Module 02: Numerical Methods

Unit 03: Ordinary Differential Equation: IVP

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Learning Objective

- To discretize first-order ordinary differential equation (ODE) along with Initial Condition (IC).

Introduction

- Ordinary Differential Equation with initial condition can be solved as Initial Value Problem with time/ time-like discretization.

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- ODE can be solved by using Finite Difference approach.
- Accuracy of the solution depends only on discretization of ODE.

General Structure of IVP

In general, first order ODE with dependent variable ϕ can be written as

$$\frac{d\phi}{dt} = \Psi(t, \phi)$$

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In general, first order ODE with dependent variable ϕ can be written as

$$\frac{d\phi}{dt} = \Psi(t, \phi)$$

subject to the initial condition

$$\phi(t_0) = \phi_0$$

where
$\Psi() =$ a general function

Numerical Discretization (Sengupta, 2013)

Integrating both sides of ODE from t_n to t_{n+1}

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt}\, dt = \int_{t_n}^{t_{n+1}} \Psi(t, \phi)\, dt$$

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Using Mean Value Theorem to evaluate the RHS of the above equation,

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_n + \theta\Delta t, \phi(t_n + \theta\Delta t))$$

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Using Mean Value Theorem to evaluate the RHS of the above equation,

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where $0 < \theta < 1$.
Different values of θ and evaluation of $\Psi(t_n + \theta \Delta t, \phi(t_n + \theta \Delta t))$ yields different numerical methods.

Truncation Error Analysis

The function ϕ at t_{n+1} can be expanded as

$$\phi(t_{n+1}) = \phi(t_n) + \underbrace{\Delta t\phi'(t_n) + \cdots + \frac{\Delta t^p}{p!}\phi^{(p)}(t_n)}_{\Delta t\Psi(t_n,\phi(t_n),\Delta t)} + \frac{\Delta t^{(p+1)}}{(p+1)!}\phi^{(p+1)}(t_n + \theta\Delta t)$$

where $0 < \theta < 1$.

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where $0 < \theta < 1$.
Thus the equation can be written as,

$$\phi(t_{n+1}) = \phi(t_n) + \Delta t\Psi(t_n, \phi(t_n), \Delta t) + \underbrace{\frac{\Delta t^{(p+1)}}{(p+1)!}\phi^{(p+1)}(t_n + \theta\Delta t)}_{\text{Truncation Error}}$$

Euler Method

For $\theta = 0$, we can write

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Or,

$$\phi(t_{n+1}) = \phi(t_n) + \Delta t \phi'(t_n) + \mathcal{O}(\Delta t^2)$$

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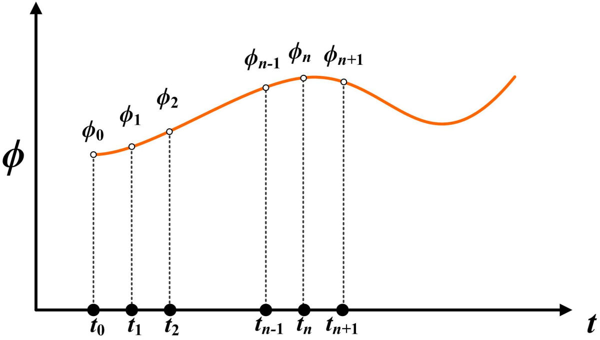
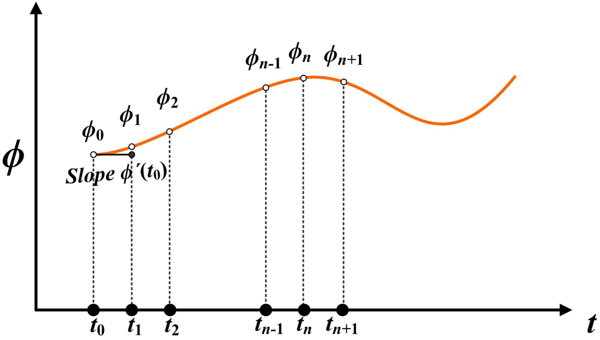
Order of Euler's method: $\mathcal{O}(\Delta t)$

