Module 04: Surface Water Hydraulics

Unit 07: Unsteady 1D Channel Flow

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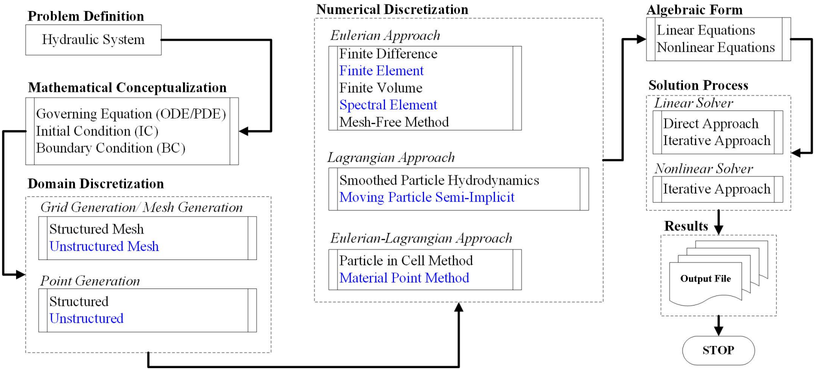
Learning Objective

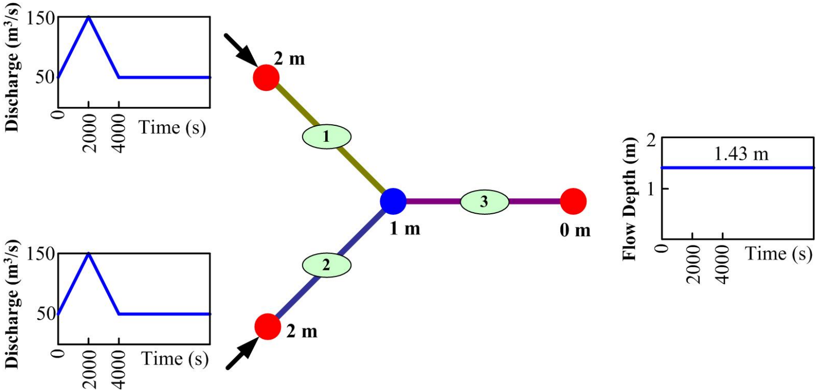
- To solve unsteady channel network problem using implicit approach.

Problem Definition to Solution



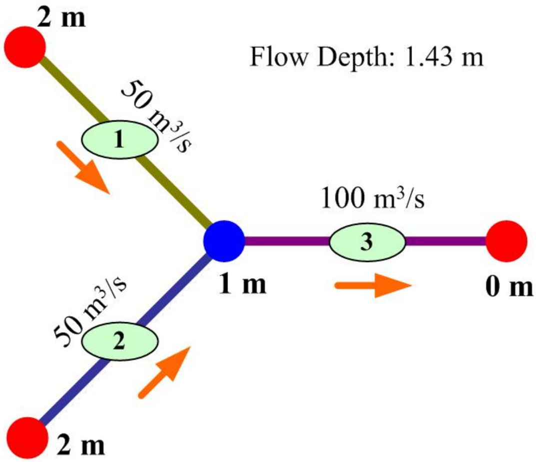
Problem Statement

Channel with Boundary Conditions

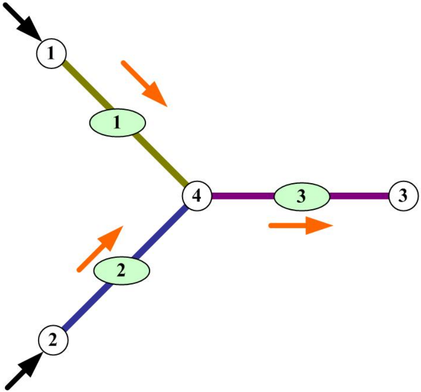


Problem Statement

Channel with Initial Conditions



Problem Statement
Configuration 1



Problem Statement

Channel Data (Zhang, 2005)

