Module 02: Numerical Methods

Unit 15: Finite Volume Method: Higher Resolution Methods

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Learning Objective

- To discretize conservation laws using Higher Resolution Methods.

Governing Equation

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \tag{1}$$

where

$\mathcal{F}_\phi$ = Flux function.

S_ϕ = Source term.

Higher Resolution Methods

Piecewise Reconstruction (LeVeque, 2002)

A piecewise linear from of cell average value ϕ_P^n can be used as

$$\tilde{\phi}^n(x, t^n) = \phi_P^n + \sigma_P^n(x - x_P) \quad \forall x \in [x_w, x_e)$$

Higher Resolution Methods

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$$\mathcal{F}_\phi = a\phi$$

where a is constant.

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$$\tilde{\phi}(x,t^{n+1}) = \tilde{\phi}(x - a\Delta t, t^n)$$

Higher Resolution Methods

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$$\tilde{\phi}(x, t^{n+1}) = \tilde{\phi}(x - a\Delta t, t^n)$$

Numerical flux function can written as

$$\bar{\mathcal{F}}_\phi(x_e, t^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t)dt$$

$$\bar{\mathcal{F}}_\phi(x_w, t^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t)dt$$

Higher Resolution Methods

$a > 0$

Numerical flux function for east face can be calculated as

$$\bar{\mathcal{F}}_\phi(x_e, t^n) = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t)dt$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\phi}(x_e, t)dt$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\phi}(x_e - a(t - t^n), t^n)dt$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\left[\phi_P^n + \sigma_P^n(x_e - a(t - t^n) - x_P)\right] dt$$

$$= a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t)$$

Higher Resolution Methods
$a > 0$

Numerical flux function for west face can be calculated as

$$\bar{\mathcal{F}}_\phi(x_w, t^n) = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t)dt$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\phi}(x_w, t)dt$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\phi}(x_w - a(t - t^n), t^n)dt$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\left[\phi_W^n + \sigma_W^n(x_w - a(t - t^n) - x_W)\right]dt$$

$$= a\phi_W^n + \frac{a}{2}\sigma_W^n(\Delta x - a\Delta t)$$

Higher Resolution Methods

$a > 0$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{\mathcal{F}}_\phi(x_e, t^n) - \bar{\mathcal{F}}_\phi(x_w, t^n)\right]$$

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or,

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x}\left[\left(a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t)\right) - \left(a\phi_W^n + \frac{a}{2}\sigma_W^n(\Delta x - a\Delta t)\right)\right]$$

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or,

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x}\left(\phi_P^n - \phi_W^n\right) - \frac{1}{2}\frac{a\Delta t}{\Delta x}(\Delta x - a\Delta t)(\sigma_P^n - \sigma_W^n)$$

Higher Resolution Methods

$a < 0$

Numerical flux function for east face can be calculated as

$$
\begin{aligned}
\bar{\mathcal{F}}_\phi(x_e, t^n) &= \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t)dt \\
&= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\phi}(x_e, t)dt \\
&= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\phi}(x_e - a(t - t^n), t^n)dt \\
&= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\left[\phi_E^n + \sigma_E^n(x_e - a(t - t^n) - x_E)\right] dt \\
&= a\phi_E^n - \frac{a}{2}\sigma_E^n(\Delta x + a\Delta t)
\end{aligned}
$$

Higher Resolution Methods
$a < 0$

Numerical flux function for west face can be calculated as

$$
\begin{aligned}
\bar{\mathcal{F}}_\phi(x_w, t^n) &= \frac{1}{\Delta t}\int\limits_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t)dt \\
&= \frac{1}{\Delta t}\int\limits_t^{t+\Delta t} a\tilde{\phi}(x_w, t)dt \\
&= \frac{1}{\Delta t}\int\limits_t^{t+\Delta t} a\tilde{\phi}(x_w - a(t - t^n), t^n)dt \\
&= \frac{1}{\Delta t}\int\limits_t^{t+\Delta t} a\left[\phi_P^n + \sigma_P^n(x_w - a(t - t^n) - x_P)\right]dt \\
&= a\phi_P^n - \frac{a}{2}\sigma_P^n(\Delta x + a\Delta t)
\end{aligned}
$$

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$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{\mathcal{F}}_\phi(x_e, t^n) - \bar{\mathcal{F}}_\phi(x_w, t^n)\right]$$

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$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x}\left[\left(a\phi_E^n - \frac{a}{2}\sigma_E^n(\Delta x + a\Delta t)\right) - \left(a\phi_P^n - \frac{a}{2}\sigma_P^n(\Delta x + a\Delta t)\right)\right]$$

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or,

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x}\left(\phi_E^n - \phi_P^n\right) - \frac{1}{2}\frac{a\Delta t}{\Delta x}(\Delta x + a\Delta t)(\sigma_E^n - \sigma_P^n)$$

