General Flow Equations

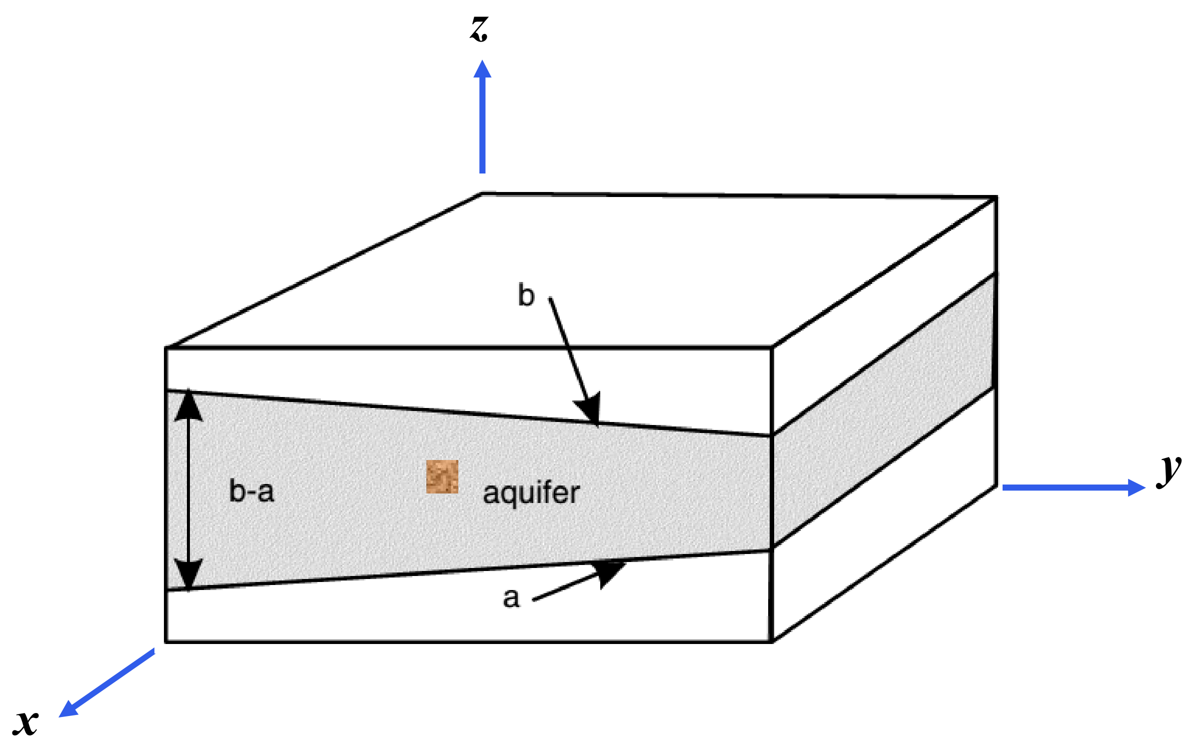
Groundwater Engineering| CE60205

Lecture: 08

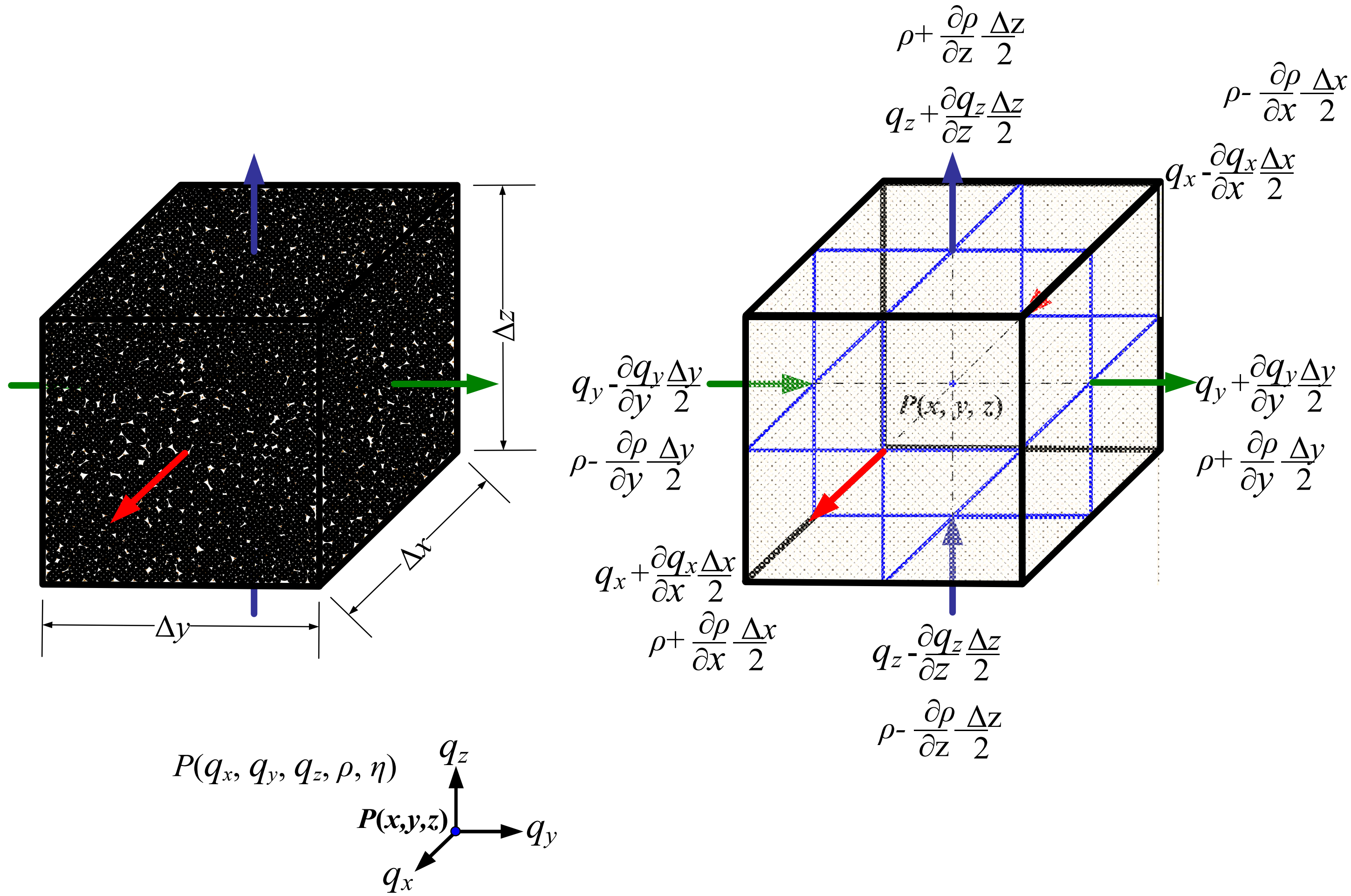
Learning Objective(s)

- **To derive the governing equation for saturated confined aquifer**

Groundwater Flow



Three-Dimensional Saturated Flow in Confined Aquifers



Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Mass inflow rate –mass outflow rate = Rate of change of mass storage with time
- Considering an elementary control volume of soil, which has the volume of (ΔxΔyΔz), the mass of groundwater, M, in this control volume is

$$M = \rho\eta\Delta x\Delta y\Delta z$$

- The inflow and outflow can be calculated for each side of the element. The mass of groundwater inflow in x-direction is

$$\rho\left(x-\frac{\Delta x}{2},y,z,t\right)q_x\left(x-\frac{\Delta x}{2},y,z,t\right)\Delta y\Delta z \approx \left(\rho-\frac{\Delta x}{2}\frac{\partial}{\partial x}(\rho)\right)\left(q_x-\frac{\Delta x}{2}\frac{\partial}{\partial x}(q_x)\right)\Delta y\Delta z$$

$$\approx \left(\rho q_x-\frac{\Delta x}{2}\frac{\partial}{\partial x}(\rho q_x)\right)\Delta y\Delta z \text{ Up to first order}$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- The mass of groundwater outflow in x-direction is

$$\rho\left(x+\frac{\Delta x}{2},y,z,t\right)q_x\left(x+\frac{\Delta x}{2},y,z,t\right)\Delta y\Delta z \approx \left(\rho q_x+\frac{\Delta x}{2}\frac{\partial}{\partial x}(\rho q_x)\right)\Delta y\Delta z$$

