Indian standards (upsurge protection)

Upsurge control = minimum of
- 1.25 times of maximum operating pressure
  - Suppose pump delivery pressure = 45 m
  - $(1.25)(45) = 56.25\,m$
- or FOS based on ultimate bursting or collapsing strength of pipeline

(b) collapsing pressure of a pipe (AWWA M11)

$$P_c = \frac{2E}{1-\mu^2}\left(\frac{e}{D}\right)^3$$

E = Youngs modulus

μ = poisson's ratio

e = pipe thickness

D = pipe dia

For example MS pipe of ~~19~~ 6 mm thick, ~~678~~ 600 mm dia

$E = 207\ GPa$, $\mu = 0.303$ for MS pipe

$$P_c = \frac{2 \times 207 \times 10^9}{(1-0.303^2)}\left(\frac{\cancel{19}\,6 \times 10^{-3}}{\cancel{678}\,0.6}\right)^3 = 4.56\ bar$$

$$= 45\,m$$

MAC CORMACK SCHEME

1 D wave equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

Predictor step (FTFS)

$$\frac{u_i^* - u_i^n}{\Delta t} + a\,\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0$$

$$u_i^* = u_i^n - a\frac{\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right)$$

↑
Provisional value at next time step or predictor value

Above mentioned step is unstable because both FD are forward steps. Therefore it has to be corrected.

Corrector step: (FTBS)

$$\frac{u_i^{**} - u_i^n}{\Delta t} + a\,\frac{u_i^* - u_{i-1}^*}{\Delta x} = 0$$

$$u_i^{**} = u_i^n - a\frac{\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right)$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^* + u_i^{**}\right)$$

Features ~~Advantages~~ of MacCormack method

1. Conditionally stable
2. 2nd order accurate
3. Explicit method
4. Can be used for linear & nonlinear

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Numerically Solving St Venant Equations Using FD methods

Prashanth Reddy Hanmaiahgari [1]

The St Venant equations given below are also referred to as the shallow water equations are assumed to describe rapidly varied flows and shocks. These equations only work for flows with hydrostatic pressure distribution.

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(QV + gA\bar{y}\right) = gA\left(S_0 - S_f\right)$$

Governing equations in terms of velocity and depth are as follows

$$\frac{\partial y}{\partial t} + D\frac{\partial V}{\partial x} + V\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial V}{\partial t} + g\frac{\partial}{\partial x}\left(\frac{V^2}{2g} + y\right) = g\left(S_0 - S_f\right)$$

Let us write the above equations in vector forms

$$\frac{\partial G}{\partial t} + A\frac{\partial G}{\partial x} = \mathrm{B}$$

$$G = \begin{pmatrix} y \\ V \end{pmatrix}, A = \begin{pmatrix} V & D \\ g & V \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 0 \\ g\left(S_0 - S_f\right) \end{pmatrix}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} y \\ V \end{pmatrix} + \begin{pmatrix} V & D \\ g & V \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} y \\ V \end{pmatrix} = \mathrm{B}$$

Eigen values of vector A written as $\begin{vmatrix} V-\lambda & D \\ g & V-\lambda \end{vmatrix} = 0$

$$\left(V-\lambda\right)^2 - gD = 0$$

$$\left(V-\lambda\right) = \pm\sqrt{gD}$$

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$$\lambda_1 = V + \sqrt{gD} \quad \text{and} \quad \lambda_2 = V - \sqrt{gD}$$

Where λ_1 and λ_2 are both real and distinct values. Therefore equations are set of Hyperbolic PDEs. Hyperbolic PDEs are kind of wave equations, only marching procedures are suitable for numerical integration of these equations. Here, governing equations are solved by using method of characteristics at the boundaries.

$$\frac{\partial y}{\partial t} + D_h \frac{\partial V}{\partial x} + V \frac{\partial y}{\partial x} = 0 \tag{1}$$

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial y}{\partial x} = g\left(S_0 - S_f\right) \tag{2}$$

$$\left(\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial y}{\partial x}\right) + \lambda\left(\frac{\partial y}{\partial t} + D_h \frac{\partial V}{\partial x} + V \frac{\partial y}{\partial x}\right) = g\left(S_0 - S_f\right)$$

$$\left(\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + \lambda D_h \frac{\partial V}{\partial x}\right) + \left(g \frac{\partial y}{\partial x} + \lambda \frac{\partial y}{\partial t} + \lambda V \frac{\partial y}{\partial x}\right) = g\left(S_0 - S_f\right)$$

$$\left(\frac{\partial V}{\partial t} + (V + \lambda D_h)\frac{\partial V}{\partial x}\right) + \lambda\left(\frac{\partial y}{\partial t} + \left(V + \frac{g}{\lambda}\right)\frac{\partial y}{\partial x}\right) = g\left(S_0 - S_f\right)$$

