Module 02: Numerical Methods

Unit 12: Finite Volume Method: Conservation Law

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Learning Objective

- To discretize conservation laws using Finite Volume Method.

One-dimensional Conservation Law

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \tag{1}$$

where

$\mathcal{F}_\phi$ = Flux function.

S_ϕ = Source term.

Conservation Laws

Conservative form

A form of conservation laws can be written as:

$$\boldsymbol{\phi}_{,t} + \boldsymbol{\mathcal{F}}_{\boldsymbol{\phi},x} = \boldsymbol{S}_{\boldsymbol{\phi}} \tag{2}$$

where

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_m \end{bmatrix}, \quad \boldsymbol{\mathcal{F}}_{\boldsymbol{\phi}} = \begin{bmatrix} \mathcal{F}_{\phi_1} \\ \mathcal{F}_{\phi_2} \\ \vdots \\ \mathcal{F}_{\phi_m} \end{bmatrix}, \quad \boldsymbol{S}_{\boldsymbol{\phi}} = \begin{bmatrix} S_{\phi_1} \\ S_{\phi_2} \\ \vdots \\ S_{\phi_m} \end{bmatrix} \tag{3}$$

Jacobian Matrix

Jacobian of Flux Function

$$\boldsymbol{A}(\phi) = \frac{\partial \boldsymbol{\mathcal{F}}_{\phi}}{\partial \boldsymbol{\phi}} = \begin{bmatrix} \frac{\partial \mathcal{F}_{\phi_1}}{\partial \phi_1} & \cdots & \frac{\partial \mathcal{F}_{\phi_1}}{\partial \phi_m} \\ \frac{\partial \mathcal{F}_{\phi_2}}{\partial \phi_1} & \cdots & \frac{\partial \mathcal{F}_{\phi_2}}{\partial \phi_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial \mathcal{F}_{\phi_m}}{\partial \phi_1} & \cdots & \frac{\partial \mathcal{F}_{\phi_m}}{\partial \phi_m} \end{bmatrix}$$

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Jacobian of Flux Function

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Non-Conservative Form

$$\boldsymbol{\phi}_{,t} + \boldsymbol{A}(\boldsymbol{\phi})\boldsymbol{\phi}_{,x} = \hat{\boldsymbol{S}}_{\boldsymbol{\phi}}$$

Eigenvalues

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Eigenvalues of the Jacobian matrix $\boldsymbol{A}$ can be obtained from the characteristic polynomial:

$$|\boldsymbol{A}(\phi) - \lambda \boldsymbol{I}| = 0$$

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Hyperbolic System

A system is hyperbolic at a point (x, t) if

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- m right eigenvectors are linearly independent

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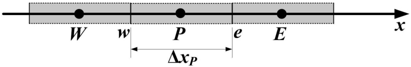
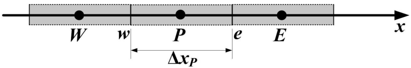
- m right eigenvectors are linearly independent

Strictly Hyperbolic: if all eigenvalues are distinct in nature

One-dimensional Conservation Law



One-dimensional Conservation Law



In Finite Volume Method, the governing equation is integrated over the element volume (in space) and time interval to form the discretized equation at node Point P.

$$\int_{t}^{t+\Delta t}\left[\int_{\Omega_P}\frac{\partial\phi}{\partial t}d\Omega\right]dt+\int_{t}^{t+\Delta t}\left[\int_{\Omega_P}\frac{\partial\mathcal{F}_{\phi}}{\partial x}d\Omega\right]dt=\int_{t}^{t+\Delta t}\left[\int_{\Omega_P}S_{\phi}d\Omega\right]dt \tag{4}$$

One-dimensional Conservation Law

The expression can be simplified as

$$\int_{t}^{t+\Delta t}\left[\int_{x_w}^{x_e}\frac{\partial \phi}{\partial t}dx\right]dt+\int_{t}^{t+\Delta t}\left[\int_{x_w}^{x_e}\frac{\partial \mathcal{F}_\phi}{\partial x}d\Omega\right]dt=\int_{t}^{t+\Delta t}\left[\int_{x_w}^{x_e}S_\phi d\Omega\right]dt$$

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This can be further simplified as

$$\left[\int_{x_w}^{x_e}\phi(x,t+\Delta t)dx-\int_{x_w}^{x_e}\phi(x,t)dx\right]+\left[\int_{t}^{t+\Delta t}\mathcal{F}_\phi(x_e,t)dt-\int_{t}^{t+\Delta t}\mathcal{F}_\phi(x_w,t)dt\right]=$$

$$\int_{t}^{t+\Delta t}\left[\int_{x_w}^{x_e}S_\phi d\Omega\right]dt$$

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Let us define

$$\phi_P^n \approx \frac{1}{\Delta x} \int_{x_w}^{x_e} \phi(x,t)dx$$

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and Numerical flux function can written as

$$\bar{\mathcal{F}}_\phi(x_e,t) = \bar{\mathcal{F}}_\phi(\phi_P^n, \phi_E^n) = \frac{1}{\Delta t} \int\limits_{t}^{t+\Delta t} \mathcal{F}_\phi(x_e,t)dt$$

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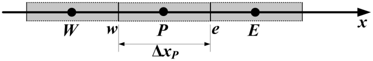
Final form of the discretization using finite volume method can be written as

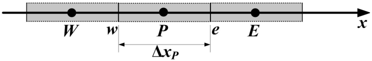
$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{\mathcal{F}}_\phi(\phi_P^n, \phi_E^n) - \bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n)\right]$$

