Module 03: Groundwater Hydraulics

Unit 03: Unsteady Two-Dimensional Flow using Finite Difference Method

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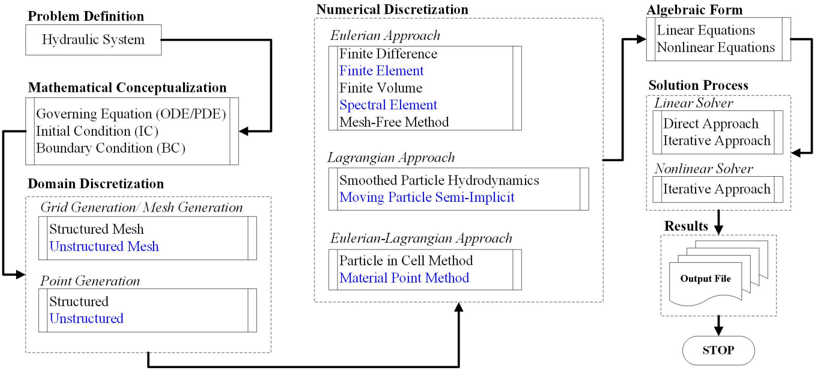
Learning Objective

- To solve unsteady state two dimensional groundwater flow equation using Finite Difference Method.

Problem Definition to Solution



Problem Definition

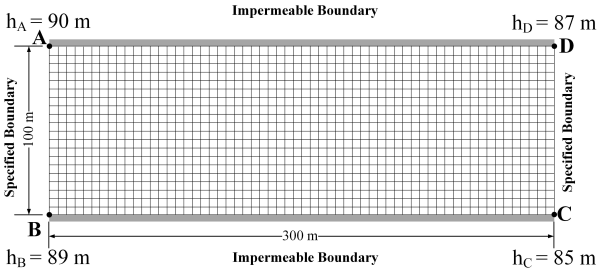


Figure: Homogeneous Aquifer System

Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega: \quad \frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

$S = 5 \times 10^{-5}$
$T = 200\ m^2/day$

Problem Definition

subject to

Initial Condition

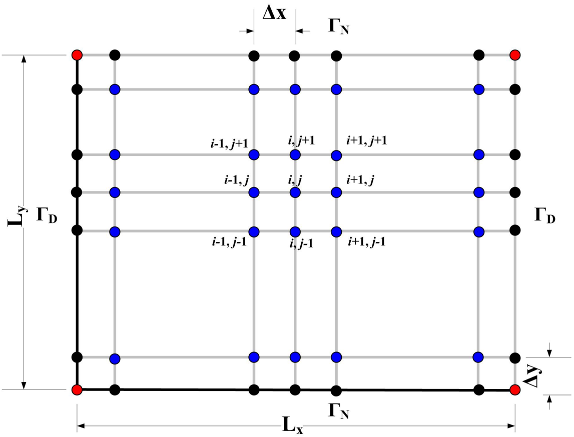
$$h(x,y,0) = h_0(x,y)$$

and

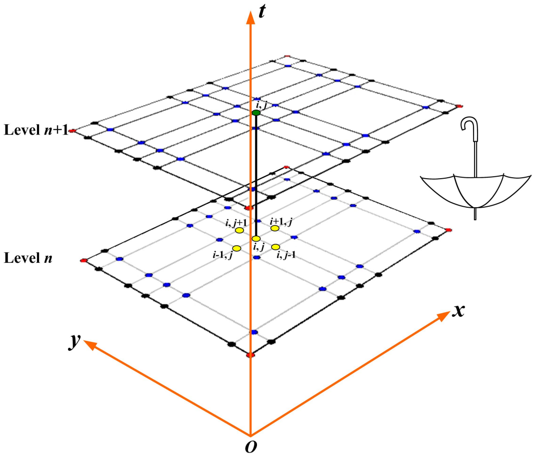
Boundary Condition

$$\begin{aligned}
\Gamma_D^1 &: \quad h(0,y,t) = h_1(y) \\
\Gamma_D^2 &: \quad h(L_x,y,t) = h_2(y) \\
\Gamma_N^3 &: \quad \left.\frac{\partial h}{\partial y}\right|_{(x,0,t)} = 0 \\
\Gamma_N^4 &: \quad \left.\frac{\partial h}{\partial y}\right|_{(x,L_y,t)} = 0
\end{aligned}$$

Domain Discretization



Space-Time Discretization
Explicit Scheme



Explicit Scheme

From Lecture 10, the discretized finite difference equation can be written as,

$$\frac{S}{T}\frac{h_{i,j}^{n+1}-h_{i,j}^{n}}{\Delta t}=\frac{h_{i-1,j}^{n}-2h_{i,j}^{n}+h_{i+1,j}^{n}}{\Delta x^2}+\frac{h_{i,j-1}^{n}-2h_{i,j}^{n}+h_{i,j+1}^{n}}{\Delta y^2} \tag{1}$$

Explicit Scheme

From Lecture 10, the discretized finite difference equation can be written as,

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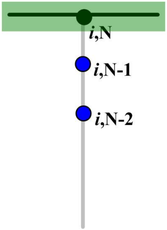
In simplified form,

$$h_{i,j}^{n+1}=\alpha_y h_{i,j-1}^{n}+\alpha_x h_{i-1,j}^{n}+\left[1-2(\alpha_x+\alpha_y)\right]h_{i,j}^{n}+\alpha_x h_{i+1,j}^{n}+\alpha_y h_{i,j+1}^{n}$$

with $\alpha_x = \frac{T\Delta t}{S\Delta x^2}$ and $\alpha_y = \frac{T\Delta t}{S\Delta y^2}$.

