$$\rho \frac{\frac{K}{\rho}}{\left(1+\frac{DK}{eE}\right)} \cdot \frac{\partial V}{\partial x} + \frac{\partial P}{\partial t} + V\frac{\partial P}{\partial x} = 0$$

$$\Rightarrow \rho a^2 \cdot \frac{\partial V}{\partial x} + \frac{\partial P}{\partial t} + V\frac{\partial P}{\partial x} = 0$$

Derivation of momentum equation

$$\frac{d}{dt}\int_v \beta\rho dv + \beta\rho(V-w)A\Big|_{outlet} - \beta\rho(V-w)A\Big|_{inlet} = 0$$

For momentum equation $\beta = V$

$$\frac{d}{dt}\int_v V\rho dv + V\rho(V-w)A\Big|_{outlet} - V\rho(V-w)A\Big|_{inlet} = \sum F$$

By applying Leibnitz rule to the first term

$$\sum F = \int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho AV)dx + \rho AV\Big|_{out}\frac{dx_2}{dt} - \rho AV\Big|_{in}\frac{dx_1}{dt} + \rho A(V-w)V\Big|_2 - \rho A(V-w)V\Big|_1$$

$$\sum F = \int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho AV)dx + (\rho AV)_2 w_2 - (\rho AV)_1 w_1 + \left[\rho A(V-w)V\right]_2 - \left[\rho A(V-w)V\right]_1$$

$$\frac{\sum F}{\Delta x} = \frac{\partial}{\partial t}(\rho AV) + \frac{(\rho AV^2)_2 - (\rho AV^2)_1}{\Delta x}$$

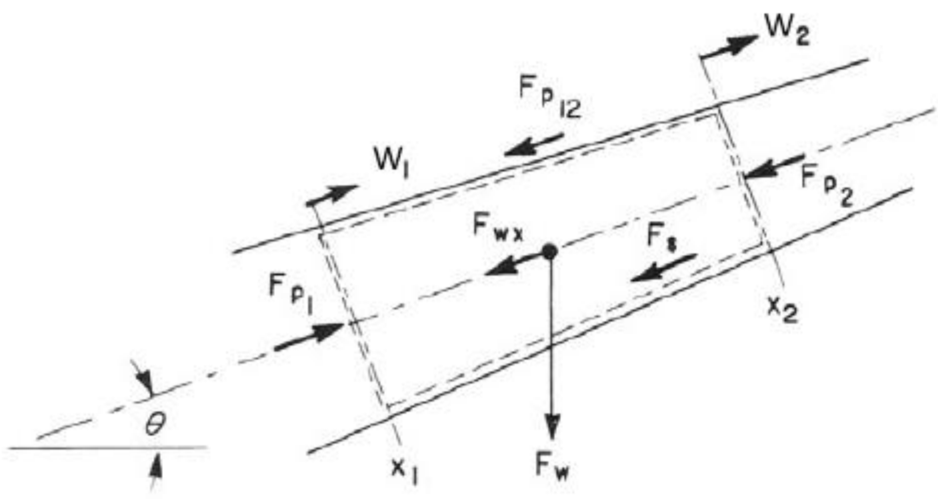

Fig. 2.10. Application of momentum equation

External forces acting are (1) pressure force, (2) gravity force and (3) shear stress. Pressure force and shear stresses are surface forces, and gravitational force is equal to body force.

Pressure forces at two sections 1and 2 are denoted as P_1A_1 , P_2A_2 respectively.

Pressure force on the converging side, $F_{P12} = \frac{1}{2}(P_1 + P_2)(A_1 - A_2)$

Gravitational force = $\gamma A(x_2 - x_1)\sin\theta$

$\theta =$ Angle between horizontal and pipe centerline

Shear force = $\tau_0 \pi D(x_2 - x_1)$

$$\sum F = P_1A_1 - P_2A_2 - \frac{1}{2}(P_1 + P_2)(A_1 - A_2) - \rho g A(x_2 - x_1)\sin\theta - \tau_0 \pi D(x_2 - x_1)$$

$$\sum F = P_1A_1 - P_2A_2 - \frac{1}{2}P_1A_1 + \frac{1}{2}P_2A_2 + \frac{1}{2}P_1A_2 - \frac{1}{2}P_2A_1 - \rho g A \cdot \Delta x \sin\theta - \tau_0 \pi D \cdot \Delta x$$

$$\frac{\sum F}{\Delta x} = \frac{(P_1 - P_2)(A_1 + A_2)}{2\Delta x} - \rho g A \sin\theta - \tau_0 \pi D$$

