# Unsaturated Flow

Geohydraulics| CE60113

Lecture:18

# Learning Objective(s)

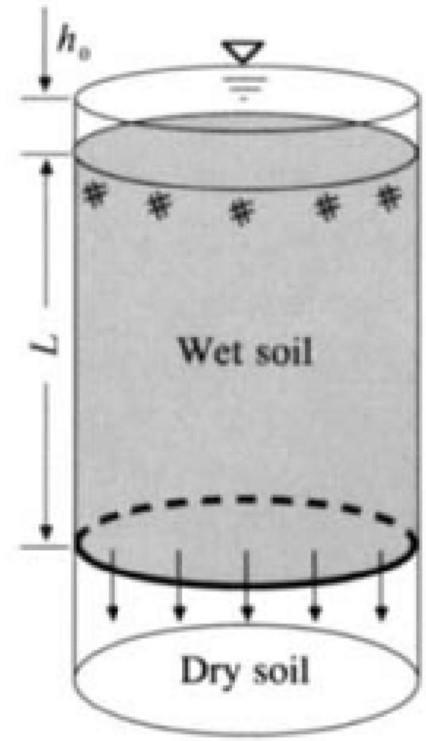
- **To estimate infiltration**

# Unsaturated Flow

- Continuity

$$F(t) = L(\eta - \theta_i)$$
$$= L\Delta\theta$$

where $\Delta\theta = \eta - \theta_i$.




- Momentum

$$q = -K\frac{\partial h}{\partial z}$$

$$f = K\left[\frac{h_1 - h_2}{z_1 - z_2}\right]$$

$$f = K\left[\frac{h_0 - (-\psi - L)}{L}\right]$$
$$\approx K\left[\frac{\psi + L}{L}\right]$$

$$f = K\left[\frac{\psi\Delta\theta + F}{F}\right]$$

$$\frac{dF}{dt} = K\left[\frac{\psi\Delta\theta + F}{F}\right]$$

# Unsaturated Flow(Contd.)

$$\left[\frac{F}{F+\psi\Delta\theta}\right]dF = K\,dt$$

$$\left[\left(\frac{F+\psi\Delta\theta}{F+\psi\Delta\theta}\right)-\left(\frac{\psi\Delta\theta}{F+\psi\Delta\theta}\right)\right]dF = K\,dt$$

$$\int_0^{F(t)}\left(1-\frac{\psi\Delta\theta}{F+\psi\Delta\theta}\right)dF = \int_0^t K\,dt$$

$$F(t)-\psi\Delta\theta\left\{\ln\left[F(t)+\psi\Delta\theta\right]-\ln\left(\psi\Delta\theta\right)\right\} = Kt$$

or

$$F(t)-\psi\Delta\theta\ \ln\left(1+\frac{F(t)}{\psi\Delta\theta}\right) = Kt$$

# Unsaturated Flow(Contd.)

- Infiltration rate

$$f(t) = K\left(\frac{\psi\,\Delta\theta}{F(t)} + 1\right)$$

- Cumulative Infiltration

$$F(t) = Kt + \psi\,\Delta\theta\,\ln\left(1 + \frac{F(t)}{\psi\,\Delta\theta}\right)$$

$$s_e = \frac{\theta - \theta_r}{\eta - \theta_r}$$

$$\Delta\theta = \eta - \theta_i = \eta - (s_e\theta_e + \theta_r)$$

$$\Delta\theta = (1 - s_e)\theta_e$$

# Unsaturated Flow(Contd.)

**TABLE 4.3.1**
**Green-Ampt infiltration parameters for various soil classes**

| Soil class | Porosity $\eta$ | Effective porosity $\theta_e$ | Wetting front soil suction head $\psi$ (cm) | Hydraulic conductivity $K$ (cm/h) |
|---|---|---|---|---|
| Sand | 0.437<br>(0.374–0.500) | 0.417<br>(0.354–0.480) | 4.95<br>(0.97–25.36) | 11.78 |
| Loamy sand | 0.437<br>(0.363–0.506) | 0.401<br>(0.329–0.473) | 6.13<br>(1.35–27.94) | 2.99 |
| Sandy loam | 0.453<br>(0.351–0.555) | 0.412<br>(0.283–0.541) | 11.01<br>(2.67–45.47) | 1.09 |
| Loam | 0.463<br>(0.375–0.551) | 0.434<br>(0.334–0.534) | 8.89<br>(1.33–59.38) | 0.34 |
| Silt loam | 0.501<br>(0.420–0.582) | 0.486<br>(0.394–0.578) | 16.68<br>(2.92–95.39) | 0.65 |
| Sandy clay loam | 0.398<br>(0.332–0.464) | 0.330<br>(0.235–0.425) | 21.85<br>(4.42–108.0) | 0.15 |
| Clay loam | 0.464<br>(0.409–0.519) | 0.309<br>(0.279–0.501) | 20.88<br>(4.79–91.10) | 0.10 |
| Silty clay loam | 0.471<br>(0.418–0.524) | 0.432<br>(0.347–0.517) | 27.30<br>(5.67–131.50) | 0.10 |
| Sandy clay | 0.430<br>(0.370–0.490) | 0.321<br>(0.207–0.435) | 23.90<br>(4.08–140.2) | 0.06 |
| Silty clay | 0.479<br>(0.425–0.533) | 0.423<br>(0.334–0.512) | 29.22<br>(6.13–139.4) | 0.05 |
| Clay | 0.475<br>(0.427–0.523) | 0.385<br>(0.269–0.501) | 31.63<br>(6.39–156.5) | 0.03 |

