Module 04: Surface Water Hydraulics

Unit 02: Gradually Varied Flow-Implicit Approach

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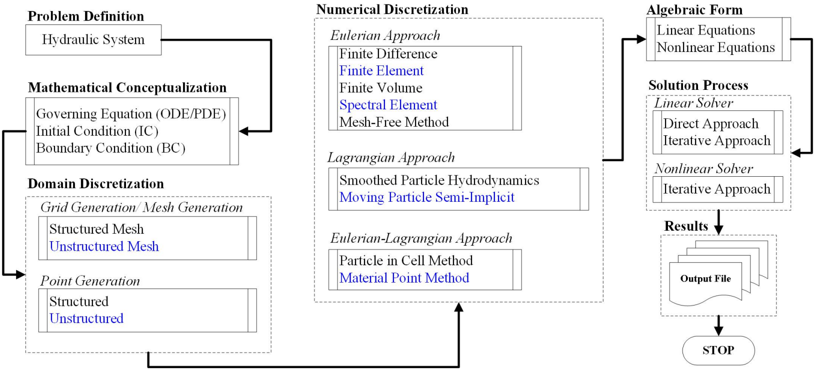
Learning Objective

- To solve gradually varied flow problem for open channels using implicit methods.

Problem Definition to Solution



Problem Definition

Governing Equation for Gradually Varied Flow in prismatic channel can be written as,

Initial Value Problem

$$\frac{dy}{dx} = \Psi(x, y) \quad \text{with} \quad \Psi(x, y) = \frac{S_0 - S_f}{1 - Fr^2} = \frac{S_0 - \frac{n^2 Q^2}{R^{4/3} A^2}}{1 - \frac{Q^2 T}{g A^3}}$$

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Initial Condition:

$$y|_{x=0} = y_0$$

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Initial Condition:

$$y|_{x=0} = y_0$$

where

$y=$ depth of flow

$S_f=$ friction slope $\left(= \frac{n^2 Q^2}{R^{4/3} A^2}\right)$

$S_0=$ bed slope

$T=$ top width

$R=$ hydraulic radius

$x=$ coordinate direction

$Fr=$ Froude number $\left(= \sqrt{\frac{Q^2 T}{g A^3}}\right)$

$Q=$ discharge

$g=$ acceleration due to gravity

$A=$ cross-sectional area

Problem Statement

Given

Channel Cross-Section Type: Rectangular

Problem Statement

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$y_0 = 0.8m$

Problem Statement

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$y_0 = 0.8m$

$B = 15m$

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$y_0 = 0.8m$

$B = 15m$

$g = 9.81m/s^2$

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Channel Cross-Section Type: Rectangular

$y_0 = 0.8m$

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$g = 9.81m/s^2$

$S_0 = 0.0008$

Problem Statement

Given

Channel Cross-Section Type: Rectangular

$y_0 = 0.8m$

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

$n = 0.015$

Problem Statement

Given

Channel Cross-Section Type: Rectangular

$y_0 = 0.8m$

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

$n = 0.015$

$L_x = 200m$

Problem Statement

Given

Channel Cross-Section Type: Rectangular

$y_0 = 0.8m$

$B = 15m$

$g = 9.81m/s^2$

$S_0 = 0.0008$

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$Q = 20m^3/s$

Required

Identify the type of GVF Profile: ?

Problem Statement

Given

Channel Cross-Section Type: Rectangular

$y_0 = 0.8m$

$B = 15m$

$g = 9.81m/s^2$

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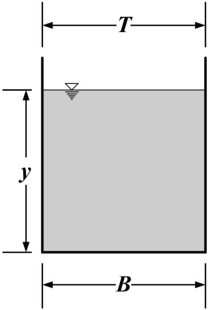
Required

Identify the type of GVF Profile: ?

Plot of the GVF Profile.

