Principles of Flow

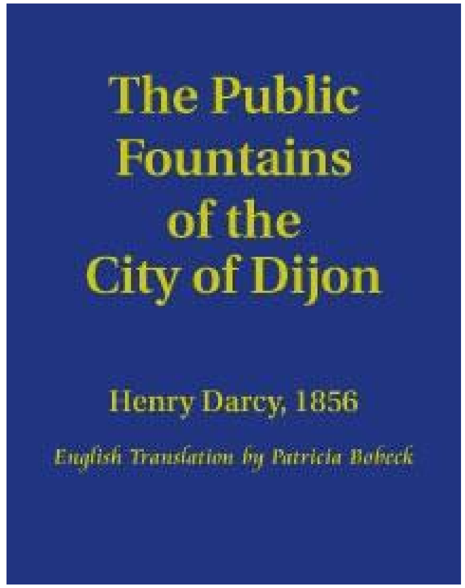
Geohydraulics| CE60113

Lecture:07

Learning Objective(s)

- **To apply Darcy's law**

Darcy’s Law


The Public Fountains of the City of Dijon
Henry Darcy, 1856
English Translation by Patricia Bobeck

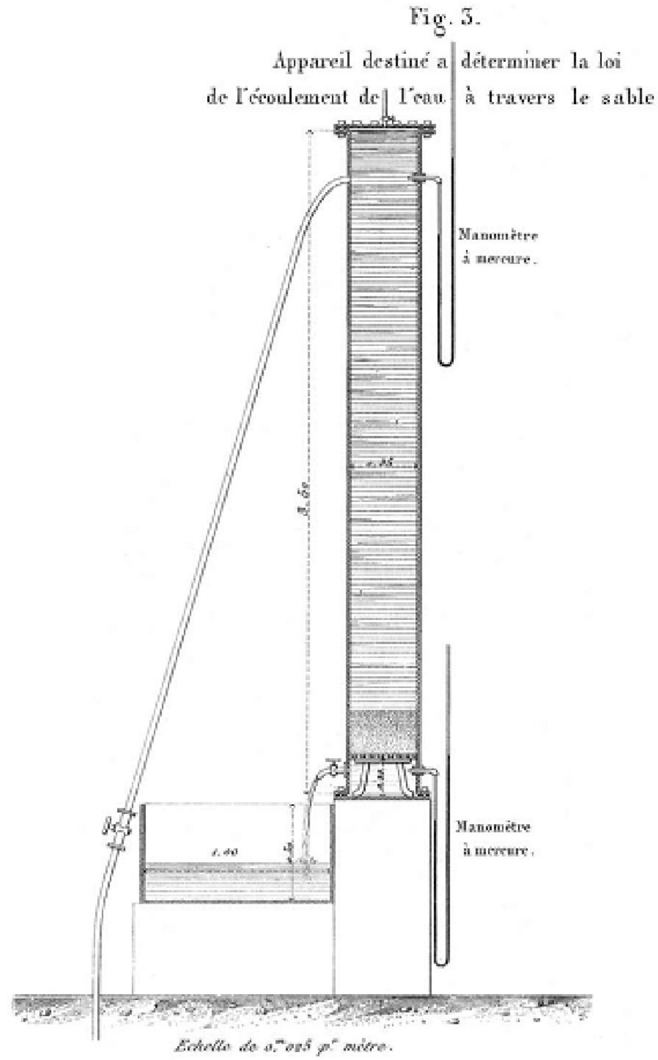
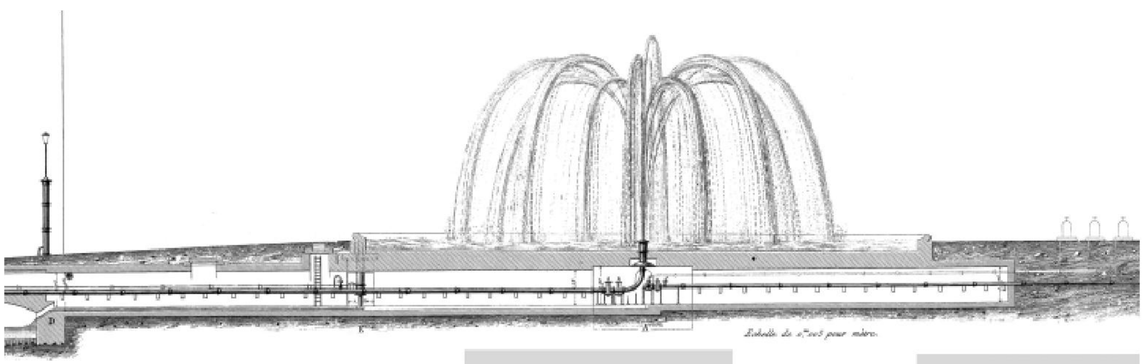
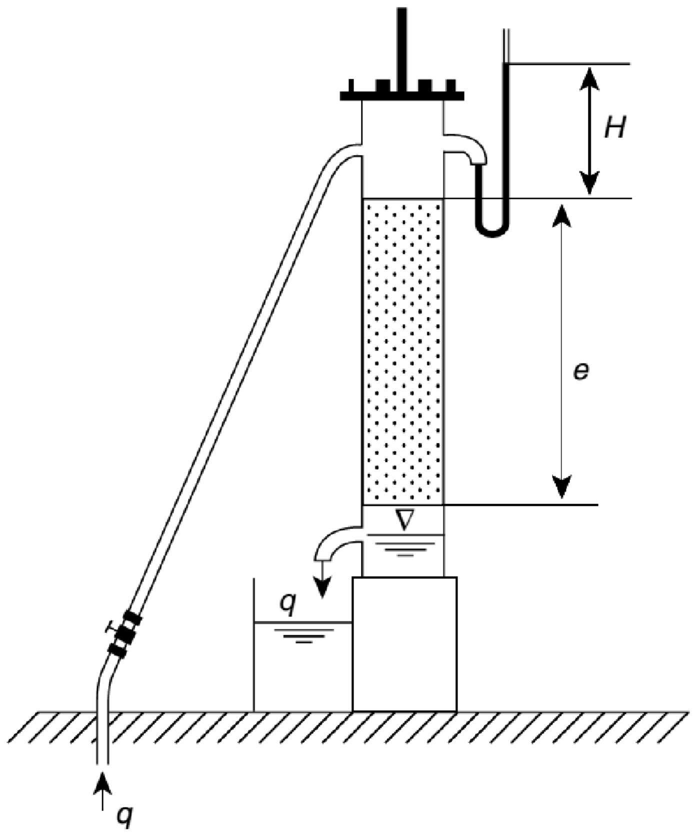