Channel	length (m)	width (m)	Side Slope		reach(m)	n	S_0	Connectivity	
			m_1	m_2				JN_1	JN_2
1	5000	50	0	0	500	0.025	0.0002	1	4
2	5000	50	0	0	500	0.025	0.0002	2	4
3	5000	100	0	0	500	0.025	0.0002	4	3

Problem Statement

Junction Data

Junction Number	Depth (m)	Discharge (m^3/s)	Bed Elevation (m)
1	-99999	2	2
2	-99999	2	2
3	1	-99999	0
4	-99999	-99999	1

Problem Statement
Junction Data

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1	-99999	2	2
2	-99999	2	2
3	1	-99999	0
4	-99999	-99999	1

Required

Plot the discharge and depth hydrographs at x = 4000 m from internal junction node in Channel reach 3 of the network.

Problem Definition

Governing Equation for unsteady 1D channel flow (St. Venant Equations) can be written as (Weiming, 2007),

Initial Boundary Value Problem

Continuity Equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{dx} = q$$

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$$\frac{\partial A}{\partial t} + \frac{\partial Q}{dx} = q$$

Momentum Equation:

$$\frac{\partial}{\partial t}\left(\frac{Q}{A}\right) + \frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{2A^2}\right) + g\frac{\partial H}{\partial x} + gS_f = 0$$

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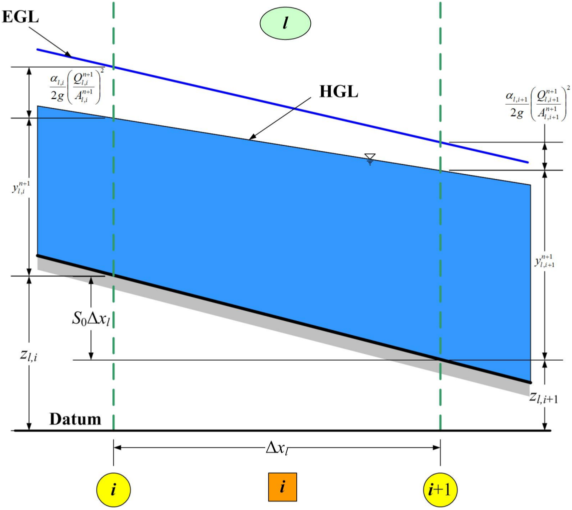
$$\frac{\partial}{\partial t}\left(\frac{Q}{A}\right) + \frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{2A^2}\right) + g\frac{\partial H}{\partial x} + gS_f = 0$$

where

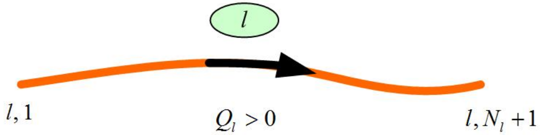
$y=$ depth of flow
$S_f=$ friction slope $\left(= \frac{n^2 Q^2}{R^{4/3} A^2}\right)$
$A=$ cross-sectional area
$q=$ lateral inflow
$z=$ elevation of the channel bottom w.r.t. datum

$H=$ water surface elevation $(= y + z)$
$\alpha=$ momentum correction factor
$Q=$ discharge
$g=$ acceleration due to gravity

