Module 02: Numerical Methods

Unit 24: Algebraic Equation: Jacobi Method

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Learning Objective

- To apply Jacobi Method for iterative solution.

Matrix Form

Full Matrix

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{r}$$

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Full Matrix

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{r}$$

$$\begin{pmatrix} \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \end{pmatrix}_{N\times N} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \\ \phi_N \end{Bmatrix}_{N\times 1} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{Bmatrix}_{N\times 1}$$

Basic Steps
Jacobi Method

The coefficient matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

where $\mathbf{L}$, $\mathbf{D}$, $\mathbf{U}$ are strictly lower triangular, diagonal, strictly upper triangular matrices respectively.

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$$(\mathbf{L} + \mathbf{D} + \mathbf{U})\boldsymbol{\phi} = \mathbf{r}$$

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$$(\mathbf{L} + \mathbf{D} + \mathbf{U})\boldsymbol{\phi} = \mathbf{r}$$

Iterative form can be written as

$$\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p})} + (\mathbf{L} + \mathbf{U})\boldsymbol{\phi}^{(\mathbf{p-1})} = \mathbf{r}$$

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$$\mathbf{D}\boldsymbol{\phi}^{(p)} + (\mathbf{L} + \mathbf{U})\boldsymbol{\phi}^{(p-1)} = \mathbf{r}$$

$$\mathbf{D}\boldsymbol{\phi}^{(p)} = -(\mathbf{L} + \mathbf{U})\boldsymbol{\phi}^{(p-1)} + \mathbf{r}$$

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Final form can be written as

$$\boldsymbol{\phi}^{(p)} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\boldsymbol{\phi}^{(p-1)} + \mathbf{D}^{-1}\mathbf{r}$$

where p is the iteration counter $(p \geq 1)$.

Basic Steps
Jacobi Method

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Iteration starts with a guess value $\boldsymbol{\phi}^{(0)}$

Jacobi Method
Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Jacobi Method
Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Coefficient matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

Jacobi Method
Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Coefficient matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

where

Strictly Lower Triangular Matrix

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{pmatrix}$$

Jacobi Method
Example

Diagonal Matrix

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

Jacobi Method
Example

Diagonal Matrix

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

Strictly Upper Triangular Matrix

$$\mathbf{U} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Jacobi Method
Example

Iteration starts with the guess value $\phi^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} & \phi_3^{(0)} & \phi_4^{(0)} & \phi_5^{(0)} \end{bmatrix}^T$$

Iteration 1:

Row 1:

$$\phi_1^{(1)} = \frac{1}{a_{11}} \left[r_1 - \sum_{\substack{j=1,\\ j\neq 1}}^{5} a_{1j}\phi_j^{(0)} \right]$$

Row 2:

$$\phi_2^{(1)} = \frac{1}{a_{22}} \left[r_2 - \sum_{\substack{j=1,\\ j\neq 2}}^{5} a_{2j}\phi_j^{(0)} \right]$$

Row 3:

$$\phi_3^{(1)} = \frac{1}{a_{33}} \left[r_3 - \sum_{\substack{j=1,\\ j\neq 3}}^{5} a_{3j}\phi_j^{(0)} \right]$$

Jacobi Method
Example

Row 4:

$$\phi_4^{(1)} = \frac{1}{a_{44}} \left[r_4 - \sum_{\substack{j=1, \\ j \neq 4}}^{5} a_{4j} \phi_j^{(0)} \right]$$

Jacobi Method
Example

Row 4:

$$\phi_4^{(1)} = \frac{1}{a_{44}} \left[r_4 - \sum_{\substack{j=1,\\ j\neq 4}}^{5} a_{4j}\phi_j^{(0)} \right]$$

Row 5:

$$\phi_5^{(1)} = \frac{1}{a_{55}} \left[r_5 - \sum_{\substack{j=1,\\ j\neq 5}}^{5} a_{5j}\phi_j^{(0)} \right]$$

Iteration 2:

Row 1:

$$\phi_1^{(2)} = \frac{1}{a_{11}} \left[r_1 - \sum_{\substack{j=1,\\ j\neq 1}}^{5} a_{1j}\phi_j{}^{(1)} \right]$$

Jacobi Method
Example

Row 2:

$$\phi_2^{(2)} = \frac{1}{a_{22}} \left[r_2 - \sum_{\substack{j=1,\\ j\neq 2}}^{5} a_{2j}\phi_j^{(1)} \right]$$

Jacobi Method
Example

Row 2:

$$\phi_2^{(2)} = \frac{1}{a_{22}} \left[r_2 - \sum_{\substack{j=1,\\ j\neq 2}}^{5} a_{2j} \phi_j{}^{(1)} \right]$$