Fig. 3.
Appareil destiné à déterminer la loi de l'écoulement de l'eau à travers le sable.
Manomètre à mercure.
Manomètre à mercure.



Darcy's Law (Contd.)



Darcy's Experiment

- For percolation through a cylinder with outflow under atmospheric pressure

$$q = Ks\frac{H + e}{e}$$

  - q=percolation flux [L^3T^{-1}]
  - K=permeability factor[LT^{-1}]
  - s=surface area of the sand filled cylinder [L^2]
  - e=length of the sand column [L]
  - H=hydraulic pressure at the upper boundary of the sand column [L]
  - $(H+e)$=hydraulic head
  - $(H+e)/e$=hydraulic gradient
- Non-steady flow equation for the condition of a falling pressure head

$$q_t = q_0 e^{(-Kt/e)}$$

  - t =time [T]
  - q_0=percolation flux [L^3T^{-1}] at t = 0

Darcy's Law (Contd.)

- Darcy found through repeated experiments with a specific sand that Q was proportional to the head difference Δh between the two manometers and inversely proportional to ($\propto$) the distance between manometers Δs:

$$Q \propto \Delta h \text{ and } Q \propto \frac{1}{\Delta s}$$

- Q is also proportional to the cross-sectional area of the column A

$$Q \propto A$$

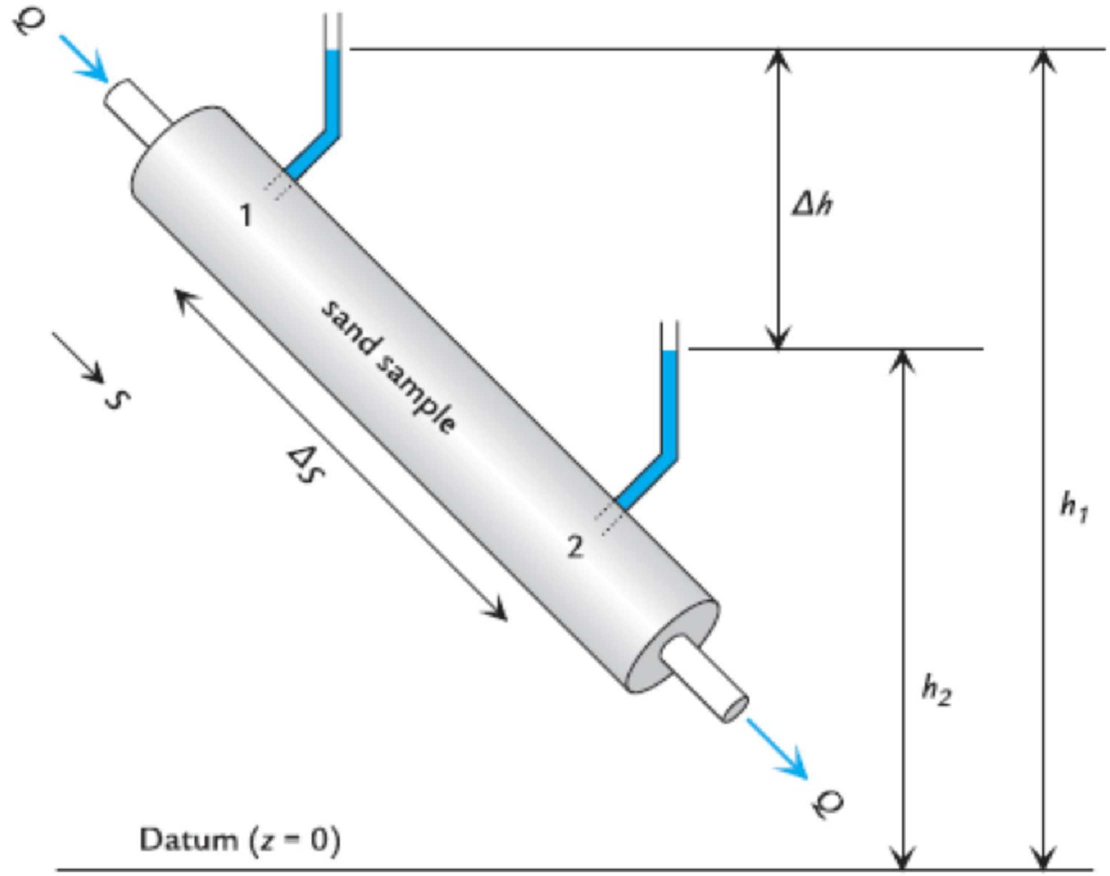
- Combining these observations **Darcy's law** for one-dimensional flow can be written as:

$$Q = -K_s \frac{\Delta h}{\Delta s} A$$

- The constant of proportionality K_s is the **hydraulic conductivity** in the s direction, a property of the geologic medium.
- Hydraulic conductivity is a measure of the ease with which a medium transmits water; higher K_s materials transmit water more easily than low K_s materials.

Darcy's Law (Contd.)

- The minus sign on the right side of Darcy's equation is necessary because head decreases in the direction of flow.
  - Q is positive and $\frac{\Delta h}{\Delta s}$ is negative
  - Q is negative and $\frac{\Delta h}{\Delta s}$ is positive

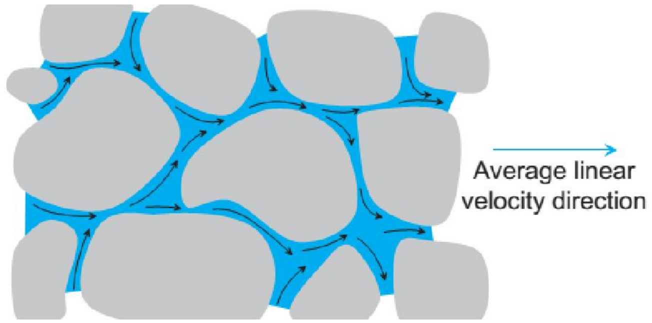


Darcy's Law (Contd.)

