Truncation error

Round off error :- store finite number decimal places.

Truncation error :- Human error due to approximation

$$\varphi(x_i+\Delta x)=\varphi(x_i)+\frac{\Delta x}{1!}\varphi'(x_i)+\frac{(\Delta x)^2}{2}\varphi''(x_i)+\frac{(\Delta x)^3}{3!}\varphi'''(x_i)$$

$$=\varphi(x_i)+\sum_{m=1}^{\infty}\frac{(\Delta x)^m}{m!}\varphi^m(x_i)$$

$$\varphi(x_i-\Delta x)=\varphi(x_i)+\sum_{m=1}^{\infty}(-1)^m\frac{(\Delta x)^m}{m!}\varphi^{(m)}(x)$$

FD approximation:

FD,
$$\varphi_i'\Big|_{FD}=\frac{\varphi_{i+1}-\varphi_i}{\Delta x}$$

$$=\frac{\varphi(x_i+\Delta x)-\varphi(x_i)}{\Delta x}$$

$$=\frac{\cancel{\varphi(x_i)}+\sum_{m=1}^{\infty}\frac{(\Delta x)^m}{m!}\varphi^{(m)}(x)-\cancel{\varphi(x_i)}}{\Delta x}$$

$$=\sum_{m=1}^{\infty}\frac{(\Delta x)^{m-1}}{m!}\varphi^{(m)}(x)$$

$$=\frac{(\Delta x)^0}{1}\varphi'(x)+\sum_{m=2}^{\infty}\frac{(\Delta x)^{m-1}}{m!}\varphi^{(m)}(x)$$

$$=\varphi'(x)+\sum_{m=2}^{\infty}\frac{(\Delta x)^{m-1}}{m!}\varphi^{(m)}(x)$$

$$=\varphi'(x)+O(\Delta x)$$

Truncation error

Exact value

$$=\varphi'(x)+\underbrace{\frac{\Delta x}{2!}\varphi^{(2)}(x)}_{\text{Leading error}}+\sum_{m=3}^{\infty}\frac{(\Delta x)^{m-1}}{m!}\varphi^{(m)}(x)$$

Truncation error

In truncation error part, term with lowest order of Δx is called leading error term. and it determines the order of truncation error.

$\therefore TE\sim O(\Delta x)$

Backward difference

$$\varphi_i'=\cancel{\varphi(x_i)}-$$

$$=\frac{\varphi_i-\varphi_{i-1}}{\Delta x}$$

$$=\frac{\varphi(x_i)-\varphi(x_i-\Delta x)}{\Delta x}$$

$$=\frac{\cancel{\varphi(x_i)}-\cancel{\varphi(x_i)}-\sum_{m=1}^{\infty}(-1)^m\frac{(\Delta x)^m}{m!}\varphi^{(m)}(x)}{\Delta x}\qquad m=0$$

$$=-(-1)^1\varphi'(x)-\sum_{m=2}^{\infty}(-1)^m\frac{(\Delta x)^{m-1}}{m!}\varphi^{(m)}(x)$$

$$=\underbrace{\varphi'(x)-\frac{\Delta x}{2}\varphi^{(2)}(x)}_{\text{Leading}}-\sum_{m=3}^{\infty}(-1)^m\frac{(\Delta x)^{m-1}}{m!}\varphi^{m}(x_i)$$

$$=\varphi'(x_i)+O(\Delta x)\ \text{Truncation error}$$

Center difference approximation

$$\varphi'(x_i) = \frac{\varphi(x_i+\Delta x) - \varphi(x_i-\Delta x)}{2\Delta x}$$

$$= \frac{\varphi_{i+1} - \varphi_{i-1}}{2\Delta x}$$

$$= \frac{\varphi(x_i) + \sum_{m=1}^{\infty} \frac{(\Delta x)^m}{m!}\varphi^{(m)}(x) - \varphi(x_i) - \sum_{m=1}^{\infty}(-1)^m \frac{(\Delta x)^m}{m!}\varphi^{(m)}(x)}{2\Delta x}$$

$$= \frac{2\Delta x\varphi'(x) + \frac{(\Delta x)^3}{3!}\varphi^{(3)}(x) + \sum_{m=4}^{\infty}\frac{(\Delta x)^m}{m!}\varphi^{m}(x) \pm \sum_{m=4}^{\infty}(-1)^m\frac{(\Delta x)^m}{m!}\varphi^{(m)}(x)}{2\Delta x}$$

$$= \varphi'(x) + \underbrace{\frac{(\Delta x)^2}{3!}\varphi'''(x)}_{\text{Leading}} + \sum_{m=2}^{\infty}\frac{(\Delta x)^{2m+1}}{(2m+1)!}\varphi^{(2m+1)}(x)$$

only for odd ($2\Delta x$) terms

It exists

$$= \underbrace{\varphi'(x)}_{\text{Exact value}} + \underbrace{\frac{(\Delta x)^2}{3!}\varphi'''(x)}_{\text{leading error}} + \sum_{m=2}^{\infty}\frac{(\Delta x)^{2m}}{(2m+1)!}\varphi^{(2m+1)}(x)$$

FD, BD → Error more

Truncation error $\sim O(\Delta x^2)$

Higher order discretization of 1st order derivative:-

Higher order (2nd order accurate, one sided 3 point stencil)

Let, $\varphi_i' = \alpha_{i-2}\varphi_{i-2} + \alpha_{i-1}\varphi_{i-1} + \alpha_i\varphi_i$

$$\varphi_i' = \alpha_{i-2}\,\varphi(x_i - 2\Delta x) + \alpha_{i-1}\varphi(x_i - \Delta x) + \alpha_i\varphi_i$$

$$= \alpha_{i-2}\left\{\varphi(x_i) + \sum_{m=1}^{\infty}(-1)^m\frac{(2\Delta x)^m}{m!}\varphi^{(m)}(x_i)\right\}$$

$$+ \alpha_{i-1}\left\{\varphi(x_i) + \sum_{m=1}^{\infty}(-1)^m\frac{(\Delta x)^m}{m!}\varphi^{(m)}(x_i)\right\}$$

$$+ \alpha_i\varphi_i$$

$$= \alpha_{i-2}\left\{\varphi_i - 2\Delta x\varphi_i' + \frac{(2\Delta x)^2}{2}\varphi_i'' - \frac{(2\Delta x)^3}{3!}\varphi_i''' + \cdots\right\}$$

$$+ \alpha_{i-1}\left\{\varphi_i - \Delta x\varphi_i' + \frac{\Delta x^2}{2}\varphi_i'' - \frac{(\Delta x)^3}{3!}\varphi_i''' + \cdots\right\}$$

$$+ \alpha_i\varphi_i$$

$$= \varphi_i(\alpha_{i-2} + \alpha_{i-1} + \alpha_i) + \varphi_i'(-2\Delta x\alpha_{i-2} - \Delta x\alpha_{i-1})$$

$$+ \varphi_i''\left(2\Delta x^2\alpha_{i-2} + \frac{\Delta x^2}{2}\alpha_{i-1}\right) + \cdots$$

Let, $\alpha_{i-2} = a$

$\alpha_{i-1} = b$

$\alpha_i = c$

$a + b + c = 0$

$-\Delta x(2a + b) = 1 \Rightarrow 2a + b = -\frac{1}{\Delta x}$

$\Delta x^2\left(2a + \frac{b}{2}\right) = 0 \Rightarrow a = \left(-b - \frac{1}{\Delta x}\right)/2$

$\Rightarrow 2a + \frac{b}{2} = 0$

$= \left(\frac{2}{\Delta x} - \frac{1}{\Delta x}\right)/2 = \frac{2}{\Delta x}$

$\Rightarrow -b - \frac{1}{\Delta x} + \frac{b}{2} = 0$

$c = -(a+b)$

$\Rightarrow \frac{b}{2} = -\frac{1}{\Delta x} \Rightarrow \boxed{b = -\frac{2}{\Delta x}}$

$= -\left(\frac{2}{\Delta x} + \frac{2}{\Delta x}\right)$

$= 0.$

We have,

$a = \alpha_i$

$b = \alpha_{i-1}$

$c = \alpha_{i-2}$

$a+b+c=0 \Rightarrow b+c = -\frac{1}{\Delta x} \Rightarrow b = -c-\frac{1}{\Delta x}$

$-\Delta x(b+c) = 1 \Rightarrow a\Delta x = 1 \Rightarrow \boxed{a = \frac{1}{\Delta x}}$

$\frac{\Delta x^2}{2}(4c+b) = 0$

$\Rightarrow \cancel{\Delta x^2}\; 4c - c - \frac{1}{\Delta x} = 0$

$\Rightarrow 3c - \frac{1}{\Delta x} = 0$

$\Rightarrow \frac{1}{\Delta x} = 3c$

$\Rightarrow \boxed{c = \frac{1}{3\Delta x}}$

$b = -\frac{1}{3\Delta x} - \frac{1}{\Delta x}$

$= \frac{4}{3\Delta x}$

$b = -4c$

then we have,

Substituating,

$\phi'_i = \frac{1}{3\Delta x}\phi_{i-2} +$

Here, equations are:-

$a+b+c=0$

$-\Delta x(2b+c) = 1 \Rightarrow 2b+c = -\frac{1}{\Delta x} \Rightarrow -8.$

$\frac{\Delta x^2}{2}(4c+b) = 0 \Rightarrow 4c+b = 0$

$a+b+c=0 \quad \cdots\cdots (i)$

$-2\Delta x c - \Delta x b = 1$

$\Rightarrow -\Delta x(2c+b) = 1$

$\Rightarrow 2c+b = -\frac{1}{\Delta x} \quad \cdots\cdots (ii)$

also, $2\Delta x^2 c + \frac{\Delta x^2}{2} \cdot b = 0$

$\Rightarrow \Delta x^2\left(2c+\frac{b}{2}\right) = 0$

$\Rightarrow 4c+b = 0 \quad \cdots\cdots (iii)$

$\Rightarrow b = -4c$

$2c - 4c = -\frac{1}{\Delta x}$

$\Rightarrow -2c = -\frac{1}{\Delta x} \Rightarrow c = \frac{1}{2\Delta x}$

Again, $b = -4 \times \frac{1}{2\Delta x} = -\frac{2}{\Delta x}$

$a = -(b+c) = -\left(-\frac{2}{\Delta x} + \frac{1}{2\Delta x}\right)$

$= -\frac{-4+1}{2\Delta x}$

$= \frac{3}{2\Delta x}$

$\therefore$ Required expression,

$\phi'_i = \frac{3}{2\Delta x}\alpha_i - \frac{2}{\Delta x}\alpha_{i-1} + \frac{1}{2\Delta x}\alpha_{i-2}$

$= \frac{3\alpha_i - 4\alpha_{i-1} + \alpha_{i-2}}{2\Delta x} + O(\Delta x^2)$

Second order derivative: (Using symmetric stencil). # Applicable for central not internal nodes.

$$\phi_i''= \lim_{\Delta x\to 0} \frac{\phi_{i+1}'-\phi_i'|_{FD}}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\frac{\phi_{i+1}-\phi_i}{\Delta x} + \frac{\phi_i - \phi_{i-1}}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{(\Delta x)^2}$$

Also, $\phi_i''|_{BD} = \frac{\phi_i'|_{FD}-\phi_{i-1}'|_{FD}}{\Delta x}$

$$= \lim_{\Delta x\to 0} \frac{\frac{\phi_{i+1}-\phi_i}{\Delta x} - \frac{\phi_i-\phi_{i-1}}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{\Delta x^2}$$

Both are giving same equation

$$= \frac{1}{(\Delta x)^2}\phi_{i+1} - \frac{2}{(\Delta x)^2}\phi_i + \frac{1}{(\Delta x)^2}\phi_{i-1}$$

$$= \frac{1}{(\Delta x)^2}\left[\phi_i + \sum_{m=1}^{\infty}\frac{(\Delta x)^m}{m!}\phi_i^{(m)}\right] - \frac{2}{(\Delta x)^2}\phi_i + \frac{1}{(\Delta x)^2}\left[\phi_i + \sum_{m=1}^{\infty}(-1)^m\frac{(\Delta x)^m}{m!}\phi_i^m\right].$$

$$= \frac{2\times}{(\Delta x)^2}\left[\frac{(\Delta x)^2}{2}\phi_i'' + \frac{(\Delta x)^4}{4!}\phi_i'''' + \frac{(\Delta x)^6}{6!}\phi_i^6 + \cdots\right]$$

$$= \phi_i'' + 2\left\{\frac{(\Delta x)^2}{4!}\phi^4(x_i) + \frac{(\Delta x)^4}{6!}\phi^6(x_i) + \cdots\right]$$

$$= \phi_i'' + \underbrace{2\sum_{m=1}^{\infty}\frac{(\Delta x)^{2m}}{2(m+1)!}\phi^{2(m+1)}(x_i)}_{\text{Truncation error}}$$

$a=4$
$d=2$
$a_m = 4+(n-1)2$
$=4+2n-2$
$=(2n+2)$

$= \phi_i'' + O(\Delta x^2)$ [Because, leading error $= \frac{\Delta x^2}{2!}\phi^4(x_i)$]

Let, $\phi_i'' = \alpha_{i+1}\phi_{i+1} + \alpha_i\phi_i + \alpha_{i-1}\phi_{i-1}$ ⇒ (# Derivation of same expression using linear combination of $\phi_{i+1}, \phi_i, \phi_{i-1}$)

$$= \alpha_{i+1}\phi(x_i+\Delta x) + \alpha_i\phi_i + \alpha_{i-1}\phi(x_i-\Delta x)$$

$$= \alpha_{i+1}\left\{\phi(x_i) + \sum_{m=1}^{\infty}\frac{(\Delta x)^m}{m!}\phi^{(m)}(x_i)\right\} + \alpha_i\phi_i + \alpha_{i-1}\left\{\phi_i + \sum_{m=1}^{\infty}(-1)^m\frac{(\Delta x)^m}{m!}\phi_i^m\right\}.$$

$$= \phi_i(\alpha_{i+1}+\alpha_i+\alpha_{i-1}) + \phi_i'(\Delta x\,\alpha_{i+1} - \Delta x\,\alpha_{i-1}) + \phi_i''\left(\frac{\Delta x^2}{2}\alpha_{i+1} + \frac{\Delta x^2}{2}\alpha_{i-1}\right) + \phi_i'''\left\{\frac{(\Delta x)^3}{3!}\alpha_{i+1} + \frac{(\Delta x)^3}{3!}\alpha_{i-1}\right\} + \cdots$$

$a+b+c=0$

$\Delta x(a-c)=0 \Rightarrow a=c$

$\frac{\Delta x^2}{2}(a+c)=1 \Rightarrow \frac{\Delta x^2}{2}\times 2a=1 \Rightarrow a=\frac{1}{\Delta x^2}=c$

$b=-2a=-2\times\frac{1}{\Delta x^2}=-\frac{2}{\Delta x^2}$

$\alpha_{i+1}=a=\frac{1}{\Delta x^2}$

$\alpha_i=b=-\frac{2}{\Delta x^2}$

$\alpha_{i-1}=c=\frac{1}{\Delta x^2}$

$\therefore \phi_i'' = \alpha_{i+1}\phi_{i+1} + \alpha_i\phi_i + \alpha_{i-1}\phi_{i-1}$

$$= \frac{1}{\Delta x^2}\phi_{i+1} - \frac{2}{\Delta x^2}\phi_i + \frac{1}{\Delta x^2}\phi_{i-1}$$

$$= \left\{\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{\Delta x^2}\right\} \text{ central difference}$$

Backward ~~difference~~:

Stencil.

One sided
m point stencil provides,
(m-1) order accurate
1st order derivative
(m-2) order 2nd order derivatives

$$\varphi_i'' = \cancel{\varphi_{i-2}}\alpha_{i-2}\varphi_{i-2} + \alpha_{i-1}\varphi_{i-1} + \alpha_i\varphi_i$$

$$= \alpha_{i-2}\left[\varphi_i + \sum_{m=1}^{\infty}\frac{(\Delta x)^m}{m!}\varphi_i^m\right] + \alpha_i\varphi_i + \alpha_{i-1}\left[\varphi_i + \sum_{i=1}^{m}(-1)^m\frac{(\Delta x)^m}{m!}\varphi_i^m\right]$$

$$= \varphi_i(\alpha_{i-2} + \alpha_i + \alpha_{i-1})$$

$$+ \varphi_i'(\Delta x\,\alpha_{i-2} - \Delta x\,\alpha_{i-1})$$

$$= \alpha_{i-2}\,\varphi(x_i - 2\Delta x) + \alpha_{i-1}\,\varphi(x_i - \Delta x) + \alpha_i\varphi_i$$

$$= \alpha_{i-2}\left[\varphi_i + \sum_{m=1}^{\infty}(-1)^m\frac{(2\Delta x)^m}{m!}\varphi_i^{(m)}\right] + \alpha_{i-1}\left[\varphi_i + \sum_{m=1}^{\infty}(-1)^m\frac{(\Delta x)^m}{m!}\varphi^m(x_i)\right] + \alpha_i\varphi_i$$

$$= \varphi_i(\alpha_{i-2} + \alpha_{i-1} + \alpha_i) + \left(-2\Delta x\,\varphi_i'\,\alpha_{i-2} - \Delta x\,\varphi_i'\,\alpha_{i-1}\right) + \left(\frac{4\Delta x^2}{2!}\alpha_{i-2}\varphi_i'' + \frac{\Delta x^2}{2!}\alpha_{i-1}\varphi_i''\right) +$$

$$+ \left(\alpha_{i-2}(-1)\frac{(2\Delta x)^3}{3!} + \alpha_{i-1}(-1)\frac{(\Delta x)^3}{3!}\right)\varphi_i''' + \dots$$

$$= \varphi_i(\alpha_{i-2} + \alpha_{i-1} + \alpha_i) + \varphi_i'(-2\Delta x\,\alpha_{i-2} - \Delta x\,\alpha_{i-1})$$

$$+ \varphi_i''\left(2\Delta x^2\alpha_{i-2} + \frac{\Delta x^2}{2}\alpha_{i-1}\right) + \left(-\alpha_{i-2}\frac{(2\Delta x)^3}{3!} - \alpha_{i-1}\frac{\Delta x^3}{3!}\right)\varphi_i''' + \dots$$

$\alpha_{i-2} = a$
$\alpha_{i-1} = b$
$\alpha_i = c$

$$\left.\begin{aligned} a + b + c &= 0 \\ \frac{\Delta x^2}{2}(4a + b) &= 1 \\ \Delta x(-2a - b) &= 0 \end{aligned}\right\} \text{Three equations}$$

$$\Rightarrow b = -2a$$

$$\frac{\Delta x^2}{2}(4a - 2a) = 1 \qquad b = -\frac{2}{\Delta x^2}$$

$$\Rightarrow \frac{\Delta x^2}{\cancel{2}} \times \cancel{2}a = 1 \qquad c = -a - b$$

$$\Rightarrow a = \frac{1}{\Delta x^2} \qquad = -\frac{1}{\Delta x^2} + \frac{2}{\Delta x^2}$$

$$= \frac{1}{\Delta x^2}$$

$$\varphi_i'' = \frac{1}{\Delta x^2}\varphi_{i-2} - \frac{2}{\Delta x^2}\varphi_{i-1} + \frac{1}{\Delta x^2}\varphi_i$$

$$\boxed{\varphi_i'' = \frac{\varphi_{i-2} - 2\varphi_{i-1} + \varphi_i}{\Delta x^2} + O(\Delta x)}$$

Lecture-15 (NPTEL-13)

Module-2 unit 9

Single valued 1-D functions discretization.
(Using finite volume method).

$\phi(x,y,z,t)$ (General variable)

$$\underbrace{\frac{\partial}{\partial t}(\Lambda_\phi \phi)}_{\text{Temporal term}} + \underbrace{\nabla\cdot(\gamma_\phi \cdot \phi u)}_{\text{Advection related.}} = \underbrace{\nabla\cdot(\Gamma_\phi \cdot \nabla\phi)}_{\text{Diffusion term}} + F_{\phi_0} + S_\phi$$

$\Gamma_\phi \rightarrow$ Tensor or coff tensor

$\Lambda_\phi \rightarrow$ Lambda
$\gamma \rightarrow$ Efsilon

$$R(x,t) \equiv \frac{\partial}{\partial t}(\Lambda_\phi \phi) + \nabla\cdot(\gamma_\phi \phi u) - \nabla\cdot(\Gamma_\phi\cdot\nabla\phi) - F_{\phi_0} - S_\phi$$

$x = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$ $\hookrightarrow$ (LHS-RHS)

Weighted integral for residuals

$$\int_\Omega \omega_l R \, d\Omega = 0 \quad \forall l$$

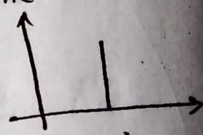
Ω (domain)

Dirac delta function

$\omega_l = \delta(x_l - x)$

At $x=l$, diract delta $=1$, otherwise, 0

$x=L \rightarrow$ dirac delta $=1$.
otherwise $=0$



In collocation method,

$$\int \delta(x_l - x) R \, d\Omega = 0$$

$\hookrightarrow$ Apply dirac delta, insted of ω_l.

$\Rightarrow R(x_l,t) = 0$

If we evaluate the residual at point L and we equate it to zero, we get the approximation, in terms of weighted residual.

In FDM, we used same method.

Used Taylor's series to approximate the derivative with the help of neighbouring points.

$$\omega_l = \begin{cases} 1, & \text{if } x \in \Omega' \\ 0, & \text{if } x \notin \Omega' \end{cases}$$

$\rightarrow$ does not belongs to sub-domain.

$$\therefore \int_\Omega \omega_l R \, d\Omega = \int_{\Omega^l} R \, d\Omega = 0$$

$\rightarrow$ R in terms of given DE's

similar to FVM $\rightarrow$ For a particular sub-domain.

Gauss Divergence theorem:-

$\Omega \rightarrow$ Volume bounded by closed surface S

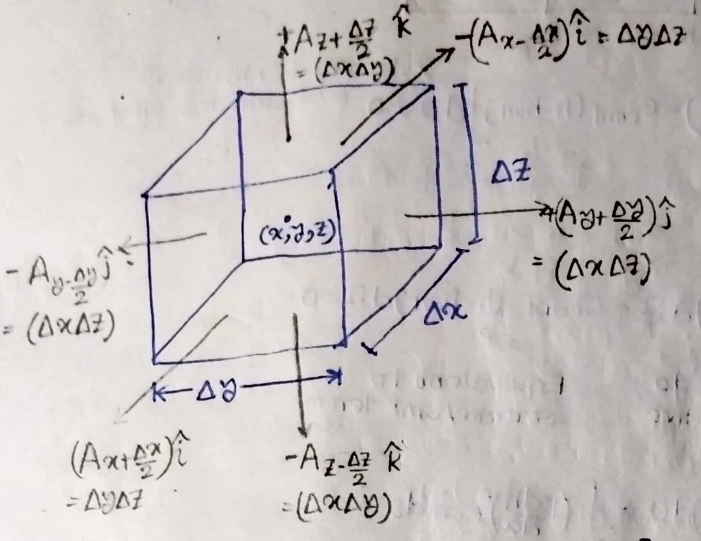
$a \rightarrow$ Arbitrary vector in Ω and on S.

$\hat{n} \rightarrow$ outward normal.

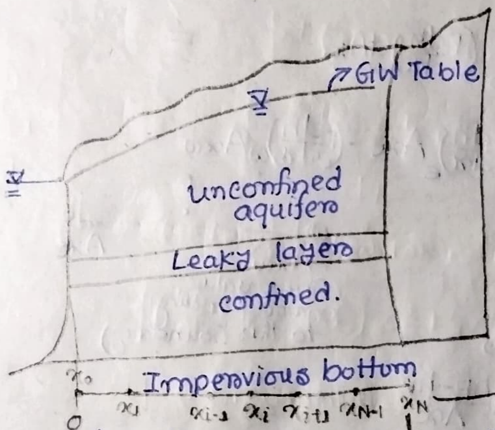
$$\iiint_\Omega \nabla\cdot\vec{a} \, d\Omega = \iint_S \vec{a}\cdot\hat{n} \, ds$$

$\hookrightarrow$ Del operator

volume integral $\longrightarrow$ surface integral.



Finite difference problems can also be sloved by Finite volume

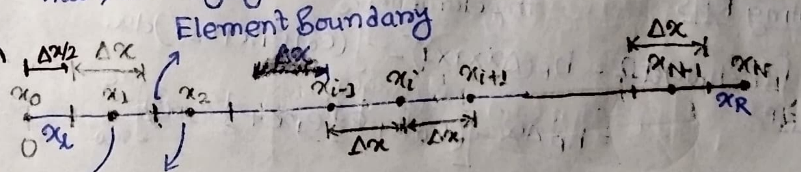


For FD approach. (Total number of elements = n).

$$\frac{d}{dx}\left(T\frac{dh}{dx}\right) = C_{conf}(h - h_{wt}) \Rightarrow \frac{d^2h}{dx^2} = \frac{C_{conf}}{T}(h - h_{wt})$$

h = head

$h_{wt} \Rightarrow$ Overlaying water table elevation $= C_0 + C_1 x + C_2 x^2$



For end points, we have $\frac{\Delta x}{2}$ length extra. For this $\frac{\Delta x}{2}$, we need seperate element.

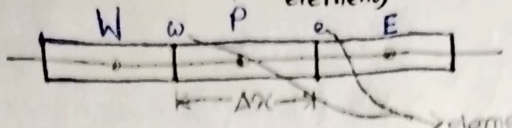
Node is defined at centre of the element.

Element boundary

For 1 to N-1, we have N-1 no. of elements. For x_L and x_R we have 2. Total number of elements = (N-1)+2 = N+1 no. of elements

This discretization model is used, because it is similar to our finite difference domain discretization

(Cell center
node for middle
element)



$$\int_{\Omega^P} \left[\frac{d}{dx}\left(T\frac{dh}{dx}\right) - C_{conf}(h-h_{wt})\right] d\Omega = 0.$$

Ω^P Represents central element

$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx}\right) d\Omega - C_{conf}\int_{\Omega^P}(h-h_{wt})\,d\Omega = 0$$

Equivalent to derivative — Equivalent to source/sink term.

Evaluating LHS:

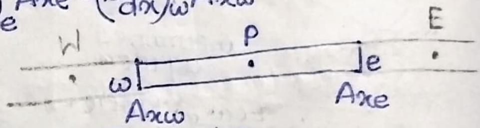
$$\int_{\Omega^P} \frac{d}{dx}\left(T\frac{dh}{dx}\right) d\Omega = \int_{S^P}\left(T\frac{dh}{dx}\right)\hat{i}\cdot\hat{n}\,ds$$

Volume integral $\rightarrow$ surface integral.

? $\hat{i}\cdot\hat{n}\,ds$ $\cos\theta$

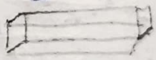
$$= \int_{S^P}\left(T\frac{dh}{dx}\right) dA_x.$$

$$= \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw}$$



For uniform grid system,

$$\left(T\frac{dh}{dx}\right)_e A_{xe} = T_e \times \frac{h_E - h_P}{\Delta x}$$ (I think $A_{xe}=1$)



(Area at element boundary)

$$\left(T\frac{dh}{dx}\right)_w A_{xe} = T_w\left(\frac{h_P - h_W}{\Delta x}\right)$$

Evaluating RHS

$$\int_{\Omega^P}(h-h_{wt})\,d\Omega = h_P\underbrace{(\Delta x \times 1 \times 1)}_{\text{Volume of the element}} - \int_{x_w}^{x_e}(C_0 + C_1 x + C_2 x^2)\,dx. \qquad = \int_{\Omega^P} h\,d\Omega$$

$$= h_P\Delta x - \left[\left(C_0 x + \frac{1}{2}C_1 x^2 + \frac{1}{3}C_2 x^3\right)\right]_{x_w}^{x_e}$$

In compact form,

$$T_e \times \frac{h_E - h_P}{\Delta x} - T_w \times \frac{h_P - h_W}{\Delta x} - C_{conf}\left\{h_P\Delta x - \left[C_0 x + \frac{1}{2}C_1 x^2 + \frac{1}{3}C_2 x^3\right]_{x_w}^{x_e}\right\}$$

$\Rightarrow$ Approximation:- $T_e = T_w$

and $x_e - x_w = \Delta x$

$$\Rightarrow \frac{h_E - 2h_P + h_W}{\Delta x} - C_{conf}\left\{h_P\Delta x - \left[C_0(x_e - x_w) + \frac{1}{2}C_1(x_e^2 - x_w^2) + \frac{1}{3}C_2(x_e^3 - x_w^3)\right]\right\} = 0$$

$$- C_{conf}\left\{\cancel{h_P\Delta x} - \left[C_0(x_e - x_w) + \frac{1}{2}C_1(x_e - x_w)(x_e^2 + x_e x_w + x_w^2) + \frac{1}{3}C_2(x_e - x_w)\right.\right.$$

$$\Rightarrow \frac{h_E - 2h_P + h_W}{\Delta x} = C_{conf}\left\{h_P\Delta x - \left[c_0(x_e - x_w) + \frac{1}{2}c_1(x_e^2 - x_w^2) + \frac{c_2}{3}(x_e^3 - x_w^3)\right]\right\}.$$

$$\Rightarrow \frac{h_E - 2h_P + h_W}{\Delta x} = C_{conf}\left\{h_P\Delta x - \left[c_0\cdot\Delta x + \frac{1}{2}c_1\,\Delta x.(x_e + x_w) + \frac{1}{3}c_2(x_e^2 + x_e x_w + x_w^2)\Delta x\right]\right\}$$

$$\Rightarrow \frac{h_E - 2h_P + h_W}{(\Delta x)^2} = C_{conf}\left\{h_P - \left[c_0 + \frac{1}{2}c_1(x_e + x_w) + \frac{1}{3}c_2(x_e^2 + x_e x_w + x_w^2)\right]\right\}$$

Now,

$x_e - x_w = \Delta x$

and $x_e + x_w = 2x_P$

(Because, all the calculations are based on nodal values)

$$= C_{conf}\left\{h_P - \left[c_0 + \frac{1}{2}c_1 \times 2x_P + \frac{1}{3}c_2\{(x_e + x_w)^2 - x_e x_w\}\right]\right.$$

$$= C_{conf}\left\{h_P - \left(c_0 + c_1 x_P + \frac{1}{3}c_2 \times \left[(2x_P)^2 - \left(x_P + \frac{\Delta x}{2}\right)\left(x_P - \frac{\Delta x}{2}\right)\right]\right)\right\}$$

$4x_P^2 - \left(x_P^2 - \frac{\Delta x^2}{4}\right)$

$= \left(3x_P^2 + \frac{\Delta x^2}{4}\right)$

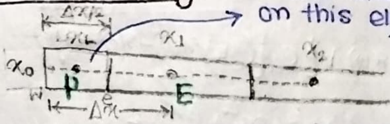
$$= C_{conf}\left\{h_P - \left(c_0 + c_1 x_P + \frac{1}{3}c_2\left(3x_P^2 + \frac{\Delta x^2}{4}\right)\right)\right\}$$

$$= C_{conf}\left\{h_P - c_0 - c_1 x_P - c_2 x_P^2 - \frac{1}{12}c_2\,\Delta x^2\right\}$$

$$= C_{conf}\left\{h_P - (c_0 + c_1 x_P + c_2 x_P^2) - \frac{1}{12}c_2\Delta x^2\right\}$$

? Extra

For left Boundary — We are interested on this element



Here (P ⇒ For which point equation is obtained)

For left most element boundary

$$\int_{\Omega^P}\frac{d}{dx}\left(T\frac{dh}{dx}\right)d\Omega - C_{conf}\int_{\Omega^P}(h - h_{wt})\,d\Omega = 0$$

$$\Rightarrow \left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_w A_{xw} = C_{conf}\left[h_P.\frac{\Delta x}{2} - \int_{x_w}^{x_e}(c_0 + c_1 x + c_2 x^2)dx\right]$$

$$\Rightarrow T_e.\frac{h_E - h_P}{3\Delta x/4} - T_w.\frac{h_P - h_0}{\Delta x/4} = C_{conf}\,h_P.\frac{\Delta x}{2} - C_{conf}\left[c_0 x + \frac{1}{2}c_1 x^2 + \frac{1}{3}c_2 x^3\right]_{x_w}^{x_e}$$

$T_e = T_W = T$

$$T\left\{\frac{4h_E - 4h_P}{3\Delta x} - \frac{4h_P - 4h_0}{\Delta x}\right\} = C_{conf}\frac{\Delta x}{2}\left\{h_P - c_0 - c_1 x_P - \frac{1}{3}c_2 \times 3x_P^2 - \frac{1}{48}c_2\Delta x^2\right\}$$

$$-C_{conf}\frac{\Delta x}{2}\left\{c_0 + \frac{1}{2}c_1(x_e + x_w) + \frac{1}{3}c_2 \times (x_e^2 + x_e x_w + x_w^2)\right\} \quad \left[\because x_e - x_w = \frac{\Delta x}{2}\right]$$

$$= c_0 + c_1 x_P + \frac{1}{3}c_2\{4x_P^2 - x_e x_w\}$$

$$\left\{4x_P^2 - \left(x_P + \frac{\Delta x}{4}\right)\left(x_P - \frac{\Delta x}{4}\right)\right\}$$

$$-C_{conf}\frac{\Delta x}{2}\left\{c_0 + c_1 x_P + \frac{c_2}{3}\left\{3x_P^2 + \frac{\Delta x^2}{16}\right\}\right\}$$

