$$J_u = \frac{1}{g}\left[k_{ut}\frac{\partial V}{\partial t} + k_{ux} sign(V)\, a \left|\frac{\partial V}{\partial x}\right|\right]$$

Method of characteristics

$$\frac{\partial H}{\partial t} + \frac{a^2}{gA}\frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + \frac{fQ|Q|}{2DA} = 0$$

Rewrite the governing equations as following

$L_1 = \frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + RQ|Q| = 0$

$L_2 = a^2\frac{\partial Q}{\partial x} + gA\frac{\partial H}{\partial t}$=0

Where R = $\frac{F}{2DA}$

Let us write linear combination of equations (1) & (2)

L= L_1+ λL_2 = 0

$(\frac{\partial Q}{\partial t} + \lambda a^2\frac{\partial Q}{\partial x}) + \lambda gA(\frac{\partial H}{\partial t} + \frac{1}{\lambda}\frac{\partial H}{\partial x}) + RQ|Q| = 0$ ……………(3)

H & Q are functions of space & time

Let us write total derivatives

$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{dx}{dt}$ ….(4)

$\frac{dH}{dt} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial x}\frac{dx}{dt}$ ……(5)

Eq.(3) can be written as

$\frac{dQ}{dt} + \lambda gA\frac{dH}{dt} + RQ|Q| = 0$ ……..(6)

$\frac{dx}{dt} = \lambda a^2$ $\quad\frac{dx}{dt} = \frac{1}{\lambda},$ $\quad\lambda = \pm\frac{1}{a}$ ………………..(7)

$\frac{dQ}{dt} + \frac{gA}{a}\frac{dH}{dt} + RQ|Q| = 0$……..(8), $\quad\frac{dx}{dt} = a$……………..(9)

$\frac{dQ}{dt} - \frac{gA}{a}\frac{dH}{dt} + RQ|Q| = 0$……..(10), $\quad\frac{dx}{dt} = -a$……………..(11)

It may be noted that Eq. (8) is valid if Eq. (9) is satisfied and Eq. (10) is valid if Eq. (11) is satisfied.

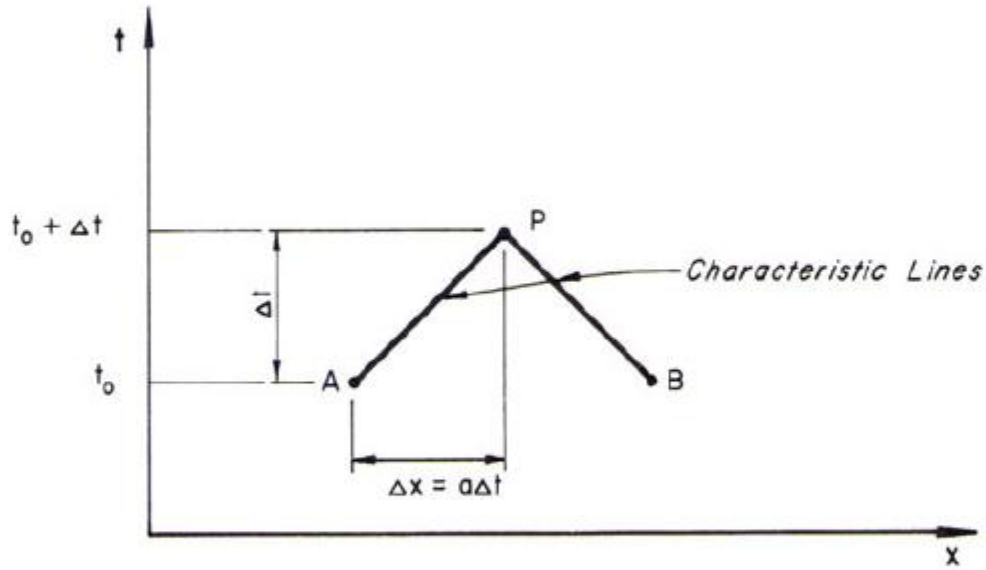

Fig 3.1. Characteristics in x, t plane

Eq. (9) & Eq. (11) are called as positive and negative characteristics lines respectively. Eq. (8) and Eq. (10) are called as compatibility equations. By means of these simple algebraic manipulations, we have eliminated the space variable, x from the governing equations and converted the equations into ordinary differential equations.