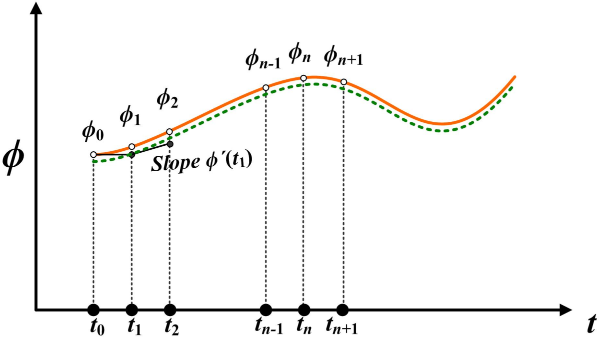
Euler Method



Euler Method



Euler Method



Backward Euler Method

For $\theta = 1$, we can write

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The function ϕ at t_n can be expanded as

$$\phi(t_n) = \phi(t_{n+1}) - \Delta t \phi'(t_{n+1}) + \underbrace{\frac{\Delta t^2}{2!}\phi''(t_{n+1})}_{\text{Leading Error}}$$

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Or,

$$\phi(t_{n+1}) = \phi(t_n) + \Delta t \phi'(t_{n+1}) + \mathcal{O}(\Delta t^2)$$

Order of Backward Euler method: $\mathcal{O}(\Delta t)$

Modified Euler Method

For $\theta = \frac{1}{2}$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \Psi \left[t_n + \frac{\Delta t}{2}, \phi\left(t_n + \frac{\Delta t}{2} \right) \right]$$

Modified Euler Method

For $\theta = \frac{1}{2}$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \Psi \left[t_n + \frac{\Delta t}{2}, \phi \left(t_n + \frac{\Delta t}{2} \right) \right]$$

$t_n + \frac{\Delta t}{2}$ is not a node and various approximations are possible.

Modified Euler Method

First Approach

If we evaluate $\phi(t_n + \frac{\Delta t}{2})$ by the Euler method, .i.e.,

$$\phi\left(t_n + \frac{\Delta t}{2}\right) = \phi\left(t_n\right) + \frac{\Delta t}{2}\Psi(t_n, \phi^n)$$

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In the next step,

$$\phi^{n+1} = \phi^n + \Delta t \Psi\left[t_n + \frac{\Delta t}{2}, \phi\left(t_n\right) + \frac{\Delta t}{2}\Psi(t_n, \phi^n)\right]$$

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In simplified form

$$\phi^{n+1} = \phi^n + K_2 + \mathcal{O}(\Delta t^3)$$

with $K_2 = \Delta t \Psi(t_n + \frac{\Delta t}{2}, \phi^n + \frac{1}{2}K_1)$ and $K_1 = \Delta t \Psi^n$.

Euler-Cauchy method
Second Approach

Using averaging approach,

$$\phi'\left(t_n + \frac{\Delta t}{2}\right) = \frac{1}{2}\left[\phi'(t_n) + \phi'(t_n + \Delta t)\right]$$

Euler-Cauchy method

Second Approach

Using averaging approach,

$$\phi'\left(t_n + \frac{\Delta t}{2}\right) = \frac{1}{2}\left[\phi'(t_n) + \phi'(t_n + \Delta t)\right]$$

With Euler approximation,

$$\phi'\left(t_n + \frac{\Delta t}{2}\right) = \frac{1}{2}\left[\Psi(t_n, \phi^n) + \Psi(t_{n+1}, \phi^n + \Delta t \Psi^n)\right]$$

Euler-Cauchy method

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The equation can be written as,

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}\left[\Psi(t_n, \phi^n) + \Psi(t_{n+1}, \phi^n + \Delta t \Psi^n)\right]$$

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In simplified form,

$$\phi^{n+1} = \phi^n + \frac{1}{2}\left[K_1 + K_2\right] + \mathcal{O}(\Delta t^3)$$

with $K_2 = \Delta t \Psi(t_{n+1}, \phi^n + K_1)$ and $K_1 = \Delta t \Psi^n$.

Runge-Kutta Methods

Individual m increments are defined as,

$$
\begin{aligned}
K_1 &= \Delta t \Psi(t_n, \phi^n) \\
K_2 &= \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\
K_3 &= \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\
&\vdots \\
K_m &= \Delta t \Psi(t_n + c_m \Delta t, \phi^n + a_{m1} K_1 + a_{m2} K_2 + \cdots + a_{m,m-1} K_{m-1})
\end{aligned}
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\end{aligned}
$$

The Runge-Kutta method is defined as weighted assembly of increments by,

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + \cdots + W_m K_m$$

The parameters c_j, a_{ij} and W_j can be obtained by matching the corresponding expansions with Taylor Series.