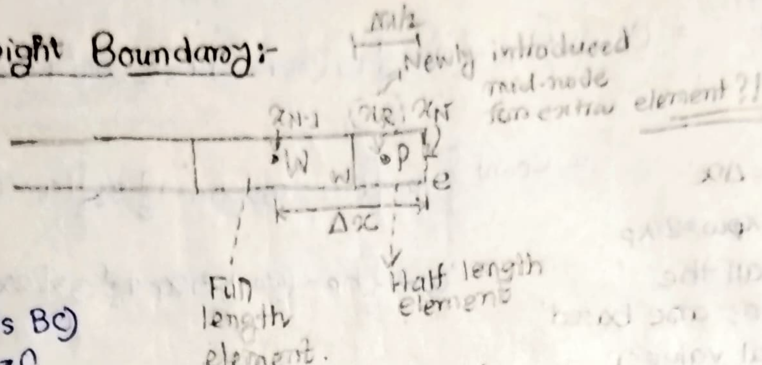
$$\Rightarrow T \times \frac{4h_E - 4h_P - 12h_P + 12h_0}{3\Delta x} = C_{conf}\frac{\Delta x}{2}\left\{h_P - (c_0 + c_1 x_P + c_2 x_P^2) - \frac{1}{48}c_2\Delta x^2\right\}$$

$$\Rightarrow \frac{4h_E - 16h_P + 12h_0}{3\Delta x^2} = \frac{C_{conf}}{2T}\left\{h_P - h_{wt}(x_P) - \frac{1}{48}c_2\Delta x^2\right\}$$

$$\Rightarrow \frac{8h_E - 32h_P}{3\Delta x^2} + \frac{24h_0}{3\Delta x^2} = \frac{C_{conf}}{T}\left\{h_P - h_{wt}(x_P) - \frac{1}{48}C_2\Delta x^2\right\}$$

$$\Rightarrow \frac{8h_E - 32h_P}{3\Delta x^2} = -\frac{8h_0}{\Delta x^2} + \frac{C_{conf}}{T}\left\{h_P - h_{wt}(x_P) - \frac{1}{48}C_2\Delta x^2\right\}$$

For right Boundary:-



(Neuman's BC)

$$\left(T\frac{dh}{dx}\right)_e A_{xe} - \left(T\frac{dh}{dx}\right)_W A_{xW} = -T_W\frac{h_P - h_W}{\left(\frac{3\Delta x}{4}\right)}$$

$$C_{conf}\int_{\Omega^P}(h - h_{wt})\,d\Omega$$

$$= C_{conf}\left[h_P\left(\frac{\Delta x}{2}\times 1\times 1\right) - \int_{x_W}^{x_e}(C_0 + C_1 x + C_2 x^2)\,dx\right]$$

$$\left\{\begin{array}{l} x_e = x_P + \frac{\Delta x}{4} \\ x_W = x_P - \frac{\Delta x}{4}\end{array}\right.$$

$$= C_{conf}\left[h_P\cdot\frac{\Delta x}{2} - \left[C_0 x + \frac{1}{2}C_1 x^2 + \frac{1}{3}C_2 x^3\right]_{x_W}^{x_e}\right]$$

$$-\frac{\Delta x}{2}\left[C_0 + \frac{1}{2}C_1(x_e + x_W) + \frac{1}{3}C_2\left\{(x_e + x_W)^2 - x_e x_W\right\}\right]$$

$$-\frac{\Delta x}{2}\left[C_0 + C_1 x_P + \frac{1}{3}C_2\left(4x_P^2 - x_P^2 + \frac{\Delta x^2}{16}\right)\right]$$

$$-\frac{\Delta x}{2}\left[C_0 + C_1 x_P + C_2 x_P^2 + \frac{C_2\Delta x^2}{48}\right]$$

$$= C_{conf}\cdot\frac{\Delta x}{2}\left[h_P - (C_0 + C_1 x_P + C_2 x_P^2) - \frac{C_2\Delta x^2}{48}\right]$$

$$T\frac{4(h_W - h_P)}{3\Delta x} = C_{conf}\cdot\frac{\Delta x}{2}\left[h_P - h_{wt}(x_P) - \frac{C_2\Delta x^2}{48}\right]$$

$$\Rightarrow \frac{8h_W - 8h_P}{3\Delta x^2} = \frac{C_{conf}}{T}\left[h_P - h_{wt}(x_P) - \frac{C_2\Delta x^2}{48}\right]$$

$$\overset{?}{=}\left\{\begin{array}{l} \left.\frac{dh}{dx}\right|_e = \alpha h_P + \beta h_E \\ \quad = \frac{h_E - h_P}{\Delta x} \\ \quad = -\frac{1}{\Delta x}h_P + \frac{1}{\Delta x}h_E \end{array}\right.$$

Lecture-14(NPTEL)

Module-2, Unit 10

FVM, Boundary Value Problem:

$$\frac{\partial(\Lambda_\varphi \varphi)}{\partial t} + \nabla\cdot(\gamma_\varphi \varphi u) = \nabla\cdot(\Gamma_\varphi \cdot \nabla\varphi) + \Gamma_{\varphi_0} + S_\varphi$$

In FDM, we have used these two terms for BVP using PDE and Boundary conditions.

2D BVP:-

$$\nabla(\Gamma\cdot\nabla\varphi) + S_\varphi(x,y) = 0$$

$$\Rightarrow \Gamma_x \frac{\partial^2\varphi}{\partial x^2} + \Gamma_y \frac{\partial^2\varphi}{\partial y^2} + S_\varphi(x,y) = 0$$

$$\Gamma = \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix}$$

$$\nabla\cdot \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix} \begin{bmatrix} \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \end{bmatrix}$$

$$= \nabla\cdot \begin{bmatrix} \Gamma_x \frac{\partial\varphi}{\partial x} \\ \Gamma_y \frac{\partial\varphi}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \Gamma_x \frac{\partial\varphi}{\partial x} \\ \Gamma_y \frac{\partial\varphi}{\partial y} \end{bmatrix}$$

$$= \Gamma_x \frac{\partial^2\varphi}{\partial x^2} + \Gamma_y \frac{\partial^2\varphi}{\partial y^2}$$

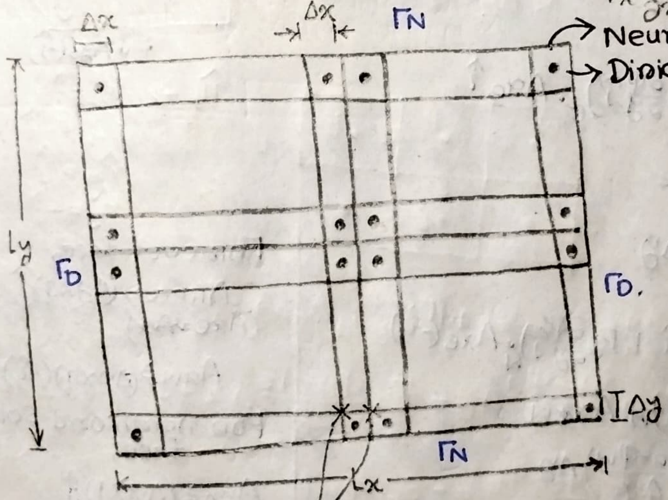
Boundary Condition:-

Γ_D^1: $\varphi(0,y) = \varphi_1$ (Left)

Γ_D^2:- $\varphi(L_x,y) = \varphi_2$ (Right)

Γ_N^3:- $\left.\frac{\partial\varphi}{\partial y}\right|_{(x,0)} = 0$ (Bottom)

Γ_N^4:- $\left.\frac{\partial\varphi}{\partial y}\right|_{(x,L_y)} = 0$ (TOP)

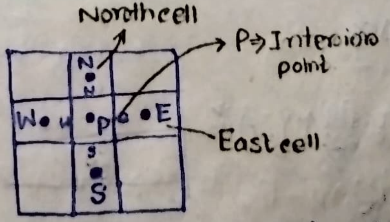


Neumann condition
Dirichlet condition
} Special attention for corner points.

For Finite difference, these were considered as nodes.

Here, we consider cell centred.

Discretization ⇒ Interior points:-



P cell have east, west, n, s faces

In FVM GE is integrated over element volume to form discretized equation over node point P which is cell centered.

Lecture-14(NPTEL)

Module-2, Unit 10

FVM, Boundary Value Problem:

$$\frac{\partial(\Lambda_\varphi \varphi)}{\partial t} + \nabla\cdot(\Upsilon_\varphi \cdot \varphi u) = \underbrace{\nabla\cdot(\Gamma_\varphi \cdot \nabla\varphi) + F_{\varphi_0} + S_\varphi}$$

In FDM, we have used these two terms for BVP using PDE and Boundary conditions.

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$$\Gamma = \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix}$$

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$$= \nabla\cdot \begin{bmatrix} \Gamma_x \frac{\partial\varphi}{\partial x} \\ \Gamma_y \frac{\partial\varphi}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \Gamma_x \frac{\partial\varphi}{\partial x} \\ \Gamma_y \frac{\partial\varphi}{\partial y} \end{bmatrix}$$

$$= \Gamma_x \frac{\partial^2\varphi}{\partial x^2} + \Gamma_y \frac{\partial^2\varphi}{\partial y^2}$$

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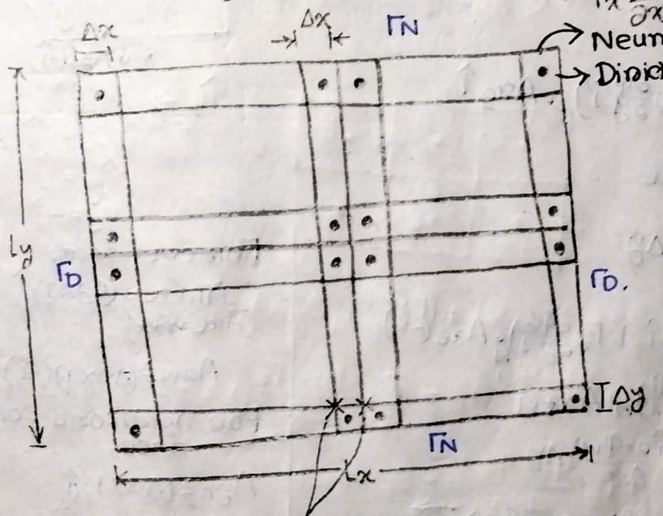

Neumann condition
Dirichlet condition
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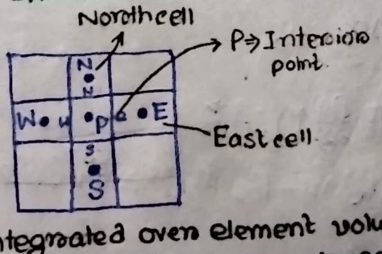
Discretization ⇒ Interior points:-



Pcell have east, west, n, s faces

In FVM GE is integrated over element volume to form discretized equation over node point P which is cell centered.

Gauss divergence theorem for this problem.

$$\int_{\Omega^P} \left[\nabla\cdot(\Gamma\cdot\nabla\varphi) + S_\varphi(x,y)\right] d\Omega$$

$$\Rightarrow \int_{\Omega^P} \nabla\cdot(\Gamma\cdot\nabla\varphi)\, d\Omega + \int_{\Omega^P} S_\varphi(x,y)\, d\Omega = 0$$

* $$\int_{\Omega^P} \nabla\cdot(\Gamma\cdot\nabla\varphi)\, d\Omega$$

$$= \int_{\Omega^P} \nabla\cdot\left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right) d\Omega$$

$$= \int_{S^P} \left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right)\cdot\hat{n}\, ds$$

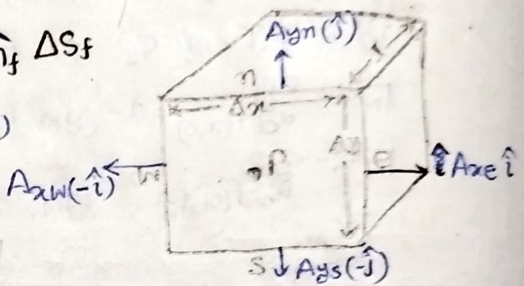
$$= \sum_{\forall f} \left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right)_f \cdot \hat{n}_f\, \Delta S_f \quad \text{.......... (1)}$$

over all faces.

Gauss Divergenee theorem,

$$\iiint_\Omega \nabla\cdot\vec{a}\, d\Omega = \iint_S \vec{a}\cdot\hat{n}\, ds$$

Volume → Surface



$\hat{n}_f \rightarrow \hat{n}_e = \hat{i}$
$\hat{n}_w = -\hat{i}$
$\hat{n}_n = \hat{j}$
$\hat{n}_s = -\hat{j}$

For east face,

$$\left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right)_e \cdot A_{xe}\hat{i}$$

$$= \left(\Gamma_x \frac{\partial\varphi}{\partial x}\right)_e A_{xe}$$

$$= \Gamma_{xe}\left(\frac{\varphi_E - \varphi_P}{\Delta x}\right)\Delta y$$

West face,

$$\left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right)_w \cdot A_{xe}(-\hat{i})$$

$$= -\Gamma_{xw}\left(\frac{\partial\varphi}{\partial x}\right)_w A_{xw}$$

$$= -\Gamma_{xw}\left(\frac{\varphi_P - \varphi_W}{\Delta x}\right)\Delta y$$

For east face,
Area ⇒ $(\Delta y \times 1)\hat{i}$
$(A_{xe} = A_x)$

$A_{xw} \Rightarrow (\Delta y \times 1)(-\hat{i})$

For north and south face,

$A_{yn} \Rightarrow (\Delta x \times 1)\hat{j}$

$A_{ys} \Rightarrow (\Delta x \times 1)(-\hat{j})$

North face,

$$\left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right)_n \cdot \vec{A}_{yn}$$

I think, this is the connect way of nepresentation

$$= \left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right)_n \cdot (\Delta x\,\hat{j})$$

$$= \Gamma_{yn}\left(\frac{\partial\varphi}{\partial y}\right)_n \Delta x$$

$$= \Gamma_{yn}\left(\frac{\varphi_N - \varphi_P}{\Delta y}\right)\Delta x$$

South face,

$$\left(\Gamma_x \frac{\partial\varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\varphi}{\partial y}\hat{j}\right)_s \cdot \vec{A}_{ys}$$

$$= -\Gamma_{ys}\left(\frac{\partial\varphi}{\partial y}\right)_s \cdot \Delta x$$

$$= -\Gamma_{ys}\,\frac{\varphi_S - \varphi_P}{\Delta y}\cdot\Delta x$$

Source term, $\int_{\Omega^P} S_\varphi(x,y)\, d\Omega = S_\varphi(x_P, y_P) \times (\Delta x \cdot \Delta y \cdot 1)$. → Volume of the element.

Putting all values to equation (1), compact form of equation,

Note:- Always forward difference has been taken.

$$\Gamma_{xe}\frac{\varphi_E-\varphi_P}{\Delta x}.\Delta y-\Gamma_{xw}\frac{\varphi_P-\varphi_W}{\Delta x}.\Delta y+\Gamma_{yn}\frac{\varphi_N-\varphi_P}{\Delta y}.\Delta x-\Gamma_{ys}\frac{(\varphi_S+\varphi_P)}{\Delta y}\Delta x$$
$$+S_\varphi(x_P,y_P)\,\Delta x\,\Delta y=0$$

Considering,

$\Gamma_{xe}=\Gamma_{xw}=\Gamma_x$

$\Gamma_{yn}=\Gamma_{ys}=\Gamma_y$

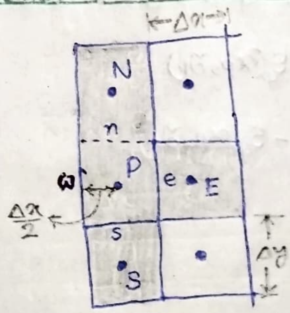
and dividing eqn by $(\Delta x.\Delta y)$, we get,

$$\Gamma_x\frac{\varphi_E-2\varphi_P+\varphi_W}{\Delta x^2}+\Gamma_y\frac{\varphi_N-2\varphi_P+\varphi_S}{\Delta y^2}+S_\varphi(x,y)=0 \;\ldots\ldots (2)$$

If we compare (2) with our original FDE, this is nothing but,

$$\Gamma_x.\frac{\varphi_{i+1,j}-2\varphi_{i,j}+\varphi_{i-1,j}}{\Delta x^2}+\Gamma_y\frac{\varphi_{i,j+1}-2\varphi_{i,j}+\varphi_{i,j-1}}{\Delta y^2}+S_\varphi\big|_{i,j}=0$$

Discretization (left boundary):-



Compact form of governing equation,

$$\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_e A_{xe}-\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_w A_{xw}+\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_n A_{yn}-\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_s A_{ys}+S_\varphi(x_P,y_P)\Delta x\Delta y=0$$

For west face, $\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_w=\Gamma_{xw}\frac{\varphi_P-\varphi_w}{\Delta x/2}$

In compact form,

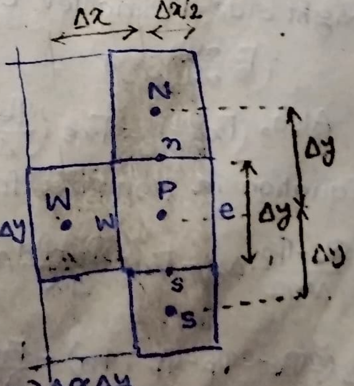
$$\Gamma_{xe}\frac{\varphi_E-\varphi_P}{\Delta x}.\Delta y-\Gamma_{xw}\frac{\varphi_P-\varphi_w}{\Delta x/2}\Delta y+\Gamma_{yn}\frac{\varphi_N-\varphi_P}{\Delta y}.\Delta x-\Gamma_{ys}\frac{\varphi_P-\varphi_S}{\Delta y}\Delta x$$
$$=-S_\varphi(x_P,y_P)\,\Delta x\Delta y.$$

Small

$$\Rightarrow\frac{\Gamma_x}{\Delta x^2}\Big[(\varphi_E-\varphi_P)-2(\varphi_P-\varphi_w)\Big]+\Gamma_y\left(\frac{\varphi_N-2\varphi_P+\varphi_S}{\Delta y^2}\right)=-S_\varphi(x_P,y_P)$$

$$\Rightarrow\Gamma_x\left(\frac{\varphi_E-3\varphi_P}{\Delta x^2}\right)+\Gamma_y\frac{\varphi_N-2\varphi_P+\varphi_S}{\Delta y^2}=-\frac{2\Gamma_x}{\Delta x^2}\varphi_w-S_\varphi(x_P,y_P)$$

Discretization (Right Boundary):-



Compact form of governing equation,

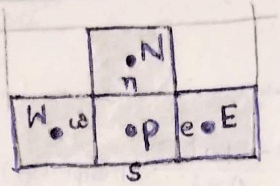
$$\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_e A_{xe}-\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_w A_{xw}$$
$$+\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_n A_{yn}-\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_s A_{ys}=-S_\varphi(x_P,y_P)\Delta x\Delta y$$

$$\Rightarrow\Gamma_{xe}\frac{\varphi_e-\varphi_P}{\Delta x/2}\Delta y-\Gamma_{xw}\frac{\varphi_P-\varphi_W}{\Delta x}\Delta y$$
$$+\Gamma_{yn}\frac{\varphi_N-\varphi_P}{\Delta y}\Delta x-\Gamma_{ys}\frac{\varphi_P-\varphi_S}{\Delta y}\Delta x=-S_\varphi(x_P,y_P)\Delta x\Delta y$$

$$\Rightarrow \frac{\Gamma_x}{\Delta x^2}\left[2(\varphi_e-\varphi_P)-(\varphi_P-\varphi_W)\right]+\frac{\Gamma_y}{\Delta y^2}(\varphi_N-2\varphi_P+\varphi_S)=-S_\varphi(x_P,y_P)$$

$$\Rightarrow \Gamma_x\left(\frac{-3\varphi_P+\varphi_W}{\Delta x^2}\right)+\Gamma_y\left(\frac{\varphi_N-2\varphi_P+\varphi_S}{\Delta y^2}\right)=-\frac{2\Gamma_x}{\Delta x^2}\varphi_e-S_\varphi(x_P,y_P)$$

Discretization Bottom Boundary:



(Neumann Boundary condition).

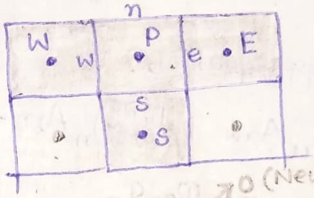
$$\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_e A_{xe}-\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_w A_{xw}+\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_n A_{yn}-\cancel{\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_s A_{ys}}^{\,0}+S_\varphi(x_P,y_P)\,\Delta x\Delta y$$

$$\Rightarrow \Gamma_{xe}\frac{\varphi_E-\varphi_P}{\Delta x}\Delta y-\Gamma_{xw}\frac{\varphi_P-\varphi_W}{\Delta x}\Delta y+\Gamma_{yn}\frac{\varphi_N-\varphi_P}{\Delta y}\Delta x-\cancel{\Gamma_{ys}\frac{\varphi_P-\varphi_S}{\Delta y}\Delta x}$$
$$+S_\varphi(x_P,y_P)\,\Delta x\Delta y=0$$

$$\Rightarrow \Gamma_x\left(\frac{\varphi_E-2\varphi_P+\varphi_W}{\Delta x^2}\right)+\frac{\Gamma_y}{\Delta y^2}(\varphi_N-2\varphi_P+\varphi_S)=-S_\varphi(x_P,y_P)$$

$$\Rightarrow \Gamma_x\left(\frac{\varphi_E-2\varphi_P+\varphi_W}{\Delta x^2}\right)+\Gamma_y\left(\frac{\varphi_N-2\varphi_P}{\Delta y^2}\right)=-\cancel{\frac{\Gamma_y}{\Delta y^2}\varphi_S}-S_\varphi(x_P,y_P)$$

Top Boundary:-



(Neumann Boundary)

$$\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_e A_{xe}-\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_w A_{xw}+\cancel{\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_n}^{\,0} A_{yn}-\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_s A_{ys}+S_\varphi(x_P,y_P)\Delta x\Delta y=0$$

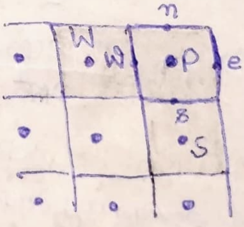
$$\Rightarrow \Gamma_{xe}\frac{\varphi_E-\varphi_P}{\Delta x}\cdot\Delta y-\Gamma_{xw}\frac{\varphi_P-\varphi_W}{\Delta x}\Delta y-\Gamma_{ys}\frac{\varphi_P-\varphi_S}{\Delta y}\Delta x+S_\varphi(x_P,y_P)\Delta x\Delta y=0$$

$$\Rightarrow \Gamma_x\left(\frac{\varphi_E-2\varphi_P+\varphi_W}{\Delta x^2}\right)+\Gamma_y\frac{\varphi_S-\varphi_P}{\Delta y^2}=-S_\varphi(x_P,y_P)$$

North-east corner:

Top ⇒ Neumann Boundary

Right side ⇒ Dirichlet Boundary.



$$\left(\Gamma_y\frac{\partial\varphi}{\partial y}\right)_n=0$$

Also, $\left(\Gamma_x\frac{\partial\varphi}{\partial x}\right)_e=\Gamma_{xe}\left(\frac{\varphi_e-\varphi_P}{\Delta x/2}\right)$

Equation in compact form:-

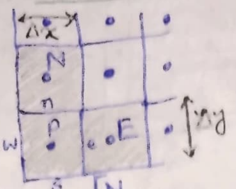
$$\Gamma_{xe}\frac{\varphi_e-\varphi_P}{\Delta x/2}\cdot\Delta y-\Gamma_{xw}\frac{\varphi_P-\varphi_W}{\Delta x}\Delta y-\Gamma_{ys}\frac{\varphi_P-\varphi_S}{\Delta y}\Delta x+S_\varphi(x_P,y_P)\Delta x\Delta y=0$$

$$\Rightarrow \frac{\Gamma_x}{\Delta x^2}(2\varphi_e-2\varphi_P-\varphi_P+\varphi_W)-\frac{\Gamma_y}{\Delta y^2}(\varphi_P-\varphi_S)+S_\varphi(x_P,y_P)=0$$

$$\Rightarrow \Gamma_x\left(\frac{-3\varphi_P+\varphi_W}{\Delta x^2}\right)-\Gamma_y\left(\frac{\varphi_P-\varphi_S}{\Delta y^2}\right)=-\frac{2\Gamma_x}{\Delta x^2}\varphi_e-S_\varphi(x_P,y_P)$$

South west node:-

Γ_D (Dirichlet BC)



Γ_N (Neumann boundary)

$\left(\Gamma_y \frac{\partial \varphi}{\partial y}\right)_s = 0.$

compact equation,

$$\left(\Gamma_x \frac{\partial \varphi}{\partial x}\right)_e A_{xe} - \left(\Gamma_x \frac{\partial \varphi}{\partial x}\right)_w A_{xw} + \left(\Gamma_y \frac{\partial \varphi}{\partial y}\right)_n A_{yn} - \cancel{\left(\Gamma_y \frac{\partial \varphi}{\partial y}\right)_s}^{\,0} A_{ys} + S_\varphi(x_p, y_p) \underset{\Delta x \Delta y}{=} 0$$

$$\Rightarrow \Gamma_{xe} \frac{\varphi_E - \varphi_P}{\Delta x} \Delta y - \Gamma_{xw} \left(\frac{\varphi_P - \varphi_w}{\Delta x/2}\right) \Delta y + \Gamma_{yn} \frac{\varphi_N - \varphi_P}{\Delta y} \Delta x + S_\varphi(x_p, y_p) \cdot \Delta x \Delta y = 0$$

$$\Rightarrow \frac{\Gamma_x}{\Delta x^2}(\varphi_E - 3\varphi_P + 2\varphi_w) + \frac{\Gamma_y}{\Delta y^2}(\varphi_N - \varphi_P) + S_\varphi(x_p, y_p) = 0$$

$$\Rightarrow \Gamma_x \frac{(\varphi_E - 3\varphi_P)}{\Delta x^2} + \Gamma_y \left(\frac{\varphi_N - \varphi_P}{\Delta y^2}\right) = -\frac{2\Gamma_x}{\Delta x^2} \varphi_w - S_\varphi(x_p, y_p)$$

NPTEL (LEC-15)

FVM - IBVP

differential eq^n with a general variable:-

$$\underbrace{\frac{\partial(\Lambda_\varphi \varphi)}{\partial t}}_{\text{Temporal}} + \nabla \cdot (\gamma_\varphi \varphi u) = \underbrace{\nabla \cdot (\Gamma_\varphi \cdot \nabla \varphi)}_{\text{Diffusive}} + F\varphi_0 + \underbrace{S_\varphi}_{\text{Source/sink}}$$

IBVP

BVP.

Problem definition:-

2-D IBVP can be written as:-

$$\Omega :- \Lambda_\varphi \frac{\partial \varphi}{\partial t} = \Gamma_x \frac{\partial^2 \varphi}{\partial x^2} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2} + S_\varphi(x, y)$$

$\Gamma = \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix}$

Γ_x, Γ_y are diffusion coefficients.

2D Tensor

$$\Omega :- \Lambda_\varphi \frac{\partial \varphi}{\partial t} = \nabla \cdot (\Gamma \cdot \nabla \varphi) + S_\varphi(x, y)$$

(In terms of divergence)

$$\boxed{\begin{aligned} \nabla \varphi &= \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} \\ \Gamma \cdot \nabla \varphi &= \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix} \begin{Bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{Bmatrix} \\ &= \Gamma_x \frac{\partial \varphi}{\partial x} \hat{i} + \Gamma_y \frac{\partial \varphi}{\partial y} \hat{j} \\ \nabla \cdot (\Gamma \cdot \nabla \varphi) &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j}\right) \cdot \left(\Gamma_x \frac{\partial \varphi}{\partial x} \hat{i} + \Gamma_y \frac{\partial \varphi}{\partial y} \hat{j}\right) \\ &= \left(\Gamma_x \frac{\partial^2 \varphi}{\partial x^2} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2}\right) \end{aligned}}$$

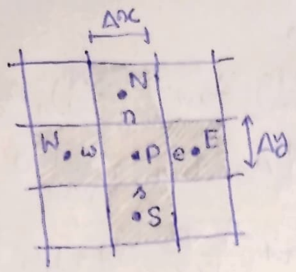
$\Rightarrow$ Trial.

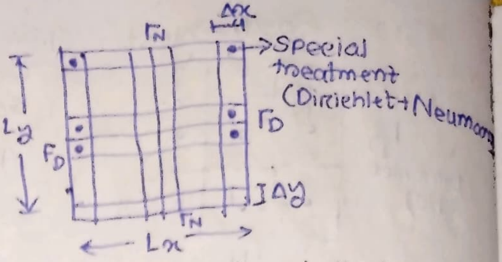
IC:- $\varphi(x, y, 0) = \varphi_0(x, y)$

Boundary condition:-

$$\left.\begin{aligned} \Gamma_D^1 &:- \varphi(0, y, t) = \varphi_1 \\ \Gamma_D^2 &:- \varphi(L_x, y, t) = \varphi_2 \end{aligned}\right\} \text{Dirichlet BC}$$

$$\left.\begin{aligned} \Gamma_N^3 &:- \left.\frac{\partial \varphi}{\partial y}\right|_{(x, 0, t)} = 0 \\ \Gamma_N^4 &:- \left.\frac{\partial \varphi}{\partial y}\right|_{(x, L_y, t)} = 0 \end{aligned}\right\} \text{Neumann B.C.}$$





In physical system, both the boundaries can be Neumann or dirichlet. But, here, we take a case where, one side is 'N' and another side 'D'.

In FVM, GE is integrated over element volume (in space) and time interval to form the discretized equation at node point P.

$$\Lambda_\varphi \frac{\partial \varphi}{\partial t} = \nabla(\Gamma.\nabla\varphi) + S_\varphi(x,y)$$

$$\Rightarrow \int_t^{t+\Delta t}\left[\int_{\Omega P} \Lambda_\varphi \frac{\partial \varphi}{\partial t} d\Omega\right]dt = \int_t^{t+\Delta t}\left[\int_{\Omega P} \nabla.(\Gamma.\nabla\varphi)d\Omega\right]dt + \int_t^{t+\Delta t}\left[\int_{\Omega P} S_\varphi(x,y)\,d\Omega\right]dt \quad \ldots\ldots (1)$$

ΩP → Element volume

Now,

Discretization for individual terms:-

(1) Temporal term:- $\int_t^{t+\Delta t}\left[\int_{\Omega P} \Lambda_\varphi \frac{\partial \varphi}{\partial t} d\Omega\right]dt$

$$= \Lambda_\varphi \int_t^{t+\Delta t} \frac{\partial}{\partial t}\left(\int_{\Omega P} \varphi\, d\Omega\right)dt$$

$$= \Lambda_\varphi \int_t^{t+\Delta t} \frac{\partial}{\partial t}\left\{\Delta\Omega_P \times \frac{1}{\Delta\Omega_P}\int_{\Omega_P} \varphi\, d\Omega\right\}dt$$

$$\varphi_P = \frac{1}{\Omega_P}\int_{\Omega P} \varphi\, d\Omega$$

φ_P ⇒ Approximated value for this domain. Spatially averaged value within the P cell.

$$= \Lambda_\varphi \int_t^{t+\Delta t} \frac{\partial}{\partial t}\{\Delta\Omega_P . \varphi_P\}dt$$

$$= \Lambda_\varphi (\varphi_P^{l+1} - \varphi_P^{l}) \underbrace{\Delta\Omega_P}_{\text{Volume of cell P}}$$

l → represents t time level

$l+1$ → " $(t+\Delta t)$ "

$$= \Lambda_\varphi (\varphi_P^{l+1} - \varphi_P^{l})(\Delta x \Delta y . 1) \quad \ldots\ldots (2)$$

(2) Discretization:- Spatial term:-

$$\int_t^{t+\Delta t}\int_{\Omega_P} \nabla.(\Gamma.\nabla\varphi)\, d\Omega\, dt = \int_t^{t+\Delta t}\int_{\Omega P} \nabla.\underbrace{\left(\Gamma_x \frac{\partial \varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial \varphi}{\partial y}\hat{j}\right)}\, d\Omega\, dt$$

(can transform to surface integral as per Gauss divergence theorem).

