Module 01: Introduction to Computational Hydraulics

Unit 02: Problem Definition and Governing Equations (GE)

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Learning Objectives

- To identify the Governing Equations for hydraulic systems

Introduction

- Governing equation defines the relationship between the variables in terms of *ordinary differential equations* (**ODE**) or *partial differential equations* (**PDE**).

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ODE

Differential Equation with **ONE** independent variable.

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ODE

Differential Equation with **ONE** independent variable.

PDE

Differential Equation with **two** or **more** independent variables.

Basic Principles

- Conservation of Mass
- Conservation of Momentum
- Conservation of Energy

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

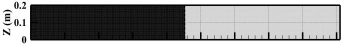
Momentum conservation equation

x-dir: $$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2}$$

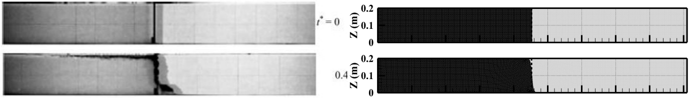
y-dir: $$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + g_y + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{3}$$

z-dir: $$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial ww}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + g_z + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{4}$$

Experiments vs. Simulations



Experiments vs. Simulations



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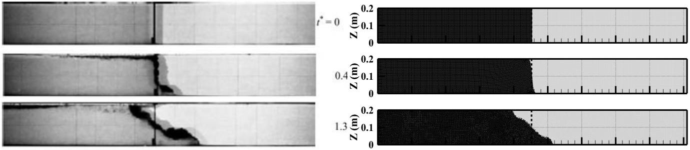

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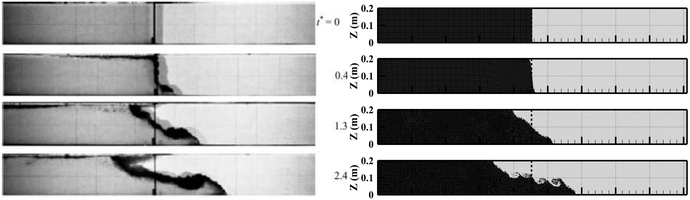

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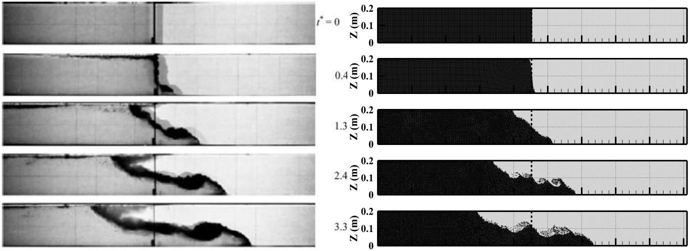

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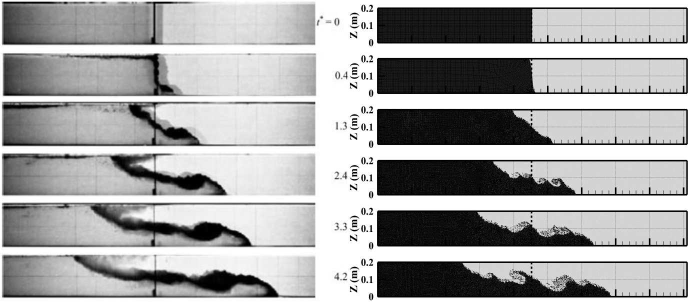

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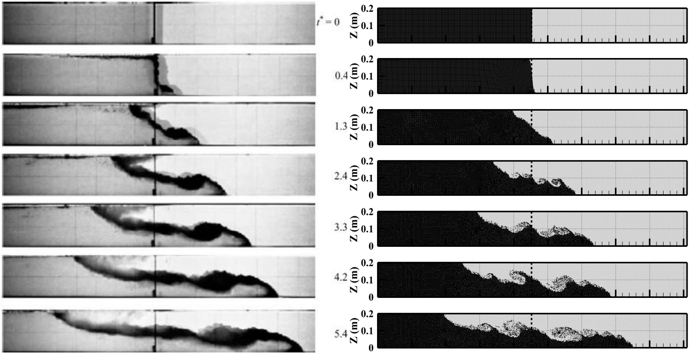