Total equation can be written as

$$\frac{\partial}{\partial t}(\rho A V) + \frac{\partial}{\partial x}(\rho A V^2) + A\frac{\partial P}{\partial x} + \rho g A \sin\theta + \tau_0 \pi D = 0$$

Substitute $\tau_0 = \frac{1}{8}\rho f V|V|$

$$V\frac{\partial}{\partial t}(\rho A) + \rho A\frac{\partial}{\partial t}(V) + V\frac{\partial}{\partial x}(\rho A V) + \rho A V\frac{\partial}{\partial x}(V) + A\frac{\partial P}{\partial x} + \rho g A \sin\theta + \frac{\rho A f V|V|}{2D} = 0$$

$$V\left[\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A V)\right] + \rho A\frac{\partial V}{\partial t} + \rho A V\frac{\partial V}{\partial x} + A\frac{\partial P}{\partial x} + \rho g A \sin\theta + \frac{\rho A f V|V|}{2D} = 0$$

$$\frac{dV}{dt} + \frac{1}{\rho}\frac{\partial P}{\partial x} + g\sin\theta + \frac{fV|V|}{2D} = 0$$

Generally PDEs are type of hyperbolic, parabolic or elliptic.

Wave equation $(u_{tt} - u_{xx} = 0)$ = Hyperbolic equation (Explicit & Implicit FD)

Heat equation $(u_t = Ku_{xx})$ = Parabolic equation (Alternate direction Implicit method)

Laplace equation, poison equation = elliptic equation (Five point method)

General differential equation

$$Au_{xx} + Bu_{xy} + CV_{yy} + Du_x + Eu_y + F = 0$$

$$B^2 - 4AC = 0\left(Parabolic\right)$$
$$B^2 - 4AC < 0\left(elliptic\right)$$
$$B^2 - 4AC > 0\left(hyperbolic\right)$$

In continuity equation and momentum equations, describe transient flows in closed conduit:

Independent variables = x, t

Dependent variables = P, V

System constants = a, ρ, f, D

Where ‘a’ is function of P but neglected. Similarly f is a function of velocity but neglected.

Classification of governing equations

CE: $$\frac{\partial P}{\partial t} + V\frac{\partial P}{\partial x} + \rho a^2 \cdot \frac{dV}{dx} = 0$$

ME: $$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} + g\sin\theta + \frac{fV|V|}{2D} = 0$$

In matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} P \\ V \end{pmatrix} + \begin{bmatrix} V & \rho a^2 \\ \frac{1}{\rho} & V \end{bmatrix}\frac{\partial}{\partial x}\begin{pmatrix} P \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ -g\sin\theta - \frac{fV|V|}{2D} \end{pmatrix}$$

$$\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} = E$$

Where $U = \begin{pmatrix} P \\ V \end{pmatrix}; B = \begin{bmatrix} V & \rho a^2 \\ \frac{1}{\rho} & V \end{bmatrix}; E = \begin{pmatrix} 0 \\ -g\sin\theta - \frac{fV|V|}{2D} \end{pmatrix}$

Eigen value (λ) & Eigen vector (V)

Determinant $|A - \lambda I| = 0$

Eigen values characterize important properties of linear transformations such as whether a system of linear equations has a unique solution or not. In many applications, eigenvalues describe physical properties of a mathematical model.

Compute Eigen values

$$\left|B-\lambda I\right|=0$$

$$\begin{bmatrix} V-\lambda & \rho a^2 \\ \dfrac{1}{\rho} & V-\lambda \end{bmatrix}=0 \Rightarrow \left(V-\lambda\right)^2 = a^2$$

$$\Rightarrow V-\lambda=\pm a$$

$$\Rightarrow \lambda = V \pm a$$

Since λ is real and distinct, governing equations (CE & ME) are type of hyperbolic partial differential equation.