Intermediate time level. $t+\theta\Delta t$

$$= \int_t^{t+\theta\Delta t}\int_{\Omega P} \nabla.\left(\Gamma_x \frac{\partial \varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial \varphi}{\partial y}\hat{j}\right) d\Omega\, dt$$

$$+ \int_{t+\theta\Delta t}^{t+\Delta t}\int_{\Omega P} \nabla.\left(\Gamma_x \frac{\partial \varphi}{\partial x} + \Gamma_y \frac{\partial \varphi}{\partial y}\hat{j}\right) d\Omega\, dt$$

$$= \int_t^{t+\theta\Delta t}\int_{S_P} \left(\Gamma_x \frac{\partial \varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial \varphi}{\partial y}\hat{j}\right).\hat{n}\, ds + \int_{t+\theta\Delta t}^{t+\Delta t}\int_{S_P} \left(\Gamma_x \frac{\partial \varphi}{\partial x} + \Gamma_y \frac{\partial \varphi}{\partial y}\right).\hat{n}\, ds$$

$\hat{n}$ → vector ⊥r to that surface

ds → Particular surface

$$= \int_t^{t+\theta\Delta t} \Big[\left(\Gamma_x \frac{\partial\phi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\phi}{\partial y}\hat{j}\right)\cdot(A_{xe}\hat{i}) + \left(\Gamma_x \frac{\partial\phi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\phi}{\partial y}\hat{j}\right)\cdot A_{xw}(-\hat{i})$$

$$+ \left(\Gamma_x \frac{\partial\phi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\phi}{\partial y}\hat{j}\right)\cdot(A_{yn}\hat{j}) + \left(\Gamma_x \frac{\partial\phi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\phi}{\partial y}\hat{j}\right) A_{ys}(-\hat{j}) \Big] dt$$

$$+ \int_{t+\theta\Delta t}^{t+\Delta t} \Big[\left(\Gamma_x \frac{\partial\phi}{\partial x}\hat{i} + \Gamma_y \frac{\partial\phi}{\partial y}\hat{j}\right)\cdot(A_{xe}\hat{i}) + \text{.... same ..} \Big] dt$$

$$= \int_t^{t+\theta\Delta t} \left\{ \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_e A_{xe} - \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_w A_{xw} + \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_n A_{yn} - \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_s A_{ys} \right\} dt$$

$$+ \int_{t+\Delta\theta t}^{t+\Delta t} \left\{ \text{...... same .....} \right\} dt$$

$$= \left\{ \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_e^l A_{xe} - \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_w^l A_{xw} + \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_n^l A_{yn} - \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_s^l A_{ys} \right\} . \left\{(t+\theta\Delta t) - t\right\}$$

$$+ \left\{ \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_e^{l+1} A_{xe} - \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_w^{l+1} A_{xw} + \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_n^{l+1} A_{yn} - \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_s^{l+1} A_{ys} \right\} . \left\{(t+\Delta t) - (t+\theta\Delta t)\right\}$$

$$= \cancel{\{\text{... same ...}\}\theta\Delta t + \{\text{... same ...}\}(1-\theta)\Delta t}$$

$$= \left\{ \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_e^l A_{xe} - \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_W^l A_{xw} + \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_n^l A_{yn} - \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_s^l A_{ys} \right\} \theta\Delta t$$

$$+ \left\{ \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_e^{l+1} A_{xe} - \left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_W^{l+1} A_{xw} + \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_n^{l+1} A_{yn} - \left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_S^{l+1} A_{ys} \right\} (1-\theta)\Delta t \quad \text{..... (3)}$$

In uniform grid system:-

Time level: t — Time level: $(t+\Delta t)$

East face:- $\left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_e^l A_{xe} = \Gamma_{xe}\left(\frac{\phi_E^l - \phi_P^l}{\Delta x}\right)\Delta y$ — $\left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_e^{l+1} A_{xe} = \Gamma_{xe}\left(\frac{\phi_E^{l+1} - \phi_P^{l+1}}{\Delta x}\right)\Delta y.$

North face:- $\left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_n^l A_{yn} = \Gamma_{yn}\left(\frac{\phi_N^l - \phi_P^l}{\Delta y}\right)\Delta x$ — $\left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_n^{l+1} A_{yn} = \Gamma_{yn}\left(\frac{\phi_N^{l+1} - \phi_P^{l+1}}{\Delta y}\right)\Delta x$

West face:- $\left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_W^l A_{xw} = \Gamma_{xw}\left(\frac{\phi_P^l - \phi_W^l}{\Delta x}\right)\Delta y$ — $\left(\Gamma_x \frac{\partial\phi}{\partial x}\right)_n^{l+1} = \Gamma_{xw}\left(\frac{\phi_P^{l+1} - \phi_W^{l+1}}{\Delta x}\right)\Delta y$

South face:- $\left(\Gamma_y \frac{\partial\phi}{\partial y}\right)_s^l A_{ys} = \Gamma_{ys}\left(\frac{\phi_P^l - \phi_S^l}{\Delta y}\right)\Delta y$ — $\left(\Gamma_y \frac{\partial\phi}{\partial y}\right)^{l+1} A_{ys} = \Gamma_{ys}\,\frac{\phi_P^{l+1} - \phi_S^{l+1}}{\Delta y}$

(3) Discretization: (Source term):

$$\int_t^{t+\Delta t}\int_{\Omega_P} S_\phi(x,y)\,d\Omega\,dt = \int_t^{t+\theta\Delta t}\int_{\Omega_P} S_\phi(x,y)\,d\Omega\,dt + \int_{t+\theta\Delta t}^{t+\Delta t}\int_{\Omega_P} S_\phi(x,y)\,d\Omega\,dt$$

$$= \int_t^{t+\theta\Delta t} \left\{S_\phi(x_P,y_P)\,\Delta x \Delta y \times 1\right\} dt + \int_{t+\theta\Delta t}^{t+\Delta t} \left\{S_\phi(x_P,y_P)\,\Delta x \Delta y \times 1\right\} dt$$

$$= S_\phi^l(x_P,y_P)\,\Delta x \Delta y . \{t+\theta\Delta t - t\} + S_\phi^{l+1}(x_P,y_P)\,\Delta x \Delta y \{t+\Delta t - t - \theta\Delta t\}$$

$$= \left\{\theta\, S_\phi^l(x_P,y_P) + (1-\theta) S_\phi^{l+1}(x_P,y_P)\right\} \Delta x \Delta y \Delta t \quad \text{..... (4)}$$

Putting values from equation (2), (3) and (4) in (1),

$$\Lambda_\varphi(\varphi_P^{l+1}-\varphi_P^l).\Delta x\Delta y =$$

$$\left[\Gamma_{xe}\left(\frac{\varphi_E^l-\varphi_P^l}{\Delta x}\right)\Delta y-\Gamma_{xw}\left(\frac{\varphi_P^l-\varphi_W^l}{\Delta x}\right)\Delta y+\Gamma_{yn}\left(\frac{\varphi_N^l-\varphi_P^l}{\Delta y}\right)\Delta x-\Gamma_{ys}\left(\frac{\varphi_P^l-\varphi_S^l}{\Delta y}\right)\Delta x\right]\theta\Delta t$$

$$+\left[\Gamma_{xe}\left(\frac{\varphi_E^{l+1}-\varphi_P^{l+1}}{\Delta x}\right)\Delta y-\Gamma_{xw}\left(\frac{\varphi_P^{l+1}-\varphi_W^{l+1}}{\Delta x}\right)\Delta y+\Gamma_{yn}\frac{(\varphi_N^{l+1}-\varphi_P^{l+1})}{\Delta y}\Delta x-\Gamma_{ys}\left(\frac{\varphi_P^{l+1}-\varphi_S^{l+1}}{\Delta y}\right)\Delta x\right](1-\theta)\Delta t$$

$$+\left[\theta\, S_\varphi^l(x_P,y_P)+(1-\theta)\,S_\varphi^{l+1}(x_P,y_P)\right]\Delta x\Delta y\Delta t \quad \ldots\ldots (4.1)$$

Dividing both sides by $(\Delta x\Delta y\Delta t)$, we have,
and taking $\Gamma_{xe}=\Gamma_{xw}=\Gamma_x$ and $\Gamma_{yn}=\Gamma_{ys}=\Gamma_y$

$$\Lambda_\varphi\frac{\varphi_P^{l+1}-\varphi_P^l}{\Delta t}$$

$$=\left[\Gamma_x\left(\frac{\varphi_E^l-2\varphi_P^l+\varphi_W^l}{\Delta x^2}\right)+\Gamma_y\left(\frac{\varphi_N^l-2\varphi_P^l+\varphi_S^l}{\Delta y^2}\right)\right]\theta$$

$$+\left[\Gamma_x\left(\frac{\varphi_E^{l+1}-2\varphi_P^{l+1}+\varphi_W^{l+1}}{\Delta x^2}\right)+\Gamma_y\left(\frac{\varphi_N^{l+1}-2\varphi_P^{l+1}+\varphi_S^{l+1}}{\Delta y^2}\right)\right](1-\theta)$$

$$+\left[\theta\, S_\varphi^l(x_P,y_P)+(1-\theta)\,S_\varphi^{l+1}(x_P,y_P)\right] \quad \ldots\ldots\ldots\ldots (5)$$

Explicit Scheme:- (when $\theta=1$),

$$\Lambda_\varphi\cdot\frac{\varphi_P^{l+1}-\varphi_P^l}{\Delta t}=\Gamma_x\frac{\varphi_E^l-2\varphi_P^l+\varphi_W^l}{\Delta x^2}+\Gamma_y\frac{\varphi_N^l-2\varphi_P^l+\varphi_S^l}{\Delta y^2}+S_\varphi^l(x_P,y_P)$$

Implicit Scheme:- (when, $\theta=0$)

$$\Lambda_\varphi\frac{\varphi_P^{l+1}-\varphi_P^l}{\Delta t}=\Gamma_x\left(\frac{\varphi_E^{l+1}-2\varphi_P^{l+1}+\varphi_W^{l+1}}{\Delta x^2}\right)+\Gamma_y\left(\frac{\varphi_N^{l+1}-2\varphi_P^{l+1}+\varphi_S^{l+1}}{\Delta y^2}\right)$$
$$+S_\varphi^{l+1}(x_P,y_P)$$

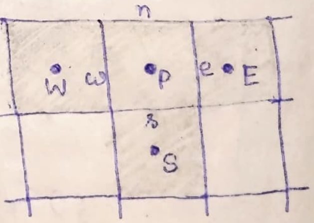
Only $l+1$ level values are there for spatial derivatives.

Crank-Nicolson Scheme ($\theta=\frac{1}{2}$):-
(Combination of l and $l+1$ level values)

$$2\Lambda_\varphi\frac{\varphi_P^{l+1}-\varphi_P^l}{\Delta t}=\Gamma_x\frac{\varphi_E^l-2\varphi_P^l+\varphi_W^l}{\Delta x^2}+\Gamma_y\frac{\varphi_N^l-2\varphi_P^l+\varphi_S^l}{\Delta y^2}$$
$$+\Gamma_x\frac{\varphi_E^{l+1}-2\varphi_P^{l+1}+\varphi_W^{l+1}}{\Delta x^2}+\Gamma_y\frac{\varphi_N^{l+1}-2\varphi_P^{l+1}+\varphi_S^{l+1}}{\Delta y^2}$$
$$+S_\varphi^l(x_P,y_P)+S_\varphi^{l+1}(x_P,y_P)$$

☑ Boundary Condition:-
(Practice)

Neumann Boundary



(i) Top Boundary:-
(Explicit scheme), $\theta=1$

$$\Lambda_\varphi(\varphi_P^{l+1}-\varphi_P^l)\Delta x\Delta y$$

$$=\left[\Gamma_{xe}\frac{\varphi_E^l-\varphi_P^l}{\Delta x}\Delta y-\Gamma_{xw}\frac{\varphi_P^l-\varphi_W^l}{\Delta x}\Delta y\right.$$

$$\left.+\cancel{\Gamma_{yn}\frac{\varphi_N^l-\varphi_P^l}{\Delta y}\Delta x}-\Gamma_{ys}\frac{\varphi_P^l-\varphi_S^l}{\Delta y}\Delta x\right]\Delta t+S_\varphi^l(x_P,y_P)\Delta x\Delta y\Delta t$$

$\to 0$ (NBC)

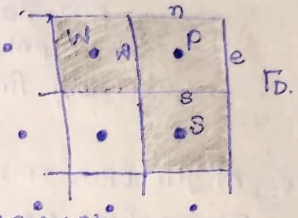
$$\frac{\Lambda_\varphi(\varphi_P^{l+1}-\varphi_P^l)}{\Delta t}=\Gamma_x\frac{\varphi_E^l-2\varphi_P^l+\varphi_W^l}{\Delta x^2}-\Gamma_y\frac{\varphi_P^l-\varphi_S^l}{\Delta y^2}+S_\varphi^l(x_P,y_P)$$

(Dividing both sides with $\Delta x\Delta y\Delta t$
and $\Gamma_{xe}=\Gamma_{xw}=\Gamma_x$, $\Gamma_{yn}=\Gamma_{ys}=\Gamma_y$)

For dirichlet Boundary, we have values specified. But, for Neumann Boundary, we can utilize nodes or cell centers next to our boundary to get information about the boundary itself.

Top right corner:-

Considering ~~explicit~~ implicit scheme:-



$\theta = 0$.

$$\Lambda_\varphi(\varphi_P^{l+1} - \varphi_P^{l})\Delta x \Delta y$$

$$= \Delta t\Big[\Gamma_x\left(\frac{\varphi_e^{l+1}-\varphi_P^{l+1}}{\Delta x/2} - \frac{\varphi_P^{l+1}-\varphi_W^{l+1}}{\Delta x}\right)\Delta y$$

$$+ \Gamma_y\Big(\underbrace{\cancel{\frac{\varphi_n^{l+1}-\varphi_P^{l+1}}{\Delta y/2}}}_{0\,(\text{Neumann})} - \frac{\varphi_P^{l+1}-\varphi_S^{l+1}}{\Delta y}\Big)\Delta x + S_\varphi^{l+1}(x_P, y_P)\,\Delta x \Delta y \Delta t$$

$$\Rightarrow \Lambda_\varphi \frac{\varphi_P^{l+1}-\varphi_P^{l}}{\Delta t} = \Gamma_x \cdot \frac{2\varphi_e^{l+1} - 3\varphi_P^{l+1} + \varphi_W^{l+1}}{\Delta x^2} + \Gamma_y \frac{-\varphi_P^{l+1} + \varphi_S^{l+1}}{\Delta y^2} + S_\varphi^{l+1}(x_P, y_P)\,\Delta x \Delta y \Delta t$$

$$\Rightarrow \Lambda_\varphi \frac{\varphi_P^{l+1}-\varphi_P^{l}}{\Delta t} - \Gamma_x\left(\frac{-3\varphi_P^{l+1}+\varphi_W^{l+1}}{\Delta x^2}\right) - \Gamma_y\left(\frac{-\varphi_P^{l+1}+\varphi_S^{l+1}}{\Delta y^2}\right) = -\Gamma_x \cdot \frac{2\varphi_e^{l+1}}{\Delta x^2} + S_\varphi^{l+1}(x_P, y_P)$$

$$\boxed{\varphi(L_x, y, t) = \varphi_1 \Rightarrow \varphi_e^{l+1} = \varphi_1} \;??$$

FVM

CONSERVATION LAW

1-D Scaler conservation law:

$$\frac{\partial \varphi}{\partial t} + \frac{\partial F_\varphi}{\partial x} = S_\varphi \ldots (6)$$

φ = General scaler variable
F_φ = Flux function
S_φ = Source term.

Conservation law, in terms of vectors:-

$$\Phi_{,t} + \mathcal{F}_{\varphi,x} = S_\varphi \ldots\ldots (7)$$

$$\Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \quad F_\varphi = \begin{bmatrix} F_{\varphi_1} \\ F_{\varphi_2} \\ \vdots \\ F_{\varphi_n} \end{bmatrix} \quad S_\varphi = \begin{bmatrix} S_{\varphi_1} \\ S_{\varphi_2} \\ \vdots \\ S_{\varphi_n} \end{bmatrix}$$

Jacobian matrix:-

$$A(\varphi) = \frac{\partial F_\varphi}{\partial \varphi} = \begin{bmatrix} \frac{\partial F_{\varphi_1}}{\partial \varphi_1} & \frac{\partial F_{\varphi_1}}{\partial \varphi_2} & \cdots & \frac{\partial F_{\varphi_1}}{\partial \varphi_m} \\ \frac{\partial F_{\varphi_2}}{\partial \varphi_1} & \frac{\partial F_{\varphi_2}}{\partial \varphi_2} & \cdots & \frac{\partial F_{\varphi_2}}{\partial \varphi_m} \\ \vdots & & & \vdots \\ \frac{\partial F_{\varphi_m}}{\partial \varphi_1} & \frac{\partial F_{\varphi_m}}{\partial \varphi_2} & \cdots & \frac{\partial F_{\varphi_m}}{\partial \varphi_m} \end{bmatrix}$$

Derivative in 1D case dictates the nature of the solution here.

Non-conservative form:

$$\Phi_{,t} + A(\Phi)\cdot\Phi_{,x} = \hat{S}_\varphi$$ ↗ modified Source/Sink function.

me

$$\boxed{\begin{aligned} &\Rightarrow \frac{\partial \varphi}{\partial t} + \frac{\partial F_\varphi}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x} = \hat{S}_\varphi \\ &\Rightarrow \frac{\partial \varphi}{\partial t} + \frac{\partial F_\varphi}{\partial x} = \hat{S}_\varphi \;\text{(Almost same)!} \end{aligned}}$$

Conservative and non-conservative form of an equation:

C. Form ⇒ It describe a physical quantity that is conserved. For advection eqⁿ:-

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0 \rightarrow$$ Conservation of mass (Rate of change of density at a point in space equals to mass inflow or outflow).

N.C form ⇒ $\frac{\partial \rho}{\partial t} + v\cdot\nabla\rho = 0$

Eqⁿ does not represents a conservation law. It emphasizes transport of density by the velocity field.

Eigenvalues of jacobian matrix:- $|A(\phi) - \lambda I| = 0.$

Eigenvalues provide information regarding speeds of propagation.

☑ **Hyperbolic system:-**

System hyperbolic at a point (x,t) if ⇒

System hyperbolic at a point (x,t) if ⇒ Jacobian matrix A has m real eigenvalues.

☑ Jacobian matrix A has m real eigenvalues.

☑ m right eigenvectors are linearly independent

Strictly Hyperbolic:-

All eigenvalues are distinct in nature.

To find eigenvalues and e-vectors

E. Values:- Let, $A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{bmatrix} = 0$$

gives, $\lambda_1 = 3,\ \lambda_2 = -5,\ \lambda_3 = 6.$

E. vectors: $(A - \lambda I)x = 0$

⇒ For $\lambda = \lambda_1 = 3$

$$\begin{bmatrix} -2-3 & 4 & 2 \\ -2 & 1-3 & 2 \\ 4 & 2 & 5-3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Putting $x = 1$, we get, first two eqⁿs

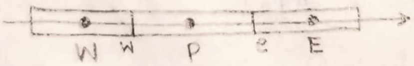
$$-5 + 4y + 2z = 0$$
$$-2 - 2y + 2z = 0$$

Solving we get, $y = -\frac{3}{2},\ z = -\frac{1}{2}$

Eigenvectors ⇒ $(1, -\frac{3}{2}, -\frac{1}{2}) \equiv (2, -3, -1)$

Similarly, get two other E. vectors for $\lambda_2 = -5, \lambda_3 = 6$.

☑ **One dimensional conservation law:-**



G.E

Integrating (6) over element volume and time interval to form discretized equation at node point P,

$$\frac{\partial \phi}{\partial t} + \frac{\partial F_\phi}{\partial x} = S_\phi$$ (Similar at P-160, for 2D).

$$\Rightarrow \int_t^{t+\Delta t} \left[\int_{\Omega_P} \frac{\partial \phi}{\partial t} d\Omega\right] dt + \int_t^{t+\Delta t} \left[\int_{\Omega_P} \frac{\partial F_\phi}{\partial x} d\Omega\right] dt = \int_t^{t+\Delta t} \left[\int_{\Omega_P} S_\phi\, d\Omega\right] dt \quad \ldots\ldots (8)$$

For elemental volume, we are considering the integration

$$\Rightarrow \int_t^{t+\Delta t}\left[\int_{x_w}^{x_e}\frac{\partial\phi}{\partial t}dx\right]dt + \int_t^{t+\Delta t}\left[\int_{x_w}^{x_e}\frac{\partial F_\phi}{\partial x}dx\right]dt = \int_t^{t+\Delta t}\left[\int_{x_w}^{x_e}S_\phi\, dx\right]dt$$

$$\Rightarrow \int_{x_w}^{x_e}\left[\int_t^{t+\Delta t}\frac{\partial\phi}{\partial t}dt\right]dx + \left[\int_t^{t+\Delta t}F_\phi(x_e,t)dt - \int_t^{t+\Delta t}F_\phi(x_w,t)dt\right] = \int_t^{t+\Delta t}\left[\int_{x_w}^{x_e}S_\phi\, dx\right]dt$$

$$\Rightarrow \left[\underbrace{\int_{x_w}^{x_e}\phi(x,t+\Delta t)dx}_{\phi_P^{n+1}} - \underbrace{\int_{x_w}^{x_e}\phi(x,t)dx}_{\phi_P^{n}}\right] + \left[\int_t^{t+\Delta t}F_\phi(x_e,t)dt - \int_t^{t+\Delta t}F_\phi(x_w,t)dt\right] = \int_t^{t+\Delta t}\left[\int_{x_w}^{x_e}S_\phi\, dx\right]dt \quad \cdots (9)$$

Temporal terms are discretized wrt time. Then averaged wrt space (Δx).

Diffusive terms are first discretized wrt space and then averaged wrt time (Δt)

Let us define, $\Delta x = x_w - x_e$

$$\bar{\phi}_P^n = \frac{1}{\Delta x}\int_{x_w}^{x_e}\phi(x,t)dx \quad \cdots (10.1)$$

Numerical flux function can be written as:-

For east face, $\bar{F}_\phi(x_e,t) = \bar{F}_\phi(\phi_P^n, \phi_E^n) = \frac{1}{\Delta t}\int_t^{t+\Delta t}F_\phi(x_e,t)dt \ldots (10.2)$

For west face, and $\bar{F}_\phi(x_w,t) = \bar{F}_\phi(\phi_W^n, \phi_P^n) = \frac{1}{\Delta t}\int_t^{t+\Delta t}F_\phi(x_w,t)dt \; (10.3)$

Final form of discretization

Putting from (10.1), (10.2), (10.3) to (9),

$$(\phi_P^{n+1} - \phi_P^n)\Delta x = -\Delta t\left\{\bar{F}_\phi(\phi_P^n,\phi_E^n) - \bar{F}_\phi(\phi_W^n,\phi_P^n)\right\}$$

We are not considering S_ϕ. Because, discretization of S_ϕ depends on its functional form and more or less it is constant.

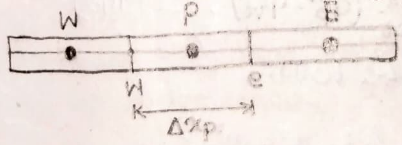
$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x}\left\{\bar{F}_\phi(\phi_P^n,\phi_E^n) - \bar{F}_\phi(\phi_W^n,\phi_P^n)\right\} \cdots (11)$$

(11) is final form. Still we need to calculate the value of flux functions.

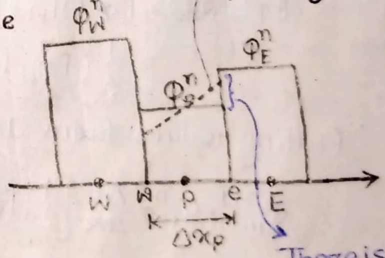
Riemann Problem:- (Conservative form).

$$\frac{\partial\phi}{\partial t} + \frac{\partial F_\phi}{\partial x} = 0 \quad \cdots (12)$$

$$\phi(x,t) = \begin{cases}\phi_P^n & \text{if } x < x_e\\ \phi_E^n & \text{if } x > x_e\end{cases}$$



Actual value may be like this. But, we consider cell-averaged quantity here.



There is a discontinuity in value at the interface

Numerical flux can be calculated by taking arithmatic average of cell-centered values.

$$\bar{F}_\phi(x_e,t) = \bar{F}_\phi(\phi_P^n,\phi_E^n) = \frac{1}{2}\left[F_{\phi P}^n + F_{\phi E}^n\right] \cdots (12.1)$$

Average flux values in both the cells

$$\bar{F}_\phi(x_w,t) = \bar{F}_\phi(\phi_W^n,\phi_P^n) = \frac{1}{2}\left[F_{\phi W}^n + F_{\phi P}^n\right] \cdots (12.2)$$

Final form of discretization {Put 12.1 and 12.2 to (11)},

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{2\Delta x}\left[F_{\phi E}^n - F_{\phi W}^n\right] \cdots (13)$$

Flux term written as,

$$F_\varphi = a\varphi \quad [a = \text{constant}]$$

$$\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{2\Delta x}\left[a\varphi_E^n - a\varphi_W^n\right] \; \text{------} \; (13.1)$$

$$\Rightarrow \varphi_i^{n+1} = \varphi_i^n - \frac{a\Delta t}{2\Delta x}\left[\varphi_{i+1}^n - \varphi_{i-1}^n\right] \; \cdots\cdots \; (13.2)$$

Error equation,

$$\varepsilon_i^{n+1} = \varepsilon_i^n - a\frac{\Delta t}{2\Delta x}\left[\varepsilon_{i+1}^n - \varepsilon_{i-1}^n\right]$$

$$\begin{cases} \varepsilon_i^n = A^n e^{\sqrt{-1}\, i \varphi_x} \\ \varepsilon_{i+1} = A^n e^{\sqrt{-1}(i+1)\varphi_x} \\ \varepsilon_{i-1} = A^n e^{\sqrt{-1}(i-1)\varphi_x} \end{cases}$$

Courant number, $C_r = \frac{a\Delta t}{\Delta x}$

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = 1 - \frac{C_r}{2}\left(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}(\varphi_x)}\right)$$

$$= 1 - \frac{C_r}{2} \times \{2\sin(\varphi_x)\} \times i \qquad \because e^{i\theta} + e^{-i\theta} = 2i\sin\theta$$

$$= 1 - C_r(\sin\varphi_x) \times i$$

$$= 1 - \sqrt{-1}\, C_r \sin\varphi_x$$

Growth factor,

$$|G|^2 = G.G^* = (1 - \sqrt{-1}\, C_r\sin\varphi_x)(1 + \sqrt{-1}\, C_r \sin\varphi_x)$$

Conjugate

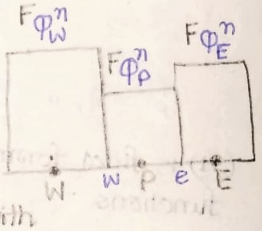
$$= 1 + C_r^2 \sin^2\varphi_x > 1$$

$\therefore$ This term is always positive. Hence, unstable.

This first hand approximation is not appropriate.

Lax-Friedrichs scheme:

me { Flux function এর জন্য হবে নাই। ← $F_{\varphi_W^n}$



$$\bar{F}_\varphi(x_e, t) = \bar{F}_\varphi(\varphi_P^n, \varphi_E^n)$$

$$= \frac{1}{2}\left(F_{\varphi_P}^n + F_{\varphi_E}^n\right) - \underbrace{\frac{\Delta x}{2\Delta t}\left(\varphi_E^n - \varphi_P^n\right)}_{1} \; \cdots\cdots \; (13.4.1)$$

Extra term is added. (compare with equation 12.1, 12.2).

$$\bar{F}_\varphi(x_w, t) = \bar{F}_\varphi(\varphi_W^n, \varphi_P^n)$$

$$= \frac{1}{2}\left(F_{\varphi_W}^n + F_{\varphi_P}^n\right) - \frac{\Delta x}{2\Delta t}\left(\varphi_P^n - \varphi_W^n\right) \; \cdots \; (14.2)$$

??

Putting values from 14.1 and 14.2 to (11),

$$\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x}\left\{\frac{1}{2}\left(F_{\varphi_E}^n - F_{\varphi_W}^n\right) - \frac{\Delta x}{2\Delta t}\left(\varphi_E^n + \varphi_W^n\right)\right\}$$

$$\Rightarrow \varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{2\Delta x}\left(F_{\varphi_E}^n - F_{\varphi_W}^n\right) + \frac{1}{2}\left(\varphi_E^n + \varphi_W^n\right) \; \text{---------} \; (15)$$

In i format,

$$\varphi_i^{n+1} = \varphi_i^n - \frac{\Delta t}{2\Delta x}\left(a\varphi_E^n - a\varphi_W^n\right) + \frac{1}{2}\left(\varphi_{i+1}^n + \varphi_{i-1}^n\right) \qquad [\because F_\varphi = a\varphi \text{ as previously assumed}]$$

$$\varphi_i^{n+1} = \varphi_i^n - \frac{a\Delta t}{2\Delta x}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) + \frac{1}{2}\left(\varphi_{i+1}^n + \varphi_{i-1}^n\right) \; \cdots\cdots \; (15.1)$$

L.F. Scheme: Numerical Diffusion:

Actual equation:- $\frac{\partial\varphi}{\partial t} + \frac{\partial F_\varphi}{\partial x} = 0$

Modified eqn:- $\frac{\partial\varphi}{\partial t} + \frac{\partial F_\varphi}{\partial x} = \beta\frac{\partial^2\varphi}{\partial x^2}$ → Virtually, we are adding some extra second order diffusion term for stabilization purpose.

?? Where, $\beta = \frac{\Delta x^2}{2\Delta t}$

For west face, diffusive flux, $F_\varphi(\varphi_W^n, \varphi_P^n)|_D = -\beta \dfrac{\varphi_P^n - \varphi_W^n}{\Delta x}$??

Error equation,

$$\varepsilon_i^{n+1} = \frac{1}{2}(\varepsilon_{i-1}^n + \varepsilon_{i+1}^n) - a\frac{\Delta t}{2\Delta x}(\varepsilon_{i+1}^n + \varepsilon_{i-1}^n)$$

? not Same as (15.1)

$$\varepsilon_i^n = A^n e^{\sqrt{-1}\, i\varphi(x)}$$

$$\varepsilon_{i+1}^n = A^n e^{\sqrt{-1}(i+1)\varphi(x)}$$

$$\varepsilon_{i-1}^n = A^n e^{\sqrt{-1}(i-1)\varphi(x)}$$

Now, $G = \dfrac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \dfrac{1}{2}\left\{\dfrac{\varepsilon_{i-1}^n}{\varepsilon_i^n} + \dfrac{\varepsilon_{i+1}^n}{\varepsilon_i^n}\right\} - \dfrac{C_r}{2}\left\{\dfrac{\varepsilon_{i+1}^n}{\varepsilon_i^n} + \dfrac{\varepsilon_{i-1}^n}{\varepsilon_i^n}\right\}$

$$= \frac{1}{2}\left\{\frac{A^n e^{\sqrt{-1}(i-1)\varphi(x)}}{A^n e^{\sqrt{-1}\, i\varphi(x)}} + \frac{A^n e^{\sqrt{-1}(i+1)\varphi(x)}}{A^n e^{\sqrt{-1}\, i\varphi(x)}}\right\} - \frac{C_r}{2}\left(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi(x)}\right)$$

$$= \frac{1}{2}\left\{e^{-\sqrt{-1}\varphi(x)} + e^{\sqrt{-1}\varphi(x)}\right\} - \frac{C_r}{2}\left(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x}\right)$$

$$= \cos\varphi_x - \frac{C_r}{2} \times 2\sqrt{-1}\sin\varphi_x$$

$$= \cos\varphi_x - \sqrt{-1}\, C_r \sin\varphi_x$$

Now, $|G|^2 = G \times G^*$

$$= (\cos\varphi_x - \sqrt{-1}\, C_r\sin\varphi_x).(\cos\varphi_x + \sqrt{-1}\, C_r\sin\varphi_x)$$

$$= \cos^2\varphi_x + C_r^2\sin^2\varphi_x$$

$$= 1 - \sin^2\varphi_x + C_r^2\sin^2\varphi_x$$

$$= 1 - (1 - C_r^2)\sin^2\varphi_x$$ → For extreme case, $\sin^2\varphi_x = 1$

$\Rightarrow 1 - (1 - C_r^2)1 = C_r^2$

The scheme is stable if $C_r < 1$

So, proper approximation for Numerical flux function is important

Upwind Approach

NPTEL: Lec 17

We cannot use arbitrary approximation for the numerical flux function. If we used averaged value of flux upto adjacent cells for the interface, we are getting unstable schemes. So, adding some extra term or virtual term in the eqn gives stability.

