General Flow Equations

Groundwater Engineering| CE60205 Lecture:09

Learning Objective(s)

- To derive the governing equation for saturated unconfined aquifer

Dupuit-Forchheimer Assumptions

- Dupuit-Forchheimer Assumptions
  - for small inclinations of the free surface of a gravity-flow system, the streamlines can be taken as horizontal
  - the velocities associated with these streamlines are proportional to the slope of the free surface but are independent of the depth

Head is independent of depth: $h(x, y, z, t) \to h(x, y, t)$

Discharge is proportional to the slope of the water table $h(x, y, t) \to H(x, y, t)$

- Aquifer flux per unit width

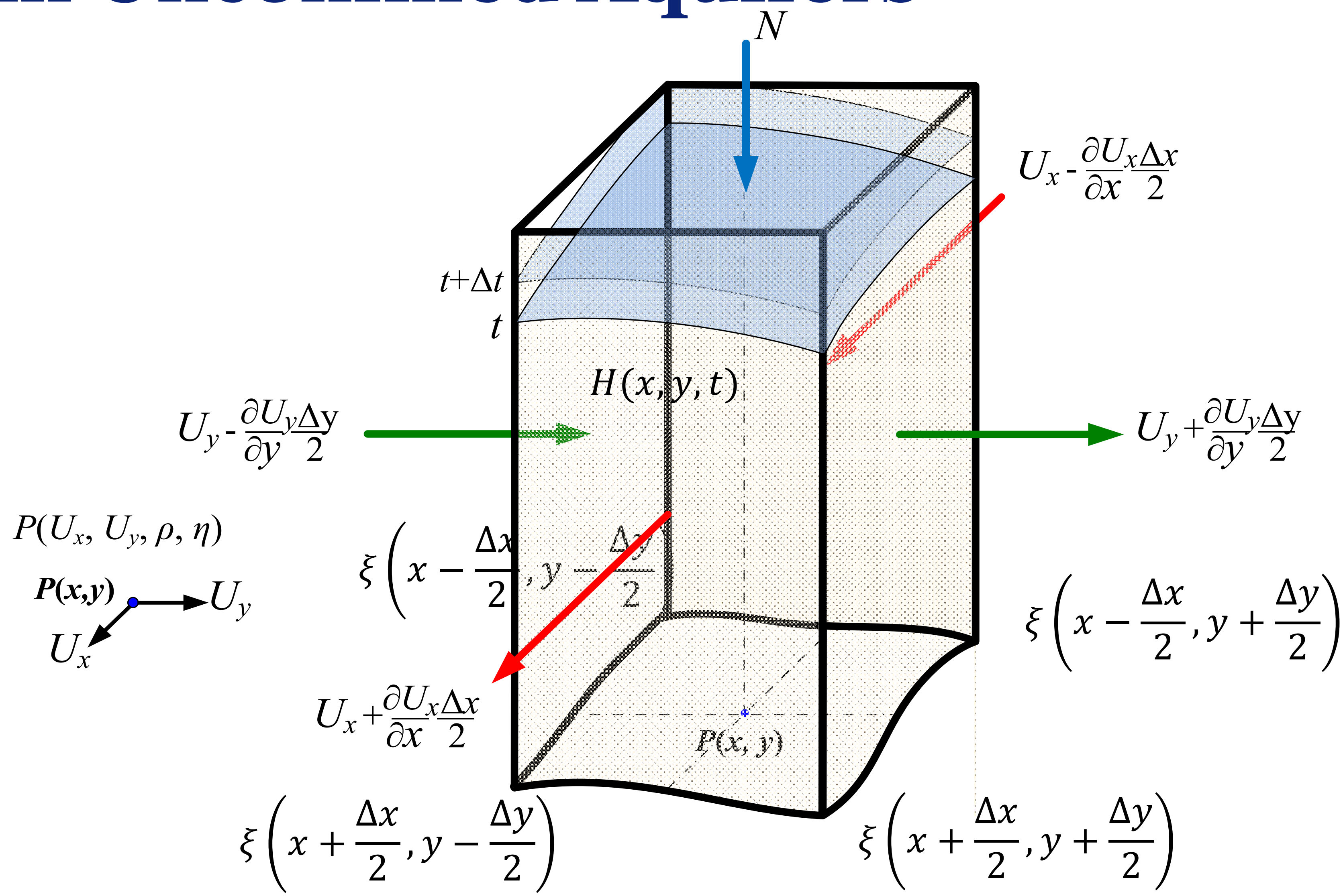
$$U_x = -K_x(H - \xi)\frac{\partial H}{\partial x}$$

$$U_y = -K_y(H - \xi)\frac{\partial H}{\partial y}$$

where ξ is the elevation of the base of the aquifer.

