1-4 Basic Waterhammer Equations

In this section, we derive the basic waterhammer equations — expressions for the velocity of pressure waves in a conduit and for the change in pressure due to instantaneous change in the flow velocity.

Let us consider the flow in a frictionless pipe (Fig. 1-1) in which a slightly compressible fluid is flowing with velocity V_o, and the initial steady state pressure head upstream of the valve is H_o. Let the flow velocity V_o be changed instantaneously at time $t = 0$ to $V_o + \Delta V$. An increase in the flow velocity ΔV and an increase in pressure ΔH are considered positive and a decrease, as negative. As a result of this change in the flow velocity, the pressure head H_o changes to $H_o + \Delta H$, the fluid density ρ_o changes to $\rho_o + \Delta\rho$, and a pressure wave of magnitude ΔH travels in the upstream direction. Let us designate the velocity of the pressure wave (commonly called *wave velocity*) by a, and, to simplify the derivation, let us assume the pipe walls are rigid, i.e., the pipe area, A, does not change due to pressure changes. In the next chapter, an expression for the wave velocity is derived in which the fluid is slightly compressible and the pipe walls are slightly deformable.

The unsteady flow of Fig. 1-1a may be converted into steady flow by superimposing velocity a in the downstream direction. This is equivalent to an observer traveling in the upstream direction with velocity a to whom the moving wave front appears as stationary (Fig. 1-1b), and the inflow and outflow velocities from the control volume are $(V_o + a)$ and $(V_o + \Delta V + a)$, respectively.

Let us consider distance, x, and velocity, V, in the downstream direction as positive. Referring to Fig. 1-1b, the time rate of change of momentum in the positive x-direction

$$\begin{aligned} &= \rho_o (V_o + a) A [(V_o + \Delta V + a) - (V_o + a)] \\ &= \rho_o (V_o + a) A \Delta V \end{aligned} \tag{1-4}$$

Neglecting friction, the resultant force, F, acting on the fluid in the control volume in the positive x-direction

$$F = \rho_o g H_o A - \rho_o g (H_o + \Delta H) A = -\rho_o g \Delta H A \tag{1-5}$$

According to Newton's second law of motion, the time rate of change of momentum is equal to the resultant force. Hence, it follows from Eqs. 1-4 and 1-5 that

$$\Delta H = -\frac{1}{g} (V_o + a) \Delta V \tag{1-6}$$

The wave velocity a in metal or concrete pipes or in the rock tunnels, is approximately 1000 m/s while typical flow velocity is about 10 m/s or less.

Therefore, V_o is significantly smaller than a and may thus be neglected. Then, Eq. 1-6 becomes

$$\Delta H = -\frac{a}{g}\Delta V \tag{1-7}$$

The negative sign on the right-hand side of Eq. 1-7 indicates that the pressure head increases (i.e., ΔH is positive) for a reduction in velocity (i.e., for negative ΔV) and vice versa. Also note that Eq. 1-7 is derived for an instantaneous velocity change at the downstream end of a pipe and for the wave front moving in the upstream direction. Proceeding similarly, it can be proven that for a velocity change at the upstream end and for the wave front to move in the downstream direction,

$$\Delta H = \frac{a}{g}\Delta V \tag{1-8}$$

Note that, unlike Eq. 1-7, there is no negative sign on the right-hand side of Eq. 1-8. This means that the pressure head in this case increases for an increase in velocity and decreases with a decrease in velocity.

If the fluid density change $\Delta\rho$ is caused by the change in pressure, Δp, then referring to Fig. 1-1b,

$$\text{Rate of mass inflow} = \rho_o A\,(V_o + a) \tag{1-9}$$

$$\text{Rate of mass outflow} = (\rho_o + \Delta\rho)\,A\,(V_o + \Delta V + a) \tag{1-10}$$