- Specific discharge or Darcy velocity

- Darcy velocity

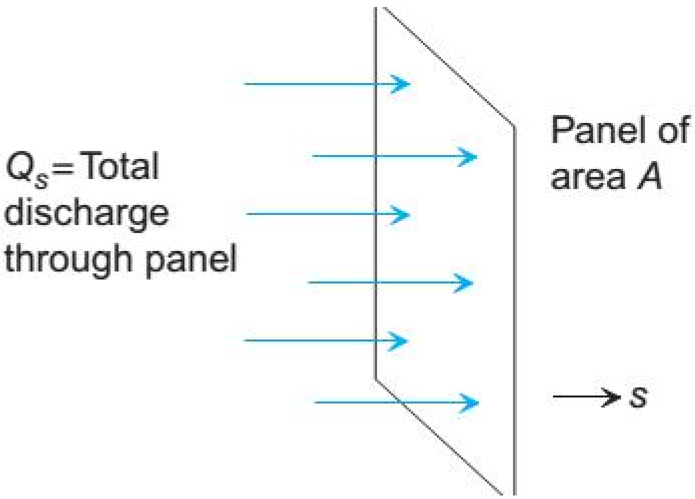
$$q_s = \frac{Q_s}{A} = -K_s \frac{dh}{ds}$$



Q_s = flow

A = total cross-sectional area through which flow occurs

- Specific discharge has units of velocity
- The specific discharge is a macroscopic concept
- It can be easily measured



Darcy's Law (Contd.)

- Darcy velocity is a fictitious velocity
  - Flow occurs across the entire cross-section of the soil sample
  - Linear flow paths assumed in Darcy's law
  - Flow actually takes place only through interconnected pore channels
- The average linear velocity of water motion is directly proportional to the specific discharge and inversely proportional to the effective porosity

$$\bar{v}_s = \frac{q_s}{\eta_e} = \frac{Q_s}{A\eta_e}$$

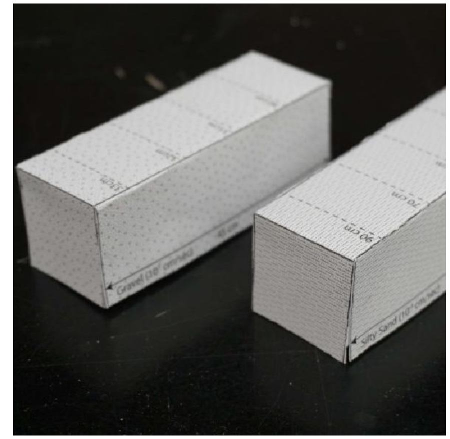
- Discharge can be written as

$$Q_s = Aq_s = A_v \bar{v}_s$$

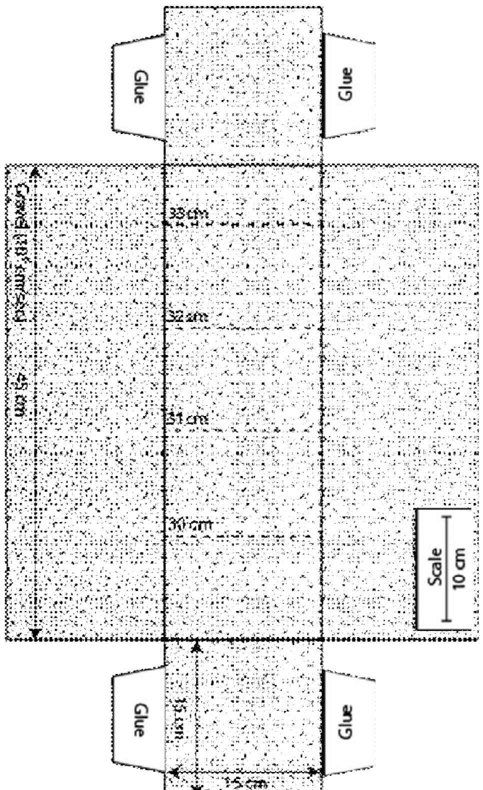
Darcy’s Law (Contd.)

