Module 02: Numerical Methods

Unit 22: Algebraic Equation: LU Decomposition Method

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Learning Objective

- To apply LU Decomposition Method for direct solution.

Matrix Form
Full Matrix

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{r}$$

Matrix Form

Full Matrix

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{r}$$

$$\begin{pmatrix} \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \end{pmatrix}_{N\times N} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \\ \phi_N \end{Bmatrix}_{N\times 1} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{Bmatrix}_{N\times 1}$$

Basic Steps
LU Decomposition

- **Decomposition:** $\mathbf{A} = \mathbf{LU}$

Basic Steps
LU Decomposition

- **Decomposition:** $\mathbf{A} = \mathbf{LU}$
- **Forward Substitution:** $\mathbf{L}\boldsymbol{\psi} = \mathbf{r}$

Basic Steps

LU Decomposition

- **Decomposition:** $\mathbf{A} = \mathbf{L}\mathbf{U}$
- **Forward Substitution:** $\mathbf{L}\boldsymbol{\psi} = \mathbf{r}$
- **Backward Substitution:** $\mathbf{U}\boldsymbol{\phi} = \boldsymbol{\psi}$

Basic Steps
LU Decomposition

- **Decomposition:** $\mathbf{A} = \mathbf{LU}$
- **Forward Substitution:** $\mathbf{L}\boldsymbol{\psi} = \mathbf{r}$
- **Backward Substitution:** $\mathbf{U}\boldsymbol{\phi} = \boldsymbol{\psi}$

Overall calculation can be presented as

$$\mathbf{L}(\mathbf{U}\boldsymbol{\phi} - \boldsymbol{\psi}) = \mathbf{LU}\boldsymbol{\phi} - \mathbf{L}\boldsymbol{\psi} = \mathbf{A}\boldsymbol{\phi} - \mathbf{r}$$

with

$$\mathbf{LU} = \mathbf{A}$$
$$\mathbf{L}\boldsymbol{\psi} = \mathbf{r}$$

Gauss Elimination
LU Decomposition

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Gauss Elimination
LU Decomposition

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Matrix form generated from forward elimination process

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r^{IV}_5 \end{Bmatrix}$$

Gauss Elimination
LU Decomposition

In the first step γ_1^2,γ_1^3,γ_1^4,γ_1^5 were multiplied for Rows 2, 3, 4, and 5 respectively.

Gauss Elimination

LU Decomposition

In the first step γ_1^2,γ_1^3,γ_1^4,γ_1^5 were multiplied for Rows 2, 3, 4, and 5 respectively. The multiplication factors can be stored as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \gamma_1^2 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ \gamma_1^3 & 0 & a''_{33} & a''_{34} & a''_{35} \\ \gamma_1^4 & 0 & 0 & a'''_{44} & a'''_{45} \\ \gamma_1^5 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix}$$

Gauss Elimination
LU Decomposition

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Similarly,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \gamma_1^2 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ \gamma_1^3 & \gamma_2^3 & a''_{33} & a''_{34} & a''_{35} \\ \gamma_1^4 & \gamma_2^4 & \gamma_3^4 & a'''_{44} & a'''_{45} \\ \gamma_1^5 & \gamma_2^5 & \gamma_3^5 & \gamma_4^5 & a^{IV}_{55} \end{pmatrix}$$

Gauss Elimination
LU Decomposition

$$\mathbf{A} \Leftarrow \mathbf{LU}$$

Gauss Elimination
LU Decomposition

$$\mathbf{A} \Leftarrow \mathbf{LU}$$

where

$$\mathbf{U} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix}$$

Gauss Elimination
LU Decomposition

$$\mathbf{A} \Leftarrow \mathbf{LU}$$

where

$$\mathbf{U} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \gamma_1^2 & 1 & 0 & 0 & 0 \\ \gamma_1^3 & \gamma_2^3 & 1 & 0 & 0 \\ \gamma_1^4 & \gamma_2^4 & \gamma_3^4 & 1 & 0 \\ \gamma_1^5 & \gamma_2^5 & \gamma_3^5 & \gamma_4^5 & 1 \end{pmatrix}$$

Gauss Elimination

Substitution Step

Forward Substitution

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{pmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Gauss Elimination
Substitution Step

Forward Substitution

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{pmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

General Algorithm

$$\psi_1 = r_1$$

Gauss Elimination
Substitution Step

Forward Substitution

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{pmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

General Algorithm

$$\psi_1 = r_1$$

$$\psi_i = r_i - \sum_{j=1}^{i-1} a_{ij}\psi_j, \quad \forall i \in \{2, 3, \ldots, N\}$$

Gauss Elimination
Substitution Step

Backward Substitution

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} \\ 0 & 0 & u_{33} & u_{34} & u_{35} \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix}$$

Gauss Elimination
Substitution Step

Backward Substitution

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} \\ 0 & 0 & u_{33} & u_{34} & u_{35} \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix}$$

General Algorithm

$$\phi_N = \frac{\psi_N}{a_{NN}}$$

Gauss Elimination
Substitution Step

Backward Substitution

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} \\ 0 & 0 & u_{33} & u_{34} & u_{35} \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix}$$

General Algorithm

$$\phi_N = \frac{\psi_N}{a_{NN}}$$

$$\phi_i = \frac{1}{a_{ii}} \left[\psi_i - \sum_{j=i+1}^{N} a_{ij}\phi_j \right], \quad \forall i \in \{N-1, N-2 \ldots, 1\}$$

Data: Matrix $\mathbf{A}$, Vector $\mathbf{r}$
Result: ϕ
Decomposition
for *k=1,n-1* **do**
    for *i=k+1,n* **do**
        $\gamma = a_{i,k} / a_{k,k}$
        $a_{i,k} = \gamma$
        for *j=k+1,n* **do**
            $a_{i,j} = a_{i,j} - \gamma \cdot a_{k,j}$
        end
    end
end
Forward Substitution
$\psi_1 = r_1$
for *i=2,n* **do**
    sum=r_i
    for *j=1,*$i-1$ **do**
        sum=sum$-a_{i,j} \cdot \psi_j$
    end
    ψ_i =sum
end
Back Substitution
$\phi_n = \psi_n / a_{n,n}$
for *i=n-1,-1,1* **do**
    sum=ψ_i
    for *j=i+1,n* **do**
        sum=sum$-a_{i,j} \cdot \phi_j$
    end
    ϕ_i =sum$/a_{i,i}$
end
return ϕ

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & -3 & 4 & 5 \\ 0 & 3 & -5 & -7 & 9 \\ 5 & -4 & 3 & -2 & 1 \\ 1 & 4 & -7 & -10 & 13 \\ -15 & 13 & 11 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 37 \\ 8 \\ 3 \\ 13 \\ 18 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & -3 & 4 & 5 \\ 0 & 3 & -5 & -7 & 9 \\ 5 & -4 & 3 & -2 & 1 \\ 1 & 4 & -7 & -10 & 13 \\ -15 & 13 & 11 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 37 \\ 8 \\ 3 \\ 13 \\ 18 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix}$$

Thank You