If the fluid is slightly compressible, the increase in the mass of control volume due to the change in fluid density is small and may be neglected. Therefore, the rate of mass inflow is equal to the rate of mass outflow. Hence,

$$\rho_o A\,(V_o + a) = (\rho_o + \Delta\rho)\,A\,(V_o + \Delta V + a) \tag{1-11}$$

which upon simplification becomes

$$\Delta V = -\frac{\Delta\rho}{\rho_o}\,(V_o + \Delta V + a) \tag{1-12}$$

Since $(V_o + \Delta V) << a$, Eq. 1-12 may be written as

$$\Delta V = -\frac{\Delta\rho}{\rho_o} a \tag{1-13}$$

The bulk modulus of elasticity, K, of a fluid is defined as [Streeter, 1966]

$$K = \frac{\Delta p}{\Delta\rho/\rho_o} \tag{1-14}$$

Hence, it follows from Eqs. 1-13 and 1-14 that

$$a = -K \frac{\Delta V}{\Delta p} \tag{1-15}$$

By utilizing Eq. 1-7 and noting that $\Delta p = \rho_o g \Delta H$, we may write this equation as

$$a = \frac{K}{a\rho_o} \tag{1-16}$$

or

$$a = \sqrt{\frac{K}{\rho_o}} \tag{1-17}$$

Note that this expression for the wave velocity is for a slightly compressible fluid confined in a rigid pipe. In the next chapter, we will discuss how this expression is modified if the pipe walls are elastic.

Example

Compute the velocity of pressure waves in a 0.5-m-diameter pipe conveying oil. Determine the pressure rise if a steady flow of 0.4 m^3/s is instantaneously stopped at the downstream end. Assume that the pipe walls are rigid, the density of the oil, $\rho = 900$ kg/m^3, and the bulk modulus of elasticity of the oil, $K = 1.5$ GPa.

Solution

$$\begin{aligned} A &= \frac{\pi}{4}(0.5)^2 = 0.196 \text{ m}^2 \\ V_o &= \frac{Q_o}{A} = \frac{0.4}{0.196} = 2.04 \text{ m/s} \\ a &= \sqrt{\frac{K}{\rho}} \\ &= \sqrt{\frac{1.5 \times 10^9}{900}} = 1291 \text{ m/s} \end{aligned}$$

Since the flow is completely stopped, $\Delta V = 0 - 2.04 = -2.04$ m/s. Therefore,

$$\begin{aligned} \Delta H &= -\frac{a}{g}\Delta V \\ &= -\frac{1291}{9.81}(-2.04) = 268.5 \text{ m} \end{aligned}$$

A positive sign for ΔH means the pressure rises as a result of this reduction in the flow velocity.

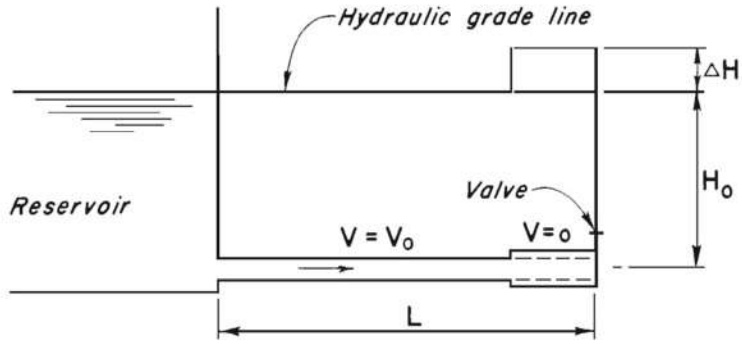
1-5 Wave Propagation

In this section, we discuss transient flow in a piping system with a constant-level reservoir at the upstream end and a valve at the downstream end (Fig. 1-2) to illustrate the propagation of a wave in a pipe and the reflections of the wave from a reservoir and from a closed valve. The pipe walls are considered elastic. Therefore, the pipe expands as the inside pressure increases and contracts as the pressure decreases.

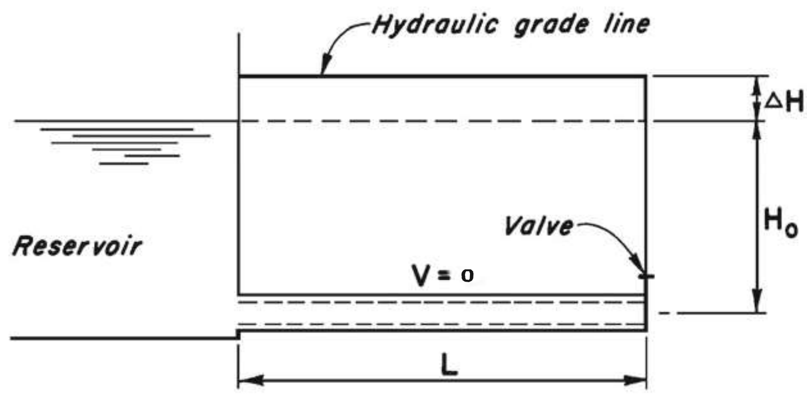
Let the flow conditions in the piping system be steady prior to instantaneous closure of the downstream valve at time $t = 0$. If the system is assumed frictionless, then the initial steady-state pressure head along the length of the pipeline is H_o. Let us consider the distance x and the velocity V as positive in the downstream direction. The upstream and downstream directions are with respect to the initial steady flow.

