Module 04: Surface Water Hydraulics

Unit 04: Steady Channel Flow: Channel Network

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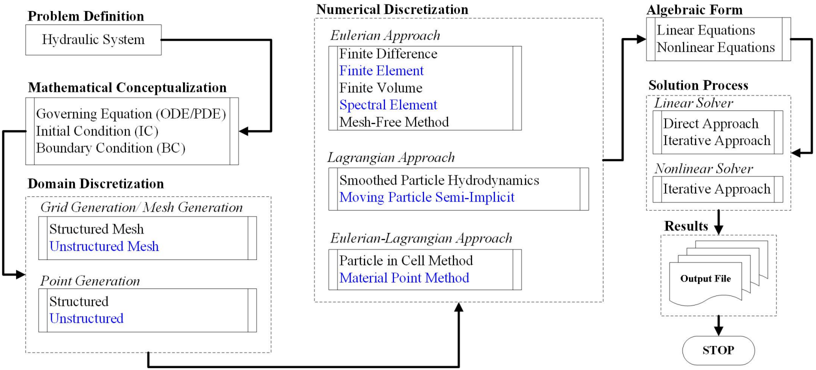
Learning Objective

- To solve steady channel flow for channel network problem using implicit method.

Problem Definition to Solution



Problem Definition

Governing Equation for Channel Flow can be written as,

Boundary Value Problem

Continuity Equation:

$$\frac{dQ}{dx} = 0$$

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Momentum Equation:

$$\frac{d\mathbb{E}}{dx} = -S_f$$

with

$$\mathbb{E} = y + z + \frac{\alpha Q^2}{2gA^2}$$

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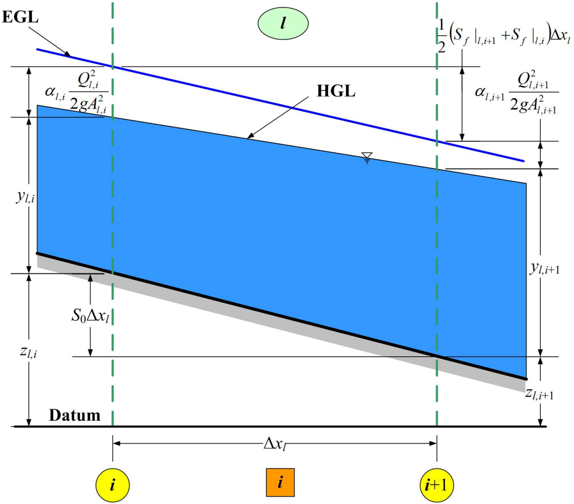
where

$y=$ depth of flow
$S_f=$ friction slope $\left(= \frac{n^2 Q^2}{R^{4/3} A^2}\right)$
$A=$ cross-sectional area
$R=$ hydraulic radius
$z=$ elevation of the channel bottom w.r.t. datum

$x=$ coordinate direction
$\alpha=$ momentum correction factor
$Q=$ discharge
$g=$ acceleration due to gravity

Channel Flow



Discretization

Continuity Equation

Continuity equation for i^{th} segment of the l^{th} channel reach can be discretized as,

$$\frac{dQ}{dx} = 0$$

$$\frac{Q_{l,i+1} - Q_{l,i}}{\Delta x_l} = 0$$

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$l =$ index for channel number
$i =$ index for different sections within a channel reach.