The numbers in parentheses below each parameter are one standard deviation around the parameter value given. *Source:* Rawls, Brakensiek, and Miller, 1983.

# Unsaturated Flow(Contd.)

- Two-layer Green-Ampt Model

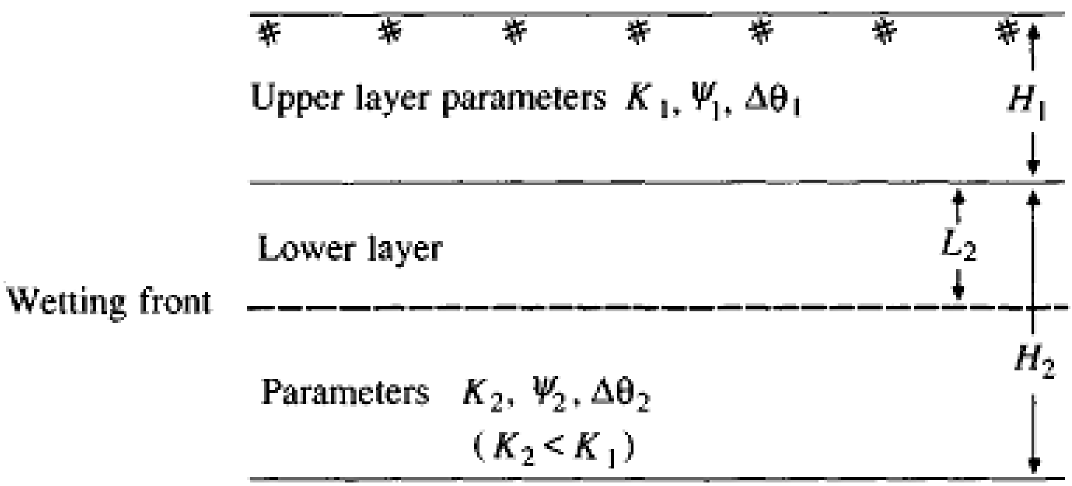



**FIGURE 4.3.4**
Parameters in a two-layer Green-Ampt model.

$$f = \frac{K_1 K_2}{H_1 K_2 + L_2 K_1}(\psi_2 + H_1 + L_2)$$

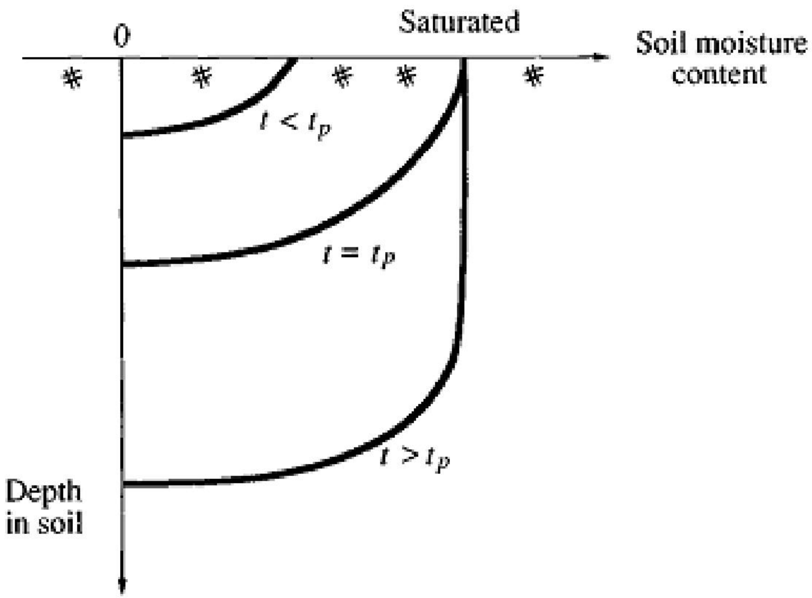
cumulative infiltration is given by

$$F = H_1 \Delta\theta_1 + L_2 \Delta\theta_2$$

$$L_2 \frac{\Delta\theta_2}{K_2} + \frac{1}{K_1 K_2}[\Delta\theta_2 H_1 K_2 - \Delta\theta_2 K_1(\psi_2 + H_1)] \ln\left[1 + \frac{L_2}{\psi_2 + H_1}\right] = t$$