- No vertical flow

Two-Dimensional Saturated Flow in Unconfined Aquifers



Two-Dimensional Saturated Flow in Unconfined Aquifers (Contd.)

- Volume inflow rate –Volume outflow rate = Rate of change of storage volume with time
- The inflow and outflow can be calculated for each side of the element. The groundwater inflow in x-direction is

$$U_x\left(x-\frac{\Delta x}{2},y,t\right)\Delta y \approx \left(U_x(x,y,t)-\frac{\partial U_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y$$

Up to first order

- The groundwater outflow in x-direction is

$$U_x\left(x+\frac{\Delta x}{2},y,t\right)\Delta y \approx U_x\Delta y+\frac{\partial U_x}{\partial x}\frac{\Delta x\Delta y}{2}$$

Two-Dimensional Saturated Flow in Unconfined Aquifers (Contd.)

- The groundwater inflow in y-direction is

$$U_y\left(x, y - \frac{\Delta y}{2}, t\right)\Delta x \approx \left(U_y(x, y, t) - \frac{\partial U_y}{\partial y}\frac{\Delta y}{2}\right)\Delta x$$

Up to first order

$$= U_y \Delta x - \frac{\partial U_y}{\partial y}\frac{\Delta y \Delta x}{2}$$

- The groundwater outflow in y-direction is

$$U_y\left(x, y + \frac{\Delta y}{2}, t\right)\Delta x \approx U_y \Delta x + \frac{\partial U_y}{\partial y}\frac{\Delta y \Delta x}{2}$$

Two-Dimensional Saturated Flow in Unconfined Aquifers (Contd.)

- Considering all equations, the total inflow minus outflow can be derived as follows:

$$\frac{\partial V}{\partial t} = -\left[\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} - N\right]\Delta x \Delta y$$

$$V = S_y V_T = S_y\big(H(x, y, t) - \xi(x, y)\big)\Delta x \Delta y$$

- The change in storage is calculated by

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left[S_y\big(H(x, y, t) - \xi(x, y)\big)\Delta x \Delta y\right] = S_y \frac{\partial H}{\partial t}\Delta x \Delta y$$

- Conservation equation can be written as

$$S_y \frac{\partial H}{\partial t} = -\left[\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} - N\right]$$

or,

$$S_y \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[K_x(H - \xi)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y(H - \xi)\frac{\partial H}{\partial y}\right] + N$$

Two-Dimensional Saturated Flow in Unconfined Aquifers (Contd.)

- Groundwater flow equation for heterogeneous, anisotropic saturated unconfined aquifer can be written as

$$S_y \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[K_x (H-\xi)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y (H-\xi)\frac{\partial H}{\partial y}\right] + N$$

- Groundwater flow equation for homogeneous, anisotropic saturated unconfined aquifer can be written as

$$S_y \frac{\partial H}{\partial t} = K_x \frac{\partial}{\partial x}\left[(H-\xi)\frac{\partial H}{\partial x}\right] + K_y \frac{\partial}{\partial y}\left[(H-\xi)\frac{\partial H}{\partial y}\right] + N$$

- Groundwater flow equation for heterogeneous, isotropic saturated unconfined aquifer can be written as

$$S_y \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[K \quad (H-\xi)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[K \quad (H-\xi)\frac{\partial H}{\partial y}\right] + N$$

- Groundwater flow equation for homogeneous, isotropic saturated unconfined aquifer can be written as

$$S_y \frac{\partial H}{\partial t} = K \quad \frac{\partial}{\partial x}\left[(H-\xi)\frac{\partial H}{\partial x}\right] + K \quad \frac{\partial}{\partial y}\left[(H-\xi)\frac{\partial H}{\partial y}\right] + N$$

Two-Dimensional Saturated Flow in Unconfined Aquifers (Contd.)