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$l =$ index for channel number
$i =$ index for different sections within a channel reach.
In simplified form for i^{th} segment of the l^{th} channel reach,

$$Q_{l,i+1} = Q_{l,i}$$

Algebraic Form
Continuity Equation

In functional form,

$$C_{l,i} = Q_{l,i+1} - Q_{l,i} = 0, \forall i \in \{1, \ldots, N_l\}$$

Algebraic Form
Continuity Equation

In functional form,

$$C_{l,i} = Q_{l,i+1} - Q_{l,i} = 0, \forall i \in \{1, \ldots, N_l\}$$

$$\frac{\partial C_{l,i}}{\partial y_{l,i}} = 0$$

$$\frac{\partial C_{l,i}}{\partial Q_{l,i}} = -1$$

$$\frac{\partial C_{l,i}}{\partial y_{l,i+1}} = 0$$

$$\frac{\partial C_{l,i}}{\partial Q_{l,i+1}} = 1$$

Discretization

Momentum Equation

Momentum equation for i^{th} segment of the l^{th} channel reach can be discretized as,

$$\frac{d\mathbb{E}}{dx} = -S_f$$

$$\frac{\mathbb{E}_{l,i+1} - \mathbb{E}_{l,i}}{\Delta x_l} = -\frac{1}{2}\left(S_f\Big|_{l,i+1} + S_f\Big|_{l,i}\right)$$

Discretization

Momentum Equation

Momentum equation for i^{th} segment of the l^{th} channel reach can be discretized as,

$$\frac{d\mathbb{E}}{dx} = -S_f$$
$$\frac{\mathbb{E}_{l,i+1} - \mathbb{E}_{l,i}}{\Delta x_l} = -\frac{1}{2}\left(S_f\big|_{l,i+1} + S_f\big|_{l,i}\right)$$

In expanded form,

$$\frac{\left(y + z + \frac{\alpha Q^2}{2gA^2}\right)_{l,i+1} - \left(y + z + \frac{\alpha Q^2}{2gA^2}\right)_{l,i}}{\Delta x_l} = -\frac{1}{2}\left[\left(\frac{n^2 Q^2}{R^{4/3} A^2}\right)_{l,i+1} + \left(\frac{n^2 Q^2}{R^{4/3} A^2}\right)_{l,i}\right]$$

Discretization

Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q^2_{l,i+1}}{A^2_{l,i+1}} - \frac{Q^2_{l,i}}{A^2_{l,i}}\right) + \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q^2_{l,i+1}}{R^{4/3}_{l,i+1} A^2_{l,i+1}} + \frac{Q^2_{l,i}}{R^{4/3}_{l,i} A^2_{l,i}}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

Discretization
Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

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Considering reverse flow situation,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q_{l,i+1}|Q_{l,i+1}|}{A_{l,i+1}^2} - \frac{Q_{l,i}|Q_{l,i}|}{A_{l,i}^2}\right) + \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q_{l,i+1}|Q_{l,i+1}|}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{Q_{l,i}|Q_{l,i}|}{R_{l,i}^{4/3} A_{l,i}^2}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

Discretization

Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q_{l,i+1}^2}{A_{l,i+1}^2} - \frac{Q_{l,i}^2}{A_{l,i}^2}\right) + \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q_{l,i+1}^2}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{Q_{l,i}^2}{R_{l,i}^{4/3} A_{l,i}^2}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

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$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q_{l,i+1}|Q_{l,i+1}|}{A_{l,i+1}^2} - \frac{Q_{l,i}|Q_{l,i}|}{A_{l,i}^2}\right) + \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q_{l,i+1}|Q_{l,i+1}|}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{Q_{l,i}|Q_{l,i}|}{R_{l,i}^{4/3} A_{l,i}^2}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

$2N_l$ non-linear equations with $2(N_l + 1)$ unknowns

Discretization

Momentum Equation

In functional form for i^{th} segment of the l^{th} channel reach,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q^2_{l,i+1}}{A^2_{l,i+1}} - \frac{Q^2_{l,i}}{A^2_{l,i}}\right) + \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q^2_{l,i+1}}{R^{4/3}_{l,i+1}A^2_{l,i+1}} + \frac{Q^2_{l,i}}{R^{4/3}_{l,i}A^2_{l,i}}\right], \quad \forall i \in \{1, \ldots, N_l\}$$

