Module 02: Numerical Methods

Unit 25: Algebraic Equation: Gauss-Seidel Method

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Learning Objective

- To apply Gauss-Seidel method for iterative solution.

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- To apply Gauss-Seidel method for iterative solution.
- To apply Successive Over Relaxation for Gauss-Seidel iteration.

Matrix Form
Full Matrix

$$\mathbf{A}\phi = \mathbf{r}$$

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Full Matrix

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{r}$$

$$\begin{pmatrix} \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \end{pmatrix}_{N\times N} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \\ \phi_N \end{Bmatrix}_{N\times 1} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{Bmatrix}_{N\times 1}$$

Basic Steps
Gauss-Seidel Method

The coefficient matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

where $\mathbf{L}$, $\mathbf{D}$, $\mathbf{U}$ are strictly lower triangular, diagonal, strictly upper triangular matrices respectively.

Basic Steps

Gauss-Seidel Method

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Overall calculation can be presented as

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Overall calculation can be presented as

$$(\mathbf{L} + \mathbf{D} + \mathbf{U})\boldsymbol{\phi} = \mathbf{r}$$

Iterative form can be written as

$$(\mathbf{L} + \mathbf{D})\boldsymbol{\phi}^{(\mathbf{p})} + \mathbf{U}\boldsymbol{\phi}^{(\mathbf{p-1})} = \mathbf{r}$$

Basic Steps
Gauss-Seidel Method

Iterative form can be written as

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Iterative form can be written as

$$(\mathbf{L} + \mathbf{D})\phi^{(p)} + \mathbf{U}\phi^{(p-1)} = \mathbf{r}$$

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Final form can be written as

$$\boldsymbol{\phi}^{(p)} = -(\mathbf{L}+\mathbf{D})^{-1}\mathbf{U}\boldsymbol{\phi}^{(p-1)} + (\mathbf{L}+\mathbf{D})^{-1}\mathbf{r}$$

where p is the iteration counter $(p \geq 1)$.

Basic Steps
Gauss-Seidel Method

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where p is the iteration counter $(p \geq 1)$.

Iteration starts with a guess value $\boldsymbol{\phi}^{(0)}$

Gauss-Seidel Method

Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Gauss-Seidel Method
Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Coefficient matrix $\mathbf{A}$ can be written as

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Gauss-Seidel Method
Example

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Coefficient matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

where

Strictly Lower Triangular Matrix

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{pmatrix}$$

Gauss-Seidel Method
Example

Diagonal Matrix

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

Gauss-Seidel Method
Example

Diagonal Matrix

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

Strictly Upper Triangular Matrix

$$\mathbf{U} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Gauss-Seidel Method
Example

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} & \phi_3^{(0)} & \phi_4^{(0)} & \phi_5^{(0)} \end{bmatrix}^T$$

Gauss-Seidel Method

Example

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

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Iteration 1:
Row 1:

$$\phi_1^{(1)} = \frac{1}{a_{11}} \left[r_1 - \sum_{j=2}^{5} a_{1j} \phi_j^{(0)} \right]$$

Gauss-Seidel Method
Example

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

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Iteration 1:
Row 1:

$$\phi_1^{(1)} = \frac{1}{a_{11}} \left[r_1 - \sum_{j=2}^{5} a_{1j} \phi_j^{(0)} \right]$$

Row 2:

$$\phi_2^{(1)} = \frac{1}{a_{22}} \left[r_2 - a_{21} \phi_1{}^{(1)} - \sum_{j=3}^{5} a_{2j} \phi_j^{(0)} \right]$$

Gauss-Seidel Method
Example

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} & \phi_3^{(0)} & \phi_4^{(0)} & \phi_5^{(0)} \end{bmatrix}^T$$

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$$\phi_2^{(1)} = \frac{1}{a_{22}} \left[r_2 - a_{21} \phi_1{}^{(1)} - \sum_{j=3}^{5} a_{2j} \phi_j^{(0)} \right]$$