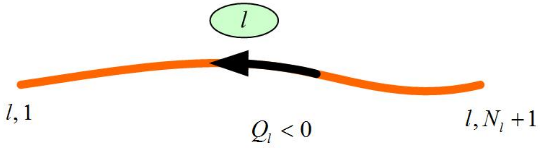
Channel Flow



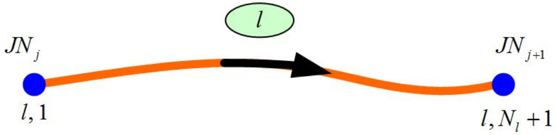
Channel Flow Conventions



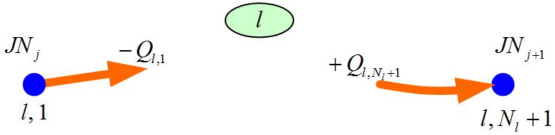
Channel Flow Conventions



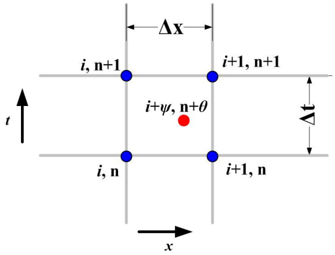
Channel Flow Conventions



Channel Flow Conventions

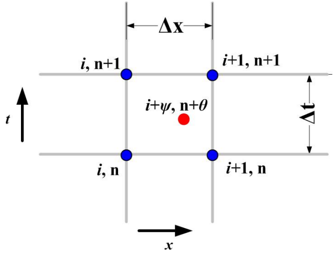


Discretization
General Variable



Discretization

General Variable



For any general variable ϕ, Preissmann scheme can be written as,

$$\phi = \theta[\psi\phi_{i+1}^{n+1} + (1-\psi)\phi_i^{n+1}] + (1-\theta)[\psi\phi_{i+1}^{n} + (1-\psi)\phi_i^{n}]$$

$$\frac{\partial \phi}{\partial t} = \psi\frac{\phi_{i+1}^{n+1} - \phi_{i+1}^{n}}{\Delta t} + (1-\psi)\frac{\phi_{i}^{n+1} - \phi_{i}^{n}}{\Delta t}$$

$$\frac{\partial \phi}{\partial x} = \theta\frac{\phi_{i+1}^{n+1} - \phi_{i}^{n+1}}{\Delta x} + (1-\theta)\frac{\phi_{i+1}^{n} - \phi_{i}^{n}}{\Delta x}$$

Discretization

Continuity Equation

The continuity equation for the i^{th} segment at the n^{th} time step of the l^{th} channel reach can be discretized with four point Preissmann scheme as,

$$
\begin{aligned}
C_{l,i}^{n,n+1} =& \frac{\psi}{\Delta t}(A_{l,i+1}^{n+1} - A_{l,i+1}^{n}) + \frac{1-\psi}{\Delta t}(A_{l,i}^{n+1} - A_{l,i}^{n}) \\
&+ \frac{\theta}{\Delta x_l}(Q_{l,i+1}^{n+1} - Q_{l,i}^{n+1}) + \frac{1-\theta}{\Delta x_l}(Q_{l,i+1}^{n} - Q_{l,i}^{n}) \\
&- \theta[\psi q_{l,i+1}^{n+1} + (1-\psi) q_{l,i}^{n+1}] - (1-\theta)[\psi q_{l,i+1}^{n} + (1-\psi) q_{l,i}^{n}] = 0
\end{aligned}
$$

Discretization
Continuity Equation

Elements of Jacobian matrix can be calculated as,

$$\frac{\partial C_{l,i}^{n,n+1}}{\partial y_{l,i}^{n+1}} = \frac{1-\psi}{\Delta t}\frac{dA}{dy}\Big|_{l,i}^{n+1}$$

$$\frac{\partial C_{l,i}^{n,n+1}}{\partial Q_{l,i}^{n+1}} = -\frac{\theta}{\Delta x_l}$$

$$\frac{\partial C_{l,i}^{n,n+1}}{\partial y_{l,i+1}^{n+1}} = \frac{\psi}{\Delta t}\frac{dA}{dy}\Big|_{l,i+1}^{n+1}$$

$$\frac{\partial C_{l,i}^{n,n+1}}{\partial Q_{l,i+1}^{n+1}} = \frac{\theta}{\Delta x_l}$$