Experiments vs. Simulations

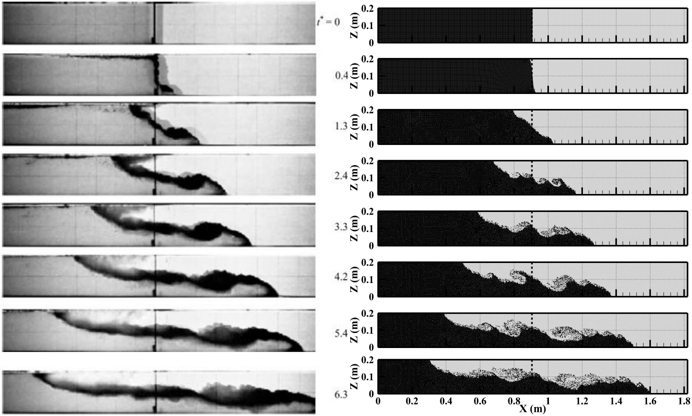
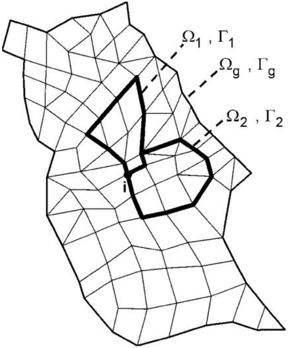


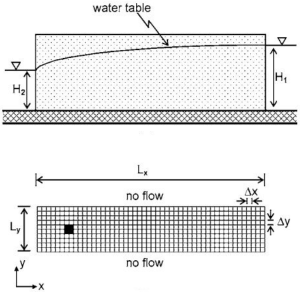
Figure: Comparison of Experimental (Lowe et al., 2005) and Numerical (Pahar and Dhar, 2016) Lock-exchange Flow

Groundwater Movement in Aquifers

Variable: h(x,y,t)



(a) Descriptive schematics of discretizations of global domain and two subdomains (Dogrul and Kadir, 2006)



(b) cross section of heterogeneous aquifer between two lakes and simulation grids (Dogrul and Kadir, 2006)

Groundwater Movement in Aquifers

Governing Equations

Depth-integrated mass conservation equation for groundwater flow can be written as,

$$S\frac{\partial h}{\partial t} + \nabla.\mathbf{q} = f \qquad (5)$$

where

S = storativity (specific yield for unconfined aquifers and storage coefficient for confined aquifers)

$h = h(x, y, t) =$ groundwater head $[L]$

$\mathbf{q} = q_x\hat{\mathbf{i}} + q_y\hat{\mathbf{j}} =$ flux $[L^2/T]$

f = source/sink term $[L/T]$

$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} =$ del operator $[1/L]$

t = time $[T]$.

Groundwater Movement in Aquifers
Governing Equations

The Darcy's law [momentum conservation equation] can be written as,

$$\mathbf{q} = -T\nabla h \tag{6}$$

where

T = aquifer transmissivity $[L^2/T]$

$$T = K[min(h, z_u) - z_b] \tag{7}$$

K = $K(x, y)$ = hydraulic conductivity $[L/T]$

z_u = top aquifer elevation $[L]$

z_b = bottom aquifer elevation $[L]$

Steady One-Dimensional Groundwater Flow

Governing Equations: ODE

Mass Conservation Equation

$$\frac{dq_x}{dx} = f \tag{8}$$

Momentum Conservation Equation

$$q_x = -T\frac{dh(x)}{dx} \tag{9}$$

Contaminant Transport

Variables: h(x,y,t), C(x,y,t)