- Groundwater flow equation for homogeneous, isotropic saturated unconfined aquifer with no bed level variation ($\xi = 0$) can be written as

$$S_y \frac{\partial H}{\partial t} = K \quad \frac{\partial}{\partial x}\left[(H-0)\frac{\partial H}{\partial x}\right] + K \quad \frac{\partial}{\partial y}\left[(H-0)\frac{\partial H}{\partial y}\right] + N$$

or,

$$S_y \frac{\partial H}{\partial t} = \frac{K}{2}\frac{\partial^2 H^2}{\partial x^2} + \frac{K}{2}\frac{\partial^2 H^2}{\partial y^2} + N$$

or,

$$\frac{2S_y}{K}\frac{\partial H}{\partial t} = \nabla^2 H^2 + \frac{2N}{K}$$

- It is called the Boussinesq equation.

- 2D Steady flow for homogeneous, isotropic saturated unconfined aquifer with no bed level variation ($\xi = 0$)

$$0 = \frac{K}{2}\frac{\partial^2 H^2}{\partial x^2} + \frac{K}{2}\frac{\partial^2 H^2}{\partial y^2} + N$$

Unconfined Aquifer

- 1D Steady flow for homogeneous, isotropic saturated unconfined aquifer with no bed level variation ($\xi = 0$)

$$0 = \frac{K}{2}\frac{\partial^2 H^2}{\partial x^2} + N$$

- Dupuit's equation for radial flow

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\cdot)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\cdot) \approx \frac{d^2}{dr^2}(\cdot) + \frac{1}{r}\frac{d}{dr}(\cdot)$$

or,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH^2}{dr}\right) = \frac{2W}{K}$$