Rectangular Cross-section



$$A = By$$
$$P = B + 2y$$
$$R = \frac{A}{P}$$
$$T = B$$

Problem Definition

Critical Depth

For critical depth, $Fr = 1$

$$Fr = \sqrt{\frac{Q^2 T}{g A^3}} = 1$$

In case of rectangular channel, $A = By$ and $T = B$

$$\sqrt{\frac{Q^2 T}{g A^3}} = 1$$

$$y_c = \left(\frac{Q^2}{g B^2}\right)^{\frac{1}{3}}$$

Problem Definition

Normal Depth

Normal depth can be calculated from Manning's equation (uniform flow),

$$Q = \frac{1}{n} R^{\frac{2}{3}} S_0^{\frac{1}{2}} A$$

In case of rectangular channel, $A = By_n$ and $P = B + 2y_n$

$$Q = \frac{1}{n} \left(\frac{By_n}{B + 2y_n} \right)^{\frac{2}{3}} S_0^{\frac{1}{2}} By_n$$

In function form,

$$G(y_n) = \frac{S_0^{\frac{1}{2}} B^{\frac{5}{3}}}{n} \left(\frac{y_n}{B + 2y_n} \right)^{\frac{2}{3}} y_n - Q = 0$$

Problem Definition

Normal Depth

From Newton-Raphson method,

$$y_n\Big|^{(p)} = y_n\Big|^{(p-1)} - \frac{G\left(y_n\Big|^{(p-1)}\right)}{G'\left(y_n\Big|^{(p-1)}\right)}$$

where

$$G'\left(y_n\right) = \frac{S_0^{\frac{1}{2}} B^{\frac{5}{3}}}{3n} \frac{y_n^{\frac{2}{3}}\left(5B + 6y_n\right)}{\left(B + 2y_n\right)^{\frac{5}{3}}}$$

Implicit Runge-Kutta Methods

The Runge-Kutta method is defined as weighted assembly of increments by,

$$y_{n+1} = y_n + \sum_{j=1}^{m} W_j K_j$$

with $$K_i = \Delta x \Psi \left(x_n + c_i^x \Delta x, y_n + \sum_{j=1}^{i} c_{ij}^y K_j \right)$$

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Complete Butcher Tableau (Butcher, 2008) can be expressed as

$$\begin{array}{c|ccccc}
c_1^x & c_{1,1}^y & c_{1,2}^y & \cdots & c_{1,m-1}^y & c_{1,m}^y \\
c_2^x & c_{2,1}^y & c_{2,2}^y & \cdots & c_{2,m-1}^y & c_{2,m}^y \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{m-1}^x & c_{m-1,1}^y & c_{m-1,2}^y & \cdots & c_{m-1,m-1}^y & c_{m-1,m}^y \\
c_m^x & c_{m,1}^y & c_{m,2}^y & \cdots & c_{m,m-1}^y & c_{m,m}^y \\
\hline
 & W_1 & W_2 & \cdots & W_{m-1} & W_m
\end{array}$$

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\hline
 & W_1 & W_2 & \cdots & W_{m-1} & W_m
\end{array}$$

In reduced matrix form

$$\begin{array}{c|c}
\mathbf{c}^x & \mathbf{C}^y \\
\hline
 & \mathbf{W}^T
\end{array}$$

Backward Euler Method

Considering Butcher Tableau as

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

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$$y_{n+1} = y_n + \Delta x \Psi(x_{n+1}, y_{n+1})$$

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Order of Backward Euler method: $\mathcal{O}(\Delta x)$

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Order of Backward Euler method: $\mathcal{O}(\Delta x)$
In function form,

$$F(y_{n+1}) = y_{n+1} - \Delta x \Psi(x_{n+1}, y_{n+1}) - y_n = 0$$

Backward Euler Method

From Newton-Raphson method,

$$y_{n+1}\Big|^{(p)} = y_{n+1}\Big|^{(p-1)} - \frac{F\left(y_{n+1}\Big|^{(p-1)}\right)}{F'\left(y_{n+1}\Big|^{(p-1)}\right)}$$