# Unsaturated Flow(Contd.)

- PONDING TIME




$$f = K\left(\frac{\psi\,\Delta\theta}{F} + 1\right)$$

**FIGURE 4.4.1**
Soil moisture profiles before, during, and after ponding occurs.

the cumulative infiltration at the ponding time $t_p$ is given by

$$F_p = i\,t_p$$

the infiltration rate by $f = i$

# Unsaturated Flow(Contd.)

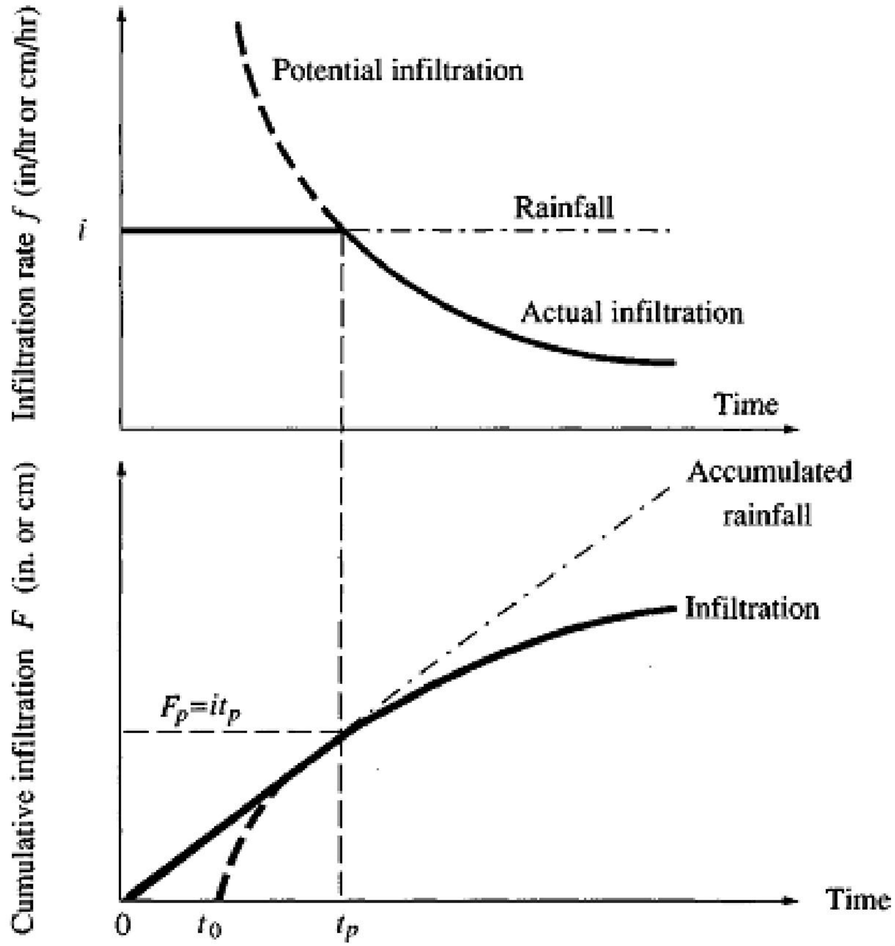



**FIGURE 4.4.2**
Infiltration rate and cumulative infiltration for ponding under constant intensity rainfall.

$$i = K\left(\frac{\psi\,\Delta\theta}{i\,t_p} + 1\right)$$

$$t_p = \frac{K\,\psi\,\Delta\theta}{i(i - K)}$$

# Unsaturated Flow(Contd.)

$$F = F_p$$

$$F_p - \psi\,\Delta\theta \ln\left(1 + \frac{F_p}{\psi\,\Delta\theta}\right) = K(t_p - t_0) \qquad t = t_p - t_0$$