Neumann Boundary Condition



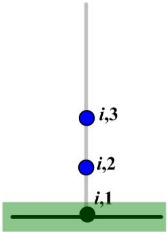
Top Boundary

Second Order Discretization

$$\frac{3h_{i,N}^{n+1} - 4h_{i,N-1}^{n+1} + h_{i,N-2}^{n+1}}{2\Delta y} = 0$$

$$h_{i,N}^{n+1} = \frac{4}{3}h_{i,N-1}^{n+1} - \frac{1}{3}h_{i,N-2}^{n+1}$$

Neumann Boundary Condition



Bottom Boundary

Second Order Discretization

$$\frac{-3h_{i,1}^{n+1} + 4h_{i,2}^{n+1} - h_{i,3}^{n+1}}{2\Delta y} = 0$$

$$h_{i,1}^{n+1} = \frac{4}{3}h_{i,2}^{n+1} - \frac{1}{3}h_{i,3}^{n+1}$$

Standard Steps

Explicit Scheme: Time-stepping Algorithm

Data: S, T, Δx, Δy, Δt, h^n at time-step n

Standard Steps

Explicit Scheme: Time-stepping Algorithm

Data: S, T, Δx, Δy, Δt, h^n at time-step n
Result: Updated h^{n+1} at time-step $n+1$

Standard Steps

Explicit Scheme: Time-stepping Algorithm

Data: S, T, Δx, Δy, Δt, h^n at time-step n
Result: Updated h^{n+1} at time-step $n+1$
while *t <end time* **do**

For interior points:

$$h_{i,j}^{n+1} = \alpha_y h_{i,j-1}^n + \alpha_x h_{i-1,j}^n + [1 - 2(\alpha_x + \alpha_y)]\, h_{i,j}^n + \alpha_x h_{i+1,j}^n + \alpha_y h_{i,j+1}^n$$

Standard Steps

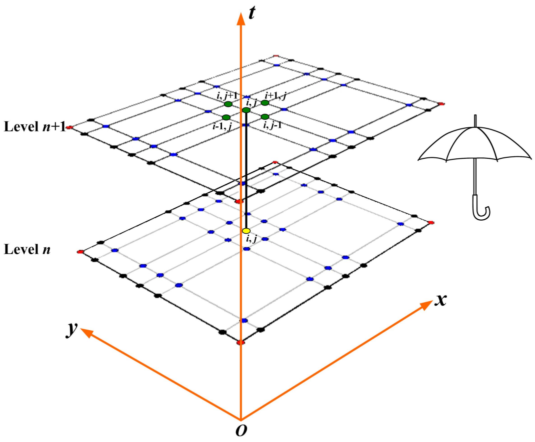
Explicit Scheme: Time-stepping Algorithm

Data: S, T, Δx, Δy, Δt, h^n at time-step n
Result: Updated h^{n+1} at time-step $n+1$
while *t <end time* **do**
For interior points:
$h_{i,j}^{n+1} = \alpha_y h_{i,j-1}^n + \alpha_x h_{i-1,j}^n + [1 - 2(\alpha_x + \alpha_y)] h_{i,j}^n + \alpha_x h_{i+1,j}^n + \alpha_y h_{i,j+1}^n$
For boundary points: Use Boundary Conditions
$n \leftarrow n + 1$
end

Stability Criteria

$(\alpha_x + \alpha_y) < \frac{1}{2}$

Implicit Scheme



Implicit Scheme

From Lecture 10, the discretized finite difference equation can be written as,

$$\frac{S}{T}\frac{h_{i,j}^{n+1}-h_{i,j}^{n}}{\Delta t}=\frac{h_{i-1,j}^{n+1}-2h_{i,j}^{n+1}+h_{i+1,j}^{n+1}}{\Delta x^2}+\frac{h_{i,j-1}^{n+1}-2h_{i,j}^{n+1}+h_{i,j+1}^{n+1}}{\Delta y^2}$$

Implicit Scheme

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$$\frac{S}{T}\frac{h_{i,j}^{n+1}-h_{i,j}^{n}}{\Delta t}=\frac{h_{i-1,j}^{n+1}-2h_{i,j}^{n+1}+h_{i+1,j}^{n+1}}{\Delta x^2}+\frac{h_{i,j-1}^{n+1}-2h_{i,j}^{n+1}+h_{i,j+1}^{n+1}}{\Delta y^2}$$

In simplified form, this can be written as

$$\alpha_y h_{i,j-1}^{n+1}+\alpha_x h_{i-1,j}^{n+1}-\left[1+2(\alpha_x+\alpha_y)\right]h_{i,j}^{n+1}+\alpha_x h_{i+1,j}^{n+1}+\alpha_y h_{i,j+1}^{n+1}=-h_{i,j}^{n}$$

with $\alpha_x = \frac{T\Delta t}{S\Delta x^2}$ and $\alpha_y = \frac{T\Delta t}{S\Delta y^2}$.

