Module 03: Groundwater Hydraulics

Unit 01: One-Dimensional Flow

Anirban Dhar

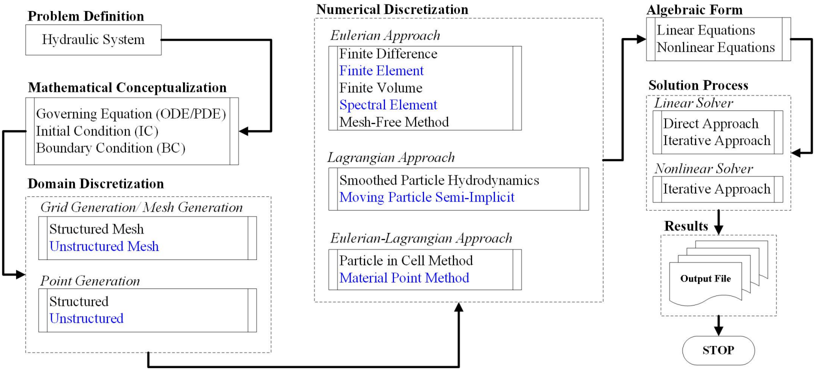
Department of Civil Engineering
Indian Institute of Technology Kharagpur, Kharagpur

National Programme for Technology Enhanced Learning (NPTEL)

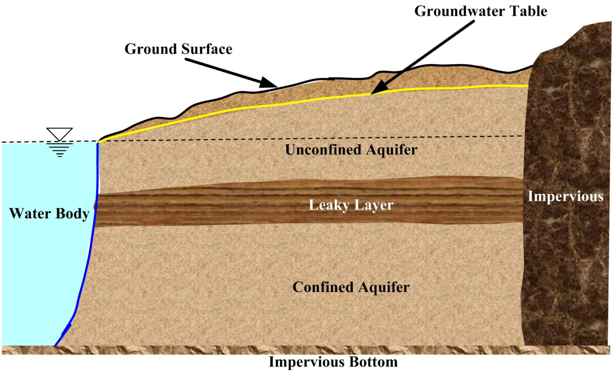
Learning Objective

- To solve one dimensional groundwater flow equation.

Problem Definition to Solution



Problem Definition



Mathematical Conceptualization

The differential equation describing the head distribution in the aquifer is given as ,

$$\frac{d^2h}{dx^2} = \frac{C_{\text{conf}}}{T}(h - h_{wt}) \tag{1}$$

where,
$h =$ head,
$T =$ aquifer transmissivity,
$C_{\text{conf}} =$ hydraulic conductivity/thickness of confining layer,
$h_{wt} =$ overlying water table elevation $(c_0 + c_1x + c_2x^2)$.

Boundary Conditions

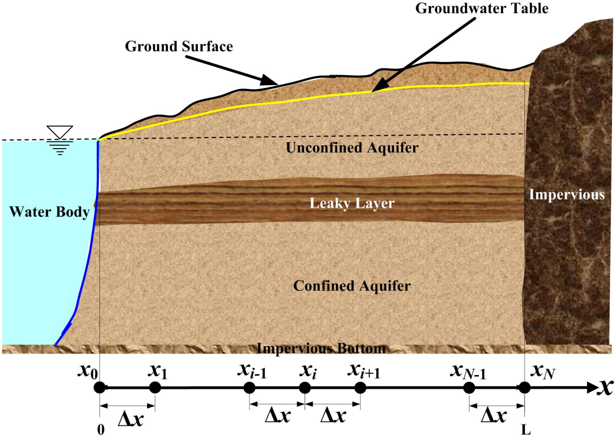
- Left Boundary is specified head/ Dirichlet boundary: $h(x = 0) = h_s$
- Right Boundary is impervious/ no-flow/ Neumann Boundary: $\frac{dh}{dx}\Big|_L = 0$

Mathematical Conceptualization

Data Values

$C_{\mathsf{conf}} = 10^{-11}$
$T = 2 \times 10^{-5}$
$c_0 = 90$
$c_1 = 0.06$
$c_2 = -0.00003$
$h_s = 90$
$L = 1000$

Domain Discretization



Numerical Discretization

Governing Equation

From Lecture 8,

Numerical Discretization
Governing Equation

From Lecture 8, the discretized governing equation for interior points:

$$\frac{h_{i-1} - 2h_i + h_{i+1}}{\Delta x^2} = \frac{C_{\text{conf}}}{T}\left[h_i - h_{wt}(x_i)\right] \quad \forall i \in \{2, \ \ldots, \ N-1\}$$