Second Order RK Method

In general terms, second order RK can be defined as,

$$
\begin{aligned}
K_1 &= \Delta t \Psi(t_n, \phi^n) \\
K_2 &= \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\
\phi^{n+1} &= \phi^n + W_1 K_1 + W_2 K_2
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\phi^{n+1} &= \phi^n + W_1 K_1 + W_2 K_2
\end{aligned}
$$

RK_2 can be presented as,

$$
\begin{aligned}
K_1 &= \Delta t \Psi(t_n, \phi^n) \\
K_2 &= \Delta t \Psi(t_n + \frac{2}{3} \Delta t, \phi^n + \frac{2}{3} K_1) \\
\phi^{n+1} &= \phi^n + \frac{1}{4}\left[K_1 + 3K_2\right] + \mathcal{O}(\Delta t^3)
\end{aligned}
$$

Third Order RK Method

In general terms, third order RK can be defined as,

$$
\begin{aligned}
K_1 &= \Delta t \Psi(t_n, \phi^n) \\
K_2 &= \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\
K_3 &= \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\
\phi^{n+1} &= \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3
\end{aligned}
$$

Third Order RK Method

In general terms, third order RK can be defined as,

$$\begin{aligned} K_1 &= \Delta t \Psi(t_n, \phi^n) \\ K_2 &= \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\ K_3 &= \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\ \phi^{n+1} &= \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3 \end{aligned}$$

RK_3 can be presented as,

$$\begin{aligned} K_1 &= \Delta t \Psi(t_n, \phi^n) \\ K_2 &= \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_1) \\ K_3 &= \Delta t \Psi(t_n + \Delta t, \phi^n - K_1 + 2K_2) \\ \phi^{n+1} &= \phi^n + \frac{1}{6} K_1 + \frac{4}{6} K_2 + \frac{1}{6} K_3 + \mathcal{O}(\Delta t^4) \end{aligned}$$

Fourth Order RK Method

In general terms, fourth order RK can be defined as,

$$\begin{aligned}
K_1 &= \Delta t \Psi(t_n, \phi^n) \\
K_2 &= \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\
K_3 &= \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\
K_4 &= \Delta t \Psi(t_n + c_4 \Delta t, \phi^n + a_{41} K_1 + a_{42} K_2 + a_{43} K_3) \\
\phi^{n+1} &= \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3 + W_4 K_4
\end{aligned}$$

Fourth Order RK Method

In general terms, fourth order RK can be defined as,

$$\begin{aligned} K_1 &= \Delta t \Psi(t_n, \phi^n) \\ K_2 &= \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\ K_3 &= \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\ K_4 &= \Delta t \Psi(t_n + c_4 \Delta t, \phi^n + a_{41} K_1 + a_{42} K_2 + a_{43} K_3) \\ \phi^{n+1} &= \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3 + W_4 K_4 \end{aligned}$$

RK_4 can be presented as,

$$\begin{aligned} K_1 &= \Delta t \Psi(t_n, \phi^n) \\ K_2 &= \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_1) \\ K_3 &= \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_2) \\ K_4 &= \Delta t \Psi(t_n + \Delta t, \phi^n + K_3) \\ \phi^{n+1} &= \phi^n + \frac{1}{6} K_1 + \frac{1}{3} K_2 + \frac{1}{3} K_3 + \frac{1}{6} K_4 + \mathcal{O}(\Delta t^5) \end{aligned}$$

Gradually Varied Flow in Open Channel

Ordinary Differential Equation

Initial Value Problem

Governing Equation:

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \tag{1}$$

Gradually Varied Flow in Open Channel

Ordinary Differential Equation

Initial Value Problem

Governing Equation:

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \tag{1}$$

Initial Condition:

$$y|_{x=0} = y_0 \tag{2}$$

$$\Psi(x, y) = \frac{S_0 - S_f}{1 - Fr^2}$$

Thank You

References

Sengupta, T. (2013). *High Accuracy Computing Methods Fluid Flows and Wave Phenomena*.