Module 03: Groundwater Hydraulics

Unit 02: Steady Two-Dimensional Flow

Anirban Dhar

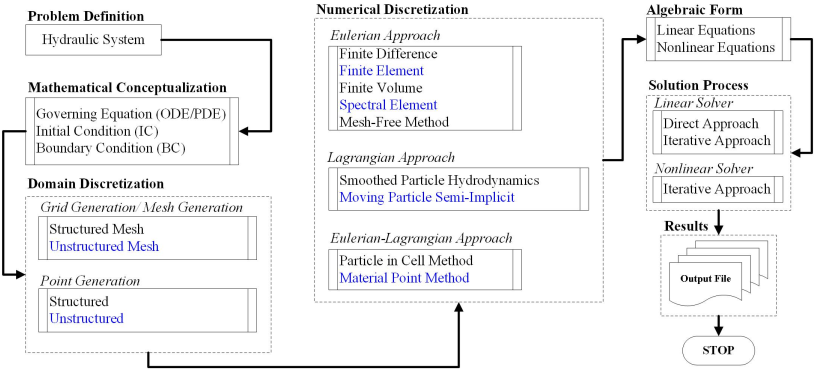
Department of Civil Engineering
Indian Institute of Technology Kharagpur, Kharagpur

National Programme for Technology Enhanced Learning (NPTEL)

Learning Objective

- To solve steady state two dimensional groundwater flow equation.

Problem Definition to Solution



Problem Definition

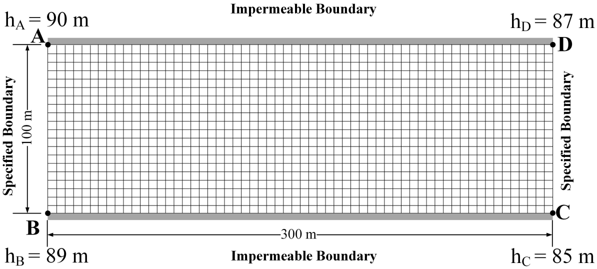


Figure: Homogeneous Aquifer System (Dhar and Patil, 2012)

Problem Definition

Governing equation

A two-dimensional BVP can be written as,

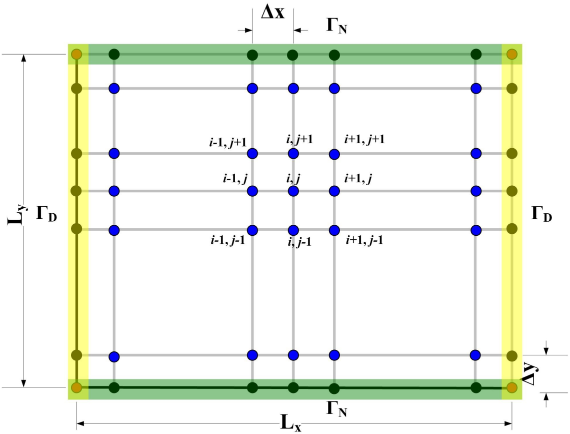
$$\Omega : \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

subject to

Boundary Condition

$$\begin{aligned}
\Gamma_D^1 : &\quad h(0, y) = h_1(y) \\
\Gamma_D^2 : &\quad h(L_x, y) = h_2(y) \\
\Gamma_N^3 : &\quad \left.\frac{\partial h}{\partial y}\right|_{(x,0)} = 0 \\
\Gamma_N^4 : &\quad \left.\frac{\partial h}{\partial y}\right|_{(x,L_y)} = 0
\end{aligned}$$

Domain Discretization



Numerical Discretization

Governing Equation

From Lecture 9,

The governing equation can be discretized as,

$$\frac{h_{i-1,j} - 2h_{i,j} + h_{i+1,j}}{\Delta x^2} + \frac{h_{i,j-1} - 2h_{i,j} + h_{i,j+1}}{\Delta y^2} = 0$$

Numerical Discretization

Governing Equation

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The equation can be rearranged as,

$$\frac{1}{\Delta y^2} h_{i,j-1} + \frac{1}{\Delta x^2} h_{i-1,j} - 2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) h_{i,j} + \frac{1}{\Delta x^2} h_{i+1,j} + \frac{1}{\Delta y^2} h_{i,j+1} = 0$$

