Module 03: Groundwater Hydraulics

Unit 04: Unsteady Two-Dimensional Flow using Finite Volume Method

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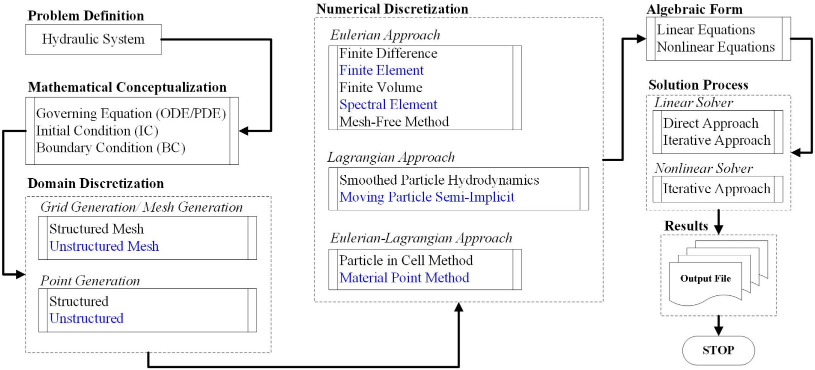
Learning Objective

- To solve unsteady state two dimensional groundwater flow equation using Finite Volume Method.

Problem Definition to Solution



Problem Definition

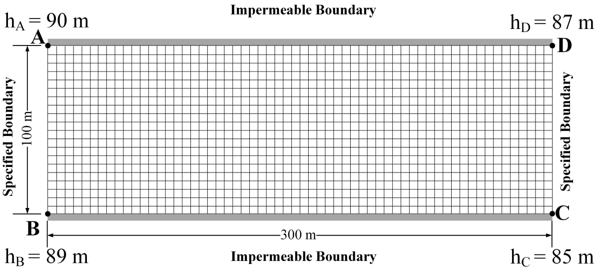


Figure: Homogeneous Aquifer System

Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega : \quad \frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

$S = 5 \times 10^{-5}$
$T = 200\ m^2/day$

Problem Definition

subject to

Initial Condition

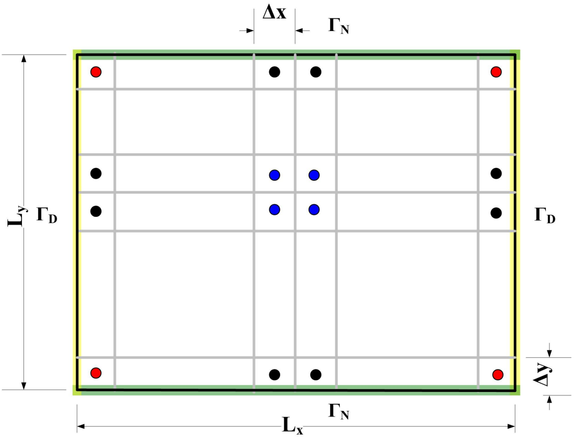
$$h(x, y, 0) = h_0(x, y)$$

and

Boundary Condition

$$\begin{aligned}
\Gamma_D^1 : &\quad h(0, y, t) = h_1(y) \\
\Gamma_D^2 : &\quad h(L_x, y, t) = h_2(y) \\
\Gamma_N^3 : &\quad \left.\frac{\partial h}{\partial y}\right|_{(x,0,t)} = 0 \\
\Gamma_N^4 : &\quad \left.\frac{\partial h}{\partial y}\right|_{(x,L_y,t)} = 0
\end{aligned}$$

Domain Discretization



Implicit Scheme

From Lecture 15,

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

For interior points,

$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1}=\frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1}=\frac{h_P^{l+1}-h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1}=\frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}$$

$$\left(\frac{\partial h}{\partial y}\right)_s^{l+1}=\frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}=\frac{h_E^{l+1}-2h_P^{l+1}+h_W^{l+1}}{\Delta x^2}+\frac{h_N^{l+1}-2h_P^{l+1}+h_S^{l+1}}{\Delta y^2}$$