Gauss-Seidel Method

Iterative Approach

From Lecture 29, iteration starts with the guess value

$$\boldsymbol{h}^{n+1}\Big|^{(0)} = \left[h_{1,1}^{n+1}\Big|^{(0)} \quad h_{1,2}^{n+1}\Big|^{(0)} \ldots \quad h_{M,N-1}^{n+1}\Big|^{(0)} \quad h_{M,N}^{n+1}\Big|^{(0)} \right]^T$$

Gauss-Seidel Method

Iterative Approach

From Lecture 29, iteration starts with the guess value

$$\boldsymbol{h}^{n+1}\Big|^{(0)} = \left[h_{1,1}^{n+1}\Big|^{(0)} \quad h_{1,2}^{n+1}\Big|^{(0)} \dots \quad h_{M,N-1}^{n+1}\Big|^{(0)} \quad h_{M,N}^{n+1}\Big|^{(0)} \right]^T$$

The Gauss-Seidel step can be written as,

$$h_{i,j}^{n+1}\Big|^{(p)} = h_{i,j}^{n+1}\Big|^{(p-1)} + \frac{1}{[-1-2(\alpha_x+\alpha_y)]} \Big[- h_{i,j}^{n} - \Big(\alpha_y h_{i,j-1}^{n+1}\Big|^{(p)} + \alpha_x h_{i-1,j}^{n+1}\Big|^{(p)} - [1+2(\alpha_x+\alpha_y)] h_{i,j}^{n+1}\Big|^{(p-1)} + \alpha_x h_{i+1,j}^{n+1}\Big|^{(p-1)} + \alpha_y h_{i,j+1}^{n+1}\Big|^{(p-1)} \Big) \Big]$$

Gauss-Seidel Method

Iterative Approach

From Lecture 29, iteration starts with the guess value

$$\boldsymbol{h}^{n+1}\big|^{(0)} = \left[h_{1,1}^{n+1}\big|^{(0)} \quad h_{1,2}^{n+1}\big|^{(0)} \ldots \quad h_{M,N-1}^{n+1}\big|^{(0)} \quad h_{M,N}^{n+1}\big|^{(0)} \right]^T$$

The Gauss-Seidel step can be written as,

$$h_{i,j}^{n+1}\big|^{(p)} = h_{i,j}^{n+1}\big|^{(p-1)} + \frac{1}{[-1-2(\alpha_x+\alpha_y)]}\Big[-h_{i,j}^{n} - \Big(\alpha_y h_{i,j-1}^{n+1}\big|^{(p)} + \alpha_x h_{i-1,j}^{n+1}\big|^{(p)} \\ - [1+2(\alpha_x+\alpha_y)]h_{i,j}^{n+1}\big|^{(p-1)} + \alpha_x h_{i+1,j}^{n+1}\big|^{(p-1)} + \alpha_y h_{i,j+1}^{n+1}\big|^{(p-1)}\Big)\Big]$$

In compact form

$$h_{i,j}^{n+1}\big|^{(p)} = h_{i,j}^{n+1}\big|^{(p-1)} + \frac{Res_{i,j}}{[-1-2(\alpha_x+\alpha_y)]}, \quad \forall (i,j) \; p \geq 1$$

Neumann Boundary Condition

Top Boundary

$$3h_{i,N}^{n+1} - 4h_{i,N-1}^{n+1} + h_{i,N-2}^{n+1} = 0$$

Bottom Boundary

$$-3h_{i,1}^{n+1} + 4h_{i,2}^{n+1} - h_{i,3}^{n+1} = 0$$

Standard Steps

Implicit Scheme: Time-stepping Algorithm

Data: S, T, Δx, Δy, Δt, h^n at time-step n

Standard Steps

Implicit Scheme: Time-stepping Algorithm

Data: S, T, Δx, Δy, Δt, h^n at time-step n
Result: Updated h^{n+1} at time-step $n+1$

Standard Steps

Implicit Scheme: Time-stepping Algorithm

Data: S, T, Δx, Δy, Δt, h^n at time-step n

Result: Updated h^{n+1} at time-step $n+1$

while $t <$ *end time* **do**

  For interior and boundary points: Solve governing equation and boundary conditions in discretized form.

  $n \leftarrow n+1$

end

List of Source Codes

Unsteady Two Dimensional Groundwater Flow

- Explicit approach
  - unsteady_2D_explicit.sci
- implicit approach
  - unsteady_2D_implicit_iterative.sci

Thank You