Algebraic Form
Momentum Equation

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} = -1 + D_1 \frac{2Q^2_{l,i}}{A^3_{l,i}} \frac{dA}{dy}\Big|_{l,i} - D_2 \left[\frac{2Q^2_{l,i}}{A^3_{l,i} R^{\frac{4}{3}}_{l,i}} \frac{dA}{dy}\Big|_{l,i} + \frac{4Q^2_{l,i}}{3A^2_{l,i} R^{\frac{7}{3}}_{l,i}} \frac{dR}{dy}\Big|_{l,i} \right]$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i}} = -D_1 \frac{2Q_{l,i}}{A^3_{l,i}} + D_2 \frac{2Q_{l,i}}{A^2_{l,i} R^{\frac{4}{3}}_{l,i}}$$

$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}} = 1 - D_1 \frac{2Q^2_{l,i+1}}{A^3_{l,i+1}} \frac{dA}{dy}\Big|_{l,i+1} - D_2 \left[\frac{2Q^2_{l,i+1}}{A^3_{l,i+1} R^{\frac{4}{3}}_{l,i+1}} \frac{dA}{dy}\Big|_{l,i+1} + \frac{4Q^2_{l,i+1}}{3A^2_{l,i+1} R^{\frac{7}{3}}_{l,i+1}} \frac{dR}{dy}\Big|_{l,i+1} \right]$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i+1}} = D_1 \frac{2Q_{l,i+1}}{A^3_{l,i+1}} + D_2 \frac{2Q_{l,i+1}}{A^2_{l,i+1} R^{\frac{4}{3}}_{l,i+1}}$$

with

$$D_1 = \frac{\alpha_l}{2g} \quad \text{and} \quad D_2 = \frac{1}{2} n_l^2 \Delta x_l$$

Algebraic Form
Momentum Equation

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} = -1 + D_1 \frac{2Q_{l,i}^2}{A_{l,i}^3} \frac{dA}{dy}\Big|_{l,i} - D_2 \left[\frac{2Q_{l,i}^2}{A_{l,i}^3 R_{l,i}^{\frac{4}{3}}} \frac{dA}{dy}\Big|_{l,i} + \frac{4Q_{l,i}^2}{3A_{l,i}^2 R_{l,i}^{\frac{7}{3}}} \frac{dR}{dy}\Big|_{l,i} \right]$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i}} = -D_1 \frac{2Q_{l,i}}{A_{l,i}^3} + D_2 \frac{2Q_{l,i}}{A_{l,i}^2 R_{l,i}^{\frac{4}{3}}}$$

$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}} = 1 - D_1 \frac{2Q_{l,i+1}^2}{A_{l,i+1}^3} \frac{dA}{dy}\Big|_{l,i+1} - D_2 \left[\frac{2Q_{l,i+1}^2}{A_{l,i+1}^3 R_{l,i+1}^{\frac{4}{3}}} \frac{dA}{dy}\Big|_{l,i+1} + \frac{4Q_{l,i+1}^2}{3A_{l,i+1}^2 R_{l,i+1}^{\frac{7}{3}}} \frac{dR}{dy}\Big|_{l,i+1} \right]$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i+1}} = D_1 \frac{2Q_{l,i+1}}{A_{l,i+1}^3} + D_2 \frac{2Q_{l,i+1}}{A_{l,i+1}^2 R_{l,i+1}^{\frac{4}{3}}}$$

with

$$D_1 = \frac{\alpha_l}{2g} \quad \text{and} \quad D_2 = \frac{1}{2} n_l^2 \Delta x_l$$