or,

$$H^2 = \frac{Wr^2}{2K} + A\ ln(r) + B$$

Agricultural Drain

- The governing equation can be written as

$$\frac{\partial^2 H^2}{\partial x^2} + \frac{2R}{K} = 0$$

- Solution of the differential equation is

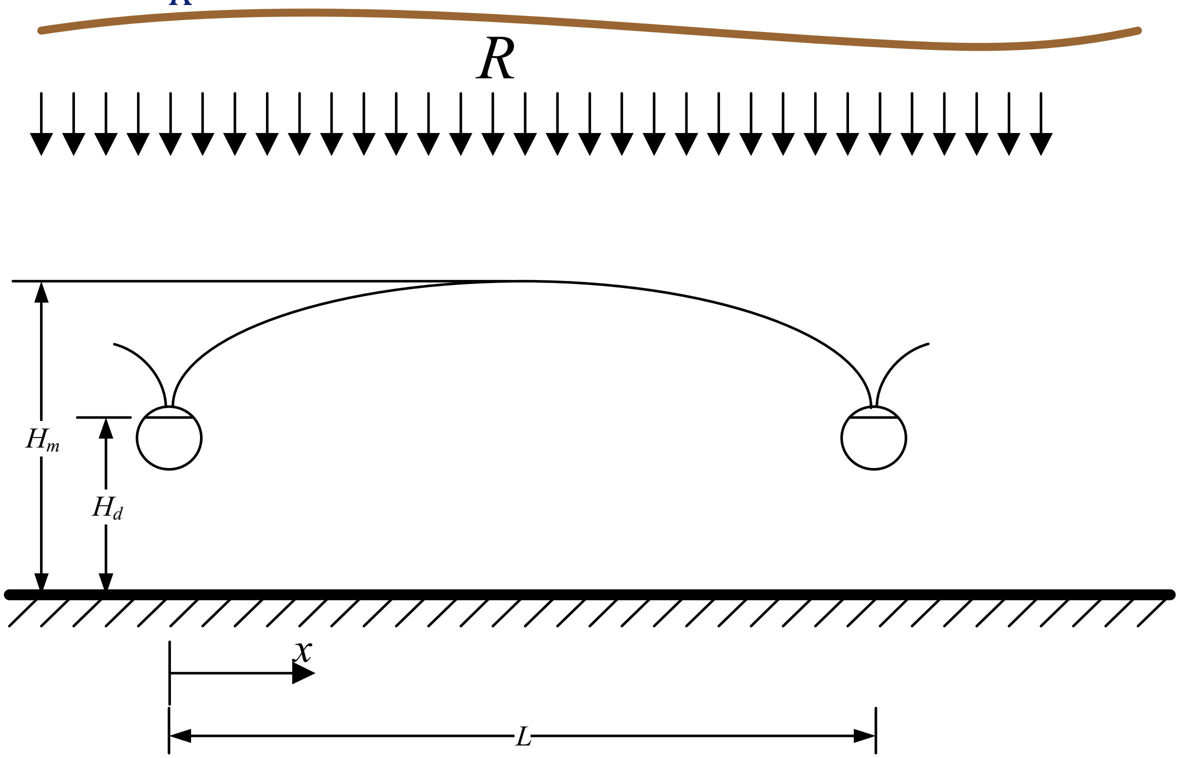
$$H^2 + \frac{Rx^2}{K} = Ax + B$$

- Boundary conditions are

$$H^2(0) = H_d^2; H^2(L) = H_d^2$$

or,

$$H^2(0) = H_d^2; \frac{dH^2}{dx}\left(\frac{L}{2}\right) = 0$$



Agricultural Drain (Contd.)

