Module 02: Numerical Methods

Unit 23: Algebraic Equation: TriDiagonal Matrix Method

Anirban Dhar

Department of Civil Engineering
Indian Institute of Technology Kharagpur, Kharagpur

National Programme for Technology Enhanced Learning (NPTEL)

Learning Objective

- To apply TriDiagonal Matrix Algorithm for direct solution.

Basic Form

$$\mathbf{A}\phi = \mathbf{r}$$

Basic Form

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{r}$$

$$\begin{pmatrix} \times & \times & & & & & \\ \times & \times & \times & & & & \\ & \times & \times & \times & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \times & \times & \times & \\ & & & & \times & \times & \times \\ & & & & & \times & \times \end{pmatrix}_{N\times N} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \\ \phi_N \end{Bmatrix}_{N\times 1} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{Bmatrix}_{N\times 1}$$

Forward Elimination

$$\begin{pmatrix} d_1 & a_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Forward Elimination

$$\begin{pmatrix} d_1 & a_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 1

$$d_1\phi_1 + a_1\phi_2 = r_1$$

Forward Elimination

$$\begin{pmatrix} d_1 & a_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 1

$$d_1\phi_1 + a_1\phi_2 = r_1$$

Division by d_1 yields

$$\phi_1 + \frac{a_1}{d_1}\phi_2 = \frac{r_1}{d_1}$$

Forward Elimination

$$\begin{pmatrix} d_1 & a_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 1

$$d_1\phi_1 + a_1\phi_2 = r_1$$

Division by d_1 yields

$$\phi_1 + \frac{a_1}{d_1}\phi_2 = \frac{r_1}{d_1}$$

Rewriting yields

$$\phi_1 + \xi_1\phi_2 = \rho_1$$

with

$$\xi_1 = \frac{a_1}{d_1}, \quad \rho_1 = \frac{r_1}{d_1}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 2

$$b_2\phi_1 + d_2\phi_2 + a_2\phi_3 = r_2$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 2

$$b_2\phi_1 + d_2\phi_2 + a_2\phi_3 = r_2$$

Multiplying b_2 with Row 1

$$b_2 \times (\phi_1 + \xi_1\phi_2 = \rho_1)$$

Forward Elimination

Row 2	$b_2\phi_1 + \quad d_2\phi_2 + a_2\phi_3 = r_2$
$b_2\times$ Row 1	$b_2\phi_1 + \quad b_2\xi_1\phi_2 \quad = b_2\rho_1$
Updated Row 2	$(d_2 - b_2\xi_1)\phi_2 + a_2\phi_3 = r_2 - b_2\rho_1$

Forward Elimination

$$
\begin{array}{r|l}
\text{Row 2} & b_2\phi_1 + \quad d_2\phi_2 + a_2\phi_3 = r_2 \\
b_2\times\ \text{Row 1} & b_2\phi_1 + \quad b_2\xi_1\phi_2 \qquad = b_2\rho_1 \\
\hline
\text{Updated Row 2} & (d_2 - b_2\xi_1)\phi_2 + a_2\phi_3 = r_2 - b_2\rho_1
\end{array}
$$

Division by $(d_2 - b_2\xi_1)$ yields

$$\phi_2 + \frac{a_2}{d_2 - b_2\xi_1}\phi_3 = \frac{r_2 - b_2\rho_1}{d_2 - b_2\xi_1}$$

Forward Elimination

$$
\begin{array}{r|l}
\text{Row } 2 & b_2\phi_1 + \quad d_2\phi_2 + a_2\phi_3 = r_2 \\
b_2 \times \text{ Row } 1 & b_2\phi_1 + \quad b_2\xi_1\phi_2 \qquad = b_2\rho_1 \\
\hline
\text{Updated Row } 2 & \quad (d_2 - b_2\xi_1)\phi_2 + a_2\phi_3 = r_2 - b_2\rho_1
\end{array}
$$

Division by $(d_2 - b_2\xi_1)$ yields

$$\phi_2 + \frac{a_2}{d_2 - b_2\xi_1}\phi_3 = \frac{r_2 - b_2\rho_1}{d_2 - b_2\xi_1}$$

Rewriting yields

$$\phi_2 + \xi_2\phi_3 = \rho_2$$

with

$$\xi_2 = \frac{a_2}{d_2 - b_2\xi_1}, \quad \rho_2 = \frac{r_2 - b_2\rho_1}{d_2 - b_2\xi_1}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 3

$$b_3\phi_2 + d_3\phi_3 + a_3\phi_4 = r_3$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 3

$$b_3\phi_2 + d_3\phi_3 + a_3\phi_4 = r_3$$

Multiplying b_3 with Row 2

$$b_3 \times (\phi_2 + \xi_2\phi_3 = \rho_2)$$

Forward Elimination

$$
\begin{array}{r|l}
\text{Row 3} & b_3\phi_2 + \qquad d_3\phi_3 + a_3\phi_4 = r_3 \\
b_3\times\ \text{Row 2} & b_3\phi_2 + \qquad b_3\xi_2\phi_3 \qquad = b_3\rho_2 \\
\hline
\text{Updated Row 3} & \qquad (d_3 - b_3\xi_2)\phi_3 + a_3\phi_4 = r_3 - b_3\rho_2
\end{array}
$$

