Module 02: Numerical Methods

Unit 09: Finite Volume Method-Overview

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Learning Objective

- To discretize the derivatives of single-valued one-dimensional functions using finite volume method.

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- To discretize the derivatives of single-valued one-dimensional functions using finite volume method.
- To derive the algebraic form using discretized ODE and BC(s).

General Equation

A form of differential equation with a general variable ϕ:

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla.(\Upsilon_\phi \phi \mathbf{u}) = \nabla.(\boldsymbol{\Gamma}_\phi.\nabla\phi) + F_{\phi_o} + S_\phi \qquad (1)$$

where

ϕ = general variable

$\Lambda_\phi,\ \Upsilon_\phi$ = problem dependent parameters

$\boldsymbol{\Gamma}_\phi$ = tensor

F_{ϕ_o} = other forces

S_ϕ = source/sink term

Background

In the Method of Weighted Residual (MWR), residual $\mathcal{R}$ (Finlayson and Scriven, 1966) can be written as,

$$\mathcal{R}(\mathbf{x}, t) \equiv \frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla.(\Upsilon_\phi \phi \mathbf{u}) - \nabla.(\mathbf{\Gamma}_\phi.\nabla\phi) - F_{\phi_o} - S_\phi$$

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The weighted integrals of the residuals set equal to zero:

$$\int_\Omega w_l \mathcal{R} d\Omega = 0, \quad \forall l$$

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The weighted integrals of the residuals set equal to zero:

$$\int_\Omega w_l \mathcal{R} d\Omega = 0, \quad \forall l$$

where w_l is prescribed weighting function. If w_l is a Dirac delta function δ such that

$$w_l = \delta(\mathbf{x}_l - \mathbf{x})$$

In collocation method,

$$\int_\Omega \delta(\mathbf{x}_l - \mathbf{x})\mathcal{R} d\Omega = 0 \Rightarrow \mathcal{R}(\mathbf{x}_l, t) = 0$$

This is similar to Finite Difference Method.

Background

In sub-domain method, the weighting function can be written as

$$w_l = \begin{cases} 1 & if \ \mathbf{x} \in \Omega^l \\ 0 & if \ \mathbf{x} \notin \Omega^l \end{cases}$$

Background

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The weighted integral can be written as,

$$\int_\Omega w_l \mathcal{R} d\Omega \Rightarrow \int_{\Omega^l} \mathcal{R} d\Omega = 0$$

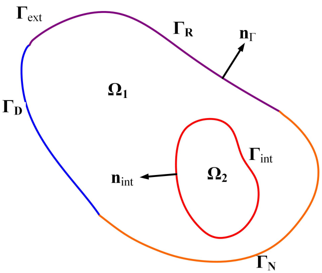
This is similar to Finite Volume Method.

Divergence Theorem

Gauss Divergence Theorem (Aris, 1990)

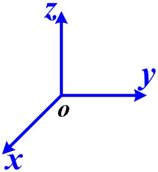
Suppose Ω is the volume bounded by a closed surface S and $\mathbf{a}$ vector field defined in Ω and on S. If S is piecewise smooth with outward normal $\hat{\mathbf{n}}$ and $\mathbf{a}$ continuously differentiable, then

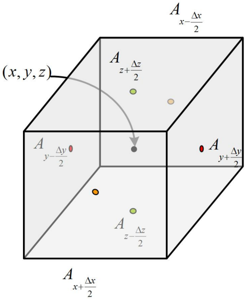
$$\iiint_{\Omega} \nabla \cdot \mathbf{a}\; d\Omega = \iint_{S} \mathbf{a} \cdot \hat{\mathbf{n}}\; dS$$



