An Explicit Finite Difference Model for Unconfined Aquifers

by Wen-sen Chu[a] and Robert Willis[b]

ABSTRACT

Most of the current simulation models for unconfined aquifers are based on the assumption that the free surface variation is small so that it can be combined with permeability to reduce the nonlinear Boussinesq equation to a linear partial differential equation (PDE). One of the most obvious reasons for using the linearization assumption is for the ease of numerical solution. This work presents a simpler alternative which permits an easy direct solution of the Boussinesq equation. A forward in time, central in space (FTCS) explicit finite difference method is used in the simulation model. The model was first validated by comparing its results with known analytical solution. It was then applied to an actual situation in which the short-term responses (from pumping) of an unconfined aquifer were simulated. The study shows that the stability of the model can be easily controlled, and because of the simple algorithm used, the code can be expeditiously developed and economically run on smaller machines. Due to the uncertainties in the calibration results, it is recommended here that more data be collected to improve the calibration before the model is used as a real-time simulation tool.

INTRODUCTION

Numerical models have been used widely in the past decade for the analysis of ground-water systems. A thorough classification and a brief introduction to most of the available ground-water models can be found in Bachmat *et al.* (1980) and Wang and Anderson (1982). Numerical techniques that have been used as solution schemes in these models include: finite difference (Remson *et al.*, 1971), finite element (Pinder and Gray, 1977), boundary integral method (Liggett and Liu, 1982), method of characteristics (Reddell and Sunada, 1970; Konikow and Bredehoeft, 1978), and random walk scheme (Prickett *et al.*, 1981). Detailed treatments of these solution methods are given in the cited references. All of these methods have been applied to a wide variety of complex hydraulic situations in the field.

Ground-water flow in shallow, unconfined aquifers can be described by the two-dimensional Boussinesq equation (Bear, 1972). The formulation is based on the Dupuit assumption that prescribes a uniformity of the velocity field over a vertical cross section (hydrostatic pressure distribution) (Eagleson, 1970). The Boussinesq equation is, however, a nonlinear second-order partial differential equation (PDE).

As discussed by Bear (1972), two methods of linearization are commonly used to approximate the solution of the Boussinesq equation. The first method assumes that the deviation from the mean water levels is small. The nonlinear model can then be reduced to a linear PDE that has the same form as the confined flow equation. In the second method, if the temporal variations of head are such that the coefficient of the derivative terms may be considered relatively constant, then the Boussinesq equation is linear in the square of the hydraulic head. The initial and boundary conditions must, however, also be linear in h^2 to preserve the linearity of the problem (Liggett, 1977). The assumptions on which the approximations are based are not always valid in the field applications.

Direct numerical solutions of the Boussinesq equation have been reported by Hornberger *et al.* (1970), Lin (1972), Yoon and Yeh (1975), and

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Reisenauer (1979). Numerical solution of unconfined ground-water flow is also introduced in Remson *et al.* (1971), Pinder and Gray (1977), and Liggett and Liu (1982). The implementation of these previously developed codes is not trivial, and the familiarity of these codes would require extensive effort from well-qualified modelers.

As part of the water resources system investigation in which the short-term response of an unconfined aquifer due to additional pumping was studied under an extremely limited budget (Willis and Chu, 1981), we decided to develop our own simple code for the investigation, rather than implementing an existing code. The code employs a forward in time, central in space (FTCS) explicit finite difference scheme for the solution of the Boussinesq equation. The scheme is not as mathematically elegant as any of the methods cited above, but as will be shown later in this paper, it is actually a very practical method to be considered in real-time (weekly or monthly) ground-water flow simulation problems. Through numerical experiments with data from a simple, as well as the real aquifer, it is found that the stability condition which restricts the size of the discrete time step can be easily controlled. The most important asset of the proposed model is the simplicity of the algorithm. The computational code was expeditiously developed and tested within two man-months, and it can be easily executed on mainframe, mini, or micro (personal) computers.