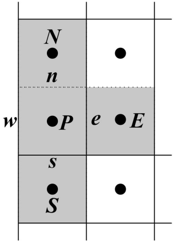
In simplified form, this can be written as

$$\alpha_y h_S^{l+1}+\alpha_x h_W^{l+1}-\left[1+2(\alpha_x+\alpha_y)\right]h_P^{l+1}+\alpha_x h_E^{l+1}+\alpha_y h_N^{l+1}=-h_P^l$$

with $\alpha_x = \frac{T\Delta t}{S\Delta x^2}$ and $\alpha_y = \frac{T\Delta t}{S\Delta y^2}$.

Boundary Conditions

Left and Right Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

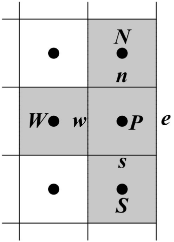
$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}-\frac{-8h_{BW}^{l+1}+9h_P^{l+1}-h_E^{l+1}}{3\Delta x}\right]\Delta y$$
$$+\left[\frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}-\frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}\right]\Delta x$$

In simplified form, this can be written as

$$\alpha_y h_S^{l+1}-\left[1+2(2\alpha_x+\alpha_y)\right]h_P^{l+1}+\frac{4}{3}\alpha_x h_E^{l+1}+\alpha_y h_N^{l+1}=-h_P^l-\frac{8}{3}\alpha_x h_{BW}^{l+1}$$

Boundary Conditions

Left and Right Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^{l}}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

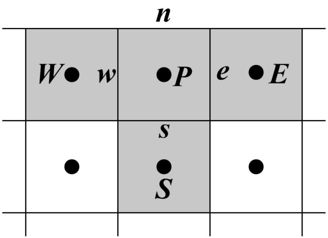
$$\frac{S}{T}\frac{h_P^{l+1}-h_P^{l}}{\Delta t}\Delta x\Delta y=\left[\frac{8h_{BE}^{l+1}-9h_P^{l+1}+h_W^{l+1}}{3\Delta x}-\frac{h_P^{l+1}-h_W^{l+1}}{\Delta x}\right]\Delta y+\left[\frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}-\frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}\right]\Delta x$$

In simplified form, this can be written as

$$\alpha_y h_S^{l+1}+\frac{4}{3}\alpha_x h_W^{l+1}-\left[1+2(2\alpha_x+\alpha_y)\right]h_P^{l+1}+\alpha_y h_N^{l+1}=-h_P^{l}-\frac{8}{3}\alpha_x h_{BE}^{l+1}$$

Boundary Conditions

Top and Bottom Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0 \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

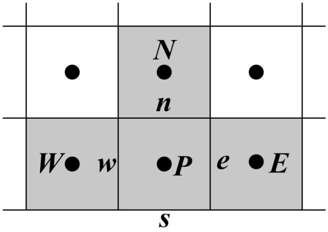
$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}-\frac{h_P^{l+1}-h_W^{l+1}}{\Delta x}\right]\Delta y+ \left[0-\frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}\right]\Delta x$$

In simplified form, this can be written as

$$\alpha_y h_S^{l+1}+\alpha_x h_W^{l+1}-\left[1+(2\alpha_x+\alpha_y)\right]h_P^{l+1}+\alpha_x h_E^{l+1}=-h_P^l$$

Boundary Conditions

Top and Bottom Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \quad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \quad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_{BS}^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

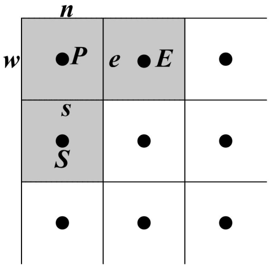
$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}-\frac{h_P^{l+1}-h_W^{l+1}}{\Delta x}\right]\Delta y+\left[\frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}-0\right]\Delta x$$

In simplified form, this can be written as

$$\alpha_x h_W^{l+1}-\left[1+(2\alpha_x+\alpha_y)\right]h_P^{l+1}+\alpha_x h_E^{l+1}+\alpha_y h_N^{l+1}=-h_P^l$$

