Module 03: Groundwater Hydraulics

Unit 05: Unsteady Flow in Unconfined Aquifer using FVM

Anirban Dhar

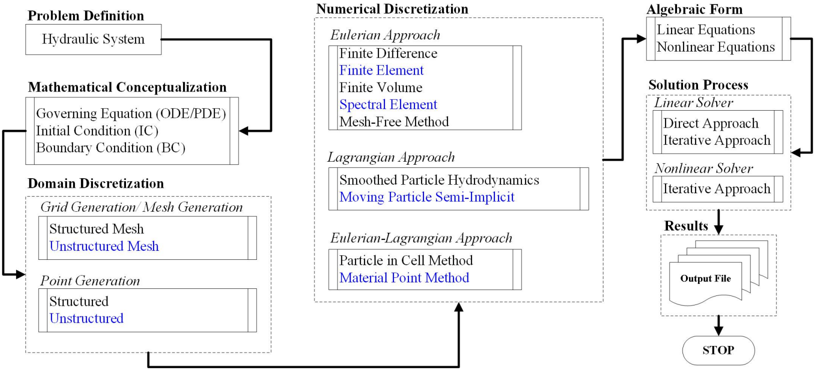
Department of Civil Engineering
Indian Institute of Technology Kharagpur, Kharagpur

National Programme for Technology Enhanced Learning (NPTEL)

Learning Objective

- To solve unsteady two dimensional groundwater flow in unconfined aquifer using Finite Volume Method.

Problem Definition to Solution



Problem Definition

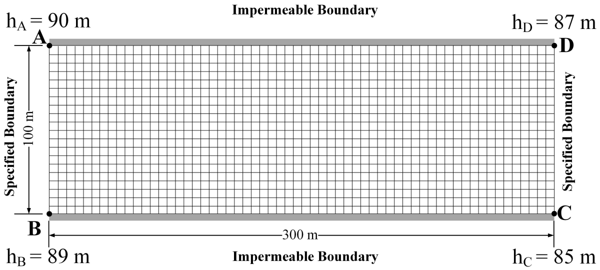


Figure: Homogeneous Isotropic System (Unconfined Aquifer)

Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega : \quad S_y \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial h}{\partial y}\right) + W$$

$S_y = 0.25$
$K = 20\ m/day$

Problem Definition

subject to

Initial Condition

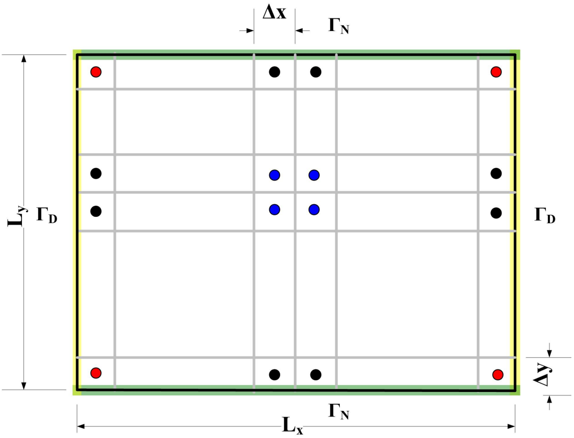
$$h(x,y,0) = h_0(x,y)$$

and

Boundary Condition

$$\begin{aligned}
\Gamma_D^1 &: \quad h(0,y,t) = h_1(y) \\
\Gamma_D^2 &: \quad h(L_x,y,t) = h_2(y) \\
\Gamma_N^3 &: \quad \frac{\partial h}{\partial y}\bigg|_{(x,0,t)} = 0 \\
\Gamma_N^4 &: \quad \frac{\partial h}{\partial y}\bigg|_{(x,L_y,t)} = 0
\end{aligned}$$

Domain Discretization



Discretization
Governing Equation

In Finite Volume Method, the governing equation is integrated over the element volume (in space) and time interval to form the discretized equation at node Point P.