The sequence of events following valve closure may be divided into four parts (Fig. 1-2) as follows:

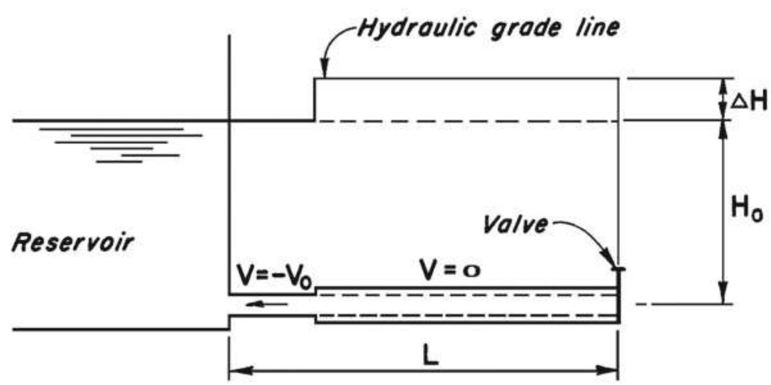
1. $0 < t \leq L/a$
   The flow velocity at the valve is reduced to zero as soon as the valve is completely closed. This increases the pressure at the valve by $\Delta H = (a/g)\, V_o$. Because of this increase in pressure, the pipe expands (the initial steady-state pipe diameter in the expanded or contracted parts of the pipe is shown by dotted lines in Fig. 1-2), the fluid is compressed which increases the fluid density, and a positive pressure wave propagates towards the reservoir. Behind this wave front, the flow velocity is zero, and the kinetic energy has been converted into elastic energy (Fig. 1-2a). If a is the velocity of the pressure wave and L is the length of the pipeline, then the wave front reaches the upstream reservoir at time $t = L/a$. At this time, along the entire length of the pipeline, the pipe is expanded, the flow velocity is zero, and the pressure head is $H_o + \Delta H$ (Fig. 1-2b).
2. $L/a < t \leq 2L/a$
   Just as the wave reaches the upstream reservoir, pressure at a section on the reservoir side is H_o while the pressure at an adjacent section in the pipe is $H_o + \Delta H$. Because of this difference in pressure, the fluid flows from the pipeline into the reservoir with velocity $-V_o$. Thus, the flow velocity at the pipe entrance is reduced from zero to $-V_o$. This causes the pressure to drop from $H_o + \Delta H$ to H_o and a negative wave travels towards the valve (Fig. 1-2c). The pressure behind this wave front (i.e., on the reservoir side) is H_o and the fluid velocity is $-V_o$. At $t = 2L/a$, the wave front reaches the closed valve, and the pressure head in the entire pipeline is H_o, and the fluid velocity is $-V_o$ (Fig. 1-2d).
3. $2L/a < t \leq 3L/a$
   Since the valve is completely closed, a negative velocity cannot be maintained at the valve. Therefore, the velocity changes instantaneously from $-V_o$ to 0, the pressure drops to $H_o - \Delta H$, and a negative wave propagates



