial \phi}{\partial x} \right)_w^{l+1} A_{xw} \right] (1-\theta) \Delta t \\
&+ \left[\left(\Gamma_y \frac{\partial \phi}{\partial y} \right)_n^{l+1} A_{yn} - \left(\Gamma_y \frac{\partial \phi}{\partial y} \right)_s^{l+1} A_{ys} \right] (1-\theta) \Delta t \\
&+ \left[\theta S_\phi^l(x_P, y_P) + (1-\theta) S_\phi^{l+1}(x_P, y_P) \right] \Delta x \Delta y \Delta t
\end{aligned}
$$

Discretization
Governing Equation

Compact Form of the equation can be written as,

$$
\begin{aligned}
&\Lambda_\phi(\phi_P^{l+1} - \phi_P^l)\Delta x \Delta y \\
&= \left[\Gamma_{xe}\frac{\phi_E^l - \phi_P^l}{\Delta x}A_{xe} - \Gamma_{xw}\frac{\phi_P^l - \phi_W^l}{\Delta x}A_{xw}\right]\theta\Delta t \\
&+ \left[\Gamma_{yn}\frac{\phi_N^l - \phi_P^l}{\Delta y}A_{yn} - \Gamma_{ys}\frac{\phi_P^l - \phi_S^l}{\Delta y}A_{ys}\right]\theta\Delta t \\
&+ \left[\Gamma_{xe}\frac{\phi_E^{l+1} - \phi_P^{l+1}}{\Delta x}A_{xe} - \Gamma_{xw}\frac{\phi_P^{l+1} - \phi_W^{l+1}}{\Delta x}A_{xw}\right](1-\theta)\Delta t \\
&+ \left[\Gamma_{yn}\frac{\phi_N^{l+1} - \phi_P^{l+1}}{\Delta y}A_{yn} - \Gamma_{ys}\frac{\phi_P^{l+1} - \phi_S^{l+1}}{\Delta y}A_{ys}\right](1-\theta)\Delta t \\
&+ \left[\theta S_\phi^l(x_P, y_P) + (1-\theta)S_\phi^{l+1}(x_P, y_P)\right]\Delta x \Delta y \Delta t
\end{aligned}
$$

Discretization
Governing Equation

Compact Form of the equation can be written as,

$$\begin{aligned}
&\Lambda_\phi(\phi_P^{l+1}-\phi_P^l)\Delta x\Delta y\\
&=\left[\Gamma_{xe}\frac{\phi_E^l-\phi_P^l}{\Delta x}\Delta y-\Gamma_{xw}\frac{\phi_P^l-\phi_W^l}{\Delta x}\Delta y\right]\theta\Delta t\\
&+\left[\Gamma_{yn}\frac{\phi_N^l-\phi_P^l}{\Delta y}\Delta x-\Gamma_{ys}\frac{\phi_P^l-\phi_S^l}{\Delta y}\Delta x\right]\theta\Delta t\\
&+\left[\Gamma_{xe}\frac{\phi_E^{l+1}-\phi_P^{l+1}}{\Delta x}\Delta y-\Gamma_{xw}\frac{\phi_P^{l+1}-\phi_W^{l+1}}{\Delta x}\Delta y\right](1-\theta)\Delta t\\
&+\left[\Gamma_{yn}\frac{\phi_N^{l+1}-\phi_P^{l+1}}{\Delta y}\Delta x-\Gamma_{ys}\frac{\phi_P^{l+1}-\phi_S^{l+1}}{\Delta y}\Delta x\right](1-\theta)\Delta t\\
&+\left[\theta S_\phi^l(x_P,y_P)+(1-\theta)S_\phi^{l+1}(x_P,y_P)\right]\Delta x\Delta y\Delta t
\end{aligned}$$