When Eigen value = 0 $\Rightarrow$ System is said to be elliptical.

Eigen values are real & distinct $\Rightarrow$ Hyperbolic

Eigen values are equal & real $\Rightarrow$ Parabolic

Governing equations are type of wave equations and describe properties of waves in a fluid.

Initial conditions correspond to steady state are required to solve the governing equations.

In steady state, $\dfrac{\partial P}{\partial t}=0, \dfrac{\partial V}{\partial t}=0$

CE: $V\dfrac{\partial P}{\partial x}+\rho a^2 \cdot \dfrac{dV}{dx}=0$

ME: $V\dfrac{\partial V}{\partial x}+\dfrac{1}{\rho}\dfrac{\partial P}{\partial x}+g\sin\theta+\dfrac{fV\left|V\right|}{2D}=0$

V and P need to be solved from the above equations and supplied to steady state CE & ME to solve V and P in $\left(x,t\right)$.

Simplified equations

In most of the problems convective acceleration terms are small hence negligible.

CE: $\dfrac{\partial P}{\partial t}+V\dfrac{\partial P}{\partial x}+\rho a^2 \cdot \dfrac{dV}{dx}=0$

ME: $\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}+g\sin\theta+\frac{fV|V|}{2D}=0$

$V\frac{\partial P}{\partial x}$ and $V\frac{\partial V}{\partial x}$ are negligible. In addition, $g\sin\theta$ is also negligible, because elevation changes does not contribute much.

CE: $\frac{\partial P}{\partial t}+\rho a^2\cdot\frac{dV}{dx}=0$

ME: $\frac{\partial V}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{fV|V|}{2D}=0$

We know that, $P=\gamma H=\rho g H$ and $V=\frac{Q}{A}$

By putting $P\,\&V$ in CE and ME, we can get:

$$\rho g\frac{\partial H}{\partial t}+\frac{\rho a^2}{A}\cdot\frac{dQ}{dx}=0 \quad \Rightarrow \frac{\partial H}{\partial t}+\frac{a^2}{gA}\cdot\frac{dQ}{dx}=0$$

$$\frac{1}{A}\frac{\partial Q}{\partial t}+\frac{1}{\rho}\rho g\frac{\partial H}{\partial x}+\frac{fQ|Q|}{2A^2D}=0\Rightarrow\frac{\partial Q}{\partial t}+gA\frac{\partial H}{\partial x}+\frac{fQ|Q|}{2AD}=0$$

Wave velocity

Wave velocity is function of bulk modulus of the fluid, mass density of the fluid, elastic properties of the conduit as well as external constraints. Elastic properties of conduit are D,e and material. External constraints include types of joints, and anchoring conditions of the pipe.

$$a=\sqrt{\frac{K}{\rho\left(1+\frac{K}{E}\psi\right)}}$$

Where K = bulk modulus of water

E = Elasticity of the pipe material

ψ = external conditions

Effect of air entrainment on wave speed

When air is present in the liquid, either as small bubbles or large volume, the wave speed decreases drastically, as a consequence, water hammer pressures are decreased.

Assumptions

Air water mixture is assumed to be uniformly distributed throughout the pipeline.

A) pipe rigidly anchored, $\sigma_1 = \mu \sigma_2$, $\varepsilon_1 = 0$ | Date: 28/2/2023 | (1)

B) For pipe with expansion joints $\sigma_1 = 0$

C) pipe anchored at one end and free to stretch longitudinally $\sigma_1 = \frac{\sigma_2}{2}$

Wave velocity in a pipe changes with type of restraint, the relation between longitudinal stress and strain varies with restraint type

$$a = \sqrt{\frac{K}{\rho\left(1 + \frac{DK}{Ee}\psi\right)}}$$

e = pipe wall thickness

ψ = non-dimensional parameter that depends on restraint type

I thin walled elastic conduit $\left(\frac{e}{D} < 0.06\right)$

(1) pipe anchorage only at the upstream end

$$\psi = \frac{5}{4} - \mu \quad \text{or} \quad (1 - 0.5\mu)$$

(2) Full pipe restraint from axial movement

$$\psi = 1 - \mu^2$$

(3) Longitudinal expansion joints along the pipeline

$$\psi = 1.0$$

(4) Rigid conduit

$$\psi = 0$$

II Thick walled elastic conduit

(i) conduit anchored against longitudinal movement throughout its length

$$\psi = \frac{2e}{D}(1+\mu)\left[\frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} - \frac{2\mu R_i^2}{R_o^2 - R_i^2}\right]$$