- The mass of groundwater inflow in y-direction is

$$\rho\left(x,y-\frac{\Delta y}{2},z,t\right)q_y\left(x,y-\frac{\Delta y}{2},z,t\right)\Delta x\Delta z \approx \left(\rho q_y-\frac{\Delta y}{2}\frac{\partial}{\partial y}(\rho q_y)\right)\Delta x\Delta z$$

- The mass of groundwater outflow in y-direction is

$$\rho\left(x,y+\frac{\Delta y}{2},z,t\right)q_y\left(x,y+\frac{\Delta y}{2},z,t\right)\Delta x\Delta z \approx \left(\rho q_y+\frac{\Delta y}{2}\frac{\partial}{\partial y}(\rho q_y)\right)\Delta x\Delta z$$

- The mass of groundwater inflow in z-direction is

$$\rho\left(x,y,z-\frac{\Delta z}{2},t\right)q_z\left(x,y,z-\frac{\Delta z}{2},t\right)\Delta x\Delta y \approx \left(\rho q_z-\frac{\Delta z}{2}\frac{\partial}{\partial z}(\rho q_z)\right)\Delta x\Delta y$$

- The mass of groundwater outflow in z-direction is

$$\rho\left(x,y,z+\frac{\Delta z}{2},t\right)q_z\left(x,y,z+\frac{\Delta z}{2},t\right)\Delta x\Delta y \approx \left(\rho q_z+\frac{\Delta z}{2}\frac{\partial}{\partial z}(\rho q_z)\right)\Delta x\Delta y$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

$$\frac{\partial M}{\partial t} = \text{Inflow} - \text{Outflow}$$

- Considering these equations, the total inflow minus outflow can be derived as follows:

$$\frac{\partial M}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z)\right]\Delta x\Delta y\Delta z$$

$$M = \rho\eta\Delta x\Delta y\Delta z$$

- The change in storage is calculated by

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}(\rho\eta\Delta x\Delta y\Delta z) = \frac{\partial}{\partial t}(\rho\eta A L_z)$$

- Change in mass

$$dM = \rho V_T(\eta\beta + \alpha)dp \text{ wtih } V_T = \Delta x\Delta y\Delta z = AL_z \text{ and } dp = \rho g dh$$

- Change in mass with respect to time

$$\frac{\partial M}{\partial t} = \rho^2 g(\eta\beta + \alpha)\frac{\partial h}{\partial t}V_T = \rho S_s \frac{\partial h}{\partial t}V_T$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Conservation equation can be written as

$$\rho S_s \frac{\partial h}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z)\right]$$

- Under incompressible condition the equation can be simplified as

$$S_s \frac{\partial h}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)$$

- In terms of piezometric head

$$\mathbf{q} = -\mathbf{K} \cdot \nabla h$$

- If x, y, z are the In principal flow directions, then

$$q_x = -K_x \frac{\partial h}{\partial x}$$
$$q_y = -K_y \frac{\partial h}{\partial y}$$
$$q_z = -K_z \frac{\partial h}{\partial z}$$

Three-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Groundwater flow equation for 3D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right)$$

- Groundwater flow equation for 3D homogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2}$$

- Groundwater flow equation for 3D heterogeneous, isotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K \quad \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \quad \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \quad \frac{\partial h}{\partial z}\right)$$

- Groundwater flow equation for 3D homogeneous, isotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = K \quad \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) = K \quad \nabla^2 h$$