Discretization

Momentum Equation

The momentum equation for the i^{th} segment at the n^{th} time step of the l^{th} channel reach can be discretized with four point Preissmann scheme as,

$$
\begin{aligned}
M_{l,i}^{n,n+1} &= \frac{\psi}{\Delta t}\left(\frac{Q_{l,i+1}^{n+1}}{A_{l,i+1}^{n+1}} - \frac{Q_{l,i+1}^{n}}{A_{l,i+1}^{n}}\right) + \frac{1-\psi}{\Delta t}\left(\frac{Q_{l,i}^{n+1}}{A_{l,i}^{n+1}} - \frac{Q_{l,i}^{n}}{A_{l,i}^{n}}\right) \\
&+ \frac{\theta}{\Delta x_l}\left[\frac{\alpha_{l,i+1}}{2}\left(\frac{Q_{l,i+1}^{n+1}}{A_{l,i+1}^{n+1}}\right)^2 - \frac{\alpha_{l,i}}{2}\left(\frac{Q_{l,i}^{n+1}}{A_{l,i}^{n+1}}\right)^2\right] \\
&+ \frac{1-\theta}{\Delta x_l}\left[\frac{\alpha_{l,i+1}}{2}\left(\frac{Q_{l,i+1}^{n}}{A_{l,i+1}^{n}}\right)^2 - \frac{\alpha_{l,i}}{2}\left(\frac{Q_{l,i}^{n}}{A_{l,i}^{n}}\right)^2\right] \\
&+ \frac{\theta g}{\Delta x_l}\left[(y_{l,i+1}^{n+1} + z_{l,i+1}) - (y_{l,i}^{n+1} + z_{l,i})\right] + \frac{(1-\theta)g}{\Delta x_l}\left[(y_{l,i+1}^{n} + z_{l,i+1}) - (y_{l,i}^{n} + z_{l,i})\right] \\
&+ \theta g\left[\psi S_f|_{l,i+1}^{n+1} + (1-\psi)S_f|_{l,i}^{n+1}\right] + (1-\theta)g\left[\psi S_f|_{l,i+1}^{n} + (1-\psi)S_f|_{l,i}^{n}\right] = 0
\end{aligned}
$$

with

$$S_f = \frac{n_m^2 Q^2}{R^{\frac{4}{3}} A^2}$$

Discretization

Momentum Equation

With reverse flow consideration the discretization can be written as,

$$
\begin{aligned}
M_{l,i}^{n,n+1} &= \frac{\psi}{\Delta t}\left(\frac{Q_{l,i+1}^{n+1}}{A_{l,i+1}^{n+1}} - \frac{Q_{l,i+1}^{n}}{A_{l,i+1}^{n}}\right) + \frac{1-\psi}{\Delta t}\left(\frac{Q_{l,i}^{n+1}}{A_{l,i}^{n+1}} - \frac{Q_{l,i}^{n}}{A_{l,i}^{n}}\right) \\
&+ \frac{\theta}{\Delta x_l}\left[\frac{\alpha_{l,i+1}}{2}\left(\frac{Q_{l,i+1}^{n+1}}{A_{l,i+1}^{n+1}}\right)^2 - \frac{\alpha_{l,i}}{2}\left(\frac{Q_{l,i}^{n+1}}{A_{l,i}^{n+1}}\right)^2\right] \\
&+ \frac{1-\theta}{\Delta x_l}\left[\frac{\alpha_{l,i+1}}{2}\left(\frac{Q_{l,i+1}^{n}}{A_{l,i+1}^{n}}\right)^2 - \frac{\alpha_{l,i}}{2}\left(\frac{Q_{l,i}^{n}}{A_{l,i}^{n}}\right)^2\right] \\
&+ \frac{\theta g}{\Delta x_l}\left[(y_{l,i+1}^{n+1} + z_{l,i+1}) - (y_{l,i}^{n+1} + z_{l,i})\right] + \frac{(1-\theta)g}{\Delta x_l}\left[(y_{l,i+1}^{n} + z_{l,i+1}) - (y_{l,i}^{n} + z_{l,i})\right] \\
&+ \theta g\left[\psi S_f|_{l,i+1}^{n+1} + (1-\psi)S_f|_{l,i}^{n+1}\right] + (1-\theta)g\left[\psi S_f|_{l,i+1}^{n} + (1-\psi)S_f|_{l,i}^{n}\right] = 0
\end{aligned}
$$