R_o and R_i are the external and internal radii of the conduit

$e = R_o - R_i$

(ii) Conduit anchored against longitudinal movement at the upstream end

$$\psi = \frac{2e}{D}\left[\frac{R_o^2 + 1{\cdot}5 R_i^2}{R_o^2 - R_i^2} + \frac{\nu(R_o^2 - 3R_i^2)}{R_o^2 - R_i^2}\right]$$

(iii) conduit with frequent ex

$$\psi = \frac{2e}{D}\left(\frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} + \mu\right)$$

III) Rock tunnel

(i) Unlined tunnel

$$\psi = \frac{e}{D}$$

Note: dont worry about e, since it will be ~~~~ cancelled out..

$$a = \sqrt{\frac{K}{\rho\left(1 + \frac{\cancel{D}K}{E\cancel{e}} \cdot \frac{\cancel{e}}{\cancel{D}}\right)}}$$

$$E = G$$

$G =$ modulus of rigidity of the rock

(ii) steel-lined tunnel

$$\psi = \frac{e}{D}\left(\frac{DE}{GD + Ee}\right)$$

$e =$ thickness of the steel-liner

$E =$ modulus of elasticity of steel

Note: A penstock encased in a concrete is also treated as steel-lined tunnel

(IV) Reinforced Concrete pipe

The reinforced concrete pipe is replaced by an equivalent steel pipe have equivalent thickness

$$e_e = E_r e_c + \frac{A_s}{l_s}$$

$e_c =$ concrete pipe thickness

A_s & l_s are the cross-sectional area and the spacing of steel bars

$E_r =$ ratio of the modulus of elasticity of concrete to that of steel

Example

Compute the wave velocity in the steel penstock of the Kootenay Canal hydroelectric powerplant, BC, Canada. The data for different segments of the penstock are listed in below table. The values of E for steel, G for concrete, and K and ρ for water are 207 GPa (E), 20.7 GPa (G), and 999 kg/m³ (ρ) respectively

$K = 2.19$ GPa

Solution : For transient analysis, the wave velocity in each segment of the penstock may be determined as follows.

Table

Pipe No	Length (m)	Diameter (m)	Wall thickness (mm)	Type of anchoring
1	244	6.771	19	Expansion coupling at one end
2	36.5	5.55	22	Encased in concrete

Pipe 1 (thin walled)

$$\psi = (1 - 0.5\mu) = \left[1 - (0.5)(0.3)\right] = 1 - 0.15 = 0.85$$

$$a = \sqrt{\frac{K}{\rho\left[1 + \frac{DK}{Ee}\psi\right]}}$$

$$\frac{D}{e} = \frac{6.71}{0.019} = 353$$

$$\frac{K}{E} = \cancel{2.19} \quad \cancel{\frac{2.07}{20.7}} = \frac{2.19 \times 10^9}{207 \times 10^9} = 0.0106$$

$$a = \sqrt{\frac{K}{\rho\left[1 + \frac{DK}{Ee}\psi\right]}}$$

$$= \sqrt{\frac{2.19 \times 10^9}{999\left[1 + (353)(0.0106)(0.85)\right]}}$$

$$= 724 \text{ m/s}$$

$\boxed{\text{pipe 2}}$ Equations for a steel-lined tunnel may be used to compute the wave velocity in pipe 2

$$\psi = \frac{e}{D}\left[\frac{DE}{GD + Ee}\right]$$

$$= \frac{22 \times 10^{-3}}{5.55}\left[\frac{5.55 \times 207 \times 10^9}{(20.7 \times 10^9 \times 5.55) + (207 \times 10^9 \times 0.022)}\right]$$

