Module 02: Numerical Methods

Unit 07: Partial Differential Equation: Numerical Stability of IBVP

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Learning Objective

- To analyze the numerical stability of the discretized PDE.

Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega : \quad \Lambda_\phi \frac{\partial \phi}{\partial t} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y)$$

Problem Definition

subject to

Initial Condition

$$\phi(x, y, 0) = \phi_0(x, y)$$

and

Boundary Condition

$$\begin{aligned}
\Gamma_D^1 : &\quad \phi(0, y, t) = \phi_1 \\
\Gamma_D^2 : &\quad \phi(L_x, y, t) = \phi_2 \\
\Gamma_N^3 : &\quad \left.\frac{\partial \phi}{\partial y}\right|_{(x,0,t)} = 0 \\
\Gamma_N^4 : &\quad \left.\frac{\partial \phi}{\partial y}\right|_{(x,L_y,t)} = 0
\end{aligned}$$

Errors

Discretization Error (Biswas, 2003)

Discretization Error =

Analytical Solution of PDE - Exact Solution of the Finite Difference Equation (obtained on a hypothetical infinite precision computer)

=Truncation Error + Error due to treatment of boundary conditions

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Round-off Error

Round-off Error (ϵ)=

Numerical Solution of the Finite Difference Equation (obtained from finite precision computer) - Exact Solution of the Finite Difference Equation (obtained on a hypothetical infinite precision computer)

Numerical Errors

- Every algorithm requires repeated operations (e.g., $\pm$, $\times$). There is an accumulation of round-off error.

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- Numerical Stability/ Instability is a property of the algorithm and discretization of PDE+BCs.

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- In time-stepping algorithm, accumulated round-off error may magnify/ reduce with every step.
- Error may increase exponentially. It is known as *Numerical Instability*.
- Numerical Stability/ Instability is a property of the algorithm and discretization of PDE+BCs.
- It does not depend on the computer used.

Stability Analysis

In stability analysis of linear PDE, we analyze only one arbitrary Fourier mode (Biswas, 2003).

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Let us consider that the error can be represented in the form of Fourier Series and single arbitrary term can be written as,

$$\epsilon_{i,j}^{n} = A^{n} e^{\sqrt{-1} i \omega_x \Delta x + \sqrt{-1} j \omega_y \Delta y}$$

where

ω_x and ω_y are wave numbers corresponding to x and y directions respectively.

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Note that, $|\epsilon_{i,j}^{n}| = |A^{n}|$.

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where

ω_x and ω_y are wave numbers corresponding to x and y directions respectively.

Note that, $|\epsilon_{i,j}^{n}| = |A^{n}|$.

In simplified form, error can be written as,

$$\epsilon_{i,j}^{n} = A^{n} e^{\sqrt{-1} i \varphi_x + \sqrt{-1} j \varphi_y}$$

where

φ_x and φ_y are phase values corresponding to x and y directions respectively.

von Neumann Stability Analysis

Define

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where G is an amplification factor. It governs the growth of the Fourier component.

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$|G| < 1 \Rightarrow$ Error reduces (Stable Scheme).

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Define

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The von Neumann Stability Condition is given by $|G| \leq 1$.

$|G| > 1 \Rightarrow$ Error grows (Unstable Scheme).

$|G| < 1 \Rightarrow$ Error reduces (Stable Scheme).

$|G| = 1 \Rightarrow$ Error remains same (Neutrally Stable Scheme).

Explicit Scheme

The discretized governing equation for IBVP with explicit scheme can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i+1,j}^{n}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j+1}^{n}}{\Delta y^2} + S_\phi\Big|_{i,j}^{n}$$

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The general variable ϕ can be written as,

$$\phi_{i,j}^n = \hat{\phi}_{i,j}^n + \epsilon_{i,j}^n$$

where
$\phi_{i,j}^n$ = Numerical solution obtained from finite precision computer
$\hat{\phi}_{i,j}^n$ = Exact discrete solution obtained on a hypothetical infinite precision computer
$\epsilon_{i,j}^n$ = Accumulated round-off error at time level n.

Explicit Scheme

The discretized governing equation for IBVP with explicit scheme can be written as,

$$\begin{aligned}
&\Lambda_\phi \frac{(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) - (\hat{\phi}_{i,j}^{n} + \epsilon_{i,j}^{n})}{\Delta t} = \\
&\Gamma_x \frac{(\hat{\phi}_{i-1,j}^{n} + \epsilon_{i-1,j}^{n}) - 2(\hat{\phi}_{i,j}^{n} + \epsilon_{i,j}^{n}) + (\hat{\phi}_{i+1,j}^{n} + \epsilon_{i+1,j}^{n})}{\Delta x^2} + \\
&\Gamma_y \frac{(\hat{\phi}_{i,j-1}^{n} + \epsilon_{i,j-1}^{n}) - 2(\hat{\phi}_{i,j}^{n} + \epsilon_{i,j}^{n}) + (\hat{\phi}_{i,j+1}^{n} + \epsilon_{i,j+1}^{n})}{\Delta y^2} \\
&+ S_\phi\Big|_{i,j}^{n}
\end{aligned} \tag{1}$$