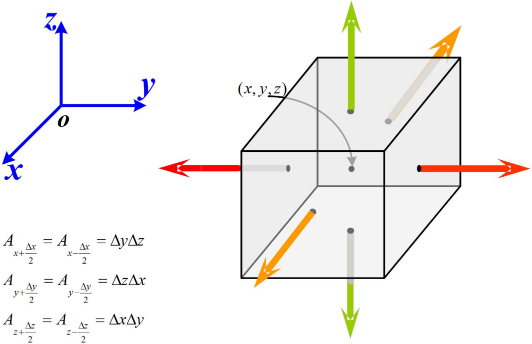
Divergence Theorem

Area Vector



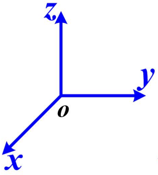


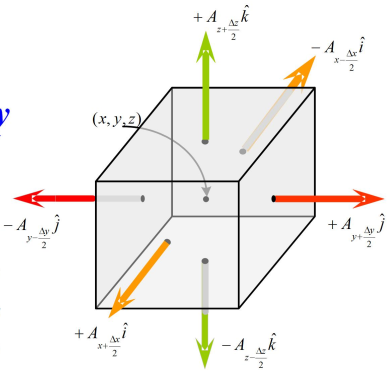
Divergence Theorem
Area Vector



$$A_{x+\frac{\Delta x}{2}} = A_{x-\frac{\Delta x}{2}} = \Delta y \Delta z$$
$$A_{y+\frac{\Delta y}{2}} = A_{y-\frac{\Delta y}{2}} = \Delta z \Delta x$$
$$A_{z+\frac{\Delta z}{2}} = A_{z-\frac{\Delta z}{2}} = \Delta x \Delta y$$

Divergence Theorem
Area Vector





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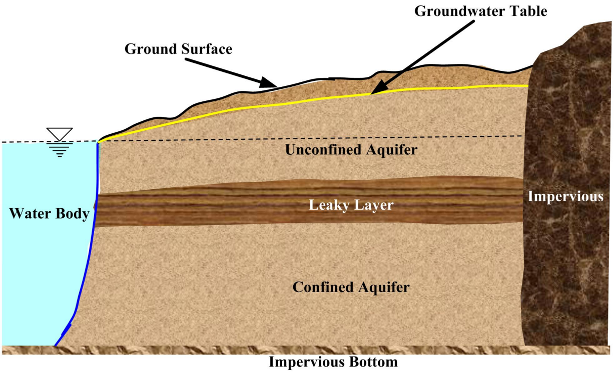
Problem Definition



Mathematical Conceptualization

The differential equation describing the head distribution in the aquifer is given as ,

$$\frac{d}{dx}\left(T\frac{dh}{dx}\right) = C_{\mathsf{conf}}(h - h_{wt}) \tag{2}$$

or,

Mathematical Conceptualization

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or,

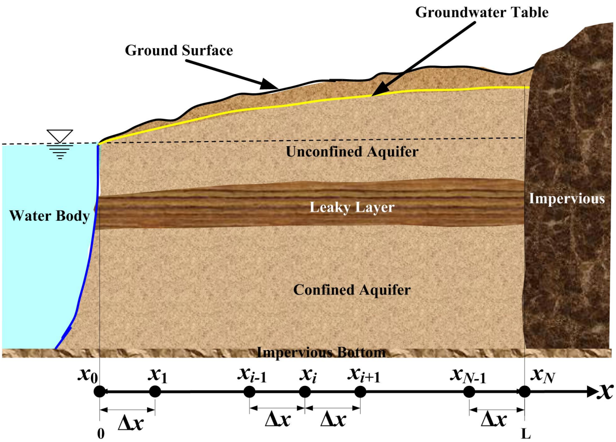
$$\frac{d^2h}{dx^2} = \frac{C_{\text{conf}}}{T}(h - h_{wt}) \tag{3}$$

where,
h = head,
T = aquifer transmissivity,
C_{conf} = hydraulic conductivity/thickness of confining layer,
h_{wt} = overlying water table elevation $(c_0 + c_1x + c_2x^2)$.