For $t > t_p$,

$$F - \psi\,\Delta\theta \ln\left(1 + \frac{F}{\psi\,\Delta\theta}\right) = K(t - t_0)$$

$$F - F_p - \psi\,\Delta\theta\left[\ln\left(\frac{\psi\,\Delta\theta + F}{\psi\,\Delta\theta}\right) - \ln\left(\frac{\psi\,\Delta\theta + F_p}{\psi\,\Delta\theta}\right)\right] = K(t - t_p)$$

$$F - F_p - \psi\,\Delta\theta \ln\left[\frac{\psi\,\Delta\theta + F}{\psi\,\Delta\theta + F_p}\right] = K(t - t_p)$$

# Unsaturated Flow(Contd.)

- Total Head

$$h = z + \psi$$

- Darcy's Law

$$\mathbf{q} = -K(\theta)\nabla h$$

- Relative Permeability

$$k_r(\theta) = \frac{K(\theta)}{K_s}$$

$$0 \leq k_r(\theta) \leq 1$$

- Expanded form of Darcy's Law

$$\begin{aligned}\mathbf{q} &= -K_s k_r(\theta)\nabla(z + \psi\,) \\ &= -K_s k_r(\theta)\mathbf{e_z} - K_s k_r(\theta)\nabla\psi\end{aligned}$$

# Unsaturated Flow(Contd.)

- Continuity Equation

$$\frac{\partial(\rho\theta)}{\partial t} + \nabla \cdot (\rho\mathbf{q}) = \rho^* q$$

- Temporal Term:

$$\frac{\partial(\rho\theta)}{\partial t} = \frac{\partial(\rho\eta S_w)}{\partial t} = \eta S_w \frac{\partial\rho}{\partial t} + \rho S_w \frac{\partial\eta}{\partial t} + \eta\rho \frac{\partial S_w}{\partial t}$$

- First Term

$$\frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial\rho}{\partial C}\frac{\partial C}{\partial t} = \rho\beta \frac{\partial p}{\partial t} + \frac{\partial\rho}{\partial C}\frac{\partial C}{\partial t}$$

- Second Term

$$\frac{\partial\eta}{\partial t} = \frac{\partial\eta}{\partial p}\frac{\partial p}{\partial t} = (1-\eta)\alpha \frac{\partial p}{\partial t}$$

- Discharge Vector

$$\mathbf{q} = -\mathbf{K} \cdot \nabla h = -\mathbf{K} \cdot \nabla(\psi + z) = -\frac{\mathbf{k}}{\mu} \cdot (\nabla p + \rho g \nabla z)$$

# Unsaturated Flow(Contd.)

- Continuity equation can be written as

$$\rho\left[\theta\beta+\frac{\theta}{\eta}(1-\eta)\alpha\right]\frac{\partial p}{\partial t}+\theta\frac{\partial\rho}{\partial C}\frac{\partial C}{\partial t}+\rho\eta\frac{\partial S_w}{\partial t}=\nabla\cdot\left[\frac{\rho\mathbf{k}}{\mu}\cdot(\nabla p+\rho g\nabla z)\right]+\rho^*q$$

- Relationship between pressure and reference pressure head

$$dp=\rho g d\psi=\rho g\frac{d\psi}{d\theta}d\theta=\rho g\frac{d\psi}{d\theta}(\eta\, dS_w+S_w\, d\eta)\approx\rho g\frac{d\psi}{d\theta}(\eta\, dS_w)\ [\because d\eta\ll dS_w]$$

- Rearranging the terms

$$\frac{1}{g}\frac{d\theta}{d\psi}\frac{\partial p}{\partial t}=\rho\eta\frac{\partial S_w}{\partial t}$$

- Substituting values

$$\rho\left[\theta\beta+\frac{\theta}{\eta}(1-\eta)\alpha\right]\frac{\partial p}{\partial t}+\theta\frac{\partial\rho}{\partial C}\frac{\partial C}{\partial t}+\frac{1}{g}\frac{d\theta}{d\psi}\frac{\partial p}{\partial t}=\nabla\cdot\left[\frac{\rho\mathbf{k}}{\mu}\cdot(\nabla p+\rho g\nabla z)\right]+\rho^*q$$

- Reference pressure head

$$\hbar=\frac{p}{\rho_0 g}$$

- Relationship between pressure head and reference pressure head

$$\rho_0 d\hbar=\rho d\psi$$

# Unsaturated Flow(Contd.)

- Continuity equation can be written as

$$\rho\left[\theta\beta+\frac{\theta}{\eta}(1-\eta)\alpha\right]\rho_0 g\frac{\partial\hbar}{\partial t}+\theta\frac{\partial\rho}{\partial C}\frac{\partial C}{\partial t}+\frac{1}{g}\frac{\rho}{\rho_0}\frac{d\theta}{d\hbar}\rho_0 g\frac{\partial\hbar}{\partial t}=\nabla\cdot\left[\frac{\rho\mathbf{k}}{\mu}\cdot(\rho_0 g\nabla\hbar+\rho g\nabla z)\right]+\rho^* q$$