Explicit Scheme

The discretized governing equation for IBVP with explicit scheme can be written as,

$$\begin{aligned}
&\Lambda_\phi \frac{(\hat{\phi}^{n+1}_{i,j} + \epsilon^{n+1}_{i,j}) - (\hat{\phi}^{n}_{i,j} + \epsilon^{n}_{i,j})}{\Delta t} = \\
&\Gamma_x \frac{(\hat{\phi}^{n}_{i-1,j} + \epsilon^{n}_{i-1,j}) - 2(\hat{\phi}^{n}_{i,j} + \epsilon^{n}_{i,j}) + (\hat{\phi}^{n}_{i+1,j} + \epsilon^{n}_{i+1,j})}{\Delta x^2} + \\
&\Gamma_y \frac{(\hat{\phi}^{n}_{i,j-1} + \epsilon^{n}_{i,j-1}) - 2(\hat{\phi}^{n}_{i,j} + \epsilon^{n}_{i,j}) + (\hat{\phi}^{n}_{i,j+1} + \epsilon^{n}_{i,j+1})}{\Delta y^2} \\
&+ S_\phi\big|^{n}_{i,j}
\end{aligned} \tag{1}$$

By definition, $\hat{\phi}$ is the exact discrete solution of the finite difference equation. Thus, the discretized finite difference equation can be written as,

$$\begin{gathered}
\Lambda_\phi \frac{\hat{\phi}^{n+1}_{i,j} - \hat{\phi}^{n}_{i,j}}{\Delta t} = \\
\Gamma_x \frac{\hat{\phi}^{n}_{i-1,j} - 2\hat{\phi}^{n}_{i,j} + \hat{\phi}^{n}_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\hat{\phi}^{n}_{i,j-1} - 2\hat{\phi}^{n}_{i,j} + \hat{\phi}^{n}_{i,j+1}}{\Delta y^2} + S_\phi\big|^{n}_{i,j}
\end{gathered} \tag{2}$$

Explicit Scheme

By subtracting Equation 2 from Equation 1, we get the error equation

$$\Lambda_\phi \frac{\epsilon_{i,j}^{n+1} - \epsilon_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\epsilon_{i-1,j}^{n} - 2\epsilon_{i,j}^{n} + \epsilon_{i+1,j}^{n}}{\Delta x^2} + \Gamma_y \frac{\epsilon_{i,j-1}^{n} - 2\epsilon_{i,j}^{n} + \epsilon_{i,j+1}^{n}}{\Delta y^2} \quad (3)$$

Explicit Scheme

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In simplified form, this can be written as,

$$\epsilon_{i,j}^{n+1} = \alpha_y \epsilon_{i,j-1}^{n} + \alpha_x \epsilon_{i-1,j}^{n} + \left[1 - 2(\alpha_x + \alpha_y)\right] \epsilon_{i,j}^{n} + \alpha_x \epsilon_{i+1,j}^{n} + \alpha_y \epsilon_{i,j+1}^{n}$$

with $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\phi \Delta x^2}$ and $\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\phi \Delta y^2}$.

Explicit Scheme

With

$$
\begin{aligned}
\epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i,j}^{n} &= A^{n} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i-1,j}^{n} &= A^{n} e^{\sqrt{-1} (i-1)\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i+1,j}^{n} &= A^{n} e^{\sqrt{-1} (i+1)\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i,j-1}^{n} &= A^{n} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} (j-1)\varphi_y} \\
\epsilon_{i,j+1}^{n} &= A^{n} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} (j+1)\varphi_y}
\end{aligned}
$$

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\epsilon_{i,j-1}^{n} &= A^{n} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} (j-1)\varphi_y} \\
\epsilon_{i,j+1}^{n} &= A^{n} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} (j+1)\varphi_y}
\end{aligned}$$

By substituting all terms in the error equation,

$$\begin{aligned}
\frac{A^{n+1}}{A^n} = \alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} + [1 - 2(\alpha_x + \alpha_y)] \\
+ \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y}
\end{aligned}$$

Explicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = 1 + 2\alpha_y(cos\varphi_y - 1) + 2\alpha_x(cos\varphi_x - 1)$$

$$G = 1 - 4\alpha_y sin^2(\frac{\varphi_y}{2}) - 4\alpha_x sin^2(\frac{\varphi_x}{2})$$