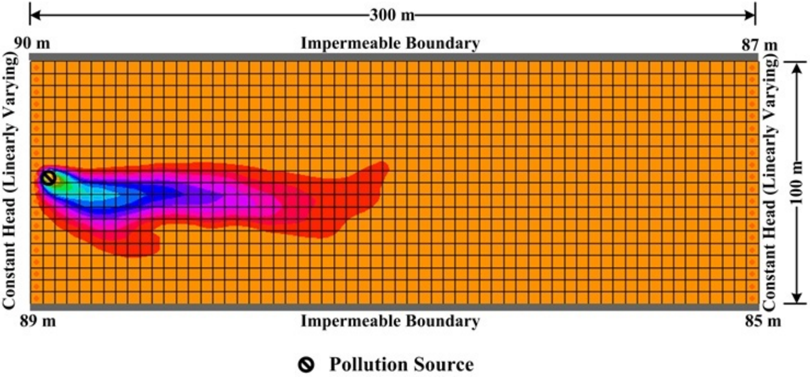


Figure: Contaminant Transport in Aquifer (Dhar and Patil, 2012)

Contaminant Transport

Fluid flow

Mass Conservation Equation

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{\partial h}{\partial y}\right)+q_s=S_s\frac{\partial h}{\partial t} \qquad (10)$$

where

$K_{xx},\ K_{yy}$ = hydraulic conductivity along x and y axes $[L/T]$

h = potentiometric head $[L]$

q_s = Volumetric flux per unit volume $[1/T]$

S_s = specific storage $[1/L]$

Contaminant Transport
Concentration Equation

Scalar Transport Equation

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x}\left(\eta D_{xx}\frac{\partial C}{\partial x} + \eta D_{xy}\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial y}\left(\eta D_{yx}\frac{\partial C}{\partial x} + \eta D_{yy}\frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial x}(\eta v_x C) - \frac{\partial}{\partial y}(\eta v_y C) + q_s C_s \quad (11)$$

where

η = effective porosity,
$D_{xx},\ D_{yy}$ = hydraulic dispersion coefficients along x and y axes $[L^2/T]$
$D_{xy},\ D_{yx}$ = hydraulic dispersion coefficients along cross-directions $[L^2/T]$
$v_x,\ v_y$ = velocity along x and y axes $[L/T]$
C_s = concentration of source $[M/L^3]$.

Channel Networks

Variables: Q(x,t), h(x,t)

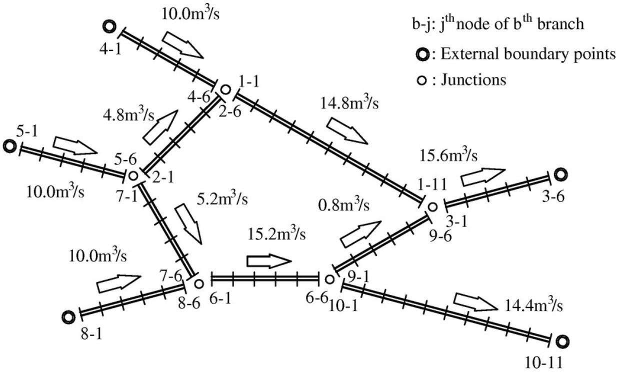


Figure: Typical channel network system (Garg and Sen, 2002)

Channel Networks
Governing Equations

Depth-integrated mass conservation equation for surface water flow can be written as,

$$\frac{\partial h}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = 0 \tag{12}$$

where

$h = h(x,t) =$ channel water depth $[L]$

$B =$ free-surface width $[L]$

$Q =$ flux $[L^3/T]$

$t =$ time $[T]$.