Higher Resolution Methods

Numerical Flux

Numerical flux values can be summarized as

$$\bar{\mathcal{F}}_\phi(x_e, t^n) = \begin{cases} a\phi_P^n + \dfrac{a}{2}\sigma_P^n(\Delta x - a\Delta t), & a > 0 \\ a\phi_E^n - \dfrac{a}{2}\sigma_E^n(\Delta x + a\Delta t), & a < 0 \end{cases}$$

or,

$$\begin{aligned} \bar{\mathcal{F}}_\phi(x_e, t^n) &= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t) \\ &= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t) \end{aligned}$$

Higher Resolution Methods
Numerical Flux

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or,

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$$\bar{\mathcal{F}}_\phi(x_w, t^n) = \begin{cases} a\phi_W^n + \dfrac{a}{2}\sigma_W^n(\Delta x - a\Delta t), & a > 0 \\ a\phi_P^n - \dfrac{a}{2}\sigma_P^n(\Delta x + a\Delta t), & a < 0 \end{cases}$$

Higher Resolution Methods
Choice of Slopes

Choice of Slopes

- Zero Slope: $\sigma_P^n = 0$ Godunov

Higher Resolution Methods

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Choice of Slopes

- Zero Slope: $\sigma_P^n = 0$ Godunov
- Centred Slope: $\sigma_P^n = \frac{\phi_E^n - \phi_W^n}{2\Delta x}$ Fromm

Higher Resolution Methods

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- Zero Slope: $\sigma_P^n = 0$ Godunov
- Centred Slope: $\sigma_P^n = \frac{\phi_E^n - \phi_W^n}{2\Delta x}$ Fromm
- Upwind Slope: $\sigma_P^n = \frac{\phi_P^n - \phi_W^n}{\Delta x}$ BeamWarming

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- Downwind Slope: $\sigma_P^n = \frac{\phi_E^n - \phi_P^n}{\Delta x}$ LaxWendroff

Higher Resolution Methods

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σ_P^n approximates the derivative $\phi_{,x}$ over the P^{th} cell.

Total Variation
TVD

Is there any limit for the Slope ?

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Total variation of a function can be defined as

$$TV(\phi) = \sum_{\forall i} |\phi_i - \phi_{i-1}|$$

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Total Variation Diminishing (TVD)

$$TV(\phi^{n+1}) \leq TV(\phi^n)$$

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Monotonicity Preserving Method

If

$$\phi_i^n \geq \phi_{i+1}^n, \quad \forall i$$

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$$TV(\phi^{n+1}) \leq TV(\phi^n)$$

Monotonicity Preserving Method

If

$$\phi_i^n \geq \phi_{i+1}^n, \quad \forall i$$

then

$$\phi_i^{n+1} \geq \phi_{i+1}^{n+1}, \quad \forall i$$

Total Variation
TVD

Is there any limit for the Slope ?

Total variation of a function can be defined as

$$TV(\phi) = \sum_{\forall i} |\phi_i - \phi_{i-1}|$$

Total Variation Diminishing (TVD)

$$TV(\phi^{n+1}) \leq TV(\phi^n)$$

Monotonicity Preserving Method

If

$$\phi_i^n \geq \phi_{i+1}^n, \quad \forall i$$

then

$$\phi_i^{n+1} \geq \phi_{i+1}^{n+1}, \quad \forall i$$

TVD is Monotonicity Preserving Method

Higher Resolution Methods
Slope-Limiter

First Order Upwind

$$\sigma_P^n = 0$$

Higher Resolution Methods

Slope-Limiter

First Order Upwind

$$\sigma_P^n = 0$$

Minmod

$$\sigma_P^n = minmod\left(\frac{\phi_P^n - \phi_W^n}{\Delta x}, \frac{\phi_E^n - \phi_P^n}{\Delta x}\right)$$

where

$$minmod(\alpha, \beta) = \begin{cases} \alpha & \text{if} \quad |\alpha| < |\beta| \quad \text{and} \quad \alpha\beta > 0 \\ \beta & \text{if} \quad |\beta| < |\alpha| \quad \text{and} \quad \alpha\beta > 0 \\ 0 & \text{if} \quad \alpha\beta \leq 0 \end{cases}$$

Higher Resolution Methods

Slope-Limiter

Superbee Limiter

$$\sigma_P^n = maxmod(\sigma_P^{(1)}, \sigma_P^{(2)})$$

Higher Resolution Methods

Slope-Limiter

Superbee Limiter

$$\sigma_P^n = maxmod(\sigma_P^{(1)}, \sigma_P^{(2)})$$

where

$$\sigma_P^{(1)} = minmod\left(\frac{\phi_E^n - \phi_P^n}{\Delta x}, 2\frac{\phi_P^n - \phi_W^n}{\Delta x}\right)$$