Forward Elimination

Row 3	$b_3\phi_2 +$	$d_3\phi_3 + a_3\phi_4$	$= r_3$
$b_3\times$ Row 2	$b_3\phi_2 +$	$b_3\xi_2\phi_3$	$= b_3\rho_2$
Updated Row 3		$(d_3 - b_3\xi_2)\phi_3 + a_3\phi_4$	$= r_3 - b_3\rho_2$

Division by $(d_3 - b_3\xi_2)$ yields

$$\phi_3 + \frac{a_3}{d_3 - b_3\xi_2}\phi_3 = \frac{r_3 - b_3\rho_2}{d_3 - b_3\xi_2}$$

Forward Elimination

$$
\begin{array}{r|llll}
\text{Row 3} & b_3\phi_2 + & d_3\phi_3 + a_3\phi_4 & = r_3 \\
b_3\times\ \text{Row 2} & b_3\phi_2 + & b_3\xi_2\phi_3 & = b_3\rho_2 \\
\hline
\text{Updated Row 3} & & (d_3 - b_3\xi_2)\phi_3 + a_3\phi_4 & = r_3 - b_3\rho_2
\end{array}
$$

Division by $(d_3 - b_3\xi_2)$ yields

$$\phi_3 + \frac{a_3}{d_3 - b_3\xi_2}\phi_3 = \frac{r_3 - b_3\rho_2}{d_3 - b_3\xi_2}$$

Rewriting yields

$$\phi_3 + \xi_3\phi_4 = \rho_3$$

with

$$\xi_3 = \frac{a_3}{d_3 - b_3\xi_2}, \quad \rho_3 = \frac{r_3 - b_3\rho_2}{d_3 - b_3\xi_2}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 4

$$b_4\phi_3 + d_4\phi_4 + a_4\phi_5 = r_4$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 4

$$b_4\phi_3 + d_4\phi_4 + a_4\phi_5 = r_4$$

Multiplying b_4 with Row 2

$$b_4 \times (\phi_3 + \xi_3\phi_4 = \rho_3)$$

Forward Elimination

Row 4	$b_4\phi_3 + \quad d_4\phi_4 + a_4\phi_5 = r_4$
$b_4\times$ Row 3	$b_4\phi_3 + \quad b_4\xi_3\phi_4 \quad = b_4\rho_3$
Updated Row 4	$(d_4 - b_4\xi_3)\phi_4 + a_4\phi_5 = r_4 - b_4\rho_3$

Forward Elimination

$$
\begin{array}{r|lcr}
\text{Row 4} & b_4\phi_3 + & d_4\phi_4 + a_4\phi_5 & = r_4 \\
b_4\times\ \text{Row 3} & b_4\phi_3 + & b_4\xi_3\phi_4 & = b_4\rho_3 \\
\hline
\text{Updated Row 4} & & (d_4 - b_4\xi_3)\phi_4 + a_4\phi_5 & = r_4 - b_4\rho_3
\end{array}
$$

Division by $(d_4 - b_4\xi_3)$ yields

$$\phi_4 + \frac{a_4}{d_4 - b_4\xi_3}\phi_5 = \frac{r_4 - b_4\rho_3}{d_4 - b_4\xi_3}$$

Forward Elimination

Row 4	$b_4\phi_3 + \quad d_4\phi_4 + a_4\phi_5 = r_4$
$b_4\times$ Row 3	$b_4\phi_3 + \quad b_4\xi_3\phi_4 \quad = b_4\rho_3$
Updated Row 4	$(d_4 - b_4\xi_3)\phi_4 + a_4\phi_5 = r_4 - b_4\rho_3$

Division by $(d_4 - b_4\xi_3)$ yields

$$\phi_4 + \frac{a_4}{d_4 - b_4\xi_3}\phi_5 = \frac{r_4 - b_4\rho_3}{d_4 - b_4\xi_3}$$

Rewriting yields

$$\phi_4 + \xi_4\phi_5 = \rho_4$$

with

$$\xi_4 = \frac{a_4}{d_4 - b_4\xi_3}, \quad \rho_4 = \frac{r_4 - b_4\rho_3}{d_4 - b_4\xi_3}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ r_5 \end{Bmatrix}$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ r_5 \end{Bmatrix}$$

