

General Flow Equations

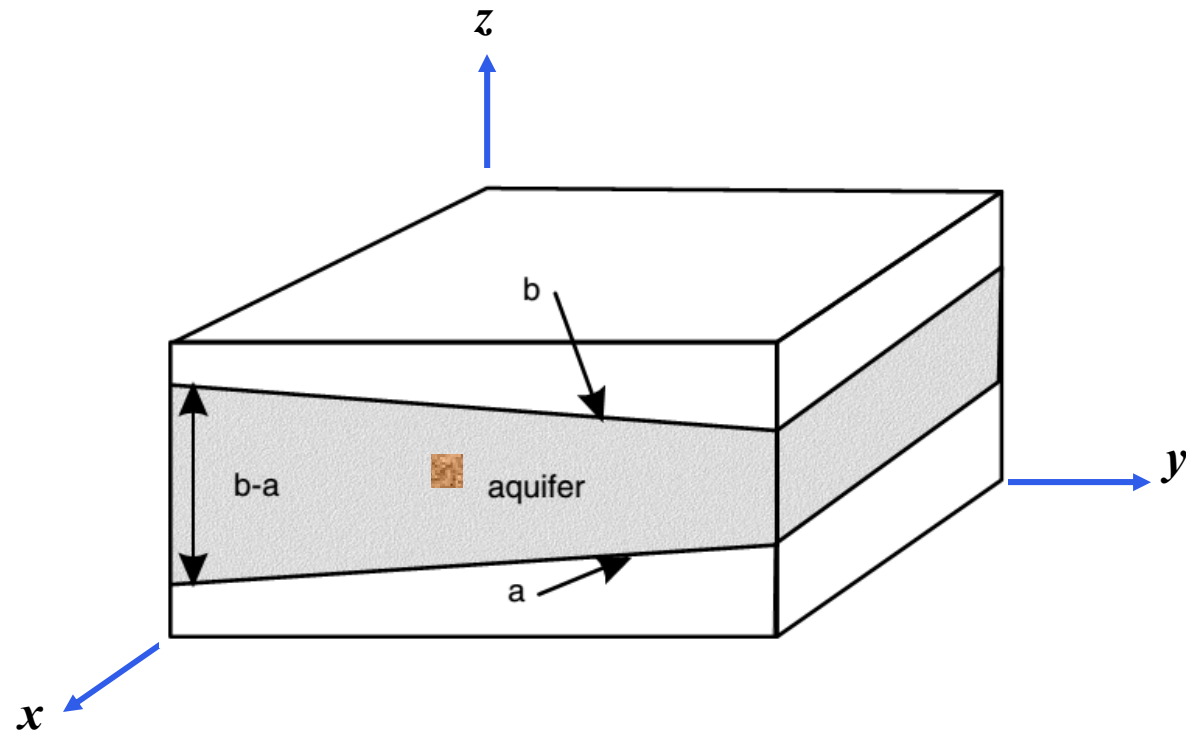
Geohydraulics| CE60113

Lecture:10

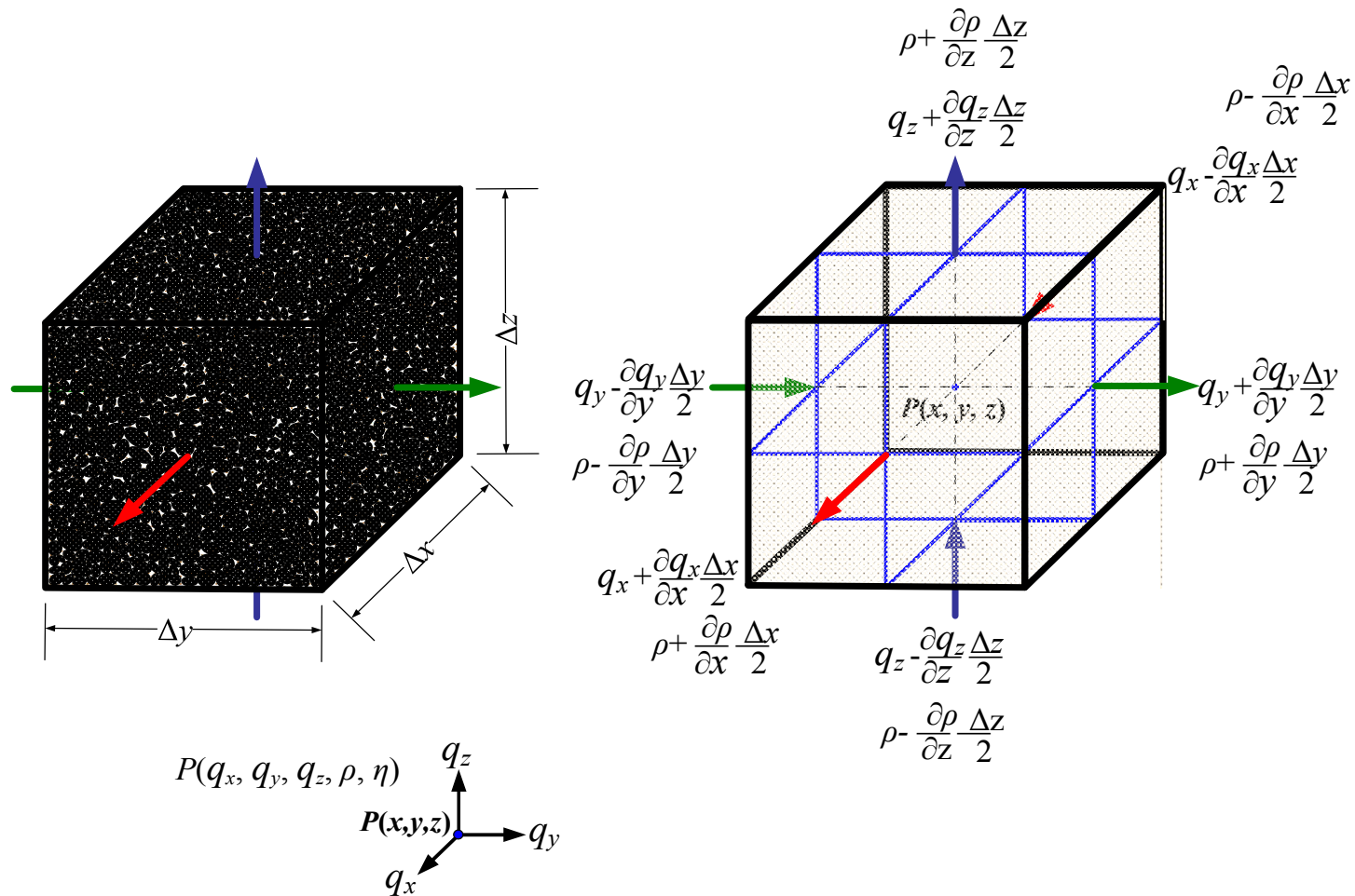
Learning Objective(s)

- To derive the governing equation for saturated confined aquifer

Groundwater Flow



Three-Dimensional Saturated Flow in Confined Aquifers



Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Mass inflow rate – mass outflow rate = Rate of change of mass storage with time
- Considering an elementary control volume of soil, which has the volume of $(\Delta x \Delta y \Delta z)$, the mass of groundwater, M , in this control volume is

$$M = \rho \eta \Delta x \Delta y \Delta z$$

- The inflow and outflow can be calculated for each side of the element. The mass of groundwater inflow in x-direction is

$$\begin{aligned} \rho \left(x - \frac{\Delta x}{2}, y, z, t \right) q_x \left(x - \frac{\Delta x}{2}, y, z, t \right) \Delta y \Delta z &\approx \left(\rho - \frac{\Delta x}{2} \frac{\partial}{\partial x} (\rho) \right) \left(q_x - \frac{\Delta x}{2} \frac{\partial}{\partial x} (q_x) \right) \Delta y \Delta z \\ &\approx \left(\rho q_x - \frac{\Delta x}{2} \frac{\partial}{\partial x} (\rho q_x) \right) \Delta y \Delta z \text{ Up to first order} \end{aligned}$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- The mass of groundwater outflow in x-direction is

$$\rho \left(x + \frac{\Delta x}{2}, y, z, t \right) q_x \left(x + \frac{\Delta x}{2}, y, z, t \right) \Delta y \Delta z \approx \left(\rho q_x + \frac{\Delta x}{2} \frac{\partial}{\partial x} (\rho q_x) \right) \Delta y \Delta z$$

- The mass of groundwater inflow in y-direction is

$$\rho \left(x, y - \frac{\Delta y}{2}, z, t \right) q_y \left(x, y - \frac{\Delta y}{2}, z, t \right) \Delta x \Delta z \approx \left(\rho q_y - \frac{\Delta y}{2} \frac{\partial}{\partial y} (\rho q_y) \right) \Delta x \Delta z$$