Darcy's Law

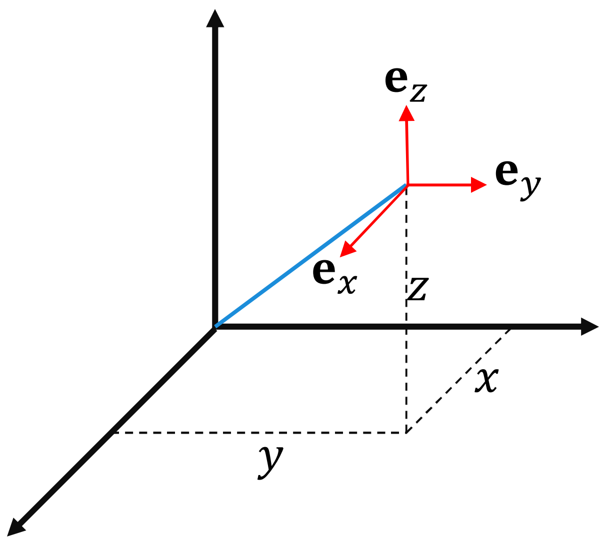
- In Cartesian coordinate system

$$\nabla(\cdot) = \frac{\partial}{\partial x}(\cdot)\mathbf{e}_x + \frac{\partial}{\partial y}(\cdot)\mathbf{e}_y + \frac{\partial}{\partial z}(\cdot)\mathbf{e}_z$$

$$\nabla^2(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot) + \frac{\partial^2}{\partial y^2}(\cdot) + \frac{\partial^2}{\partial z^2}(\cdot)$$

$$\nabla \cdot \mathbf{q} = \frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)$$

- If x, y, z are the principal flow directions, then



$$q_x = -K_x \frac{\partial h}{\partial x}$$

$$q_y = -K_y \frac{\partial h}{\partial y}$$

$$q_z = -K_z \frac{\partial h}{\partial z}$$

Darcy's Law (Contd.)

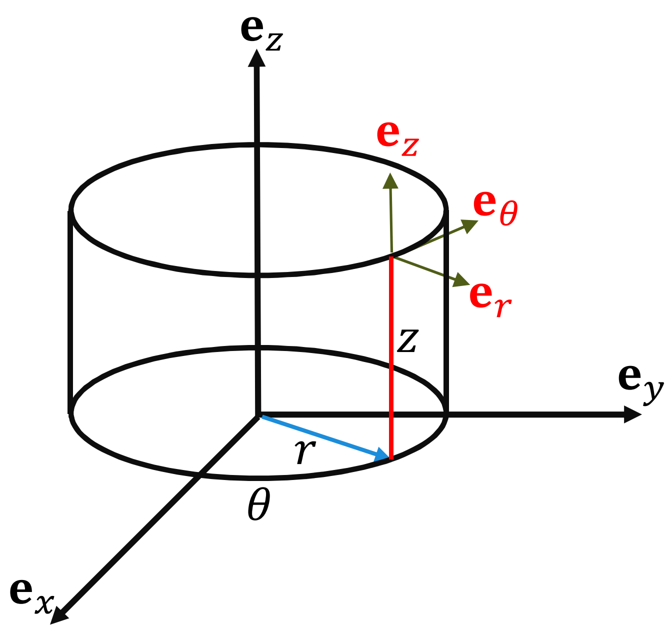
- In Cylindrical coordinate system

$$\nabla(\cdot) = \frac{\partial}{\partial r}(\cdot)\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}(\cdot)\mathbf{e}_\theta + \frac{\partial}{\partial z}(\cdot)\mathbf{e}_z$$

$$\nabla^2(\cdot) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\cdot)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\cdot) + \frac{\partial^2}{\partial z^2}(\cdot)$$

$$\nabla \cdot \mathbf{q} = \frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta) + \frac{\partial}{\partial z}(q_z)$$

- If r, θ, z are the principal flow directions, then



$$q_r = -K_r\frac{\partial h}{\partial r}$$

$$q_\theta = -\frac{K_\theta}{r}\frac{\partial h}{\partial \theta}$$

$$q_z = -K_z\frac{\partial h}{\partial z}$$

Groundwater Flow Equation

- Groundwater flow equation for 3D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) - W$$

or,

$$S_s \frac{\partial h}{\partial t} = \nabla \cdot (\mathbf{K} \cdot \nabla h) - Q$$