Objective:- To discretize conservation laws using upwind method.

$$\frac{\partial\varphi}{\partial t} + \frac{\partial F_\varphi}{\partial x} = S_\varphi$$

Approximation of flux for linear case:-

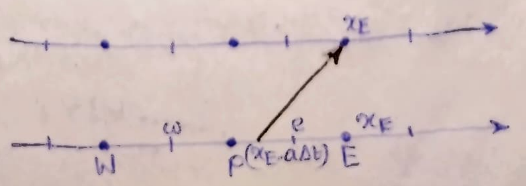
$$F_\varphi = a\varphi \quad \ldots \ldots (16)$$

Cell averaged value can be approximated as:-

$$\varphi_P^{n+1} \approx \varphi(x_P, t+\Delta t) = \varphi(x_P - a\Delta t, t) \quad \ldots (16.1)$$

$\varphi_P^n \approx \varphi(x_P, t)$

$$\varphi_E^{n+1} \approx \varphi(x_P, t+\Delta t) = \varphi(x_E - a\Delta t, t) \quad \ldots (16.2)$$



Future time level is represented as convex combination of nodal val

when $a>0$.

?

$$\varphi_P^{n+1} = \frac{a\Delta t}{\Delta x}\varphi_W^n + \left(1-\frac{a\Delta t}{\Delta x}\right)\varphi_P^n$$

$$\varphi_E^{n+1} = \frac{a\Delta t}{\Delta x}\varphi_P^n + \left(1-\frac{a\Delta t}{\Delta x}\right)\varphi_E^n$$

Summation of the weight functions = 1

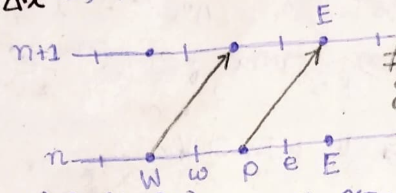
Rearranging, we get,

$$\boxed{\begin{aligned}\varphi_P^{n+1} &= \varphi_P^n - \frac{a\Delta t}{\Delta x}(\varphi_P^n - \varphi_W^n) \quad \cdots\cdots (17.1)\\ \varphi_E^{n+1} &= \varphi_E^n - \frac{a\Delta t}{\Delta x}(\varphi_E^n - \varphi_P^n) \quad \cdots\cdots (17.2)\end{aligned}}$$

CFL condition for $a>0$,

$0 \leq \frac{a\Delta t}{\Delta x} \leq 1$ (Derivation from finite difference approximation for stability criteria)

☑ If $\frac{a\Delta t}{\Delta x}=1$, $\Rightarrow a\Delta t = \Delta x$



Information directly got transferred from point P (n^{th} level) to point E ($(n+1)^{th}$ level).

$(x_P - a\Delta t = x_P - \Delta x = x_W)$ $(x_E - a\Delta t = x_E - \Delta x = x_P)$

me

We know, information from nth level at point $(x_E - a\Delta t)$ travels to x_E of $(n+1)^{th}$ level.

That means $\varphi_E^{n+1} = \varphi(x_E - a\Delta t, t)$

$$\varphi(x_E, t+\Delta t) = \varphi(x_E - a\Delta t, t)$$

$$\varphi^{n+1}(x_E) = \varphi^n(x_E - a\Delta t)$$

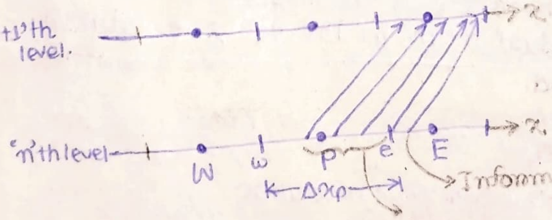
Similarly, if we put $\frac{a\Delta t}{\Delta x}=1$ in (17.1) and (17.2)

$$\varphi_P^{n+1} = \varphi_W^n$$

$$\varphi_E^{n+1} = \varphi_P^n$$

. Upwind method

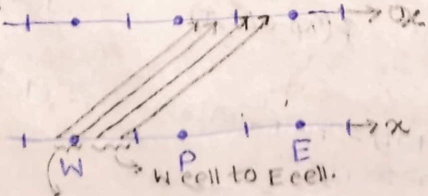
(Time stepping): $(a>0)$



Information travelling from E cell to E cell

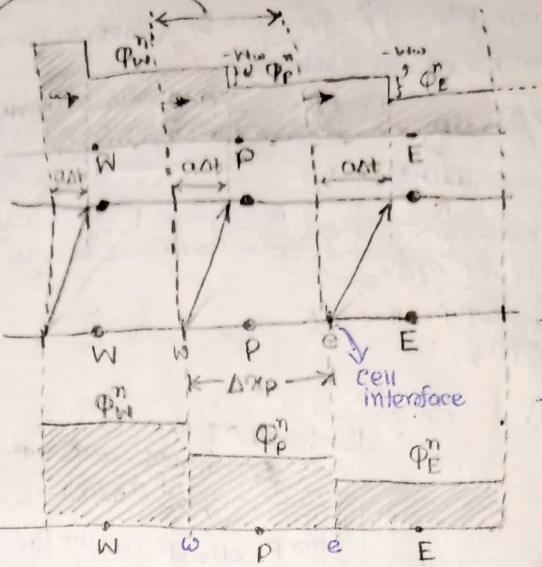
Information travelling from P cell to E cell

If we increase the time steps (This depends on CFL condition. i.e $\frac{a\Delta t}{\Delta x}$ value)



W cell to E cell.

W cell to P cell



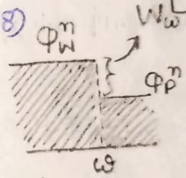
Information from the cell interface is getting transfered to the next cell.
and we are getting the average value.

| W_w | = - W_w
Because, W_w is itself -ve in this case.

☑ Upwind method:- Wave propagation point of view:-

- at face w, Jump, $\boxed{W_w = \varphi_P^n - \varphi_W^n}$ ....(18)

  (conceptualized as wave moving into P cell at velocity a)

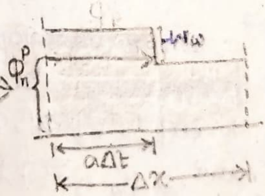


- Wave modifies the value of φ by $-W_w$ at each point.
- Wave velocity = a

  Distance covered by wave at each time step = $a\Delta t$
  $= \frac{a\Delta t}{\Delta x}$ fraction of a cell.

$$\boxed{\varphi_P^{n+1} = \varphi_P^n - \frac{a\Delta t}{\Delta x} W_w} \quad \text{.....(19)}$$

Explanation at figure



Total area inside the cell:-
$= -W_w \cdot a\Delta t + (\varphi_P^n \cancel{-W_w})\Delta x$

Average value of φ at (n+1)th level =
$$\frac{-W_w \cdot a\Delta t + (\varphi_P^n \cancel{-W_w})\Delta x}{\Delta x}$$
$$= -\frac{a\Delta t}{\Delta x} W_w + \varphi_P^n \cancel{-W_w}$$
$$= \varphi_P^n \cancel{-W_w} - \left(\frac{a\Delta t}{\Delta x}\right) W_w$$

Putting the value of W_w from (18) to (19),

$$\varphi_P^{n+1} = \varphi_P^n - \frac{a\Delta t}{\Delta x}(\varphi_P^n - \varphi_W^n)$$
$$= \varphi_P^n - \frac{\Delta t}{\Delta x}(a\varphi_P^n - a\varphi_W^n) \quad \text{[From (16)]}$$
$$= \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{F}_\varphi(\varphi_P^n, \varphi_E^n) - \bar{F}_\varphi(\varphi_W^n, \varphi_P^n)\right]$$

যা থাকবে সেটিকে ও তার সামনের cell থেকে নিতে হবে। (when a>0)

For w face, information is travelling from W to P

a>0
$a\varphi_P^n = \bar{F}_\varphi(\varphi_P^n, \varphi_E^n)$
$a\varphi_W^n = \bar{F}_\varphi(\varphi_W^n, \varphi_P^n)$ ....(20.1)

Information is travelling from W to P and P to E.

Similarly, if a<0,

$$a\varphi_P^n = \bar{F}_\varphi(\varphi_W^n, \varphi_P^n)$$
$$a\varphi_E^n = \bar{F}_\varphi(\varphi_P^n, \varphi_E^n) \quad \text{......(20.2)}$$

Because information travels from (W ← P) to from and (P ← E) to from for a<0

At face e, the jump $W_e = \varphi_E^n - \varphi_P^n$ can be conceptualized as wave moving from P cell at a velocity a.

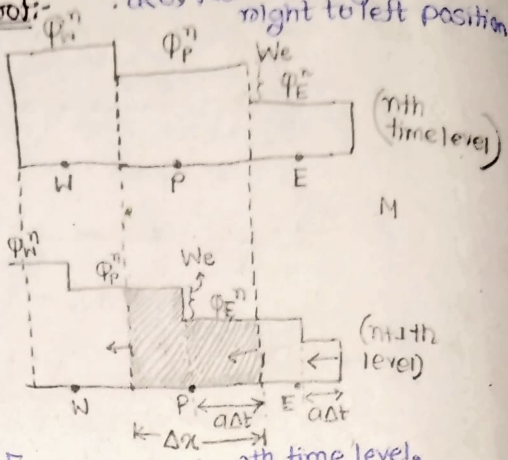
$$\boxed{\varphi_P^{n+1} = \varphi_P^n - \frac{a\Delta t}{\Delta x} W_e} \quad \cdots (20.3)$$

(Me) # যখন -ve দিকে a travell করছে তখন φ_E^n থেকে P cell এ ঢুকে যাচ্ছে।

$$\text{me}\left\{ \varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x}(a\varphi_E^n - a\varphi_P^n) \right. \cdots (20.4)$$

me

Proof:- ∵ a<0, wave moving from right to left position.



For P cell, at $(n+1)^{th}$ time level,

$$\text{Area} = (\varphi_P^n - W_e)a\Delta t + \varphi_P^n(\Delta x - a\Delta t)$$
$$= \varphi_P^n a\Delta t - W_e a\Delta t + \varphi_P^n \Delta x - \varphi_P^n a\Delta t$$

Average value,

$$\varphi_P^{n+1} = \varphi_P^n - W_e \cdot \frac{a\Delta t}{\Delta x} \quad \left(\frac{\text{Area}}{\Delta x}, \text{ হবে যাই}\right)$$

CFL condition for a<0:

If we donot consider the absolute value of a,

$$-1 \leq \frac{a\Delta t}{\Delta x} \leq 0$$

[Opposite to original criteria $0 \leq \frac{a\Delta t}{\Delta x} \leq 1$

In both cases, combined form can be written as:- $0 \leq \frac{|a|\Delta t}{\Delta x} \leq 1$

General form of numerical flux can be written as:-

$$\boxed{\begin{aligned} F_\varphi(\varphi_P^n, \varphi_E^n) &= a^- \varphi_E^n + a^+ \varphi_P^n \\ F_\varphi(\varphi_W^n, \varphi_P^n) &= a^- \varphi_P^n + a^+ \varphi_W^n \end{aligned}}$$

where, $a^+ = \max(a, 0)$

$a^- = \min(a, 0)$

...... (21.1) and (21.2)

→ Interpretation:-

When $a>0$, $a^+ = a$, $a^- = 0$

$\therefore a\varphi_P^n = F_\varphi(\varphi_P^n, \varphi_E^n)$

and $a\varphi_W^n = F(\varphi_W^n, \varphi_P^n)$

When $a<0$,

$a^+ = 0$, $a^- = a$

$a\varphi_E^n = F_\varphi(\varphi_P^n, \varphi_E^n)$

$a\varphi_P^n = F_\varphi(\varphi_W^n, \varphi_P^n)$

These gives the explanation for eqn (20.1) & (20.2)

The final form,

$$\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x}\left[a^-(\varphi_E^n - \varphi_P^n) + a^+(\varphi_P^n - \varphi_W^n)\right] \cdots\cdots (22)$$

If $a>0$,

$$\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x} a(\varphi_P^n - \varphi_W^n) \Rightarrow \text{See (17.1) and (19)}$$

If $a<0$,

$$\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x} a(\varphi_E^n - \varphi_P^n) \Rightarrow \text{Same as (20.3) and (20.4).}$$

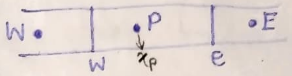
Module 2, Unit 15 (NPTEL, lec-19)

Finite Volume Method:
Higher resolution methods:-

Objective: To discretize conservation laws using higher resolution methods

G.E :- $\frac{\partial \varphi}{\partial t} + \frac{\partial F_\varphi}{\partial x} = S_\varphi$ Here, $\varphi(x,t)$

1D general cell.



Piecewise linear form of cell average value φ_P^n can be used as:-

"Concept of piecewise reconstruction"

$$\boxed{\tilde{\varphi}^n(x,t^n) = \varphi_P^n + \sigma_P^n(x-x_P)} \quad \forall x \in [x_w, x_e) \quad \cdots\cdots\cdots (23)$$

('φ tilde' depends on x and present time level.)

(cell centered value)

Means the approximation is valid within 'P' cell only.

We now,

flux term, $F_\varphi = a\varphi$

Solution for future time level,

$$\tilde{\varphi}(x,t^{n+1}) = \tilde{\varphi}(x-a\Delta t, t^n)$$

Numerical flux function can be written as:-

Numerical flux function at east face.

$$\bar{F}_\varphi(x_e,t)^n = \frac{1}{\Delta t}\int_t^{t+\Delta t} F_\varphi(x_e,t)\,dt$$

$$\bar{F}_\varphi(x_w,t)^n = \frac{1}{\Delta t}\int_t^{t+\Delta t} F_\varphi(x_w,t)\,dt$$

When $a>0$,

Numerical flux function for east face can be written as:-

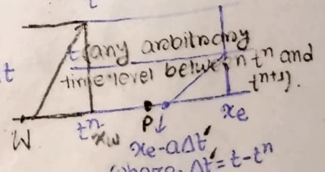
$$\bar{F}_\varphi(x_e,t^n) = \frac{1}{\Delta t}\int_t^{t+\Delta t} F_\varphi(x_e,t)\,dt$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\varphi}(x_e,t)\,dt$$

In case of east face, flow of information is from P cell. Piecewise approximation is in the form of x_p (eqn 23)

Similarly, (see next page), for west face, flow of information is from W cell. So, piecewise approximation is in the form of x_W.

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\varphi}(\underbrace{x_e-a(t-t^n)}_{x}, \underbrace{t^n}_{t^n})\,dt$$



(Here, information is moving from t^n to t level)

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a[\varphi_P^n + \varphi_P^n\, x_e - a(t-t^n)$$

$$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a[\varphi_P^n + \sigma_P^n(x_e-a(t-t^n)-x_P)]\,dt$$

According to (23)

$$\boxed{\text{i.e } \tilde{\varphi}^n(x,t^n) = \varphi_P^n + \sigma_P^n(x-x_P)}$$

Here, $x = x_e - a(t-t^n)$

me

$$= \cancel{\text{a}} \frac{1}{\Delta t}\, a[\varphi_P^n + \sigma_P^n(x_e - a(t-t^n) - x_P)]\,\Delta t$$

$$= a\varphi_P^n + a\sigma_P^n[x_e - x_P - a(t-t^n)]$$

?

$$= a\varphi_P^n + a\sigma_P^n\left[\frac{\Delta x}{2} - \frac{a\Delta t}{2}\right]$$

$= \left[\because x_e - x_P = \frac{\Delta x}{2} \text{ and } t-t^n = \frac{\Delta t}{2}\right]$

I think, t is middle at t^n and t^{n+1}

$= a\varphi_P^n + \frac{a\sigma_P^n}{2}(\Delta x - a\Delta t)$

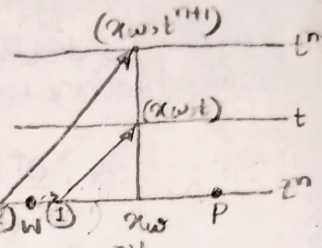
Numerical flux for west face,

$\bar{F}_\varphi(x_w, t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} F_\varphi(x_w, t)\,dt$

$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\tilde{\varphi}(x_w, t)\,dt$

$= \cancel{\frac{1}{\Delta t}\int_t^{t+\Delta t}}$

t is any arbitrary time level between t^n and t^{n+1}. Used to evaluate the integration.



$t^{n+1} - t^n = \Delta t$

$t - t^n = \Delta t'$

me $\left\{ \begin{array}{l} ① = x_w - a\Delta t' = \{x_w - a(t - t^n)\} \\ ② = (x_w - a\Delta t) \\ \therefore\ \tilde{\varphi}(x_w - a\Delta t', t^n) = \tilde{\varphi}(x_w, t) \\ \text{similarly, } \tilde{\varphi}(x_w - a\Delta t, t^n) = \tilde{\varphi}(x_w, t^{n+1}) \end{array} \right.$

$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\,\tilde{\varphi}(x_w - a\Delta t', t^n)\,dt$

$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\,\tilde{\varphi}(x_w - a(t - t^n), t^n)\,dt$

$= \cancel{\frac{1}{\Delta t}\int_t^{t+\Delta t} a[\varphi_P^n + \sigma_P^n[x_w - a(t - t^n) - x_P]]\,dt}$

$\tilde{\varphi}(x, t^n) = \varphi_W^n + \sigma_W^n(x - x_W)$

$\forall x \in [x_w, x_e]$

$= \frac{1}{\Delta t}\int_t^{t+\Delta t} a\,[\varphi_W^n + \sigma_W^n(x_w - a(t - t^n) - x_W)]\,dt$ → Equation 23 is modified to this one for the west face.

me $\left\{ = \frac{1}{\Delta t}\, a\left[\varphi_W^n + \sigma_W^n(x_w - x_W) - \sigma_W^n . a . \frac{\Delta t}{2}\right]\Delta t \right.$

→ For integration, Average value for $t - t^n$ $= \frac{(t^n - t^n) + (t^{n+1} - t^n)}{2} = \frac{\Delta t}{2}$ } Me

$= a\varphi_W^n + \sigma_W^n a\frac{\Delta x}{2} - \sigma_W^n . a^2 \frac{\Delta t}{2}$

$= a\varphi_W^n + \sigma_W^n \frac{a}{2}(\Delta x - a\Delta t).$

To discretize conservation law using Godunov approach:

1-D scalar conservation law, $\frac{\partial \phi}{\partial t} + \frac{\partial F_\phi}{\partial x} = S_\phi$

Flux term, $F_\phi = a\phi$

Basic structure of Gudonov method is **Reconstruct-Evolve-Average** algorithm.

? Step1:- we reconstruct a piecewise polynomial from cell average value ϕ_P^n, as,

$$\boxed{\tilde{\phi}^n(x, t^n) = \phi_P^n} \quad \forall x \in [x_w, x_e) \quad \text{.......} (24)$$

(Note difference from 23)

Step2: Evolve the hyperbolic equation with base condition/initial condition to obtain $\tilde{\phi}^{n+1}(x, t+\Delta t)$ at future time level $t+\Delta t$.

Step 3: Average the polynomial function at cell level to obtain cell average value at future time $t+\Delta t$

$$\boxed{\phi_P^{n+1} = \frac{1}{\Delta x}\int_{x_w}^{x_e} \tilde{\phi}(x, t^{n+1})\,dx}$$

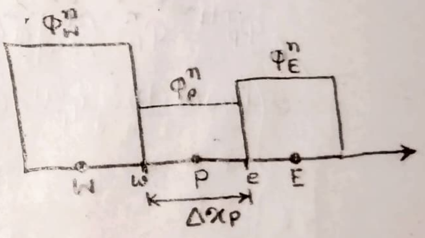
Steps are repeated at every time level.

$\tilde{\phi}(x, t^n)$ is constant over each time interval $t^n < t < t^{n+1}$.

Riemann problem: (eq^n 12)

$$\frac{\partial \phi}{\partial t} + \frac{\partial F_\phi}{\partial x} = 0$$

$$\phi(x,t) = \begin{cases} \phi_P^n & \text{if } x < x_e \\ \phi_E^n & \text{if } x > x_e. \end{cases}$$

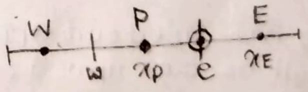


Flux function $F_\phi(\tilde{\phi}(x,t))$ at cell face depends on the exact solution $\tilde{\phi}(x,t)$ of the riemann problem along t axis. Considering local co-ordinate:-

Riemann Problem for east face,

$$\boxed{\tilde{\phi}(x,t) = \phi_e\left(\frac{x - x_e}{t - t^n}\right)}, \quad x_P \le x \le x_E, \quad t^n \le t \le t^{n+1} \quad \text{........} (25)$$

where, $\xi = \left(\frac{x - x_e}{t - t^n}\right)$



For the west face, Riemann problem,

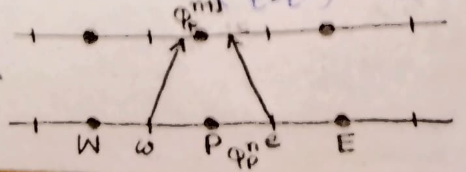
$$\tilde{\phi}(x,t) = \phi_w\left(\frac{x - x_w}{t - t^n}\right), \quad x_w \le x \le x_P \quad t^n \le t \le t^{n+1} \quad \text{......} (26)$$

From Riemann's problem, (eq^n 26 and 25),

$$\tilde{\phi}(x_w, t) = \phi_w\left(\frac{x_w - x_w}{t - t^n}\right) = \phi_w(0) \quad \text{with } t^n \le t \le t^{n+1} \quad \text{.....} (27.1)$$

$$\tilde{\phi}(x_e, t) = \phi_e\left(\frac{x_e - x_e}{t - t^n}\right) = \phi_e(0), \quad \text{with } t^n \le t \le t^{n+1} \quad \text{.....} (27.2)$$

? !!



Characteristics lines should not intersect each other.

Numerical flux values can be written as:-

$$\bar{F}_\varphi(x_e,t) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F_\varphi(\tilde{\varphi}(x_e,t))\,dt$$

Fixed, Variable

(10·1),

$$\bar{F}_\varphi(x_e,t) = \bar{F}_\varphi(\varphi_P^n, \varphi_E^n) = \frac{1}{\Delta t}\int_t^{t+\Delta t} F_\varphi(x_e,t)\,dt$$

Average numerical flux at east face

$$\overset{?}{=} \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F_\varphi(\varphi_e(0)) \quad \text{\# (From 27·2)}$$

Eq(11) and Eq(12·1),

$$\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{F}_\varphi(x_e,t) - \bar{F}_\varphi(x_w,t)\right]$$

$$= F_\varphi(\varphi_e(0)) \quad \text{------ (28·1)}$$

$$\bar{F}_\varphi(x_w,t) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F_\varphi(\tilde{\varphi}(x_w,t))\,dt$$

flux for each piecewise values?? } me

$$= \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F_\varphi(\varphi_w(0))\,dt \quad \text{(\# From equation 27·1)}$$

$$= F_\varphi(\varphi_w(0)) \quad \text{------ (28·2)}$$

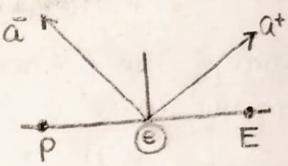
If $F_\varphi = a\varphi$, numerical flux can be written as:-

$$F_\varphi(\varphi_e(0)) = \bar{a}\varphi_E^n + a^+\varphi_P^n \quad \text{------ (29·1)}$$

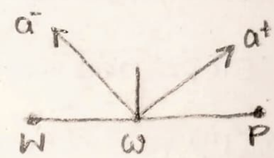
$$F_\varphi(\varphi_w(0)) = \bar{a}\varphi_P^n + a^+\varphi_W^n \quad \text{------ (29·2)}$$

where, $a^+ = \max(a, 0)$ $\quad \bar{a} = \min(a, 0)$

Same as upwind approach.



me
$$F_\varphi(\varphi_e(0)) = \bar{F}_\varphi(x_e,t) \quad \text{... (From 28·1)}$$
$$= \bar{F}_\varphi(\varphi_P^n, \varphi_E^n) \quad \text{.... (From 12·1)}$$
$$= \bar{a}\varphi_E^n + a^+\varphi_P^n \quad \text{.... (From 20·1)}$$



$$\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{a}(\varphi_E^n - \varphi_P^n) + a^+(\varphi_P^n - \varphi_W^n)\right] \quad \text{----- (30)}$$

me

From (11), $\varphi_P^{n+1} = \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{F}_\varphi(\varphi_P^n, \varphi_E^n) - \bar{F}_\varphi(\varphi_W^n, \varphi_P^n)\right]$

$$= \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{F}_\varphi(x_e,t) - \bar{F}_\varphi(x_w,t)\right] \quad \text{...... (From 12·1)}$$

$$= \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{F}_\varphi(\varphi_e(0)) - \bar{F}_\varphi(\varphi_w(0))\right] \quad \text{.... (From 28·1 \& 28·2)}$$

$$= \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{a}\varphi_E^n + a^+\varphi_P^n - \bar{a}\varphi_P^n - a^+\varphi_W^n\right] \quad \text{.... (From 29·1 and 29·2)}$$

$$= \varphi_P^n - \frac{\Delta t}{\Delta x}\left[\bar{a}(\varphi_E^n - \varphi_P^n) + a^+(\varphi_P^n - \varphi_W^n)\right]$$

We have used our approximation. But, getting similar result as our upwind approach (1st order upwind approach)

∴ Gudonov method in basic form is essentially 1st order upwind approach... (see eqn (22))

Module 2, Unit-20

NPTEL-24, LEC-21

Matrix structure and Scilab:

☑ To identify kind of matrix structures generated from discretization.

$A\varphi = r$

const. coff ↙ ↘ variable vector.

if $A(\varphi).\varphi = r$

This form is non-linear. Because, we are multiplying coff's with variable vector.

$$\underbrace{\begin{pmatrix} a_{11}(\varphi) & \cdots & a_{1N}(\varphi) \\ \vdots & \ddots & \vdots \\ a_{N1}(\varphi) & \cdots & a_{NN}(\varphi) \end{pmatrix}}_{N\times N} \underbrace{\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{N-1} \\ \varphi_N \end{Bmatrix}}_{N\times 1} = \underbrace{\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{Bmatrix}}_{N\times 1}$$

Compatibility should be there in terms of matrix dimension.

For linear
- → Direct approach (Gauss elimination, LU decomposition)
- → Indirect/Iterative approach (Jacobi's method, Gauss Siedel

For non-linear form,

Take, $A\varphi - r = F$

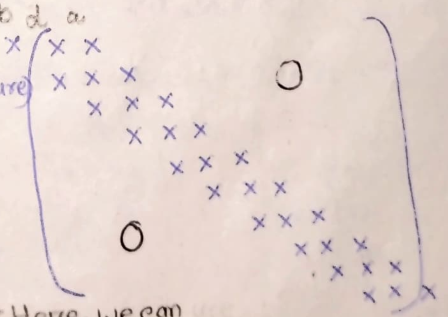
where, in vector form, $F = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$ → Taylor series / → Optimization approach.

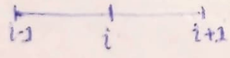
One non-linear equation converts the whole system to non-linear system.

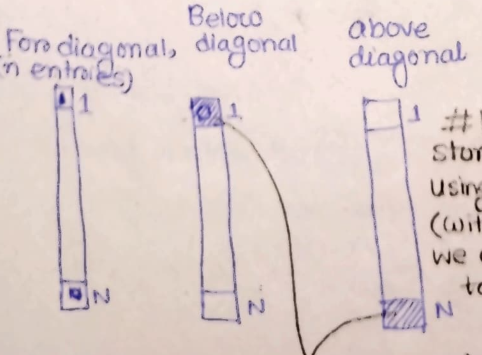
☑ when support domain size = size of the domain.
We should consider all the points.
Structure of A → having all the entries (non-zero)

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

☑ Matrix form: Banded matrix
(Tri-diagonal structure)



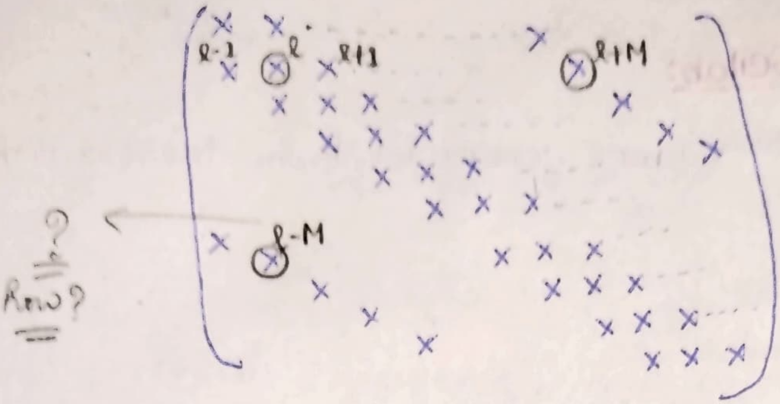


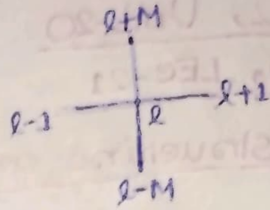


Here, we can store matrix information using only three column vector. (with only 3N number of entries we can represent same information to the calculation process)

You can put zero ('0') here But, during calculation process, we are not using that.

Penta diagonal structure:

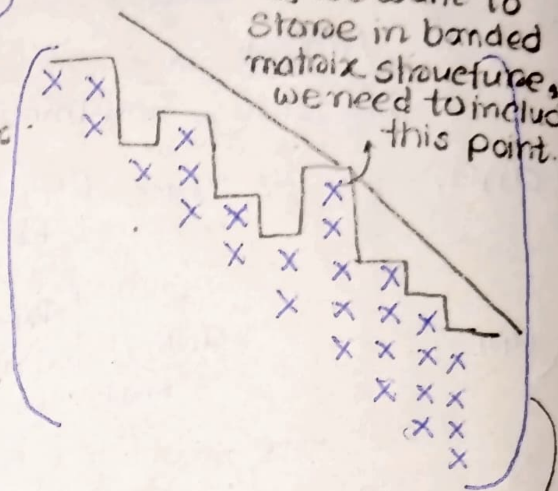




Skyline matrix structure:

If the points are symmetric in nature, or information that is available for the lower portion of diagonal is symmetric in nature, we can represent our information with skyline structure.

If we want to store in banded matrix structure, we need to include this point.



Depending on requirement of algorithm, we can store in different small column vector.

Console,
Semote
Clc → clear the console
Clear → clear variable related information.

Compare two values '=='
Different or not equal <>

x=rand(10,1)
y=rand(10,1)
plot(x,y) → show ~~points with~~ connected lines
plot(x,y,'o') → shows only point ∘∘∘ → Like this
plot(x,y,'*')
Polot(x,y,'o-') → shows points and lines.

Module 2, Unit-21:-

Algebraic equation:- Gauss-Elimination method

NPTEL-25.

For direct solution.

$$A\phi = r$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & & & & \vdots \\ a_{31} & & & & \vdots \\ a_{41} & & & & \\ a_{51} & \cdots & & & a_{55} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

Row 2:-

Let, $\gamma_1^2 = \frac{a_{21}}{a_{11}}$

$\gamma_1^2 \rightarrow$ Gamma is a factor multiplied with row 1 for the calculation of row 2.

Row 2:- $a_{21}\phi_1 + a_{22}\phi_2 + \cdots\cdots + a_{25}\phi_5 = r_2$

$\gamma_1^2 \times$ Row 1:- $a_{21}\phi_1 + \gamma_1^2 a_{12}\phi_2 + \cdots\cdots + \gamma_1^2 a_{15}\phi_5 = \gamma_1^2 r_1$

Updated row 2:- $a_{22}'\phi_2 + \cdots\cdots + a_{25}'\phi_5 = r_2'$

$$a_{22}' = a_{22} - \gamma_1^2 a_{12}$$
$$a_{23}' = a_{23} - \gamma_1^2 a_{13}$$
$$a_{24}' = a_{24} - \gamma_1^2 a_{14}$$
$$a_{25}' = a_{25} - \gamma_1^2 a_{15}$$
$$r_2' = r_2 - \gamma_1^2 r_1. \qquad \text{---------- (31)}$$

Similarly, updating row 3, 4, 5,

$$\text{Row 3} \dashrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22}' & a_{23}' & a_{24}' & a_{25}' \\ 0 & a_{32}' & a_{33}' & a_{34}' & a_{35}' \\ 0 & a_{42}' & a_{43}' & a_{44}' & a_{45}' \\ 0 & a_{52}' & a_{53}' & a_{54}' & a_{55}' \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2' \\ r_3' \\ r_4' \\ r_5' \end{Bmatrix}$$

Now, we need to eliminate these terms also.