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P} S_y \frac{\partial h}{\partial t} d\Omega\right] dt = \int_{t}^{t+\Delta t}\left[\int_{\Omega_P} \nabla \cdot \mathbf{F} d\Omega\right] dt + \int_{t}^{t+\Delta t}\left[\int_{\Omega_P} W d\Omega\right] dt$$

with

$$\begin{aligned}
\mathbf{F} &= [f_x \quad f_y] \\
f_x &= K_x h \frac{\partial h}{\partial x} \\
f_y &= K_y h \frac{\partial h}{\partial y}
\end{aligned}$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t} \left[\int_{\Omega_P} S_y \frac{\partial h}{\partial t} d\Omega \right] dt$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t} \left[\int_{\Omega_P} S_y \frac{\partial h}{\partial t} d\Omega \right] dt$$

$$= S_y \int_{t}^{t+\Delta t} \frac{\partial}{\partial t} \left(\int_{\Omega_P} h d\Omega \right) dt$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P} S_y \frac{\partial h}{\partial t} d\Omega\right] dt$$

$$= S_y \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(\int_{\Omega_P} h d\Omega\right) dt$$

$$= S_y \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(h_P \Delta\Omega_P\right) dt$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P} S_y \frac{\partial h}{\partial t} d\Omega\right] dt$$

$$= S_y \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(\int_{\Omega_P} h d\Omega\right) dt$$

$$= S_y \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(h_P \Delta\Omega_P\right) dt$$

$$= S_y (h_P^{l+1} - h_P^{l}) \Delta\Omega_P$$

Discretization

Governing Equation: Temporal Term

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P} S_y \frac{\partial h}{\partial t} d\Omega\right] dt$$

$$= S_y \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(\int_{\Omega_P} h d\Omega\right) dt$$

$$= S_y \int_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left(h_P \Delta\Omega_P\right) dt$$

$$= S_y (h_P^{l+1} - h_P^{l}) \Delta\Omega_P$$

$$= S_y (h_P^{l+1} - h_P^{l}) \Delta x \Delta y$$

Discretization

Governing Equation: Spatial Term

$$\int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot\mathbf{F}d\Omega\; dt = \int_{t}^{t+\Delta t}\int_{\Omega^P} \nabla\cdot\left(f_x\hat{i}+f_y\hat{j}\right)d\Omega\; dt$$

$$= \left[(f_x)_e^{l+1}A_{xe} - (f_x)_w^{l+1}A_{xw} + (f_y)_n^{l+1}A_{yn} - (f_y)_s^{l+1}A_{ys}\right]\Delta t$$

$$= \left[\left(K_x h\frac{\partial h}{\partial x}\right)_e^{l+1}A_{xe} - \left(K_x h\frac{\partial h}{\partial x}\right)_w^{l+1}A_{xw} + \left(K_y h\frac{\partial h}{\partial y}\right)_n^{l+1}A_{yn} - \left(K_y h\frac{\partial h}{\partial y}\right)_s^{l+1}A_{ys}\right]\Delta t$$

Discretization

Governing Equation

In a uniform grid system,

$$\begin{aligned} A_{xe} = A_{xw} = \Delta y \\ A_{yn} = A_{ys} = \Delta x \end{aligned} \tag{1}$$

Discretization
Governing Equation

In a uniform grid system,

$$A_{xe} = A_{xw} = \Delta y$$
$$A_{yn} = A_{ys} = \Delta x \tag{1}$$

Source Term:

$$\int_{t}^{t+\Delta t} \int_{\Omega^P} W(x,y) d\Omega \; dt = W(x_P, y_P) \Delta x \Delta y \Delta t \tag{2}$$

Discretization

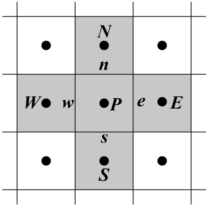
Governing Equation

Compact Form of the equation can be written as,

$$
\begin{aligned}
& S_y(h_P^{l+1} - h_P^l)\Delta x \Delta y \\
& = \left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} \Delta y - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1} \Delta y\right] \Delta t \\
& + \left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} \Delta x - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1} \Delta x\right] \Delta t \\
& + W(x_P, y_P)\Delta x \Delta y \Delta t
\end{aligned}
$$