- The mass of groundwater outflow in y-direction is

$$\rho \left(x, y + \frac{\Delta y}{2}, z, t \right) q_y \left(x, y + \frac{\Delta y}{2}, z, t \right) \Delta x \Delta z \approx \left(\rho q_y + \frac{\Delta y}{2} \frac{\partial}{\partial y} (\rho q_y) \right) \Delta x \Delta z$$

- The mass of groundwater inflow in z-direction is

$$\rho \left(x, y, z - \frac{\Delta z}{2}, t \right) q_z \left(x, y, z - \frac{\Delta z}{2}, t \right) \Delta x \Delta y \approx \left(\rho q_z - \frac{\Delta z}{2} \frac{\partial}{\partial z} (\rho q_z) \right) \Delta x \Delta y$$

- The mass of groundwater outflow in z-direction is

$$\rho \left(x, y, z + \frac{\Delta z}{2}, t \right) q_z \left(x, y, z + \frac{\Delta z}{2}, t \right) \Delta x \Delta y \approx \left(\rho q_z + \frac{\Delta z}{2} \frac{\partial}{\partial z} (\rho q_z) \right) \Delta x \Delta y$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

$$\frac{\partial M}{\partial t} = \text{Inflow} - \text{Outflow}$$

- Considering these equations, the total inflow minus outflow can be derived as follows:

$$\frac{\partial M}{\partial t} = - \left[\frac{\partial}{\partial x} (\rho q_x) + \frac{\partial}{\partial y} (\rho q_y) + \frac{\partial}{\partial z} (\rho q_z) \right] \Delta x \Delta y \Delta z$$

$$M = \rho \eta \Delta x \Delta y \Delta z$$

- The change in storage is calculated by

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} (\rho \eta \Delta x \Delta y \Delta z) = \frac{\partial}{\partial t} (\rho \eta A L_z)$$

- Change in mass

$$dM = \rho V_T (\eta \beta + \alpha) dp \text{ with } V_T = \Delta x \Delta y \Delta z = A L_z \text{ and } dp = \rho g dh$$

- Change in mass with respect to time

$$\frac{\partial M}{\partial t} = \rho^2 g (\eta \beta + \alpha) \frac{\partial h}{\partial t} V_T = \rho S_s \frac{\partial h}{\partial t} V_T$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Conservation equation can be written as

$$\rho S_s \frac{\partial h}{\partial t} = - \left[\frac{\partial}{\partial x} (\rho q_x) + \frac{\partial}{\partial y} (\rho q_y) + \frac{\partial}{\partial z} (\rho q_z) \right]$$

- Under incompressible condition the equation can be simplified as

$$S_s \frac{\partial h}{\partial t} = - \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right)$$

- In terms of piezometric head

$$\mathbf{q} = -\mathbf{K} \cdot \nabla h$$

- If x, y, z are the In principal flow directions, then

$$q_x = -K_x \frac{\partial h}{\partial x}$$
$$q_y = -K_y \frac{\partial h}{\partial y}$$
$$q_z = -K_z \frac{\partial h}{\partial z}$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Groundwater flow equation for 3D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial h}{\partial z} \right)$$

- Groundwater flow equation for 3D homogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2}$$

- Groundwater flow equation for 3D heterogeneous, isotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(K \frac{\partial h}{\partial z} \right)$$

- Groundwater flow equation for 3D homogeneous, isotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = K \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) = K \nabla^2 h$$

Darcy's Law

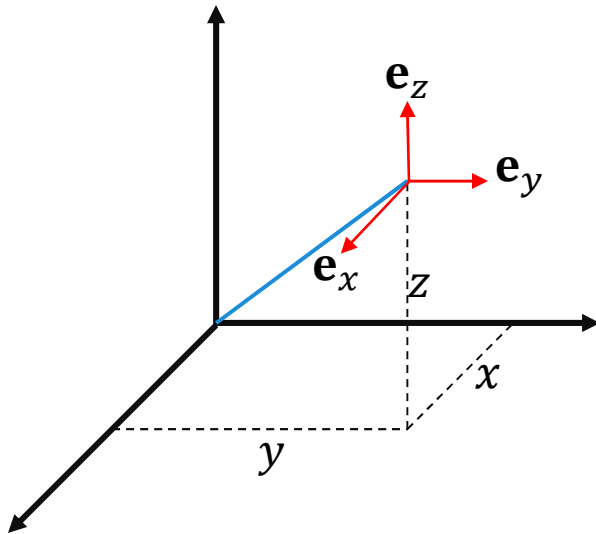
- In Cartesian coordinate system

$$\nabla(\cdot) = \frac{\partial}{\partial x}(\cdot)\mathbf{e}_x + \frac{\partial}{\partial y}(\cdot)\mathbf{e}_y + \frac{\partial}{\partial z}(\cdot)\mathbf{e}_z$$