Channel Networks

Governing Equations

The momentum conservation equation can be written as,

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h}{\partial x} + gA(S_f - S_0) = 0 \qquad (13)$$

where

g = acceleration due to gravity $[L/T^2]$

A = cross-sectional area $[L^2]$

S_f = friction slope

S_0 = bed slope

Surface Flooding

Variable: h(x,y,t)

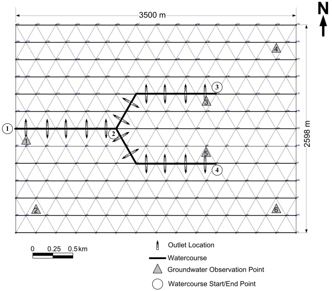


Figure: Initial Condition (Biswas, 2016)

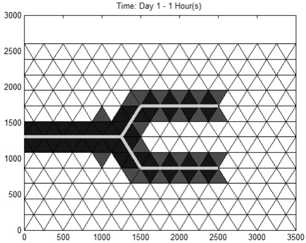
$$A\frac{dh}{dt} = Q \qquad (14)$$

A = c/s area of the plot $[L^2]$

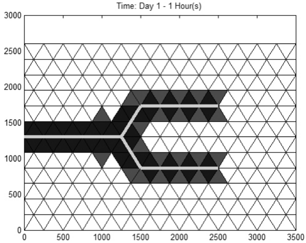
h = depth of water $[L]$

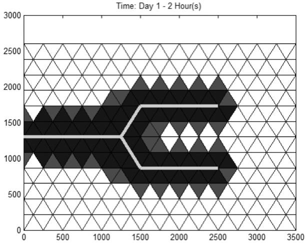
Q = net inflow of the water to the plot $[L^3/T]$

Surface Flooding

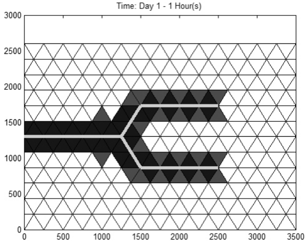


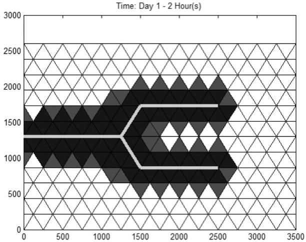
Surface Flooding

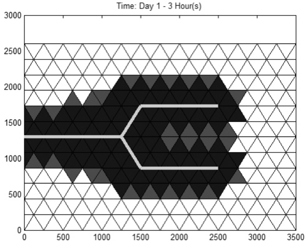




Surface Flooding







Surface Flooding

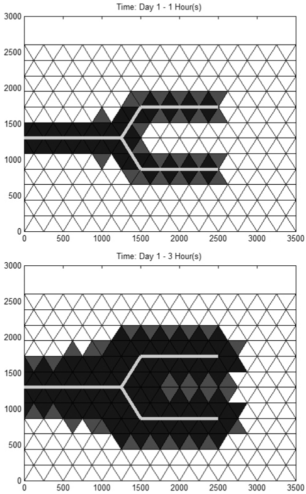


Figure: Surface flooding (Biswas, 2016)

Open Channel Flow
Hydraulic jump

Variables: u(x,z,t), w(x,z,t)

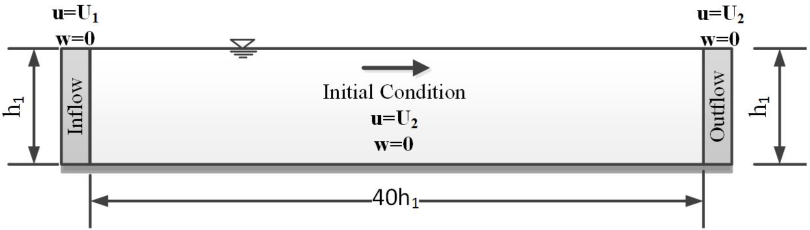
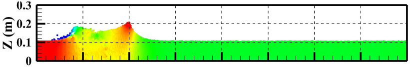


Figure: Initial condition of hydraulic jump (Pahar and Dhar, 2017)

