Module 02: Numerical Methods

Unit 08: Partial Differential Equation: Numerical Stability of One-Dimensional PDE

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Learning Objective

- To analyze the numerical stability of discretized one-dimensional conservation law.

General Equation

A form of differential equation with a general variable ϕ:

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla.(\Upsilon_\phi \phi \mathbf{u}) = \nabla.(\mathbf{\Gamma}_\phi.\nabla\phi) + F_{\phi_o} + S_\phi \quad (1)$$

where

ϕ = general variable

$\Lambda_\phi,\ \Upsilon_\phi$ = problem dependent parameters

$\mathbf{\Gamma}_\phi$ = tensor

F_{ϕ_o} = other forces

S_ϕ = source/sink term

One-dimensional Conservation Law

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \tag{2}$$

where

$\mathcal{F}_\phi$ = Flux function (amount of ϕ that passes at the abscissa x per unit time due to displacement of ϕ). $\mathcal{F}_\phi(\phi, x, t)$ does not depend on derivatives of ϕ w.r.t. space/time.

S_ϕ = Source term (amount of ϕ that appears per unit time per unit volume irrespective of the amount transported via flux).

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For example,

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For example,

$$\mathcal{F}_\phi = u\phi \Rightarrow \textbf{Allowed}$$

$$\mathcal{F}_\phi = -\Gamma_x \frac{\partial \phi}{\partial x} \Rightarrow \textbf{Not Allowed}$$

One-dimensional Conservation Law

Non-Conservative form (Guinot, 2010)

$$\frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial x} = \hat{S}_{\phi} \tag{3}$$

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where

$$\lambda = \frac{\partial \mathcal{F}_\phi}{\partial \phi}$$

$$\hat{S}_\phi = S_\phi - \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_{\phi=\text{constant}}$$

Explicit Upwind Scheme

Conservative Form

Governing Equation

$$\left.\frac{\partial \phi}{\partial t}\right|_i^n + \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_i^n = \left.S_\phi\right|_i^n \tag{4}$$

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Time Discretization

$$\frac{\partial \phi}{\partial t} = \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + \mathcal{O}(\Delta t)$$

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$$\frac{\partial \mathcal{F}_\phi}{\partial x} = \begin{cases} \dfrac{{\mathcal{F}_\phi}_i^n - {\mathcal{F}_\phi}_{i-1}^n}{\Delta x} + \mathcal{O}(\Delta x) & if \quad \lambda_i^n > 0 \\ \dfrac{{\mathcal{F}_\phi}_{i+1}^n - {\mathcal{F}_\phi}_i^n}{\Delta x} + \mathcal{O}(\Delta x) & if \quad \lambda_i^n \leq 0 \end{cases}$$

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Source Term

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Stability Analysis

- The von Neumann Stability analysis can be performed for linear equations. Let us consider that the flux term can be written as,

$$\mathcal{F}_\phi = a\phi$$

where a is constant.Thus,

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- The error can be represented in the form of Fourier Series and single arbitrary term can be written as,

$$\epsilon_i^n = A^n e^{\sqrt{-1} i \omega_x \Delta x}$$

where ω_x is wave number corresponding to x direction.

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- In simplified form, error can be written as,

$$\epsilon_i^n = A^n e^{\sqrt{-1} i \varphi_x}$$

where φ_x is the phase value corresponding to x direction.

Stability Analysis

$$\phi_i^{n+1} = \begin{cases} \phi_i^n + a\dfrac{\Delta t}{\Delta x}(\phi_{i-1}^n - \phi_i^n) + \Delta t {S_\phi}_i^n & if \quad a > 0 \\ \phi_i^n + a\dfrac{\Delta t}{\Delta x}(\phi_i^n - \phi_{i+1}^n) + \Delta t {S_\phi}_i^n & if \quad a < 0 \end{cases}$$

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The discretized governing equation for IVP with explicit scheme can be written as,

$$(\hat{\phi}_i^{n+1} + \epsilon_i^{n+1}) = \begin{cases} (\hat{\phi}_i^n + \epsilon_i^n) + a\dfrac{\Delta t}{\Delta x}[(\hat{\phi}_{i-1}^n + \epsilon_{i-1}^n) - (\hat{\phi}_i^n + \epsilon_i^n)] + \Delta t {S_\phi}_i^n & if \quad a > 0 \\ (\hat{\phi}_i^n + \epsilon_i^n) + a\dfrac{\Delta t}{\Delta x}[(\hat{\phi}_i^n + \epsilon_i^n) - (\hat{\phi}_{i+1}^n + \epsilon_{i+1}^n)] + \Delta t {S_\phi}_i^n & if \quad a < 0 \end{cases}$$

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$$\epsilon_{i+1}^n = A^n e^{\sqrt{-1}(i+1)\varphi_x}$$

Stability Analysis

$$G = \frac{\epsilon_i^{n+1}}{\epsilon_i^n} = \begin{cases} 1 + a\dfrac{\Delta t}{\Delta x}\left(\dfrac{\epsilon_{i-1}^n}{\epsilon_i^n} - 1\right) & if \quad a > 0 \\ 1 + a\dfrac{\Delta t}{\Delta x}\left(1 - \dfrac{\epsilon_{i+1}^n}{\epsilon_i^n}\right) & if \quad a < 0 \end{cases}$$

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where $Cr = |a|\frac{\Delta t}{\Delta x}$.