- Defining modified compressibilities of media and water as

$$\alpha' = (1-\eta)\alpha\rho_0 g$$
$$\beta' = \beta\rho_0 g$$

- Final form can be written as

$$\frac{\rho}{\rho_0}\left[\theta\beta'+\frac{\theta}{\eta}\alpha'+\frac{d\theta}{d\hbar}\right]\frac{\partial\hbar}{\partial t}+\frac{\theta}{\rho_0}\frac{\partial\rho}{\partial C}\frac{\partial C}{\partial t}=\nabla\cdot\left[\frac{\rho g\mathbf{k}}{\mu}\cdot\left(\nabla\hbar+\frac{\rho}{\rho_0}\nabla z\right)\right]+\frac{\rho^*}{\rho_0}q$$

- In compact form

$$\frac{\rho}{\rho_0}F\frac{\partial\hbar}{\partial t}+\frac{\theta}{\rho_0}\frac{\partial\rho}{\partial C}\frac{\partial C}{\partial t}=\nabla\cdot\left[\mathbf{K}\cdot\left(\nabla\hbar+\frac{\rho}{\rho_0}\nabla z\right)\right]+\frac{\rho^*}{\rho_0}q$$

where

$$\frac{\rho}{\rho_0}=a_0+a_1C+a_2C^2+a_3C^3$$
$$\frac{\mu}{\mu_0}=b_0+b_1C+b_2C^2+b_3C^3$$

# Unsaturated Flow(Contd.)

- In case of seawater intrusion

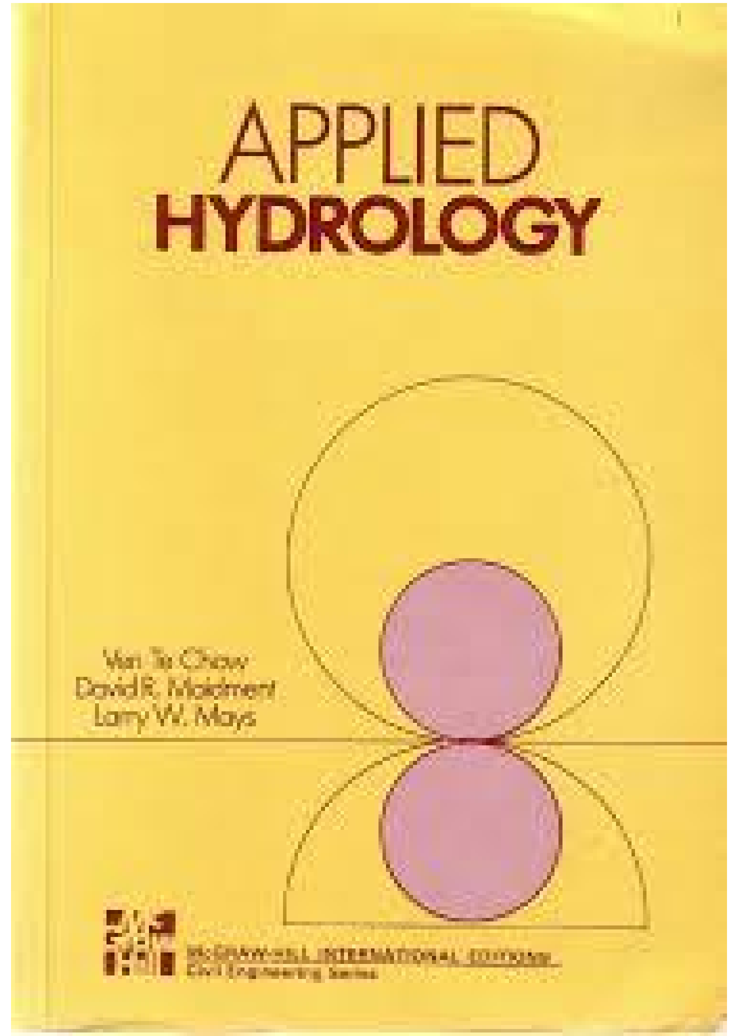
$$\frac{\rho}{\rho_0} = 1 + \epsilon C$$

where

$$\epsilon = \frac{\rho_{max}}{\rho_0} - 1$$

# Learning Strategy

Chapter 4: Subsurface Water



# Thank you