Module 02: Numerical Methods

Unit 05: Partial Differential Equation: BVP

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Learning Objectives

- To discretize the derivatives of single-valued multi-dimensional functions using finite difference approximations.

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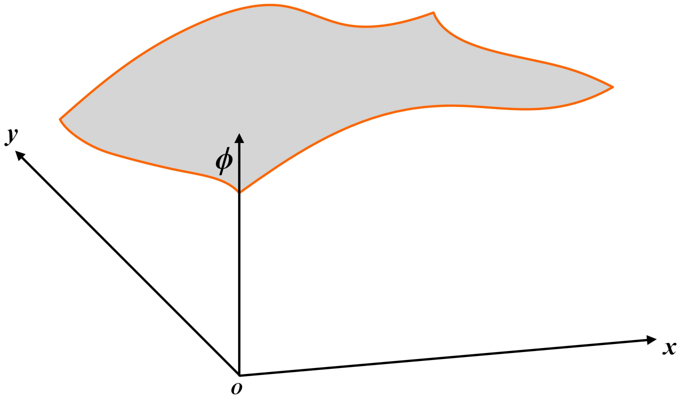
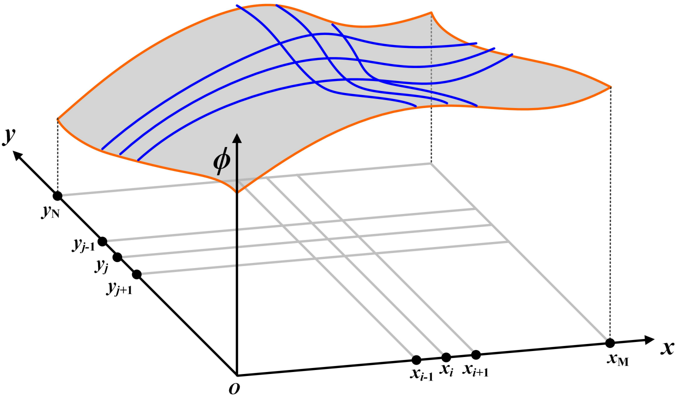
- To discretize the derivatives of single-valued multi-dimensional functions using finite difference approximations.
- To derive the algebraic form using discretized PDE and BCs.

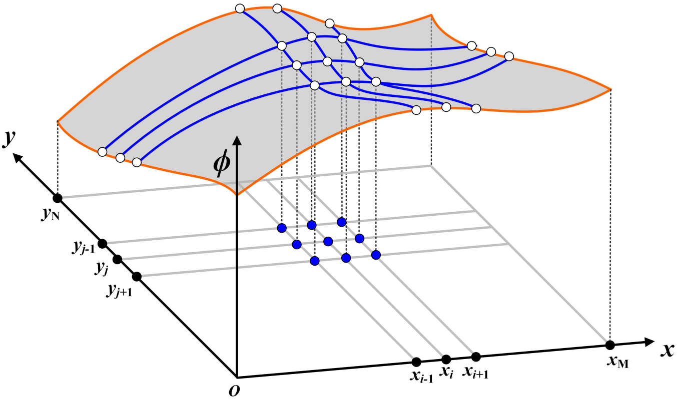
Finite Difference



Finite Difference



Finite Difference



Taylor Series

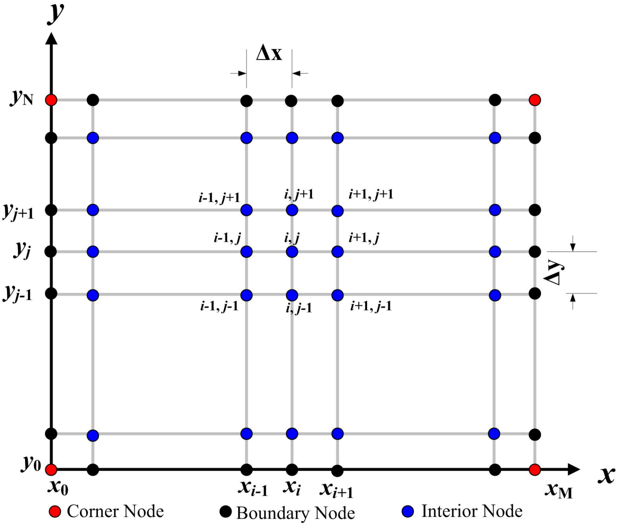
Taylor series expansion for a function with two independent variables can be expressed as,