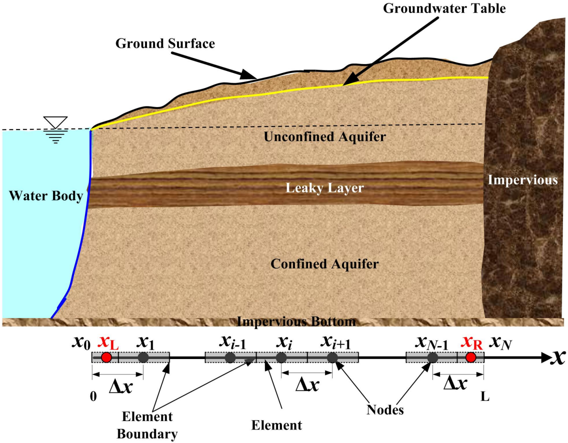
Boundary Conditions

- Left Boundary is specified head/ Dirichlet boundary: $h(x=0) = h_s$
- Right Boundary is impervious/ no-flow/ Neumann Boundary: $\frac{dh}{dx}\Big|_L = 0$

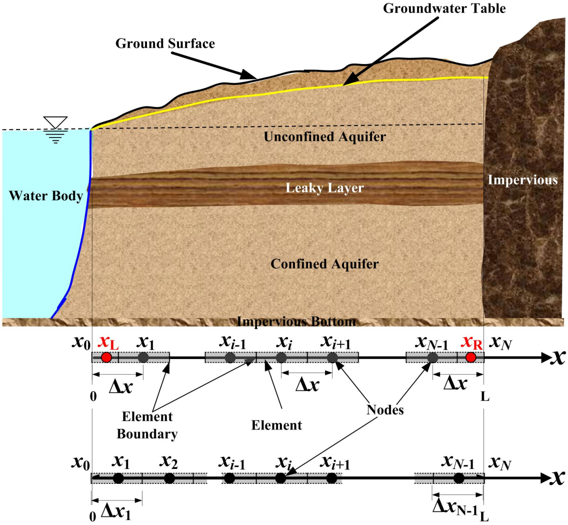
Domain Discretization (Finite Difference)



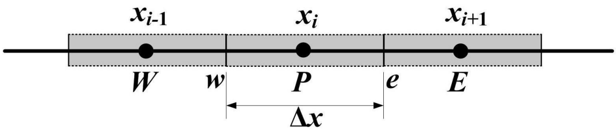
Domain Discretization (Finite Volume)



Domain Discretization (Finite Volume)



Discretization
Governing Equation

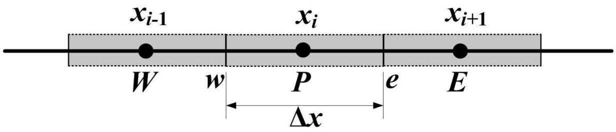


In Finite Volume Method, the governing equation is integrated over the element volume to form the discretized equation at node Point P.

$$\int\limits_{\Omega^P} \left[\frac{d}{dx}\left(T \frac{dh}{dx} \right) - C_{\text{conf}}(h - h_{wt}) \right] d\Omega = 0 \tag{4}$$

Discretization

Governing Equation



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$$\int_{\Omega^P} \left[\frac{d}{dx}\left(T\frac{dh}{dx} \right) - C_{\text{conf}}(h - h_{wt}) \right] d\Omega = 0 \tag{4}$$

or,

$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx} \right) d\Omega - C_{\text{conf}} \int_{\Omega^P} (h - h_{wt}) d\Omega = 0 \tag{5}$$

Discretization

Governing Equation

$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx}\right) d\Omega = \int_{S^P} \left(T\frac{dh}{dx}\right) \hat{i}\cdot\hat{\mathbf{n}}dS$$
$$= \int_{S^P} \left(T\frac{dh}{dx}\right) dA_x \qquad (6)$$
$$= \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw}$$

Discretization

Governing Equation

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$$= \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw}$$

In a uniform grid system,

$$\left(T\frac{dh}{dx}\right)_e A_{xe} = T_e \frac{h_E - h_P}{\Delta x}$$

$$\left(T\frac{dh}{dx}\right)_w A_{xw} = T_w \frac{h_P - h_W}{\Delta x} \qquad (7)$$