Numerical Discretization

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The equation can be further simplified as,

$$b_i h_{i-1} + d_i h_i + a_i h_{i+1} = r_i$$

Numerical Discretization

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The equation can be further simplified as,

$$b_i h_{i-1} + d_i h_i + a_i h_{i+1} = r_i$$

where the coefficients are given by, $b_i = \frac{1}{\Delta x^2}$, $d_i = -\left(\frac{C_{\text{conf}}}{T} + \frac{2}{\Delta x^2}\right)$, $a_i = \frac{1}{\Delta x^2}$ and $r_i = -\frac{C_{\text{conf}}}{T} h_{wt}(x_i)$

Numerical Discretization

Boundary Conditions

Left Boundary

$$h_0 = h_s \tag{2}$$

In general equation format,
$b_0 = 0$, $d_0 = 1$, $a_0 = 0$ and $r_0 = h_s$

Numerical Discretization

Boundary Conditions

Left Boundary

$$h_0 = h_s \tag{2}$$

In general equation format,
$b_0 = 0$, $d_0 = 1$, $a_0 = 0$ and $r_0 = h_s$

Right Boundary

First Order Discretization

$$\frac{h_N - h_{N-1}}{\Delta x} = 0 \tag{3}$$

In general equation format,
$b_N = -\frac{1}{\Delta x}$, $d_N = \frac{1}{\Delta x}$, $a_N = 0$ and $r_N = 0$

Fictitious Point Method

Using the boundary condition, the discretized governing equation can be written as,

$$b_N h_{N-1} + d_N h_N + a_N h_{N-1} = r_N$$

Fictitious Point Method

Using the boundary condition, the discretized governing equation can be written as,

$$b_N h_{N-1} + d_N h_N + a_N h_{N-1} = r_N$$

This can be simplified as,

$$(b_N + a_N) h_{N-1} + d_N h_N = r_N$$

where the coefficients are given by, $b_N = \frac{1}{\Delta x^2}$, $d_N = -\left(\frac{C_{\text{conf}}}{T} + \frac{2}{\Delta x^2}\right)$, $a_N = \frac{1}{\Delta x^2}$ and $r_N = -\frac{C_{\text{conf}}}{T} h_{wt}(x_N)$

Accuracy of Boundary Condition

Right Boundary

Second Order Discretization

$$\frac{3h_N - 4h_{N-1} + h_{N-2}}{2\Delta x} = 0 \tag{4}$$

In general equation format,
$b_N = -\frac{4}{2\Delta x}$, $d_N = \frac{3}{2\Delta x}$, $a_N = 0$ and $r_N = 0$
$e_N = \frac{1}{2\Delta x}$

Algebraic Form

$$\underbrace{\begin{bmatrix} d_0 & a_0 & & & & & & \\ b_1 & d_1 & a_1 & & & & & \\ & b_2 & d_2 & a_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \mathbf{0} & \\ & & \mathbf{0} & & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & b_{N-2} & d_{N-2} & a_{N-2} & \\ & & & & & & b_{N-1} & d_{N-1} & a_{N-1} \\ & & & & & & & b_N & d_N \end{bmatrix}}_{\mathbf{A}_{(N+1)\times(N+1)}} \underbrace{\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h_{N-1} \\ h_{N-2} \\ h_N \end{bmatrix}}_{\mathbf{h}_{(N+1)\times 1}} = \underbrace{\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ r_{N-1} \\ r_{N-2} \\ r_N \end{bmatrix}}_{\mathbf{r}_{(N+1)\times 1}}$$

Solution can be obtained as,

$$\mathbf{A}\mathbf{h} = \mathbf{r} \rightarrow \mathbf{h} = \mathbf{A}^{-1}\mathbf{r} \tag{5}$$