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The von Neumann Stability condition

$$|1 - 4\alpha_y sin^2(\frac{\varphi_y}{2}) - 4\alpha_x sin^2(\frac{\varphi_x}{2})| \leq 1$$

$$-1 \leq 1 - 4\alpha_y sin^2(\frac{\varphi_y}{2}) - 4\alpha_x sin^2(\frac{\varphi_x}{2}) \leq 1$$

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Two Cases:

- $sin(\frac{\varphi_x}{2}) = 0$ and $sin(\frac{\varphi_y}{2}) = 0 \Rightarrow G = 1$

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- $sin(\frac{\varphi_x}{2}) = 0$ and $sin(\frac{\varphi_y}{2}) = 0 \Rightarrow G = 1$
- $sin(\frac{\varphi_x}{2}) = 1$ and $sin(\frac{\varphi_y}{2}) = 1 \Rightarrow G = 1 - 4(\alpha_x + \alpha_y) \Rightarrow (\alpha_x + \alpha_y) \leq \frac{1}{2}$

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$$G = \frac{A^{n+1}}{A^n} = 1 + 2\alpha_y(cos\varphi_y - 1) + 2\alpha_x(cos\varphi_x - 1)$$

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$$|1 - 4\alpha_y sin^2(\frac{\varphi_y}{2}) - 4\alpha_x sin^2(\frac{\varphi_x}{2})| \leq 1$$
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Two Cases:

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Explicit scheme is Conditionally Stable.

Implicit Scheme

The discretized governing equation for IBVP with implicit scheme can be written as,

$$
\begin{aligned}
&\Lambda_\phi \frac{(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) - (\hat{\phi}_{i,j}^{n} + \epsilon_{i,j}^{n})}{\Delta t} = \\
&\Gamma_x \frac{(\hat{\phi}_{i-1,j}^{n+1} + \epsilon_{i-1,j}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i+1,j}^{n+1} + \epsilon_{i+1,j}^{n+1})}{\Delta x^2} + \\
&\Gamma_y \frac{(\hat{\phi}_{i,j-1}^{n+1} + \epsilon_{i,j-1}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i,j+1}^{n+1} + \epsilon_{i,j+1}^{n+1})}{\Delta y^2} \\
&+ S_\phi\big|_{i,j}^{n+1}
\end{aligned}
\tag{4}
$$

Implicit Scheme

The discretized governing equation for IBVP with implicit scheme can be written as,

$$
\begin{aligned}
&\Lambda_\phi \frac{(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) - (\hat{\phi}_{i,j}^{n} + \epsilon_{i,j}^{n})}{\Delta t} = \\
&\Gamma_x \frac{(\hat{\phi}_{i-1,j}^{n+1} + \epsilon_{i-1,j}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i+1,j}^{n+1} + \epsilon_{i+1,j}^{n+1})}{\Delta x^2} + \\
&\Gamma_y \frac{(\hat{\phi}_{i,j-1}^{n+1} + \epsilon_{i,j-1}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i,j+1}^{n+1} + \epsilon_{i,j+1}^{n+1})}{\Delta y^2} \\
&+ S_\phi\big|_{i,j}^{n+1}
\end{aligned}
\tag{4}
$$

By definition, $\hat{\phi}$ is the exact discrete solution of the finite difference equation. Thus, the discretized finite difference equation can be written as,

$$
\begin{gathered}
\Lambda_\phi \frac{\hat{\phi}_{i,j}^{n+1} - \hat{\phi}_{i,j}^{n}}{\Delta t} = \\
\Gamma_x \frac{\hat{\phi}_{i-1,j}^{n+1} - 2\hat{\phi}_{i,j}^{n+1} + \hat{\phi}_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\hat{\phi}_{i,j-1}^{n+1} - 2\hat{\phi}_{i,j}^{n+1} + \hat{\phi}_{i,j+1}^{n+1}}{\Delta y^2} + S_\phi\big|_{i,j}^{n+1}
\end{gathered}
\tag{5}
$$

Implicit Scheme

By subtracting Equation 5 from Equation 4, we get the error equation

$$\Lambda_\phi \frac{\epsilon_{i,j}^{n+1} - \epsilon_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\epsilon_{i-1,j}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\epsilon_{i,j-1}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i,j+1}^{n+1}}{\Delta y^2} \tag{6}$$

Implicit Scheme

By subtracting Equation 5 from Equation 4, we get the error equation

$$\Lambda_\phi \frac{\epsilon_{i,j}^{n+1} - \epsilon_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\epsilon_{i-1,j}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\epsilon_{i,j-1}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i,j+1}^{n+1}}{\Delta y^2} \tag{6}$$

In simplified form, this can be written as,

$$\alpha_y \epsilon_{i,j-1}^{n+1} + \alpha_x \epsilon_{i-1,j}^{n+1} - \left[1 + 2(\alpha_x + \alpha_y)\right] \epsilon_{i,j}^{n+1} + \alpha_x \epsilon_{i+1,j}^{n+1} + \alpha_y \epsilon_{i,j+1}^{n+1} = -\epsilon_{i,j}^{n}$$

with $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\phi \Delta x^2}$ and $\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\phi \Delta y^2}$.