Discretization

Governing Equation

$$\int_{\Omega^P} (h - h_{wt})d\Omega = h_P \Delta x - \int_{x_w}^{x_e} (c_0 + c_1 x + c_2 x^2)dx$$

$$= h_P \Delta x - \left[c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right]_{x_w}^{x_e}$$

Discretization
Governing Equation

$$\int_{\Omega^P} (h - h_{wt}) d\Omega = h_P \Delta x - \int_{x_w}^{x_e} (c_0 + c_1 x + c_2 x^2) dx$$

$$= h_P \Delta x - \left[c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right]_{x_w}^{x_e}$$

Compact form of the equation

$$T_e \frac{h_E - h_P}{\Delta x} - T_w \frac{h_P - h_W}{\Delta x} = C_{\text{conf}} h_P \Delta x - C_{\text{conf}} \left[c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right]_{x_w}^{x_e}$$

Discretization

Governing Equation

With $T_e = T_w = T$ and $x_e - x_w = \Delta x$,

$$\frac{h_E - 2h_P + h_W}{\Delta x^2} = \frac{C_{\text{conf}}}{T}\left[h_P - \left(c_0 + \frac{1}{2}c_1(x_e + x_w) + \frac{1}{3}c_2(x_e^2 + x_e x_w + x_w^2)\right)\right]$$

Discretization

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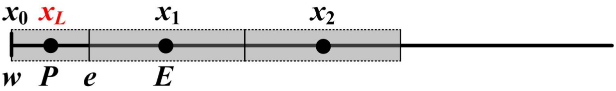
In simplified form,

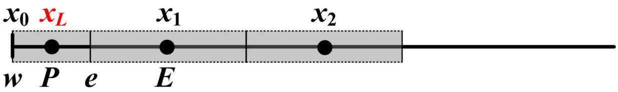
$$\frac{h_E - 2h_P + h_W}{\Delta x^2} = \frac{C_{\text{conf}}}{T}\left[h_P - h_{wt}(x_P) - \frac{1}{12}\Delta x^2\right]$$

Discretization
Left Boundary



$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx}\right) d\Omega = \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw} \tag{8}$$

Discretization
Left Boundary



$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx}\right) d\Omega = \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw} \tag{8}$$

$$\begin{aligned} \left(T\frac{dh}{dx}\right)_e A_{xe} &= T_e \frac{h_E - h_P}{3\Delta x/4} \\ \left(T\frac{dh}{dx}\right)_w A_{xw} &= T_w \frac{h_P - h_0}{\Delta x/4} \end{aligned} \tag{9}$$

Discretization
Left Boundary

$$\int_{\Omega^P} (h - h_{wt}) d\Omega = h_P \frac{\Delta x}{2} - \int_{x_w}^{x_e} (c_0 + c_1 x + c_2 x^2) dx$$

$$= h_P \frac{\Delta x}{2} - \left[c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right]_{x_w}^{x_e}$$

Discretization

Left Boundary

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Compact form of the equation

$$T_e\frac{h_E - h_P}{3\Delta x/4} - T_w\frac{h_P - h_0}{\Delta x/4} = C_{\text{conf}}h_P\frac{\Delta x}{2} - C_{\text{conf}}\left[c_0 x + \frac{1}{2}c_1 x^2 + \frac{1}{3}c_2 x^3\right]_{x_w}^{x_e}$$

Discretization

Left Boundary

With $T_e = T_w = T$ and $x_e - x_w = \Delta x/2$,

$$\frac{4h_E - 16h_P}{3\Delta x} = -\frac{4h_0}{\Delta x} + \frac{\Delta x}{2}\frac{C_{\text{conf}}}{T}\left[h_P - \left(c_0 + \frac{1}{2}c_1(x_e + x_w) + \frac{1}{3}c_2(x_e^2 + x_e x_w + x_w^2)\right)\right]$$