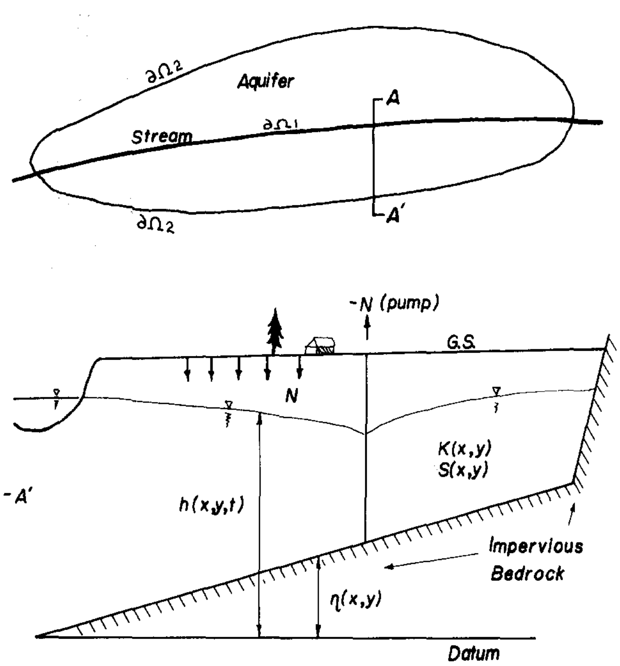

Fig. 1. Stream-aquifer system.

MODEL GOVERNING EQUATION

The vertically averaged governing equation of an unconfined, inhomogeneous and isotropic stream aquifer system (Figure 1) can be written as (Bear, 1972):

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[K(h-\eta)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[K(h-\eta)\frac{\partial h}{\partial y}\right] \pm \sum_{i\epsilon\Omega} \delta(x-x_i)(y-y_i)N \qquad (1)$$

where $h(x, y, t)$ is the hydraulic head (ground-water level); $S(x, y)$ is the storage coefficient; $K(x, y)$ is the permeability; $\eta(x, y)$ is the elevation of the aquifer bottom; $N(x, y, t)$ is the recharge and/or pumpage rate occurring in the domain Ω; and δ is the Dirac delta function.

The initial and boundary conditions for the problem are:

$$h(x, y, 0) = H(x, y) \text{ for all } x, y\epsilon\Omega \qquad (2)$$

$$h(x, y, t) = HQ(x, y, t) \text{ for } x, y\epsilon\partial\Omega_1 \qquad (3)$$

$$\left.\frac{\partial h}{\partial n}\right|_{\partial\Omega_2} = 0 \qquad (4)$$

where $HQ(x, y, t)$ is the held head values along the stream boundary $\partial\Omega_1$; and n is the normal direction to the no-flow boundary $\partial\Omega_2$. The actual values for H and HQ will be presented later in the application.

MODEL SOLUTION SCHEME

The simplest numerical solution scheme to the nonlinear governing equation [equation (1)] is the forward in time, central in space (FTCS) explicit finite difference method (Richtmyer and Morton, 1967). The finite difference formulation for the governing equation can be written as (Figure 2):

$$S_{ij}\frac{h_{ij}^{t+1}-h_{ij}^{t}}{\Delta t} = \frac{1}{\Delta x}\left[\bar{K}_{ij}^{x}\,\overline{(h_{ij}^{t}-\eta_{ij})}^{x}\left(\frac{h_{i+1j}^{t}-h_{ij}^{t}}{\Delta x}\right) - \bar{\bar{K}}_{ij}^{x}\,\overline{\overline{(h_{ij}^{t}-\eta_{ij})}}^{x}\left(\frac{h_{ij}^{t}-h_{i-1j}^{t}}{\Delta x}\right)\right] + \frac{1}{\Delta y}\left[\bar{K}_{ij}^{y}\,\overline{(h_{ij}^{t}-\eta_{ij})}^{y}\cdot\left(\frac{h_{ij+1}^{t}-h_{ij}^{t}}{\Delta y}\right) - \bar{\bar{K}}_{ij}^{y}\,\overline{\overline{(h_{ij}^{t}-\eta_{ij})}}^{y}\left(\frac{h_{ij}^{t}-h_{ij-1}^{t}}{\Delta y}\right)\right] + \sum_{ij}\delta_{ij}N_{ij} \qquad (5)$$

where the averaging operators for a given function f_{ij} are defined by:

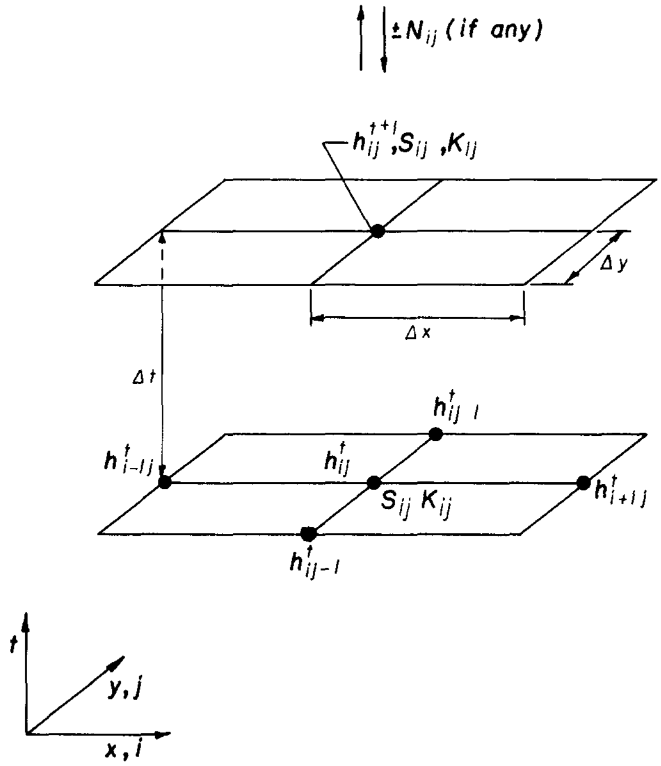

Fig. 2. Finite difference grid structure.

$$\overline{f}^x_{ij} = \tfrac{1}{2}(f_{i+1j} + f_{ij}) \tag{6}$$

$$\overline{\overline{f}}^x_{ij} = \tfrac{1}{2}(f_{ij} + f_{i-1j}) \tag{7}$$

$$\overline{f}^y_{ij} = \tfrac{1}{2}(f_{ij+1} + f_{ij}) \tag{8}$$

$$\overline{\overline{f}}^y_{ij} = \tfrac{1}{2}(f_{ij} + f_{ij-1}) \tag{9}$$

Because of the explicit structure, there is only one unknown, h^{t+1}_{ij} in equation (5). The unknowns at every time step can be directly solved. Also because of the explicit structure, the FTCS method is only conditionally stable (Richtmyer and Morton, 1967). Due to the amplification of its Fourier modes, the solution is not stable (bounded) unless the following condition is satisfied:

$$\frac{\Delta t\, \overline{K(h-\eta)}}{(\Delta \ell)^2 S} \leqslant \tfrac{1}{4} \tag{10}$$

where $\Delta \ell = \Delta x = \Delta y$ for square grid; and $\overline{K(h-\eta)}$ is the linearized $K(h-\eta)$ at a particular point in the domain. Note, however, that equation (10) should not be used as an exact measure of stability for nonlinear equations because $\overline{K(h-\eta)}$ in equation (10) changes over time and space (see Richtmyer and Morton, 1967, pp. 205-206). Since the unknown variable h and aquifer properties K and S appear in the equation, the exact stability condition in a given field application can only be determined by repeated simulations. It is shown in the next sections that the determination and control of the stability condition in the model is easily done in actual model application.

CODE VALIDATION

To check for coding errors, and to verify the numerical properties of the solution scheme, the model was first tested with a simple hypothetical aquifer. The hypothetical aquifer is a rectangular unconfined aquifer (Figure 3) with Dirichlet boundary condition $h(0, y, t) = 0$ on the left edge, and Neumann boundary conditions $\partial h/\partial x = \partial h/\partial y = 0$ on the remaining boundary sides. This narrow aquifer is designed for symmetric solutions in the x-direction which can then be verified with available one-dimensional analytical solutions. Assuming that (1) S and K are constant, and (2) no sinks or sources are in the domain, the analytical solution for the given unconfined aquifer can be written as (Bear, 1972, Section 8.2):

$$h(x, y', t) = \frac{h_0(x)}{\{1 + [\dfrac{\beta K\, h_0(L)\, t}{SL^2}]\}} \tag{11}$$

where $h_0(x)$ is an x-symmetric initial condition; y' is any given y value in the domain; L is the length of the aquifer; β is a constant (the value is approximately 1.12). The initial condition chosen for the test of the model is:

$$h_0(x) = h(x, y', 0) = 2\sqrt{x} \tag{12}$$

With an assumed K of 200 ft/day, S of 0.15, $\Delta x = \Delta y = 500$ ft, and the maximum probable head of 126 ft in the test aquifer [equation (12)], the

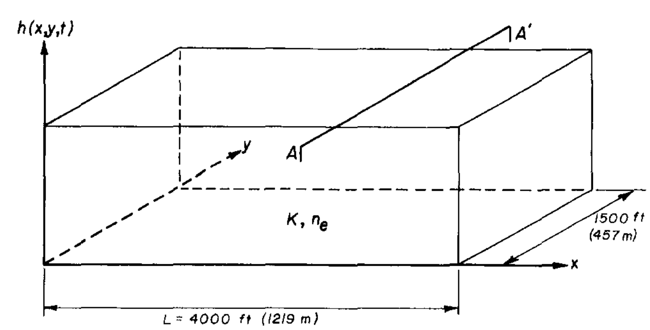

Fig. 3. Hypothetical aquifer configuration.