Discretization

Governing Equation

Compact Form of the equation can be written as,

$$
\begin{aligned}
&\Lambda_\phi \frac{\phi_P^{l+1} - \phi_P^l}{\Delta t} \\
&= \theta \left[\Gamma_{xe} \frac{\phi_E^l - \phi_P^l}{\Delta x^2} - \Gamma_{xw} \frac{\phi_P^l - \phi_W^l}{\Delta x^2} \right] \\
&+ \theta \left[\Gamma_{yn} \frac{\phi_N^l - \phi_P^l}{\Delta y^2} - \Gamma_{ys} \frac{\phi_P^l - \phi_S^l}{\Delta y^2} \right] \\
&+ (1-\theta) \left[\Gamma_{xe} \frac{\phi_E^{l+1} - \phi_P^{l+1}}{\Delta x^2} - \Gamma_{xw} \frac{\phi_P^{l+1} - \phi_W^{l+1}}{\Delta x^2} \right] \\
&+ (1-\theta) \left[\Gamma_{yn} \frac{\phi_N^{l+1} - \phi_P^{l+1}}{\Delta y^2} - \Gamma_{ys} \frac{\phi_P^{l+1} - \phi_S^{l+1}}{\Delta y^2} \right] \\
&+ \left[\theta S_\phi^l(x_P, y_P) + (1-\theta) S_\phi^{l+1}(x_P, y_P) \right]
\end{aligned}
$$

Discretization
Governing Equation

Compact Form of the equation can be written as,

$$
\begin{aligned}
&\Lambda_\phi \frac{\phi_P^{l+1} - \phi_P^l}{\Delta t} \\
&= \theta \left[\Gamma_x \frac{\phi_E^l - 2\phi_P^l + \phi_W^l}{\Delta x^2} + \Gamma_y \frac{\phi_N^l - 2\phi_P^l + \phi_S^l}{\Delta y^2} \right] \\
&+ (1-\theta) \left[\Gamma_x \frac{\phi_E^{l+1} - 2\phi_P^{l+1} + \phi_W^{l+1}}{\Delta x^2} + \Gamma_y \frac{\phi_N^{l+1} - 2\phi_P^{l+1} + \phi_S^{l+1}}{\Delta y^2} \right] \\
&+ \left[\theta S_\phi^l(x_P, y_P) + (1-\theta) S_\phi^{l+1}(x_P, y_P) \right]
\end{aligned}
$$

With $\Gamma_{xe} = \Gamma_{xw} = \Gamma_x$,$\Gamma_{yn} = \Gamma_{ys} = \Gamma_y$

θ-Schemes

Explicit Scheme ($\theta = 1$)

$$\Lambda_\phi \frac{\phi_P^{l+1} - \phi_P^l}{\Delta t} = \Gamma_x \frac{\phi_E^l - 2\phi_P^l + \phi_W^l}{\Delta x^2} + \Gamma_y \frac{\phi_N^l - 2\phi_P^l + \phi_S^l}{\Delta y^2} + S_\phi^l(x_P, y_P)$$

θ-Schemes

Explicit Scheme ($\theta = 1$)

$$\Lambda_\phi \frac{\phi_P^{l+1} - \phi_P^l}{\Delta t} = \Gamma_x \frac{\phi_E^l - 2\phi_P^l + \phi_W^l}{\Delta x^2} + \Gamma_y \frac{\phi_N^l - 2\phi_P^l + \phi_S^l}{\Delta y^2} + S_\phi^l(x_P, y_P)$$

Implicit Scheme ($\theta = 0$)

$$\Lambda_\phi \frac{\phi_P^{l+1} - \phi_P^l}{\Delta t} = \Gamma_x \frac{\phi_E^{l+1} - 2\phi_P^{l+1} + \phi_W^{l+1}}{\Delta x^2} + \Gamma_y \frac{\phi_N^{l+1} - 2\phi_P^{l+1} + \phi_S^{l+1}}{\Delta y^2} + S_\phi^{l+1}(x_P, y_P)$$