Row 3:

$$\phi_3^{(1)} = \frac{1}{a_{33}} \left[r_3 - \sum_{j=1}^{2} a_{3j} \phi_j{}^{(1)} - \sum_{j=4}^{5} a_{3j} \phi_j^{(0)} \right]$$

Gauss-Seidel Method
Example

Row 4:

$$\phi_4^{(1)} = \frac{1}{a_{44}} \left[r_4 - \sum_{j=1}^{3} a_{4j} \phi_j{}^{(1)} - a_{45} \phi_5^{(0)} \right]$$

Gauss-Seidel Method
Example

Row 4:

$$\phi_4^{(1)} = \frac{1}{a_{44}} \left[r_4 - \sum_{j=1}^{3} a_{4j} \phi_j^{(1)} - a_{45} \phi_5^{(0)} \right]$$

Row 5:

$$\phi_5^{(1)} = \frac{1}{a_{55}} \left[r_5 - \sum_{j=1}^{4} a_{5j} \phi_j^{(1)} \right]$$

Iteration 2:
Row 1:

$$\phi_1^{(2)} = \frac{1}{a_{11}} \left[r_1 - \sum_{j=2}^{5} a_{1j} \phi_j^{(1)} \right]$$

Gauss-Seidel Method
Example

Row 2:

$$\phi_2^{(2)} = \frac{1}{a_{22}} \left[r_2 - a_{21} \phi_1^{(2)} - \sum_{j=3}^{5} a_{2j} \phi_j^{(1)} \right]$$

Gauss-Seidel Method
Example

Row 2:

$$\phi_2^{(2)} = \frac{1}{a_{22}} \left[r_2 - a_{21} {\phi_1}^{(2)} - \sum_{j=3}^{5} a_{2j} \phi_j^{(1)} \right]$$

Row 3:

$$\phi_3^{(2)} = \frac{1}{a_{33}} \left[r_3 - \sum_{j=1}^{2} a_{3j} {\phi_j}^{(2)} - \sum_{j=4}^{5} a_{3j} \phi_j^{(1)} \right]$$

Gauss-Seidel Method
Example

Row 2:

$$\phi_2^{(2)} = \frac{1}{a_{22}}\left[r_2 - a_{21}\phi_1{}^{(2)} - \sum_{j=3}^{5} a_{2j}\phi_j^{(1)}\right]$$

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$$\phi_3^{(2)} = \frac{1}{a_{33}}\left[r_3 - \sum_{j=1}^{2} a_{3j}\phi_j{}^{(2)} - \sum_{j=4}^{5} a_{3j}\phi_j^{(1)}\right]$$

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Gauss-Seidel Method
General Algorithm

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \left[\phi_1^{(0)} \quad \phi_2^{(0)} \dots \quad \phi_{N-1}^{(0)} \quad \phi_N^{(0)}\right]^T$$

Gauss-Seidel Method

General Algorithm

Iteration starts with the guess value $\phi^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} \dots & \phi_{N-1}^{(0)} & \phi_N^{(0)} \end{bmatrix}^T$$

$$\phi_i^{(p)} = \frac{1}{a_{ii}} \left[r_i - \sum_{j=1}^{i-1} a_{ij} \phi_j{}^{(p)} - \sum_{j=i+1}^{N} a_{ij} \phi_j^{(p-1)} \right], \quad \forall i \in \{1, \dots, N\}, p \geq 1$$

Gauss-Seidel Method
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By adding and subtracting $\phi_i^{(p-1)}$ in right hand side

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{1}{a_{ii}} \left[r_i - \sum_{j=1}^{i-1} a_{ij} \phi_j^{(p)} - a_{ii} \phi_i^{(p-1)} - \sum_{j=i+1}^{N} a_{ij} \phi_j^{(p-1)} \right], \quad \forall i, p \geq 1$$

Gauss-Seidel Method
General Algorithm

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In compact form

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{a_{ii}}, \quad \forall i, p \geq 1$$

Gauss-Seidel Method

Stopping Criterion

Residual Error in a particular iteration can be expressed as

$$\boldsymbol{\varepsilon}^{(p)} = \mathbf{A}\boldsymbol{\phi}^{(p)} - \mathbf{r}$$

