



## Module 02: Numerical Methods

### Unit 21: Algebraic Equation: Gauss Elimination Method

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## Learning Objective

- To apply Gauss Elimination Method for direct solution.



# Matrix Form

## Full Matrix

$$\mathbf{A}\phi = \mathbf{r}$$



# Matrix Form

## Full Matrix

$$\mathbf{A}\phi = \mathbf{r}$$

$$\begin{pmatrix} \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \end{pmatrix}_{N \times N} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \\ \phi_N \end{pmatrix}_{N \times 1} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{pmatrix}_{N \times 1}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

### Row 2

Let  $\gamma_1^2 = \frac{a_{21}}{a_{11}}$

Row 2	$a_{21}\phi_1 + a_{22}\phi_2 + a_{23}\phi_3 + a_{24}\phi_4 + a_{25}\phi_5 = r_2$
$\gamma_1^2 \times$ Row 1	$a_{21}\phi_1 + \gamma_1^2 a_{12}\phi_2 + \gamma_1^2 a_{13}\phi_3 + \gamma_1^2 a_{14}\phi_4 + \gamma_1^2 a_{15}\phi_5 = \gamma_1^2 r_1$
Updated Row 2	$a'_{22}\phi_2 + a'_{23}\phi_3 + a'_{24}\phi_4 + a'_{25}\phi_5 = r'_2$

with

$$a'_{22} = a_{22} - \gamma_1^2 a_{12}, \quad a'_{23} = a_{23} - \gamma_1^2 a_{13}$$

$$a'_{24} = a_{24} - \gamma_1^2 a_{14}, \quad a'_{25} = a_{25} - \gamma_1^2 a_{15}$$

$$r'_2 = r_2 - \gamma_1^2 r_1$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

### Row 3

Let  $\gamma_1^3 = \frac{a_{31}}{a_{11}}$

Row 3	$a_{31}\phi_1 + a_{32}\phi_2 + a_{33}\phi_3 + a_{34}\phi_4 + a_{35}\phi_5 = r_3$
$\gamma_1^3 \times$ Row 1	$a_{31}\phi_1 + \gamma_1^3 a_{12}\phi_2 + \gamma_1^3 a_{13}\phi_3 + \gamma_1^3 a_{14}\phi_4 + \gamma_1^3 a_{15}\phi_5 = \gamma_1^3 r_1$
Updated Row 2	$a'_{32}\phi_2 + a'_{33}\phi_3 + a'_{34}\phi_4 + a'_{35}\phi_5 = r'_3$

with

$$a'_{32} = a_{32} - \gamma_1^3 a_{12}, \quad a'_{33} = a_{33} - \gamma_1^3 a_{13}$$

$$a'_{34} = a_{34} - \gamma_1^3 a_{14}, \quad a'_{35} = a_{35} - \gamma_1^3 a_{15}$$

$$r'_3 = r_3 - \gamma_1^3 r_1$$





# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & a'_{32} & a'_{33} & a'_{34} & a'_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r'_3 \\ r_4 \\ r_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & a'_{32} & a'_{33} & a'_{34} & a'_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r'_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

**Row 4**

Let  $\gamma_1^4 = \frac{a_{41}}{a_{11}}$

Row 4	$a_{41}\phi_1 + a_{42}\phi_2 + a_{43}\phi_3 + a_{44}\phi_4 + a_{45}\phi_5 = r_4$
$\gamma_1^4 \times$ Row 1	$a_{41}\phi_1 + \gamma_1^4 a_{12}\phi_2 + \gamma_1^4 a_{13}\phi_3 + \gamma_1^4 a_{14}\phi_4 + \gamma_1^4 a_{15}\phi_5 = \gamma_1^4 r_1$
Updated Row 2	$a'_{42}\phi_2 + a'_{43}\phi_3 + a'_{44}\phi_4 + a'_{45}\phi_5 = r'_4$

with

$$\begin{aligned} a'_{42} &= a_{42} - \gamma_1^4 a_{12}, & a'_{43} &= a_{43} - \gamma_1^4 a_{13} \\ a'_{44} &= a_{44} - \gamma_1^4 a_{14}, & a'_{45} &= a_{45} - \gamma_1^4 a_{15} \\ r'_4 &= r_4 - \gamma_1^4 r_1 \end{aligned}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & a'_{32} & a'_{33} & a'_{34} & a'_{35} \\ 0 & a'_{42} & a'_{43} & a'_{44} & a'_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r'_3 \\ r'_4 \\ r_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & a'_{32} & a'_{33} & a'_{34} & a'_{35} \\ 0 & a'_{42} & a'_{43} & a'_{44} & a'_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r'_3 \\ r'_4 \\ r_5 \end{Bmatrix}$$