Algebraic Form

$$
\underbrace{\begin{bmatrix}
d_0 & a_0 & & & & & & \\
b_1 & d_1 & a_1 & & & & & \\
 & b_2 & d_2 & a_2 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & \ddots & \mathbf{0} & \\
 & \mathbf{0} & & & \ddots & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & \ddots \\
 & & & & & b_{N-2} & d_{N-2} & a_{N-2} & \\
 & & & & & & b_{N-1} & d_{N-1} & a_{N-1} \\
 & & & & & & e_N & b_N & d_N
\end{bmatrix}}_{\mathbf{A}_{(N+1)\times(N+1)}}
\underbrace{\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h_{N-1} \\ h_{N-2} \\ h_N \end{bmatrix}}_{\mathbf{h}_{(N+1)\times 1}}
=
\underbrace{\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ r_{N-1} \\ r_{N-2} \\ r_N \end{bmatrix}}_{\mathbf{r}_{(N+1)\times 1}}
$$

Solution can be obtained as,

$$\mathbf{Ah} = \mathbf{r} \rightarrow \mathbf{h} = \mathbf{A}^{-1}\mathbf{r} \tag{6}$$

Matrix Form

$$
\begin{bmatrix}
\bullet & \bullet & & & & & & & & \\
\color{blue}{\times} & \color{red}{\times} & \color{green}{\times} & & & & & & & \\
& \color{blue}{\times} & \color{red}{\times} & \color{green}{\times} & & & & & & \\
& & \color{blue}{\ddots} & \color{red}{\ddots} & \color{green}{\ddots} & & & & & \\
& & & \color{blue}{\ddots} & \color{red}{\ddots} & \color{green}{\ddots} & & 0 & & \\
& & 0 & & \color{blue}{\ddots} & \color{red}{\ddots} & \color{green}{\ddots} & & & \\
& & & & & \color{blue}{\ddots} & \color{red}{\ddots} & \color{green}{\ddots} & & \\
& & & & & & \color{blue}{\times} & \color{red}{\times} & \color{green}{\times} & \\
& & & & & & & \color{blue}{\times} & \color{red}{\times} & \color{green}{\times} \\
& & & & & & & \bullet & \bullet & \bullet
\end{bmatrix}
$$

Comment on Convergence

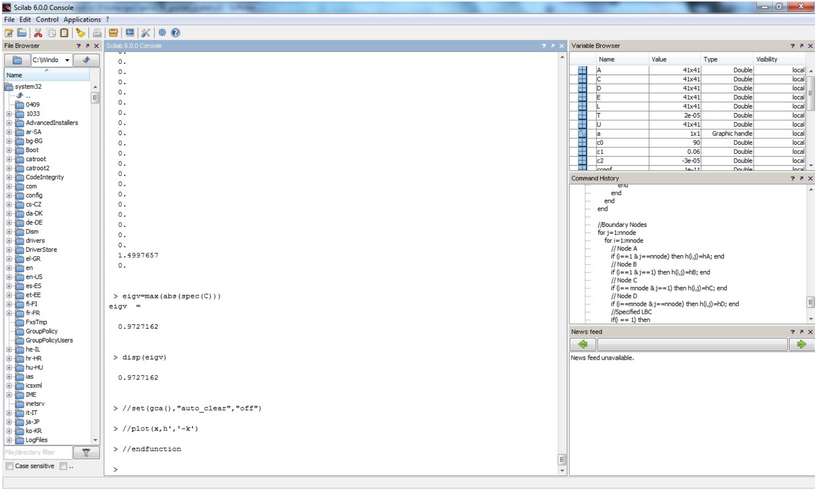
From Lecture 29, coefficient matrix of the iterative step can be used to calculate spectral radius.

Spectral radius

$$\rho\left(-\mathbf{D}^{-\mathbf{1}}(\mathbf{L}+\mathbf{U})\right) = max\{|\lambda_1|\ldots|\lambda_N|\} < 1$$

where $|\lambda_1|\ldots|\lambda_N|$ are eigenvalues of the matrix.

Spectral Radius



Spectral Radius

```
  0.
  0.
  0.
  0.
  1.4997657
  0.


 > eigv=max(abs(spec(C)))
eigv  =

  0.9727162


 > disp(eigv)

  0.9727162


 > //set(gca(),"auto_clear","off")

 > //plot(x,h','-k')

 > //endfunction

 >
```

List of Source Codes

One Dimensional Groundwater Flow

- Full matrix with 2 point/ 3 point BC implementation using Gauss elimination
  - gw1d_fd_gausselim.sci
- Banded matrix with 2 point using TDMA
  - gw1d_fd_tdma.sci
- Full matrix with 2 point/ 3 point BC implementation using Gauss Seidel
  - gw1d_fd_gseidel.sci
  - gw1d_fd_gseidel_scaled.sci

Thank You