Original equations (1) & (2) are valid for any value of x and t; the transformed equations however valid along the characteristic lines. Multiply Eqs. (8) & (10) by $'dt'$ and integrating along the characteristics AP & BP, we obtain

$\int_A^p dQ + \frac{gA}{a}\int_A^p dH + R\int_A^p Q|Q|dt = 0$……(12)

$R\int_A^p Q|Q|dt = RQ_A|Q_A|(t_p - t_A) = RQ_A|Q_A|\Delta t$ ……..(13)

Q Remains constant from A to P for the evaluation of this term, Eq. (12) becomes

$Q_p - Q_A + \frac{gA}{a}(H_p - H_A) + R\Delta t Q_A|Q_A| = 0$ …..(14)

In Eq. (14) Q_p & H_p are unknowns.

Eq. (14) is exact except Q remains constant from A to P for the evaluation of this term in Eq. (13) and approximation of the term. Eq. (14) is a first order approximation which yields satisfactory results for typical existing problems we face. The first order approximation may unsatisfactory if the friction term becomes very large. To counter this problem, we have to use a shorter computation Δt, or use higher order approximation or an iterative procedure to evaluate the friction term.

Similarly processing equation along the negative characteristics line i.e. Eq. (10)

$Q_p - Q_B + \frac{gA}{a}(H_p - H_B) + R\Delta t Q_B|Q_B| = 0$..............(16)

Eqs. (14) & (16) may be written as

$Q_p = C_p - C_a H_p$ ....(17)

$Q_p = C_n + C_a H_p$ ....(18)

$C_p = Q_A + \frac{gA}{a} H_A - R\Delta t Q_A|Q_A|$ ....(19)

$C_n = Q_B + \frac{gA}{a} H_B - R\Delta t Q_B|Q_B|$.........(20)

$C_a = \frac{gA}{a}$ ....(21)

Eq. (17) is valid along the positive characteristics line.

Eq. (18) is valid only along negative characteristics line. C_p and C_a are constant for one time step. C_a is constant for all the time steps in Eq. (17) and Eq. (18) unknowns are Q_p and H_p, can be determine by solving Eqs. (17) & (18) simultaneously.

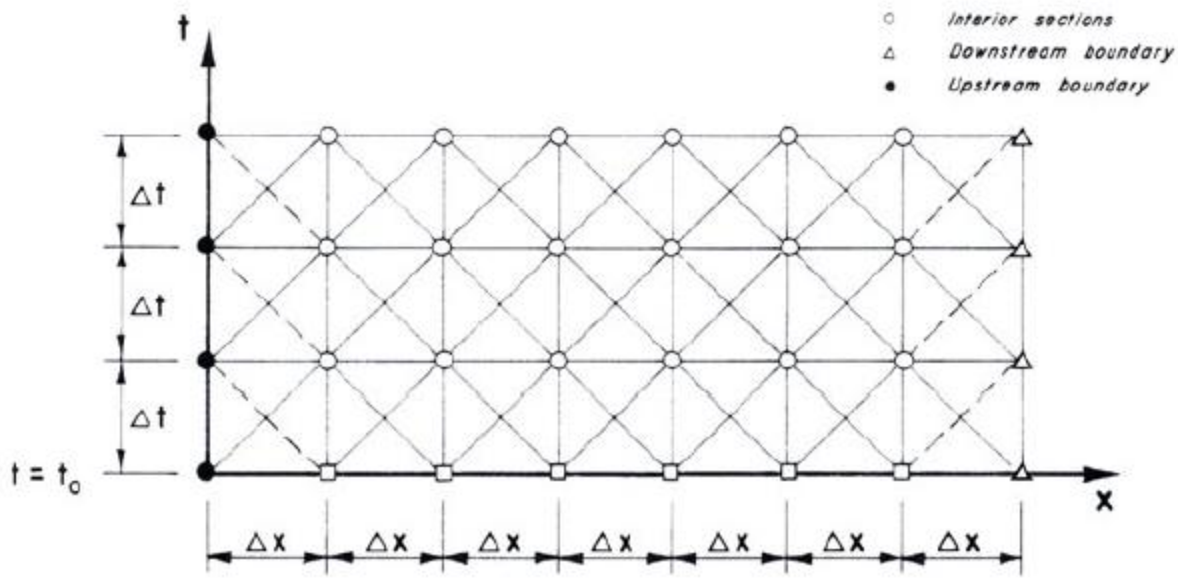

Fig 3.2 Method of characteristics grid

Boundary Conditions

Constant level upstream reservoir:

To determine transient state head and discharge at the boundaries, we need special boundary conditions. Special boundary condition need to be solved with either PCE & NCE.