**Row 5**

Let  $\gamma_1^5 = \frac{a_{51}}{a_{11}}$

$$\begin{array}{l|l} \text{Row 5} & a_{51}\phi_1 + a_{52}\phi_2 + a_{53}\phi_3 + a_{54}\phi_4 + a_{55}\phi_5 = r_5 \\ \gamma_1^5 \times \text{Row 1} & a_{51}\phi_1 + \gamma_1^5 a_{12}\phi_2 + \gamma_1^5 a_{13}\phi_3 + \gamma_1^5 a_{14}\phi_4 + \gamma_1^5 a_{15}\phi_5 = \gamma_1^5 r_1 \\ \hline \text{Updated Row 2} & a'_{52}\phi_2 + a'_{53}\phi_3 + a'_{54}\phi_4 + a'_{55}\phi_5 = r'_5 \end{array}$$

with

$$a'_{52} = a_{52} - \gamma_1^5 a_{12}, \quad a'_{53} = a_{53} - \gamma_1^5 a_{13}$$

$$a'_{54} = a_{54} - \gamma_1^5 a_{14}, \quad a'_{55} = a_{55} - \gamma_1^5 a_{15}$$

$$r'_5 = r_5 - \gamma_1^5 r_1$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & a'_{32} & a'_{33} & a'_{34} & a'_{35} \\ 0 & a'_{42} & a'_{43} & a'_{44} & a'_{45} \\ 0 & a'_{52} & a'_{53} & a'_{54} & a'_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r'_3 \\ r'_4 \\ r'_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & a'_{32} & a'_{33} & a'_{34} & a'_{35} \\ 0 & a'_{42} & a'_{43} & a'_{44} & a'_{45} \\ 0 & a'_{52} & a'_{53} & a'_{54} & a'_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r'_3 \\ r'_4 \\ r'_5 \end{Bmatrix}$$

### Row 3

Let  $\gamma_2^3 = \frac{a'_{32}}{a'_{22}}$

$$\begin{array}{l|l} \text{Row 3} & a'_{32}\phi_2 + a'_{33}\phi_3 + a'_{34}\phi_4 + a'_{35}\phi_5 = r'_3 \\ \gamma_2^3 \times \text{Row 2} & a'_{32}\phi_2 + \gamma_2^3 a'_{23}\phi_3 + \gamma_2^3 a'_{24}\phi_4 + \gamma_2^3 a'_{25}\phi_5 = \gamma_2^3 r'_2 \\ \hline \text{Updated Row 3} & a''_{33}\phi_3 + a''_{34}\phi_4 + a''_{35}\phi_5 = r''_3 \end{array}$$

with

$$\begin{aligned} a''_{33} &= a'_{33} - \gamma_2^3 a'_{23}, & a''_{34} &= a'_{34} - \gamma_2^3 a'_{24} \\ a''_{35} &= a'_{35} - \gamma_2^3 a'_{25}, & r''_3 &= r'_3 - \gamma_2^3 r'_2 \end{aligned}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & a'_{42} & a'_{43} & a'_{44} & a'_{45} \\ 0 & a'_{52} & a'_{53} & a'_{54} & a'_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'_4 \\ r'_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & a'_{42} & a'_{43} & a'_{44} & a'_{45} \\ 0 & a'_{52} & a'_{53} & a'_{54} & a'_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'_4 \\ r'_5 \end{Bmatrix}$$