We know $\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot \frac{dx}{dt}$

Let $\frac{dx}{dt} = V + \lambda D_h$

Similarly $\frac{dy}{dt} = \frac{\partial y}{\partial t} + \left(V + \frac{g}{\lambda}\right)\frac{\partial y}{\partial x}$; where $\frac{dx}{dt} = V + \frac{g}{\lambda}$

$$\frac{dV}{dt} + \lambda \frac{dy}{dt} = g\left(S_0 - S_f\right)$$

$$\frac{dx}{dt} = V + \frac{g}{\lambda} = V + \lambda D_h \tag{3}$$

$$\lambda^2 = \frac{g}{D_h}; \; \lambda = \pm\sqrt{\frac{g}{D_h}} = \pm\sqrt{\frac{gT}{A}}$$

Celerity of a gravity wave $C = \sqrt{\frac{gA}{T}}$

Therefore we can write $\lambda_1 = \frac{g}{C}; \quad \lambda_2 = -\frac{g}{C}$ (4)

Therefore we can write above equations as

$$\frac{dV}{dt} + \frac{g}{C} \cdot \frac{dy}{dt} = g\left(S_0 - S_f\right) \tag{5}$$

$$\frac{dx}{dt} = V + C \tag{6}$$

$$\frac{dV}{dt} - \frac{g}{C} \cdot \frac{dy}{dt} = g\left(S_0 - S_f\right) \tag{7}$$

$$\frac{dx}{dt} = V - C \tag{8}$$

It may be noted that Eq. (5) is valid if Eq. (6) is satisfied and Eq. (7) is valid if Eq. (8) is satisfied.

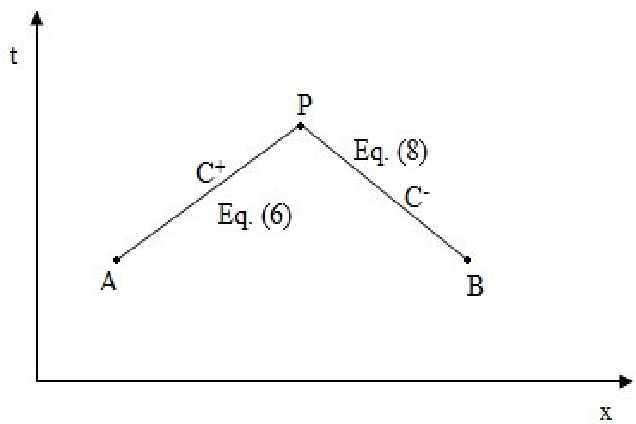


Fig. 1 Characteristics lines

Eq. (6) and Eq. (8) are called as positive and negative characteristics lines respectively. Eq. (5) & (7) are called as compatibility equations. By means of these simple algebraic manipulations, we have eliminated the space variable, x from the governing equations and converted the equations into ordinary differential equations.

Original equations (1) & (2) are valid for any value of x and t; the transformed equations however valid along the characteristics lines.

Multiply Eq. (5) & (7) by dt and integrating along the characteristics AP and BP, we obtain

$$\int_A^P dV + \int_A^P \frac{g}{C} dy = g\int_A^P \left(S_0 - S_f\right) dt$$

And $$\int_B^P dV - \int_B^P \frac{g}{C} dy = g\int_B^P \left(S_0 - S_f\right) dt$$

Where V and y are unknowns

$$\left(V_P - V_A\right) + \left(\frac{g}{C}\right)_A \left(y_P - y_A\right) = g\left(S_0 - S_f\right)_A \left(t_P - t_A\right)$$

$$\Rightarrow V_P = V_A - \left(\frac{g}{C}\right)_A \left(y_P - y_A\right) + g\left(S_0 - S_f\right)_A \left(t_P - t_A\right)$$

And

$$\left(V_P - V_B\right) - \left(\frac{g}{C}\right)_B \left(y_P - y_B\right) = g\left(S_0 - S_f\right)_B \left(t_P - t_B\right)$$

$$\Rightarrow V_P = V_B + \left(\frac{g}{C}\right)_B \left(y_P - y_B\right) + g\left(S_0 - S_f\right)_B \left(t_P - t_B\right)$$