Discretization

Governing Equation: Interior Cells

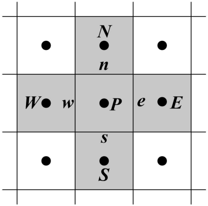


In a uniform grid system for interior cells,
East Face:

$$\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} = K_{xe} h_e \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} = K_{xe} \frac{h_E^{l+1} + h_P^{l+1}}{2} \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x}$$

Discretization

Governing Equation: Interior Cells



In a uniform grid system for interior cells,

East Face:

$$\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} = K_{xe} h_e \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} = K_{xe} \frac{h_E^{l+1} + h_P^{l+1}}{2} \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x}$$

West Face:

$$\left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1} = K_{xw} h_w \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x} = K_{xw} \frac{h_P^{l+1} + h_W^{l+1}}{2} \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

Discretization

Governing Equation: Interior Cells

North Face:

$$\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} = K_{yn} h_n \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} = K_{yn} \frac{h_N^{l+1} + h_P^{l+1}}{2} \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y}$$

Discretization

Governing Equation: Interior Cells

North Face:

$$\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} = K_{yn} h_n \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} = K_{yn} \frac{h_N^{l+1} + h_P^{l+1}}{2} \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y}$$

South Face:

$$\left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1} = K_{ys} h_s \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y} = K_{ys} \frac{h_P^{l+1} + h_S^{l+1}}{2} \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Discretization

Governing Equation: Interior Cells

North Face:

$$\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} = K_{yn} h_n \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} = K_{yn} \frac{h_N^{l+1} + h_P^{l+1}}{2} \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y}$$

South Face:

$$\left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1} = K_{ys} h_s \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y} = K_{ys} \frac{h_P^{l+1} + h_S^{l+1}}{2} \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Compact Form of the equation can be written as,

$$\begin{aligned}
& S_y(h_P^{l+1} - h_P^l)\Delta x \Delta y \\
&= \left[\frac{K_{xe}}{2} \frac{(h_E^{l+1})^2 - (h_P^{l+1})^2}{\Delta x} \Delta y - \frac{K_{xw}}{2} \frac{(h_P^{l+1})^2 - (h_W^{l+1})^2}{\Delta x} \Delta y\right] \Delta t \\
&+ \left[\frac{K_{yn}}{2} \frac{(h_N^{l+1})^2 - (h_P^{l+1})^2}{\Delta y} \Delta x - \frac{K_{ys}}{2} \frac{(h_P^{l+1})^2 - (h_S^{l+1})^2}{\Delta y} \Delta x\right] \Delta t \\
&+ W(x_P, y_P)\Delta x \Delta y \Delta t
\end{aligned}$$

Discretization

Governing Equation: Interior Cells

Compact Form of the equation can be written as,

$$
\begin{aligned}
& h_P^{l+1} - h_P^l \\
& = \frac{K_x \Delta t}{2S_y} \frac{(h_E^{l+1})^2 - (h_P^{l+1})^2}{\Delta x^2} - \frac{K_x \Delta t}{2S_y} \frac{(h_P^{l+1})^2 - (h_W^{l+1})^2}{\Delta x^2} \\
& + \frac{K_y \Delta t}{2S_y} \frac{(h_N^{l+1})^2 - (h_P^{l+1})^2}{\Delta y^2} - \frac{K_y \Delta t}{2S_y} \frac{(h_P^{l+1})^2 - (h_S^{l+1})^2}{\Delta y^2} \\
& + \frac{W(x_P, y_P)}{S_y} \Delta t
\end{aligned}
$$

In simplified form, this can be written as,

$$
\begin{aligned}
\alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 - [2(\alpha_x + \alpha_y)] (h_P^{l+1})^2 - h_P^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 = \\
- h_P^l - \frac{W(x_P, y_P)\Delta t}{S_y}
\end{aligned}
$$

with

$$
\alpha_x = \frac{K_x \Delta t}{2S_y \Delta x^2} \quad \alpha_y = \frac{K_y \Delta t}{2S_y \Delta y^2}
$$