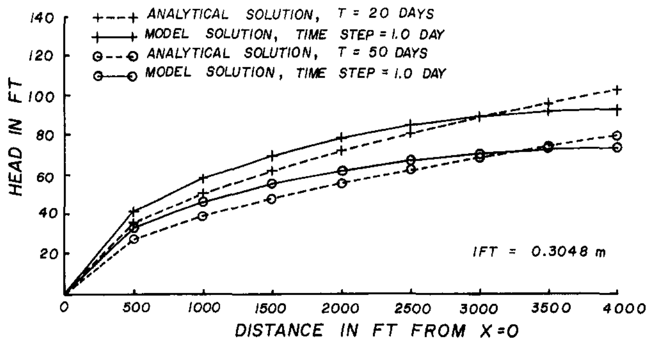

Fig. 4. Comparison between analytical solutions and model results.

largest time step allowed according to equation (10) is 0.37 day. However, it was found from numerical experiments that the model is stable with time steps of up to one day. This shows that even with known maximum values of K and $(h - \eta)$, the stability conditions given by equation (10) are still not exact. It is obviously more efficient to use a time step of one day, rather than 0.37 day given by the stability condition.

Representative model results using a time step of one day are shown with the analytical solutions in Figure 4. To test for consistency of the code (Richtmyer and Morton, 1967), results using different time steps (within the stability constraint) were also compared, and found to be almost identical.

With the stability and consistency tests completed, the model is then applied to an actual unconfined aquifer. The application results are reported next.

FIELD APPLICATION

As part of the water resources system investigation study, the model was applied to simulate the ground-water response (due to additional pumping) in Blue Lake Aquifer in Northern California (Figure 5). Blue Lake Aquifer is an unconfined, alluvial aquifer system formed by deposition from the Mad River Basin. The aquifer is underlain by the Franciscan basement formation, and its material compositions range from fine sand to gravel. Clay lenses and faults have also been found throughout the basin (Willis and Chu, 1981). The aquifer provides water through five Ranney collector wells operated by the Humboldt Bay Municipal Water District (HBMWD) which serves the surrounding municipalities and service districts. Despite its important water-supply role, there is virtually no long-term observation data.

Because of the dense locations of the pumping wells (each having a designed capacity of 15 MGD) and the hydraulic interaction between the aquifer and the river, the validity of the assumptions used in linearizing the Boussinesq equation becomes questionable. The use of equation (1) is more appropriate in this aquifer.

The primary objective of the application is to explore the possibility of using the proposed model as a simulation tool for short-term (weekly and daily) as well as long-term (monthly) operations. The secondary objective of the application is to gain some insight into the capability and response of the aquifer system for future planning purposes (Willis and Chu, 1981).

To accommodate the finite difference method, the aquifer system was discretized using a uniform nodal spacing of 528 ft (Figure 6). The depth of the aquifer varies from 40 ft to 80 ft, and the storage coefficient is between 10 to 15 percent. Permeability K was initially set to 268 ft/day throughout the domain. These values were determined from cross-sectional data, and scattered well log data available at the HBMWD office (Willis and Chu, 1981). All river stages and pumping rates were set equal to known monthly averaged values (steady-state) in the testing runs. The river stages were obtained by back-water computation from the only gaging station at Essex (Figure 6). Good hydraulic connection between the river and the aquifer was assumed [equation (3)].

