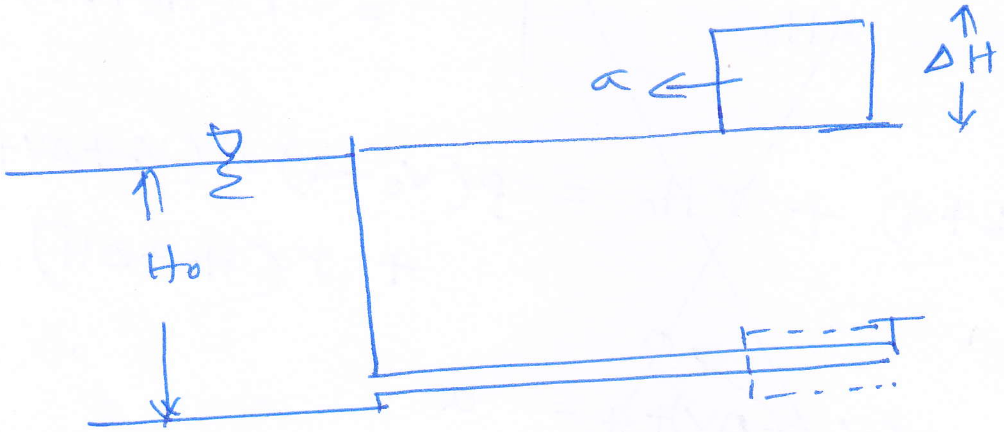
Pressure wave velocity and water hammer pressure rise of slightly compressible fluid in a rigid pipe



ρ $\qquad$ $\rho+\Delta\rho$

V_0+a $\qquad$ $V_0+\Delta V+a$

A $\qquad$ A

H $\qquad$ $H_0+\Delta H$

Apply continuity equation.

$$V_0+\Delta V = 0 \quad \text{(valve is closed)}$$

$$\rho\,(V_0+a)\,A = (\rho+\Delta\rho)\,a\,A$$

divide above equation $\rho a A$

$$\frac{V_0}{a}+1 = 1+\frac{\Delta\rho}{\rho}$$

$$\boxed{\frac{V_0}{a} = \frac{\Delta\rho}{\rho}} \rightarrow (1)$$

Apply momentum equation

$$\rho_1 Q V_{1x} + P_1 A_1 = \rho_2 Q V_{2x} + P_2 A_2$$

$$Q = (V_0 + a) A$$

$$P_1 = \gamma H_0$$

$$V_{x_1} = \frac{V_0 + a}{A}$$

$$Q = (V_0 + a) A$$

$$P_2 = \gamma (H_0 + \Delta H)$$

$$V_{x_2} = \frac{V_0 + \Delta V + a}{A}$$

$$\rho (V_0 + a) A (V_0 + a) + \gamma H_0 A = \rho (V_0 + a) A (V_0 + \Delta V + a) + \gamma (H_0 + \Delta H) A$$

$$\rho (V_0 + a) A \left[V_0 + a - V_0 - \Delta V - a \right] = \gamma (H_0 + \Delta H - H_0) A$$