Implicit Scheme

With

$$
\begin{aligned}
\epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i,j}^{n} &= A^{n} e^{\sqrt{-1} i\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i-1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i+1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1} j\varphi_y} \\
\epsilon_{i,j-1}^{n+1} &= A^{n+1} e^{\sqrt{-1} i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\
\epsilon_{i,j+1}^{n+1} &= A^{n+1} e^{\sqrt{-1} i\varphi_x + \sqrt{-1}(j+1)\varphi_y}
\end{aligned}
$$

Implicit Scheme

With

$$\epsilon_{i,j}^{n+1} = A^{n+1} e^{\sqrt{-1} i \varphi_x + \sqrt{-1} j \varphi_y}$$
$$\epsilon_{i,j}^{n} = A^{n} e^{\sqrt{-1} i \varphi_x + \sqrt{-1} j \varphi_y}$$
$$\epsilon_{i-1,j}^{n+1} = A^{n+1} e^{\sqrt{-1} (i-1) \varphi_x + \sqrt{-1} j \varphi_y}$$
$$\epsilon_{i+1,j}^{n+1} = A^{n+1} e^{\sqrt{-1} (i+1) \varphi_x + \sqrt{-1} j \varphi_y}$$
$$\epsilon_{i,j-1}^{n+1} = A^{n+1} e^{\sqrt{-1} i \varphi_x + \sqrt{-1} (j-1) \varphi_y}$$
$$\epsilon_{i,j+1}^{n+1} = A^{n+1} e^{\sqrt{-1} i \varphi_x + \sqrt{-1} (j+1) \varphi_y}$$

By substituting all terms in the error equation,

$$\frac{A^{n+1}}{A^n}\left[\alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} - [1 + 2(\alpha_x + \alpha_y)] + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y}\right] = -1$$

Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(cos\varphi_y - 1) + 2\alpha_x(cos\varphi_x - 1)}$$

Implicit Scheme

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$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(cos\varphi_y - 1) + 2\alpha_x(cos\varphi_x - 1)}$$

$$G = \frac{1}{1 + 4\alpha_y sin^2(\frac{\varphi_y}{2}) + 4\alpha_x sin^2(\frac{\varphi_x}{2})}$$

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$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(cos\varphi_y - 1) + 2\alpha_x(cos\varphi_x - 1)}$$

$$G = \frac{1}{1 + 4\alpha_y sin^2(\frac{\varphi_y}{2}) + 4\alpha_x sin^2(\frac{\varphi_x}{2})}$$

The von Neumann Stability condition

$$\left|\frac{1}{1 + 4\alpha_y sin^2(\frac{\varphi_y}{2}) + 4\alpha_x sin^2(\frac{\varphi_x}{2})}\right| \leq 1$$

Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(cos\varphi_y - 1) + 2\alpha_x(cos\varphi_x - 1)}$$

$$G = \frac{1}{1 + 4\alpha_y sin^2(\frac{\varphi_y}{2}) + 4\alpha_x sin^2(\frac{\varphi_x}{2})}$$

The von Neumann Stability condition

$$\left| \frac{1}{1 + 4\alpha_y sin^2(\frac{\varphi_y}{2}) + 4\alpha_x sin^2(\frac{\varphi_x}{2})} \right| \leq 1$$

Two Cases:

- $sin(\frac{\varphi_x}{2}) = 0$ and $sin(\frac{\varphi_y}{2}) = 0 \Rightarrow G = 1$

Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(cos\varphi_y - 1) + 2\alpha_x(cos\varphi_x - 1)}$$

$$G = \frac{1}{1 + 4\alpha_y sin^2(\frac{\varphi_y}{2}) + 4\alpha_x sin^2(\frac{\varphi_x}{2})}$$

The von Neumann Stability condition

$$\left|\frac{1}{1 + 4\alpha_y sin^2(\frac{\varphi_y}{2}) + 4\alpha_x sin^2(\frac{\varphi_x}{2})}\right| \leq 1$$

Two Cases:

- $sin(\frac{\varphi_x}{2}) = 0$ and $sin(\frac{\varphi_y}{2}) = 0 \Rightarrow G = 1$
- $sin(\frac{\varphi_x}{2}) = 1$ and $sin(\frac{\varphi_y}{2}) = 1 \Rightarrow G = \frac{1}{1+4\alpha_y+4\alpha_x} < 1$

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Implicit scheme is Unconditionally Stable.

Thank You

References

Biswas, G. (2003). *Computational Fluid Flow and Heat Transfer*. Narosa Publishing House, New Delhi.