According to equation (10), the largest Δt that can be used for $\Delta \ell = 528$ ft, $S = 0.15$, $K = 268$ ft/day, and the maximum aquifer depth of 80 ft, is

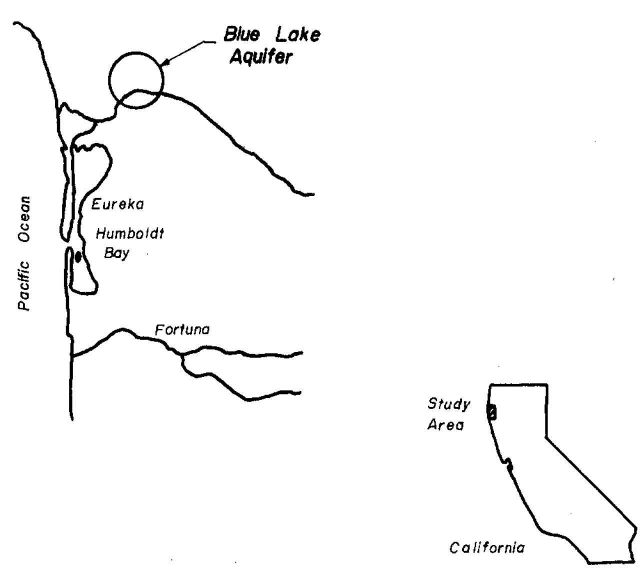

Fig. 5. Study area location.

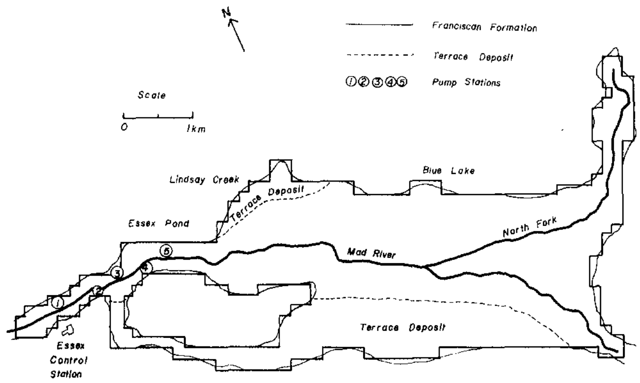

Fig. 6. Blue Lake Aquifer.

Table 1. Some Selected Pumping Schedules of Blue Lake Aquifer System

Date	Wells	*Daily operating hours*	Q^{*} *MGD*	Q *cfs*	Q^{**} *ft/day*
May average, 1981	1	17.41	4.18	6.48	2.10
	2	10.12	2.43	3.77	1.17
	3	4.32	1.56	2.42	0.75
	4	5.90	1.42	2.20	0.68
	5	5.87	2.10	3.27	1.10

* Million gallons per day.
** Pumping per finite difference grid area, model input.

0.5 day. Preliminary simulations with the test data verified that the model would not produce stable solutions in 50 time steps when Δt was greater than 0.5 day.

After the initial runs assuming homogeneous conditions, the model was tested using more realistic parameters. Because there are no observation wells within the study area, the model cannot be fully calibrated. Due to budgetary and time constraints, no rigorous data collection effort was attempted in the study (Willis and Chu, 1981). Instead, water-table data were gathered from a number of privately-owned service wells in the aquifer during the month of May 1981. Due to the nonsynoptic nature of the water-table data and the lack of transient river stage data in the month of May 1981, only a steady-state calibration was performed.

The steady-state calibration assumed that the available water-table data remained constant over time during the month of May 1981. These data were then compared with the model steady-state solutions as various permeability values were tried to reflect the characteristics of the aquifer more realistically. The model steady-state solutions were obtained by continuously running the model with the known (constant) monthly pumping schedule (given in Table 1) and steady-state river stages constructed from back-water computation. To simplify the adjustment process, the storage coefficient was left unchanged. Except from the river, recharge to the aquifer was assumed to be zero.

One of the better comparisons between the model solutions and the service well data is shown in Figure 7. The particular permeability map used for the comparisons is shown in Figure 8. From Figure 7, it can be seen that the matching between the model solutions and service well data is quite good except at certain wells near the river.

The actual locations of the service wells where the comparisons were poor (see Figure 7) are all located within one grid length (528 ft) from the river. Yet the observed water tables at those wells are all (except one) consistently 20 to 30 feet higher than calculated river stages. On the other

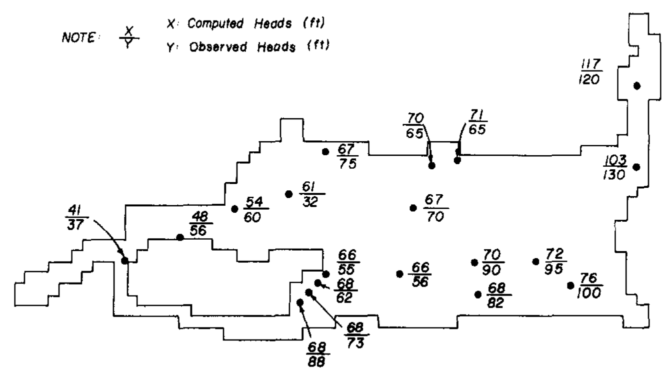

Fig. 7. Comparison between model simulation results and observed data.