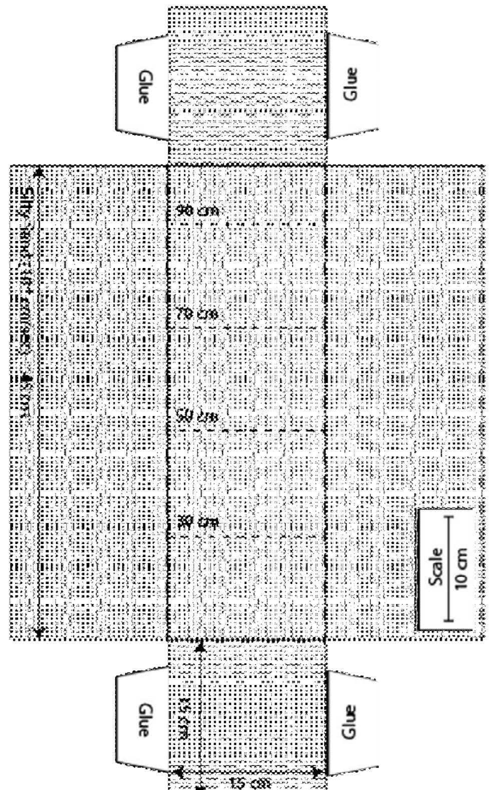
- Darcy’s law holds for
  - Saturated flow and unsaturated flow
  - Steady-state and transient flow
  - Flow in aquifers and aquitards
  - Flow in homogeneous and heterogeneous systems
  - Flow in isotropic or anisotropic media
  - Flow in rocks and granular media

Home Lab



- Foldable Aquifer Project -http://aquifer.geology.buffalo.edu/
- Paper aquifer model
  - Darcy Columns
  - Objectives: To explore the relationship between discharge (*Q*), specific discharge (*q*), and average linear velocity (*v*) in homogenous aquifers.



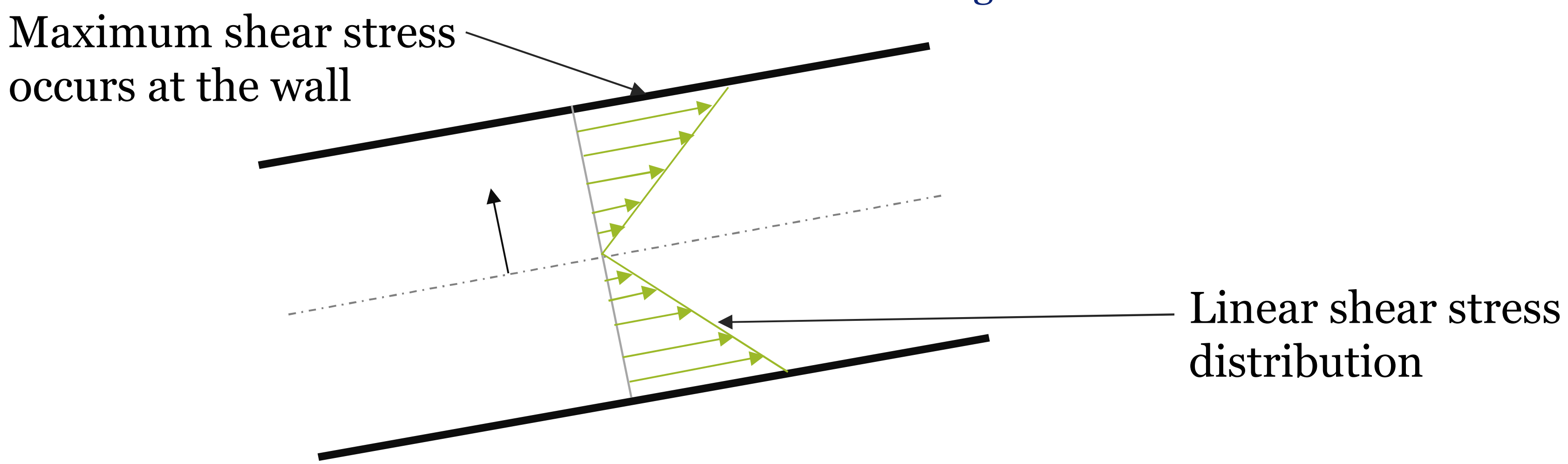


Home Lab (Contd.)

- The problem is based on water flow through two Darcy columns. The first column is filled with a gravel that has a hydraulic conductivity of 10^{1} cm/sec, and the second is filled with silty sand with a hydraulic conductivity of 10^{-3} cm/sec. The equipotential lines are shown on the top of the models represented as dashed lines with their associated hydraulic heads. Note that each column has a different distribution of equipotential lines. The total length of the columns are 45 cm and a scale bare is shown lower right corner of the back panel of the model. Using these models please answer the following questions.
- A. Based on the two Darcy's columns provided quantify the difference in groundwater discharge [cm^3/sec] between the Gravel column and the Silty Sand column.
- B. Assuming the gravel column has an effective porosity of 0.30 and the silty sand column has an effective porosity of 0.25. Determine which column has a higher average linear velocity and by how much.
- C. Explain which of the following column would make a better aquifer.