where

$$F'\left(y_{n+1}\right) = 1 - \Delta x\left[\left(1 - \frac{Q^2}{B^2 g y_{n+1}^3}\right)^{-1}\left[\left(\frac{2n^2Q^2}{B^2y_{n+1}^3}\right)\left(\frac{By_{n+1}}{B+2y_{n+1}}\right)^{-\frac{4}{3}} + \left(\frac{4n^2Q^2}{3B^2y_{n+1}^2}\right)\left(\frac{By_{n+1}}{B+2y_{n+1}}\right)^{-\frac{7}{3}}\left(\frac{B}{B+2y_{n+1}} - \frac{2By_{n+1}}{(B+2y_{n+1})^2}\right)\right] - \left(\frac{3Q^2}{B^2gy_{n+1}^4}\right)\left(1 - \frac{Q^2}{B^2gy_{n+1}^3}\right)^{-2}\left[S_0 - \left(\frac{n^2Q^2}{B^2y_{n+1}^2}\right)\left(\frac{By_{n+1}}{B+2y_{n+1}}\right)^{-\frac{4}{3}}\right]\right]$$

Implicit Runge-Kutta

Increments can be written as

$$
\begin{aligned}
K_i &= \Delta x \Psi \left(x_n + c_i^x \Delta x, y_n + \sum_{j=1}^{i} c_{ij}^y K_j \right) \\
&= \Delta x \Psi \left(x_n + c_i^x \Delta x, y_n + \sum_{j=1}^{i-1} c_{ij}^y K_j + c_{ii}^y K_i \right) \\
&= \Delta x \Psi \left(x_n + \delta_x, y_n + \delta_y + c_{ii}^y K_i \right)
\end{aligned}
$$

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Increments can be written as

$$\begin{aligned} K_i &= \Delta x \Psi \left(x_n + c_i^x \Delta x, y_n + \sum_{j=1}^{i} c_{ij}^y K_j \right) \\ &= \Delta x \Psi \left(x_n + c_i^x \Delta x, y_n + \sum_{j=1}^{i-1} c_{ij}^y K_j + c_{ii}^y K_i \right) \\ &= \Delta x \Psi \left(x_n + \delta_x, y_n + \delta_y + c_{ii}^y K_i \right) \end{aligned}$$

$$\text{with} \quad \delta_x = c_i^x \Delta x \quad \text{and} \quad \delta_y = \sum_{j=1}^{i-1} c_{ij}^y K_j$$

The the multivariate function $\Psi()$ can be expanded as

$$\Psi \left(x_n + \delta_x, y_n + \delta_y + c_{ii}^y K_i \right) = \Psi \left(x_n + \delta_x, y_n + \delta_y \right) + c_{ii}^y \Psi' \left(x_n + \delta_x, y_n + \delta_y \right) K_i$$

Implicit Runge-Kutta

By combining all the expressions

$$K_i = \Delta x \left[\Psi\left(x_n + \delta_x, y_n + \delta_y\right) + c_{ii}^{y} \Psi'\left(x_n + \delta_x, y_n + \delta_y\right) K_i \right]$$

Implicit Runge-Kutta

By combining all the expressions

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In implicit compact form it can be written as

$$K_i = \Delta x \left[1 - c_{ii}^{y}\Delta x \Psi'\left(x_n + \delta_x, y_n + \delta_y\right)\right]^{-1} \Psi\left(x_n + \delta_x, y_n + \delta_y\right)$$

where

$$\Psi'(x, y) = \left(1 - \frac{Q^2}{B^2 g y^3}\right)^{-1} \left[\left(\frac{2n^2Q^2}{B^2y^3}\right)\left(\frac{By}{B+2y}\right)^{-\frac{4}{3}} + \left(\frac{4n^2Q^2}{3B^2y^2}\right)\left(\frac{By}{B+2y}\right)^{-\frac{7}{3}}\left(\frac{B}{B+2y} - \frac{2By}{(B+2y)^2}\right)\right] - \left(\frac{3Q^2}{B^2gy^4}\right)\left(1 - \frac{Q^2}{B^2gy^3}\right)^{-2}\left[S_0 - \left(\frac{n^2Q^2}{B^2y^2}\right)\left(\frac{By}{B+2y}\right)^{-\frac{4}{3}}\right]$$

Second Order RK Method (RK2)