$$\nabla^2(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot) + \frac{\partial^2}{\partial y^2}(\cdot) + \frac{\partial^2}{\partial z^2}(\cdot)$$

$$\nabla \cdot \mathbf{q} = \frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)$$

- If x, y, z are the principal flow directions, then



$$q_x = -K_x \frac{\partial h}{\partial x}$$

$$q_y = -K_y \frac{\partial h}{\partial y}$$

$$q_z = -K_z \frac{\partial h}{\partial z}$$

Darcy's Law (Contd.)

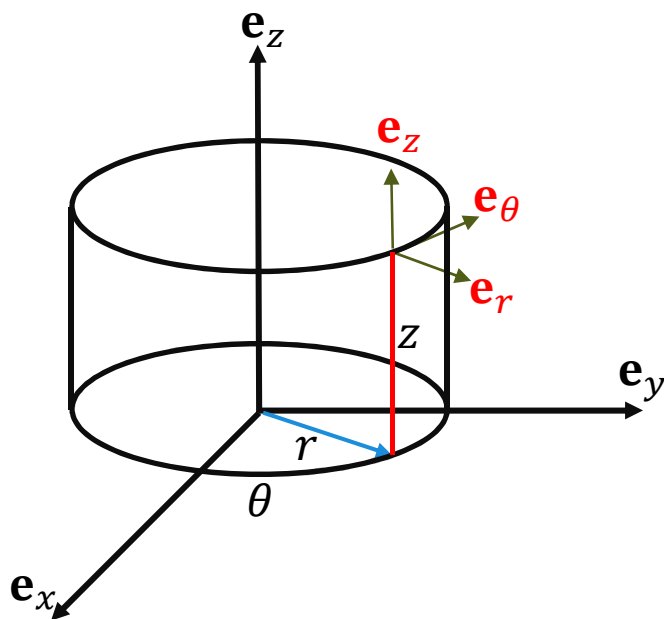
- In Cylindrical coordinate system

$$\nabla(\cdot) = \frac{\partial}{\partial r}(\cdot)\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}(\cdot)\mathbf{e}_\theta + \frac{\partial}{\partial z}(\cdot)\mathbf{e}_z$$

$$\nabla^2(\cdot) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\cdot)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\cdot) + \frac{\partial^2}{\partial z^2}(\cdot)$$

$$\nabla \cdot \mathbf{q} = \frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta) + \frac{\partial}{\partial z}(q_z)$$

- If r, θ, z are the principal flow directions, then



$$q_r = -K_r \frac{\partial h}{\partial r}$$

$$q_\theta = -\frac{K_\theta}{r} \frac{\partial h}{\partial \theta}$$

$$q_z = -K_z \frac{\partial h}{\partial z}$$

Groundwater Flow Equation

- Groundwater flow equation for 3D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial h}{\partial z} \right) - W$$

or,

$$S_s \frac{\partial h}{\partial t} = \nabla \cdot (\mathbf{K} \cdot \nabla h) - Q$$

$h(x, y, z, t)$

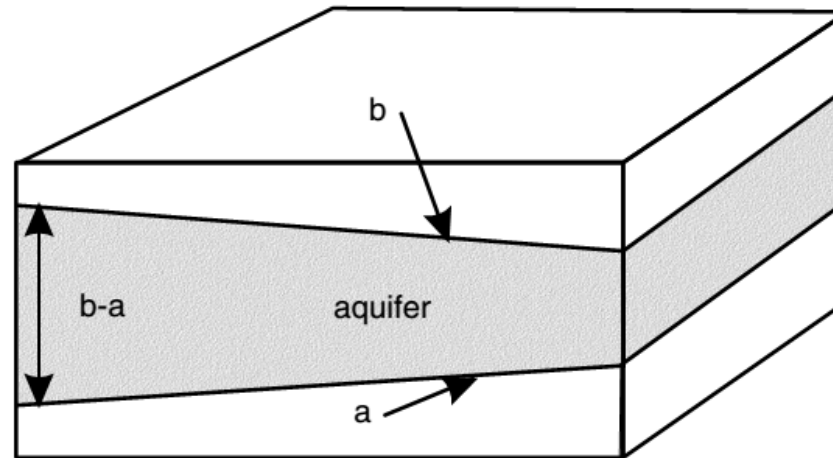
In Cylindrical coordinate system groundwater flow equation for 3D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(K_r r \frac{\partial h}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{K_\theta}{r} \frac{\partial h}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial h}{\partial z} \right) - W$$