Let us write $C_A = \left(\frac{g}{C}\right)_A, C_B = \left(\frac{g}{C}\right)_B$

$$V_P = C_P - C_A y_P \tag{9}$$

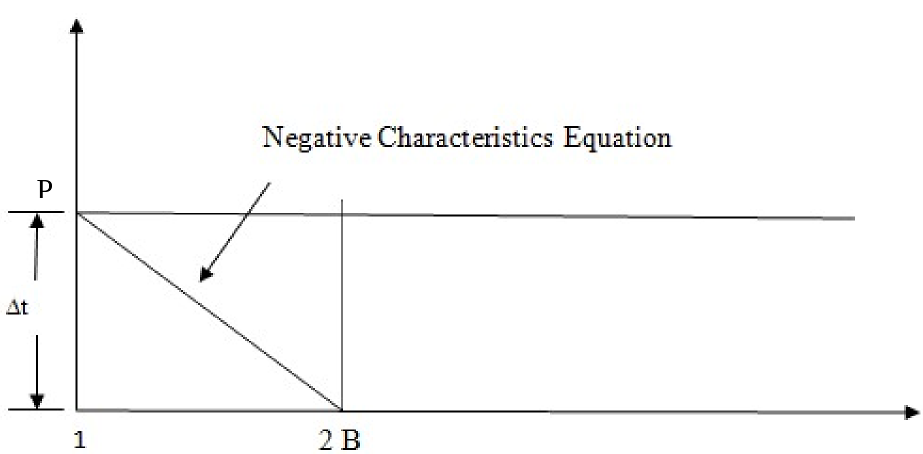
$$V_P = C_n + C_B y_P \tag{10}$$

Where $C_P = V_A + C_A y_A + g\left(S_0 - S_f\right)_A \left(t_P - t_A\right)$

and $C_n = V_B - C_B y_B + g\left(S_0 - S_f\right)_B \left(t_P - t_B\right)$

Equations (9) and (10) are final algebraic equations which are valid along positive and negative characteristic equations respectively.

Upstream Boundary Condition:



Negative Characteristics Equation

$$V_P = C_n + C_B y_P$$

$$C_n = V(2) - C_B y(2) + g\left(S_0 - S_{f2}\right) \cdot \Delta t$$

$$S_{f2} = \frac{nQ(2)\left|Q(2)\right|}{(A(2))^2\left(R_H(2)\right)^{4/3}}$$

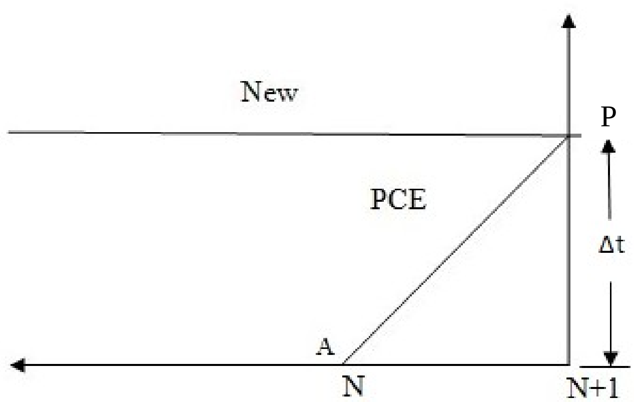
$$C_B = \sqrt{\frac{gT(2)}{A(2)}}$$

$y_{new}(1) = y_0$

$V_P = C_n + C_B y_P; Q_P = V_P A_P$

$A_P = A_{new}$

Downstream Boundary Conditions:



$$Q_{new}(N+1) = 0$$

$$C_A = \sqrt{\frac{gT(N)}{A(N)}}$$

$$S_{fA} = \frac{nQ(N) \cdot \left|Q(N)\right|}{(A(N))^2 \cdot \left(R_H(N)\right)^{4/3}}$$

$$C_P = \frac{Q(N)}{A(N)} + C_A y(N) + g\left(S_0 - S_{fA}\right) \cdot \Delta t$$

We know that $V_P = C_P - C_A y_P$, but here $V_P = 0$.