Function and Jacobian

In the form of function discretized form can be written as,

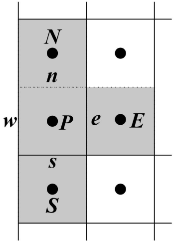
$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_y(h_S^{l+1})^2 + \alpha_x(h_W^{l+1})^2 - [2(\alpha_x+\alpha_y)]\,(h_P^{l+1})^2 - h_P^{l+1}$$
$$+\,\alpha_x(h_E^{l+1})^2 + \alpha_y(h_N^{l+1})^2 - \left[-h_P^l - \frac{W(x_P, y_P)\Delta t}{S_y}\right] = 0$$

Elements of Jacobian matrix can be calculated as

$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$
$$J_W^m = \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1}$$
$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -1 - 4(\alpha_x+\alpha_y)h_P^{l+1}$$
$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = 2\alpha_x h_E^{l+1}$$
$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

Boundary Conditions

Left Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x \frac{\left(h_E^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta x^2} - K_x h_{BW}^{l+1} \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x^2}\right] + \left[K_y \frac{\left(h_N^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta y^2} - K_y \frac{\left(h_P^{l+1}\right)^2 - \left(h_S^{l+1}\right)^2}{2\Delta y^2}\right]$$

In simplified form, this can be written as

$$\alpha_y (h_S^{l+1})^2 - \left[\alpha_x + 2\alpha_y\right](h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BW}^{l+1}\right] h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 = -h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2$$

Function and Jacobian

In the form of function discretized form can be written as,

$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_y (h_S^{l+1})^2 - [\alpha_x + 2\alpha_y]\,(h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BW}^{l+1}\right] h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2\right] = 0$$

Elements of Jacobian matrix can be calculated as

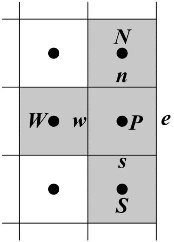
$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$

$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -\left[1 + 6\alpha_x h_{BW}^{l+1}\right] - 2\,[\alpha_x + 2\alpha_y]\,h_P^{l+1}$$

$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = \frac{2}{3}\alpha_x h_{BW}^{l+1} + 2\alpha_x h_E^{l+1}$$

$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

Boundary Conditions
Right Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x h_{BE}^{l+1} \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x^2} - K_x \frac{\left(h_P^{l+1}\right)^2 - \left(h_W^{l+1}\right)^2}{2\Delta x^2}\right] + \left[K_y \frac{\left(h_N^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta y^2} - K_y \frac{\left(h_P^{l+1}\right)^2 - \left(h_S^{l+1}\right)^2}{2\Delta y^2}\right]$$

In simplified form, this can be written as

$$\alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - \left[\alpha_x + 2\alpha_y\right] (h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BE}^{l+1}\right] h_P^{l+1} + \alpha_y (h_N^{l+1})^2 = -h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2$$

Function and Jacobian

In the form of function discretized form can be written as,

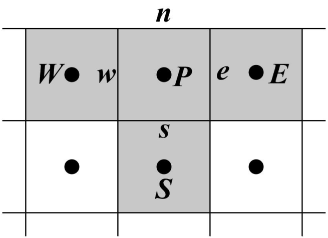
$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_y(h_S^{l+1})^2 + \alpha_x(h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - [\alpha_x + 2\alpha_y]\,(h_P^{l+1})^2$$
$$- \left[1 + 6\alpha_x h_{BE}^{l+1}\right] h_P^{l+1} + \alpha_y(h_N^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x(h_{BE}^{l+1})^2\right] = 0$$

Elements of Jacobian matrix can be calculated as

$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$
$$J_W^m = \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1} + \frac{2}{3}\alpha_x h_{BE}^{l+1}$$
$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -\left[1 + 6\alpha_x h_{BE}^{l+1}\right] - 2\,[\alpha_x + 2\alpha_y]\,h_P^{l+1}$$
$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

Boundary Conditions

Top Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0 \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x \frac{\left(h_E^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta x^2} - K_x \frac{\left(h_P^{l+1}\right)^2 - \left(h_W^{l+1}\right)^2}{2\Delta x^2}\right] + \left[0 - K_y \frac{\left(h_P^{l+1}\right)^2 - \left(h_S^{l+1}\right)^2}{2\Delta y^2}\right]$$