Row 4

Let  $\gamma_2^4 = \frac{a'_{42}}{a'_{22}}$

$$\begin{array}{l|l} \text{Row 4} & a'_{42}\phi_2 + a'_{43}\phi_3 + a'_{44}\phi_4 + a'_{45}\phi_5 = r'_4 \\ \gamma_2^4 \times \text{Row 2} & a'_{42}\phi_2 + \gamma_2^4 a'_{23}\phi_3 + \gamma_2^4 a'_{24}\phi_4 + \gamma_2^4 a'_{25}\phi_5 = \gamma_2^4 r'_2 \\ \hline \text{Updated Row 4} & a''_{43}\phi_3 + a''_{44}\phi_4 + a''_{45}\phi_5 = r''_4 \end{array}$$

with

$$\begin{aligned} a''_{43} &= a'_{43} - \gamma_2^4 a'_{23}, & a''_{44} &= a'_{44} - \gamma_2^4 a'_{24} \\ a''_{45} &= a'_{45} - \gamma_2^4 a'_{25}, & r''_4 &= r'_4 - \gamma_2^4 r'_2 \end{aligned}$$





# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & a''_{43} & a''_{44} & a''_{45} \\ 0 & a'_{52} & a'_{53} & a'_{54} & a'_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r''_4 \\ r'_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & a''_{43} & a''_{44} & a''_{45} \\ 0 & a'_{52} & a'_{53} & a'_{54} & a'_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r''_4 \\ r'_5 \end{Bmatrix}$$

### Row 5

Let  $\gamma_2^5 = \frac{a'_{52}}{a'_{22}}$

Row 5	$a'_{52}\phi_2 + a'_{53}\phi_3 + a'_{54}\phi_4 + a'_{55}\phi_5 = r'_5$
$\gamma_2^5 \times$ Row 2	$a'_{52}\phi_2 + \gamma_2^5 a'_{23}\phi_3 + \gamma_2^5 a'_{24}\phi_4 + \gamma_2^5 a'_{25}\phi_5 = \gamma_2^5 r'_2$
Updated Row 5	$a''_{53}\phi_3 + a''_{54}\phi_4 + a''_{55}\phi_5 = r''_5$

with

$$\begin{aligned} a''_{53} &= a'_{53} - \gamma_2^5 a'_{23}, & a''_{54} &= a'_{54} - \gamma_2^5 a'_{24} \\ a''_{55} &= a'_{55} - \gamma_2^5 a'_{25}, & r''_5 &= r'_5 - \gamma_2^5 r'_2 \end{aligned}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & a''_{43} & a''_{44} & a''_{45} \\ 0 & 0 & a''_{53} & a''_{54} & a''_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r''_4 \\ r''_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & a''_{43} & a''_{44} & a''_{45} \\ 0 & 0 & a''_{53} & a''_{54} & a''_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r''_4 \\ r''_5 \end{Bmatrix}$$

Similarly,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & a'''_{54} & a'''_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r'''_5 \end{Bmatrix}$$



# Basic Steps

## Forward Elimination

Similarly,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r^{IV}_5 \end{Bmatrix}$$



# Basic Steps

## Back Substitution

Row 5 (Last Row)

$$\phi_5 = \frac{r_5^{IV}}{a_{55}^{IV}}$$



# Basic Steps

## Back Substitution

Row 5 (Last Row)

$$\phi_5 = \frac{r_5^{IV}}{a_{55}^{IV}}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r^{IV}_5 \end{Bmatrix}$$



# Basic Steps

## Back Substitution

### Row 4

$$a_{44}''' \phi_4 + a_{45}''' \phi_5 = r_4'''$$

Rewriting yields

$$\phi_4 = \frac{1}{a_{44}'''} [r_4''' - a_{45}''' \phi_5]$$





# Basic Steps

## Back Substitution

### Row 4

$$a_{44}''' \phi_4 + a_{45}''' \phi_5 = r_4'''$$

Rewriting yields

$$\phi_4 = \frac{1}{a_{44}'''} [r_4''' - a_{45}''' \phi_5]$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22}' & a_{23}' & a_{24}' & a_{25}' \\ 0 & 0 & a_{33}'' & a_{34}'' & a_{35}'' \\ 0 & 0 & 0 & a_{44}''' & a_{45}''' \\ 0 & 0 & 0 & 0 & a_{55}^{IV} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2' \\ r_3'' \\ r_4''' \\ r_5^{IV} \end{Bmatrix}$$



# Basic Steps

## Back Substitution

### Row 3

$$a''_{33}\phi_3 + a''_{34}\phi_4 + a''_{35}\phi_5 = r''_3$$

Rewriting yields

$$\phi_3 = \frac{1}{a''_{33}} [r''_3 - a''_{34}\phi_4 - a''_{35}\phi_5]$$



# Basic Steps

## Back Substitution

### Row 3

$$a''_{33}\phi_3 + a''_{34}\phi_4 + a''_{35}\phi_5 = r''_3$$

Rewriting yields

$$\phi_3 = \frac{1}{a''_{33}} [r''_3 - a''_{34}\phi_4 - a''_{35}\phi_5]$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r^{IV}_5 \end{Bmatrix}$$