(a) Time, $t = \epsilon$

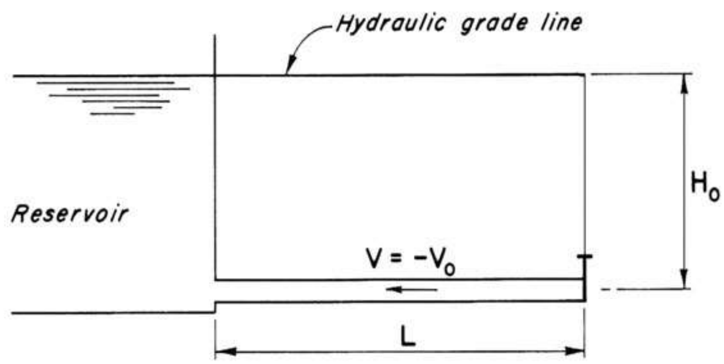


(b) Time, $t = \frac{L}{a}$

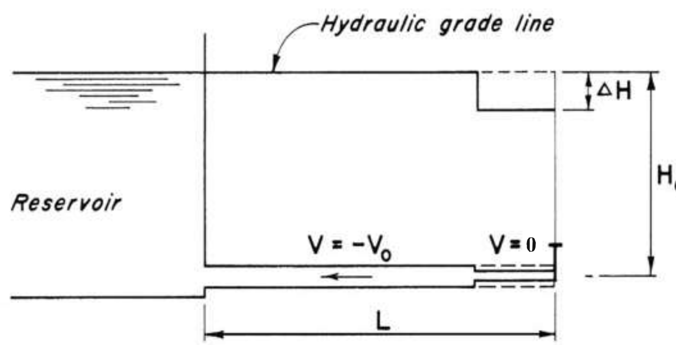


(c) Time, $t = \frac{L}{a} + \epsilon$

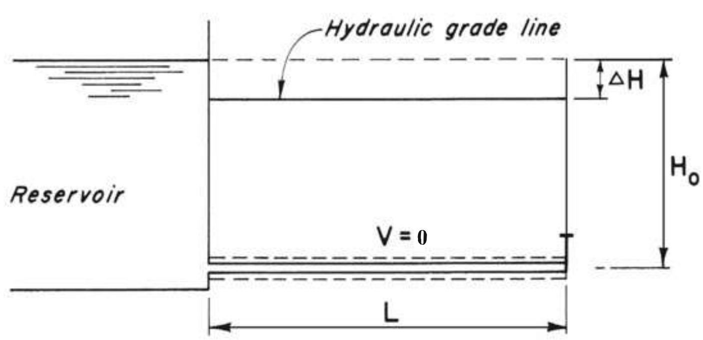
Fig. 1-2. Propagation and Reflection of pressure waves.



(d) **Time,** $t = \frac{2L}{a}$

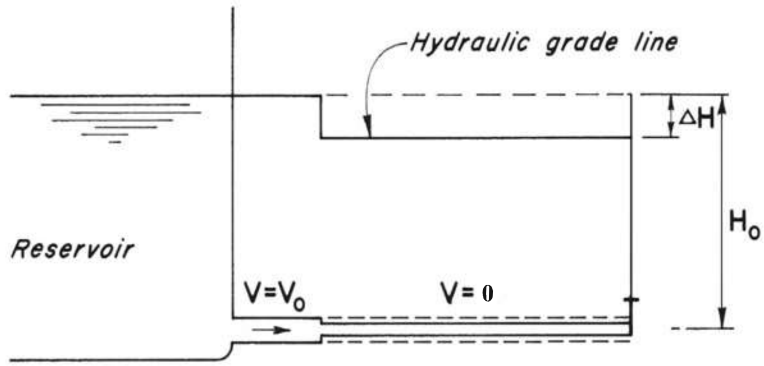


(e) **Time,** $t = \frac{2L}{a} + \epsilon$

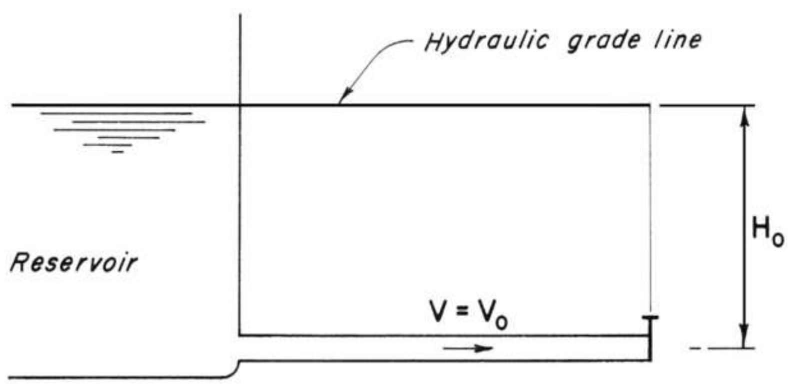


(f) **Time,** $t = \frac{3L}{a}$

Fig. 1-2. *(Continued)*



(g) Time, $t = \frac{3L}{a} + \epsilon$



(h) Time, $t = \frac{4L}{a}$

Fig. 1-2. *(Continued)*

towards the reservoir (Fig. 1-2e). Behind this wave front, the pressure is $H_o - \Delta H$ and the fluid velocity is zero. The wave front reaches the upstream reservoir at time $t = 3L/a$, the pressure head in the entire pipeline is $H_o - \Delta H$, and the fluid velocity is zero (Fig. 1-2f).