- Boundary condition set-I

$$H^2(0) = B = H_d^2$$

$$H^2(L) = H_d^2 = -\frac{RL^2}{K} + AL + H_d^2$$

$$A = \frac{RL}{K}$$

- The water table elevation is given by

$$H^2 = H_d^2 + \frac{Rx}{K}(L - x)$$

- Boundary condition set-II

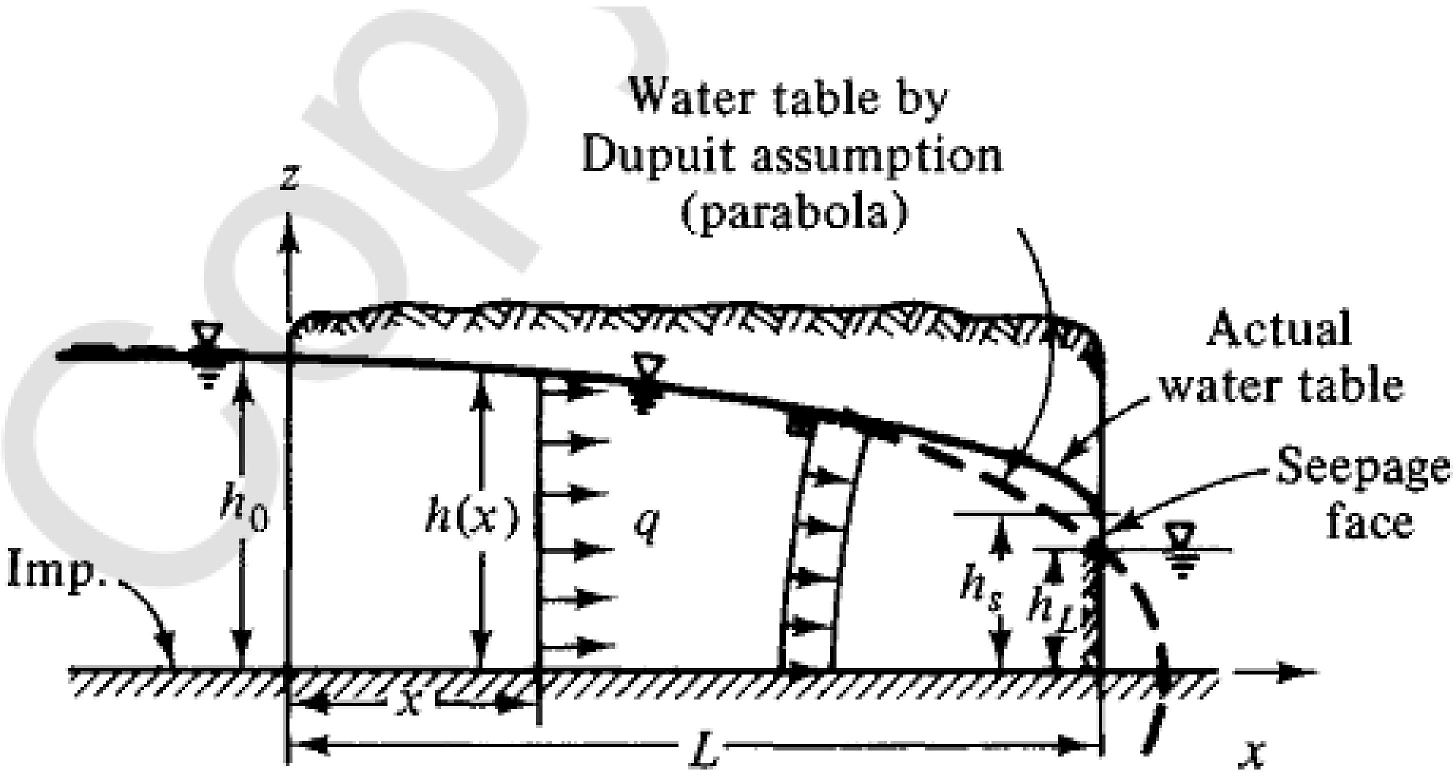
$$H^2(0) = B = H_d^2$$

$$\frac{dH^2}{dx}\left(\frac{L}{2}\right) = -\frac{2R\left(\frac{L}{2}\right)}{K} + A = 0 \Rightarrow A = \frac{RL}{K}$$

- The maximum saturated thickness

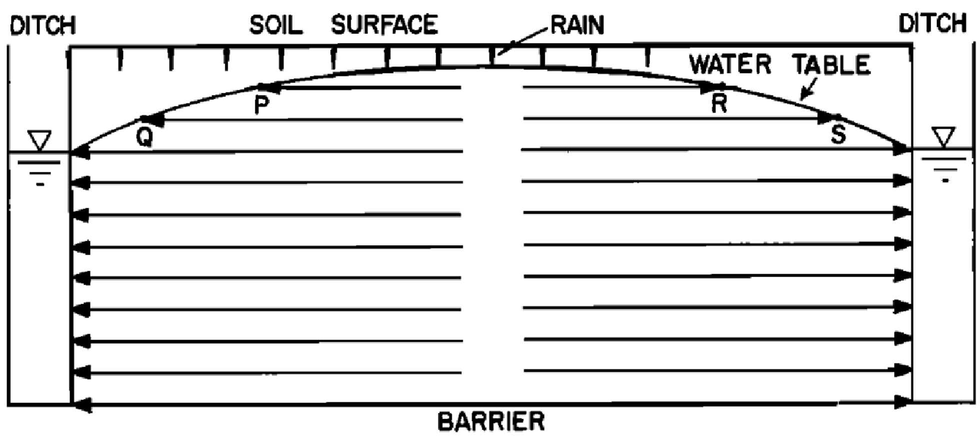
$$H_m = \sqrt{H_d^2 + \frac{RL^2}{4K}}$$

Dupuit-Forchheimer Discharge Formula



Paradoxes in Dupuit-Forchheimer Seepage Theory

- two Dupuit assumptions on which it rests are 'entirely contradictory to the implications of Darcy's law'.
- many problem solutions based on the Dupuit assumptions compare favorably with those of more rigorous method
- correct solutions for the discharge, but not the free surface



. **Horizontal flow lines as (improperly) conceived for Dupuit-Forchheimer flow.**

VOL. 3, NO. 2 WATER RESOURCES RESEARCH SECOND QUARTER 1967

Explanation of Paradoxes in Dupuit-Forchheimer Seepage Theory[1]

DON KIRKHAM

Free-surface Condition

- At every point within the considered domain and on its boundaries, the piezometric head is defined as:

$$h(x, y, z, t) = z + p(x, y, z, t)/\gamma$$

- On the phreatic surface, the pressure is atmospheric, assumed zero, *i.e.*, $p = 0$.