Gauss-Seidel Method
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Maximum Absolute Error:

$$\max_{i \in 1,\ldots,N} |\varepsilon_i^{(p)}| \leq \varepsilon_{max}$$

Gauss-Seidel Method

Stopping Criterion

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Maximum Absolute Error:

$$\max_{i \in 1,\ldots,N} |\varepsilon_i^{(p)}| \leq \varepsilon_{max}$$

Root Mean Square Error:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\varepsilon_i^{(p)}\right)^2} \leq \varepsilon_{max}$$

Convergence Criteria
Diagonal Dominance

Diagonal Dominance:

$$|a_{ii}| = \sum_{\substack{j=1,\\ j\neq i}}^{N} |a_{ij}|$$

Convergence Criteria

Diagonal Dominance

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$$|a_{ii}| = \sum_{\substack{j=1, \\ j\neq i}}^{N} |a_{ij}|$$

$$\exists l : |a_{ll}| > \sum_{\substack{j=1, \\ j\neq l}}^{N} |a_{lj}|$$

Convergence Criteria
Diagonal Dominance

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Weak Diagonal Dominance:

$$|a_{ii}| \geq \sum_{\substack{j=1,\\ j\neq i}}^{N} |a_{ij}|$$

Convergence Criteria

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Weak Diagonal Dominance:

$$|a_{ii}| \geq \sum_{\substack{j=1,\\ j\neq i}}^{N} |a_{ij}|$$

Strict Diagonal Dominance:

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{N} |a_{ij}|$$

Gauss-Seidel Method (GS)

Successive Over-Relaxation (SOR)

Convergence can be achieved by increasing or reducing the step size

$$\phi^{(p)} - \phi^{(p-1)} = \omega \left[\phi_{GS}^{(p)} - \phi^{(p-1)} \right]$$

Gauss-Seidel Method (GS)

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In iterative form

$$\phi^{(p)} = \omega \phi_{GS}^{(p)} + (1-\omega)\phi^{(p-1)}$$

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In iterative form

$$\phi^{(p)} = \omega \phi_{GS}^{(p)} + (1-\omega)\phi^{(p-1)}$$

Gauss-Seidel approximation can be written as

$$\mathbf{D}\boldsymbol{\phi}_{\mathbf{GS}}^{(\mathbf{p})} = -\mathbf{L}\boldsymbol{\phi}^{(\mathbf{p})} - \mathbf{U}\boldsymbol{\phi}^{(\mathbf{p-1})} + \mathbf{r}$$

Gauss-Seidel Method (GS)

Successive Over-Relaxation (SOR)

By combining expressions

$$
\begin{aligned}
\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p})} &= \omega\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p})}_{\mathbf{GS}} + (\mathbf{1}-\omega)\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p-1})} \\
&= \omega\left[-\mathbf{L}\boldsymbol{\phi}^{(\mathbf{p})} - \mathbf{U}\boldsymbol{\phi}^{(\mathbf{p-1})} + \mathbf{r}\right] + (1-\omega)\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p-1})} \\
&= -\omega\mathbf{L}\boldsymbol{\phi}^{(\mathbf{p})} + (\mathbf{1}-\omega)\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p-1})} - \omega\mathbf{U}\boldsymbol{\phi}^{(\mathbf{p-1})} + \omega\mathbf{r}
\end{aligned}
$$

Gauss-Seidel Method (GS)

Successive Over-Relaxation (SOR)

By combining expressions

$$\begin{aligned}\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p})} &= \omega\mathbf{D}\boldsymbol{\phi}_{\mathbf{GS}}^{(\mathbf{p})} + (\mathbf{1}-\omega)\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p-1})} \\ &= \omega\left[-\mathbf{L}\boldsymbol{\phi}^{(\mathbf{p})} - \mathbf{U}\boldsymbol{\phi}^{(\mathbf{p-1})} + \mathbf{r}\right] + (1-\omega)\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p-1})} \\ &= -\omega\mathbf{L}\boldsymbol{\phi}^{(\mathbf{p})} + (\mathbf{1}-\omega)\mathbf{D}\boldsymbol{\phi}^{(\mathbf{p-1})} - \omega\mathbf{U}\boldsymbol{\phi}^{(\mathbf{p-1})} + \omega\mathbf{r}\end{aligned}$$