In simplified form, this can be written as

$$\alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 - \left[2\alpha_x + \alpha_y\right](h_P^{l+1})^2 - h_P^{l+1} + \alpha_x (h_E^{l+1})^2 = -h_P^l$$

Function and Jacobian

In the form of function discretized form can be written as,

$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 - [2\alpha_x + \alpha_y]\,(h_P^{l+1})^2 - h_P^{l+1} + \alpha_x (h_E^{l+1})^2 - \left[-h_P^l\right] = 0$$

Elements of Jacobian matrix can be calculated as

$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$

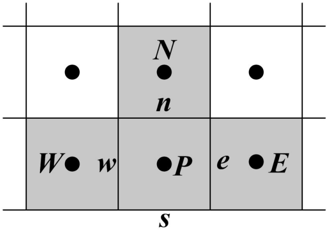
$$J_W^m = \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1}$$

$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -1 - 2\,[2\alpha_x + \alpha_y]\,h_P^{l+1}$$

$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = 2\alpha_x h_E^{l+1}$$

Boundary Conditions

Bottom Boundary



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_{BS}^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x \frac{\left(h_E^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta x^2} - K_x \frac{\left(h_P^{l+1}\right)^2 - \left(h_W^{l+1}\right)^2}{2\Delta x^2}\right] + \left[K_y \frac{\left(h_N^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta y^2} - 0\right]$$

In simplified form, this can be written as

$$\alpha_x (h_W^{l+1})^2 - \left[2\alpha_x + \alpha_y\right](h_P^{l+1})^2 - h_P^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 = -h_P^l$$

Function and Jacobian

In the form of function discretized form can be written as,

$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_x (h_W^{l+1})^2 - [2\alpha_x + \alpha_y]\,(h_P^{l+1})^2 - h_P^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 - \left[-h_P^l\right] = 0$$

Elements of Jacobian matrix can be calculated as

$$J_W^m = \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1}$$

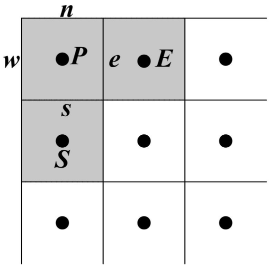
$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -1 - 2\,[2\alpha_x + \alpha_y]\,h_P^{l+1}$$

$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = 2\alpha_x h_E^{l+1}$$

$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

Boundary Conditions

N-W Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \quad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0 \quad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x \frac{\left(h_E^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta x^2} - K_x h_{BW}^{l+1} \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x^2}\right] + \left[0 - K_y \frac{\left(h_P^{l+1}\right)^2 - \left(h_S^{l+1}\right)^2}{2\Delta y^2}\right]$$

In simplified form, this can be written as

$$\alpha_y (h_S^{l+1})^2 - [\alpha_x + \alpha_y]\,(h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BW}^{l+1}\right] h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1} + \alpha_x (h_E^{l+1})^2 = -h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2$$

Function and Jacobian

In the form of function discretized form can be written as,

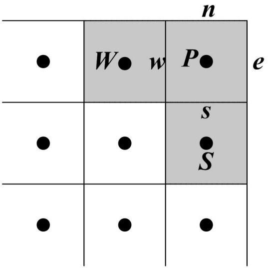
$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_y (h_S^{l+1})^2 - [\alpha_x + \alpha_y](h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BW}^{l+1}\right] h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1}$$
$$+ \alpha_x (h_E^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2\right] = 0$$

Elements of Jacobian matrix can be calculated as

$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$
$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -\left[1 + 6\alpha_x h_{BW}^{l+1}\right] - 2[\alpha_x + \alpha_y] h_P^{l+1}$$
$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = \frac{2}{3}\alpha_x h_{BW}^{l+1} + 2\alpha_x h_E^{l+1}$$