Let, $\gamma_2^3 = \frac{a_{32}'}{a_{22}'}$

Row 3:- $a_{32}'\phi_2 + a_{33}'\phi_3 + a_{34}'\phi_4 + a_{35}'\phi_5 = r_3'$

$\gamma_2^3 \times$ Row 2:- $a_{32}'\phi_2 + \gamma_2^3 a_{23}'\phi_3 + \gamma_2^3 a_{24}'\phi_4 + \gamma_2^3 a_{25}'\phi_5 = r_2'\gamma_2^3$

Updated row 3:- $a_{33}''\phi_3 + a_{34}''\phi_4 + a_{35}''\phi_5 = r_3''$

↳ Results $\rightarrow$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22}' & a_{23}' & a_{24}' & a_{25}' \\ 0 & 0 & a_{33}'' & a_{34}'' & a_{35}'' \\ 0 & a_{42}' & a_{43}' & a_{44}' & a_{45}' \\ 0 & \cdots & & & a_{55} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \vdots \\ \phi_5 \end{pmatrix} = \begin{Bmatrix} r_1 \\ r_2' \\ r_3'' \\ r_4 \\ r_5 \end{Bmatrix}$$

Doing similar to Row 4 and row 5,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22}' & a_{23}' & a_{24}' & a_{25}' \\ 0 & 0 & a_{33}'' & a_{34}'' & a_{35}'' \\ 0 & 0 & a_{43}'' & a_{44}'' & a_{45}'' \\ 0 & 0 & a_{53}'' & a_{54}'' & a_{55}'' \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2' \\ r_3'' \\ r_4'' \\ r_5'' \end{Bmatrix}$$

Similarly:-

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a''''_{55} \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r''''_5 \end{Bmatrix}$$

So, now $2 \to 3 \to 4 \to 5$. Called forward elimination

Now substitute $5 \to 4 \to 3 \to 2 \to 1$, backward substitution.

Back substitution:-

$$\phi_5 = \frac{r''''_5}{a''''_{55}} = \frac{r^{IV}_5}{a^{IV}_{55}}$$

$$\phi_4 = \frac{1}{a'''_{44}}(r'''_4 - a'''_{45}\phi_5)$$

$$\phi_3 = \frac{1}{a''_{33}}(r''_3 - a''_{34}\phi_4 - a''_{35}\phi_5)$$

$$\phi_2 = \frac{1}{a'_{22}}(r'_2 - a'_{23}\phi_3 - a'_{24}\phi_4 - a'_{25}\phi_5) \quad \text{-----} \quad (32)$$

(Page-184, Youtube)

☐ Algorithm:-

Data: Matrix A, Vector r

Result: ϕ

Forward elimination:-

```
for k=1:N-1 do
    for i=k+1,n do
        γ = a_i,k/a_k,k
        for j=k+1,n do
            a_ij = a_ij - γ.a_kj
        end
        r_i = r_i - γ.r_k
    end
end.
```

Let, n=3

For k=1,

$k=1,\ i=2,\ \gamma = a_{21}/a_{11}$
$j=2,\ a_{22} = a_{22} - \gamma a_{12}$
$r_2 = r_2 - \gamma r_1$

$k=1,\ i=2,\ \gamma = a_{21}/a_{11}$
$j=3,\ a_{23} = a_{23} - \gamma . a_{13},\ r_2 = r_2 - \gamma r_1$

$k=1,\ i=3,\ \gamma = a_{31}/a_{11}$
$j=2,\ a_{32} = a_{32} - \gamma . a_{12}$
$r_3 = r_3 - \gamma r_1$

$k=1,\ i=3,\ \gamma = a_{31}/a_{11}$
$j=3,\ a_{33} = a_{33} - \gamma . a_{13}$

$k=2,\ i=3,\ \gamma = a_{32}/a_{22}$
$j=3,\ a_{33} = a_{33} - \gamma a_{23}$

Back Substitution:

```
φ_n = r_n/a_n,n.
for i=n-1,-1,1 do
    Sum = r_i
    for j=i+1,n do
        Sum = Sum - a_ij . φ_j
    end
    φ_i = Sum / a_i,i
end
return φ
```

Consider $n=3$,

Forward elimination

$k=1$	"	"	"	$k=2$
$i=2$	"	$i=3$	"	$i=3$
$r=a_{21}/a_{11}$	"	$r=a_{31}/a_{11}$	"	$r=a_{32}/a_{22}$
$j=2$	$j=3$	$j=2$	$j=3$	$j=3$
$a_{22}=a_{22}-ra_{12}$	$a_{23}=a_{23}-ra_{13}$	$a_{32}=a_{32}-ra_{12}$	$a_{33}=a_{33}-ra_{13}$	$a_{33}=a_{33}-ra_{23}$
$r_2=r_2-rr_1$	"	$r_3=r_3-rr_1$	"	$r_3=r_3-rr_2$

Backward Substitution:-

$\phi_3 = r_3/a_{33}$;

$i=2$	$i=1$
$sum=r_2$	$sum=r_1$
$j=3$	$j=2$
$Sum=r_2-a_{23}\phi_3$	$Sum=r_1-a_{12}\phi_2$
$\phi_2=\dfrac{r_2-a_{23}\phi_3}{a_{22}}$	$\phi_1=\dfrac{r_1-a_{12}\phi_2}{a_{11}}$.

Problems for gauss elimination:

Problems
- If first term is zero (i.e $a_{11}=0$), 2nd row cannot be multiplied by a_{21}/a_{11}.
- Round of error
- Ill-conditioned system:- If we change a particular variable with a slight value, if big change occurs in the system, called ill-conditioned system.

Solutions: (i) **Pivoting:** We can reorder the rows of a matrix in such a way that largest element in a given column (known as pivot element) is placed in the diagonal position of that column. It allows stable and accurate computation during inversion, multiplication and solution

Example :- $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \xrightarrow[\text{to}]{\text{Changes}} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

(ii) **Scaling:**

Divide a particular row/column/diagonal/all element with a large value to maintain the order of equation.

Row scaling ⇒ To adjust the relative importance of different variables

Column " ⇒ Data preprocessing → to ensure different columns has similar magnitude and range.

Diagonal " ⇒ Preserve eigenvalues, determinant.

Uniform " ⇒ Adjust overall magnitude of matrix.

Module 2, Unit-22. (NPTEL-26.

Algebraic Equation, LU Decomposition Method

Direct method:-

$$A\phi = r$$

Decomposition:- $A = LU$ $\Rightarrow$ $LU\phi = r \Rightarrow L(U\phi) = r$ ..... (33)

$\Rightarrow L\psi = r$

Forward substitution:- $L\psi = r$

Backward substitution:- $\psi = U\phi$

Overall calculation is represented as:-

$$L(U\phi - \psi) = LU\phi - L\psi$$
$$= A\phi - r$$

Explanation using an example:- (Youtube ⇒ Gajendra Purohit)

$x+5y+z=14$
$2x+y+3z=13$
$3x+y+4z=17$

In AX=B form,

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{Bmatrix} 14 \\ 13 \\ 17 \end{Bmatrix}$$

$A = LU$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$u_{11}=1$ $\quad$ $u_{12}=5$ $\quad$ $u_{13}=1$

$l_{21}u_{11} = 2$ $\quad$ $l_{21}u_{12}+u_{22}=1$ $\quad$ $l_{21}u_{13}+u_{23}=3$

$\Rightarrow l_{21}\times 1=2$ $\quad$ $\Rightarrow 2\times 5+u_{22}=1$ $\quad$ $\Rightarrow 2\times 1+u_{23}=3$

$\Rightarrow l_{21}=2$ $\quad$ $\Rightarrow u_{22}=-9.$ $\quad$ $\Rightarrow u_{23}=1.$

$l_{31}u_{11}=3$ $\quad$ $l_{31}u_{12}+l_{32}u_{22}=1$ $\quad$ $l_{31}u_{13}+l_{32}u_{23}+u_{33}=4$

$\Rightarrow l_{31}=3$ $\quad$ $\Rightarrow 3\times 5+l_{32}\times(-9)=1$ $\quad$ $\Rightarrow 3\times 1+\frac{14}{9}\times 1+u_{33}=4$

$\Rightarrow l_{32}=\frac{14}{9}$ $\quad$ $\Rightarrow u_{33}=1-\frac{14}{9}=-\frac{5}{9}$

$$\therefore A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix}$$

$\therefore AX=B$

$LUX=B$ $\quad$ let, $UX=Y$

$LY=B$

From,

$$LY=B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 14 \\ 13 \\ 17 \end{Bmatrix}$$ ⇒ Forward substitution

$$\Rightarrow \begin{Bmatrix} y_1 \\ 2y_1+y_2 \\ 3y_1+\frac{14}{9}y_2+y_3 \end{Bmatrix} = \begin{Bmatrix} 14 \\ 13 \\ 17 \end{Bmatrix}$$

$y_1 = 14$

$2\times 14 + y_2 = 13$

$\Rightarrow y_2 = -15$

$3\times 14 + \frac{14}{9}\times(-15) + y_3 = 17$

$\Rightarrow y_3 = -5/3$

Now, we have, $UX = Y$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 14 \\ -15 \\ -5/3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} x+5y+z \\ -9y+z \\ -\frac{5}{9}z \end{Bmatrix} = \begin{Bmatrix} 14 \\ -15 \\ -5/3 \end{Bmatrix}$$

$-\frac{5}{9}z = -\frac{5}{3}$ $\Rightarrow z = 3$

$-9y + 3 = -15$ $\Rightarrow -9y = -18$ $\Rightarrow y = 2$

$x + 5\times 2 + 3 = 14$ $\Rightarrow x = 1$

$\therefore$ Solution, $\{x\} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

Back to the video.

$A = LU$

Matrix generated from forward elimination process,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ 0 & 0 & a''_{33} & a''_{34} & a''_{35} \\ 0 & 0 & 0 & a'''_{44} & a'''_{45} \\ 0 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} = \begin{pmatrix} r_1 \\ r'_2 \\ r''_3 \\ r'''_4 \\ r^{IV}_5 \end{pmatrix}$$

In the first step, $\gamma_1^2, \gamma_1^3, \gamma_1^4, \gamma_1^5$ were multiplied with row 1 for row 2,3,4,5 respectively.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \gamma_1^2 & a'_{22} & & & a'_{25} \\ \gamma_1^3 & 0 & a''_{33} & & a''_{35} \\ \gamma_1^4 & 0 & 0 & a'''_{44} & a''_{45} \\ \gamma_1^5 & 0 & 0 & 0 & a^{IV}_{55} \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \gamma_1^2 & a'_{22} & a'_{23} & a'_{24} & a'_{25} \\ \gamma_1^3 & \gamma_2^3 & a''_{33} & a''_{34} & a''_{35} \\ \gamma_1^4 & \gamma_2^4 & \gamma_3^4 & a'''_{44} & a'''_{45} \\ \gamma_1^5 & \gamma_2^5 & \gamma_3^5 & \gamma_4^5 & a^{IV}_{55} \end{pmatrix}$$

$A = LU$

$$U = \begin{pmatrix} a_{11} & \cdots & & & a_{15} \\ & a'_{22} & & & a'_{25} \\ & & a''_{33} & & a''_{35} \\ & 0 & & a'''_{44} & a'''_{45} \\ & & & & a^{IV}_{55} \end{pmatrix} \qquad L = \begin{pmatrix} 1 & & & & 0 \\ \gamma_1^2 & 1 & & & \\ \gamma_1^3 & \gamma_2^3 & 1 & & \\ \gamma_1^4 & \gamma_2^4 & \gamma_3^4 & 1 & \\ \gamma_1^5 & \gamma_2^5 & \gamma_3^5 & \gamma_4^5 & 1 \end{pmatrix}$$

Forward substitution:- $L\Psi = r$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

$\psi_1 = r_1 \cdots\cdots (34.1)$

$l_{21}\psi_1 + \psi_2 = r_2$

$\Rightarrow \psi_2 = r_2 - l_{21}\psi_1 \cdots (34.2)$

$l_{31}\psi_1 + l_{32}\psi_2 + \psi_3 = r_3$

$\Rightarrow \psi_3 = r_3 - (l_{31}\psi_1 + l_{32}\psi_2) \cdots\cdots (34.3)$

$\psi_4 = r_4 - (l_{41}\psi_1 + l_{42}\psi_2 + l_{43}\psi_3) \cdots\cdots (34.4)$

$\psi_5 = r_5 - (l_{51}\psi_1 + l_{52}\psi_2 + l_{53}\psi_3 + l_{54}\psi_4) \cdots\cdots (34.5)$

General algorithm:-

$$\psi_1 = r_1$$

$$\psi_i = r_i - \sum_{j=1}^{i-1} a_{ij}\psi_j \cdots\cdots (34)$$

In general form, (other than $n=1$)

$\psi_1 = r_1,\quad \psi_i = r_i - \sum_{j=1}^{n-1} a_{ij}\psi_j$

$\forall i \in \{2, 3, \ldots\ldots N\}$

It goes upto $(n-1)$

Backward Substitution:- $U\phi = \psi$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} \\ 0 & 0 & u_{33} & u_{34} & u_{35} \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{Bmatrix}$$

$$\phi_N = \frac{\psi_N}{a_{NN}}$$

$$\phi_i = \frac{1}{a_{ii}}\left[\psi_i - \sum_{j=i+1}^{N} a_{ij}\phi_j\right], \quad \forall i \in \{N-1, N-2, \ldots 1\} \cdots\cdots (35)$$

$a_{55}\phi_5 = \psi_5$

$\boxed{a \text{ or } u}$ considered same here.

$\Rightarrow \phi_5 = \frac{\psi_5}{a_{55}} \cdots (35.1)$

$a_{44}\phi_4 + a_{45}\phi_5 = \psi_4$

$\Rightarrow \phi_4 = \frac{1}{a_{44}}(\psi_4 - a_{45}\phi_5) \cdots\cdots (35.2)$

$a_{33}\phi_3 + a_{34}\phi_4 + a_{35}\phi_5 = \psi_3$

$\Rightarrow \phi_3 = \frac{1}{a_{33}}(\psi_3 - (a_{34}\phi_4 + a_{35}\phi_5)) \cdots (35.3)$

$a_{22}\phi_2 + a_{23}\phi_3 + a_{24}\phi_4 + a_{25}\phi_5 = \psi_2$

$\Rightarrow \phi_2 = \frac{1}{a_{22}}\left[\psi_2 - (a_{23}\phi_3 + a_{24}\phi_4 + a_{25}\phi_5)\right] \cdots (35.4)$

$\phi_1 = \frac{1}{a_{11}}(\psi_1 - (a_{12}\phi_2 + a_{13}\phi_3 + a_{14}\phi_4 + a_{15}\phi_5)) \cdots (35.5)$

So, $\phi_i = \frac{1}{a_{ii}}\left[\psi_i - \sum_{j=i+1}^{N} a_{ij}\phi_j\right]$ established.

Data: Matrix A, Vector r

Result q

Decomposition:-

for $k=1, N-1$ do

for $i=k+1, n$ do

$\gamma = a_{i,k}/a_{k,k}$

$a_{ik} = \gamma$

for $j=k+1, n$ do

$a_{i,j} = a_{i,j} - \gamma . a_{k,j}$

end

end

end.

Forward substitution:-

$K=1$

$i=2$

$\gamma = A(2,1)/A(1,1) = \frac{a_{21}}{a_{11}}$

$a_{21} = \frac{a_{21}}{a_{11}}$

$j=2$

$A(2,2) = A(2,2) - \gamma . A(1,2)$

$\Rightarrow a_{22} = a_{22} - \gamma . a_{12}$

$n=4$ A → coff matrix

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{matrix}$$

$$Aug = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & r_1 \\ a_{21} & a_{22} & a_{23} & r_2 \\ a_{31} & a_{32} & a_{33} & r_3 \end{array}\right]$$

lets a be a 4x4 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$B = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$j=1$
$i=2$
$m = Aug(2,1)/Aug(1,1)$
$= a_{21}/a_{11}$
$Aug(2,:) = Aug(2,:) - m * Aug(1,:)$
⇓
Result:- 2nd row,
$[0 \; a'_{22} \; a'_{23} \; r'_2]$ → হল

$j=1$
$i=3$
$m = Aug(3,1)/Aug(1,1)$
$= a_{31}/a_{11}$
$Aug(3,:) = Aug(3,:) - m * Aug(1,:)$
⇓
3rd row,
$[0 \; a'_{32} \; a'_{33} \; r'_3]$ হল.

$j=2$
$i=3$
$m = Aug(3,2)/Aug(2,2)$
$= a_{32}/a_{22}$
$Aug(3,:) = Aug(3,:) - m * Aug(2,:)$
⇓
3rd row
$[0 \; 0 \; a''_{33} \; r''_3]$ করা হল

complete Augmented matrix is prepared now.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & r_1 \\ 0 & a'_{22} & a'_{23} & r'_2 \\ 0 & 0 & a''_{33} & r''_3 \end{bmatrix}$$

$x(3) = Aug(3,4)/Aug(3,3)$
$= r'''_3/a'''_{33}$.

For loop:

$K=2$
$x(2) = \{Aug(2,4) - Aug(2,3)\}/Aug(2,2) = (r'_2 - a'_{23})\frac{1}{a'_{22}}$

$K=1$
$x(1) = \{Aug(1,4) - Aug(1,2) - Aug(1,3)\}/Aug(1,1)$
$= (r_1 - a_{12} - a_{13})/a_{11}$

for K=

$x(K) \to$ sumt(K)

4x4 matrix

x_3 → sumterm(3) = a_{34}
sumterm(2) = $a_{23} + a_{24}$
sumterm(1) = $a_{12} + a_{13} + a_{14}$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & r_1 \\ & a'_{22} & a'_{23} & a'_{24} & r'_2 \\ & & a''_{33} & a''_{34} & r''_3 \\ & & & a'''_{44} & r'''_4 \end{matrix}$$

K=N-1

K=3
i=4
Sumterm(4) = 0 + a_{34}

K=2
i=3
Sumterm(2) = 0 + $a_{23} + a_{24}$

K=3, Sum(3) = Aug(3,4)
K=2, Sum(2) = Aug(2,4) + Aug(2,3)

K=1:N Let, n=5

K=1,

U(1, 1:5)= A(1, 1:5) − L(1, 1:

LU decomposition method

Gauss elimination ⇒ computational cost $O(n^3)$ → need $3^3=27$ operations

LU Decomposition ⇒ " " $O(n^2)$ → Need $3^2=9$ operations

If 3x3 matrix,

LU decomposition preferred.

A=LU

LUX=B.

$\boxed{\begin{matrix} UX=Y \\ LY=B. \end{matrix}}$ → Need to solve this system.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cancel{l_{11}}^{1} & 0 & 0 \\ l_{21} & \cancel{l_{22}}^{1} & 0 \\ l_{31} & l_{32} & \cancel{l_{33}}^{1} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Dollitle's method → set diagonals of L matrix as 1.

Crout's method

Cholesky's method.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cancel{l_{11}}u_{11} & \cancel{l_{11}u_{12}}\ u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} \quad \text{......(36)}$$

Code:

1st part

```
A= input('Enter your coff matrix:');
N= length(A);  B=input('Enter source vector?');
L=zeros(N,N);
U=zeros(N,N);
for a=1:N
    L(a,a)=1; #all the diagonal terms of L should be 1.
end
```

1st row of U and 1st column of L are defined

```
U(1,:)=A(1,:)    # u11=a11, u12=a12, u13=a13.
L(:,1)= A(:,1)/U(1,1)  #(l11=a11/u11, l21=a21/u11, l31=a31/u11)
```

$l_{11} = \frac{a_{11}}{u_{11}} = 1$

L(:,1) → 1st column; A(:,1) → 1st column; U(1,1) → u_{11}

Now, use nested loops to evaluate remaining rows of U and Columns of L.

$$U = \begin{bmatrix} u_{11}=a_{11} & u_{12}=a_{12} & u_{13}=a_{13} \\ 0 & u_{22}=a_{22}-l_{21}u_{12} & u_{23}=a_{23}-l_{21}u_{13} \\ 0 & 0 & u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23} \end{bmatrix}$$

$$l = \begin{bmatrix} l_{11}=\frac{a_{11}}{u_{11}}=\frac{a_{11}}{a_{11}}=1 & 0 & 0 \\ l_{21}=\frac{a_{21}}{u_{11}} & 1 & 0 \\ l_{31}=\frac{a_{31}}{u_{11}} & l_{32}=\frac{1}{u_{22}}(a_{32}-l_{31}u_{12}) & 1 \end{bmatrix}$$

2nd part

Evaluate L(except

```
for i=2:N
    for j=i:N
        U(i,j)= A(i,j) - L(i,1:i-1)*U(1:i-1,j);
    end

    for K=i+1:N
        L(K,i)=A(K,i)- L(K,1:i-1)*U(1:i-1,i)/U(i,i) ;
    end
end
L,U
```

3rd part

Y

```
Y=zeros
Y(1)= B(1)/L(1,1);      LY=B.
for k=2:N
    y(K)=(B(K) - L(K,1:K-1)*
              Y(1:K-1))/L(K,K)
end
Y
```

Matlab itself performs the summation

$LY=B$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Forward Substitution

$\boxed{y_1 = b_1}$

$l_{21}y_1 + y_2 = b_2$

$\Rightarrow \boxed{y_2 = b_2 - l_{21}y_1}$

$l_{31}y_1 + l_{32}y_2 + y_3 = b_3$

$\Rightarrow \boxed{y_3 = b_3 - l_{31}y_1 - l_{32}y_2}$

4th part

X

```
X=zeros(N,1)
x(N)=y(N)/U(N,N)
for K=N-1,-1:1,
    x(K)=(yK- U(K,K+1:N)*
              X(K+1:N))/U(K,K)
end
X
```

$UX=Y$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Backward Substitution

$u_{33}x_3 = y_3$

$\Rightarrow \boxed{x_3 = y_3/u_{33}}$

$u_{22}x_2 + u_{23}x_3 = y_2$

$\Rightarrow \boxed{x_2 = \frac{1}{u_{22}}(y_2 - u_{23}x_3)}$

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$

$\Rightarrow \boxed{x_1 = \frac{1}{u_{11}}(y_1 - u_{12}x_2 - u_{13}x_3)}$

For 2nd part:

$i=2$
$j=2$
$u(2,2) = A(2,2) - L(2,1) * U(1,2)$
$= a_{22} - l_{21} u_{12}$

$i=3$
$j=3$
$u(3,3) = A(3,3) - L(3,1) \times U(1,3) - L(3,2) \times U(2,3)$
$= \{a_{33} - l_{31} \times u_{13} - l_{32} u_{23}\} / u_{33}$

$K=3$,
$L(3,2) = (A(3,2) - L(3,1) * U(1,2)) / U(2,2)$
$= (a_{32} - l_{31} u_{12}) / u_{22}$

Consider a 4x4 matrix,

$\boxed{A\varphi = r}$ → Main equation:- $AX = B$.

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$A = LU$
$LUX = B$
$LY = B$
$UX = Y$

$A = LU$
$\cancel{LU\varphi = r}$
$\cancel{U\varphi = Y}$
$\cancel{LY = r}$

$$= \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$A = \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} & l_{11}u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} & l_{21}u_{14} + l_{22}u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} & l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44}u_{44} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} = \frac{a_{11}}{l_{11}} & u_{12} = \frac{a_{12}}{l_{11}} & u_{13} = \frac{a_{13}}{l_{11}} & u_{14} = \frac{a_{14}}{l_{11}} \\ 0 & u_{22} = \frac{a_{22} - l_{21}u_{12}}{l_{22}} & u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} & u_{24} = \frac{a_{24} - l_{21}u_{14}}{l_{22}} \\ 0 & 0 & u_{33} = \frac{a_{33} - l_{31}u_{13} - l_{32}u_{23}}{l_{33}} & u_{34} = \frac{a_{34} - l_{31}u_{14} - l_{32}u_{24}}{l_{33}} \\ 0 & 0 & 0 & u_{44} = \frac{a_{44} - l_{41}u_{14} - l_{42}u_{24} - l_{43}u_{34}}{l_{44}} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} = \frac{a_{11}}{u_{11}} = 1 & 0 & 0 & 0 \\ l_{21} = \frac{a_{21}}{u_{11}} & 1 & 0 & 0 \\ l_{31} = \frac{a_{31}}{u_{11}} & l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} & 1 & 0 \\ l_{41} = \frac{a_{41}}{u_{11}} & l_{42} = \frac{a_{42} - l_{41}u_{12}}{u_{22}} & l_{43} = \frac{a_{43} - l_{41}u_{13} - l_{42}u_{23}}{u_{33}} & 1 \end{bmatrix}$$

$LY = B$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} \frac{b_1}{l_{11}} \\ \frac{1}{l_{22}}(b_2 - l_{21}y_1) \\ \frac{1}{l_{33}}(b_3 - l_{31}y_1 - l_{32}y_2) \\ \frac{1}{l_{44}}(b_4 - l_{41}y_1 - l_{42}y_2 - l_{43}y_3) \end{Bmatrix}$$

Now, $UX = Y$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{u_{11}}(y_1 - u_{12}x_2 - u_{13}x_3 - u_{14}x_4) \\ \frac{1}{u_{22}}(y_2 - u_{23}x_3 - u_{24}x_4) \\ \frac{1}{u_{33}}(y_3 - x_4 u_{34}) \\ \frac{y_4}{u_{44}} \end{Bmatrix}$$

length ⇒ length of longest array dimension.
vector ⇒ Number of elements.

$A\phi = R$

$x =$ phi

$y =$ psi

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$x_1 = \frac{1}{a_{11}}(r_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4)$

-Aug i

Tridiagonal matrix method:

NPTEL-27 Class-24

Dummy value to Keep the diagonal consistent.

One extra point

$A\phi = r$

$$\begin{pmatrix} (x) & x & x & & & \\ x & x & x & & 0 & \\ & x & x & x & & \\ & & x & x & x & \\ & 0 & & x & x & x \\ & & & & x & x \end{pmatrix}_{N\times N} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{Bmatrix}_{N\times 1} = \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{Bmatrix}_{N\times 1}$$

diagonal term

d

a above diagonal

b below diagonal

A matrix into 3 different column vector.

Try to reduce the storage requirement.

$$b_1 \begin{bmatrix} d_1 & a_1 & & & \\ b_2 & d_2 & a_2 & & \\ 0 & b_3 & d_3 & a_3 & \\ & & b_4 & d_4 & a_4 \\ & & & b_5 & d_5 \end{bmatrix} a_5 \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_5 \end{Bmatrix}$$

$b_1 = 0$ $a_5 = 0$

Row 1: $d_1\phi_1 + a_1\phi_2 = r_1$.

$\Rightarrow \phi_1 + \frac{a_1}{d_1}\phi_2 = \frac{r_1}{d_1}$.

$\Rightarrow \boxed{\phi_1 + \xi_1\phi_2 = \rho_1.}$

where, $\xi_1 = \frac{a_1}{d_1}$, $\rho_1 = \frac{r_1}{d_1}$

Row-2:-

$b_2 \times$ row1

$\Rightarrow b_2 \times (\phi_1 + \xi_1\phi_2 = \rho_1)$

In gauss elimination, we have unchanged or undisturbed row1. But, in tridiagonal algorithm, same forward elimination is used but, multiplying with disturbed one.

Row2: $b_2\phi_1 + d_2\phi_2 + a_2\phi_3 = r_2$

$b_2\times$row1: $b_2\phi_1 + b_2\xi_1\phi_2 = b_2\rho_1$

Updated row2:- $(d_2 - b_2\xi_1)\phi_2 + a_2\phi_3 = r_2 - b_2\rho_1$.

$\Rightarrow \phi_2 + \frac{a_2}{d_2 - b_2\xi_1}\cdot\phi_3 = \frac{r_2 - b_2\rho_1}{d_2 - b_2\xi_1}$.

Our objective is to convert the diagonal term with value = 1 as coff.

and we are storing this thing.

If this value is too small, it will introduce numerical error to the system.

$\xi_2 = \frac{a_2}{d_2 - b_2\xi_1}$ $\rho_2 = \frac{r_2 - b_2\rho_1}{d_2 - b_2\xi_1}$

$\therefore \boxed{\phi_2 + \xi_2\phi_3 = \rho_2}$

For r vector we can store this ρ_2 value directly.

We will not store ξ. Because, our objective is to reduce the storage. We will utilize our A vector for storage of ξ_2 directly row 3:-

$b_3\phi_1 + \phi_2 d_3 + \phi_3 a_3 =$

$0\times\phi_1 + b_3\phi_2 + d_3\phi_3 + a_3\phi_4 = r_3$

row 3 $\Rightarrow$ $b_3\phi_2 + d_3\phi_3 + a_3\phi_4 = r_3$

$b_3\times$row2 $\Rightarrow$ $b_3\phi_2 + b_3\xi_2\phi_3 \quad = b_3\rho_2 \quad b_3(\phi_2 + \xi_2\phi_3 = \rho_2)$

updated row3 $\Rightarrow$ $(d_3 - b_3\xi_2)\phi_3 + a_3\phi_4 = r_3 - b_3\rho_2$

$$\Rightarrow \phi_3 + \underbrace{\frac{a_3}{d_3 - b_3\xi_2}}_{\xi_3}\phi_4 = \underbrace{\frac{r_3 - b_3\rho_2}{d_3 - b_3\xi_2}}_{\rho_3}$$

Rewriting, $\boxed{\phi_3 + \xi_3\phi_4 = \rho_3.}$

row-4:-

row4: $b_4\phi_3 + d_4\phi_4 + a_4\phi_5 = r_4$

$b_4\times$row3: $b_4\phi_3 \pm b_4\xi_3\phi_4 \quad = b_4\rho_3$

Updated row4: $(d_4 - b_4\xi_3)\phi_4 + a_4\phi_5 = r_4 - b_4\rho_3$

$$\Rightarrow \phi_4 + \underbrace{\frac{a_4}{d_4 - b_4\xi_3}}_{\xi_4}\phi_5 = \underbrace{\frac{r_4 - b_4\rho_3}{d_4 - b_4\xi_3}}_{\rho_4}$$

$\therefore \boxed{\phi_4 + \xi_4\phi_5 = \rho_4}$

Row-5: row5: $b_5\phi_4 + d_5\phi_5 \quad = r_5$

$b_5\times$row4: $b_5\phi_4 + b_5\xi_4\phi_5 = b_5\rho_4$

Updated row5: $(d_5 - b_5\xi_4)\phi_5 = (r_5 - b_5\rho_4)$

$$\Rightarrow \phi_5 = \left(\frac{r_5 - b_5\rho_4}{d_5 - b_5\xi_4}\right)$$

$$\Rightarrow \phi_5 + \xi_5 0 = \underbrace{\frac{r_5 - b_5\rho_4}{d_5 - b_5\xi_4}}_{\rho_5}$$

$\Rightarrow \boxed{\phi_5 = \rho_5}$

$\therefore$ updated matrix structure,

$$\begin{pmatrix} 1 & \xi_1 & 0 & 0 & 0 \\ 0 & 1 & \xi_2 & 0 & 0 \\ 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 1 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{Bmatrix}$$

Backward substitution:- $\phi_5 = \rho_5$

$\phi_4 = \rho_4 - \xi_4\phi_5$

So, similarly we get, other terms ϕ_3, ϕ_2, ϕ_1 by backward substitution.

Algorithm:

Data: Vector b, d, a, r

Result: ϕ.

$\underbrace{b, d, g}$ 3N vector 3N×1 · $\underset{N\times 1}{\phi}$ $\underset{N\times 1}{r}$.

(See code in video)

Forward elimination:-

$a_1 = a_1/d_1$

$r_1 = r_1/d_1$.

for $i = 2, N-1$ do → for nth row, term in coff matrix = 1.

$fact = d_i - b_i \cdot a_{i-1}$.

$a_i = a_i/fact$

$r_i = (r_i - b_i \cdot r_{i-1})/fact$.

end

$r_n = (r_n - b_n \cdot r_{n-1})/(d_n - b_n \cdot a_{n-1})$ → at the end point, we are explicitly calculating r_n. Because, a_n has already 0 value. (a_n, matrix এর বাইরে আছে).

Backward substitution:

$\phi_n = r_n$

for $i = n-1, -1, 1$ do

$\phi_i = \underbrace{r_i - a_i}_{\text{This is storing the } \xi \text{ values.}} \cdot \underbrace{\phi_{i+1}}$

end

return ϕ

→ Because, only upper diagonal is available. Lower diagonal we eliminated.

Example:

Module 2, Unit 24

Jacobi's method:

NPTEL 28. Class-25

(Gauss elimination, LU Decomposition, Tri-diagonal matrix algorithm) ↳ Direct methods

Iterative technique.
Starting from a guess value.

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}_{N\times N} \begin{Bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{Bmatrix}_{N\times 1} = \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{Bmatrix}_{N\times 1}.$$

In L-U decomposition,

$A = LU \Rightarrow$ Essentially valid for multiplication.

For Jacobi's method,

$$\boxed{A = L + D + U} \quad \cdots (1)$$

Strictly lower, Diagonal matrix, Strictly upper

$\therefore (L+D+U)\varphi = r$

Iterative form is written as:-

$$\boxed{D\varphi^{(P)} + (L+U)\varphi^{(P-1)} = r} \quad \cdots (2)$$

⇒ [Here, D, L, U, r are constant. every iteration φ is varying].

This value will be evaluated for the guess value. i.e $p=0$ level.

Previous iteration value
P → Iteration counter
Guess level ⇒ P=0
So, valid for $P \geq 1$.

In direct approach, we changed A matrix into different forms. Here, this is not changed, only divided.

$$\Rightarrow D\varphi^{(P)} + (L+U)\varphi^{(P-1)} = r$$

$$\Rightarrow D\varphi^{(P)} = -(L+U)\varphi^{(P-1)} + r$$

$$\Rightarrow \boxed{\varphi^{(P)} = -D^{-1}(L+U)\varphi^{(P-1)} + D^{-1}r} \quad \cdots (3)$$

where, P = iteration counter.