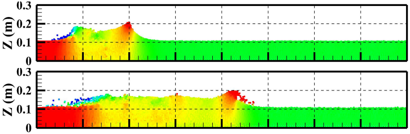
Open Channel Flow

Hydraulic jump



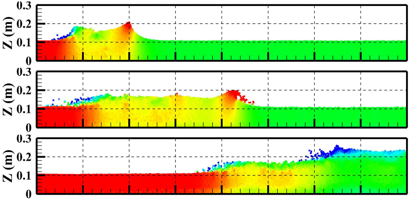
Open Channel Flow

Hydraulic jump



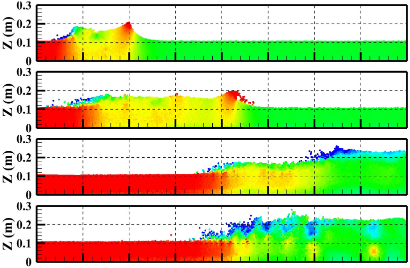
Open Channel Flow

Hydraulic jump



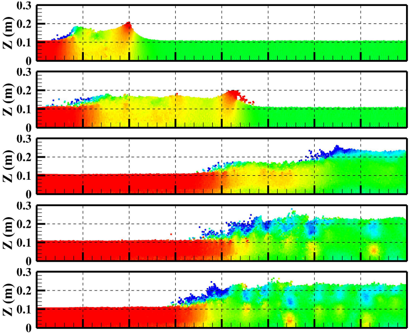
Open Channel Flow

Hydraulic jump



Open Channel Flow

Hydraulic jump



Open Channel Flow

Hydraulic jump

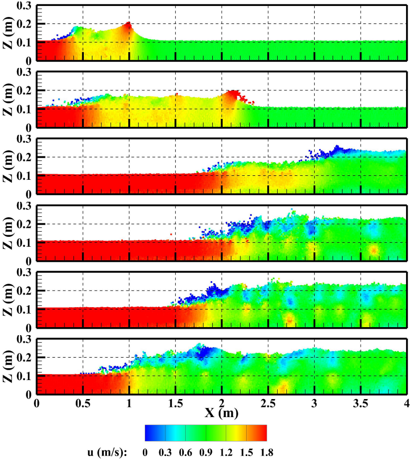


Figure: Velocity evolutions of hydraulic jump (Pahar and Dhar, 2017)

Open Channel Flow

Hydraulic jump

Mass conservation equation

$$\nabla.\mathbf{u} = 0 \tag{15}$$

Momentum conservation equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{g} + \nu\nabla^2\mathbf{u} + \frac{1}{\rho}\nabla.\tau \tag{16}$$

where

$\frac{D}{Dt}$ = total derivative

P = fluid pressure

$\frac{\tau}{\rho}$ = sub-particle-scale tensor

ν = viscosity

Pressurized Conduits

Variables: H(x,t), Q(x,t)

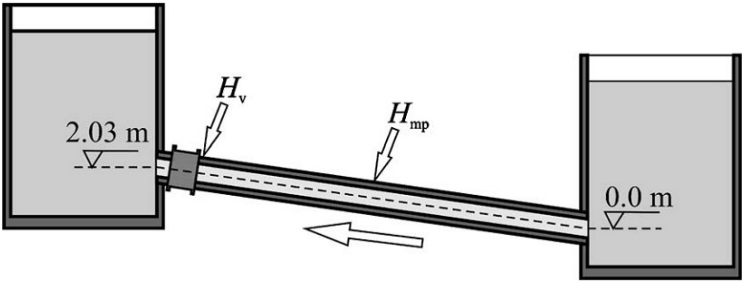


Figure: Connection between two reservoirs (Skific et al., 2010)

Pressurized Conduits
Governing Equations

Depth-integrated mass conservation equation for pressurized conduit can be written as,

$$\frac{\partial H}{\partial t} + \frac{c^2}{gA}\frac{\partial Q}{\partial x} = 0 \tag{17}$$

where

$H = H(x,t) =$ piezometric head $[L]$

$c =$ wave speed $[L^3/T]$

$Q =$ discharge $[L^3/T]$

$t =$ time $[T]$.