For general channel cross-section,

$$\frac{dR}{dy} = \frac{T}{P} - \frac{R}{P}\frac{dP}{dy}$$

Discretization
Boundary Condition

For subcritical flows,

$$y_{l,N_l+1} = y_d$$
$$DB_{l,N_l+1} = y_{l,N_l+1} - y_d$$

Discretization

Boundary Condition

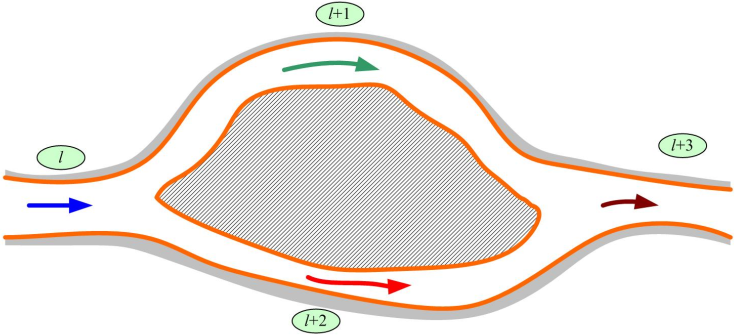
For subcritical flows,

$$y_{l,N_l+1} = y_d$$
$$DB_{l,N_l+1} = y_{l,N_l+1} - y_d$$

Elements of Jacobian Matrix can be written as,

$$\frac{\partial DB_{l,N_l+1}}{\partial y_{l,N_l}} = 0$$
$$\frac{\partial DB_{l,N_l+1}}{\partial Q_{l,N_l}} = 0$$
$$\frac{\partial DB_{l,N_l+1}}{\partial y_{l,N_l+1}} = 1$$
$$\frac{\partial DB_{l,N_l+1}}{\partial Q_{l,N_l+1}} = 0$$

Channel Network



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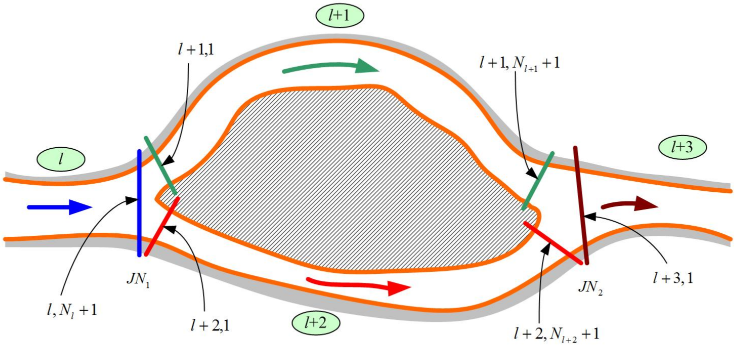

Channel Networks

Internal Boundary condition

The junction conditions can be written as,

Mass conservation

$$\sum Q_I = \sum Q_O$$

where

Q_I = channel discharge at inflow branch and Q_O = channel discharge at outflow branch

Energy conservation

$$y_{l,N_l+1} + z_{l,N_l+1} = y_{l+1,1} + z_{l+1,1} = y_{l+2,1} + z_{l+2,1}$$

Channel Networks

Internal Boundary condition

Junction 1

$$JC_{JN_1,1} = Q_{l,N_l+1} - Q_{l+1,1} - Q_{l+2,1} = 0$$
$$JC_{JN_1,2} = y_{l,N_l+1} - y_{l+1,1} + z_{l,N_l+1} - z_{l+1,1} = 0$$
$$JC_{JN_1,3} = y_{l,N_l+1} - y_{l+2,1} + z_{l,N_l+1} - z_{l+2,1} = 0$$

$$\frac{\partial JC_{JN_1,1}}{\partial Q_{l,N_l+1}} = 1 \quad \frac{\partial JC_{JN_1,1}}{\partial Q_{l+1,1}} = -1$$
$$\frac{\partial JC_{JN_1,1}}{\partial Q_{l+2,1}} = -1$$
$$\frac{\partial JC_{JN_1,2}}{\partial y_{l,N_l+1}} = 1 \quad \frac{\partial JC_{JN_1,2}}{\partial y_{l+1,1}} = -1$$
$$\frac{\partial JC_{JN_1,3}}{\partial y_{l,N_l+1}} = 1 \quad \frac{\partial JC_{JN_1,3}}{\partial y_{l+2,1}} = -1$$

