Module 01: Introduction to Computational Hydraulics

Unit 04: Classification of Differential Equations

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Learning Objective

- To classify the *Differential Equations* based on physical behavior, completeness of problem definition and linearity.

Classification based on Physical Behavior

Equilibrium Problem

"Problems in which a solution of a given PDE is desired in a closed domain subject to a prescribed set of boundary conditions" (Tannehill et al., 1997).

- Also known as **Jury problems**.

Classification based on Physical Behavior

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- Generally Steady state problems.

Classification based on Physical Behavior

Equilibrium Problem

"Problems in which a solution of a given PDE is desired in a closed domain subject to a prescribed set of boundary conditions" (Tannehill et al., 1997).

- Also known as **Jury problems**.
- Generally Steady state problems.
- Solution is always smooth even if there is disturbance.

Equilibrium Problem

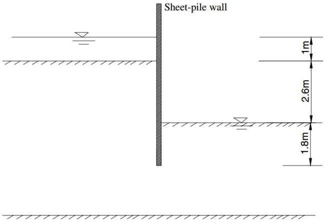

Figure: Cross-section of a foundation pit (Jie et al., 2004)

Equilibrium Problem

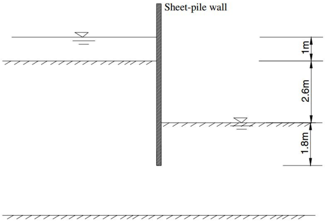

Figure: Cross-section of a foundation pit (Jie et al., 2004)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{1}$$

$$\begin{aligned} h\big|_{\Gamma_D} &= h_0(x,y) \\ K\frac{dh}{dx}\big|_{\Gamma_N} &= 0 \end{aligned} \tag{2}$$

Classification based on Physical Behavior

Marching Problem

Problems in which a solution of a given differential equation is desired in an open domain subject to a prescribed set of initial and boundary conditions.

- Generally transient/transient-like problems

Classification based on Physical Behavior

Marching Problem

Problems in which a solution of a given differential equation is desired in an open domain subject to a prescribed set of initial and boundary conditions.

- Generally transient/transient-like problems
- Not all marching problems are unsteady.

Classification based on Completeness of Problem Definition

Well-Posed Problem

- Solution of the problem exists

Classification based on Completeness of Problem Definition

Well-Posed Problem

- Solution of the problem exists
- Solution of the problem is unique

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- Solution of the problem exists
- Solution of the problem is unique
- Solution depends continuously on data and parameters

Classification based on Completeness of Problem Definition

Well-Posed Problem

- Solution of the problem exists
- Solution of the problem is unique
- Solution depends continuously on data and parameters

Ill-Posed Problem

- Not well-posed problems

Classifications based on Linearity

Linear

Groundwater equation for confined aquifer

Classifications based on Linearity

Linear

Groundwater equation for confined aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) \tag{3}$$

Classifications based on Linearity

Linear

Groundwater equation for confined aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) \tag{3}$$

Non-linear

Classifications based on Linearity

Linear

Groundwater equation for confined aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} = \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) \tag{3}$$

Non-linear

Momentum conservation equation for surface water

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h}{\partial x} + gA(S_f - S_0) = 0 \tag{4}$$

Classification of Second Order PDE

A second order PDE in two co-ordinates x and y for a general variable ϕ can be written as,

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G = 0 \quad (5)$$

where the coefficients $A,\ B,\ C,\ D,\ E,\ F$, and G are functions of $x,\ y$ or constants.

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where the coefficients $A,\ B,\ C,\ D,\ E,\ F$, and G are functions of $x,\ y$ or constants. Highest partial derivatives determine the nature of the equation. The characteristic equation can be written as,

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \quad (6)$$

Depending on sign of discriminant $(B^2 - 4AC)$ equations are classified.

Classification of Second Order PDE

Parabolic Equations

Parabolic: $B^2 - 4AC = 0$

Classification of Second Order PDE

Parabolic Equations

Parabolic: $B^2 - 4AC = 0$

Transient One-dimensional groundwater flow equation in confined aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \tag{7}$$

Classification of Second Order PDE

Parabolic Equations

Parabolic: $B^2 - 4AC = 0$

Transient One-dimensional groundwater flow equation in confined aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \tag{7}$$

Here, $A = 0$, $B = 0$, $C = -1$ and $B^2 - 4AC = 0$.

Classification of Second Order PDE

Elliptic Equations

Elliptic: $B^2 - 4AC < 0$

Classification of Second Order PDE

Elliptic Equations

Elliptic: $B^2 - 4AC < 0$

Steady two-dimensional groundwater flow equation in confined aquifer (Laplace equation)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{8}$$

Classification of Second Order PDE

Elliptic Equations

Elliptic: $B^2 - 4AC < 0$

Steady two-dimensional groundwater flow equation in confined aquifer (Laplace equation)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{8}$$

Here, $A = 1$, $B = 0$, $C = 1$ and $B^2 - 4AC = -4 < 0$.

Classification of Second Order PDE

Hyperbolic Equations

Hyperbolic: $B^2 - 4AC > 0$

Classification of Second Order PDE

Hyperbolic Equations

Hyperbolic: $B^2 - 4AC > 0$

One-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{9}$$

Classification of Second Order PDE

Hyperbolic Equations

Hyperbolic: $B^2 - 4AC > 0$

One-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{9}$$

Here, $A = -1$, $B = 0$, $C = 1$ and $B^2 - 4AC = 4 > 0$.