Pressurized Conduits

Governing Equations

The momentum conservation equation can be written as,

$$\frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + J_s + J_u = 0 \tag{18}$$

where

g = acceleration due to gravity $[L/T^2]$

A = cross-sectional area $[L^2]$

J_s = steady friction loss

J_u = unsteady friction loss

Pressurized Conduits

Governing Equations

Steady Friction Loss

$$J_s = \frac{f_s Q|Q|}{2DA} \tag{19}$$

where

f_s = Darcy-Weisbach friction factor

D = pipe diameter $[L]$

Unsteady Friction Loss

$$J_u = \frac{k}{2}(Q_t + c\Phi_A |Q_x|) \tag{20}$$

where

$\Phi_A = sqn(Q)$

k = Brunone friction coefficient $[L]$

Pressurized Conduits

Variables: p(x,t), q(x,t)

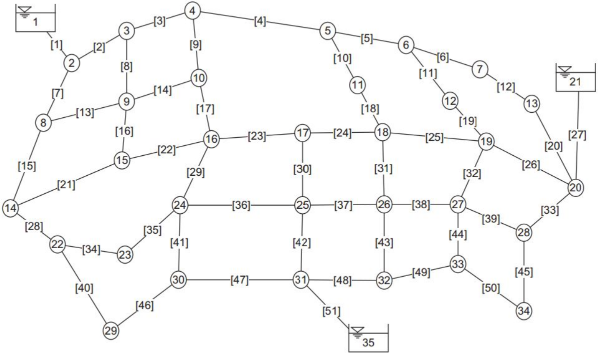


Figure: Pipe Networks (Zecchin et al., 2009)

Pressurized Conduits

Governing equations

Mass Conservation equation

$$\frac{\partial q_j}{\partial t} + \frac{A_j}{\rho}\frac{\partial p_j}{\partial x} + \tau_j(q_j) = 0, \;\; x \in [0, l_j], \quad j \in \Lambda \qquad (21)$$

where

p_j = pressure along the line j

q_j = flow along the line j

c_j = fluid line wave speed for pipe j

A_j = cross-sectional area for pipe j

l_j = length of pipe j

Λ = set of fluid lines

Pressurized Conduits

Governing equations

Momentum Conservation equation

$$\frac{\partial p_j}{\partial t} + \frac{\rho c_j^2}{A_j}\frac{\partial q_j}{\partial x} + \tau_j(q_j) = 0, \; x \in [0, l_j], \quad j \in \Lambda \qquad (22)$$

where

τ_j = cross-sectional frictional resistance

Surface water-groundwater interaction

Variables: $\zeta^s(x,t)$, $p^s(x,t)$, $\zeta^g(x,t)$, $p^g(x,t)$,

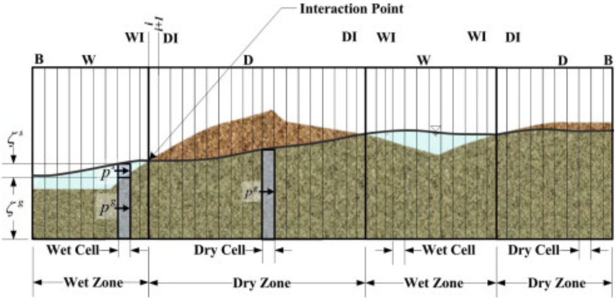


Figure: Conceptual representation of dry cell-wet cell theory (Pahar and Dhar, 2014)