Hagen-Poiseuille Flow

- Laminar flow in a round tube is called Poiseuille flow or Hagen-Poiseuille flow.



$$\tau = \mu \frac{dV}{dy}$$

- y is the distance from the pipe wall. Let us consider $y = r_0 - r$

$$\tau = \mu \frac{dV}{dy} = \mu \frac{dV}{dr}\frac{dr}{dy} = -\mu \frac{dV}{dr}$$

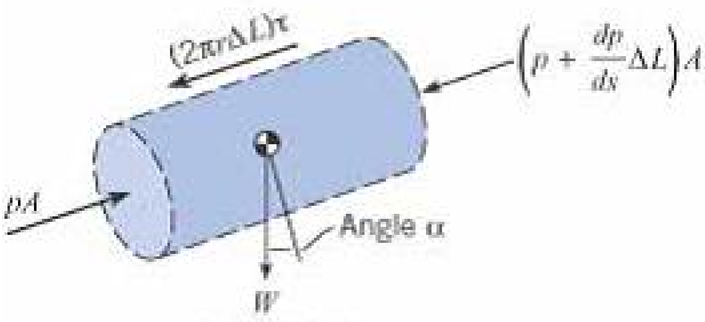
Hagen-Poiseuille Flow (Contd.)

- Considering force balance for the control volume

$$pA - \left(p + \frac{dp}{ds}\Delta L\right)A - W sin\alpha - (2\pi r \Delta L)\tau = 0$$

where $W = \gamma A \Delta L$ and $sin\alpha = \Delta z/\Delta L$

$$\tau = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right]$$



Force diagram

Shear stress distribution

$$\left(\frac{2\mu}{r}\right)\frac{dV}{dr} = \frac{d}{ds}(p + \gamma z)$$

This can be true iff

$$Constant = \frac{d}{ds}(p + \gamma z) = \frac{\Delta(p + \gamma z)}{\Delta L} = \frac{\gamma \Delta h}{\Delta L}$$

Combining

$$\frac{dV}{dr} = \left(\frac{r}{2\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

$$V = \left(\frac{r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) + C$$

Hagen-Poiseuille Flow (Contd.)

No-slip condition: velocity of the fluid at wall is zero

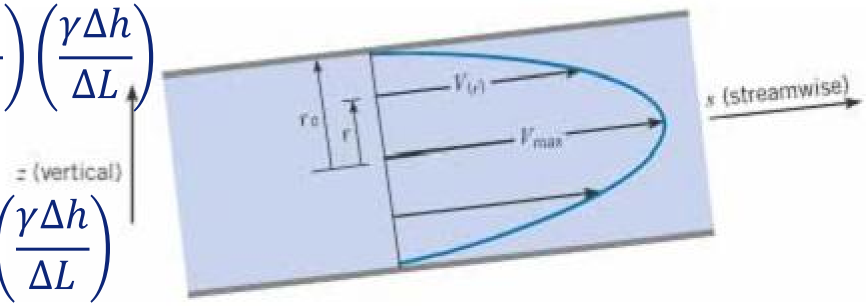
$$V(r = r_0) = 0$$

Thus constant C

$$C = -\left(\frac{r_0^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

Velocity can be expressed as

$$V(r) = -\left(\frac{r_0^2 - r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$



Maximum velocity

$$V_{max} = -\left(\frac{r_0^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

Combining maximum velocity

$$V(r) = V_{max}\left(1 - \frac{r^2}{r_0^2}\right)$$

Hagen-Poiseuille Flow (Contd.)