Numerical Discretization

Governing Equation

From Lecture 9,
The governing equation can be discretized as,

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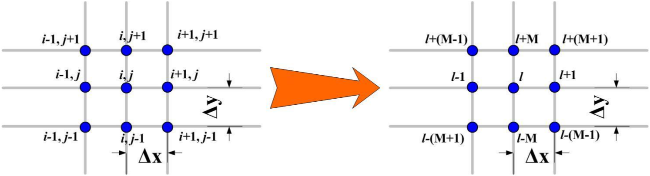
In simplified form, this can be written as

$$\alpha_y h_{i,j-1} + \alpha_x h_{i-1,j} - 2\left(\alpha_x + \alpha_y\right) h_{i,j} + \alpha_x h_{i+1,j} + \alpha_y h_{i,j+1} = 0$$

with $\alpha_x = \frac{1}{\Delta x^2}$ and $\alpha_y = \frac{1}{\Delta y^2}$.

Single Index Approach

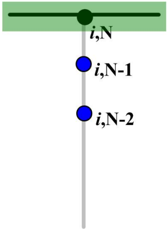
$$l = i + (j-1) \times M$$



With single index notation, the equation can be written as,

$$\alpha_y h_{l-M} + \alpha_x h_{l-1} - 2\left(\alpha_x + \alpha_y\right) h_l + \alpha_x h_{l+1} + \alpha_y h_{l+M} = 0$$

Neumann Boundary Condition

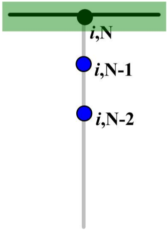


Top Boundary

Second Order Discretization

$$\frac{3h_{i,N} - 4h_{i,N-1} + h_{i,N-2}}{2\Delta y} = 0 \tag{1}$$

Neumann Boundary Condition



Top Boundary

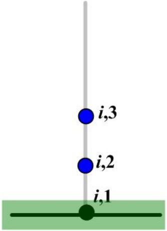
Second Order Discretization

$$\frac{3h_{i,N} - 4h_{i,N-1} + h_{i,N-2}}{2\Delta y} = 0 \tag{1}$$

In single index notation format,

$$\frac{3h_l - 4h_{l-M} + h_{l-2M}}{2\Delta y} = 0 \tag{2}$$

Neumann Boundary Condition

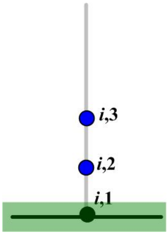


Bottom Boundary

Second Order Discretization

$$\frac{-3h_{i,1} + 4h_{i,2} - h_{i,3}}{2\Delta y} = 0 \tag{3}$$

Neumann Boundary Condition



Bottom Boundary

Second Order Discretization

$$\frac{-3h_{i,1} + 4h_{i,2} - h_{i,3}}{2\Delta y} = 0 \tag{3}$$

In single index notation format,

$$\frac{-3h_{l} + 4h_{l+M} - h_{l+2M}}{2\Delta y} = 0 \tag{4}$$

Gauss-Seidel Method

Iterative Approach

From Lecture 29, iteration starts with the guess value $\mathbf{h}^{(\mathbf{0})}$

$$\boldsymbol{h}^{(0)} = \left[h_{1,1}^{(0)} \quad h_{1,2}^{(0)} \ldots \quad h_{M,N-1}^{(0)} \quad h_{M,N}^{(0)}\right]^T$$

Gauss-Seidel Method
Iterative Approach

From Lecture 29, iteration starts with the guess value $\mathbf{h}^{(\mathbf{0})}$

$$\boldsymbol{h}^{(0)} = \left[h_{1,1}^{(0)} \quad h_{1,2}^{(0)} \ldots \quad h_{M,N-1}^{(0)} \quad h_{M,N}^{(0)}\right]^T$$

$$h_{i,j}^{(p)} = \frac{1}{\left[-2\left(\alpha_x + \alpha_y\right)\right]}\left[0 - \left(\alpha_y h_{i,j-1}^{(p)} + \alpha_x h_{i-1,j}^{(p)} + \alpha_x h_{i+1,j}^{(p-1)} + \alpha_y h_{i,j+1}^{(p-1)}\right)\right]$$

Gauss-Seidel Method

Iterative Approach

From Lecture 29, iteration starts with the guess value $\mathbf{h}^{(0)}$

$$\boldsymbol{h}^{(0)} = \left[h_{1,1}^{(0)} \quad h_{1,2}^{(0)} \dots \quad h_{M,N-1}^{(0)} \quad h_{M,N}^{(0)}\right]^T$$

$$h_{i,j}^{(p)} = \frac{1}{\left[-2\left(\alpha_x + \alpha_y\right)\right]}\left[0 - \left(\alpha_y h_{i,j-1}^{(p)} + \alpha_x h_{i-1,j}^{(p)} + \alpha_x h_{i+1,j}^{(p-1)} + \alpha_y h_{i,j+1}^{(p-1)}\right)\right]$$