Iteration starts Guess value $\varphi^{(0)}$.

Example:-

$$\begin{pmatrix} a_{11} & \cdots & a_{15} \\ \vdots & & \vdots \\ a_{51} & \cdots & a_{55} \end{pmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_5 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_5 \end{Bmatrix}$$

A = L+D+U

$$L = \begin{pmatrix} a_{21} & & & 0 \\ a_{31} & a_{32} & & \\ a_{41} & a_{42} & a_{43} & \\ a_{51} & a_{52} & a_{53} & a_{54} \end{pmatrix} \quad D = \begin{pmatrix} a_{11} & & & & 0 \\ & a_{22} & & & \\ & & a_{33} & & \\ 0 & & & a_{44} & \\ & & & & a_{55} \end{pmatrix} \quad U = \begin{pmatrix} a_{21} & a_{22} & a_{2} \\ a_{12} & a_{13} & a_{14} \\ & b_{23} & b_{24} \\ & & a_{34} \end{pmatrix}$$

Iteration starts from guess value $\varphi^{(0)}$

$$\varphi^{(0)} = \begin{Bmatrix} \varphi_1^{(0)} \\ \varphi_2^{(0)} \\ \varphi_3^{(0)} \\ \varphi_4^{(0)} \\ \varphi_5^{(0)} \end{Bmatrix}$$

Inverse of $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 1/a_{11} & 0 & 0 \\ 0 & 1/a_{22} & 0 \\ 0 & 0 & 1/a_{33} \end{pmatrix}$

Expanding eq^n(3),

$$\varphi^{(p)} = -D^{-1}(L+U)\varphi^{(p-1)} + D^{-1}r.$$

For 1st iteration, $p=1$, $p-1=0$.

$$\varphi^{(1)} = -D^{-1}(L+U)\varphi^{(0)} + D^{-1}r.$$

$$\Rightarrow \begin{Bmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \\ \varphi_3^{(1)} \\ \varphi_4^{(1)} \\ \varphi_5^{(1)} \end{Bmatrix} = -\begin{pmatrix} 1/a_{11} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{33}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a_{44}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a_{55}} \end{pmatrix} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{pmatrix} \begin{Bmatrix} \varphi_1^{(0)} \\ \varphi_2^{(0)} \\ \varphi_3^{(0)} \\ \varphi_4^{(0)} \\ \varphi_5^{(0)} \end{Bmatrix}$$

$$+ \begin{pmatrix} \frac{1}{a_{11}} & & & & \\ & \frac{1}{a_{22}} & & 0 & \\ & & \frac{1}{a_{33}} & & \\ & 0 & & \frac{1}{a_{44}} & \\ & & & & \frac{1}{a_{55}} \end{pmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

$\cancel{\varphi_1^{(1)} = -\frac{1}{a_{11}}(a_{12}+a_{13}+a_{14}+a_{15})}$

$$\begin{Bmatrix} \varphi_1^{(1)} \\ \vdots \\ \varphi_5^{(1)} \end{Bmatrix} = -\begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{a_{14}}{a_{11}} & \frac{a_{15}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \frac{a_{24}}{a_{22}} & \frac{a_{25}}{a_{22}} \\ \cdots & & 0 & & \\ \cdots & & & 0 & \\ \frac{a_{51}}{a_{55}} & \cdots & & \frac{a_{54}}{a_{55}} & 0 \end{pmatrix} \begin{Bmatrix} \varphi_1^{(0)} \\ \vdots \\ \varphi_5^{(0)} \end{Bmatrix} + \begin{pmatrix} \frac{1}{a_{11}} & & & & \\ & \frac{1}{a_{22}} & & & \\ & & \frac{1}{a_{33}} & & \\ & & & \frac{1}{a_{44}} & \\ & & & & \frac{1}{a_{55}} \end{pmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix}$$

row1:- $\varphi_1^{(1)} = -\frac{1}{a_{11}}\left(a_{12}\varphi_2^{(0)} + a_{13}\varphi_3^{(0)} + a_{14}\varphi_4^{(0)} + a_{15}\varphi_5^{(0)}\right) + \frac{1}{a_{11}}r_1$

$$= \frac{1}{a_{11}}\left\{r_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{5} a_{1j}\varphi_j^{(0)}\right\}$$

$j \rightarrow$ varies 1 to 5 only 1 excluded.

row2:- $\varphi_2^{(1)} = \frac{1}{a_{22}}\left\{r_2 - \left(a_{21}\varphi_1^{(0)} + a_{23}\varphi_3^{(0)} + a_{24}\varphi_4^{(0)} + a_{25}\varphi_5^{(0)}\right)\right\}$

$$= \frac{1}{a_{22}}\left\{r_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{5} a_{2j}\varphi_j\right\}$$

Similarly, row 3,4,5:-

$$\varphi_5^{(1)} = \frac{1}{a_{55}}\left\{r_5 - \sum_{\substack{j=1 \\ j\neq 5}}^{5} a_{5j}\varphi_j^{(0)}\right\}$$

Iteration 2:-

$$\phi_1^{(2)} = \frac{1}{a_{11}}\left[r_1 - \sum_{\substack{j=1\\ j\neq 1}}^{5} a_{1j}\phi_j^{(1)}\right]$$

$\vdots$

$$\phi_5^{(2)} = \frac{1}{a_{55}}\left[r_5 - \sum_{\substack{j=1\\ j\neq 5}}^{5} a_{5j}\phi_j^{(1)}\right]$$

General Algorithm:-

$$\phi^{(0)} = \left[\phi_1^{(0)}\ \phi_2^{(0)} \cdots\cdots \phi_{N-1}^{(0)}\ \phi_N^{(0)}\right]^T$$

For ith row and pth iteration.

$$\boxed{\phi_i^{(P)} = \frac{1}{a_{ii}}\left[r_i - \sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij}\phi_j^{(P-1)}\right]},\ \forall i\in\{1,2,\cdots N\},\ P\geq 1 \cdots\cdots (4)$$

Compare

$$\phi_1^{(1)} = \frac{1}{a_{11}}\left[r_1 - \{a_{12}\phi_2^{(0)} + a_{13}\phi_3^{(0)} + a_{14}\phi_4^{(0)} + a_{15}\phi_5^{(0)}\}\right]$$

By adding and subtracting $\phi_i^{(P-1)}$ in right hand side in (4), (was not there. Because, $j\neq i$)

$$\phi_i^{(P)} = \underline{\underline{\phi_i^{(P-1)}}} + \frac{1}{a_{ii}}\left[r_i \pm \underline{\underline{a_{ii}\phi_i^{(P-1)}}} - \sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij}\phi_j^{(P-1)}\right]$$

$$= \phi_i^{(P-1)} + \frac{1}{a_{ii}}\left[r_i - \sum_{j=1}^{i-1} a_{ij}\phi_j^{(P-1)} - a_{ii}\phi_i^{(P-1)} - \sum_{j=i+1}^{N} a_{ij}\phi_j^{(P-1)}\right]\ \forall i,\ P\geq 1.$$

In compact form,

$$\boxed{\phi_i^{(P)} = \phi_i^{(P-1)} + \frac{Res_i}{a_{ii}}}\quad \forall i,\ P\geq 1$$

Where, Residual = RHS − LHS

!(me) ← $= \{r\} - \{$Terms associated with A and $\phi\}$

Residual error in a particular iteration:-

$$\mathcal{E}^{(P)} = [A]\phi^{(P)} - \{r\}$$

To check where we need to check the iteration to be stopped.

Absolute max error (MAE)

$$\max_{i\in 1,\ldots N} |\mathcal{E}_i^{(P)}| \leq \mathcal{E}_{max}$$

Root mean square error:- (RMSE)

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\mathcal{E}_i^{(P)}\right)^2} \leq \mathcal{E}_{max}$$

Convergence criteria

Diagonal dominance:- (Without diagonal dominance, convergence is not possible)

Weak diagonal dominance:-

$$|a_{ii}| \geq \sum_{\substack{j=1\\ j\neq i}}^{N} |a_{ij}|$$

← for some rows

Value of diagonal terms should be more than sum of the non-diagonal terms.

Strict diagonal dominance:-

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{N} |a_{ij}|$$

← For all the rows

Gauss Seidal method

NPTEL-29, Class-26

Iterative method like Jacobi's method

⇒ Apply successive over relaxation for Gauss Seidal method.

$$A\varphi = r$$

$$\begin{pmatrix} \cdot \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix}$$

$$A = L+D+U$$

$$(L+D+U)\varphi = r$$

$$\Rightarrow \boxed{(L+D)\varphi^{(p)} + U\varphi^{(p-1)} = r} \quad \ldots\ldots (36)$$

For Jacobi's method, it was $D\varphi^{(p)} + (L+U)\varphi^{(p-1)} = r$

$$\Rightarrow \boxed{\varphi^{(p)} = -(L+D)^{-1}U\varphi^{(p-1)} + (L+D)^{-1}r} \quad \ldots (37)$$

$p \rightarrow$ Iteration counter $(p \geq 1)$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ \vdots & & & \vdots \\ a_{51} & \cdots & \cdots & a_{55} \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_5 \end{pmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_5 \end{Bmatrix}$$

Iteration 1:

Row 1: $\varphi_1^{(1)} = \frac{1}{a_{11}}\left[r_1 - \sum_{j=2}^{5} a_{1j}\varphi_j(0)\right]$ → Same as jacobi

Row-2: $\varphi_2^{(1)} = \frac{1}{a_{22}}\left[r_2 - a_{21}\varphi_1^{(1)} - \sum_{j=3}^{5} a_{2j}\varphi_j^{(0)}\right]$

Row-3: $\varphi_3^{(1)} = \frac{1}{a_{33}}\left[r_3 - \sum_{j=1}^{2} a_{3j}\varphi_j^{(1)} - \sum_{j=4}^{5} a_{3j}\varphi_j^{(0)}\right]$

Row-4: $\varphi_4^{(1)} = \frac{1}{a_{44}}\left[r_4 - \sum_{j=1}^{3} a_{4j}\varphi_j^{(1)} - a_{45}\varphi_5^{(0)}\right]$

$$= \frac{1}{a_{44}}\left[r_4 - \underbrace{a_{41}\varphi_1^{(1)} - a_{42}\varphi_2^{(1)} - a_{43}\varphi_3^{(1)}}_{} - a_{45}\varphi_5^{(0)}\right]$$

Coff Terms in lower triangular matrix would have present iteration values.

Row 5: $\varphi_5^{(1)} = \frac{1}{a_{55}}\left(r_5 - \sum_{j=1}^{4} a_{5j}\varphi_j^{(1)}\right)$

Iteration 2

$$\varphi_1^{(2)} = \frac{1}{a_{11}}\left[r_1 - \sum_{j=2}^{5} a_{1j}\varphi_j^{(1)}\right]$$

$$\varphi_2^{(2)} = \frac{1}{a_{22}}\left[r_2 - \underbrace{a_{21}\varphi_1^{(2)}}_{\text{updated value}} - \sum_{j=3}^{5} a_{2j}\varphi_j^{(1)}\right]$$

$$\varphi_3^{(2)} = \frac{1}{a_{33}}\left[r_3 - \sum_{j=1}^{2} a_{3j}\varphi_j^{(2)} - \sum_{j=4}^{5} a_{3j}\varphi_j^{(1)}\right]$$

Row 4:- $\phi_4^{(2)} = \frac{1}{a_{44}}\left[r_4 - \sum_{j=1}^{3} a_{4j}\phi_j^{(2)} - \sum_{j} a_{45}\phi_j^{(1)}\right]$

Row 5:- $\phi_5^{(2)} = \frac{1}{a_{55}}\left(r_5 - \sum_{j=1}^{4} a_{5j}\phi_j^{(2)}\right)$

Generalize the algorithm:

$$\phi^{(0)} = \left[\phi_1^{(0)}\ \phi_2^{(0)} \cdots\ \phi_{N-1}^{(0)}\ \phi_N^{(0)}\right]^T$$

$$\phi_i^{(P)} = \frac{1}{a_{ii}}\left[r_i - \sum_{j=1}^{i-1} a_{ij}\phi_j^{(P)} - \sum_{j=i+1}^{N} a_{ij}\phi_j^{(P-1)}\right], \forall i \in \{1, \cdots N\}, P \geq 1. \quad \cdots\cdots (38)$$

ith row of pth iteration, we have.

By adding and subtracting $\phi_i^{(P-1)}$ in the RHS,

$$\boxed{\phi_i^{(P)} = \phi_i^{(P-1)} + \frac{1}{a_{ii}}\left[\underbrace{r_i - \sum_{j=1}^{i-1} a_{ij}\phi_j^{(P)}}_{\text{RHS}} \underbrace{- a_{ii}\phi_i^{(P-1)} - \sum_{j=i+1}^{N} a_{ij}\phi_j^{(P-1)}}_{\text{LHS}}\right]} \cdots (39)$$

In compact form,

$$\phi_i^{(P)} = \phi_i^{(P-1)} + \frac{Res_i}{a_{ii}} \quad \forall i, P \geq 1 \cdots\cdots (40) \cdots\cdots (39.1)$$

Here, some values are updated form.
Some are at non-updated form.
But, in jacobi's method, all were non-updated.

☑ Residual error for a particular iteration,

$$\{\varepsilon^{(P)}\} = [A]\{\phi^{(P)}\} - \{r\}$$

Max absolute error:-

$$\max_{i \in 1, 2, \ldots N} |\varepsilon_i^{(P)}| \leq \varepsilon_{max}$$

Residual error term of for any row (ranges from 1st to N) for pth iteration.

Root mean square error:-

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\varepsilon_i^{(P)})^2} \leq \varepsilon_{max}$$

* Similarly, diagonal dominance is required for convergence of solution.

Gauss Seidal method:

☑ Successive- Over relaxation: (Another method):

Convergence can be achieved by increasing or reducing the step size

$$\phi^{(P)} - \phi^{(P-1)} = \omega\left[\phi_{GS}^{(P)} - \phi^{(P-1)}\right] \quad (39.2)$$

Here, ω is an assumed constant

$\phi_{GS}^{(P)}$ obtained from gauss-siedel

$\phi^{(P-1)}$ Obtained from previous iteration.

In iterative form,

$$\phi^{(P)} = \omega\phi_{GS}^{(P)} + (1-\omega)\phi^{(P-1)} \cdots\cdots (40)$$

G.S approximation written as:- (see eq 36)

$$(L+D)\phi^{(P)} + U\phi^{(P-1)} = r$$

$$\Rightarrow D\phi_{GS}^{(P)} = -L\phi^{(P)} - U\phi^{(P-1)} + r \cdots\cdots (41)$$

Dx (40),

$$D\varphi^{(P)} = \omega D\varphi^{(P)}_{GS} + (1-\omega) D\,\varphi^{(P-1)}$$

↳ Put this from (41)

$$= \omega(-L\varphi^{(P)} - U\varphi^{(P-1)} + r) + (1-\omega) D.\varphi^{(P-1)}$$

$$= -\omega L\varphi^{(P)} + (1-\omega).D\varphi^{(P-1)} - \omega U\varphi^{(P-1)} + \omega r$$

Rearrangement:- (Terms connected to P and (P-1)th iterations are seperated)

$$(D+\omega L)\,\varphi^{(P)} = [(1-\omega)D - \omega U]\varphi^{(P-1)} + \omega r$$

Finally, $$\varphi^{(P)} = (D+\omega L)^{-1}[(1-\omega)D - \omega U]\varphi^{(P-1)} + \omega(D+\omega L)^{-1} r \;\ldots\ldots (42)$$

General algorithm:

From (40), (39.1)

compact form of G.S method.

$$\varphi^{(P)}_{i,GS} = \varphi_i^{(P-1)} + \frac{Res_i}{a_{ii}} \quad \forall i, P \geq 1.$$

Convergence can be achieved by increasing or reducing the step size,

$$\underbrace{\varphi_i^{(P)} - \varphi_i^{(P-1)}}_{\text{This difference is for Successive over relaxation method!?}} = \omega \underbrace{\left[\varphi^{(P)}_{i,GS} - \varphi_i^{(P-1)}\right]}_{\text{For G.S method!}} \quad \text{(From 39.2)}$$

$$\Rightarrow \varphi_i^{(P)} = \varphi_i^{(P-1)} + \omega.\left(\frac{Res_i}{a_{ii}}\right)$$

From eq^n (39.1), $\varphi^{P}_{i,GS} - \varphi_i^{(P-1)} = \frac{Res_i}{a_{ii}}$

Rearrangement,

$$\boxed{\varphi_i^{(P)} = \varphi_i^{(P-1)} + \frac{Res_i}{(a_{ii}/\omega)}}, \quad \text{for } 0<\omega<2 \;\ldots\ldots\ldots (43)$$

SOR method is modified GS method to meet the convergence!!

$0<\omega<1 \Rightarrow$ under relaxation → For under relaxation, we are decreasing the $\frac{Res_i}{a_{ii}}$ term. or we can say, increasing the diagonal term a_{ii}. $\because \frac{a_{ii}}{\omega} > a_{ii}$ for under relaxation.

$1<\omega<2 \Rightarrow$ Over relaxation.

So, changing ω value, we can control the convergence for Gauss-Seidal over-relaxation method.

Res = RHS - LHS

Newton- Raphson Method

NPTEL-30

Iterative solution for non-linear systems of equations.

$$A(\phi)\,\phi = r$$

$$\begin{pmatrix} a_{11}(\phi) & \cdots & a_{1N}(\phi) \\ \vdots & & \vdots \\ a_{N1}(\phi) & \cdots & a_{NN}(\phi) \end{pmatrix}_{N\times N} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{Bmatrix}$$

To so non-linear problems
- → Based on Taylor series
- → " " Optimisation.

Newton- Raphson ⇒ Based on Taylor series expansion.

$$\underbrace{[A(\phi)]\{\phi\}}_{LHS} - \underbrace{\{r\}}_{RHS} = 0$$

Can be written as:-

$$\begin{Bmatrix} F_1(\phi_1, \ldots, \phi_N) \\ F_2(\phi_1, \ldots, \phi_N) \\ F_3(\phi_1, \ldots, \phi_N) \\ \vdots \\ F_{N-2}(\phi_1, \ldots, \phi_N) \\ F_{N-1}(\phi_1, \ldots, \phi_N) \\ F_N(\phi_1, \ldots, \phi_N) \end{Bmatrix} = \begin{Bmatrix} F_1(\phi) \\ F_2(\phi) \\ \vdots \\ F_{N-2}(\phi) \\ F_{N-1}(\phi) \\ F_N(\phi) \end{Bmatrix} = 0$$

$$F(\phi) = 0$$

$$[\phi] = [\phi_1, \phi_2, \ldots, \phi_{N-1}, \phi_N]^T$$

$$[F(\phi)] = [F_1(\phi)\ F_2(\phi) \ldots F_{N-1}(\phi)\ F_N(\phi)]^T$$

Taylor series expansion of ith function,

$$F_i(\phi+\Delta\phi) = F_i(\phi_1+\Delta\phi_1, \phi_2+\Delta\phi_2, \ldots, \phi_N+\Delta\phi_N)$$

Here, ϕ and $\Delta\phi$ both are vectors. (I think, ith term of $[F(\phi)]$ vector)

$$= F_i(\phi_1, \phi_2, \ldots \phi_N) + \sum_{j=1}^{N} \frac{\partial F_i(\phi_1, \ldots \phi_N)}{\partial \phi_j} \cdot \Delta\phi_j + O(\Delta\phi_1^2, \ldots \Delta\phi_N^2)$$

$$= F_i(\phi) + \sum_{j=1}^{N} \frac{\partial F_i(\phi)}{\partial \phi_j} \Delta\phi_j \qquad F_i(\phi) \to F_i(\phi_1, \phi_2, \ldots \phi_N)$$

$$\Rightarrow \boxed{F_i(\phi+\Delta\phi) = F_i(\phi) + \sum_{j=1}^{N} \frac{\partial F_i(\phi)}{\partial \phi_j} \Delta\phi_j} \quad \ldots\ldots (44)$$

We have N numbers of equations and N variables.

$$F_1(\phi+\Delta\phi) = F_1(\phi) + \frac{\partial F_1(\phi)}{\partial \phi_1}\Delta\phi_1 + \frac{\partial F_1(\phi)}{\partial \phi_2}\Delta\phi_2 + \cdots + \frac{\partial F_1(\phi)}{\partial \phi_N}\Delta\phi_N$$

$$F_2(\phi+\Delta\phi) = F_2 + \frac{\partial F_2}{\partial \phi_1}\Delta\phi_1 + \frac{\partial F_2}{\partial \phi_2}\Delta\phi_2 + \cdots + \frac{\partial F_2}{\partial \phi_N}\Delta\phi_N.$$

$$F_N(\phi+\Delta\phi) = F_N + \frac{\partial F_N}{\partial \phi_1}\Delta\phi_1 + \frac{\partial F_N}{\partial \phi_2}\Delta\phi_2 + \cdots + \frac{\partial F_N}{\partial \phi_N}\cdot\Delta\phi_N.$$

Combining these expansions, we have,

$$\begin{Bmatrix} F_1(\phi+\Delta\phi) \\ F_2(\phi+\Delta\phi) \\ \vdots \\ F_N(\phi+\Delta\phi) \end{Bmatrix} = \begin{Bmatrix} F_1(\phi) \\ F_2(\phi) \\ \vdots \\ F_N(\phi) \end{Bmatrix} + \underbrace{\begin{pmatrix} \frac{\partial F_1}{\partial \phi_1} & \frac{\partial F_1}{\partial \phi_2} & \cdots & \frac{\partial F_1}{\partial \phi_N} \\ \frac{\partial F_2}{\partial \phi_1} & \frac{\partial F_2}{\partial \phi_2} & & \frac{\partial F_2}{\partial \phi_N} \\ \vdots & & & \vdots \\ \frac{\partial F_N}{\partial \phi_1} & \frac{\partial F_N}{\partial \phi_2} & \cdots & \frac{\partial F_N}{\partial \phi_N} \end{pmatrix}}_{N\times N} \begin{Bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \vdots \\ \Delta\phi_N \end{Bmatrix}$$

This is called Jacobian matrix. Containing 1st order derivatives.

In vector/matrix form,

$$\vec{F}(\phi+\Delta\phi) = \vec{F}(\phi) + [J(\phi)]\,\vec{\Delta\phi} \quad \ldots\ldots (45)$$

Assume $(\phi+\Delta\phi)$ as the solution,

$$\vec{F}(\phi+\Delta\phi) = 0$$

$\therefore\ 0 = \vec{F}(\phi) + [J(\phi)]\,\Delta\vec{\phi}$ (Putting value to equation 45)

$\Rightarrow [\vec{J}(\phi)][\vec{\Delta\phi}] = -[\vec{F}(\phi)]$

This is equivalent to our equation $A\phi = r$

$\Rightarrow [\Delta\phi] = -[J(\phi)]^{-1}[F(\phi)]$

At a particular point ϕ, we can apply this increment.

$\therefore$ Iterative form can be written as:-

$$[\Delta\phi]^{(p)} = -[J(\phi^{(p-1)})]^{-1}[F(\phi^{(p-1)})] \quad \ldots\ldots (46)$$

Say for 1st iteration, $[\Delta\phi^{(1)}]$ $\Rightarrow$ when $p=1$, we can start from our guess value $\phi^{(0)}$ at $p=1$.

$\Rightarrow [\Delta\phi^{(1)}] = -[J(\phi^{(0)})]^{-1}[F(\phi^{(0)})]$

Where, p = iteration number (≥ 1)

Explicitly, it can be written as:- $\phi^{(p)} = \phi^{(p-1)} + \Delta\phi^{(p)}$

$$\Rightarrow \boxed{[\phi^{(p)}] = [\phi^{(p-1)}] - [J(\phi^{(p-1)})]^{-1}[F(\phi^{(p-1)})]} \text{ (From 46)} \quad \ldots\ldots (47)$$

Iteration starts from the guess value $\phi^{(0)}$

$$[\phi^{(0)}] = [\phi_1^{(0)}\ \phi_2^{(0)} \ldots \phi_{N-1}^{(0)}\ \phi_N^{(0)}]^T$$

Eq (47) in matrix form:-

$$\begin{Bmatrix} \phi_1^{(p)} \\ \phi_2^{(p)} \\ \vdots \\ \phi_N^{(p)} \end{Bmatrix} = \begin{Bmatrix} \phi_1^{(p-1)} \\ \phi_2^{(p-1)} \\ \vdots \\ \phi_N^{(p-1)} \end{Bmatrix} - \begin{bmatrix} \left(\frac{\partial F_1}{\partial \phi_1}\right)^{(p-1)} & \cdots & \left(\frac{\partial F_1}{\partial \phi_N}\right)^{(p-1)} \\ \vdots & & \\ \left(\frac{\partial F_N}{\partial \phi_1}\right)^{(p-1)} & \cdots & \left(\frac{\partial F_N}{\partial \phi_N}\right)^{(p-1)} \end{bmatrix}^{-1} \begin{Bmatrix} F_1(\phi^{(p-1)}) \\ F_2(\phi^{(p-1)}) \\ \vdots \\ F_N(\phi^{(p-1)}) \end{Bmatrix}$$

Residual error $\Rightarrow$ LHS-RHS

$\Rightarrow [A(\phi)]\{\phi\}-\{r\}=\{F(\phi)\}$

Residual error:- in a particular iteration, $\Rightarrow$ From equation (45),

$[F(\phi+\Delta\phi)]=[F(\phi)]+[J(\phi)][\Delta\phi]$

0 (Assumed for next iteration)

? why $F(\phi)$ is not residual?

$\hookrightarrow [A(\phi)]\{\phi\}-\{r\}=[F(\phi)]$

$$\varepsilon^{(p)}=[\Delta\phi^{(p)}]=[\phi^{(p)}]-[\phi^{(p-1)}]$$
$$=-[J(\phi^{(p-1)})]^{-1}[F(\phi^{(p-1)})]$$

$\Rightarrow [F(\phi)]=-[J(\phi)][\Delta\phi]$

$\Rightarrow [\Delta\phi]=-[J(\phi)]^{-1}[F(\phi)]$

why $[\Delta\phi]$?

Root MSE:-
$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\varepsilon_i^{(p)}\right)^2}\leq \varepsilon_{max}$$

Module-3, Unit-1
Groundwater Hydraulics, 1D flow:
Class-28, NPTEL-31

To solve 1-D groundwater flow equation.

(i) Problem definition
(Hydraulic System)

↓

(ii) Mathematical Conceptualization
GE, IC, BC

↓

(iii) Domain Discretization
(a) Grid/mess generation (Structured/ unstructured mess)

(b) Point generation (structured/ unstructured

↓

(iv) Numerical Discretization.

(a) Eulerian approach
(FDM, FEM, FVM, Spectral element, Mess-free method)

(b) Lagrangian approach
(Smoothed particle hydrodynamics, Moving particle semi-implicit).

(c) Euler-Lagrangian approach
(Particle in cell method, material point method).

↓

(v) Algebraic form
Linear Equations $A\phi=r$ $A\to$ constant coff
Non " " $A\to A(\phi)$

↓

Solution process:

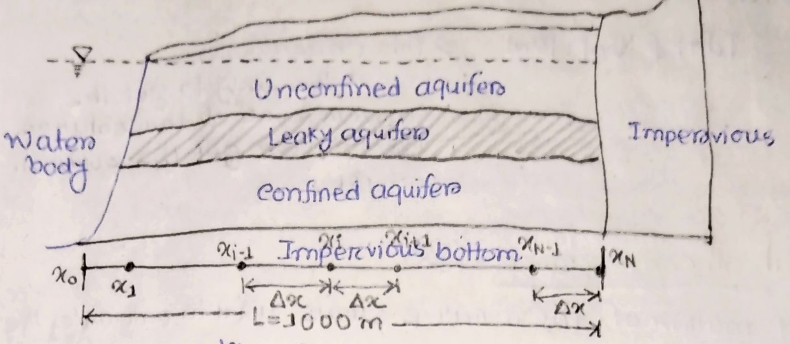
Linear Solver { Direct approach → LU decomposition, Tri-diagonal matrix method, Gauss elimination.
Iterative " → Jacobi, Gauss siedel, Gauss-siedel (with successive over relaxation.

Non-linear Solver { Iterative approach → (N-R method)

↓

Results (output file)

FDM, FVM, MFM → Discretization is almost same. Only at Boundary, some deviations.



$$\frac{d^2h}{dx^2} = \frac{C_{conf}}{T}(h - h_{wt}) \quad \ldots\ldots (48)$$

$$C_{conf} = \frac{\text{Hydraulic conductivity}}{\text{Thickness of confining layer}}$$

h_{wt} = Overlaying water table elevation $= C_0 + C_1 x + C_2 x^2$

Left Boundary:- Specified head/ Dirichlet Boundary $\Rightarrow h(x=0) = h_s$

Right " :- impervious/No flow/Neuman " $\Rightarrow \left.\frac{dh}{dx}\right|_{x=L} = 0$.

Data:-

$C_{conf} = 10^{-11}$

$T = 2\times10^{-5}$

$C_0 = 90$

$C_1 = 0.06$

$C_2 = -0.00003$

$h_s = 90$ m

$L = 1000$ m.

Interior nodes,

$$\frac{h_{i-1} - 2h_i + h_{i+1}}{\Delta x^2} = \frac{C_{conf}}{T}\left[h_i - h_{wt}(x_i)\right] \quad \forall i \in \{2, \ldots\ldots N-1\}$$

Because, in scilab, we can store array values from 1 to N, not from zero

1 2 N-1 N

Equation further simplified as:-

$$\underbrace{\frac{1}{\Delta x^2}}_{b_i} h_{i-1} + \underbrace{\left(-\frac{2}{\Delta x^2} - \frac{C_{conf}}{T}\right)}_{d_i} h_i + \underbrace{\frac{1}{\Delta x^2}}_{a_i} h_{i+1} = \underbrace{-\frac{C_{conf}}{T} h_{wt}(x_i)}_{r_i}$$

$\Rightarrow b_i h_{i-1} + d_i h_i + a_i h_{i+1} = r_i$

for code $r_i = -\frac{C_{conf}}{T}\left\{C_0 + C_1 x(\text{node}) + C_2 x(\text{node})^2\right\}$

Right boundary:-

Second-order discretization:-

$$\frac{3h_N - 4h_{N-1} + h_{N-2}}{2\Delta x} = 0 \quad \text{(See page-135)} \Rightarrow \text{For derivation} \ldots\ldots (49)$$

Here, $b_N = -\frac{4}{2\Delta x}$, $d_N = \frac{3}{2\Delta x}$, $a_N = 0$, $r_N = 0$

& $e_N = \frac{1}{2\Delta x}$

Sum of the coff of the non diagonal terms, $\left|-\frac{4}{2\Delta x}\right| + \left|\frac{1}{2\Delta x}\right| = \frac{2.5}{\Delta x}$

coff of diagonal term $= \left|\frac{3}{2\Delta x}\right| = 1.5/\Delta x$.

$\therefore \frac{2.5}{\Delta x} > \frac{1.5}{\Delta x}$.

∴ Diagonal dominance is not satisfied.

From graph, seeing the B.C itself we can test the nature correctness of the solution.

$|d_N| \ngtr |b_N| + |a_N|$ → This criteria is not the way to get the convergence of the solution. But still we get the solution. Why?!

Comment on convergence:-

Coff matrix of the iterative step is used to calculate the spectral radius.

see page 193, Eq(3).

$\rho(-D^{-1}(L+U)) = \max\{|\lambda_1| \ldots |\lambda_n|\} < 1$ → If all eigenvalues are less than 1, then our solutions will converge.

Where, $|\lambda_1| \ldots |\lambda_n|$ are eigenvalues of the matrix.

Spectral radius means the maximum eigenvalue of $[-D^{-1}(L+U)]$ matrix.

These are 'strictly' Lower and upper triangular matrix.

Now if maximum eigenvalue becomes more than 1, convergence condition will not be satisfied. Now, we need to solve the 'Scaling' problem.

We are divingding interior equation by Δx^2, converging to small numbers and boundary(right hand) by $2\Delta x$. So multiply interior equations by Δx^2 and RH boundary eq^n by $2\Delta x$.
→ i.e last row.
other rows.

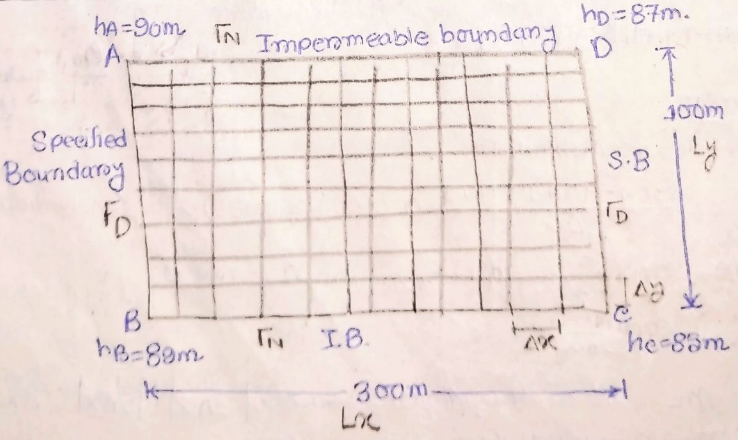
After scaling, we will get eigenvalue as less than 1.