Surface water-groundwater interaction

Surface water

Mass conservation equation

$$\frac{\partial \zeta^s}{\partial t} + \frac{\partial p^s}{\partial x} = 0 \tag{23}$$

where

ζ^s = surface water height $[L]$

p^s = surface water unit discharge $[L^2/T]$

Momentum conservation equation

$$\frac{\partial p^s}{\partial t} + \frac{\partial}{\partial x}\left(\beta_m U p^s\right) = -gH\frac{\partial \zeta^s}{\partial x} - \frac{g(p^s)^2}{H^2C^2} + \nu\frac{\partial^2 p^s}{\partial x^2} \tag{24}$$

β_m = momentum correction factor

ν = kinematic viscosity $[L^2/T]$

$$p_s = \int_0^{\zeta^s} u dz = UH \tag{25}$$

U = average surface water velocity

Surface water-groundwater interaction

Groundwater

Mass conservation equation

$$\eta\frac{\partial \zeta^g}{\partial t}+\frac{\partial p^g}{\partial x}=0 \tag{26}$$

where

η = effective porosity

ζ^g = ground water height $[L]$

p^g = ground water discharge $[L^2/T]$

Momentum conservation equation

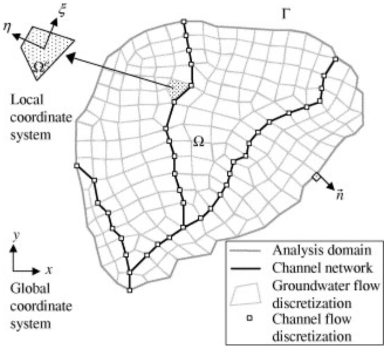
$$p^g=-K\zeta^g\frac{\partial \zeta^g}{\partial x} \tag{27}$$

where

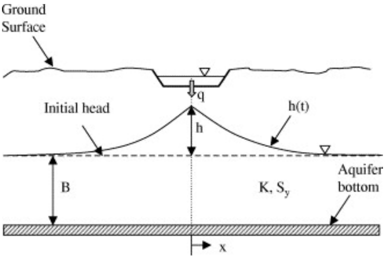
K = hydraulic conductivity $[L/T]$

Surface water-groundwater interaction

Variables: $h_s(x,t)$, $Q(x,t)$, $h_g(x,y,t)$



(a) Coupled modeling domain (Gunduz and Aral, 2005)



(b) Stream aquifer interaction (Gunduz and Aral, 2005)

Surface water-groundwater interaction

Governing Equations: Channel flow

Depth-integrated mass conservation equation for channel flow can be written as,

Mass Conservation Equation

$$\frac{\partial s_c(A + A_0)}{\partial t} + \frac{\partial Q}{\partial x} - q_L = 0 \quad (28)$$

Momentum Conservation Equation

$$\frac{\partial s_m Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\beta Q^2}{A}\right) + gA\frac{\partial h_L}{\partial x} + gA(S_f + S_e) + M_L = 0 \quad (29)$$

Surface water-groundwater interaction

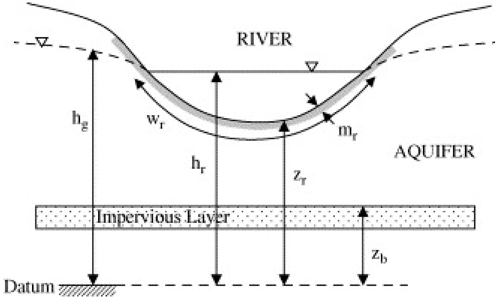


Figure: Channel flow/groundwater flow interaction (Gunduz and Aral, 2005)

