Module 02: Numerical Methods

Unit 14: Finite Volume Method: Godunov Approach

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Learning Objective

- To discretize conservation laws using Godunov method.

Governing Equation

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \tag{1}$$

where

$\mathcal{F}_\phi$ = Flux function.

S_ϕ = Source term.

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S_ϕ = Source term.

Let us consider that the flux term can be written as,

$$\mathcal{F}_\phi = a\phi$$

where a is constant.

Godunov Method

REA Algorithm (LeVeque, 2002)

Reconstruct-Evolve-Average

- Reconstruct a piecewise polynomial from cell average value ϕ_P^n as

$$\tilde{\phi}^n(x, t^n) = \phi_P^n \quad \forall x \in [x_w, x_e)$$

Godunov Method

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- Evolve the hyperbolic equation with base condition to obtain $\tilde{\phi}^n(x, t + \Delta t)$ at future time $t + \Delta t$.

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- Evolve the hyperbolic equation with base condition to obtain $\tilde{\phi}^n(x, t+\Delta t)$ at future time $t+\Delta t$.
- Average the polynomical function at cell level to obtain cell average value at future time $t+\Delta t$ as

$$\phi_P^{n+1} = \frac{1}{\Delta x}\int_{x_w}^{x_e} \tilde{\phi}(x,t^{n+1})dx$$

Godunov Method

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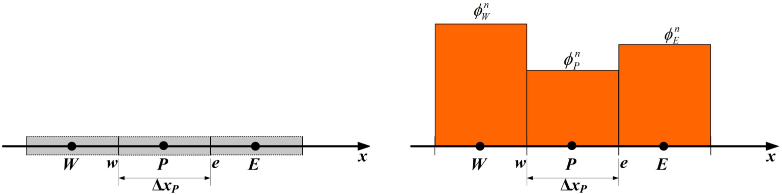
- Average the polynomical function at cell level to obtain cell average value at future time $t + \Delta t$ as

$$\phi_P^{n+1} = \frac{1}{\Delta x} \int\limits_{x_w}^{x_e} \tilde{\phi}(x, t^{n+1}) dx$$

Steps are repeated at every time level.

$\tilde{\phi}^n(x, t_n)$ is constant over time interval $t_n < t < t_{n+1}$

Riemann Problem
Conservative Form



Riemann Problem

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = 0$$

$$\phi(x,t) = \begin{cases} \phi_P^n & \text{if} \quad x < x_e \\ \phi_E^n & \text{if} \quad x > x_e \end{cases}$$

Godunov Method

$\mathcal{F}_\phi\left(\tilde{\phi}(x,t)\right)$ at cell face depends on the exact solution $\tilde{\phi}(x,t)$ of the Riemann problem along the taxis. Considering local coordinates

$$\tilde{\phi}(x,t) = \phi_e\left(\frac{x-x_e}{t-t^n}\right), \quad x_P \le x \le x_E, \quad t^n \le t \le t^{n+1}$$

Godunov Method

From Riemann problems:

$\tilde{\phi}(x_w, t) = \phi_w\left(\frac{x_w - x_w}{t - t^n}\right) = \phi_w(0)$ with $\quad t^n \leq t \leq t^{n+1}$

Godunov Method

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$\tilde{\phi}(x_w, t) = \phi_w\left(\frac{x_w - x_w}{t - t^n}\right) = \phi_w(0)$ with $\quad t^n \leq t \leq t^{n+1}$

$\tilde{\phi}(x_e, t) = \phi_e\left(\frac{x_e - x_e}{t - t^n}\right) = \phi_e(0)$ with $\quad t^n \leq t \leq t^{n+1}$

Godunov Method

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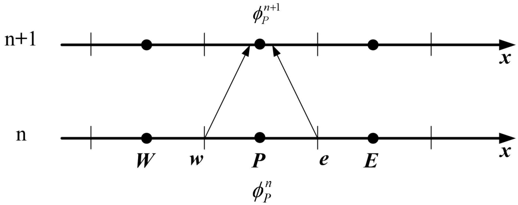

Godunov Method

Numerical flux values can be written as

$$\bar{\mathcal{F}}_\phi(x_e,t) = \frac{1}{\Delta t}\int\limits_{t^n}^{t^{n+1}} \mathcal{F}_\phi\left(\tilde{\phi}(x_e,t)\right)dt = \frac{1}{\Delta t}\int\limits_{t^n}^{t^{n+1}} \mathcal{F}_\phi\left(\phi_e(0)\right)dt = \mathcal{F}_\phi\left(\phi_e(0)\right)$$

$$\bar{\mathcal{F}}_\phi(x_w,t) = \frac{1}{\Delta t}\int\limits_{t^n}^{t^{n+1}} \mathcal{F}_\phi\left(\tilde{\phi}(x_w,t)\right)dt = \frac{1}{\Delta t}\int\limits_{t^n}^{t^{n+1}} \mathcal{F}_\phi\left(\phi_w(0)\right)dt = \mathcal{F}_\phi\left(\phi_w(0)\right)$$

Godunov Method

If $\mathcal{F}_\phi = a\phi$, then numerical flux can be written as,

$$\mathcal{F}_\phi\left(\phi_e(0)\right) = a^- \phi_E^n + a^+ \phi_P^n$$
$$\mathcal{F}_\phi\left(\phi_w(0)\right) = a^- \phi_P^n + a^+ \phi_W^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

Godunov Method

If $\mathcal{F}_\phi = a\phi$, then numerical flux can be written as,

$$\mathcal{F}_\phi\left(\phi_e(0)\right) = a^-\phi_E^n + a^+\phi_P^n$$
$$\mathcal{F}_\phi\left(\phi_w(0)\right) = a^-\phi_P^n + a^+\phi_W^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x}\left[a^-(\phi_E^n - \phi_P^n) + a^+(\phi_P^n - \phi_W^n)\right]$$

This is same as first order upwind approach.

Thank You

References

Guinot, V. (2010). *Scalar Hyperbolic Conservation Laws in One Dimension of Space*, pages 1–53. ISTE.

LeVeque, R. J. (2002). *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press, Cambridge.