Steady 2-D flow	(Groundwater
NPTEL-32, Class-29	Hydraulics)

Previously $h(x)$

Now, $h(x,y)$

Problem definition:



Homogenous, isotropic, confined, $K_{xx}\frac{d^2\phi}{dx^2}+K_{yy}\frac{\partial^2\phi}{\partial y^2}+\frac{N}{b}-\frac{S}{b}\frac{\partial\phi}{\partial t}$

Homogenous isotropic aquifer system, DE is Laplace equation.

GE: Two dimensional BVP,

$$\frac{d^2h}{dx^2}+\frac{d^2h}{dy^2}=0$$

BC's:-

(Left boundary) $\Gamma_D^1 = h(0,y) = h_1(y)$

(Right ,,) $\Gamma_D^2 = h(L_x,y) = h_2(y)$

(Bottom boundary) $\Gamma_N^3 : \left.\frac{\partial h}{\partial y}\right|_{(x,0)}=0$

(Top boundary) $\Gamma_N^4 : \left.\frac{\partial h}{\partial y}\right|_{(x,L_y)}=0$

2D groundwater flow equation in non-isotropic, non-homogenous media,

$$\frac{\partial}{\partial x}\left[k(x,y)\frac{\partial h}{\partial x}\right]+\frac{\partial}{\partial y}\left[k(x,y)\frac{\partial h}{\partial y}\right]=S(x,y)$$

$S \rightarrow$ source, sink term representing external/internal water supply or demand.

Governing equation discretized as:-

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0$$

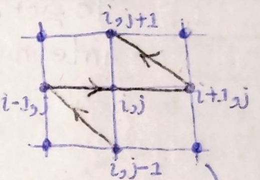
$$\Rightarrow \frac{h_{i+1,j}-2h_{i,j}+h_{i-1,j}}{\Delta x^2}+\frac{h_{i,j+1}-2h_{i,j}+h_{i,j-1}}{\Delta y^2}=0$$

$$\Rightarrow \frac{1}{\Delta y^2}h_{i,j-1}+\frac{1}{\Delta x^2}h_{i-1,j}-2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)h_{i,j}+\frac{1}{\Delta x^2}h_{i+1,j}+\frac{1}{\Delta y^2}h_{i,j+1}=0$$

Rough:

$$\frac{\partial^2 h}{\partial x^2}=\frac{\partial}{\partial n}\left(\frac{\partial h}{\partial x}\right)=\frac{h'_{i+1}-h'_i}{\Delta x}=\left(\frac{h_{i+1}-h_i}{\Delta x}-\frac{h_i-h_{i-1}}{\Delta x}\right)/\Delta x=\frac{h_{i+1}-2h_i+h_{i-1}}{\Delta x^2}$$

may be done by Taylor series also. But, here, we don't need truncation error.

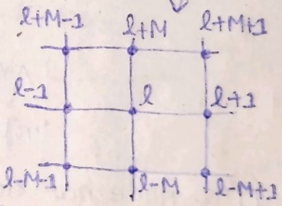


$$\Rightarrow \alpha_y h_{i,j-1}+\alpha_x h_{i-1,j}-2(\alpha_x+\alpha_y)h_{i,j}+\alpha_x h_{i+1,j}+\alpha_y h_{i,j+1}=0 \quad \text{------ (50)}$$

(where, $\alpha_x=\frac{1}{\Delta x^2}$, $\alpha_y=\frac{1}{\Delta y^2}$)

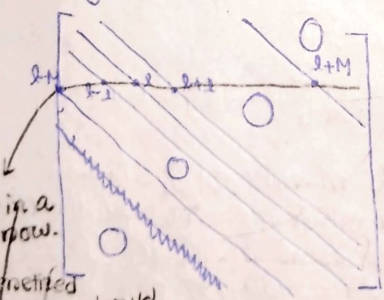
(50) Using single index notation,

$$\alpha_y h_{l-M}+\alpha_x h_{l-1}-2(\alpha_x+\alpha_y)h_l+\alpha_x h_{l+1}+\alpha_y h_{l+M}=0 \quad \text{.... (51)}$$



(single index notation)

- Coefficient matrix would have a pentadiagonal structure.

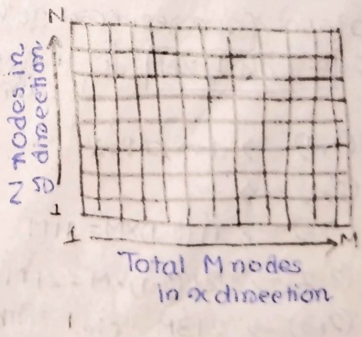


?!

This is in a single row.

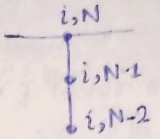
In discretized domain, they should not be in single row.

But, in matrix, it represents the head equation for a point. So, in same row.



Total M nodes in x direction

Top Neuman boundary,



$$\left.\frac{\partial h}{\partial y}\right|_{(x,ly)} = 0$$

$$\Rightarrow \left.\frac{\partial h}{\partial y}\right|_{(i,N)} = 0$$

derivative wrt y, 2nd ordinate will change only.

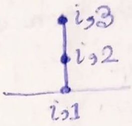
$$\Rightarrow \frac{3h_{i,N} - 4h_{i,N-1} + h_{i,N-2}}{2\Delta y} = 0 \quad \text{------} \quad (52)$$

(see page 145)

In single index notation,

$$\frac{3h_l - 4h_{l-M} + h_{l-2M}}{2\Delta y} = 0 \quad \text{......} \quad (52.1)$$

Bottom Neumann boundary,

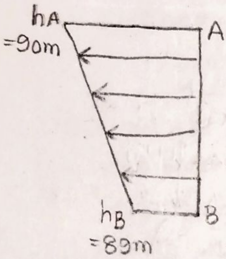


why opposite signs are used? than (52)

$$\frac{\overset{-}{3}h_{i,1} \overset{+}{4}h_{i,2} \overset{-}{}h_{i,3}}{2\Delta y} = 0 \quad \text{....} \quad (52.2)$$

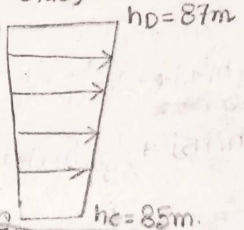
$$\Rightarrow \frac{-3h_l + 4h_{l+M} - h_{l+2M}}{2\Delta y} = 0 \quad \text{-----} \quad (52.3)$$

At left side,



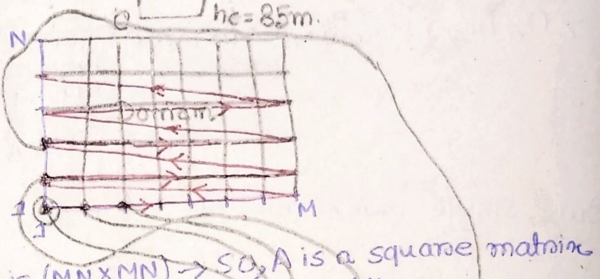
To get any intermediate values,

At right side,



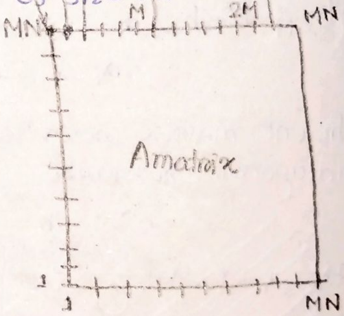
M Nodes in x direction

N nodes in y ,,



[A] → size is (MN×MN) → So, A is a square matrix of size MN.

{h} → size is (MN×1)



If a node has double index notation (i,j) and single index notation (represented as l) will be $\boxed{l = i + (j-1) \times M}$ [M = No. of nodes in y direction].

$(1,1) \longrightarrow 1 + 0 \times M = 1$

$(2,1) \not\longrightarrow$

$(1,2) \longrightarrow 1 + (2-1) \times M = 1 + M$

$(2,2) \rightarrow 2 + (2-1) \times M = 2 + M$

$(2,3) \rightarrow 2 + 3M$ etc.

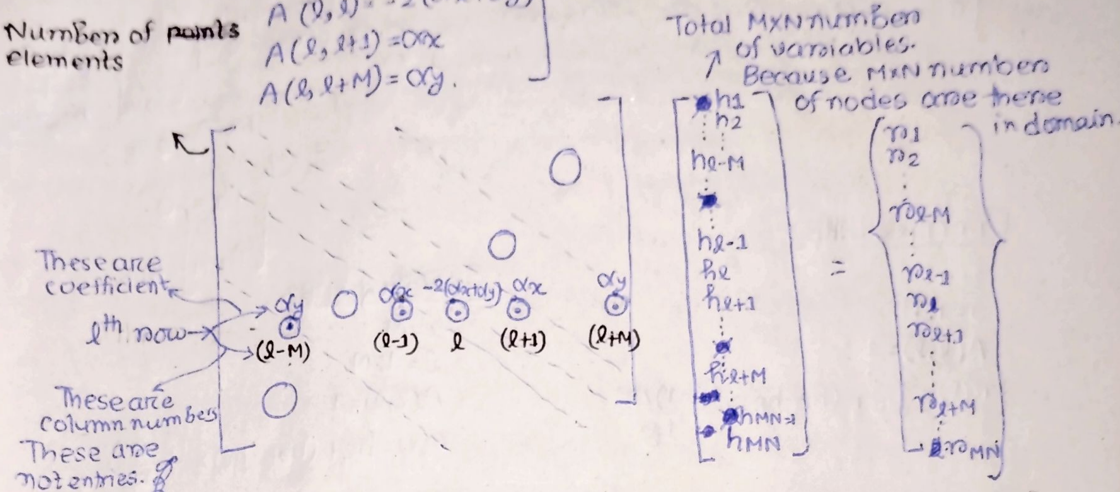
'A' matrix এর প্রতিটি row এক একটি node এর equation কে represent করে। এবারে প্রতিটি node এর equation এর জন্য চারপাশের node থেকে information দরকার। তাই A matrix এ MN গুলো entry থাকে এবং domain থেকে A matrix এ data transfer করার জন্য single index notation ব্যবহার করা হয়।

let us consider a point in domain whose single index notation is l.

So, lth row of coff matrix A will give the equation for that node,

Number of points elements

$$\left.\begin{aligned} &A(l, l-M) = \alpha_y \\ &A(l, l-1) = \alpha_x \\ &A(l, l) = -2(\alpha_x + \alpha_y) \\ &A(l, l+1) = \alpha_x \\ &A(l, l+M) = \alpha_y \end{aligned}\right\} \rightarrow \text{These coffs are for interior nodes}$$



Equation for lth row,

$$\alpha_y h_{l-M} + \alpha_x h_{l-1} - (\alpha_x+\alpha_y)2 h_l + \alpha_x h_{l+1} + \alpha_y h_{l+M} = r_l \quad \dots (53)$$

Is this the head value of that lth node?!

For confined aquifer,

$$K_{xx}\frac{\partial^2 h}{\partial x^2} + K_{yy}\frac{\partial^2 h}{\partial y^2} + \underbrace{\frac{N}{b}}_{\text{Recharge/ Source/sink term}} = \frac{S}{b}\cancel{\frac{\partial h}{\partial t}}^{\,0} \quad \text{(considering steady state)}$$

As per G.E,

$$\frac{d^2h}{dx^2} + \frac{d^2h}{dy^2} = 0,$$

Is this valid for interior nodes only.

For other cases, $\frac{d^2h}{dx^2} + \frac{d^2h}{dy^2} = h$?

For node A,

$(i,j) = (1, N)$

$l = 1 + (N-1)M$

$A(l,l) = 1$

$r_l = h_A = 90\,m.$

for lth or $\{1+(N-1)M\}^{th}$ row,

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 \dots & l \dots & MN \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \dots & 1 \dots & 0 \end{bmatrix} \end{array} \begin{Bmatrix} h_1 \\ \vdots \\ h_l \\ \vdots \\ h_{MN} \end{Bmatrix} = \begin{Bmatrix} r_1 \\ \vdots \\ r_l = h_A \\ \vdots \\ r_{MN} \end{Bmatrix}$$

$\Rightarrow h_l = h_A$

$\Rightarrow h_l = 90\,m.$

For node B,

$(i,j) = (1,1)$

$l = 1$

$A(l,l) = 1$

$r_l = h_B = 89\,m.$

Node C,

$(i,j) = (M,1)$

$l = M$

$A(l,l) = 1$

$r_l = h_C = 85\,m$

Node D,

$(i,j) = (M,N)$

$l = M + (N-1)M = MN$

$A(l,l) = 1$

$r_l = h_D = 87\,m.$

For interior points,

$A(l, l-M) = \alpha_y$

$A(l, l-1) = \alpha_x$

$A(l, l) = -2(\alpha_x + \alpha_y)$

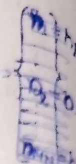
$A(l, l+1) = \alpha_x$

$A(l, l+M) = \alpha_y$.

Each row in [A] matrix is calculated from represents the equation for one node.

Single index notation number of that node = Row number in [A] matrix.

$$\begin{array}{ccccccccc} & l-M & & l-1 & l & l+1 & & l+M & \\ [0 \cdots & \alpha_y & \cdots 0 \cdots & \alpha_x & -2(\alpha_x+\alpha_y) & \alpha_x & \cdots 0 \cdots & \alpha_y & \cdots 0] \end{array} \begin{Bmatrix} h_1 \\ \vdots \\ h_l \\ \vdots \\ h_{MN} \end{Bmatrix} = \begin{Bmatrix} r_1 \\ \vdots \\ r_l \\ \vdots \\ r_{MN} \end{Bmatrix} \begin{matrix} = h_B \\ \\ = 0 \\ \\ = h_D \end{matrix}$$



Specified LHB:-

$i = 1$

$j = 1:N$

$A(l, l) = 1$

$r(l) = h_B + (h_A - h_B) \times \frac{(j-1)\Delta y}{L_y}$

Specified RHB:-

$i = M$

$j = 1:N$

$A(l, l) = 1$

$r(l) = h_C + (h_D - h_C) \times \frac{(j-1)\Delta y}{L_y}$

Neuman Bottom Boundary ⟶ From eq^n (52.3),

$-3h_l + 4h_{l+M} - h_{l+2M} = 0$

$i = 1:M$,

$j = 1$.

$A(l, l) = -3$

$A(l, l+M) = 4$

$A(l, l+2M) = -1$

$r(l) = 0$

From Eq^n (52), $3h_{i,N} - 4h_{i,N-1} + h_{i,N-2} = 0$

$\Rightarrow 3h_l - 4h_{l-M} + h_{l-2M} = 0$

Neumann Top boundary:-

$i = 1:M$

$j = N$

$A(l, l) = 3$

$A(l, l-M) = -4$

$A(l, l-2M) = 1$

$r(l) = 0$

☐ Storage requirement is a major issue if we want to store the full matrix. To avoid the storing of full matrix, we would use Gauss-Seidel method now,

G.S method: Iterative Approach:

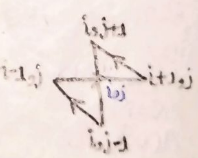
For Interior nodes:-

Iteration starts with guess value $[h^{(0)}]$.

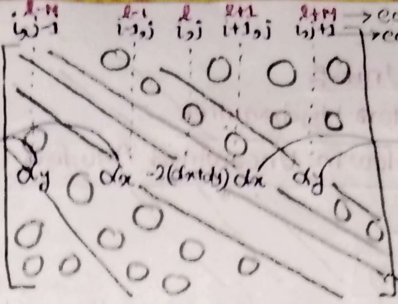
$[h^{(0)}] = [h_{1,1}^{(0)} \; h_{1,2}^{(0)} \cdots\cdots h_{M,N-1}^{(0)} \; h_{M,N}^{(0)}]^T$

From eq^n (50),

$\alpha_y h_{i,j-1} + \alpha_x h_{i+1,j} - 2(\alpha_x + \alpha_y) h_{i,j} + \alpha_x h_{i-1,j} + \alpha_y h_{i,j+1} = 0$



$$\Rightarrow h_{i,j}^{(P)} = \frac{1}{[-2(\alpha_x+\alpha_y)]} \left[0 - \left(\alpha_y h_{i,j-1}^{(P)} + \alpha_x h_{i+1,j}^{(P)} + \alpha_x h_{i-1,j}^{(P-1)} + \alpha_y h_{i,j+1}^{(P-1)}\right)\right] \quad \cdots\cdots (54)$$



These two are strictly at lower triangle (uses the present iteration values) or updated values

These are at strictly upper triangle. Uses, previous iteration values.

Main diagonal

Now,
adding and subtracting $h_{i,j}^{(P-1)}$ in right hand side,

$$h_{i,j}^{(P)} = h_{i,j}^{(P-1)} + \frac{1}{[-2(\alpha_x+\alpha_y)]}\Big[-\alpha_y h_{i,j-1}^{(P)} - \alpha_x h_{i-1,j}^{(P)} + 2(\alpha_x+\alpha_y)h_{i,j}^{(P-1)} - \alpha_x h_{i+1,j}^{(P-1)} - \alpha_y h_{i,j+1}^{(P-1)}\Big]$$

For Eq^n(50),

$$\text{Residual} = \text{RHS}-\text{LHS} = 0 - \Big[\alpha_y h_{i,j-1}^{(P)} + \alpha_x h_{i-1,j}^{(P)} - 2(\alpha_x+\alpha_y)h_{i,j}^{(P-1)} + \alpha_x h_{i+1,j}^{(P)} + \alpha_y h_{i,j+1}^{(P-1)}\Big]$$

So, in compact form,

$$h_{i,j}^{(P)} = h_{i,j}^{(P-1)} + \frac{\text{Res}_{i,j}}{[-2(\alpha_x+\alpha_y)]} \quad \forall (i,j) \quad P \geq 1 \quad \cdots\cdots (55)$$

#Interesting to note that, we have not constructed the A matrix. We have only multiplied $\alpha_x, \alpha_y, -2(\alpha_x+\alpha_y)$ these terms here and only stored $h(i,j)$ values.

Top Neumann Boundary:

Second order discretization:-

This is central coefficient

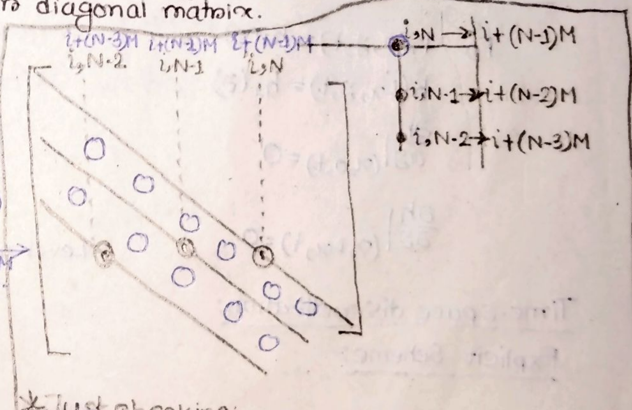
$$\frac{3h_{i,N} - 4h_{i,N-1} + h_{i,N-2}}{2\Delta y} = 0$$

$$h_{i,N}^{(P)} = \frac{1}{3}\left(4h_{i,N-1}^{(P-1)} - h_{i,N-2}^{(P-1)}\right)$$

Here, we have this equation for (i,N)th node. So, coff of $h_{i,N}$ will be in main diagonal. So, coff of $h_{i,N-2}$ and $h_{i,N-1}$ will be in lower diagonal matrix.

Subtracting and Adding $h_{i,N}^{(P-1)}$ at the RHS,

$$h_{i,N}^{(P)} = h_{i,N}^{(P-1)} + \frac{1}{3}\left(-h_{i,N-2}^{(P)} + 4h_{i,N-1}^{(P)} - 3h_{i,N}^{(P-1)}\right) \cdots\cdots (56)$$

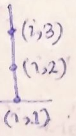


Bottom Neumann Boundary:

2nd order discretization,

$$\frac{-3h_{i,1} + 4h_{i,2} - h_{i,3}}{2\Delta y} = 0$$

at diagonal



$$\Rightarrow h_{i,1} = -\frac{1}{3}\left(-4h_{i,2}^{(P-1)} + h_{i,3}^{(P-1)}\right)$$

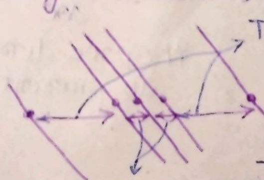
at upper triangular matrix.

Adding and subtracting $h_{i,N-1}$ at RHS, we get,

$$h_{i,1} = h_{i,1}^{(P-1)} - \frac{1}{3}\left(3h_{i,1}^{(P-1)} - 4h_{i,2}^{(P-1)} + h_{i,3}^{(P-1)}\right) \cdots (57)$$

*Just checking, the position for full [A] matrix. Not for Gauss-Seidel method.

For pentadiagonal structure,



These two are M entries apart. (Where, M is no. of nodes in x dir. in the domain)

These are side by side entries.

Module-3, Unit-5

Groundwater Hydraulics

Unsteady flow in Unconfined Aquifer using FVM:

NPTEL-35,

$\boxed{h(x,y,t)}$

In unconfined aquifer flow, flow is only considered as in horizontal direction. Because, there is no confining layer to restrict vertical flow, water can move laterally towards discharge areas (river, pond) in response to hydraulic gradient. In confined aquifer, confining layer restricts the vertical movement of water, water is in high pressure. Groundwater can flow in any direction.

\# In unconfined

$$\frac{S}{T}\frac{\partial h}{\partial t} = \nabla^2 h$$

We can use this type of framework for solving our steady-state problem. We can start with a arbitrary initial value and we can go upto certain time and whether the variation is still there. Obviously, for steady state problem if you utilize unsteady state framework you should get the same result. This $\frac{\partial h}{\partial t}$ term will become very close to zero and we can solve this problem.

2-D IBVP:-

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

$S = 5\times10^{-5}$ (Storativity)

$T = 200\ m^2/day.$

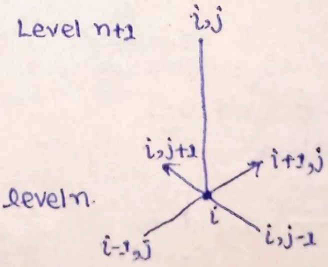
IC:- $h(x,y,0) = h_0(x,y)$

Γ_D^1 $h(0,y,t) = h_1(y)$

$h(L_x,y,t) = h_2(y)$

$\left.\frac{\partial h}{\partial y}\right|_{(x,0,t)} = 0$

$\left.\frac{\partial h}{\partial y}\right|_{(x,L_x,t)} = 0$



Time-space discretization:

Explicit Scheme:-

Discretized finite difference equation:-

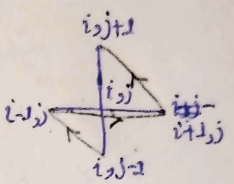
$$\frac{S}{T}\frac{h^{n+1}_{i,j} - h^{n}_{i,j}}{\Delta t} = \frac{h^{n}_{i+1,j} - 2h^{n}_{i,j} + h^{n}_{i-1,j}}{\Delta x^2} + \frac{h^{n}_{i,j-1} - 2h^{n}_{i,j} + h^{n}_{i,j+1}}{\Delta y^2}$$

\# Space derivatives are evaluated as present time level.

In simplified form, we have,

$$h_{i,j}^{n+1} = h_{i,j}^{n} + \underbrace{\frac{T\Delta t}{S\Delta x^2}}_{\alpha x}\left(h_{i-1,j}^{n} - 2h_{i,j}^{n} + h_{i+1,j}^{n}\right) + \underbrace{\frac{T\Delta t}{S\Delta y^2}}_{\alpha y}\left(h_{i,j-1}^{n} - 2h_{i,j}^{n} + h_{i,j+1}^{n}\right)$$

$$\therefore h_{i,j}^{n+1} = \alpha y\, h_{i,j-1}^{n} + \alpha x\, h_{i-1,j}^{n}$$
$$+ \left[1-2(\alpha x+\alpha y)\right] h_{i,j}^{n} + \alpha x\, h_{i+1,j} + \alpha y\, h_{i,j+1}$$



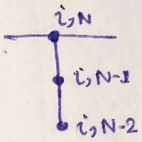
Neumann Boundary Condition:-

All values are evaluated at future time level.

Top Boundary:-

$$\frac{3h_{i,N}^{n+1} - 4h_{i,N-1}^{n+1} + h_{i,N-2}^{n+1}}{2\Delta y} = 0$$

$$\Rightarrow h_{i,N}^{n+1} = \frac{4}{3}h_{i,N-1}^{n+1} - \frac{1}{3}h_{i,N-2}^{n+1}$$

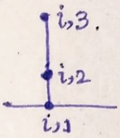


Bottom boundary:

~~$-3h_{i,N}^{n+1} + 4h_{i,N-1}$~~

$$\frac{-3h_{i,1}^{n+1} + 4h_{i,2}^{n+1} - h_{i,3}^{n+1}}{2\Delta y} = 0$$

$$h_{i,1}^{n+1} = \frac{4}{3}h_{i,2}^{n+1} - \frac{1}{3}h_{i,3}^{n+1}$$



Standard step:-

Explicit Scheme: Time-stepping Algorithm:

Data:- $S, T, \Delta x, \Delta y, \Delta t, h^n$ at time-step n.

Result: updated h^{n+1} at time step (n+1)

while t<end time do

for interior points

$$h_{i,j}^{n+1} = \alpha y\, h_{i,j-1}^{n} + \alpha x\, h_{i-1,j}^{n} + \left[1-2(\alpha x+\alpha y)\right] h_{i,j}^{n} + \alpha x\, h_{i+1,j}^{n} + \alpha y\, h_{i,j+1}^{n}$$

For boundary points: Use boundary conditions.

$n \leftarrow n+1$

end

Stability Criteria:

$$(\alpha x + \alpha y) < \frac{1}{2}$$

~~tmax~~ → Time max, may be 5 days!

salpha → sumalpha

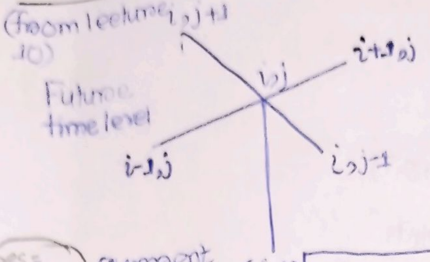
NPTEL-33

Unsteady 2D flow using F.D

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

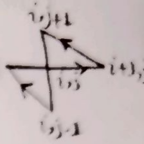
Implicit formulation:-

(from lecture 10)



Discretized finite difference equation in simplified form

$$\frac{S}{T}\frac{h_{i,j}^{n+1} - h_{i,j}^{n}}{\Delta t} = \frac{h_{i+1,j}^{n+1} - 2h_{i,j}^{n+1} + h_{i-1,j}^{n+1}}{\Delta x^2} + \frac{h_{i,j+1}^{n+1} - 2h_{i,j}^{n+1} + h_{i,j-1}^{n+1}}{\Delta y^2}$$



res = RHS-LHS

current time level (i,j)

$$\boxed{\alpha_y h_{i,j-1}^{n+1} + \alpha_x h_{i-1,j}^{n+1} - [1+2(\alpha_x+\alpha_y)]\, h_{i,j}^{n+1} + \alpha_x h_{i+1,j}^{n+1} + \alpha_y h_{i,j+1}^{n+1} = -h_{i,j}^{n}} \quad \dots (58)$$

I think, → Residue =(RHS-LHS) (Discretized portion should be always in LHS portion) → Actually, formula is, res = discretized portion - non-discretized portion.

$\boxed{res = a+2b+c = d}$

$\frac{1}{2}\{d-(a+2b+c)\} \Rightarrow -a-2b-c=-d$

$res \Rightarrow \frac{1}{-2}\{-d-(a-2b-c)\}$

where, $\alpha_x = \frac{T\Delta t}{S\Delta x^2}$ and $\alpha_y = \frac{T\Delta t}{S\Delta y^2}$

res ≠ $d = a+2b+c$

$\frac{1}{2}\{a+2b+c-d\}$

From lecture 29, iteration starts with the guess value,

$$h^{n+1}\Big|^{(0)} = \left[h_{1,1}^{n+1}\Big|^{(0)}\ h_{1,2}^{n+1}\Big|^{(0)} \cdots\cdots h_{M,N-1}^{n+1}\Big|^{(0)}\ h_{M,N}^{n+1}\Big|^{(0)}\right]^T$$

Gauss Seidal step, (From (39), we have,

$$h_{i,j}^{n+1}\Big|^{(P)} = \cancel{h_{i,j}^{n+1}\Big|^{(P)}+}$$

$$= h_{i,j}^{n+1}\Big|^{(P-1)} + \frac{1}{-[1+2(\alpha_x+\alpha_y)]}\left[-h_{i,j}^{n} - \left(\alpha_y h_{i,j-1}^{n+1}\Big|^{(P)} + \alpha_x h_{i-1,j}^{n+1}\Big|^{(P)} - [1-2(\alpha_x+\alpha_y)]\, h_{i,j}^{n+1}\Big|^{(P-1)} + \alpha_x h_{i+1,j}^{n+1}\Big|^{(P-1)} + \alpha_y h_{i,j+1}^{n+1}\Big|^{(P-1)}\right)\right]$$

In compact form,

$$\boxed{h_{i,j}^{n+1}\Big|^{(P)} = h_{i,j}^{n+1}\Big|^{(P-1)} + \frac{Res_{i,j}}{[-1-2(\alpha_x+\alpha_y)]}}, \quad \forall (i,j),\ P \geq 1 \quad \dots\dots (59)$$

For top boundary,

$$\boxed{3h_{i,N}^{n+1} - 4h_{i,N-1}^{n+1} + h_{i,N-2}^{n+1} = 0}$$

For bottom boundary,

$$\boxed{-3h_{i,1}^{n+1} + 4h_{i,2}^{n+1} - h_{i,3}^{n+1} = 0} \quad \dots\dots (60)$$

Standard steps:

Data:- $S, T, \Delta x, \Delta y, \Delta t, h^n$ at time-step n

Result:- Updated h^{n+1} at time-step $(n+1)$.

while t<end time do,

Inside one space loop

while t<end time do,

For interior and boundary points:- Solve governing equations and boundary conditions in discretized form.

$n \leftarrow n+1$

end

Outside one time loop.

In explicit scheme, first we need to solve the interior points. Then, we need to ~~solve~~ update the boundary points.

?! Here, we need to solve these simultaneously.

Groundwater Hydraulics:-

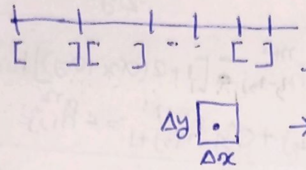
Unsteady Two dimensional flow using FVM

NPTEL-34

For confined aquifer-

Previously,

$h(x) \longrightarrow h(x,y) \longrightarrow h(x,y,t)$

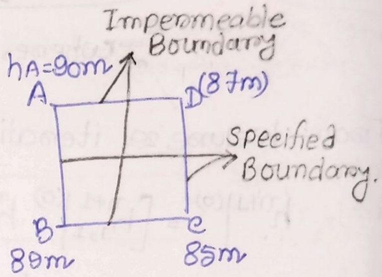


FD→ node points.

→FVM→ 1D cells w

→ For 2 dimensions

Quasi Steady state approach.



IBVP $\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$

(Homogenous, isotropic, confined aquifer condition)

$S = 5\times10^{-5}$

$T = 200\,m^2/day.$

Problem definition:-

IC:- $h(x,y,0) = h_0(x,y)$

#In transient problem, we need to exactly specify the actual initial condition in the field. But, if we are solving the steady state problem, we can start with any arbitrary initial guess for the steady-state condition. If the guess value is close to actual value, number of iterations should be relatively small.