$$\rho (V_0 + a) \left[-\Delta V \right] = \rho g \Delta H$$

$$\rho (V_0 + a) (-\Delta V) = \Delta p$$

$$V_0 << a$$

$$\boxed{\Delta p = -\rho a \Delta V} \rightarrow (2)$$

$$\Delta H = \frac{-a \Delta V}{g} \rightarrow (3)$$

for sudden closure of valve

$$\Delta V = -V_0$$

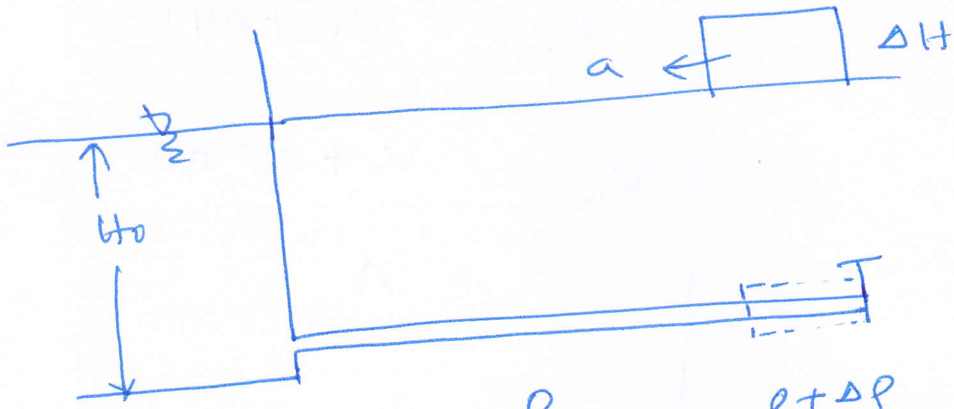
$$\boxed{\Delta H = \frac{a V_0}{g}} \rightarrow (4)$$

from the definition of bulk modulus

$$K = \frac{\Delta p}{\frac{\Delta \rho}{\rho}} \Rightarrow \Delta p = K \frac{\Delta \rho}{\rho}$$

$$\boxed{\Delta p = \rho a V_0} \rightarrow (5) \Rightarrow \rho a V_0 = K \frac{\Delta \rho}{\rho} \Rightarrow \boxed{a = \sqrt{\frac{K}{\rho}}} \rightarrow (6)$$

Pressure wave velocity of slightly compressible fluid in non-rigid pipe



ρ	$\rho+\Delta\rho$
V_0+a	$(V_0+\Delta V+a) = a$
A	$A+\Delta A$
H_0	$H_0+\Delta H$

Apply continuity equation

$$\rho(V_0+a)A = (\rho+\Delta\rho)\,a\,(A+\Delta A)$$

divide above equation by $\rho a A$

$$\frac{V_0}{a}+1 = \left(1+\frac{\Delta\rho}{\rho}\right)\left(1+\frac{\Delta A}{A}\right)$$

$$\Rightarrow \quad 1+\frac{V_0}{a} = 1+\frac{\Delta\rho}{\rho}+\frac{\Delta A}{A}+\cancel{\frac{\Delta\rho}{\rho}\frac{\Delta A}{A}}^{\,0} \rightarrow (7)$$

from compressibility $\quad K = \dfrac{\Delta p}{(\Delta\rho/\rho)}$

$$\Rightarrow \quad \frac{\Delta\rho}{\rho} = \frac{\Delta p}{K} \rightarrow (8)$$

from Eq.(5) $\quad V_0 = \dfrac{\Delta p}{\rho a} \rightarrow (9)$

substitute (8) & (9) in (7)

$$1 + \frac{V_0}{a} = 1 + \frac{\Delta p}{K} + \frac{\Delta A}{A} \rightarrow (10)$$

Substitute (9) in (10) $\Rightarrow$ $\boxed{1 + \frac{\Delta \rho}{\rho a^2} = 1 + \frac{\Delta p}{K} + \frac{\Delta A}{A}} \rightarrow (10)$

from hoop stress

$$\sigma = \frac{\Delta p \cdot (D/2)}{t}$$

t = thickness

σ = circumferential stress

stress = Young's modulus × strain

$$\sigma = E \frac{\Delta D}{D}$$

strain $= \frac{\pi \Delta D}{\pi D}$

$$E \frac{\Delta D}{D} = \frac{\Delta p \cdot D}{2t}$$

$$\boxed{\frac{\Delta D}{D} = \frac{D}{2Et} \cdot \Delta p} \rightarrow (11)$$

$$A = \frac{\pi D^2}{4} \Rightarrow \Delta A = \frac{2\pi D \, dD}{4}$$

$$\frac{\Delta A}{A} = \frac{2 \, dD}{D} = 2 \frac{\Delta D}{D}$$

$$\frac{\Delta A}{A} = 2 \cdot \frac{D}{2Et} \Delta p = \frac{D}{Et} \Delta p$$

$$\boxed{\frac{\Delta A}{A} = \frac{D}{Et} \Delta p} \rightarrow (12)$$

from (10)

$$1 + \frac{\Delta p}{\rho a^2} = 1 + \frac{\Delta p}{K} + \frac{D}{Et} \Delta p$$