$$= 1410 \text{ m/s}$$

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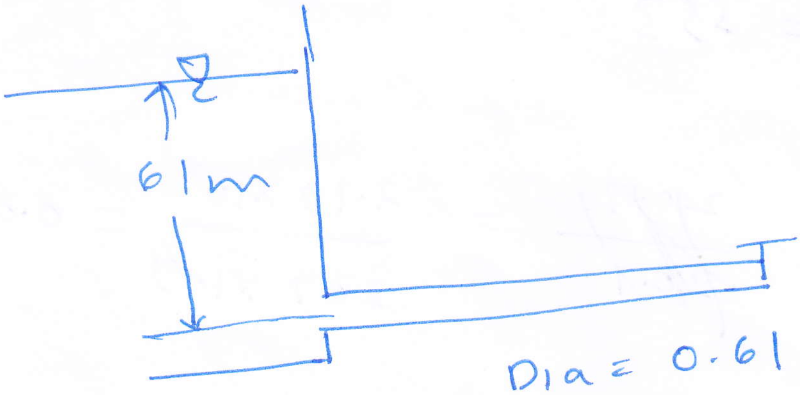

$E = 207$ GPa

~~$G = 207$ GPa~~

$K = 2.19$ GPa

$\rho = 1000$ kg/m³

Dia = 0.61 m

$e = 6.35$ mm

determine wavespeed for following cases

(a) anchored only at u/s end

(b) pipe restrained from axial movement

(c) longitudinal expansion joints

(d) rigid pipe

(a) $a = \sqrt{\dfrac{K/\rho}{\left(1 + \dfrac{DK}{eE}\cdot\psi\right)}}$

$\psi = 1 - (0.5)(0.3) = 0.85$

$\dfrac{DK}{eE} = 1.02$

$a = \sqrt{\dfrac{2.19\times10^9}{10^3(1 + 1.02(0.85))}} = 1083.96$ m/s

(b) $a = \sqrt{\dfrac{K}{\rho\left(1 + \dfrac{DK}{eE}(1 - \cancel{\mu}^2)\right)}}$

$\psi = 1 - 0.3^2$

$=$

$= 1067$ m/s

(c) $a = \sqrt{\dfrac{K}{\rho\left(1 + \dfrac{DK}{eE}(1.0)\right)}} = 1042$ m/s

If steady state water velocity is 2 m/s and valve closed suddenly determine pressure rise in above three conditions

$$\Delta H = -\frac{a}{g}\Delta V$$

(a) $\Delta H = -\frac{1084}{9.81}(-2) = 221\ m$

$\Delta p = (9810)(221) = 2.17 \times 10^6\ Pa$

(b) $\Delta H = -\frac{1067(-2)}{9.81} = 217.5\ m$

$\Delta p = (9810)(217.5) = 2.13 \times 10^6\ Pa$

(c) $\Delta H = -\frac{1042(-2)}{9.81} = 212.4\ m$

$\Delta p = (9810)(212.4) = 2.08 \times 10^6$

Now calculate the change in pipe wall stress caused by these head increases for all three types of restraints

case (a) anchored at u/s end

$$\Delta \sigma_2 = \frac{\Delta p D}{2e}$$

$$\Delta \sigma_1 = \frac{1}{2}\Delta \sigma_2$$

$$\Delta \sigma_2 = \frac{2.17 \times 10^6 \times 0.61}{2 \times 6.35 \times 10^{-3}} = 1.04 \times 10^8\ Pa$$

$$\Delta \sigma_1 = 0.52 \times 10^8\ Pa$$

anchored at both ends

(b) $$\Delta\sigma_2 = \frac{\Delta p D}{2e}$$

$$\Delta\sigma_1 = \nu\,\Delta\sigma_2$$

$$\Delta\sigma_2 = \frac{(2.13\times10^6)(0.61)}{2(6.35\times10^{-3})} = 1.02\times10^8\ Pa$$

$$\Delta\sigma_1 = (0.3)(1.02\times10^8) = 0.306\times10^8\ Pa$$

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expansion joints throughout the pipe length

(c) $$\Delta\sigma_2 = \frac{\Delta p D}{2e}$$

$$\Delta\sigma_1 = 0$$

$$\Delta\sigma_2 = \frac{(2.08\times10^6)(0.61)}{2(6.35\times10^{-3})} = 1.0\times10^8\ Pa$$

$$\Delta\sigma_1 = 0$$