Module 04: Surface Water Hydraulics

Unit 01: Gradually Varied Flow

Anirban Dhar

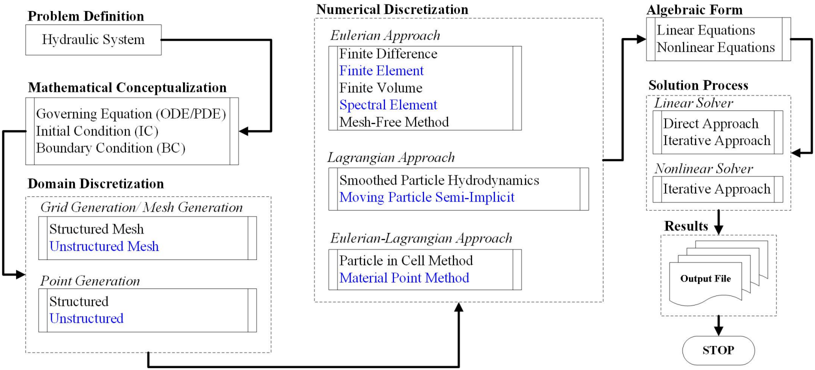
Department of Civil Engineering
Indian Institute of Technology Kharagpur, Kharagpur

National Programme for Technology Enhanced Learning (NPTEL)

Learning Objective

- To solve gradually varied flow problem for open channels.

Problem Definition to Solution



Problem Definition

Governing Equation for Gradually Varied Flow in prismatic channel can be written as,

Initial Value Problem

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \tag{1}$$

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Initial Condition:

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where
$y=$ depth of flow
$x=$ coordinate direction
$S_0=$ bed slope
$S_f=$ friction slope $= \left(\frac{n^2 Q^2}{R^{4/3} A^2}\right)$
$Fr=$ Froude number $= \left(\sqrt{\frac{Q^2 T}{g A^3}}\right)$
$Q=$ discharge
$T=$ top width
$g=$ acceleration due to gravity
$R=$ hydraulic radius
$A=$ cross-sectional area.

Problem Definition
Gradually Varied Flow in Open Channel

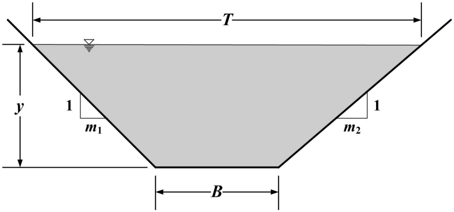
From Lecture 7, in general format

$$\frac{dy}{dx} = \Psi(x, y)$$

where

$$\begin{aligned}\Psi(x, y) &= \frac{S_0 - S_f}{1 - Fr^2} \\ &= \frac{S_0 - \frac{n^2 Q^2}{R^{4/3} A^2}}{1 - \frac{Q^2 T}{g A^3}}\end{aligned}$$

Trapezoidal Cross-section



$$A = By + \frac{1}{2}(m_1 + m_2)y^2$$

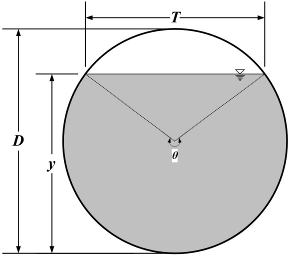
$$P = B + \left(\sqrt{1+m_1^2} + \sqrt{1+m_2^2}\right) y$$

$$R = \frac{A}{P}$$

$$T = B + (m_1 + m_2)y$$

where $P =$ wetted perimeter.

Circular Cross-section



$$A = \frac{1}{8}(\theta - \sin\theta)D^2$$

$$P = \frac{1}{2}\theta D$$

$$R = \frac{A}{P}$$

$$T = D\sin\left(\frac{\theta}{2}\right)$$

Problem Definition

Critical Depth

For critical depth, $Fr = 1$

$$Fr = \sqrt{\frac{Q^2T}{gA^3}} = 1$$

In case of rectangular channel, $A = By$ and $T = B$

$$\sqrt{\frac{Q^2T}{gA^3}} = 1$$

$$y_c = \left(\frac{Q^2}{gB^2}\right)^{\frac{1}{3}}$$

Problem Definition
Normal Depth

Normal depth can be calculated from Manning's equation (uniform flow),

$$Q = \frac{1}{n} R^{\frac{2}{3}} S_0^{\frac{1}{2}} A$$

In case of rectangular channel, $A = By_n$ and $P = B + 2y_n$

$$Q = \frac{1}{n} \left(\frac{By_n}{B + 2y_n} \right)^{\frac{2}{3}} S_0^{\frac{1}{2}} By_n$$

In function form,

$$G(y_n) = \frac{S_0^{\frac{1}{2}} B^{\frac{5}{3}}}{n} \left(\frac{y_n}{B + 2y_n} \right)^{\frac{2}{3}} y_n - Q = 0$$