Row 5 (Last Row)

$$b_5\phi_4 + d_5\phi_5 = r_5$$

Forward Elimination

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ r_5 \end{Bmatrix}$$

Row 5 (Last Row)

$$b_5\phi_4 + d_5\phi_5 = r_5$$

Row 5	$b_5\phi_4 + \quad d_5\phi_5 = r_5$
$b_5\times$ Row 4	$b_5\phi_4 + \quad b_5\xi_4\phi_5 = b_5\rho_4$
Updated Row 5	$(d_5 - b_5\xi_4)\phi_5 = r_5 - b_5\rho_4$

Forward Elimination

Division by $(d_5 - b_5\xi_4)$ yields

$$\phi_5 = \frac{r_5 - b_5\rho_4}{d_5 - b_5\xi_4}$$

Forward Elimination

Division by $(d_5 - b_5\xi_4)$ yields

$$\phi_5 = \frac{r_5 - b_5\rho_4}{d_5 - b_5\xi_4}$$

Rewriting yields

$$\phi_5 = \rho_5$$

with

$$\rho_5 = \frac{r_5 - b_5\rho_4}{d_5 - b_5\xi_4}$$

Forward Elimination

Division by $(d_5 - b_5\xi_4)$ yields

$$\phi_5 = \frac{r_5 - b_5\rho_4}{d_5 - b_5\xi_4}$$

Rewriting yields

$$\phi_5 = \rho_5$$

with

$$\rho_5 = \frac{r_5 - b_5\rho_4}{d_5 - b_5\xi_4}$$

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{Bmatrix}$$

Backward Substitution

Row 5 (Last Row)

$$\phi_5 = \rho_5$$

Backward Substitution

Row 5 (Last Row)

$$\phi_5 = \rho_5$$

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{Bmatrix}$$

Backward Substitution

Row 4

$$\phi_4 + \xi_4 \phi_5 = \rho_4$$

Backward Substitution

Row 4

$$\phi_4 + \xi_4 \phi_5 = \rho_4$$

Rewriting yields

$$\phi_4 = \rho_4 - \xi_4 \phi_5$$

Backward Substitution

Row 4

$$\phi_4 + \xi_4 \phi_5 = \rho_4$$

Rewriting yields

$$\phi_4 = \rho_4 - \xi_4 \phi_5$$

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{Bmatrix}$$

Backward Substitution

Row 3

$$\phi_3 + \xi_3 \phi_4 = \rho_3$$

Backward Substitution

Row 3

$$\phi_3 + \xi_3\phi_4 = \rho_3$$

Rewriting yields

$$\phi_3 = \rho_3 - \xi_3\phi_4$$

Backward Substitution

Row 3

$$\phi_3 + \xi_3\phi_4 = \rho_3$$

Rewriting yields

$$\phi_3 = \rho_3 - \xi_3\phi_4$$

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{Bmatrix}$$

Backward Substitution

Row 2

$$\phi_2 + \xi_2 \phi_3 = \rho_2$$

Backward Substitution

Row 2

$$\phi_2 + \xi_2 \phi_3 = \rho_2$$

Rewriting yields

$$\phi_2 = \rho_2 - \xi_2 \phi_3$$

Backward Substitution

Row 2

$$\phi_2 + \xi_2\phi_3 = \rho_2$$

Rewriting yields

$$\phi_2 = \rho_2 - \xi_2\phi_3$$

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{Bmatrix}$$

Backward Substitution

Row 1

$$\phi_1 + \xi_1 \phi_2 = \rho_1$$

Backward Substitution

Row 1

$$\phi_1 + \xi_1 \phi_2 = \rho_1$$

Rewriting yields

$$\phi_1 = \rho_1 - \xi_1 \phi_2$$

Backward Substitution

Row 1

$$\phi_1 + \xi_1 \phi_2 = \rho_1$$

Rewriting yields

$$\phi_1 = \rho_1 - \xi_1 \phi_2$$

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{Bmatrix}$$

Thomas Algorithm

Data: Vector $\mathbf{b}$, $\mathbf{d}$, $\mathbf{a}$, $\mathbf{r}$
Result: ϕ
Forward Elimination
$a_1 = a_1/d_1$
$r_1 = r_1/d_1$
for $i = 2, n-1$ **do**
    $fact = d_i - b_i \cdot a_{i-1}$
    $a_i = a_i/fact$
    $r_i = (r_i - b_i \cdot r_{i-1})/fact$
end
$r_n = (r_n - b_n \cdot r_{n-1})/(d_n - b_n \cdot a_{n-1})$
Backward Substitution
$\phi_n = r_n$
for $i = n-1, -1, 1$ **do**
    $\phi_i = r_i - a_i \cdot \phi_{i+1}$
end
return ϕ

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{Bmatrix}$$

Thank You