4. $3L/a < t \leq 4L/a$
   As soon as this negative wave reaches the reservoir, an unbalanced condi-

tion is created again at the upstream end. Now the pressure is higher on the reservoir side than at an adjacent section in the pipeline. Therefore, the

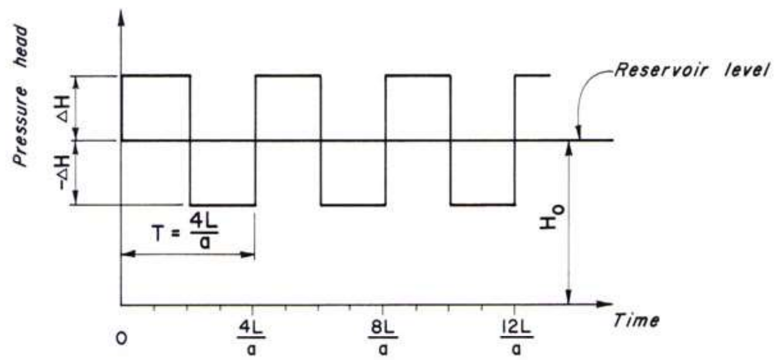


Fig. 1-3. Pressure variation at valve; friction losses neglected.

fluid now flows from the reservoir into the pipeline with velocity V_o, and the pressure head increases to H_o (Fig. 1-2g). At time $t = 4L/a$ the wave front reaches the downstream valve, the pressure head in the entire pipeline is H_o, and the flow velocity is V_o. Thus, the conditions in the pipeline at this time are the same as those during the initial steady-state conditions except that the valve is now closed (Fig. 1-2h).

Since the valve is completely closed, the preceding sequence of events starts again at $t = 4L/a$. Figure 1-2 illustrates the sequence of events along the pipeline, while Fig. 1-3 shows the variation of pressure at the valve with time. Since we assumed the system to be frictionless, this process continues with the conditions repeating at an interval of $4L/a$. The interval at which conditions are repeated is termed the *theoretical period* of the pipeline. In a real system, however, pressure waves are dissipated due to energy losses as the waves propagate back and forth in the pipeline, and the fluid becomes stationary after some time. The variation of pressure at the valve with time is shown in Fig. 1-4 if the friction losses are taken into consideration.

1-6 Wave Reflection and Transmission

In the previous section we discussed the propagation of pressure waves in a pipe and the reflection of the wave from a reservoir and from a closed valve. In this section, we introduce the concept of reflection and transmission coefficients which defines the magnitude and the sign of reflected and transmitted

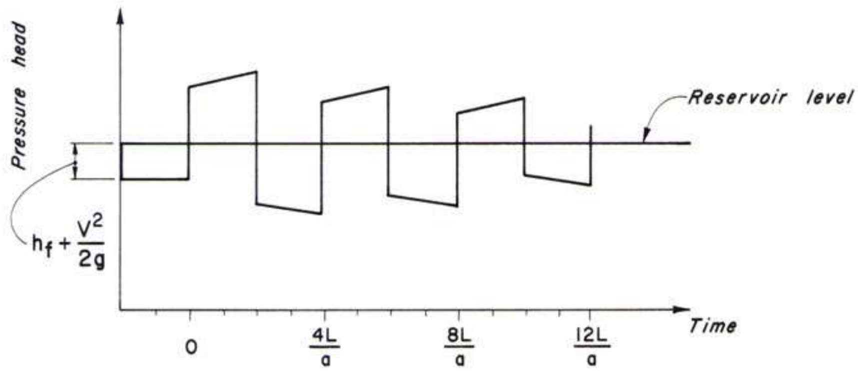


Fig. 1-4. Pressure variation at valve; friction losses considered.

waves from or at a boundary. For simplification, we assume no energy loss as a wave is reflected from or is transmitted at a boundary.

Let us designate an incident pressure wave approaching a boundary as F and the wave reflected from the boundary as f. Then, the reflection coefficient, r, is defined as $r = f/F$. The pressure in a positive pressure wave is higher behind the wave front than ahead of the wave front while the pressure in a negative wave is lower behind the wave front than ahead of the wave front.