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$$G = \frac{\epsilon_i^{n+1}}{\epsilon_i^n} = \begin{cases} \underbrace{[(1 - Cr) + Cr\cos\varphi_x]}_{\text{Re G}} + \sqrt{-1}\underbrace{[-Cr\sin\varphi_x]}_{\text{Im G}} & if \quad a > 0 \\ \underbrace{[(1 - Cr) + Cr\cos\varphi_x]}_{\text{Re G}} + \sqrt{-1}\underbrace{[Cr\sin\varphi_x]}_{\text{Im G}} & if \quad a < 0 \end{cases}$$

Courant-Friedrichs-Lewy Condition

The modulus of amplification factor can be written as,

$$\begin{aligned}|G|^2 &= [(1 - Cr) + Cr\cos\varphi_x]^2 + [Cr\sin\varphi_x]^2 \\ &= 1 + 4Cr(Cr - 1)\sin^2\frac{\varphi_x}{2}\end{aligned}$$

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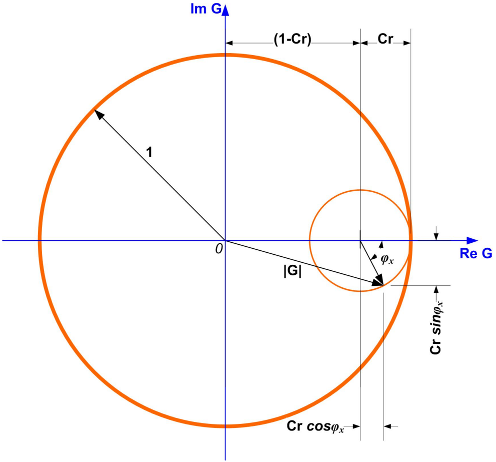
CFL Condition: $0 < Cr \leq 1$
Explicit Scheme is Conditionally Stable.

Courant-Friedrichs-Lewy Condition



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Governing Equation

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The discretized governing equation for IVP with implicit scheme can be written as,

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$Cr = |a|\frac{\Delta t}{\Delta x}$

$$G = \begin{cases} 1 + CrG(e^{-\sqrt{-1}\varphi_x} - 1) & if \quad a > 0 \\ 1 + CrG(e^{\sqrt{-1}\varphi_x} - 1) & if \quad a < 0 \end{cases}$$

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$$G = \begin{cases} [1 - Cr(e^{-\sqrt{-1}\varphi_x} - 1)]^{-1} & if \quad a > 0 \\ [1 - Cr(e^{\sqrt{-1}\varphi_x} - 1)]^{-1} & if \quad a < 0 \end{cases}$$

Stability Analysis

$a > 0$

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$$|G|^2 = \frac{1}{(1 + Cr - Cr\cos\varphi_x) + \sqrt{-1}(Cr\sin\varphi_x)}\frac{1}{(1 + Cr - Cr\cos\varphi_x) - \sqrt{-1}(Cr\sin\varphi_x)}$$

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$$G = [1 - Cr(e^{-\sqrt{-1}\varphi_x} - 1)]^{-1}$$

$$G = \frac{1}{1 - Cr(\cos\varphi_x - \sqrt{-1}\sin\varphi_x - 1)}$$

$$|G|^2 = G.G^*$$

$$|G|^2 = \frac{1}{(1 + Cr - Cr\cos\varphi_x) + \sqrt{-1}(Cr\sin\varphi_x)}\,\frac{1}{(1 + Cr - Cr\cos\varphi_x) - \sqrt{-1}(Cr\sin\varphi_x)}$$

$$|G|^2 = \frac{1}{(1 + Cr - Cr\cos\varphi_x)^2 + (Cr\sin\varphi_x)^2}$$

Stability Analysis

$a > 0$

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$$|G|^2 = \frac{1}{(1 + Cr - Cr\cos\varphi_x)^2 + (Cr\sin\varphi_x)^2}$$

$$|G|^2 = \frac{1}{1 + 4Cr(Cr + 1)\sin^2\frac{\varphi_x}{2}}$$

Thus $|G| < 1$ even for extreme conditions.

Stability Analysis

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$$|G|^2 = \frac{1}{(1 + Cr - Cr\cos\varphi_x)^2 + (Cr\sin\varphi_x)^2}$$

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Thus $|G| < 1$ even for extreme conditions.
Implicit Scheme is Unconditionally Stable.

Thank You

References

Guinot, V. (2010). *Scalar Hyperbolic Conservation Laws in One Dimension of Space*, pages 1–53. ISTE.