$$\phi(x+\Delta x, y+\Delta y) = \sum_{\eta_x=0}^{\infty}\sum_{\eta_y=0}^{\infty} \frac{\Delta x^{\eta_x}\Delta y^{\eta_y}}{\eta_x!\eta_y!}\frac{\partial^{\eta_x+\eta_y}\phi(x,y)}{\partial x^{\eta_x}\partial y^{\eta_y}}$$

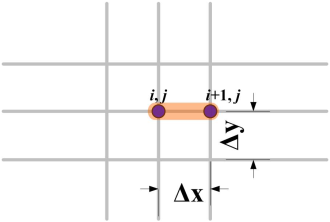
$$= \phi(x,y) + \Delta x\frac{\partial \phi}{\partial x} + \Delta y\frac{\partial \phi}{\partial y} +$$

$$\frac{1}{2!}\left[\Delta x^2\frac{\partial^2\phi}{\partial x^2} + 2\Delta x\Delta y\frac{\partial^2\phi}{\partial x\partial y} + \Delta y^2\frac{\partial^2\phi}{\partial y^2}\right] + \cdots$$

Grid Points


y
Δx
y_N
y_{j+1}
y_j
y_{j-1}
y_0
i-1, j+1
i, j+1
i+1, j+1
i-1, j
i, j
i+1, j
i-1, j-1
i, j-1
i+1, j-1
Δy
x_0
x_{i-1}
x_i
x_{i+1}
x_M
x
Corner Node
Boundary Node
Interior Node

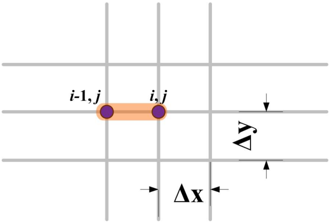
Finite Difference Approximations



First Order Forward Difference w.r.t. x

$$\left.\frac{\partial \phi}{\partial x}\right|_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} + \mathcal{O}(\Delta x) \qquad (1)$$

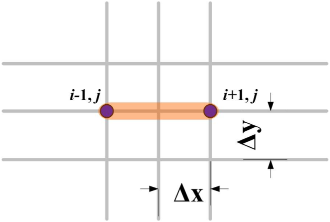
Finite Difference Approximations



First Order Backward Difference w.r.t. x

$$\left.\frac{\partial \phi}{\partial x}\right|_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} + \mathcal{O}(\Delta x) \tag{2}$$

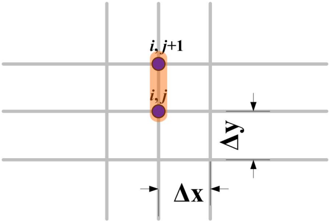
Finite Difference Approximations



Second Order Center Difference w.r.t. x

$$\left.\frac{\partial \phi}{\partial x}\right|_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} + \mathcal{O}(\Delta x^2) \quad (3)$$

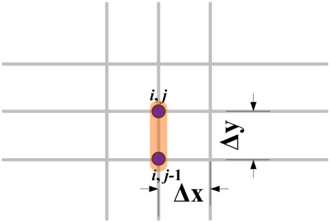
Finite Difference Approximations



First Order Forward Difference w.r.t. y

$$\left.\frac{\partial \phi}{\partial y}\right|_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y} + \mathcal{O}(\Delta y) \qquad (4)$$

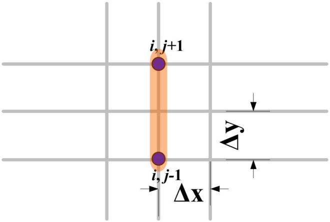
Finite Difference Approximations



First Order Backward Difference w.r.t. y

$$\left.\frac{\partial \phi}{\partial y}\right|_{i,j} = \frac{\phi_{i,j} - \phi_{i,j-1}}{\Delta y} + \mathcal{O}(\Delta y) \quad (5)$$