$$h(x, y, z, t) = z$$

Free surface function

$$F_1(x, y, z, t) = z - h(x, y, z, t) = 0$$

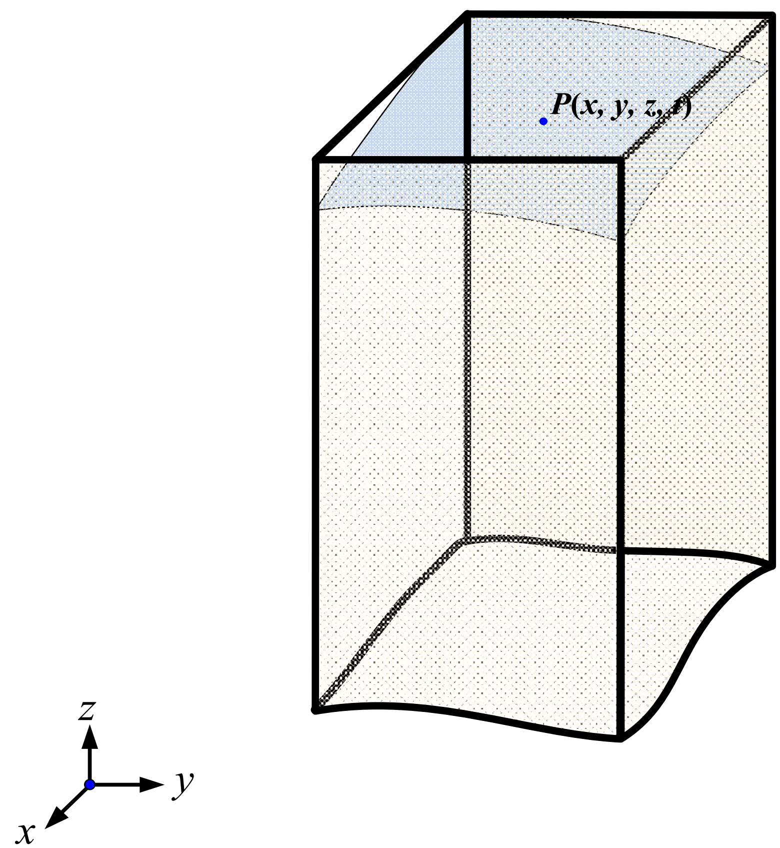
- The free surface elevation at a given (x, y) location

$$z = H(x, y, t)$$

Free surface function

$$F_2(x, y, z, t) = z - H(x, y, t) = 0$$

- F_1 and F_2 are different functions, but they represent the same free surface



Free-surface Condition (Contd.)

- F_1 does not vary on the surface as it move

$$\frac{DF_1}{Dt} = \frac{D}{Dt}\big(z - h(x,y,z,t)\big)$$
$$= \frac{\partial}{\partial t}\big(z - h(x,y,z,t)\big) + \mathbf{v}_{wt} \cdot \nabla\big(z - h(x,y,z,t)\big)$$
$$= -\frac{\partial h}{\partial t} - v_{wtx}\frac{\partial h}{\partial x} - v_{wty}\frac{\partial h}{\partial y} + v_{wtz}\left(1 - \frac{\partial h}{\partial z}\right)$$
$$= 0$$

Equation can be written as

$$-\eta_e\frac{\partial h}{\partial t} - \eta_e v_{wtx}\frac{\partial h}{\partial x} - \eta_e v_{wty}\frac{\partial h}{\partial y} + \eta_e v_{wtz}\left(1 - \frac{\partial h}{\partial z}\right) = 0$$

- The BC on the moving surface is obtained from the mass balance condition on that surface

$$(\mathbf{q} - \mathbf{N}).\,\mathbf{n} = \eta_e\,\mathbf{v}_{wt}.\,\mathbf{n}$$

where η_e is the drainable porosity or effective porosity

$$\mathbf{N} = -N\mathbf{e_z}$$

Free-surface Condition (Contd.)

- In terms of piezometric head

$$\mathbf{q} = -\mathbf{K} \cdot \nabla h$$

- If x, y, z are the In principal flow directions, then

$$q_x = -K_x \frac{\partial h}{\partial x}, q_y = -K_y \frac{\partial h}{\partial y}, q_z = -K_z \frac{\partial h}{\partial z}$$

By combining all terms

$$-\eta_e \frac{\partial h}{\partial t} - \eta_e v_{wtx} \frac{\partial h}{\partial x} - \eta_e v_{wty} \frac{\partial h}{\partial y} + \eta_e v_{wtz} \left(1 - \frac{\partial h}{\partial z}\right) = 0$$

Or,

$$-\eta_e \frac{\partial h}{\partial t} + K_x \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} + K_y \frac{\partial h}{\partial y} \frac{\partial h}{\partial y} + \left(-K_z \frac{\partial h}{\partial z} + N\right)\left(1 - \frac{\partial h}{\partial z}\right) = 0$$

Or,

$$\eta_e \frac{\partial h}{\partial t} = K_x \left(\frac{\partial h}{\partial x}\right)^2 + K_y \left(\frac{\partial h}{\partial y}\right)^2 + K_z \left(\frac{\partial h}{\partial z}\right)^2 - (K_z + N) \frac{\partial h}{\partial z} + N$$

Free-surface Condition (Contd.)