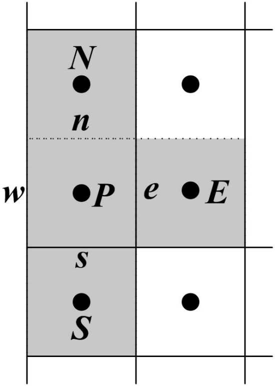
θ-Schemes

Crank-Nicolson Scheme ($\theta = \frac{1}{2}$)

$$
\begin{aligned}
2\Lambda_\phi \frac{\phi_P^{l+1} - \phi_P^l}{\Delta t} = {} & \Gamma_x \frac{\phi_E^l - 2\phi_P^l + \phi_W^l}{\Delta x^2} + \Gamma_y \frac{\phi_N^l - 2\phi_P^l + \phi_S^l}{\Delta y^2} \\
& + \Gamma_x \frac{\phi_E^{l+1} - 2\phi_P^{l+1} + \phi_W^{l+1}}{\Delta x^2} + \Gamma_y \frac{\phi_N^{l+1} - 2\phi_P^{l+1} + \phi_S^{l+1}}{\Delta y^2} \\
& + S_\phi^l(x_P, y_P) + S_\phi^{l+1}(x_P, y_P)
\end{aligned}
$$

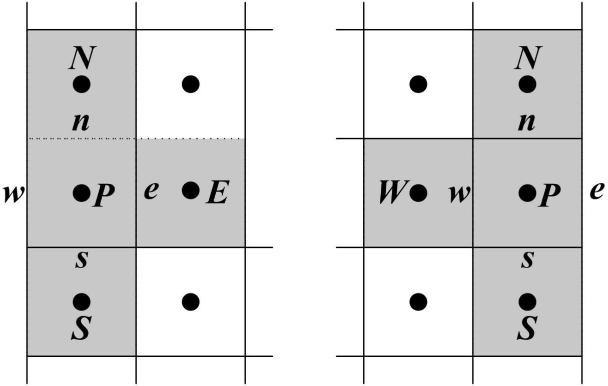
Boundary Conditions

Left and Right Boundary

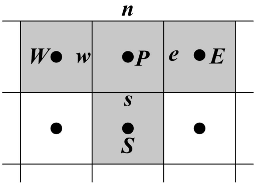


Boundary Conditions

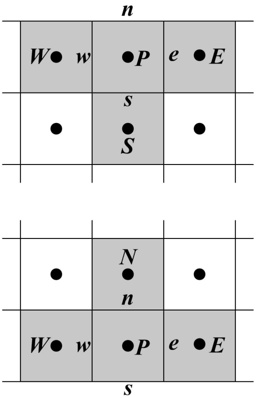
Left and Right Boundary



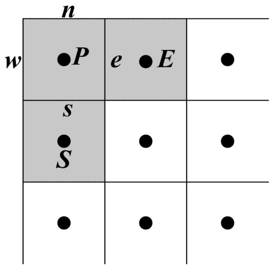
Boundary Conditions
Top and Bottom Boundary



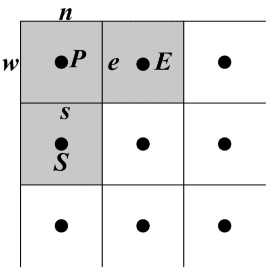
Boundary Conditions
Top and Bottom Boundary

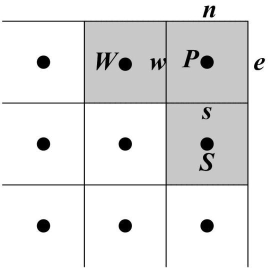


Boundary Conditions
Corners

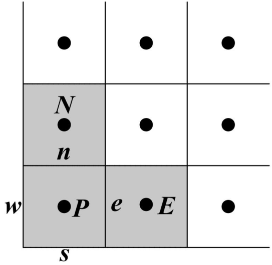


Boundary Conditions
Corners



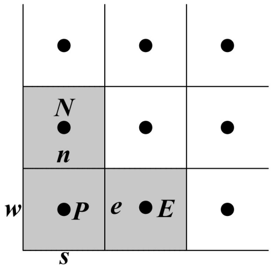


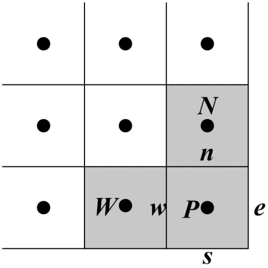
Boundary Conditions
Corners



Boundary Conditions

Corners





Thank You

References

Versteeg, H. and Malalasekra, W. (2008). *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Pearson, New Delhi.