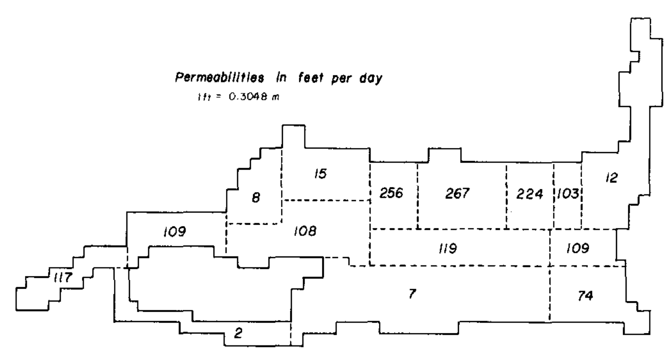

Fig. 8. Estimated permeability map.

hand, the model solutions at these same points closely followed the river stage because of the good hydraulic connection assumed. The higher observed water tables could have been the result of: (1) perched ground-water table over clay lenses; (2) local fault formation; or (3) some other vertical impervious barriers. Because of the lack of soil and river stage data near these locations, none of the above speculated causes could be positively identified in this preliminary study. However, as a result of these discrepancies, a strong recommendation for data collection was made to HBMWD in the planned second phase of the study (Willis and Chu, 1981).

The steady-state aquifer response obtained from running the model with the permeability data shown in Figure 8 and the monthly averaged pumping schedule and river stage of May 1981 is shown in Figure 9.

The model was also extensively tested with known transient pumping schedules from HBMWD operation records and assumed transient river stages. The results from all of the runs indicated that the model can be used as an effective simulation tool for actual HBMWD operations. The model was found to be stable under all of the simulated transient operating conditions with the time step of 0.5 day.

Computationally, the central memory requirement for the Blue Lake Aquifer grid dimension of 61 × 33 on a CYBER 170-750 computer system is 29 K of 60-bit words. Execution time (excluding graphic output production) for a 100-step simulation (50 days) is 11 CYBER CPU seconds.

SUMMARY

A simulation model for transient two-dimensional vertically averaged flow in an unconfined aquifer was successfully developed. The model solves the Boussinesq equation governing the water-table variation in the aquifer. The nonlinear partial differential equation is solved by the simplest finite difference scheme—the forward in time, central in space explicit method without any matrix solver. Although the method is known to have a fairly restrictive stability condition, it was shown that the condition is not a significant limiting factor for short-term (months) real-time simulations.

The accuracy of the present code was tested by comparing results with an analytical solution. All of the head values predicted by the model compared closely to the analytical solution. The model was also tested using realistic data for Blue Lake Aquifer in Northern California. The aquifer parameters are determined partly from a small number of well logs and cross-sectional surveys, and partly from water-level data taken from small, privately-owned service wells.

Through the comparison of model simulation results and limited observations, it was possible to identify certain "trouble spots" in the aquifer. The results indicate that besides the river stage and water-table information, special attention should be directed toward data collection in the southeastern portion of the aquifer. This exercise showed that the use of simulation models does not have to be constrained by limited data, and that models can be used as an aid to identify locations where additional data should be collected.

Perhaps the most valuable aspect of the proposed model is its simplicity. This model can actually be coded with minimum in-house manpower and resources, and can be easily implemented on mini or even personal computers.

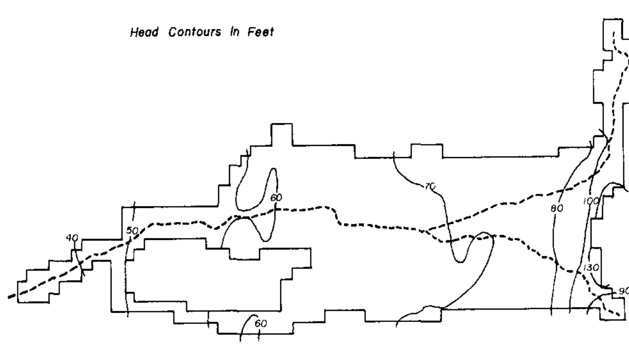

Fig. 9. Simulated head contour using monthly averaged pumping schedule of May 1981.

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