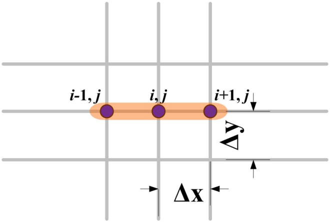
Finite Difference Approximations



Second Order Center Difference w.r.t. y

$$\left.\frac{\partial \phi}{\partial y}\right|_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y} + \mathcal{O}(\Delta y^2) \qquad (6)$$

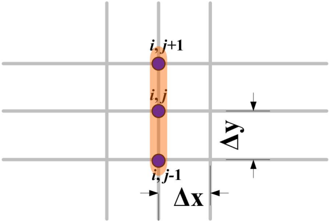
Finite Difference Approximations



Second Order Center Difference w.r.t. x

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{7}$$

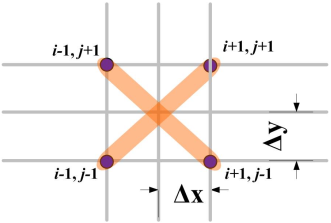
Finite Difference Approximations



Second Order Center Difference w.r.t. y

$$\left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j} = \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta y^2) \tag{8}$$

Finite Difference Approximations



Second Order Center Mixed Difference w.r.t. x and y

$$\left.\frac{\partial^2 \phi}{\partial x \partial y}\right|_{i,j} = \frac{\phi_{i+1,j+1} + \phi_{i-1,j-1} - \phi_{i-1,j+1} - \phi_{i+1,j-1}}{4\Delta x \Delta y} + \mathcal{O}(\Delta x^2, \Delta y^2) \quad (9)$$

General Equation

A form of differential equation with a general variable ϕ:

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla.(\Upsilon_\phi \phi \mathbf{u}) = \nabla.(\mathbf{\Gamma}_\phi.\nabla\phi) + F_{\phi_o} + S_\phi \tag{10}$$

where

ϕ = general variable

$\Lambda_\phi,\ \Upsilon_\phi$ = problem dependent parameters

$\mathbf{\Gamma}_\phi$ = tensor

F_{ϕ_o} = other forces

S_ϕ = source/sink term

Problem Definition

Governing equation

A two-dimensional BVP can be written as,

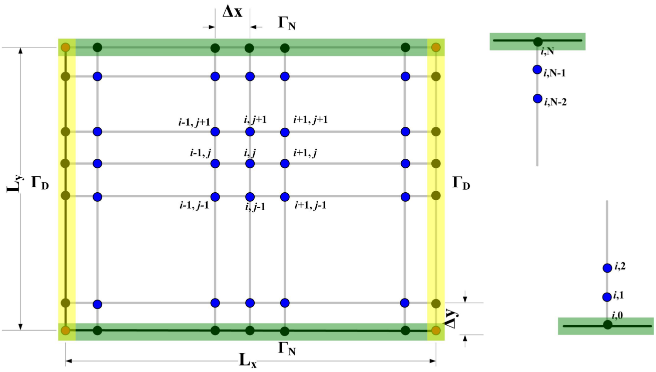
$$\Omega : \quad \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y) = 0$$

subject to

Boundary Condition

$$\begin{aligned}
\Gamma_D^1 : &\quad \phi(0, y) = \phi_1 \\
\Gamma_D^2 : &\quad \phi(L_x, y) = \phi_2 \\
\Gamma_N^3 : &\quad \left.\frac{\partial \phi}{\partial y}\right|_{(x,0)} = 0 \\
\Gamma_N^4 : &\quad \left.\frac{\partial \phi}{\partial y}\right|_{(x,L_y)} = 0
\end{aligned}$$

Domain Discretization



Numerical Discretization
Governing Equation

The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi\big|_{i,j}$$

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The equation can be rearranged as,

$$\frac{\Gamma_y}{\Delta y^2}\phi_{i,j-1} + \frac{\Gamma_x}{\Delta x^2}\phi_{i-1,j} - 2\left(\frac{\Gamma_x}{\Delta x^2} + \frac{\Gamma_y}{\Delta y^2}\right)\phi_{i,j} + \frac{\Gamma_x}{\Delta x^2}\phi_{i+1,j} + \frac{\Gamma_y}{\Delta y^2}\phi_{i,j+1} = -S_\phi\big|_{i,j}$$