# Basic Steps

## Back Substitution

### Row 2

$$a'_{22}\phi_2 + a'_{23}\phi_3 + a'_{24}\phi_4 + a'_{25}\phi_5 = r'_2$$

Rewriting yields

$$\phi_2 = \frac{1}{a'_{22}} [r'_2 - a'_{23}\phi_3 - a'_{24}\phi_4 - a'_{25}\phi_5]$$



# Basic Steps

## Back Substitution

### Row 2

$$a'_{22}\phi_2 + a'_{23}\phi_3 + a'_{24}\phi_4 + a'_{25}\phi_5 = r'_2$$

Rewriting yields

$$\phi_2 = \frac{1}{a'_{22}} [r'_2 - a'_{23}\phi_3 - a'_{24}\phi_4 - a'_{25}\phi_5]$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r^{IV}_5 \end{Bmatrix}$$



# Basic Steps

## Back Substitution

### Row 1

$$a_{11}\phi_1 + a_{12}\phi_2 + a_{13}\phi_3 + a_{14}\phi_4 + a_{15}\phi_5 = r_1$$

Rewriting yields

$$\phi_1 = \frac{1}{a_{11}} [r_1 - a_{12}\phi_2 - a_{13}\phi_3 - a_{14}\phi_4 - a_{15}\phi_5]$$



# Basic Steps

## Back Substitution

### Row 1

$$a_{11}\phi_1 + a_{12}\phi_2 + a_{13}\phi_3 + a_{14}\phi_4 + a_{15}\phi_5 = r_1$$

Rewriting yields

$$\phi_1 = \frac{1}{a_{11}} [r_1 - a_{12}\phi_2 - a_{13}\phi_3 - a_{14}\phi_4 - a_{15}\phi_5]$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r^{IV}_5 \end{Bmatrix}$$



# Algorithm

**Data:** Matrix  $A$ , Vector  $r$

**Result:**  $\phi$

**Forward Elimination**

```
for k=1,n-1 do
    for i=k+1,n do
         $\gamma = a_{i,k} / a_{k,k}$ 
        for j=k+1,n do
             $a_{i,j} = a_{i,j} - \gamma \cdot a_{k,j}$ 
        end
         $r_i = r_i - \gamma \cdot r_k$ 
    end
end
```

**end**

**Back Substitution**

$\phi_n = r_n / a_{n,n}$

```
for i=n-1,-1,1 do
    sum =  $r_i$ 
    for j=i+1,n do
        sum = sum -  $a_{i,j} \cdot \phi_j$ 
    end
     $\phi_i = \text{sum} / a_{i,i}$ 
end
```

**end**

**return**  $\phi$





# Associated Problems and Solutions

## Problems

- Division by Zero



# Associated Problems and Solutions

## Problems

- Division by Zero
- Round-off errors



# Associated Problems and Solutions

## Problems

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- Ill-Conditioned system



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## Solutions

- Pivoting



# Associated Problems and Solutions

## Problems

- Division by Zero
- Round-off errors
- Ill-Conditioned system

## Solutions

- Pivoting
- Scaling



## Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$



## Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

**Solution:**

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{Bmatrix}$$



## Example

$$\begin{pmatrix} 1 & 2 & -3 & 4 & 5 \\ 0 & 3 & -5 & -7 & 9 \\ 5 & -4 & 3 & -2 & 1 \\ 1 & 4 & -7 & -10 & 13 \\ -15 & 13 & 11 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 37 \\ 8 \\ 3 \\ 13 \\ 18 \end{Bmatrix}$$





## Example

$$\begin{pmatrix} 1 & 2 & -3 & 4 & 5 \\ 0 & 3 & -5 & -7 & 9 \\ 5 & -4 & 3 & -2 & 1 \\ 1 & 4 & -7 & -10 & 13 \\ -15 & 13 & 11 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 37 \\ 8 \\ 3 \\ 13 \\ 18 \end{Bmatrix}$$

**Solution:**

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix}$$



# Thank You