The friction slope

$$S_f = \frac{n_m^2 Q|Q|}{R^{\frac{4}{3}} A^2}$$

Discretization

Momentum Equation: Jacobian Matrix

Elements of Jacobian matrix can be calculated as,

$$
\begin{aligned}
\frac{\partial M_{l,i}^{n,n+1}}{\partial y_{l,i}^{n+1}} = & -\frac{1-\psi}{\Delta t}\frac{Q_{l,i}^{n+1}}{(A_{l,i}^{n+1})^2}\frac{dA}{dy}\Big|_{l,i}^{n+1} + \frac{\theta\alpha_{l,i}}{\Delta x_l}\frac{(Q_{l,i}^{n+1})^2}{(A_{l,i}^{n+1})^3}\frac{dA}{dy}\Big|_{l,i}^{n+1} - \frac{\theta g}{\Delta x_l} \\
& -\theta(1-\psi)gn_{m,l}^2\left[\frac{2Q_{l,i}^{n+1}|Q_{l,i}^{n+1}|}{(R_{l,i}^{n+1})^{\frac{4}{3}}(A_{l,i}^{n+1})^3}\frac{dA}{dy}\Big|_{l,i}^{n+1} + \frac{4Q_{l,i}^{n+1}|Q_{l,i}^{n+1}|}{3(R_{l,i}^{n+1})^{\frac{7}{3}}(A_{l,i}^{n+1})^2}\frac{dR}{dy}\Big|_{l,i}^{n+1}\right]
\end{aligned}
$$

$$
\frac{\partial M_{l,i}^{n,n+1}}{\partial Q_{l,i}^{n+1}} = \frac{1-\psi}{\Delta t}\frac{1}{A_{l,i}^{n+1}} - \frac{\theta\alpha_{l,i}}{\Delta x_l}\frac{Q_{l,i}^{n+1}}{(A_{l,i}^{n+1})^2} + 2\theta(1-\psi)gn_{m,l}^2\frac{|Q_{l,i}^{n+1}|}{(R_{l,i}^{n+1})^{\frac{4}{3}}(A_{l,i}^{n+1})^2}
$$

Discretization

Momentum Equation: Jacobian Matrix

$$\frac{\partial M_{l,i}^{n,n+1}}{\partial y_{l,i+1}^{n+1}} = -\frac{\psi}{\Delta t}\frac{Q_{l,i+1}^{n+1}}{(A_{l,i+1}^{n+1})^2}\frac{dA}{dy}\Big|_{l,i+1}^{n+1} - \frac{\theta\alpha_{l,i+1}}{\Delta x_l}\frac{(Q_{l,i+1}^{n+1})^2}{(A_{l,i+1}^{n+1})^3}\frac{dA}{dy}\Big|_{l,i+1}^{n+1} + \frac{\theta g}{\Delta x_l}$$

$$- \theta\psi g n_{m,l}^2\left[\frac{2Q_{l,i+1}^{n+1}|Q_{l,i+1}^{n+1}|}{(R_{l,i+1}^{n+1})^{\frac{4}{3}}(A_{l,i+1}^{n+1})^3}\frac{dA}{dy}\Big|_{l,i+1}^{n+1} + \frac{4Q_{l,i+1}^{n+1}|Q_{l,i+1}^{n+1}|}{3(R_{l,i+1}^{n+1})^{\frac{7}{3}}(A_{l,i+1}^{n+1})^2}\frac{dR}{dy}\Big|_{l,i+1}^{n+1}\right]$$

$$\frac{\partial M_{l,i}^{n,n+1}}{\partial Q_{l,i+1}^{n+1}} = \frac{\psi}{\Delta t}\frac{1}{A_{l,i+1}^{n+1}} + \frac{\theta\alpha_{l,i+1}}{\Delta x_l}\frac{Q_{l,i+1}^{n+1}}{(A_{l,i+1}^{n+1})^2} + 2\theta\psi g n_{m,l}^2\frac{|Q_{l,i+1}^{n+1}|}{(R_{l,i+1}^{n+1})^{\frac{4}{3}}(A_{l,i+1}^{n+1})^2}$$