$$\frac{1}{\rho a^2} = \frac{1}{K} + \frac{D}{Et}$$

$$\frac{1}{\rho a^2} = \frac{1}{K'}$$ where $\frac{1}{K'} = \frac{1}{K} + \frac{D}{Et}$

$$\frac{1}{K'} = \frac{Et + DK}{KEt}$$

$$K' = \frac{KEt}{Et + DK}$$

$$a = \sqrt{\frac{K'}{\rho}} \rightarrow (13)$$

$$a = \sqrt{\frac{KEt}{\rho(Et + DK)}} \rightarrow (14)$$

pressure wave velocity of compressible fluid in an elastic pipe

$$a=\sqrt{\frac{K}{\rho_0}}$$

This formula is applicable only for slightly compressible fluid in a rigid pipe.

Reynolds Transport Theorem (RTT)

$$\frac{dB_{sys}}{dt}=\frac{d}{dt}\int_v \beta\rho dv+\left(\beta\rho AV_s\right)_{out}-\left(\beta\rho AV_s\right)_{in}$$

$B=$ extensive property

$\beta=$ intensive property

$v=$ Total volume

Note that the velocity V_s is with respect to the control surface since it accounts for the inflow or outflow from the control volume.

For fixed control volume, $V_s=$ fluid flow velocity V. If control volume is moving with 'w' velocity then $V_s=V-w$

$$\frac{dB_{sys}}{dt}=\frac{d}{dt}\int_v \beta\rho dv+\left[\beta\rho A\left(V-w\right)\right]_{out}-\left[\beta\rho A\left(V-w\right)\right]_{in}$$

Derivation of Continuity Equation

Assumptions

1) Flow is slightly compressible
2) Conduit walls are elastic
3) Flow is one directional.

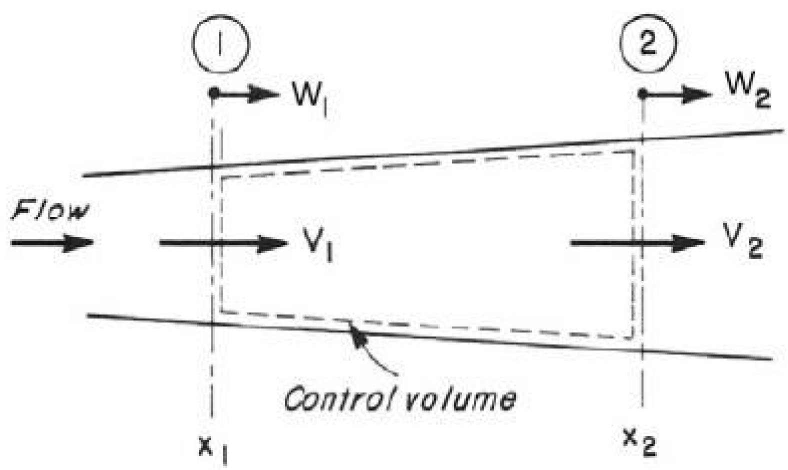


Fig. 2.9. Application of continuity equation

Control surfaces are moving with w_1 and w_2 velocities, that means control volume is either contracting or expanding. Rapid expansion and contraction are taken into account in this derivation.

Applying RTT, $\beta = 1$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho A dx + \rho_2 A_2 \left(V_2 - w_2\right) - \rho_1 A_1 \left(V_1 - w_1\right) = 0$$

Then apply Leibnitz's rule to the first term on the LHS.

We know, Leibnitz theorem

$$\frac{d}{dt}\int_{f_1(t)}^{f_2(t)} F\left(x,t\right)dx = \int_{f_1(t)}^{f_2(t)} \frac{\partial}{\partial t} F\left(x,t\right)dx + F\left(f_2\left(t\right),t\right)\frac{df_2}{dt} - F\left(f_1\left(t\right),t\right)\frac{df_1}{dt}$$

Therefore first term on LHS can be written as

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho A dx = \int \frac{\partial}{\partial t}\left(\rho A\right)dx + \rho_2 A_2 \frac{dx_2}{dt} - \rho_1 A_1 \frac{dx_1}{dt}$$