Higher Resolution Methods

Slope-Limiter

Superbee Limiter

$$\sigma_P^n = maxmod(\sigma_P^{(1)}, \sigma_P^{(2)})$$

where

$$\sigma_P^{(1)} = minmod\left(\frac{\phi_E^n - \phi_P^n}{\Delta x}, 2\frac{\phi_P^n - \phi_W^n}{\Delta x}\right)$$

$$\sigma_P^{(2)} = minmod\left(2\frac{\phi_E^n - \phi_P^n}{\Delta x}, \frac{\phi_P^n - \phi_W^n}{\Delta x}\right)$$

Flux Limiter

$$
\begin{aligned}
\bar{\mathcal{F}}_\phi(x_e, t^n) &= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t) \\
&= a^+\phi_P^n + a^-\phi_E^n + \frac{1}{2}|a|\left(1 - \left|\frac{a\Delta t}{\Delta x}\right|\right)\Psi(\theta_e^n)(\phi_E^n - \phi_P^n)
\end{aligned}
$$

Flux Limiter

$$\bar{\mathcal{F}}_\phi(x_e, t^n) = a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t)$$
$$= a^+\phi_P^n + a^-\phi_E^n + \frac{1}{2}|a|\left(1 - \left|\frac{a\Delta t}{\Delta x}\right|\right)\Psi(\theta_e^n)(\phi_E^n - \phi_P^n)$$

where

$$\theta_e^n = \begin{cases} \dfrac{\phi_P^n - \phi_W^n}{\phi_E^n - \phi_P^n} & \text{for} \quad a \geq 0 \\ \dfrac{\phi_E^n - \phi_{EE}^n}{\phi_E^n - \phi_P^n} & \text{for} \quad a \leq 0 \end{cases}$$

Flux Limiter

$$
\begin{aligned}
\bar{\mathcal{F}}_\phi(x_w, t^n) &= a^+\phi_W^n + a^-\phi_P^n + \frac{a^+}{2}\sigma_W^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_P^n(\Delta x + a\Delta t) \\
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\end{aligned}
$$

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where

$$\theta_w^n = \begin{cases} \dfrac{\phi_W^n - \phi_{WW}^n}{\phi_P^n - \phi_W^n} & \text{for} \quad a \geq 0 \\ \dfrac{\phi_E^n - \phi_P^n}{\phi_P^n - \phi_W^n} & \text{for} \quad a \leq 0 \end{cases}$$

Flux Limiter

Linear Models

- Upwind: $\Psi(\theta) = 0$

Flux Limiter

Linear Models

- Upwind: $\Psi(\theta) = 0$
- Lax-Wendroff: $\Psi(\theta) = 1$

Flux Limiter

Linear Models

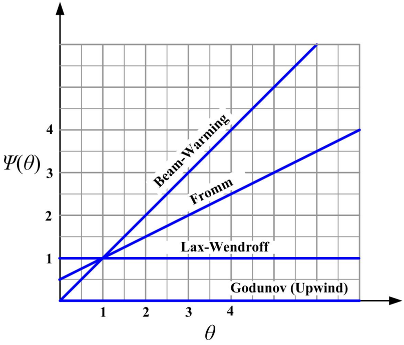
- Upwind: $\Psi(\theta) = 0$
- Lax-Wendroff: $\Psi(\theta) = 1$
- Beam-Warming: $\Psi(\theta) = \theta$

Flux Limiter

Linear Models

- Upwind: $\Psi(\theta) = 0$
- Lax-Wendroff: $\Psi(\theta) = 1$
- Beam-Warming: $\Psi(\theta) = \theta$
- Fromm: $\Psi(\theta) = \frac{1}{2}(1 + \theta)$

Flux Limiter



Flux Limiter

High-resolution Limiter Models

- minmod: $\Psi(\theta) = minmod(1, \theta)$

Flux Limiter

High-resolution Limiter Models

- minmod: $\Psi(\theta) = minmod(1, \theta)$
- superbee: $\Psi(\theta) = max(0, min(1, 2\theta), min(2, \theta))$

Flux Limiter

High-resolution Limiter Models

- minmod: $\Psi(\theta) = minmod(1, \theta)$
- superbee: $\Psi(\theta) = max(0, min(1, 2\theta), min(2, \theta))$
- MC: $\Psi(\theta) = max(0, min((1 + \theta)/2, 2, 2\theta))$

Flux Limiter

High-resolution Limiter Models

- minmod: $\Psi(\theta) = minmod(1, \theta)$
- superbee: $\Psi(\theta) = max(0, min(1, 2\theta), min(2, \theta))$
- MC: $\Psi(\theta) = max(0, min((1+\theta)/2, 2, 2\theta))$
- van Leer: $\Psi(\theta) = \frac{\theta+|\theta|}{1+|\theta|}$

Thank You

References

Guinot, V. (2010). *Scalar Hyperbolic Conservation Laws in One Dimension of Space*, pages 1–53. ISTE.

LeVeque, R. J. (2002). *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press, Cambridge.