Boundary Conditions

N-W Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0 \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

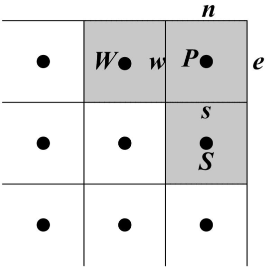
$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}-\frac{-8h_{BW}^{l+1}+9h_P^{l+1}-h_E^{l+1}}{3\Delta x}\right]\Delta y + \left[0-\frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}\right]\Delta x$$

In simplified form, this can be written as

$$\alpha_y h_S^{l+1}-\left[1+(4\alpha_x+\alpha_y)\right]h_P^{l+1}+\frac{4}{3}\alpha_x h_E^{l+1}=-h_P^l-\frac{8}{3}\alpha_x h_{BW}^{l+1}$$

Boundary Conditions

N-E Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

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Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

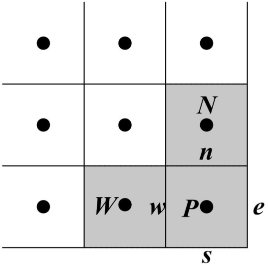
$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\frac{8h_{BE}^{l+1}-9h_P^{l+1}+h_W^{l+1}}{3\Delta x}-\frac{h_P^{l+1}-h_W^{l+1}}{\Delta x}\right]\Delta y$$
$$+\left[0-\frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}\right]\Delta x$$

In simplified form, this can be written as

$$\alpha_y h_S^{l+1}+\frac{4}{3}\alpha_x h_W^{l+1}-\left[1+(4\alpha_x+\alpha_y)\right]h_P^{l+1}=-h_P^l-\frac{8}{3}\alpha_x h_{BE}^{l+1}$$

Boundary Conditions

S-E Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_{BS}^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

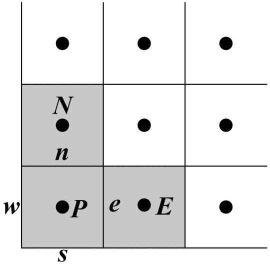
$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\frac{8h_{BE}^{l+1}-9h_P^{l+1}+h_W^{l+1}}{3\Delta x}-\frac{h_P^{l+1}-h_W^{l+1}}{\Delta x}\right]\Delta y+\left[\frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}-0\right]\Delta x$$

In simplified form, this can be written as

$$\frac{4}{3}\alpha_x h_W^{l+1}-\left[1+(4\alpha_x+\alpha_y)\right]h_P^{l+1}+\alpha_y h_N^{l+1}=-h_P^l-\frac{8}{3}\alpha_x h_{BE}^{l+1}$$

Boundary Conditions

S-W Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_{BS}^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

Implicit Scheme

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1}-\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y+\left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1}-\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x$$

$$\frac{S}{T}\frac{h_P^{l+1}-h_P^l}{\Delta t}\Delta x\Delta y=\left[\frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}-\frac{-8h_{BW}^{l+1}+9h_P^{l+1}-h_E^{l+1}}{3\Delta x}\right]\Delta y$$
$$+\left[\frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}-0\right]\Delta x$$

In simplified form, this can be written as

$$-\left[1+(4\alpha_x+\alpha_y)\right]h_P^{l+1}+\frac{4}{3}\alpha_x h_E^{l+1}+\alpha_y h_N^{l+1}=-h_P^l-\frac{8}{3}\alpha_x h_{BW}^{l+1}$$

General Form

In general form, the governing equation including boundary conditions can be written as,

$$a_S h_S^{l+1} + a_W h_W^{l+1} + a_P h_P^{l+1} + a_E h_E^{l+1} + a_N h_N^{l+1} = r_P$$

Source Code

Unsteady Two Dimensional Groundwater Flow with Finite Volume Method

- Without coefficient matrix using Gauss Seidel
  - unsteady_2D_fvm_conf_implicit_iterative.sci

Thank You