Discretization

Left Boundary

With $T_e = T_w = T$ and $x_e - x_w = \Delta x/2$,

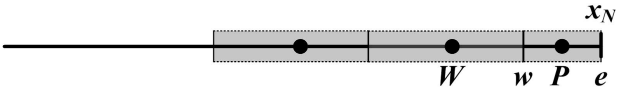
$$\frac{4h_E - 16h_P}{3\Delta x} = -\frac{4h_0}{\Delta x} + \frac{\Delta x}{2}\frac{C_{\text{conf}}}{T}\left[h_P - \left(c_0 + \frac{1}{2}c_1(x_e + x_w) + \frac{1}{3}c_2(x_e^2 + x_e x_w + x_w^2)\right)\right]$$

In simplified form,

$$\frac{8h_E - 32h_P}{3\Delta x^2} = -\frac{8h_0}{\Delta x^2} + \frac{C_{\text{conf}}}{T}\left[h_P - h_{wt}(x_P) - \frac{1}{48}\Delta x^2\right]$$

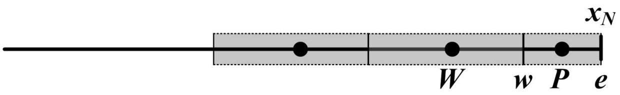
Discretization

Right Boundary



$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx}\right) d\Omega = \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw} \tag{10}$$

Discretization
Right Boundary



$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx}\right) d\Omega = \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw} \quad (10)$$

$$\left(T\frac{dh}{dx}\right)_e A_{xe} = 0$$
$$\left(T\frac{dh}{dx}\right)_w A_{xw} = T_w \frac{h_P - h_W}{3\Delta x/4} \quad (11)$$

Discretization

Right Boundary

$$\int_{\Omega^P} (h - h_{wt}) d\Omega = h_P \frac{\Delta x}{2} - \int_{x_w}^{x_e} (c_0 + c_1 x + c_2 x^2) dx$$

$$= h_P \frac{\Delta x}{2} - \left[c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right]_{x_w}^{x_e}$$

Discretization

Right Boundary

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$$= h_P \frac{\Delta x}{2} - \left[c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right]_{x_w}^{x_e}$$

Compact form of the equation

$$-T_w \frac{h_P - h_W}{3\Delta x/4} = C_{\text{conf}} h_P \frac{\Delta x}{2} - C_{\text{conf}} \left[c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right]_{x_w}^{x_e}$$

Discretization

Right Boundary

With $T_e = T_w = T$ and $x_e - x_w = \Delta x/2$,

$$-\frac{4h_P - 4h_w}{3\Delta x} = \frac{\Delta x}{2}\frac{C_{\text{conf}}}{T}\left[h_P - \left(c_0 + \frac{1}{2}c_1(x_e + x_w) + \frac{1}{3}c_2(x_e^2 + x_e x_w + x_w^2)\right)\right]$$

Discretization

Right Boundary

With $T_e = T_w = T$ and $x_e - x_w = \Delta x/2$,

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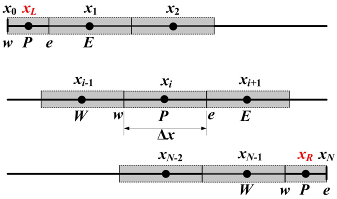
In simplified form,

$$\frac{8h_W - 8h_P}{3\Delta x^2} = \frac{C_{\text{conf}}}{T}\left[h_P - h_{wt}(x_P) - \frac{1}{48}\Delta x^2\right]$$

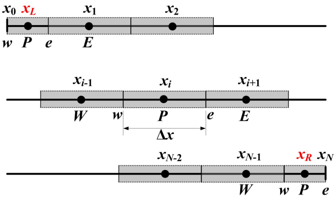
Discretization

Neumann Boundary



Discretization

Neumann Boundary



Neumann Boundary

$$\left.\frac{dh}{dx}\right|_{x=L} = 0$$

$$\frac{15h_N - 16h_R + h_{N-1}}{3\Delta x} = 0$$

Thank You

References

Aris, R. (1990). *Computational Fluid Flow and Heat Transfer*. Dover Publications Inc, New York.

Finlayson, B. A. and Scriven, L. E. (1966). The Method of Weighted Residuals-A Review. *Applied Mechanics Reviews*, 19(9):735–748.