Channel Networks

Internal Boundary condition

Junction 2

$$JC_{JN_2,1} = Q_{l+3,1} - Q_{l+1,N_{l+1}+1} - Q_{l+2,N_{l+2}+1} = 0$$

$$JC_{JN_2,2} = y_{l+3,1} - y_{l+1,N_{l+1}+1} + z_{l+3,1} - z_{l+1,N_{l+1}+1} = 0$$

$$JC_{JN_2,3} = y_{l+3,1} - y_{l+2,N_{l+2}+1} + z_{l+3,1} - z_{l+2,N_{l+2}+1} = 0$$

Algebraic Form
Subcritical flow

In general form, continuity equation including boundary condition can be written as,

$$\frac{\partial C_{l,i}}{\partial y_{l,i}}\Delta y_{l,i} + \frac{\partial C_{l,i}}{\partial Q_{l,i}}\Delta Q_{l,i} + \frac{\partial C_{l,i}}{\partial y_{l,i+1}}\Delta y_{l,i+1} + \frac{\partial C_{l,i}}{\partial Q_{l,i+1}}\Delta Q_{l,i+1} = -C_{l,i}$$

$$\frac{\partial M_{l,i}}{\partial y_{l,i}}\Delta y_{l,i} + \frac{\partial M_{l,i}}{\partial Q_{l,i}}\Delta Q_{l,i} + \frac{\partial M_{l,i}}{\partial y_{l,i+1}}\Delta y_{l,i+1} + \frac{\partial M_{l,i}}{\partial Q_{l,i+1}}\Delta Q_{l,i+1} = -M_{l,i},$$

$$\forall i \in \{1, \ldots, N_l\}$$

For subcritical flow,

$$\Delta Q_{1,1} = -UB_{l,N_l+1}$$
$$\Delta y_{l,N_l+1} = -DB_{l,N_l+1}$$

Jacobian Matrix Structure

Subcritical flow

$$
\begin{bmatrix}
\bullet & \bullet & & & & & & & & \\
\color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} & & & & & & \\
\color{red}{\times} & \color{red}{\times} & \color{red}{\times} & \color{red}{\times} & & & & & & \\
& & \color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} & & & & \\
& & \color{red}{\times} & \color{red}{\times} & \color{red}{\times} & \color{red}{\times} & & & & \\
& & & & \color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} & & \\
& & & & \color{red}{\times} & \color{red}{\times} & \color{red}{\times} & \color{red}{\times} & & \\
& & & & & & \color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} & \color{blue}{\times} \\
& & & & & & \color{red}{\times} & \color{red}{\times} & \color{red}{\times} & \color{red}{\times} \\
& & & & & & & & \bullet & \bullet
\end{bmatrix}
$$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

Problem Statement
Channels in Series

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Channel Cross-Section Type: Rectangular

$B = 15m$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

$L_{x2} = 100m$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

$L_{x2} = 100m$

$Q = 20m^3/s$

Problem Statement

Channels in Series

Given

Channel Cross-Section Type: Rectangular

$B = 15m$

$g = 9.81m/s^2$

$S_{01} = 0.0004$

$S_{02} = 0.0008$

$n_1 = 0.01$

$n_2 = 0.015$

$L_{x1} = 100m$

$L_{x2} = 100m$

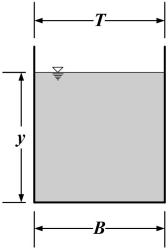
$Q = 20m^3/s$

$y_d = 0.60m$

Required

Estimate the flow depth across the channels in series.

Rectangular Cross-section



$$A = By$$
$$P = B + 2y$$
$$R = \frac{A}{P}$$
$$T = B$$
$$\frac{dR}{dy} = \frac{B^2}{(B + 2y)^2}$$

List of Source Codes

Channel Flow

- Channels network
  - steady_1D_channel_network.sci

Thank You