Eigenvalue based Classification

A general second order PDE with N independent variables $(x_1,\ x_2 ... x_N)$ can be represented as,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial \phi}{\partial x_i} + c\phi + d = 0 \tag{10}$$

where

$a_{ij},\ b_i,\ c\ =$ functions of $x_1,\ x_2, ..., x_N$

Assumptions:

- $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$
- $A_\lambda = [a_{ij}]$ is symmetric.

Eigenvalue based Classification

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Assumptions:

- $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$
- $A_\lambda = [a_{ij}]$ is symmetric.

The eigenvalues of A_λ are values of λ that satisfy the equation

$$|A_\lambda - \lambda I| = 0 \tag{11}$$

Eigenvalue based Classification

The equation can be classified based on the sign of eigenvalues $(\lambda_1,\ \lambda_2,\ ...,\ \lambda_N)$ of matrix A_λ as

- Parabolic Equation: one or more zero eigenvalues $(\lambda_i\ =\ 0)$

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- Parabolic Equation: one or more zero eigenvalues $(\lambda_i\ =\ 0)$
- Elliptic Equation: non-zero and with same sign $(\lambda_i > 0,\ \forall i$ or $\lambda_i < 0,\ \forall i)$

Eigenvalue based Classification

The equation can be classified based on the sign of eigenvalues $(\lambda_1,\ \lambda_2,\ ...,\ \lambda_N)$ of matrix A_λ as

- Parabolic Equation: one or more zero eigenvalues $(\lambda_i\ =\ 0)$
- Elliptic Equation: non-zero and with same sign $(\lambda_i > 0,\ \forall i$ or $\lambda_i < 0,\ \forall i)$
- Hyperbolic Equation: non-zero and all but one are with same sign

$$\lambda_i > 0,\ i \in\ \{1,\ 2,\ ...,\ N\}\backslash\{j\}$$
$$\lambda_j < 0$$

Eigenvalue based Classification

The equation can be classified based on the sign of eigenvalues $(\lambda_1,\ \lambda_2,\ \dots,\ \lambda_N)$ of matrix A_λ as

- Parabolic Equation: one or more zero eigenvalues $(\lambda_i\ =\ 0)$
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- Hyperbolic Equation: non-zero and all but one are with same sign

$$\lambda_i > 0,\ i \in\ \{1,\ 2,\ \dots,\ N\}\backslash\{j\}$$
$$\lambda_j < 0$$

or

$$\lambda_i < 0,\ i \in\ \{1,\ 2,\ \dots,\ N\}\backslash\{j\}$$
$$\lambda_j > 0$$

Parabolic Equations

Transient 1D Groundwater Equation in Confined Aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \tag{12}$$

Parabolic Equations

Transient 1D Groundwater Equation in Confined Aquifer

$$\frac{S}{T}\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \tag{12}$$

$$A_\lambda = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -1 < 0$

Elliptic Equations

Steady 2D Groundwater Equation in Confined Aquifer

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{13}$$

Elliptic Equations

Steady 2D Groundwater Equation in Confined Aquifer

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{13}$$

$$A_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 1 > 0$, $\lambda_2 = 1 > 0$

Hyperbolic Equations

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{14}$$

Hyperbolic Equations

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{14}$$

$$A_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 1 > 0$, $\lambda_2 = -1 < 0$

Let us consider a form of differential equation with a general variable ϕ:

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla.(\Upsilon_\phi \phi \mathbf{u}) = \nabla.(\mathbf{\Gamma}_\phi.\nabla\phi) + F_{\phi_o} + S_\phi \qquad (15)$$

where

ϕ = general variable

$\Lambda_\phi,\ \Upsilon_\phi$ = problem dependent parameters

$\mathbf{\Gamma}_\phi$ = tensor

F_{ϕ_o} = other forces

S_ϕ = source/sink term

$\Lambda_\phi = 1$, $\phi = \rho =$ constant, $\Upsilon_\phi = 1$, $\mathbf{\Gamma}_\phi = \mathbf{0}$, $F_{\phi_o} = 0$, $S_\phi = 0$

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{16}$$

$\Lambda_\phi = 1$, $\phi = \rho =$ constant, $\Upsilon_\phi = 1$, $\mathbf{\Gamma}_\phi = \mathbf{0}$, $F_{\phi_o} = 0$, $S_\phi = 0$

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{16}$$

$\Lambda_\phi = \rho$, $\phi = u$, $\Upsilon_\phi = \rho$, $\mathbf{\Gamma}_\phi = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$, $F_{\phi_o} = -\frac{\partial P}{\partial x} + \rho g_x$, $S_\phi = 0$

Momentum conservation equation

x-dir: $$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{17}$$

Contaminant Transport

Concentration Equation

$$\Lambda_\phi = 1,\ \phi = \eta C,\ \Upsilon_\phi = 1,\ \mathbf{\Gamma}_\phi = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix},\ F_{\phi_o} = 0,\ S_\phi = q_s C_s$$

Scalar Transport Equation

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x}\left(\eta D_{xx}\frac{\partial C}{\partial x} + \eta D_{xy}\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial y}\left(\eta D_{yx}\frac{\partial C}{\partial x} + \eta D_{yy}\frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial x}(\eta v_x C) - \frac{\partial}{\partial y}(\eta v_y C) + q_s C_s \tag{18}$$

Thank You

References

Jie, Y., Jie, G., Mao, Z., and Li, G. (2004). Seepage analysis based on boundary-fitted coordinate transformation method. *Computers and Geotechnics*, 31(4):279 – 283.

Tannehill, J. C., Anderson, D., and Pletcher, R. H. (1997). *Computational Fluid Mechanics and Heat Transfer, Second Edition*. Taylor and Francis.