Total equation can be written as

$$\int \frac{\partial}{\partial t}\left(\rho A\right)dx + \rho_2 A_2 w_2 - \rho_1 A_1 w_1 + \rho_2 A_2 \left(V_2 - w_2\right) - \rho_1 A_1 \left(V_1 - w_1\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho A\right)\Delta x + \rho_2 A_2 V_2 - \rho_1 A_1 V_1 = 0$$

$$\frac{\partial}{\partial t}\left(\rho A\right) + \frac{\partial}{\partial x}\left(\rho A V\right) = 0$$

Expansion of terms inside the parenthesis

$$A\frac{\partial \rho}{\partial t} + \rho\frac{\partial A}{\partial t} + \rho A\frac{\partial V}{\partial x} + \rho V\frac{\partial A}{\partial x} + AV\frac{\partial \rho}{\partial x} = 0$$

Divide it by $\left(\rho A\right)$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{A}\frac{\partial A}{\partial t} + \frac{\partial V}{\partial x} + \frac{V}{A}\frac{\partial A}{\partial x} + \frac{V}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{A}\frac{dA}{dt} + \frac{\partial V}{\partial x} = 0$$

We need to convert the above equation interms of pressure and velocity V.

The bulk modulus of Elasticity, $K = \dfrac{dP}{\dfrac{d\rho}{\rho_0}} = -\dfrac{dP}{\dfrac{\Delta V}{V_0}}$

The equation may be written as $\frac{d\rho}{dt} = \rho \frac{dP}{Kdt}$

For circular conduits, $A = \pi R^2$, $R =$ radius.

$$\frac{dA}{dt} = 2\pi R \frac{dR}{dt}$$

The above equation may be written as

$$\frac{dA}{dt} = 2\pi R^2 \frac{1}{R}\frac{dR}{dt}$$

$\frac{dR}{R} = d\varepsilon$ Where $\varepsilon =$ Strain

$$\frac{1}{A}\frac{dA}{dt} = 2 \cdot \frac{d\varepsilon}{dt}$$

For circular conduits, $\varepsilon = \frac{\sigma_h - \sigma_a \mu}{E}$

Where $\sigma_h =$ hoops stress, $\sigma_a =$ axial stress and $\mu =$ poison ratio.

Pipe has expansion joints, therefore $\sigma_a = 0$.

So, $\varepsilon = \frac{\sigma_h}{E}$

Hoop stress is defined as, $\sigma_h = \frac{PD}{2e}$

By taking the time derivative on both sides, we can get

$$\frac{d\sigma_h}{dt} = \frac{P}{2e}\frac{dD}{dt} + \frac{D}{2e}\frac{dP}{dt}$$

$\sigma_h = E\varepsilon$, where E is a constant.

$$E\frac{d\varepsilon}{dt} = \frac{P}{2e}\frac{dD}{dt} + \frac{D}{2e}\frac{dP}{dt}$$

We know that, $\frac{1}{A}\frac{dA}{dt} = 2 \cdot \frac{d\varepsilon}{dt} \quad \Rightarrow \frac{1}{\frac{\pi D^2}{4}} \cdot \frac{1}{4}\frac{d\left(\pi D^2\right)}{dt} = 2\frac{d\varepsilon}{dt}$

$$\frac{1}{D}\frac{dD}{dt}=\frac{d\varepsilon}{dt}$$

$$E\frac{d\varepsilon}{dt}=\frac{P}{2e}D\frac{d\varepsilon}{dt}+\frac{D}{2e}\frac{dP}{dt}$$

$$\frac{d\varepsilon}{dt}=\frac{\frac{D}{2e}\frac{dP}{dt}}{E-\frac{PD}{2e}}$$

We already know that, $\frac{1}{A}\frac{dA}{dt}=2\cdot\frac{d\varepsilon}{dt}$

$$\frac{1}{2A}\frac{dA}{dt}=\frac{\frac{D}{2e}\frac{dP}{dt}}{E-\frac{PD}{2e}}$$

Continuity equation

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{1}{A}\frac{dA}{dt}+\frac{\partial V}{\partial x}=0$$

$$\frac{d\rho}{dt}=\frac{\rho}{K}\frac{dP}{dt}$$

$$\frac{1}{\rho}\frac{\rho}{K}\frac{dP}{dt}+\left(\frac{\frac{D}{e}}{E-\frac{PD}{2e}}\right)\cdot\frac{dP}{dt}+\frac{\partial V}{\partial x}=0$$