Problem Definition

Normal Depth

From Newton-Raphson method,

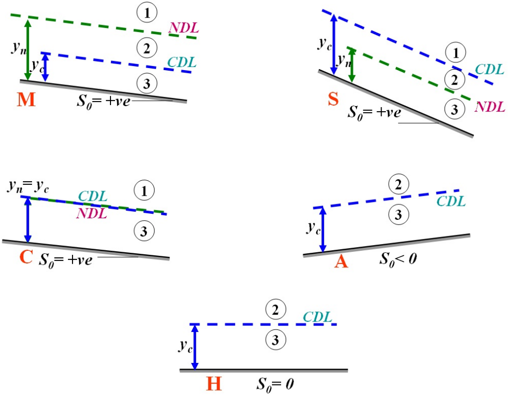
$$y_n\Big|^{(p)} = y_n\Big|^{(p-1)} - \frac{G\left(y_n\Big|^{(p-1)}\right)}{G'\left(y_n\Big|^{(p-1)}\right)}$$

where

$$G'\left(y_n\right) = \frac{S_0^{\frac{1}{2}} B^{\frac{5}{3}}}{3n} \frac{y_n^{\frac{2}{3}}\left(5B + 6y_n\right)}{\left(B + 2y_n\right)^{\frac{5}{3}}}$$

Problem Definition

Possible Flow Conditions



Problem Definition

Summary of the Flow Conditions

Channel Category	Symbol	Characteristic Condition	Remark
Mild slope	M	$y_n > y_c$	Subcritical flow at normal depth
Steep slope	S	$y_n < y_c$	Supercritical flow at normal depth
Critical Slope	C	$y_n = y_c$	Critical flow at normal depth
Horizontal Bed Slope	H	$S_0 = 0$	Cannot sustain uniform flow
Adverse slope	A	$S_0 < 0$	Cannot sustain uniform flow

Forward Euler Method

Euler Method

$$y_{n+1} = y_n + \Delta x \Psi(x_n, y_n)$$

Order of Euler's method: $\mathcal{O}(\Delta x)$

Modified Euler Method

First Approach

Modified Euler Method

$$y_{n+1} = y_n + K_2$$

with

$$K_2 = \Delta x \Psi \left(x_n + \frac{\Delta x}{2}, y_n + \frac{1}{2} K_1 \right)$$

$$K_1 = \Delta x \Psi (x_n, y_n)$$

Modified Euler Method

First Approach

Modified Euler Method

$$y_{n+1} = y_n + K_2$$

with

$$K_2 = \Delta x \Psi \left(x_n + \frac{\Delta x}{2}, y_n + \frac{1}{2} K_1 \right)$$

$$K_1 = \Delta x \Psi (x_n, y_n)$$

Order of Modified Euler method: $\mathcal{O}(\Delta x^2)$

Euler-Cauchy method

Second Approach

Modified Euler Method

$$y_{n+1} = y_n + \frac{1}{2}\left[K_1 + K_2\right]$$

with

$$K_2 = \Delta x \Psi(x_n + \Delta x, y_n + K_1)$$
$$K_1 = \Delta x \Psi(x_n, y_n)$$

Order of Modified Euler method: $\mathcal{O}(\Delta x^2)$

Second Order RK Method (RK2)

RK2

$$y_{n+1} = y_n + \frac{1}{4}\left[K_1 + 3K_2\right]$$

with

$$K_1 = \Delta x \Psi(x_n, y_n)$$

$$K_2 = \Delta x \Psi(x_n + \frac{2}{3}\Delta x, y_n + \frac{2}{3}K_1)$$

Order of RK2 method: $\mathcal{O}(\Delta x^2)$

Third Order RK Method (RK3)

RK3

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

with

$$K_1 = \Delta x \Psi(x_n, y_n)$$

$$K_2 = \Delta x \Psi(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}K_1)$$

$$K_3 = \Delta x \Psi(x_n + \Delta x, y_n - K_1 + 2K_2)$$

Order of RK3 method: $\mathcal{O}(\Delta x^3)$

Fourth Order RK Method (RK4)

RK_4 can be presented as,

RK4

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

with

$$K_1 = \Delta x \Psi(x_n, y_n)$$

$$K_2 = \Delta x \Psi(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}K_1)$$

$$K_3 = \Delta x \Psi(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}K_2)$$

$$K_4 = \Delta x \Psi(x_n + \Delta x, y_n + K_3)$$

Order of RK3 method: $\mathcal{O}(\Delta x^4)$

List of Source Codes

Gradually Varied Flow

- Forward Euler approach
  - forward_euler.sci
- Modified Euler approach
  - modified_euler_1st.sci
  - modified_euler_2nd.sci
- RK2 approach
  - RK2.sci
- RK3 approach
  - RK3.sci
- RK4 approach
  - RK4.sci

Thank You