Constant-Level Reservoir

A large lake, reservoir, tank or other storage facility may be considered to have a constant level if its water surface level remains unchanged irrespective of the changes in the flow in the pipeline connected to the facility. The reflected pressure wave, f, from a constant-level reservoir is equal in magnitude to that of the incident wave, F, but is of the opposite sign. Therefore, the reflection coefficient for a constant-level reservoir, $r = -1$ (Fig. 1-5). For example, a 10-m positive pressure wave is reflected from a reservoir as a 10-m negative pressure wave.

The reflection of a velocity wave at a reservoir has the same magnitude and the same sign as that of the incident wave.

Dead End

At a dead end or at a fully closed valve, a reflected pressure wave has the same magnitude and the same sign as that of the incident wave (Fig. 1-6), i.e., the refelction coefficient, $r = 1$. Thus, if a 10-m pressure wave is approaching a dead end with the initial pressure head of 100 m, the pressure increases to

110 m as the wave arrives at the dead end and then increases to 110+10 = 120 m after the pressure wave is reflected from the dead end.

Series Junction

A junction of two pipes having different diameters, wall thicknesses, wave velocities, and/or friction factors is called a series junction. A pressure wave F traveling in one of the series pipes, say pipe 1, is reflected back into pipe 1 and another wave, f_s is transmitted into the second pipe, say pipe 2 (Fig. 1-7). The refelection coefficient, r and transmission coefficient, s, for this junction are:

$$r = \frac{f}{F} = \frac{\dfrac{A_1}{a_1} - \dfrac{A_2}{a_2}}{\dfrac{A_1}{a_1} + \dfrac{A_2}{a_2}}$$

$$s = \frac{f_s}{F} = \frac{\dfrac{2A_1}{a_1}}{\dfrac{A_1}{a_1} + \dfrac{A_2}{a_2}} \tag{1-18}$$

in which A = pipe cross-sectional area; a = wave velocity; and subscripts 1 and 2 refer to the quantities for pipe 1 and 2, respectively. Note that f_s is a positive wave since the diameter of pipe 1 is larger than that of pipe 2. If the diameter of pipe 2 is larger than that of pipe 1, then f_s will be a negative wave.

Branching Junction

At the junction, pipe 1 branches into pipe 2 and pipe 3. The incident wave, F, in pipe 1 is reflected back as f from the junction into pipe 1 and a transmitted wave f_s is transmitted into pipe 2 and pipe 3. The reflection and transmission coefficiets for a branching junction are:

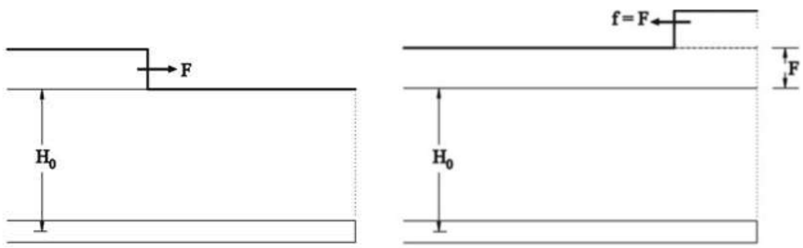
$$r = \frac{f}{F} = \frac{\dfrac{A_1}{a_1} - \dfrac{A_2}{a_2} - \dfrac{A_3}{a_3}}{\dfrac{A_1}{a_1} + \dfrac{A_2}{a_2} + \dfrac{A_3}{a_3}}$$

$$s = \frac{f_s}{F} = \frac{\dfrac{2A_1}{a_1}}{\dfrac{A_1}{a_1} + \dfrac{A_2}{a_2} + \dfrac{A_3}{a_3}} \tag{1-19}$$

Similar equations for r and s may be written for a branching junction of two incoming pipes and one outgoing pipe.

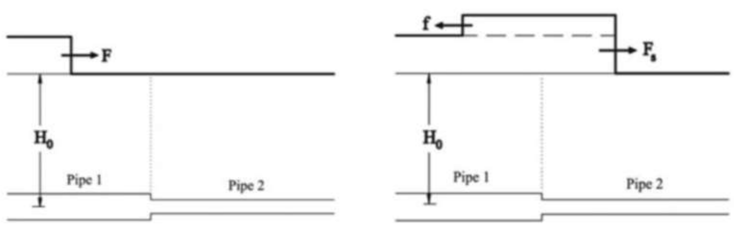
(a) Incident pressure wave **(b) Reflected wave**

Fig. 1-5. Incident and reflected waves at a reservoir.