- Average velocity

$$V_{avg} = \frac{1}{A}\iint V dA$$

$$V_{avg} = \frac{1}{A}\iint V dA = \frac{1}{\pi r_0^2}\int_0^{r_0} -\left(\frac{r_0^2 - r^2}{4\mu}\right)\left(\frac{\gamma\Delta h}{\Delta L}\right)(2\pi r dr) = -\left(\frac{r_0^2}{8\mu}\right)\left(\frac{\gamma\Delta h}{\Delta L}\right)$$

$$V_{avg} = -\left(\frac{r_0^2}{8\mu}\right)\left(\frac{\gamma\Delta h}{\Delta L}\right) = \frac{1}{2}V_{max}$$

Darcy's law & General Fluid Flow

- Specific discharge q
- *Hagen-Poiseuille Flow*

$$V_{avg} = -\left(\frac{r_0^2}{8\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

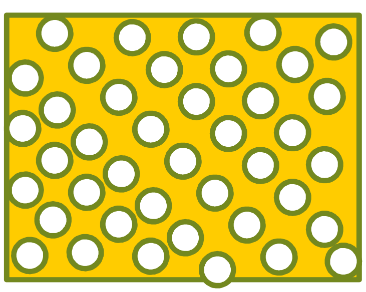
- Specific discharge q

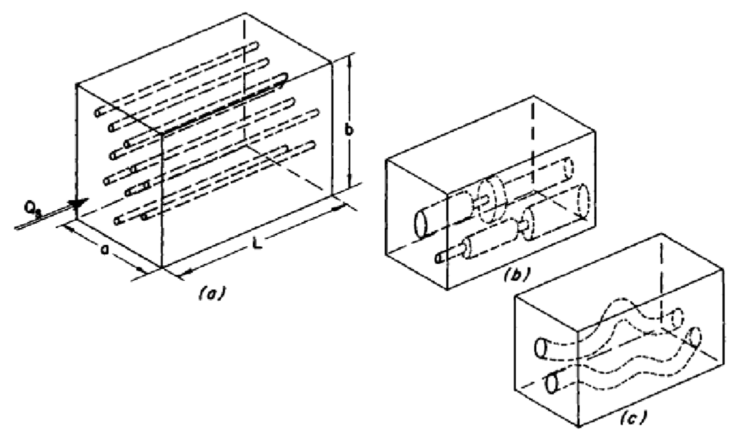
$$V_{avg} = \frac{q}{n}$$

$$q = nV_{avg} = -n\left(\frac{r_0^2}{8\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) = -\frac{\rho g}{\mu}\frac{n\, r_0^2}{8}\frac{\Delta h}{\Delta L} = -K\,\frac{\Delta h}{\Delta L}$$

$$K = \frac{\rho g}{\mu}\frac{n\, r_0^2}{8} = \frac{\rho g}{\mu} k$$

$$k = \frac{n\, r_0^2}{8}$$



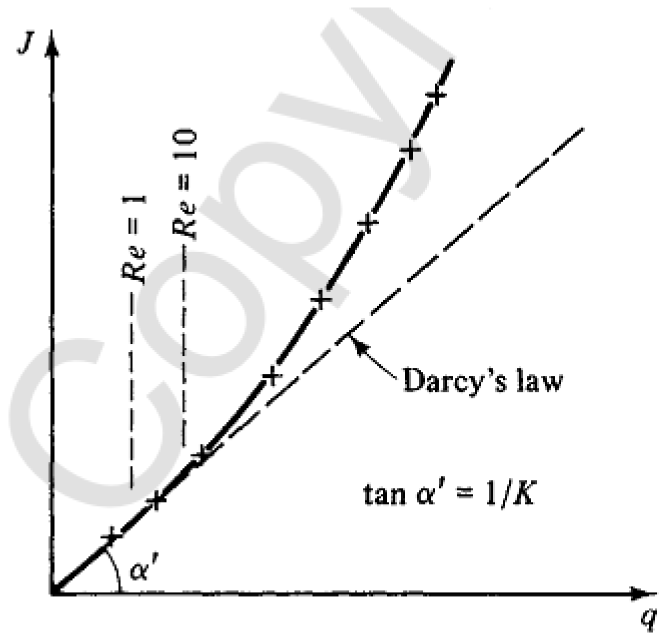


Validity of Darcy's Law

- Validity of Darcy's Law
  - ignored kinetic energy (low velocity)
  - assumed laminar flow
- Reynolds Number for the flow

$$N_R = \frac{\rho q d_{10}}{\mu}$$

  - q = Specific discharge
  - d_{10} = effective grain size diameter
- Darcy's Law is valid for $N_R < 1$



Learning Strategy

Chapter 3: Principles of Flow

Thank you