Numerical Discretization
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In simplified form, this can be written as

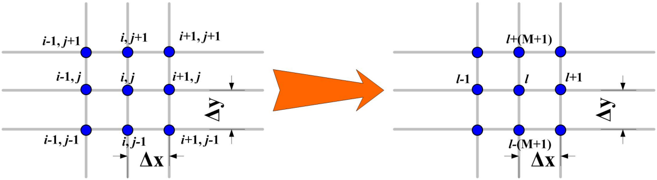
$$\alpha_y\phi_{i,j-1} + \alpha_x\phi_{i-1,j} - 2\left(\alpha_x + \alpha_y\right)\phi_{i,j} + \alpha_x\phi_{i+1,j} + \alpha_y\phi_{i,j+1} = -S_\phi\big|_{i,j}$$

with $\alpha_x = \frac{\Gamma_x}{\Delta x^2}$ and $\alpha_y = \frac{\Gamma_y}{\Delta y^2}$.

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$



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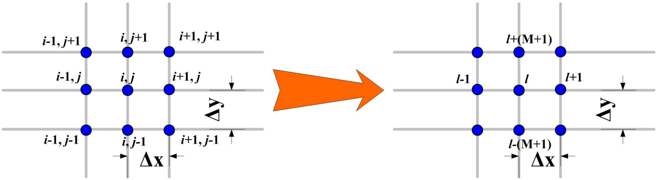
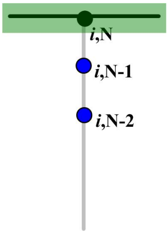


With single index notation, the equation can be written as,

$$\alpha_y \phi_{l-(M+1)} + \alpha_x \phi_{l-1} - 2\left(\alpha_x + \alpha_y\right)\phi_l + \alpha_x \phi_{l+1} + \alpha_y \phi_{l+(M+1)} = -S_\phi\Big|_{i,j}$$

Neumann Boundary Condition

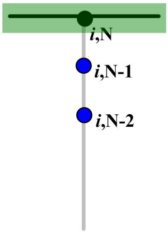


Top Boundary

Second Order Discretization

$$\frac{3\phi_{i,N} - 4\phi_{i,N-1} + \phi_{i,N-2}}{2\Delta y} + \mathcal{O}(\Delta y^2) = 0 \tag{11}$$

Neumann Boundary Condition



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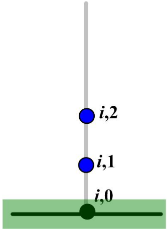
In single index notation format,

$$\frac{3\phi_{l} - 4\phi_{l-(M+1)} + \phi_{l-2(M+1)}}{2\Delta y} = 0 \tag{12}$$

Neumann Boundary Condition

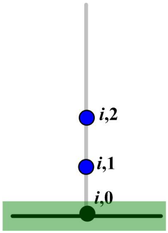


Bottom Boundary

Second Order Discretization

$$\frac{-3\phi_{i,0} + 4\phi_{i,1} - \phi_{i,2}}{2\Delta y} + \mathcal{O}(\Delta y^2) = 0 \tag{13}$$

Neumann Boundary Condition



Bottom Boundary

Second Order Discretization

$$\frac{-3\phi_{i,0} + 4\phi_{i,1} - \phi_{i,2}}{2\Delta y} + \mathcal{O}(\Delta y^2) = 0 \quad (13)$$

In single index notation format,

$$\frac{-3\phi_l + 4\phi_{l+(M+1)} - \phi_{l+2(M+1)}}{2\Delta y} = 0 \quad (14)$$

Matrix Form

$$
\begin{bmatrix}
\times & \times & & & & \times & & & & \\
\times & \times & \times & & & & \times & & 0 & \\
& \times & \times & \times & & & & \times & & \\
& & \times & \times & \times & & 0 & & \times & \\
& & & \times & \times & \times & & & & \times \\
\times & & & & \times & \times & \times & & & \\
& \times & & 0 & & \times & \times & \times & & \\
& & \times & & & & \times & \times & \times & \\
& 0 & & \times & & & & \times & \times & \times \\
& & & & \times & & & & \times & \times
\end{bmatrix}
$$

Thank You