- F_2 does not vary on the surface as it move

$$\frac{DF_2}{Dt} = \frac{D}{Dt}\left(z - H(x,y,t)\right)$$

$$= \frac{\partial}{\partial t}\left(z - H(x,y,t)\right) + \mathbf{v}_{wt} \cdot \nabla\left(z - H(x,y,t)\right)$$

$$= -\frac{\partial H}{\partial t} - v_{wtx}\frac{\partial H}{\partial x} - v_{wty}\frac{\partial H}{\partial y} + v_{wtz}\frac{\partial z}{\partial z}$$

$$= 0$$

Equation can be written as

$$-\eta_e\frac{\partial H}{\partial t} - \eta_e v_{wtx}\frac{\partial H}{\partial x} - \eta_e v_{wty}\frac{\partial H}{\partial y} + \eta_e v_{wtz} = 0$$

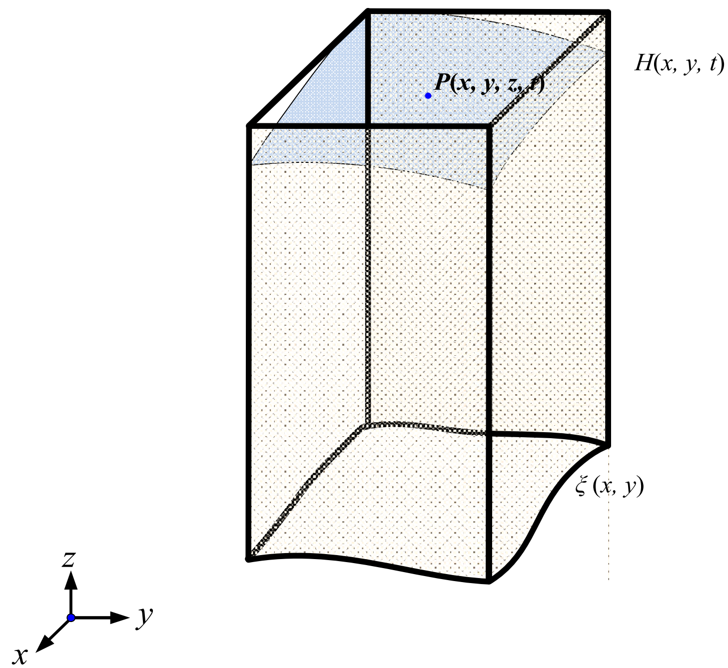
By combining all terms

$$\eta_e\frac{\partial H}{\partial t} = K_x\frac{\partial h}{\partial x}\frac{\partial H}{\partial x} + K_y\frac{\partial h}{\partial y}\frac{\partial H}{\partial y} - K_z\frac{\partial h}{\partial z} + N$$

- For very small slope of the phreatic surface

$$\eta_e\frac{\partial H}{\partial t} = -K_z\frac{\partial h}{\partial z} + N$$

Reduction in Dimensionality



- Vertical Integration of the Flow Equation

$$\int_{\xi(x,y)}^{H(x,y,t)} \left(\nabla \cdot \mathbf{q} + S_s \frac{\partial h}{\partial t} \right) dz = 0$$

Reduction in Dimensionality (Contd.)

- Using Leibniz integral rule

$$\int_{\xi(x,y)}^{H(x,y,t)} \nabla\cdot\mathbf{q}\,dz = \int_{\xi}^{H}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)dz = \frac{\partial}{\partial x}\int_{\xi}^{H} q_x\,dz - q_x|_{z=H}\frac{\partial H}{\partial x} + q_x|_{z=\xi}\frac{\partial \xi}{\partial x}$$
$$+\frac{\partial}{\partial y}\int_{\xi}^{H} q_y\,dz - q_y|_{z=H}\frac{\partial H}{\partial y} + q_y|_{z=\xi}\frac{\partial \xi}{\partial y} + q_z|_{z=H} - q_z|_{z=\xi}$$

- In general

$$\int_{\xi(x,y)}^{H(x,y,t)} \nabla\cdot\mathbf{q}\,dz = \nabla_{xy}\cdot\int_{\xi(x,y)}^{H(x,y,t)} \mathbf{q}_{xy}\,dz + \mathbf{q}|_{z=H}\cdot\nabla(z-H) - \mathbf{q}|_{z=\xi}\cdot\nabla(z-\xi)$$

where

$$\nabla_{xy}(\cdot) \equiv \frac{\partial}{\partial x}(\cdot)\mathbf{e}_x + \frac{\partial}{\partial y}(\cdot)\mathbf{e}_y$$

Reduction in Dimensionality (Contd.)