$$\left(\frac{1}{K}+\frac{1}{\frac{eE}{D}-\frac{P}{2}}\right)\cdot\frac{dP}{dt}+\frac{\partial V}{\partial x}=0\,,\qquad \text{Where } \frac{P}{2}<<\frac{eE}{D}$$

$$\frac{1}{K}\left(1+\frac{DK}{eE}\right)\cdot\frac{dP}{dt}+\frac{\partial V}{\partial x}=0$$

$$\frac{K}{\left(1+\frac{DK}{eE}\right)}\cdot\frac{\partial V}{\partial x}+\frac{dP}{dt}=0$$

$$\rho \frac{\frac{K}{\rho}}{\left(1+\frac{DK}{eE}\right)} \cdot \frac{\partial V}{\partial x} + \frac{\partial P}{\partial t} + V\frac{\partial P}{\partial x} = 0$$

$$\Rightarrow \rho a^2 \cdot \frac{\partial V}{\partial x} + \frac{\partial P}{\partial t} + V\frac{\partial P}{\partial x} = 0$$

Derivation of momentum equation

$$\frac{d}{dt}\int_v \beta\rho dv + \beta\rho\left(V-w\right)A\Big|_{outlet} - \beta\rho\left(V-w\right)A\Big|_{inlet} = 0$$

For momentum equation $\beta = V$

$$\frac{d}{dt}\int_v V\rho dv + V\rho\left(V-w\right)A\Big|_{outlet} - V\rho\left(V-w\right)A\Big|_{inlet} = \sum F$$

By applying Leibnitz rule to the first term

$$\sum F = \int_{x_1}^{x_2} \frac{\partial}{\partial t}\left(\rho AV\right)dx + \rho AV\Big|_{out}\frac{dx_2}{dt} - \rho AV\Big|_{in}\frac{dx_1}{dt} + \rho A\left(V-w\right)V\Big|_2 - \rho A\left(V-w\right)V\Big|_1$$

$$\sum F = \int_{x_1}^{x_2} \frac{\partial}{\partial t}\left(\rho AV\right)dx + \left(\rho AV\right)_2 w_2 - \left(\rho AV\right)_1 w_1 + \left[\rho A\left(V-w\right)V\right]_2 - \left[\rho A\left(V-w\right)V\right]_1$$

$$\frac{\sum F}{\Delta x} = \frac{\partial}{\partial t}\left(\rho AV\right) + \frac{\left(\rho AV^2\right)_2 - \left(\rho AV^2\right)_1}{\Delta x}$$

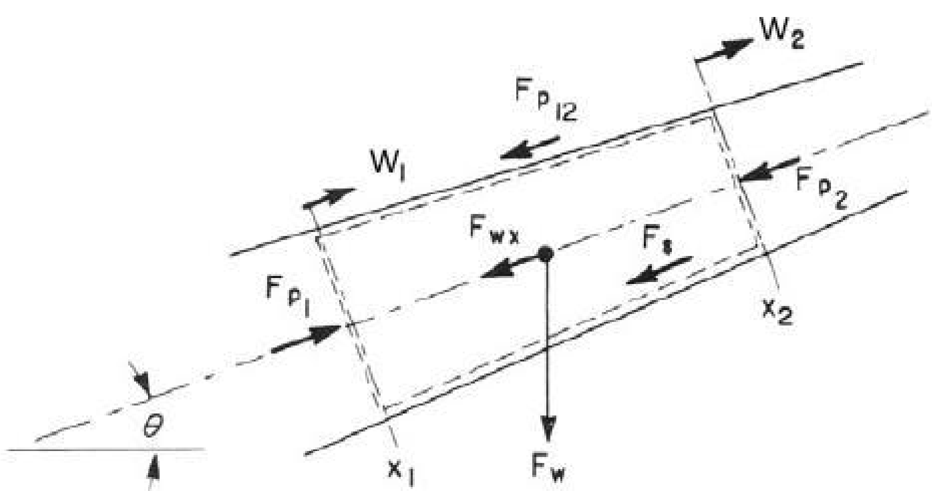


Fig. 2.10. Application of momentum equation