Rearrangement yields

$$(\mathbf{D}+\omega\mathbf{L})\boldsymbol{\phi}^{(\mathbf{p})} = \left[(\mathbf{1}-\omega)\mathbf{D} - \omega\mathbf{U}\right]\boldsymbol{\phi}^{(\mathbf{p-1})} + \omega\mathbf{r}$$

Finally in matrix form

$$\boldsymbol{\phi}^{(p)} = (\mathbf{D}+\omega\mathbf{L})^{-\mathbf{1}}\left[(\mathbf{1}-\omega)\mathbf{D} - \omega\mathbf{U}\right]\boldsymbol{\phi}^{(\mathbf{p-1})} + \omega(\mathbf{D}+\omega\mathbf{L})^{-\mathbf{1}}\mathbf{r}$$

Gauss-Seidel Method
General Algorithm

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} \dots & \phi_{N-1}^{(0)} & \phi_N^{(0)} \end{bmatrix}^T$$

By adding and subtracting $\phi_i^{(p-1)}$ in right hand side

$$\phi_{i,GS}^{(p)} = \phi_i^{(p-1)} + \frac{1}{a_{ii}} \left[r_i - \sum_{j=1}^{i-1} a_{ij} \phi_j^{(p)} - a_{ii}\phi_i^{(p-1)} - \sum_{j=i+1}^{N} a_{ij} \phi_j^{(p-1)} \right], \quad \forall i, p \geq 1$$

Gauss-Seidel Method

General Algorithm

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In compact form

$$\phi_{i,GS}^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{a_{ii}}, \quad \forall i, p \geq 1$$

Gauss-Seidel Method
General Algorithm

Iteration starts with the guess value $\boldsymbol{\phi}^{(0)}$

$$\boldsymbol{\phi}^{(0)} = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} \dots & \phi_{N-1}^{(0)} & \phi_N^{(0)} \end{bmatrix}^T$$

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In compact form

$$\phi_{i,GS}^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{a_{ii}}, \quad \forall i, p \geq 1$$

Convergence can be achieved by increasing or reducing the step size

$$\phi_i^{(p)} - \phi_i^{(p-1)} = \omega \left[\phi_{i,GS}^{(p)} - \phi_i^{(p-1)} \right]$$

Gauss-Seidel Method

General Algorithm

Final form can be written as

$$\begin{aligned}\phi_i^{(p)} &= \phi_i^{(p-1)} + \omega\left[\phi_{i,GS}^{(p)} - \phi_i^{(p-1)}\right] \\ &= \phi_i^{(p-1)} + \omega\frac{Res_i}{a_{ii}}\end{aligned}$$

Gauss-Seidel Method
General Algorithm

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Rearrangement yields

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{(a_{ii}/\omega)}, \quad 0 < \omega < 2$$

Gauss-Seidel Method
General Algorithm

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Rearrangement yields

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{(a_{ii}/\omega)}, \quad 0 < \omega < 2$$

- $0 < \omega < 1 \Rightarrow$ Under-relaxation

Gauss-Seidel Method

General Algorithm

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Rearrangement yields

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{(a_{ii}/\omega)}, \quad 0 < \omega < 2$$

- $0 < \omega < 1 \Rightarrow$ Under-relaxation
- $1 < \omega < 2 \Rightarrow$ Over-relaxation

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 & 0 \\ 0 & -4 & 3 & -2 & 0 \\ 0 & 0 & -7 & -10 & 13 \\ 0 & 0 & 0 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -9 \\ -7 \\ 4 \\ -26 \end{Bmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 & 0 \\ 0 & -4 & 3 & -2 & 0 \\ 0 & 0 & -7 & -10 & 13 \\ 0 & 0 & 0 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -9 \\ -7 \\ 4 \\ -26 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix}$$

Thank You