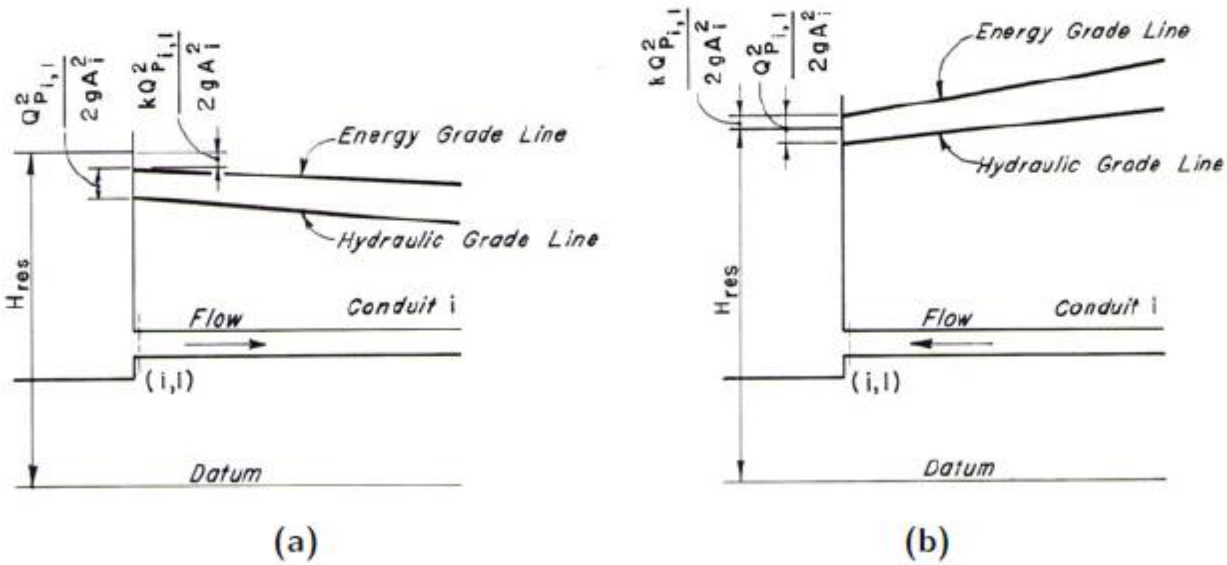

Fig 3.3 Constant level upstream reservoir

Conduit is numbered from i, nodes are numbered from 1 to $n + 1$.

$Q_p = C_n + C_a H_p$

Entrance losses (h_e) = $\frac{kQ^2{}_{p_1}}{2gA^2}$

K= entrance loss coefficient = ½

$H_{p1} = H_{res} - (1 + k)\ \frac{Q^2{}_{p_1}}{2gA^2}$……(1)

$Q_{p_1} = C_n + C_a H_{p1}$….(2)

Let us substitute (2) in 1

$H_{p1} = \frac{Q_{p_1} - C_n}{C_a}$

$Q_{p_1} - C_n = C_a H_{res} - C_a(1 + k)\frac{Q^2{}_{p_1}}{2gA^2}$

Let $\frac{C_a(1+k)}{2gA^2} = k_1$

$k_1 Q^2{}_{p_1} + Q_{p_1} - (C_n + C_a H_{res}) = 0$

$$Q_{p_1} = \frac{-1 \pm \sqrt{1 + 4k_1(C_n \ + \ C_a H_{res})}}{2k_1}$$

After determining Q_{p_1}, $H_{p1} = \frac{Q_{p_1} - C_n}{C_a}$

Constant level downstream reservoir:

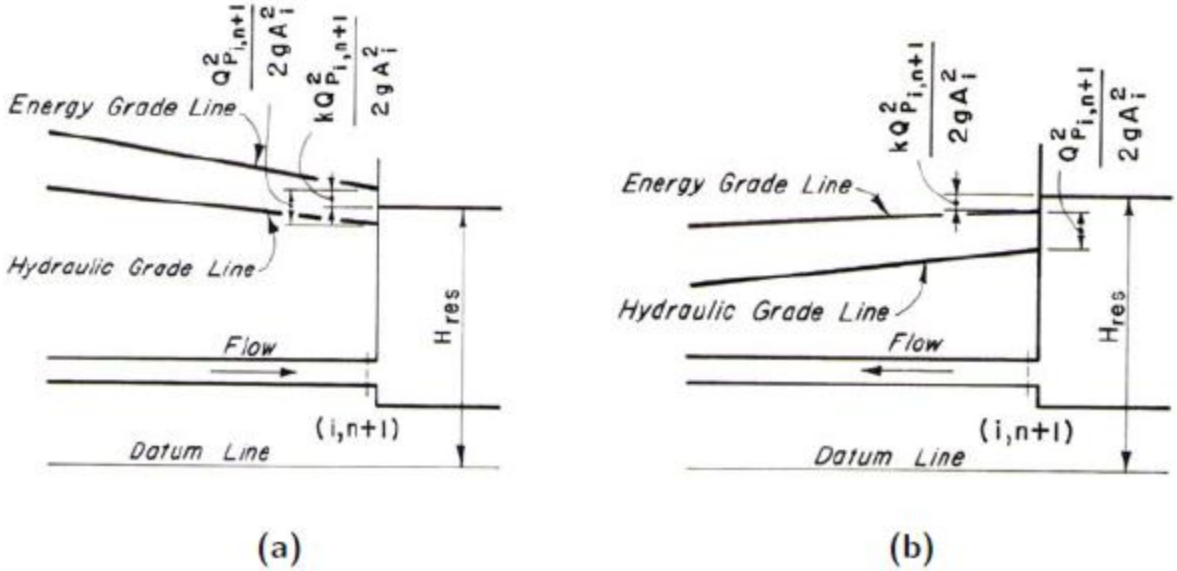

Fig 3.4 Constant level downstream reservoir

$$\begin{aligned} H_{P_{n+1}} &= H_{res} + \frac{KQ^2_{P_{n+1}}}{2gA^2} - \frac{Q^2_{P_{n+1}}}{2gA^2} \\ &= H_{res} + \frac{Q^2_{P_{n+1}}}{2gA^2}(K-1) \\ &= H_{res} - \frac{Q^2_{P_{n+1}}}{2gA^2}(1-K) \end{aligned}$$

$$Q_{P_{n+1}} = C_P - C_a H_{P_{n+1}}$$

$$Q_{P_{n+1}} = C_P - C_a \left[H_{res} - \frac{Q_{P_{n+1}}^2}{2gA^2}(1-K) \right]$$

$$\text{Let} \quad K_2 = \frac{C_a(1-K)}{2gA^2}$$

$$Q_{P_{n+1}} = C_P - C_a H_{res} + K_2 Q_{P_{n+1}}^2$$

$$K_2 Q_{P_{n+1}}^2 - Q_{P_{n+1}} + (C_P - C_a H_{res}) = 0$$

$$Q_{P_{n+1}} = \frac{1 - \sqrt{1 - 4K_2(C_P - C_a H_{res})}}{2K_2}$$

$$H_{P_{n+1}} = H_{res} - \frac{Q_{P_{n+1}}^2}{2gA^2}(1-K)$$

Dead end

Dead end is at the downstream end.

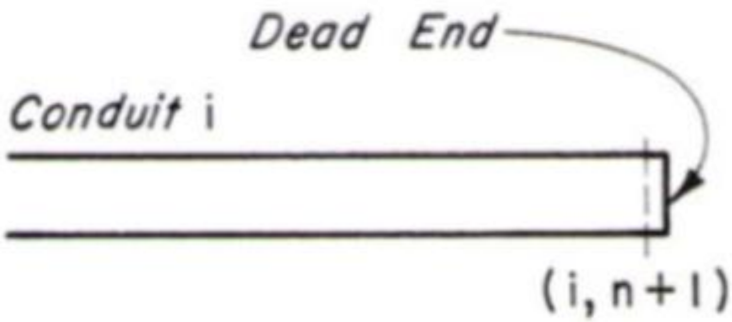

Fig 3.5 Dead end at the downstream

$$Q_{P_{n+1}} = C_P - C_a H_{P_{n+1}}$$

$$Q_{P_{n+1}} = 0 \Rightarrow H_{P_{n+1}} = \frac{C_P}{C_a}$$

Downstream valve

The condition imposed by a valve boundary which is a relationship between the head and discharge through the valve.

Steady state flow through a valve discharging into air.