Boundary Conditions

N-E Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0 \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x h_{BE}^{l+1} \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x^2} - K_x \frac{\left(h_P^{l+1}\right)^2 - \left(h_W^{l+1}\right)^2}{2\Delta x^2}\right] + \left[0 - K_y \frac{\left(h_P^{l+1}\right)^2 - \left(h_S^{l+1}\right)^2}{2\Delta y^2}\right]$$

In simplified form, this can be written as

$$\alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - \left[\alpha_x + \alpha_y\right](h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BE}^{l+1}\right] h_P^{l+1} = -h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2$$

Function and Jacobian

In the form of function discretized form can be written as,

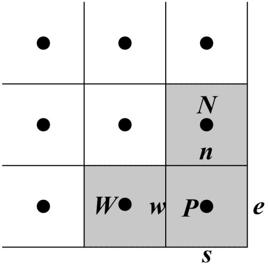
$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - [\alpha_x + \alpha_y]\,(h_P^{l+1})^2$$
$$- \left[1 + 6\alpha_x h_{BE}^{l+1}\right] h_P^{l+1} - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2\right] = 0$$

Elements of Jacobian matrix can be calculated as

$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$
$$J_W^m = \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1} + \frac{2}{3}\alpha_x h_{BE}^{l+1}$$
$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -\left[1 + 6\alpha_x h_{BE}^{l+1}\right] - 2\,[\alpha_x + \alpha_y]\, h_P^{l+1}$$

Boundary Conditions

S-E Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_{BS}^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x h_{BE}^{l+1} \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x^2} - K_x \frac{\left(h_P^{l+1}\right)^2 - \left(h_W^{l+1}\right)^2}{2\Delta x^2}\right] + \left[K_y \frac{\left(h_N^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta y^2} - 0\right]$$

In simplified form, this can be written as

$$\alpha_x (h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - \left[\alpha_x + \alpha_y\right](h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BE}^{l+1}\right] h_P^{l+1} + \alpha_y (h_N^{l+1})^2 = -h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2$$

Function and Jacobian

In the form of function discretized form can be written as,

$$F_m\left(\mathbf{h^{l+1}}\right) = \alpha_x (h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - [\alpha_x + \alpha_y]\,(h_P^{l+1})^2 \\ - \left[1 + 6\alpha_x h_{BE}^{l+1}\right] h_P^{l+1} + \alpha_y (h_N^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2\right] = 0$$

Elements of Jacobian matrix can be calculated as

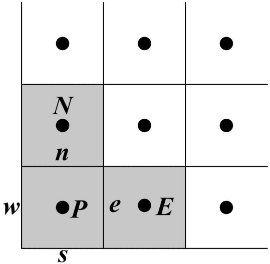
$$J_W^m = \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1} + \frac{2}{3}\alpha_x h_{BE}^{l+1}$$

$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -\left[1 + 6\alpha_x h_{BE}^{l+1}\right] - 2\,[\alpha_x + \alpha_y]\, h_P^{l+1}$$

$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

Boundary Conditions

S-W Corner



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_{BS}^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

Implicit Scheme

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \frac{1}{\Delta x}\left[\left(K_x h \frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h \frac{\partial h}{\partial x}\right)_w^{l+1}\right] + \frac{1}{\Delta y}\left[\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = \left[K_x \frac{\left(h_E^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta x^2} - K_x h_{BW}^{l+1} \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x^2}\right] + \left[K_y \frac{\left(h_N^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{2\Delta y^2} - 0\right]$$

In simplified form, this can be written as

$$-\left[\alpha_x + \alpha_y\right](h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BW}^{l+1}\right] h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 = -h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2$$

Function and Jacobian

In the form of function discretized form can be written as,

$$F_m\left(\mathbf{h^{l+1}}\right) = -\left[\alpha_x + \alpha_y\right](h_P^{l+1})^2 - \left[1 + 6\alpha_x h_{BW}^{l+1}\right] h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1}$$
$$+ \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2\right] = 0$$

Elements of Jacobian matrix can be calculated as

$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -\left[1 + 6\alpha_x h_{BW}^{l+1}\right] - 2\left[\alpha_x + \alpha_y\right] h_P^{l+1}$$
$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = \frac{2}{3}\alpha_x h_{BW}^{l+1} + 2\alpha_x h_E^{l+1}$$
$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