$$h(x, y, z, t)$$

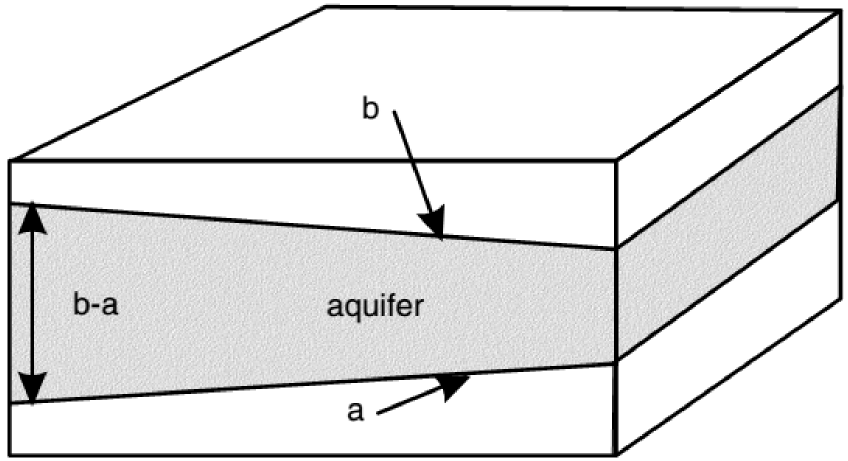
In Cylindrical coordinate system groundwater flow equation for 3D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S_s \frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(K_r r \frac{\partial h}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{K_\theta}{r} \frac{\partial h}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) - W$$

$$h(r, \theta, z, t)$$

$$h(r, z, t)$$

Reduction in Dimensionality



- Vertical Integration of the Flow Equation

$$\int_{a(x,y,t)}^{b(x,y,t)} \left(\nabla \cdot \mathbf{q} + S_s \frac{\partial h}{\partial t} + W \right) dz = 0$$

Reduction in Dimensionality (Contd.)

- Using Leibniz integral rule

$$\int_{a(x,y,t)}^{b(x,y,t)} \nabla \cdot \mathbf{q}\, dz = \int_a^b \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) dz = \frac{\partial}{\partial x} \int_a^b q_x\, dz - q_x|_{z=b} \frac{\partial b}{\partial x} + q_x|_{z=a} \frac{\partial a}{\partial x}$$

$$+ \frac{\partial}{\partial y} \int_a^b q_y\, dz - q_y|_{z=b} \frac{\partial b}{\partial y} + q_y|_{z=a} \frac{\partial a}{\partial y} + q_z|_{z=b} - q_z|_{z=a}$$

- In general

$$\int_{a(x,y,t)}^{b(x,y,t)} \nabla \cdot \mathbf{q}\, dz = \nabla_{xy} \cdot \int_{a(x,y,t)}^{b(x,y,t)} \mathbf{q}_{xy}\, dz + \mathbf{q}|_{z=b} \cdot \nabla(z-b) - \mathbf{q}|_{z=a} \cdot \nabla(z-a)$$

where

$$\nabla_{xy}(\cdot) \equiv \frac{\partial}{\partial x}(\cdot)\mathbf{e}_x + \frac{\partial}{\partial y}(\cdot)\mathbf{e}_y$$

Reduction in Dimensionality (Contd.)

- In terms of piezometric head

$$\mathbf{q} = -\mathbf{K} \cdot \nabla h$$

$$\int_{a(x,y,t)}^{b(x,y,t)} \mathbf{q}_{xy}\, dz = -\overline{\mathbf{K}} \cdot \int_{a}^{b} \nabla_{xy} h\, dz$$

$$= -\overline{\mathbf{K}} \cdot \left[\nabla_{xy} \int_{a}^{b} h\, dz - h|_{z=b} \nabla_{xy} b + h|_{z=a} \nabla_{xy} a \right]$$

Reduction in Dimensionality (Contd.)