Hence $y_P = \frac{C_P}{C_A}$

$y_{new}(N+1) = y_P; A_{new}(N+1) = Compute$

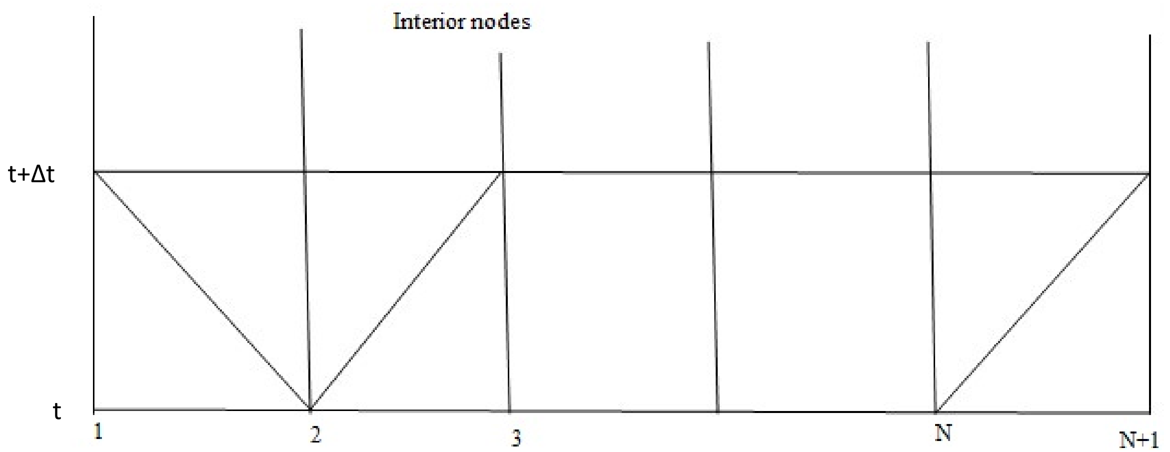
<u>Interior nodes:</u>

Mac Cormack scheme (predictor and corrector method) is used to solve governing equations at interior nodes The Mac Cormack scheme is an explicit, two step predictor and corrector scheme that is second order accurate in space and time and is capable of capturing the shocks without separating them.

Governing equations

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(QV + gA\bar{y}\right) = gA\left(S_0 - S_f\right)$$



Predictor step:

$$\frac{\partial f}{\partial t} = \frac{f_i^* - f_i^k}{\Delta t}$$

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^k - f_i^k}{\Delta x}$$

(NOTE: *f* can be *Q* and *A*)

Corrector step:

$$\frac{\partial f}{\partial t} = \frac{f_i^{**} - f_i^k}{\Delta t}$$

$$\frac{\partial f}{\partial x} = \frac{f_i^* - f_{i-1}^*}{\Delta x}$$

$$Q_i^{k+1} = \frac{1}{2}\left(Q_i^* + Q_i^{**}\right)$$

$$A_i^{k+1} = \frac{1}{2}\left(A_i^* + A_i^{**}\right)$$

Let us write $\frac{Q^2}{A} + gA\bar{y} = E$ in momentum equation

Determine $E(i), E(i-1), S_f(i), S_f(i-1)$. Iterate i=2 to N

Predictor step:

Continuity equation

$$\frac{A_i^* - A_i^k}{\Delta t} + \frac{Q_{i+1}^k - Q_i^k}{\Delta x} = 0$$

$$A_i^* = A_i^k - \Delta t.\frac{Q_{i+1}^k - Q_i^k}{\Delta x}$$

Similarly from momentum equation

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = gA\left(S_0 - S_f\right)$$

$$\frac{Q_i^* - Q_i^k}{\Delta t} + \frac{E_{i+1}^k - E_i^k}{\Delta x} = gA_i^k\left(S_0 - S_{fi}^{\ k}\right)$$

$$Q_i^* = Q_i^k - \Delta t.\frac{E_{i+1}^k - E_i^k}{\Delta x} + gA_i^k\left(S_0 - S_{fi}^{\ k}\right).\Delta t$$

Determine y_i^* and S_{fi}^*. Do this for all nodes from 2 to N.

Corrector step method:

Continuity equation

$$\frac{A_i^{**} - A_i^k}{\Delta t} + \frac{Q_i^* - Q_{i-1}^*}{\Delta x}$$

$$A_i^{**} = A_i^k - \Delta t \frac{Q_i^* - Q_{i-1}^*}{\Delta x}$$

Similarly momentum equation

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = gA\left(S_0 - S_f\right)$$

$$\frac{Q_i^{**} - Q_i^k}{\Delta t} + \frac{E_i^* - E_{i-1}^*}{\Delta x} = gA_i^*\left(S_0 - S_{fi}^*\right)$$

$$Q_i^{**} = Q_i^k - \Delta t \frac{E_i^* - E_{i-1}^*}{\Delta x} + \Delta t g A_i^*\left(S_0 - S_{fi}^*\right)$$

Do this for all nodes from 2 to N.

$$Q_i^{k+1} = \frac{1}{2}\left(Q_i^* + Q_i^{**}\right)$$

$$A_i^{k+1} = \frac{1}{2}\left(A_i^* + A_i^{**}\right)$$

$$V_i^{k+1} = \frac{Q_i^{k+1}}{A_i^{k+1}}$$

Similarly determine y_i^{k+1}.

Stability Criteria

Finally check for the stability and determine $V_{\max}$ and $C_{\max}$.

The MacCormack scheme and method of characteristics is stable if CFL (Courant Friedrich Lewy) is less than or equal to 1. For the computation to be stable, this condition must be satisfied at each grid point during every computational interval.

Then find
$$dt = \frac{dx}{V_{\max} + C_{\max}}$$

In computations minimum value of dt among all the grid points should be chosen.