General Form

In general form, the governing equation including boundary conditions can be written as,

$$J_S^m \Delta h_S^{l+1} + J_W^m \Delta h_W^{l+1} + J_P^m \Delta h_P^{l+1} + J_E^m \Delta h_E^{l+1} + J_N^m \Delta h_N^{l+1} = -F_m \left(\mathbf{h^{l+1}}\right)$$

General Form

In general form, the governing equation including boundary conditions can be written as,

$$J_S^m \Delta h_S^{l+1} + J_W^m \Delta h_W^{l+1} + J_P^m \Delta h_P^{l+1} + J_E^m \Delta h_E^{l+1} + J_N^m \Delta h_N^{l+1} = -F_m \left(\mathbf{h}^{\mathbf{l+1}}\right)$$

Iterative form can be written as

$$\begin{aligned} & J_S^m \left[h_S^{l+1}\Big|^{(p)} - h_S^{l+1}\Big|^{(p-1)} \right] + J_W^m \left[h_W^{l+1}\Big|^{(p)} - h_W^{l+1}\Big|^{(p-1)} \right] + J_P^m \left[h_P^{l+1}\Big|^{(p)} - h_P^{l+1}\Big|^{(p-1)} \right] \\ & + J_E^m \left[h_E^{l+1}\Big|^{(p)} - h_E^{l+1}\Big|^{(p-1)} \right] + J_N^m \left[h_N^{l+1}\Big|^{(p)} - h_N^{l+1}\Big|^{(p-1)} \right] \\ & = -F_m \left(h_S^{l+1}\Big|^{(p)}, h_W^{l+1}\Big|^{(p)}, h_P^{l+1}\Big|^{(p-1)}, h_E^{l+1}\Big|^{(p-1)}, h_N^{l+1}\Big|^{(p-1)} \right) \end{aligned}$$

General Form

In general form, the governing equation including boundary conditions can be written as,

$$J_S^m \Delta h_S^{l+1} + J_W^m \Delta h_W^{l+1} + J_P^m \Delta h_P^{l+1} + J_E^m \Delta h_E^{l+1} + J_N^m \Delta h_N^{l+1} = -F_m \left(\mathbf{h^{l+1}}\right)$$

Iterative form can be written as

$$\begin{aligned}
& J_S^m \left[h_S^{l+1}\Big|^{(p)} - h_S^{l+1}\Big|^{(p-1)} \right] + J_W^m \left[h_W^{l+1}\Big|^{(p)} - h_W^{l+1}\Big|^{(p-1)} \right] + J_P^m \left[h_P^{l+1}\Big|^{(p)} - h_P^{l+1}\Big|^{(p-1)} \right] \\
& + J_E^m \left[h_E^{l+1}\Big|^{(p)} - h_E^{l+1}\Big|^{(p-1)} \right] + J_N^m \left[h_N^{l+1}\Big|^{(p)} - h_N^{l+1}\Big|^{(p-1)} \right] \\
& = -F_m \left(h_S^{l+1}\Big|^{(p)}, h_W^{l+1}\Big|^{(p)}, h_P^{l+1}\Big|^{(p-1)}, h_E^{l+1}\Big|^{(p-1)}, h_N^{l+1}\Big|^{(p-1)} \right)
\end{aligned}$$

Final iterative form can be written as

$$h_P^{l+1}\Big|^{(p)} = h_P^{l+1}\Big|^{(p-1)} + \frac{Res}{J_P^m}$$

with

$$Res = -F_m - \left[J_S^m \Delta h_S^{l+1}\Big|^{(p)} + J_W^m \Delta h_W^{l+1}\Big|^{(p)} + J_E^m \Delta h_E^{l+1}\Big|^{(p-1)} + J_N^m \Delta h_N^{l+1}\Big|^{(p-1)} \right]$$

Source Code

Unsteady Two Dimensional Unconfined Groundwater Flow with Finite Volume Method

- Without coefficient matrix using Gauss Seidel
  - unsteady_2D_fvm_unconf_implicit_iterative.sci

Thank You