(a) Incident wave **(b) Reflected wave**

Fig. 1-6. Incident and reflected waves at a dead end.



(a) Incident pressure wave **(b) Reflected and transmitted waves**

Fig. 1-7. Incident, reflected, and transmitted waves at a series junction.

1-7 Transient Flow Analysis

For analysis, transient flow may be classified as closed-conduit flow, open-channel or free-surface flow and combined free-surface-pressurized flow. We consider each of these flows in the following paragraphs.

The transient flow in closed-conduits may be analyzed by using the distributed-system or lumped-system approach. In the distributed-system approach, transient phenomenon is in the form of traveling waves. Examples are transients in water-supply pipes, in power plant conduits, and in gas-transmission lines. In the lumped-system approach, any change in the flow conditions is assumed to travel instantaneously throughout the system, i.e., the fluid is considered as a solid body. An example of such a system is the slow oscillations of water level in a surge tank following a load change on the turbines in a hydroelectric power plant.

Mathematically speaking, the transients in a distributed system are represented by partial differential equations, whereas the transients in a lumped system are described by ordinary differential equations. As discussed in Chapter 8, the system may be analyzed as a lumped system if $\omega L/a$ is significantly less than 1, [Chaudhry, 1970]; otherwise, the system should be analyzed as a distributed system. In the preceding expression, ω = frequency of the flow oscillations, L = length of the pipeline, and a = wave velocity.

Transients in an open channel may be classified based on the time rate at which they occur: gradual, such as flood waves in rivers, and rapid, such as surges in power canals. If the wave front in the rapidly varied flow is steep, it is referred to as a *bore*.

Free-surface flow may become pressurized due to priming of the conduit during the transient-state conditions. Such flows are called combined free-surface-pressurized flows. Examples of such flows are the flow in a sewer following a large rainstorm, and the flow in the tailrace tunnel of a hydroelectric power plant following rapid acceptance of load on turbines.

1-8 Causes of Transients

As defined previously, the intermediate-stage flow when the flow conditions change from one steady state to another, is termed *transient-state flow*. In other words, the transient conditions are initiated whenever the steady state conditions are disturbed. Such a disturbance may be caused by planned or accidental changes in the settings of the control equipment of a man-made system or by changes in the inflow or outflow of a natural system.

Common causes of transients in engineering systems are:

Opening, closing, or “chattering” of valves in a pipeline;
Starting or stopping the pumps in a pumping system;
Starting-up a hydraulic turbine, accepting or rejecting load;

Vibrations of the vanes of a runner or an impeller;
Sudden opening or closing the control gates of a canal;
Failure of a dam, and
Sudden increases in the river or sewer inflow due to a flash storm.

1-9 System Design and Operation

No generalized procedures are presently available to design a system directly that gives an acceptable transient response. Therefore, following trial-and-error approach is employed.

The system layout and parameters are first selected, and the system is analyzed for transients caused by various possible operating conditions. If the system response is unacceptable, e.g., the maximum and minimum pressures are not within the prescribed limits, then the system layout and/or parameters are changed, or various control devices are provided and the system is analyzed again. This procedure is repeated until a desired response is obtained. For a particular system, several different control devices may be suitable for transient control. If possible it may be economical in some cases to modify the operating conditions, or the acceptable response. However, the final objective is to have an *overall* economical system that yields an acceptable response.

The system is designed for the normal operating conditions expected to occur during its life. And, similarly, it is essential that the system be operated strictly according to the operating guidelines. Failure to do so may cause spectacular accidents and result in extensive property damage and many times in loss of life [Rocard, 1937; Jaeger, 1948; Bonin, 1960; Jaeger, 1963; Kerensky, 1965-1966; Pulling, 1976; Trenkle, 1979; Serkiz, 1983].

If the data for a system are not precisely known, e.g., wave speed, friction factors, reservoir levels, etc., then the system should be analyzed for the expected range of various variables. This is usually referred to as sensitivity analysis. For example, such an analysis may be done by varying the variable by $\pm 10\%$.

During the commissioning of a newly built system or after major modifications have been completed on an existing project, the system should be tested for possible operating conditions. To avoid accidents and failures, it is usually advisable to conduct the tests in a progressive manner. For example, if there are four parallel pumping-sets on a pipeline, the tests for power failure should begin with one pumping-set and progressively increase to all four.

1-10 Accidents and Incidents

In this section, a number of photographs of failures caused by transients are presented.