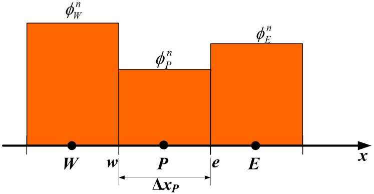
Riemann Problem

Conservative Form



Riemann Problem
Conservative Form





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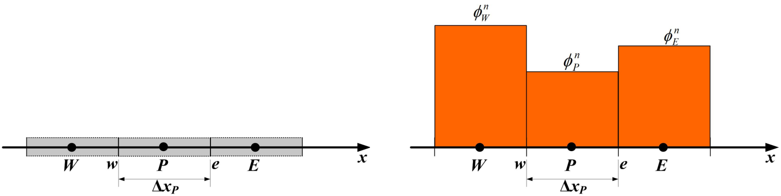

Riemann Problem

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = 0$$

$$\phi(x,t) = \begin{cases} \phi_P^n & \text{if} \quad x < x_e \\ \phi_E^n & \text{if} \quad x > x_e \end{cases}$$

Unstable Flux

Numerical flux can be calculated by taking arithmetic average of cell centred values

$$\bar{\mathcal{F}}_\phi(x_e,t) = \bar{\mathcal{F}}_\phi(\phi_P^n,\phi_E^n) = \frac{1}{2}\left[\mathcal{F}^n_{\phi_P} + \mathcal{F}^n_{\phi_E}\right]$$

$$\bar{\mathcal{F}}_\phi(x_w,t) = \bar{\mathcal{F}}_\phi(\phi_W^n,\phi_P^n) = \frac{1}{2}\left[\mathcal{F}^n_{\phi_W} + \mathcal{F}^n_{\phi_P}\right]$$

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Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{2\Delta x}\left[\mathcal{F}^n_{\phi_E} - \mathcal{F}^n_{\phi_W}\right]$$

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Let us consider that the flux term can be written as,

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or,

$$\phi_i^{n+1} = \phi_i^n - a\frac{\Delta t}{2\Delta x}(\phi_{i+1}^n - \phi_{i-1}^n)$$

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The error equation can be written as

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With $Cr = a\frac{\Delta t}{\Delta x}$

$$\begin{aligned} G = \frac{\epsilon_i^{n+1}}{\epsilon_i^n} &= 1 - \frac{Cr}{2}(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x}) \\ &= 1 - \sqrt{-1}Cr\sin\varphi_x \end{aligned}$$

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$$\begin{aligned} |G|^2 = G.G^* &= (1 - \sqrt{-1}Cr\sin\varphi_x).(1 + \sqrt{-1}Cr\sin\varphi_x) \\ &= 1 + Cr^2\sin^2\varphi_x \quad > 1 \end{aligned}$$

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The scheme is unstable.

Lax-Friedrichs Scheme

Numerical flux can be calculated by taking arithmetic average of cell centred values (LeVeque, 2002):

$$\bar{\mathcal{F}}_\phi(x_e,t) = \bar{\mathcal{F}}_\phi(\phi_P^n,\phi_E^n) = \frac{1}{2}\left[\mathcal{F}_{\phi_P}^n + \mathcal{F}_{\phi_E}^n\right] - \frac{\Delta x}{2\Delta t}(\phi_E^n - \phi_P^n)$$

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Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \frac{1}{2}(\phi_W^n + \phi_E^n) - \frac{\Delta t}{2\Delta x}\left[\mathcal{F}_{\phi_E}^n - \mathcal{F}_{\phi_W}^n\right]$$

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Lax-Friedrichs Scheme

Numerical Diffusion

Actual Equation

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Lax-Friedrichs Scheme

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Modified Equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = \beta \frac{\partial^2 \phi}{\partial x^2}$$

where $\beta = \frac{\Delta x^2}{2\Delta t}$

Lax-Friedrichs Scheme

Numerical Diffusion

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Modified Equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = \beta \frac{\partial^2 \phi}{\partial x^2}$$

where $\beta = \frac{\Delta x^2}{2\Delta t}$

$$\bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n)\big|_D = -\beta \frac{\phi_P^n - \phi_W^n}{\Delta x}$$

Lax-Friedrichs Scheme

The error equation can be written as

$$\varepsilon_i^{n+1} = \frac{1}{2}(\varepsilon_{i-1}^n + \varepsilon_{i+1}^n) - a\frac{\Delta t}{2\Delta x}(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$

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With $Cr = a\frac{\Delta t}{\Delta x}$

$$G = \frac{\epsilon_i^{n+1}}{\epsilon_i^n} = \frac{1}{2}(e^{\sqrt{-1}\varphi_x} + e^{-\sqrt{-1}\varphi_x}) - \frac{Cr}{2}(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x})$$

$$= \cos\varphi_x - \sqrt{-1}Cr\sin\varphi_x$$

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$$\begin{aligned} |G|^2 = G.G^* &= (\cos\varphi_x - \sqrt{-1}Cr\sin\varphi_x).(\cos\varphi_x + \sqrt{-1}Cr\sin\varphi_x) \\ &= \cos^2\varphi_x + Cr^2\sin^2\varphi_x \\ &= 1 - (1 - Cr^2)\sin^2\varphi_x \end{aligned}$$

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$$= \cos\varphi_x - \sqrt{-1}Cr\sin\varphi_x$$

$$|G|^2 = G.G^* = (\cos\varphi_x - \sqrt{-1}Cr\sin\varphi_x).(\cos\varphi_x + \sqrt{-1}Cr\sin\varphi_x)$$
$$= \cos^2\varphi_x + Cr^2\sin^2\varphi_x$$
$$= 1 - (1 - Cr^2)\sin^2\varphi_x$$

The scheme is stable if $Cr < 1$.

Thank You

References

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