By adding and subtracting $h_{i,j}^{(p-1)}$ in right hand side

$$h_{i,j}^{(p)} = h_{i,j}^{(p-1)} + \frac{1}{\left[-2\left(\alpha_x + \alpha_y\right)\right]}\left[-\alpha_y h_{i,j-1}^{(p)} - \alpha_x h_{i-1,j}^{(p)} + 2\left(\alpha_x + \alpha_y\right) h_{i,j}^{(p-1)} - \alpha_x h_{i+1,j}^{(p-1)} - \alpha_y h_{i,j+1}^{(p-1)}\right]$$

Gauss-Seidel Method

Iterative Approach

From Lecture 29, iteration starts with the guess value $\mathbf{h}^{(0)}$

$$\boldsymbol{h}^{(0)} = \begin{bmatrix} h_{1,1}^{(0)} & h_{1,2}^{(0)} \dots & h_{M,N-1}^{(0)} & h_{M,N}^{(0)} \end{bmatrix}^T$$

$$h_{i,j}^{(p)} = \frac{1}{[-2(\alpha_x + \alpha_y)]}\left[0 - \left(\alpha_y h_{i,j-1}^{(p)} + \alpha_x h_{i-1,j}^{(p)} + \alpha_x h_{i+1,j}^{(p-1)} + \alpha_y h_{i,j+1}^{(p-1)}\right)\right]$$

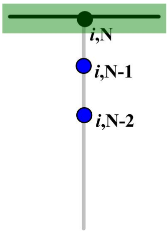
By adding and subtracting $h_{i,j}^{(p-1)}$ in right hand side

$$h_{i,j}^{(p)} = h_{i,j}^{(p-1)} + \frac{1}{[-2(\alpha_x + \alpha_y)]}\left[-\alpha_y h_{i,j-1}^{(p)} - \alpha_x h_{i-1,j}^{(p)} + 2(\alpha_x + \alpha_y) h_{i,j}^{(p-1)} - \alpha_x h_{i+1,j}^{(p-1)} - \alpha_y h_{i,j+1}^{(p-1)}\right]$$

In compact form

$$h_{i,j}^{(p)} = h_{i,j}^{(p-1)} + \frac{Res_{i,j}}{[-2(\alpha_x + \alpha_y)]}, \quad \forall (i,j)\ p \geq 1$$

Neumann Boundary Condition



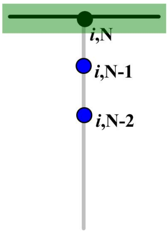
Top Boundary

Second Order Discretization

$$\frac{3h_{i,N} - 4h_{i,N-1} + h_{i,N-2}}{2\Delta y} = 0 \qquad (5)$$

$$h_{i,N}^{(p)} = h_{i,N}^{(p-1)} + \frac{1}{3}\left[-h_{i,N-2}^{(p)} + 4h_{i,N-1}^{(p)} - 3h_{i,N}^{(p-1)}\right]$$

Neumann Boundary Condition



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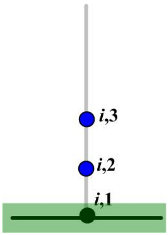
$$h_{i,N}^{(p)} = h_{i,N}^{(p-1)} + \frac{1}{3}\left[-h_{i,N-2}^{(p)} + 4h_{i,N-1}^{(p)} - 3h_{i,N}^{(p-1)}\right]$$

Neumann Boundary Condition



Bottom Boundary

Second Order Discretization

$$\frac{-3h_{i,1} + 4h_{i,2} - h_{i,3}}{2\Delta y} = 0 \tag{6}$$

$$h_{i,1}^{(p)} = h_{i,1}^{(p-1)} + \frac{1}{-3}\left[3h_{i,1}^{(p-1)} - 4h_{i,2}^{(p-1)} + h_{i,3}^{(p-1)}\right]$$

List of Source Codes

Steady Two Dimensional Groundwater Flow

- Full matrix using Gauss elimination
  - laplace_2D.sci
- Without coefficient matrix using Gauss Seidel
  - laplace_2D_iterative.sci

Thank You

References

Dhar, A. and Patil, R. S. (2012). Multiobjective design of groundwater monitoring network under epistemic uncertainty. *Water Resources Management*, 26(7):1809–1825.