Discretization

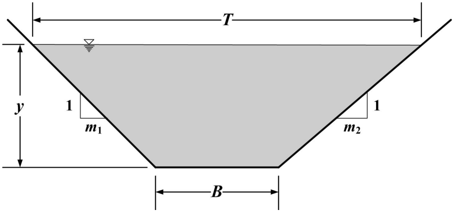
Momentum Equation: Jacobian Matrix

$$\frac{\partial M_{l,i}^{n,n+1}}{\partial y_{l,i+1}^{n+1}} = -\frac{\psi}{\Delta t}\frac{Q_{l,i+1}^{n+1}}{(A_{l,i+1}^{n+1})^2}\frac{dA}{dy}\Big|_{l,i+1}^{n+1} - \frac{\theta\alpha_{l,i+1}}{\Delta x_l}\frac{(Q_{l,i+1}^{n+1})^2}{(A_{l,i+1}^{n+1})^3}\frac{dA}{dy}\Big|_{l,i+1}^{n+1} + \frac{\theta g}{\Delta x_l}$$
$$- \theta\psi g n_{m,l}^2\left[\frac{2Q_{l,i+1}^{n+1}|Q_{l,i+1}^{n+1}|}{(R_{l,i+1}^{n+1})^{\frac{4}{3}}(A_{l,i+1}^{n+1})^3}\frac{dA}{dy}\Big|_{l,i+1}^{n+1} + \frac{4Q_{l,i+1}^{n+1}|Q_{l,i+1}^{n+1}|}{3(R_{l,i+1}^{n+1})^{\frac{7}{3}}(A_{l,i+1}^{n+1})^2}\frac{dR}{dy}\Big|_{l,i+1}^{n+1}\right]$$

$$\frac{\partial M_{l,i}^{n,n+1}}{\partial Q_{l,i+1}^{n+1}} = \frac{\psi}{\Delta t}\frac{1}{A_{l,i+1}^{n+1}} + \frac{\theta\alpha_{l,i+1}}{\Delta x_l}\frac{Q_{l,i+1}^{n+1}}{(A_{l,i+1}^{n+1})^2} + 2\theta\psi g n_{m,l}^2\frac{|Q_{l,i+1}^{n+1}|}{(R_{l,i+1}^{n+1})^{\frac{4}{3}}(A_{l,i+1}^{n+1})^2}$$

$2N_l$ non-linear equations with $2(N_l+1)$ unknowns (discharge + flow-depth)

Trapezoidal Cross-section



$$A = By + \frac{1}{2}(m_1 + m_2)y^2$$

$$P = B + \left(\sqrt{1 + m_1^2} + \sqrt{1 + m_2^2}\right) y$$

$$R = \frac{A}{P}$$

$$T = B + (m_1 + m_2)y$$

where P = wetted perimeter.