Surface water-groundwater interaction

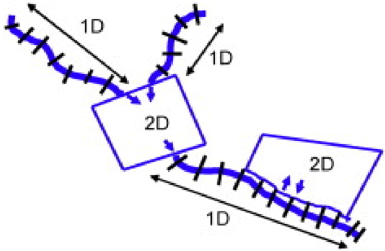
Groundwater

Mass Conservation Equation

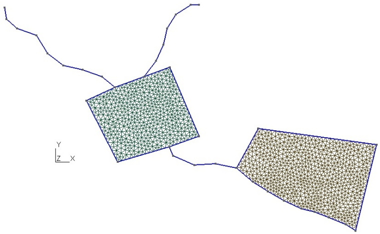
$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[(h_g - z_b)K_{xx}\frac{\partial h_g}{\partial x} + (h_g - z_b)K_{xy}\frac{\partial h_g}{\partial y}\right] \\
&+\frac{\partial}{\partial y}\left[(h_g - z_b)K_{yx}\frac{\partial h}{\partial x} + (h_g - z_b)K_{yy}\frac{\partial h}{\partial y}\right] + \sum_{k=1}^{n_w}[Q_{w_k}\delta(x - x_k)\delta(y - y_k)] + \\
&\sum_{m=1}^{n_r}\left[\int_0^1 q_{L_m}\sqrt{\left(\frac{dg_{x_m}}{dt}\right)^2 + \left(\frac{dg_{y_m}}{dt}\right)^2} \times \delta(x - g_x(t))\delta(y - g_{y_m}(t))dt\right] \\
&+I = S_y\frac{\partial h}{\partial t}
\end{aligned}
\tag{30}
$$

1D-2D integrated system

Variables: $h_c(x,t)$, $Q_c(x,t)$, $h_f(x,y,t)$, $u_f(x,y,t)$, $v_f(x,y,t)$



(a) Integrated 1D-2D simulations with lateral and flow direction connections (Blade et al., 2012)



(b) Discretization of computational domain

1D-2D integrated system

One-dimensional Governing Equations

Depth-integrated mass and momentum conservation equations for surface water flow can be written as,

$$\mathbf{U_{1D}}_{,t} + \mathbf{F_{1D}}_{,x} = \mathbf{H_{1D}} \tag{31}$$

$$\mathbf{U_{1D}} = \begin{bmatrix} A \\ Q \end{bmatrix} \tag{32}$$

$$\mathbf{F_{1D}} = \begin{bmatrix} Q \\ \frac{Q^2}{A} + gI_1 \end{bmatrix} \tag{33}$$

$$\mathbf{H_{1D}} = \begin{bmatrix} q_l \\ gI_2 + gA(S_0 - S_f) \end{bmatrix} \tag{34}$$

1D-2D integrated system

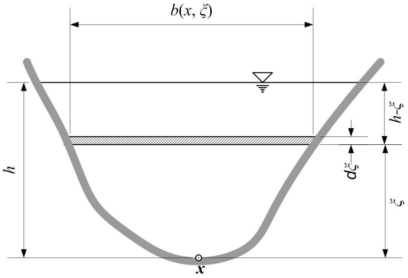


Figure: Channel cross-section

$$I_1 = \int_0^h (h - \xi) b(x, \xi) d\xi \tag{35}$$

$$I_2 = \int_0^h (h - \xi) \frac{\partial b(x, \xi)}{\partial x} d\xi \tag{36}$$

1D-2D integrated system

Two-dimensional Governing Equations

Depth-integrated mass and momentum conservation equations for surface water flow can be written as,

$$\mathbf{U_{2D}}_{,t} + \nabla \mathbf{F_{2D}} = \mathbf{H_{2D}} \tag{37}$$

$$\mathbf{U_{2D}} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} \tag{38}$$

$$\mathbf{F_{2D}} = \begin{bmatrix} hu & hv \\ hu^2 + \frac{gh^2}{2} & huv \\ huv & hv^2 + \frac{gh^2}{2} \end{bmatrix} \tag{39}$$

$$\mathbf{H_{2D}} = \begin{bmatrix} i_R \\ gh(S_{0x} - S_{fx}) \\ gh(S_{0y} - S_{fy}) \end{bmatrix} \tag{40}$$

Thank You

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