$h(r, \theta, z, t)$

$$h(r, z, t)$$

Reduction in Dimensionality



- Vertical Integration of the Flow Equation

$$\int_{a(x,y,t)}^{b(x,y,t)} \left(\nabla \cdot \mathbf{q} + S_s \frac{\partial h}{\partial t} + W \right) dz = 0$$

Reduction in Dimensionality (Contd.)

- Using Leibniz integral rule

$$\begin{aligned} \int_{a(x,y,t)}^{b(x,y,t)} \nabla \cdot \mathbf{q} \, dz &= \int_a^b \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) dz = \frac{\partial}{\partial x} \int_a^b q_x \, dz - q_x|_{z=b} \frac{\partial b}{\partial x} + q_x|_{z=a} \frac{\partial a}{\partial x} \\ &\quad + \frac{\partial}{\partial y} \int_a^b q_y \, dz - q_y|_{z=b} \frac{\partial b}{\partial y} + q_y|_{z=a} \frac{\partial a}{\partial y} + q_z|_{z=b} - q_z|_{z=a} \end{aligned}$$

- In general

$$\int_{a(x,y,t)}^{b(x,y,t)} \nabla \cdot \mathbf{q} \, dz = \nabla_{xy} \cdot \int_{a(x,y,t)}^{b(x,y,t)} \mathbf{q}_{xy} \, dz + \mathbf{q}|_{z=b} \cdot \nabla(z-b) - \mathbf{q}|_{z=a} \cdot \nabla(z-a)$$

where

$$\nabla_{xy}(\cdot) \equiv \frac{\partial}{\partial x}(\cdot) \mathbf{e}_x + \frac{\partial}{\partial y}(\cdot) \mathbf{e}_y$$

Reduction in Dimensionality (Contd.)

- In terms of piezometric head

$$\begin{aligned} \mathbf{q} &= -\mathbf{K} \cdot \nabla h \\ \int_{a(x,y,t)}^{b(x,y,t)} \mathbf{q}_{xy} dz &= -\bar{\mathbf{K}} \cdot \int_a^b \nabla_{xy} h dz \\ &= -\bar{\mathbf{K}} \cdot \left[\nabla_{xy} \int_a^b h dz - h|_{z=b} \nabla_{xy} b + h|_{z=a} \nabla_{xy} a \right] \end{aligned}$$

Reduction in Dimensionality (Contd.)

- Time derivative term

$$\begin{aligned} \int_{a(x,y,t)}^{b(x,y,t)} S_s \frac{\partial h}{\partial t} dz &= \bar{S}_s \int_a^b \frac{\partial h}{\partial t} dz \\ &= \bar{S}_s \left[\frac{\partial}{\partial t} \int_a^b h dz - h|_{z=b} \frac{\partial b}{\partial t} + h|_{z=a} \frac{\partial a}{\partial t} \right] \end{aligned}$$

- Combining all the terms

$$\begin{aligned} -\nabla_{xy} \cdot \left[\bar{\mathbf{K}} \cdot \left(\nabla_{xy} \int_a^b h dz - h|_{z=b} \nabla_{xy} b + h|_{z=a} \nabla_{xy} a \right) \right] &+ \mathbf{q}|_{z=b} \cdot \nabla(z-b) - \mathbf{q}|_{z=a} \cdot \nabla(z-a) \\ &+ \bar{S}_s \left[\frac{\partial}{\partial t} \int_a^b h dz - h|_{z=b} \frac{\partial b}{\partial t} + h|_{z=a} \frac{\partial a}{\partial t} \right] + \int_a^b W dz = 0 \end{aligned}$$

Reduction in Dimensionality (Contd.)