Trapezoidal Section

For trapezoidal channel cross-section,

$$\frac{dA}{dy} = B + (m_1 + m_2)y$$

Trapezoidal Section

For trapezoidal channel cross-section,

$$\frac{dA}{dy} = B + (m_1 + m_2)y$$

$$\frac{dR}{dy} = \frac{T}{P} - \frac{R}{P}\frac{dP}{dy}$$

with

$$T = B + (m_1 + m_2)y$$

$$P = B + \left(\sqrt{1+m_1^2} + \sqrt{1+m_2^2}\right) y$$

$$R = \frac{A}{P}$$

$$\frac{dP}{dy} = \left(\sqrt{1+m_1^2} + \sqrt{1+m_2^2}\right)$$

Algebraic Form

In general form, continuity and momentum equations can be written as,

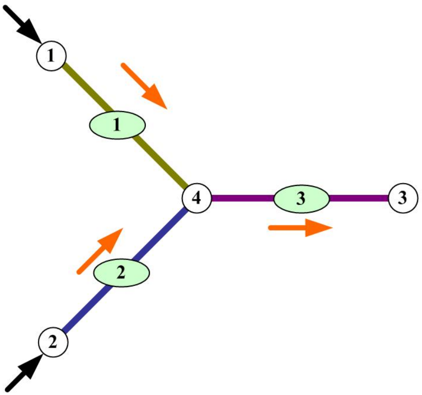
$$\frac{\partial C_{l,i}^{n,n+1}}{\partial y_{l,i}^{n+1}}\Delta y_{l,i}^{n+1}+\frac{\partial C_{l,i}^{n,n+1}}{\partial Q_{l,i}^{n+1}}\Delta Q_{l,i}^{n+1}+\frac{\partial C_{l,i}^{n,n+1}}{\partial y_{l,i+1}^{n+1}}\Delta y_{l,i+1}^{n+1}+\frac{\partial C_{l,i}^{n,n+1}}{\partial Q_{l,i+1}^{n+1}}\Delta Q_{l,i+1}^{n+1}=-C_{l,i}^{n,n+1}$$

$$\frac{\partial M_{l,i}^{n,n+1}}{\partial y_{l,i}^{n+1}}\Delta y_{l,i}^{n+1}+\frac{\partial M_{l,i}^{n,n+1}}{\partial Q_{l,i}^{n+1}}\Delta Q_{l,i}^{n+1}+\frac{\partial M_{l,i}^{n,n+1}}{\partial y_{l,i+1}^{n+1}}\Delta y_{l,i+1}^{n+1}+\frac{\partial M_{l,i}^{n,n+1}}{\partial y_{l,i+1}^{n+1}}\Delta Q_{l,i+1}^{n+1}=-M_{l,i}^{n,n+1},$$

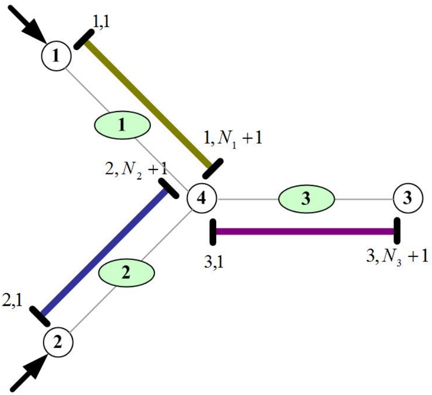
$\forall i \in \{1,\ldots,N_l\}$

Problem Statement

Configuration 1



Problem Statement
Configuration 1



Program Implementation
Configuration 1

$$\text{chl\_inf} = \begin{bmatrix} 1 & 5000 & 50 & 0 & 0 & 50 & 0.025 & 0.0002 & 1 & 4 \\ 2 & 5000 & 50 & 0 & 0 & 50 & 0.025 & 0.0002 & 2 & 4 \\ 3 & 5000 & 100 & 0 & 0 & 50 & 0.025 & 0.0002 & 4 & 3 \end{bmatrix}$$

Program Implementation
Configuration 1

$$\text{chl\_inf} = \begin{bmatrix} 1 & 5000 & 50 & 0 & 0 & 50 & 0.025 & 0.0002 & 1 & 4 \\ 2 & 5000 & 50 & 0 & 0 & 50 & 0.025 & 0.0002 & 2 & 4 \\ 3 & 5000 & 100 & 0 & 0 & 50 & 0.025 & 0.0002 & 4 & 3 \end{bmatrix}$$

$$\text{jun\_inf} = \begin{bmatrix} -99999 & 2 & 2 \\ -99999 & 2 & 2 \\ 1 & -99999 & 0 \\ -99999 & -99999 & 1 \end{bmatrix} \quad \text{jun\_con} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 3 & 3 & -1 & -2 \end{bmatrix}$$

List of Source Code

Channel Flow with Reverse Flow

- Channel Network with Configuration 1
  - unsteady_1D_channel_network_with_reverse_cfg1.sci

Thank You

References

Weiming, W. (2007). *Computational River Dynamics*. Taylor & Francis, London, UK.

Zhang, Y. (2005). Simulation of open channel network flows using finite element approach. *Communications in Nonlinear Science and Numerical Simulation*, 10(5):467 – 478.