Considering Butcher Tableau as

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

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RK2

$$y_{n+1} = y_n + K_1$$

$$\text{with} \quad K_1 = \Delta x \Psi \left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}K_1 \right)$$

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Order of RK2 method: $\mathcal{O}(\Delta x^2)$

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with $\quad K_1 = \Delta x \Psi \left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}K_1 \right)$

Order of RK2 method: $\mathcal{O}(\Delta x^2)$
Semi-Implicit Equation can be written as

$$K_1 = \Delta x \left[1 - \frac{1}{2}\Delta x \Psi' \left(x_n + \frac{1}{2}\Delta x, y_n \right) \right]^{-1} \Psi \left(x_n + \frac{1}{2}\Delta x, y_n \right)$$

Fourth Order RK Method (RK4)

Considering Butcher Tableau as

$$
\begin{array}{c|cc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
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\end{array}
$$

Fourth Order RK Method (RK4)

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RK4

$$y_{n+1} = y_n + \frac{1}{2}K_1 + \frac{1}{2}K_2$$

with

$$K_1 = \Delta x \Psi\left(x_n + c_1^x \Delta x, y_n + c_{11}^y K_1 + c_{12}^y K_2\right)$$

$$K_2 = \Delta x \Psi\left(x_n + c_2^x \Delta x, y_n + c_{21}^y K_1 + c_{22}^y K_2\right)$$

Fourth Order RK Method (RK4)

Considering Butcher Tableau as

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\begin{array}{c|cc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
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$$

RK4

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{2}K_1 + \frac{1}{2}K_2 \\
\text{with} \quad K_1 &= \Delta x \Psi\left(x_n + c_1^x \Delta x, y_n + c_{11}^y K_1 + c_{12}^y K_2\right) \\
K_2 &= \Delta x \Psi\left(x_n + c_2^x \Delta x, y_n + c_{21}^y K_1 + c_{22}^y K_2\right)
\end{aligned}
$$

Order of RK4 method: $\mathcal{O}(\Delta x^4)$

Fourth Order RK Method (RK4)

Expanded form of the increment equations can be written as

$$K_1 = \Delta x \left[\Psi \left(x_n + c_1^x \Delta x, y_n \right) + (c_{11}^y K_1 + c_{12}^y K_2) \Psi' \left(x_n + c_1^x \Delta x, y_n \right) \right]$$
$$K_2 = \Delta x \left[\Psi \left(x_n + c_2^x \Delta x, y_n \right) + (c_{21}^y K_1 + c_{22}^y K_2) \Psi' \left(x_n + c_2^x \Delta x, y_n \right) \right]$$

Fourth Order RK Method (RK4)

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$$K_2 = \Delta x \left[\Psi\left(x_n + c_2^x \Delta x, y_n\right) + (c_{21}^y K_1 + c_{22}^y K_2)\Psi'\left(x_n + c_2^x \Delta x, y_n\right)\right]$$

By rearranging the expressions

$$\begin{bmatrix} 1 - \Delta x c_{11}^y \Psi'\left(x_n + c_1^x \Delta x, y_n\right) & -\Delta x c_{12}^y \Psi'\left(x_n + c_1^x \Delta x, y_n\right) \\ -\Delta x c_{21}^y \Psi'\left(x_n + c_2^x \Delta x, y_n\right) & 1 - \Delta x c_{22}^y \Psi'\left(x_n + c_2^x \Delta x, y_n\right) \end{bmatrix} \begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix} = \begin{bmatrix} \Delta x \Psi\left(x_n + c_1^x \Delta x, y_n\right) \\ \Delta x \Psi\left(x_n + c_2^x \Delta x, y_n\right) \end{bmatrix}$$

List of Source Codes

Gradually Varied Flow-Implicit Approach

- Backward Euler approach
  - backward_euler.sci
- RK2 approach
  - RK2_implicit.sci
- RK4 approach
  - RK4_implicit.sci

Thank You

References

Butcher, J. C. (2008). *Numerical Methods for Ordinary Differential Equations*. John Wiley & Sons, Ltd, West Sussex, England.