- Let us consider the vertical average values as

$$\bar{h} \equiv \frac{1}{(b-a)} \int_a^b h \, dz = \frac{1}{l} \int_a^b h \, dz$$

and

$$\bar{W} \equiv \frac{1}{(b-a)} \int_a^b W \, dz = \frac{1}{l} \int_a^b W \, dz$$

- Combining all the terms

$$\begin{aligned} -\nabla_{xy} \cdot [\bar{\mathbf{K}} \cdot (\nabla_{xy} l \bar{h} - \bar{h} \nabla_{xy} l)] + \mathbf{q}|_{z=b} \cdot \nabla(z-b) - \mathbf{q}|_{z=a} \cdot \nabla(z-a) \\ + \bar{S}_s \left[\frac{\partial(l\bar{h})}{\partial t} - \bar{h} \frac{\partial l}{\partial t} \right] + l\bar{W} = 0 \end{aligned}$$

Or,

$$-\nabla_{xy} \cdot [l\bar{\mathbf{K}} \cdot \nabla_{xy} \bar{h}] + \mathbf{q}|_{z=b} \cdot \nabla(z-b) - \mathbf{q}|_{z=a} \cdot \nabla(z-a) + l\bar{S}_s \frac{\partial \bar{h}}{\partial t} + l\bar{W} = 0$$

Two-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Storage Coefficient

$$S \equiv l\bar{S}_s$$

- Transmissivity Tensor

$$\mathbf{T} \equiv l\bar{\mathbf{K}}$$

In 2D the governing equation can be written as

$$S \frac{\partial \bar{h}}{\partial t} + q_T + q_B + q_{ext} = \nabla_{xy} \cdot [\mathbf{T} \cdot \nabla_{xy} \bar{h}]$$

where

$$\begin{aligned} q_T &= \mathbf{q}|_{z=b} \cdot \nabla(z - b) \\ q_B &= -\mathbf{q}|_{z=a} \cdot \nabla(z - a) \\ q_{ext} &= l\bar{W} \end{aligned}$$

Two-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Storage Coefficient

$$S \equiv l\bar{S}_s$$

- Transmissivity Tensor

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In 2D the governing equation can be written as

$$S \frac{\partial \bar{h}}{\partial t} + q_T + q_B + q_{ext} = \nabla_{xy} \cdot [\mathbf{T} \cdot \nabla_{xy} \bar{h}]$$

where

$$\begin{aligned} q_T &= \mathbf{q}|_{z=b} \cdot \nabla(z-b) \\ q_B &= -\mathbf{q}|_{z=a} \cdot \nabla(z-a) \\ q_{ext} &= l\bar{W} \end{aligned}$$

Two-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Groundwater flow equation for 2D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(T_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(T_y \frac{\partial h}{\partial y} \right)$$

- Groundwater flow equation for 2D homogeneous, anisotropic saturated confined aquifer can be written as

$$S \frac{\partial h}{\partial t} = T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2}$$

- Groundwater flow equation for 2D heterogeneous, isotropic saturated confined aquifer can be written as

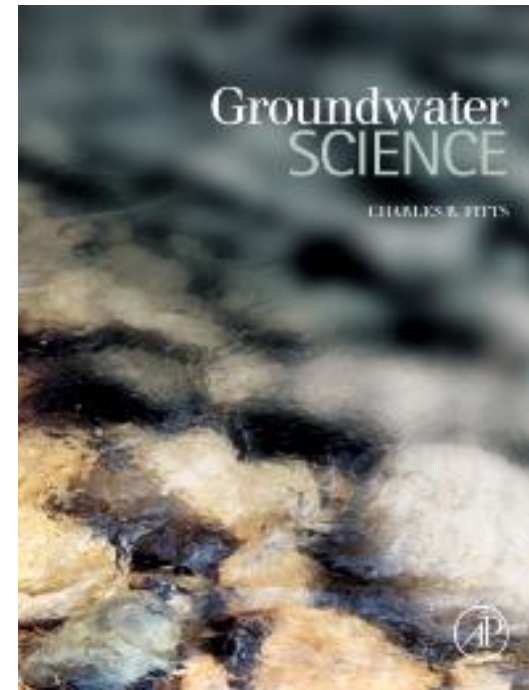
$$S \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(T \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(T \frac{\partial h}{\partial y} \right)$$

- Groundwater flow equation for 2D homogeneous, isotropic saturated confined aquifer can be written as

$$S \frac{\partial h}{\partial t} = T \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right)$$

Learning Strategy

Chapter 6: Deformation, Storage, and General Flow Equations



Thank you