Row 3:

$$\phi_3^{(2)} = \frac{1}{a_{22}} \left[r_3 - \sum_{\substack{j=1,\\ j\neq 3}}^{5} a_{3j} \phi_j{}^{(1)} \right]$$

Jacobi Method
Example

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$$\phi_3^{(2)} = \frac{1}{a_{22}} \left[r_3 - \sum_{\substack{j=1,\\ j\neq 3}}^{5} a_{3j} \phi_j{}^{(1)} \right]$$

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$$\phi_2^{(2)} = \frac{1}{a_{22}}\left[r_2 - \sum_{\substack{j=1,\\ j\neq 2}}^{5} a_{2j}\phi_j^{(1)}\right]$$

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Row 5:

$$\phi_5^{(2)} = \frac{1}{a_{22}}\left[r_5 - \sum_{j=1,}^{5} a_{5j}\phi_j^{(1)}\right]$$

Jacobi Method
General Algorithm

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} \dots & \phi_{N-1}^{(0)} & \phi_N^{(0)} \end{bmatrix}^T$$

Jacobi Method
General Algorithm

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} \dots & \phi_{N-1}^{(0)} & \phi_N^{(0)} \end{bmatrix}^T$$

$$\phi_i^{(p)} = \frac{1}{a_{ii}} \left[r_i - \sum_{\substack{j=1, \\ j \neq i}}^{N} a_{ij} \phi_j^{(p-1)} \right], \quad \forall i \in \{1, \dots, N\}, p \geq 1$$

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By adding and subtracting $\phi_i^{(p-1)}$ in right hand side

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{1}{a_{ii}} \left[r_i - \sum_{j=1}^{i-1} a_{ij}\phi_j{}^{(p-1)} - a_{ii}\phi_i^{(p-1)} - \sum_{j=i+1}^{N} a_{ij}\phi_j^{(p-1)} \right], \quad \forall i, p \geq 1$$

Jacobi Method
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Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} \dots & \phi_{N-1}^{(0)} & \phi_N^{(0)} \end{bmatrix}^T$$

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By adding and subtracting $\phi_i^{(p-1)}$ in right hand side

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{1}{a_{ii}} \left[r_i - \sum_{j=1}^{i-1} a_{ij}\phi_j^{(p-1)} - a_{ii}\phi_i^{(p-1)} - \sum_{j=i+1}^{N} a_{ij}\phi_j^{(p-1)} \right], \quad \forall i, p \geq 1$$

In compact form

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{a_{ii}}, \quad \forall i, p \geq 1$$

Jacobi Method
Stopping Criterion

Residual Error in a particular iteration can be expressed as

$$\boldsymbol{\varepsilon}^{(p)} = \mathbf{A}\boldsymbol{\phi}^{(p)} - \mathbf{r}$$

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Maximum Absolute Error:

$$\max_{i \in 1,\ldots,N} \left|\varepsilon_i^{(p)}\right| \leq \varepsilon_{max}$$

Jacobi Method
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Residual Error in a particular iteration can be expressed as

$$\boldsymbol{\varepsilon}^{(p)} = \mathbf{A}\boldsymbol{\phi}^{(p)} - \mathbf{r}$$

Maximum Absolute Error:

$$\max_{i \in 1,\ldots,N} \left|\varepsilon_i^{(p)}\right| \leq \varepsilon_{max}$$

Root Mean Square Error:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\varepsilon_i^{(p)}\right)^2} \leq \varepsilon_{max}$$

Convergence Criteria

Diagonal Dominance

Diagonal Dominance:

$$|a_{ii}| = \sum_{\substack{j=1,\\ j\neq i}}^{N} |a_{ij}|$$

$$\exists l : |a_{ll}| > \sum_{\substack{j=1,\\ j\neq l}}^{N} |a_{lj}|$$

Weak Diagonal Dominance:

$$|a_{ii}| \geq \sum_{\substack{j=1,\\ j\neq i}}^{N} |a_{ij}|$$

Strict Diagonal Dominance:

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{N} |a_{ij}|$$

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 & 0 \\ 0 & -4 & 3 & -2 & 0 \\ 0 & 0 & -7 & -10 & 13 \\ 0 & 0 & 0 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -9 \\ -7 \\ 4 \\ -26 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 & 0 \\ 0 & -4 & 3 & -2 & 0 \\ 0 & 0 & -7 & -10 & 13 \\ 0 & 0 & 0 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -9 \\ -7 \\ 4 \\ -26 \end{Bmatrix}$$

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Thank You