- Time derivative term

$$\int_{a(x,y,t)}^{b(x,y,t)} S_s \frac{\partial h}{\partial t} dz = \bar{S}_s \int_a^b \frac{\partial h}{\partial t} dz$$

$$= \bar{S}_s \left[\frac{\partial}{\partial t} \int_a^b h\, dz - h|_{z=b} \frac{\partial b}{\partial t} + h|_{z=a} \frac{\partial a}{\partial t} \right]$$

- Combining all the terms

$$-\nabla_{xy} \cdot \left[\bar{\mathbf{K}} \cdot \left(\nabla_{xy} \int_a^b h\, dz - h|_{z=b} \nabla_{xy} b + h|_{z=a} \nabla_{xy} a \right) \right] + \mathbf{q}|_{z=b} \cdot \nabla(z-b) - \mathbf{q}|_{z=a} \cdot \nabla(z-a)$$

$$+\bar{S}_s \left[\frac{\partial}{\partial t} \int_a^b h\, dz - h|_{z=b} \frac{\partial b}{\partial t} + h|_{z=a} \frac{\partial a}{\partial t} \right] + \int_a^b W\, dz = 0$$

Reduction in Dimensionality (Contd.)

- Let us consider the vertical average values as

$$\bar{h} \equiv \frac{1}{(b-a)}\int_{a}^{b} h\,dz = \frac{1}{l}\int_{a}^{b} h\,dz$$

and

$$\bar{W} \equiv \frac{1}{(b-a)}\int_{a}^{b} W\,dz = \frac{1}{l}\int_{a}^{b} W\,dz$$

- Combining all the terms

$$-\nabla_{xy}\cdot\left[\bar{\mathbf{K}}\cdot\left(\nabla_{xy}l\bar{h} - \bar{h}\nabla_{xy}l\right)\right] + \mathbf{q}|_{z=b}\cdot\nabla(z-b) - \mathbf{q}|_{z=a}\cdot\nabla(z-a)$$
$$+\bar{S}_s\left[\frac{\partial\left(l\bar{h}\right)}{\partial t} - \bar{h}\frac{\partial l}{\partial t}\right] + l\bar{W} = 0$$

Or,

$$-\nabla_{xy}\cdot\left[l\bar{\mathbf{K}}\cdot\nabla_{xy}\bar{h}\right] + \mathbf{q}|_{z=b}\cdot\nabla(z-b) - \mathbf{q}|_{z=a}\cdot\nabla(z-a) + l\bar{S}_s\frac{\partial\bar{h}}{\partial t} + l\bar{W} = 0$$

Two-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Storage Coefficient

$$S \equiv l\overline{S_s}$$

- Transmissivity Tensor

$$\mathbf{T} \equiv l\overline{\mathbf{K}}$$

In 2D the governing equation can be written as

$$S\frac{\partial \bar{h}}{\partial t} + q_T + q_B + q_{ext} = \nabla_{xy} \cdot \left[\mathbf{T} \cdot \nabla_{xy}\bar{h}\right]$$

where

$$q_T = \mathbf{q}|_{z=b} \cdot \nabla(z-b)$$
$$q_B = -\mathbf{q}|_{z=a} \cdot \nabla(z-a)$$
$$q_{ext} = l\overline{W}$$

Two-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Storage Coefficient

$$S \equiv l\overline{S_s}$$

- Transmissivity Tensor

$$\mathbf{T} \equiv l\overline{\mathbf{K}}$$

In 2D the governing equation can be written as

$$S\frac{\partial \bar{h}}{\partial t} + q_T + q_B + q_{ext} = \nabla_{xy} \cdot \left[\mathbf{T} \cdot \nabla_{xy}\bar{h}\right]$$

where

$$q_T = \mathbf{q}|_{z=b} \cdot \nabla(z - b)$$
$$q_B = -\mathbf{q}|_{z=a} \cdot \nabla(z - a)$$
$$q_{ext} = l\overline{W}$$

Two-Dimensional Saturated Flow in Confined Aquifers (Contd.)

- Groundwater flow equation for 2D heterogeneous, anisotropic saturated confined aquifer can be written as

$$S \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right)$$

- Groundwater flow equation for 2D homogeneous, anisotropic saturated confined aquifer can be written as

$$S \frac{\partial h}{\partial t} = T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2}$$

- Groundwater flow equation for 2D heterogeneous, isotropic saturated confined aquifer can be written as

$$S \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T \frac{\partial h}{\partial y}\right)$$

- Groundwater flow equation for 2D homogeneous, isotropic saturated confined aquifer can be written as

$$S \frac{\partial h}{\partial t} = T \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)$$

Learning Strategy

Chapter 6: Deformation, Storage, and General Flow Equations

Thank you