- Free surface function $F_2(x, y, z, t) = z - H(x, y, t) = 0$
- F_2 does not vary on the surface as it move $\frac{DF_2}{Dt} = \frac{D}{Dt}\left(z - H(x, y, t)\right) = 0$

$$\eta_e \frac{\partial(z - H)}{\partial t} + (\mathbf{q} - \mathbf{N}) \cdot \nabla(z - H) = 0$$

Or,

$$\mathbf{q}|_{z=H} \cdot \nabla(z - H) = \eta_e \frac{\partial H}{\partial t} - N$$

- Let us consider the vertical average values as

$$\overline{q_x} \equiv \frac{1}{(H - \xi)} \int_{\xi}^{H} q_x dz = \frac{1}{l} \int_{\xi}^{H} q_x dz = \frac{U_x}{l}$$

$$\overline{q_y} \equiv \frac{1}{(H - \xi)} \int_{\xi}^{H} q_y dz = \frac{1}{l} \int_{\xi}^{H} q_y dz = \frac{U_y}{l}$$

$$\bar{h} \equiv \frac{1}{(H - \xi)} \int_{\xi}^{H} h dz = \frac{1}{l} \int_{\xi}^{H} h dz$$

Reduction in Dimensionality (Contd.)

- Time derivative term

$$\int_{\xi(x,y)}^{H(x,y,t)} S_s \frac{\partial h}{\partial t} dz = \bar{S_s} \int_{\xi}^{H} \frac{\partial h}{\partial t} dz$$

$$= \bar{S_s}\left[\frac{\partial}{\partial t}\int_{\xi}^{H} h\, dz - h|_{z=H}\frac{\partial H}{\partial t} + h|_{z=\xi}\frac{\partial \xi}{\partial t}\right]$$

$$= \bar{S_s}\left[l\frac{\partial \bar{h}}{\partial t} + \bar{h}\frac{\partial l}{\partial t} - h|_{z=H}\frac{\partial H}{\partial t}\right]$$

$$= \bar{S_s}\left[l\frac{\partial \bar{h}}{\partial t} + \bar{h}\frac{\partial (H-\xi)}{\partial t} - h|_{z=H}\frac{\partial H}{\partial t}\right] = \bar{S_s} l \frac{\partial \bar{h}}{\partial t}\ [\because \bar{h} = H]$$

Reduction in Dimensionality (Contd.)

- Combining all the terms

$$\nabla_{xy} \cdot \int_{\xi(x,y)}^{H(x,y,t)} \mathbf{q}_{xy}\, dz + \mathbf{q}|_{z=H} \cdot \nabla(z - H) - \mathbf{q}|_{z=\xi} \cdot \nabla(z - \xi) + \bar{S}_s l \frac{\partial \bar{h}}{\partial t} = 0$$

Or,

$$\nabla_{xy} \cdot \mathbf{U} + \mathbf{q}|_{z=H} \cdot \nabla(z - H) - \mathbf{q}|_{z=\xi} \cdot \nabla(z - \xi) + \bar{S}_s l \frac{\partial \bar{h}}{\partial t} = 0$$

- Non-leaky condition: $q_B = -\mathbf{q}|_{z=\xi} \cdot \nabla(z - \xi) = 0$
- Free surface condition: $q_T = \mathbf{q}|_{z=H} \cdot \nabla(z - H) \; = \eta_e \frac{\partial H}{\partial t} - N$

$$(\eta_e + \bar{S}_s l) \frac{\partial H}{\partial t} = -\nabla_{xy} \cdot \mathbf{U} + N$$

$$S_y \frac{\partial H}{\partial t} = -\nabla_{xy} \cdot \mathbf{U} + N$$

$\because \eta_e \gg \bar{S}_s l$ and $\eta_e = S_y$

Thank you