$\Gamma_D^1 : h(0,y,t) = h_1(y)$

$\Gamma_D^2 : h(L_x,y,t) = h_2(y)$

$\Gamma_N^3 : \left.\frac{\partial h}{\partial y}\right|_{(x,0,t)} = 0$

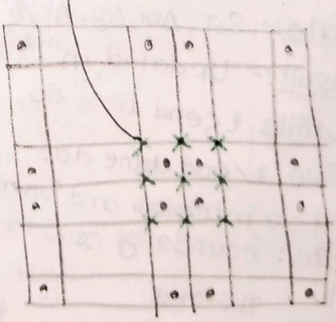
$\Gamma_N^4 \quad \left.\frac{\partial h}{\partial y}\right|_{(x,L_y,t)} = 0$

x Finite difference → Nodes

Finite volume → cells

9 cases are there

4 corners + 4 boundaries + 1 interior.



$$\frac{S}{T}\frac{\partial h}{\partial t} = \nabla^2 h \Rightarrow \iint\limits_{\Omega^t}^{t+\Delta t} \frac{S}{T}\frac{\partial h}{\partial t}\, d\Omega\, dt = \iint\limits_{\Omega_p t}^{t+\Delta t} \nabla\cdot(\nabla h)\, d\Omega\, dt$$

$\left.\begin{array}{l} l \to t \\ l \to t+\Delta t \end{array}\right\}$ Time level

Temporal term:-

R.-160

$$\int\limits_{\Omega^p}\int\limits_{t}^{t+\Delta t} \frac{S}{T}\frac{\partial h}{\partial t}\, d\Omega dt \rightarrow$$

Think, যদি সরাসরি লিখতে চাই,

$$\rightarrow \frac{S}{T}(h_p^{l+1} - h_p^{l})\Delta x \Delta y \,1$$

$$= \frac{S}{T}\int\limits_{t}^{t+\Delta t}\int\limits_{\Omega^p} \frac{\partial}{\partial t}(h d\Omega) dt$$

$$= \frac{S}{T}\int\limits_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left\{\Delta\Omega_p \times \frac{1}{\Delta\Omega_p}\int\limits_{\Omega_p} h d\Omega\right\} dt$$

$$= \frac{S}{T}\int\limits_{t}^{t+\Delta t} \frac{\partial}{\partial t}\left\{\Delta\Omega_p \cdot h_p\right\} dt$$

$h_p \Rightarrow$ spatially averaged value within the t cell.

$$= \frac{S}{T}\Delta\Omega_p (h_p^{l+1} - h_p^{l})$$

Spatial term:-

$$\int\limits_{t}^{t+\Delta t}\int\limits_{S} (\nabla h)\hat{n}\, ds\, dt$$

$$= \int\limits_{t}^{t+\Delta t} \left(\frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j}\right)\hat{n}\, ds\, dt$$

$$= \left(\frac{\partial h}{\partial x}\right)_e^{l+1} A_{xe} - \left(\frac{\partial h}{\partial x}\right)_w^{l+1} A_{xw} + \left(\frac{\partial h}{\partial y}\right)_n^{l+1} A_{yn} - \left(\frac{\partial h}{\partial y}\right)_s^{l+1} A_{ys}$$

$$= \left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1} - \left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y - \left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1} - \left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x.$$

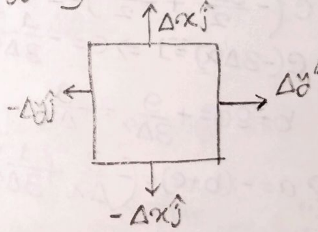
Gauss divergence theorem,

$$\iiint\limits_{\Omega} \nabla\cdot\vec{a}\, d\Omega = \iint\limits_{S} \vec{a}\,\hat{n}\, ds$$

(volume integral) $\rightarrow$ (surface integral).

Four faces $\rightarrow$ four surface integral,

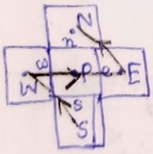
$\left.\begin{array}{l} A_{xe}\hat{i} \\ -A_{xw}\hat{i} \\ A_{yn}\hat{j} \\ -A_{ys}\hat{j} \end{array}\right\}$ Four faces and their area with directions



Implicit Scheme:

Completely discretized equation is:-

$$\boxed{\frac{S}{T}\frac{h_p^{l+1} - h_p^{l}}{\Delta t}\Delta x \Delta y = \left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1} - \left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y + \left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1} - \left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x} \quad \cdots (61)$$



$$\frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x} \qquad \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}.$$

Substituting these values,

$$\frac{S}{T}\frac{h_P^{l+1} - h_P^{l}}{\Delta t} = \frac{h_E^{l+1} - 2h_P^{l+1} + h_W^{l+1}}{\Delta x^2} + \frac{h_N^{l+1} - 2h_P^{l+1} + h_S^{l+1}}{\Delta y^2}$$

$$\Rightarrow \frac{T\Delta t}{S\Delta x^2}(h_E^{l+1} + h_W^{l+1}) + \frac{T\Delta t}{S\Delta y^2}(h_N^{l+1} + h_S^{l+1}) - \{1 + 2(\alpha_x + \alpha_y)\}h_P^{l+1} = -h_P^{l}$$

$$\Rightarrow \alpha_y h_S^{l+1} + \alpha_x h_W^{l+1} - \{1 + 2(\alpha_x + \alpha_y)\}h_P^{l+1} + \alpha_x h_E^{l+1} + \alpha_y h_N^{l+1} = -h_P^{l} \quad \cdots\cdots (62)$$

$\hookrightarrow$ Applicable for interior nodes.

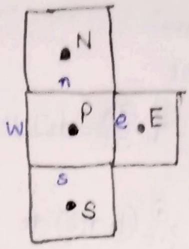
④

Boundary condition: Left Boundary:-

$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}$$

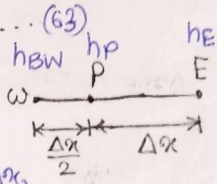
$$\left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}$$



For the west face, we have,

Let, $\left.\frac{\partial h}{\partial x}\right|_w^{l+1} = \cancel{\alpha h_{BW}+\beta h_P+\delta}$

$$=(a\,h_{BW}+b\,h_P+c.\,h_E) \quad \cdots\cdots (63)$$



Now, $h_{BW}=h_i=h(x_i)$

$h_P = h_{i+1/2} = \cancel{(h_i+\frac{1}{2}\Delta x)} = h(x_i+\frac{\Delta x}{2})$

$h_E = h_{i+3/2} = \cancel{(h_i+\frac{3}{2}\Delta x)} = h(x_i+\frac{3}{2}\Delta x)$

$$h_{i+1/2} = h_i + \left.\frac{\partial h}{\partial x}\right|_i \frac{\Delta x}{2} + \left.\frac{\partial^2 h}{\partial x^2}\right|_i \left(\frac{\Delta x}{2}\right)^2 + \cdots\cdots$$

$$h_{i+\frac{3}{2}} = h_i + \left.\frac{\partial h}{\partial x}\right|_i \left(\frac{3\Delta x}{2}\right) + \left.\frac{\partial^2 h}{\partial x^2}\right|_i \left(\frac{3\Delta x}{2}\right)^2 + \cdots\cdots$$

From (63),

$a+b+c=0$ (coff of h_i)

$b\times\frac{\Delta x}{2}+c\times\frac{3\Delta x}{2}=1$ (coff. of $\left.\frac{\partial h}{\partial x}\right|_i$)

$b\times\left(\frac{\Delta x}{2}\right)^2+c\times\left(\frac{3\Delta x}{2}\right)^2=0$

$\Rightarrow \frac{b}{4}+\frac{9c}{4}=0 \Rightarrow b+9c=0 \Rightarrow b=-9c$

$-9c.\frac{\Delta x}{2}+c.\frac{3\Delta x}{2}=1$

$\Rightarrow c\left(-\frac{9\Delta x}{2}+\frac{3\Delta x}{2}\right)=1$

$\Rightarrow c(-3\Delta x)=1 \Rightarrow c=-\frac{1}{3\Delta x}$

$b=9c=+\frac{9}{3\Delta x}=+\frac{3}{\Delta x}$

$\Rightarrow a=-(b+c)=\left(-\frac{3}{\Delta x}+\frac{1}{3\Delta x}\right)=\frac{-9+1}{3\Delta x}=-\frac{8}{3\Delta x}$

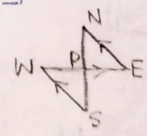
Now, $\left.\frac{\partial h}{\partial x}\right|_w^{l+1} = -\frac{8}{3\Delta x}h_{BW}+\frac{3}{\Delta x}h_P-\frac{1}{3\Delta x}h_E$

$$\boxed{\left.\frac{\partial h}{\partial x}\right|_w^{l+1} = \frac{-8h_{BW}+9h_P-h_E}{3\Delta x}} \quad \cdots\cdots (64)$$

Discretized equation for western face, (from 61),

$$\frac{S}{T}\,\frac{h_P^{l+1}-h_P^{l}}{\Delta t}\,\Delta x\Delta y = \left[\cancel{\left(\frac{\partial h}{\partial x}\right)}\frac{h_E^{l+1}-h_P^{l+1}}{\Delta x}-\frac{-8h_{BW}^{l+1}+9h_P^{l+1}-h_E^{l+1}}{3\Delta x}\right]\Delta y$$

$$+\left[\frac{h_N^{l+1}-h_P^{l+1}}{\Delta y}-\frac{h_P^{l+1}-h_S^{l+1}}{\Delta y}\right]\Delta x$$



$$\Rightarrow -h_P^{l}+h_P^{l+1} = \frac{T\Delta t}{S\Delta x^2}\left[\frac{4}{3}h_E^{l+1}-4h_P^{l+1}+\frac{8}{3}h_{BW}^{l+1}\right]+\frac{T\Delta t}{S\Delta y^2}\left[h_N^{l+1}-2h_P^{l+1}+h_S^{l+1}\right]$$

$$\Rightarrow \alpha_y h_S^{l+1}+(-4\alpha_x-2\alpha_y-1)h_P^{l+1}+\frac{4}{3}\alpha_x h_E^{l+1}+\alpha_y h_N^{l+1}$$

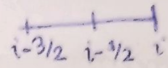
$$=-h_P^{l}-\frac{8}{3}\alpha_x h_{BW}^{l+1}$$

$$\Rightarrow \alpha_y h_S^{l+1} - [1+2(2\alpha_x+\alpha_y)]\, h_P^{l+1} + \frac{4}{3}\alpha_x h_E^{l+1} + \alpha_y h_N^{l+1} = -h_P^{l} - \frac{8}{3}\alpha_x h_{BW}^{l+1}.$$

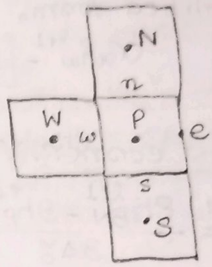
↳ # Eq^n for west boundary cell. ...... (65)

No term for h_W^{l+1} is there.

Right Boundary: $\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$



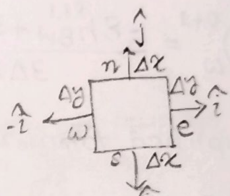
$$\Rightarrow \iiint \frac{S}{T}\frac{\partial h}{\partial t}\, d\Omega\, dt = \int_t^{t+\Delta t}\int_\Omega \nabla(\nabla h)\, d\Omega\, dt$$



$$\Rightarrow \frac{S}{T}\int_t^{t+\Delta t} \frac{\partial}{\partial t}\left\{\Omega_P \times \frac{1}{\Omega_P}\int h\, d\Omega\right\} dt$$

$$= \int_t^{t+\Delta t} (\nabla h)\hat{n}\, ds\, dt$$

$$\Rightarrow \frac{S}{T\Delta t}\Delta\Omega_P (h_P^{l+1} - h_P^{l}) = \left(\frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j}\right)\hat{n}\, ds\, \Delta t$$

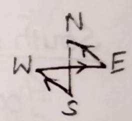


$$\Rightarrow \frac{S}{T\Delta t}(h_P^{l+1} - h_P^{l})\,\Delta x\Delta y = \left[\left(\frac{\partial h}{\partial x}\right)_e^{l+1} - \left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta x + \left[\left(\frac{\partial h}{\partial y}\right)_n^{l+1} - \left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta y$$

$$\boxed{\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x}}$$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \qquad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

$$\Rightarrow \frac{S}{T\Delta t}(h_P^{l+1} - h_P^{l})\,\Delta x\Delta y = \left\{\frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x} - \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}\right\}\Delta y + \left\{\frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} - \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}\right\}\Delta x$$



$$\Rightarrow h_P^{l+1} - h_P^{l} = \frac{T\Delta t}{S\Delta x^2}\left\{\frac{8}{3}h_{BE}^{l+1} - 4h_P^{l+1} + \frac{4}{3}h_W^{l+1}\right\} + \frac{T\Delta t}{S\Delta y^2}\left\{h_N^{l+1} - 2h_P^{l+1} + h_S^{l+1}\right\}$$

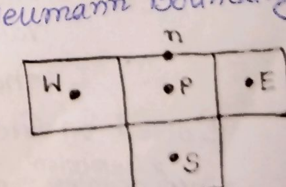
$$\Rightarrow -h_P^{l} - \frac{8}{3}\alpha_x h_{BE}^{l+1} = \alpha_y h_S^{l+1} + \frac{4}{3}\alpha_x h_W^{l+1} + (-1-4\alpha_x-2\alpha_y)\, h_P^{l+1} + \alpha_y h_N^{l+1}$$

$$\Rightarrow \alpha_y h_S^{l+1} + \frac{4}{3}\alpha_x h_W^{l+1} - [1+2(2\alpha_x+\alpha_y)]\, h_P^{l+1} + \alpha_y h_N^{l+1} = -h_P^{l} - \frac{8}{3}\alpha_x h_{BE}^{l+1} \quad \ldots\ldots (66)$$

Obviously, there is no h_E^{l+1} term in this case.

Top boundary:-

Here, $\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0$ (Neumann Boundary).



Also, $\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x}$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

⑥

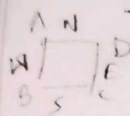
In simplified form,

$$\alpha_y h_S^{l+1} + \alpha_x h_W^{l+1} - [1+(2\alpha_x+\alpha_y)]h_P^{l+1} + \alpha_x h_E^{l+1} = -h_P^{l} \quad \cdots\cdots (67)$$

(No special term. i.e h_{BN}^{l+1}, because zero Neumann Boundary).

Bottom boundary:

In simplified form,



$$\alpha_x h_W^{l+1} - [1+(2\alpha_x+\alpha_y)]h_P^{l+1} + \alpha_x h_E^{l+1} + \alpha_y h_N^{l+1} = -h_P^{l} \quad \cdots (68)$$

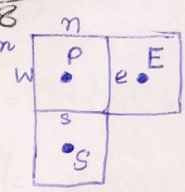
North-west corner: (A)

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0 \text{ (zero Neumann Boundary).}$$

$$\rightarrow 8h_{BN} = 9h_P - h_S$$

$$\rightarrow h_{BN} = \frac{9h_P - h_S}{8}$$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_e^{l+1}}{3\Delta x}.$$



In simplified form, we have,

$$\alpha_y h_S^{l+1} - [1+(4\alpha_x+\alpha_y)]h_P^{l+1} + \frac{4}{3}\alpha_x h_E^{l+1} = -h_P^{l} - \frac{8}{3}\alpha_x h_{BW}^{l+1} \quad \cdots (69)$$

যখন unknown point এর node number বেশি হবে, তখন, $= \frac{8-9+1}{3}$

যখন কম হবে, তখন $\frac{-8+9-1}{3}$

North east corner: (D)

$$\alpha_y h_S^{l+1} + \frac{4}{3}\alpha_x h_W^{l+1} - [1+(4\alpha_x+\alpha_y)]h_P^{l+1} = -h_P^{l} - \frac{8}{3}\alpha_x h_{BE}^{l+1} \quad \cdots\cdots (69)$$

South-east corner:- (C)

$$\frac{4}{3}\alpha_x h_W^{l+1} - [1+(4\alpha_x+\alpha_y)]h_P^{l+1} + \alpha_y h_N^{l+1} = -h_P^{l} - \frac{8}{3}\alpha_x h_{BE}^{l+1} \quad \cdots\cdots (70)$$

South-west corner:- (B)

$$-[1+(4\alpha_x+\alpha_y)]h_P^{l+1} + \frac{4}{3}\alpha_x h_E^{l+1} + \alpha_y h_N^{l+1} = -h_P^{l} - \frac{8}{3}\alpha_x h_{BW}^{l+1} \quad \cdots\cdots (71)$$

General form:-

In general form, the governing equation including the boundary conditions can be written as:-

$$a_S h_S^{l+1} + a_W h_W^{l+1} + a_P h_P^{l+1} + a_E h_E^{l+1} + a_N h_N^{l+1} = r_P$$

↳ # LHS ⇒ Pentagdiagonal structure

RHS ⇒ Known quantity ⇒ It may be known time level (h_P^{l}) value or ($h_{BN}^{l+1}, h_{BS}^{l+1}, h_{BE}^{l+1}, h_{BW}^{l+1}$) i.e boundary values, that are known.

Solved by Gauss siedel iterative techniques.

$$h_P^{l+1}\Big|^{(P)} = h_P^{l+1}\Big|^{(P-1)} + \omega\frac{Res}{a_P}$$

(P) → Iteration counter

NPTEL:-36

Surface Water Hydraulics

Gradually Varied Flow:

To solve GVF problem in open channels

channel flow → 1D

Flow depth $y = f(x)$ ⇒ For this course particularly.

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \quad \text{.......} \quad (72)$$

S_0 → Bed slope, S_f ⇒ Energy slope.

IC:- $y|_{x=0} = y_0$ ⇒ We may think this condition like zero-time condition. If flow depth is specified at a particular section of the channel, we may determine the variation of y with x in the channel.

S_f = friction slope

$$= \frac{n^2 Q^2}{R^{4/3} A^2}$$

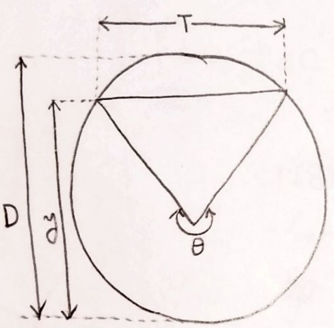
Fr = Froude's number $= \sqrt{\frac{Q^2 T}{g A^3}}$.

General format, $\frac{dy}{dx} = \psi(x, y)$ → # Previously we saw this format in lec7:- $\frac{d\phi}{dt} = \psi(t, \phi)$

$$\psi(x,y) = \frac{S_0 - S_f}{1 - Fr^2}$$

$$= \frac{S_0 - \frac{n^2 Q^2}{R^{4/3} A^2}}{1 - \frac{Q^2 T}{g A^3}} \quad \text{--------} \quad (73)$$

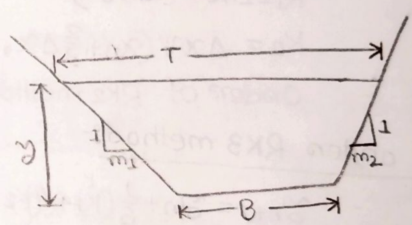
We can express this equation as $\psi(y)$ only; no x.



$A = \frac{1}{8}(\theta - \sin\theta) D^2$

$P = \frac{1}{2}\theta D$

$R = \frac{A}{P}$, $T = D\sin\left(\frac{\theta}{2}\right)$



$A = By + \frac{1}{2}(m_1 + m_2) y^2$

$P = B + \left(\sqrt{1 + y_1^2} + \sqrt{1 + y_2^2}\right) y.$

$R = \frac{A}{P}$,

$T = B + (m_1 + m_2) y.$

Critical depth,

$$y_c = \left(\frac{Q^2}{g B^2}\right)^{1/3}$$

Normal depth from manning's equation, $Q = \frac{1}{n} A R^{2/3} S_0^{1/2}$

Rectangular channel, $A = B y_n$, $P = B + 2y_n$

$$Q = \frac{1}{n}\left(\frac{B y_n}{B + 2y_n}\right)^{2/3} S_0^{1/2} B y_n \quad \text{......} \quad (74)$$

$$\frac{y_n^{5/3}}{(B+2y_n)^{2/3}}$$

$$= \frac{(B+2y_n)^{2/3} \cdot \frac{5}{3} \cdot y_n^{2/3} -}{(B+2y_n)^{4/3}}$$

In function form,

$$G(y_n) = \frac{S_0^{1/2} B^{5/3}}{n}\left(\frac{y_n}{B + 2y_n}\right)^{2/3} y_n - Q = 0 \quad \text{.......} \quad (75)$$

⑧

Newton Raphson's method,

$$y_n^{(p)} = y_n^{(p-1)} - \frac{G(y_n^{(p-1)})}{G'(y_n^{(p-1)})} \quad \text{......(75.1)}$$

Where, $G'(y_n) = \frac{S_0^{1/2} B^{5/3}}{3n} \cdot \frac{y_n^{2/3}(5B+6y_n)}{(B+2y_n)^{5/3}}$ ...... (76)

May be $\frac{dQ}{dy_n} = 0$ (∵ Q is not a function of y_n)

Euler's method,

$$y_{n+1} = y_n + \Delta x \psi(x_n, y_n) \quad \text{......(77)}$$

Order of Euler's method, $O(\Delta x)$.

Modified Euler's method:

$$y_{n+1} = y_n + k_2 \quad \text{......(78)}$$

$k_2 = \Delta x \psi(x_n + \frac{\Delta x}{2}, y_n + \frac{1}{2}k_1)$

$k_1 = \Delta x \psi(x_n, y_n)$.

Order: $O(\Delta x^2)$

Euler Cauchy method:

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \quad \text{......(79)}$$

$k_2 = \Delta x \psi(x_n + \Delta x, y_n + k_1)$

$k_1 = \Delta x \psi(x_n, y_n)$

Order: $O(\Delta x^2)$

2nd order Rk2:

$$y_{n+1} = y_n + \frac{1}{4}(k_1 + 3k_2) \quad \text{......(80)}$$

$k_1 = \Delta x \psi(x_n, y_n)$

$k_2 = \Delta x \psi(x_n + \frac{2}{3}\Delta x, y_n + \frac{2}{3}k_1)$

Order of RK2 method: $O(\Delta x^2)$.

3rd order RK3 method:-

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3) \quad \text{......(81)}$$

$k_1 = \Delta x \psi(x_n, y_n)$

$k_2 = \Delta x \psi(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1)$

$k_3 = \Delta x \psi(x_n + \Delta x, y_n - k_1 + 2k_2)$

Order: $O(\Delta x^3)$.

Fourth order RK method (RK4)

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{......(82)}$$

$k_1 = \Delta x \psi(x_n, y_n)$

$k_2 = \Delta x \psi(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1)$

$k_3 = \Delta x \psi(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_2)$

$k_4 = \Delta x \psi(x_n + \Delta x, y_n + k_3)$

Identify the type of GVF profile :-

Gauss elimination

$[\cancel{m},A, \cancel{n}C.A] = size(A)$

Data:- $Q = 20\,m^3/s$
$S_0 = 0.0008$
$n = 0.015$
$B = 15m$
$L_x = 210m$
$y_0 = 0.8m$
$g = 9.81\,m/s^2$

why absolute error is taken as 1? at 1st step?

Lecture-35

Groundwater Hydraulics

Unsteady flow in unconfined aquifer using FVM (2D)

The equations are non-linear.

Homogenous, isotropic system.

Confined aquifer flow ⇒ Storativity (S)

Unconfined aquifer flow ⇒ Specific yeild (S_y)

Governing Equation: 2-D (in space) IBVP can be written as:-

$$S_y\frac{\partial h}{\partial t} = \underbrace{\frac{\partial}{\partial x}\left(K_x h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h\frac{\partial h}{\partial y}\right)}_{\text{Nonlinearity comes due to these terms.}} + W \quad \text{-------- (83)}$$

$S_y = 0.25$

$K_x = K_y = K = 20\,m/day.$

In case of confined aquifer, $\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$

BC and IC: Same as Page-2.

1 (Interior cell) + Boundary cell (4) + Corner cell (4)

In FVM, GE is integrated over the element volume (in space) and time interval to form the discretized equation at node point P.

$$\int_t^{t+\Delta t}\left[\int_{\Omega_P} S_y\frac{\partial h}{\partial t}d\Omega\right]dt = \int_t^{t+\Delta t}\left[\int_{\Omega_P}\nabla\cdot\vec{F}\,d\Omega\right]dt + \int_t^{t+\Delta t}\left[\int_{\Omega_P} W\,d\Omega\right]dt \quad \text{----- (84)}$$

$\vec{F} = [f_x\ f_y]$

$f_x = K_x h\frac{\partial h}{\partial x}$

$f_y = K_y h\frac{\partial h}{\partial y}$

$F = f_x\hat{i} + f_y\hat{j}$ (Sir)

Now, $\nabla\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).(f_x\hat{i} + f_y\hat{j})$

$= \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left(K_x h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y . h\frac{\partial h}{\partial y}\right)$

Discretization: GE

Temporal term:

$$\int_t^{t+\Delta t}\left[\int_{\Omega P} S_y\frac{\partial h}{\partial t}d\Omega\right]dt$$

$$= S_y\int_t^{t+\Delta t}\frac{\partial}{\partial t}\left(\int_{\Omega P} h\,d\Omega\right)dt$$

We consider, there is no variation of h within the volume of cell Ω^P. So, $h = h_P$ (constant value)

$$= S_y\int_t^{t+\Delta t}\frac{\partial}{\partial t}(h_P\Delta\Omega_P)dt$$

$$= S_y(h_P^{l+1} - h_P^l)\Delta\Omega_P$$

$l \to t$, t^{th} time level.
$l+1 \to (t+1)^{th}$ time level.

$$= S_y(h_P^{l+1} - h_P^l)\Delta x\Delta y.$$

$\Delta\Omega_P = \Delta x.\Delta y.1$

---- (85)

Spatial term:-

$$\int_t^{t+\Delta t}\int_{\Omega_P} \nabla\cdot F \, d\Omega\, dt \qquad F = f_x\hat{i} + f_y\hat{j}$$

$$= \int_t^{t+\Delta t}\int_{\Omega_P} \nabla\cdot(f_x\hat{i}+f_y\hat{j})\, d\Omega\, dt$$

Gauss divergence theorem:-

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot d\vec{s}$$

$$= \int_t^{t+\Delta t}\left[\int_S (f_x\hat{i}+f_y\hat{j})\,d\vec{s}\right]dt \rightarrow (F\cdot d\vec{A})_f \rightarrow \text{for all faces.}$$

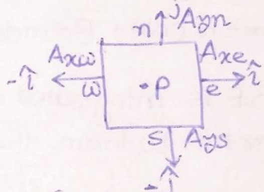
Why all values are taken at $(l+1)$ time step??

$$= \{(f_x\hat{i}+f_y\hat{j})_e^{l+1}\cdot(A_{xe}\hat{i}) + (f_x\hat{i}+f_y\hat{j})_w^{l+1}\cdot(A_{xw}(-\hat{i})) + (f_x\hat{i}+f_y\hat{j})_n^{l+1}\cdot(A_{yn}\hat{j}) + (f_x\hat{i}+f_y\hat{j})_s^{l+1}\cdot(A_{ys}(-\hat{j}))\}\cdot\Delta t$$

$$= (f_x)_e^{l+1}A_{xe} - (f_x)_w^{l+1}A_{xw} + (f_x)_n^{l+1}A_{yn} - (f_x)_s^{l+1}A_{ys}$$

$$= \left(K_x h\frac{\partial h}{\partial x}\right)_e^{l+1}\Delta y - \left(K_x h\frac{\partial h}{\partial x}\right)_w^{l+1}\Delta y + \left(K_y h\frac{\partial h}{\partial y}\right)_n^{l+1}\Delta x - \left(K_x h\frac{\partial h}{\partial x}\right)_s^{l+1}\Delta x \quad \text{---- (86)}$$

[For uniform grid system, $A_{xe} = A_{xw} = \Delta y$ and $A_{yn} = A_{ys} = \Delta x$]



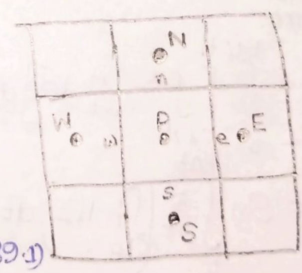
Source/Sink Term:-

$$\int_t^{t+\Delta t}\int_{\Omega_P} W(x,y)\,d\Omega\,dt = W(x_P, y_P)\,\Delta x\,\Delta y\,\Delta t \quad \text{---- (87)}$$

$\therefore$ compact form of the equation (From 84, 85, 86, 87)

$$S_y(h_P^{l+1} - h_P^{l})\,\Delta x\,\Delta y = \left[\left[\left(K_x h\frac{\partial h}{\partial x}\right)_e^{l+1} - \left(K_x h\frac{\partial h}{\partial x}\right)_w^{l+1}\right]\Delta y\right]\Delta t$$
$$+\left[\left[\left(K_y h\frac{\partial h}{\partial y}\right)_n^{l+1} - \left(K_x h\frac{\partial h}{\partial y}\right)_s^{l+1}\right]\Delta x\right]\Delta t$$
$$+ W(x_P, y_P)\,\Delta x\,\Delta y\,\Delta t \quad \text{..... (88)}$$

GE: Interior cells:



East face,

$$\left(K_x h\frac{\partial h}{\partial x}\right)_e^{l+1} = K_{xe}h_e \times \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x}$$

at east face

K_x at east face

$$= K_{xe}\times\frac{h_E^{l+1} + h_P^{l+1}}{2}\times\frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \quad \text{..(89.1)}$$

Average of h_E and h_P.

West face,

$$\left(K_x h\frac{\partial h}{\partial x}\right)_w^{l+1} = K_{xw}h_w\,\frac{h_P^{l+1} - h_w^{l+1}}{\Delta x}$$

$$= K_{xw}\times\frac{h_P^{l+1} + h_W^{l+1}}{2}\times\frac{h_P^{l+1} - h_W^{l+1}}{\Delta x} \quad \text{..... (89.2)}$$

North face:

$$\left(K_y h \frac{\partial h}{\partial y}\right)_n^{l+1} = K_{yn} \times \frac{h_P^{l+1} + h_N^{l+1}}{2} \times \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y} \quad \cdots\cdots (89.3)$$

South face:-

$$\left(K_y h \frac{\partial h}{\partial y}\right)_s^{l+1} = K_{ys} \times \frac{h_P^{l+1} + h_S^{l+1}}{2} \times \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y} \quad \cdots\cdots (89.4)$$

Putting these values to (88),

Compact form:-

$$S_y \left(h_P^{l+1} - h_P^{l}\right) \Delta x \Delta y = \left[\frac{K_{xe}}{2} \cdot \frac{\left(h_E^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{\Delta x} \cdot \Delta y - \frac{K_{xw}}{2} \frac{\left(h_P^{l+1}\right)^2 - \left(h_W^{l+1}\right)^2}{\Delta x} \Delta y \right] \Delta t$$

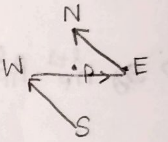
$$+ \left[\frac{K_{yn}}{2} \frac{\left(h_N^{l+1}\right)^2 - \left(h_P^{l+1}\right)^2}{\Delta y} \cdot \Delta x - \frac{K_{ys}}{2} \frac{\left(h_P^{l+1}\right)^2 - \left(h_S^{l+1}\right)^2}{\Delta y} \cdot \Delta x \right] \Delta t$$

$$+ W(x_P, y_P)\, \Delta x . \Delta y . \Delta t. \quad \cdots\cdots (90)$$

Again, compact form of (90),

$$h_P^{l+1} - h_P^{l} = \underbrace{\frac{K_x \Delta t}{2 S_y \Delta x^2}}_{\alpha_x} \left\{ \frac{\left(h_E^{l+1}\right)^2 - 2\left(h_P^{l+1}\right)^2 + \left(h_W^{l+1}\right)^2}{\cancel{\Delta x^2}} \right\} + \underbrace{\frac{K_y \Delta t}{2 S_y \Delta y^2}}_{\alpha_y} \left\{ \frac{\left(h_N^{l+1}\right)^2 - 2\left(h_P^{l+1}\right)^2 + \left(h_S^{l+1}\right)^2}{\cancel{\Delta y^2}} \right\}$$

$$+ \frac{W(x_P, y_P)}{S_y} \Delta t$$



$$\boxed{\begin{aligned} &\alpha_y \left(h_S^{l+1}\right)^2 + \alpha_x \left(h_W^{l+1}\right)^2 - \left[2(\alpha_x + \alpha_y)\right] \left(h_P^{l+1}\right)^2 - h_P^{l+1} \\ &+ \alpha_x \left(h_E^{l+1}\right)^2 + \alpha_y \left(h_N^{l+1}\right)^2 = - h_P^{l} - \frac{W(x_P, y_P) . \Delta t}{S_y} \end{aligned}} \quad \cdots\cdots (91)$$

where, $\alpha_x = \frac{K_x \Delta t}{2 S_y \Delta x^2}$, $\alpha_y = \frac{K_y \Delta t}{2 S_y \Delta y^2}$

Non-linear equation need to be solved by non-linear (h^2 is there) technique. Newton Raphson method.

In function discretized form, (91) becomes,

$$F_m\left(h^{l+1}\right) = \alpha_y \left(h_S^{l+1}\right)^2 + \cdots\cdots\cdots + \alpha_y \left(h_N^{l+1}\right)^2 - \underbrace{\left[- h_P^{l} - \frac{W(x_P, y_P)\Delta t}{S_y} \right]}_{\text{Constant. Because, we are considering these unknowns at } l+1 \text{ time level.}} = 0$$

Elements of Jacobian matrix,

☑ For interior nodes:-

This entries only for a single interior node. Each node of the domain captures one row of the Jacobian matrix.

$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$

$$J_W^m = \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1}$$

$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -1 - 4(\alpha_x + \alpha_y) h_P^{l+1} \quad \cdots (92)$$

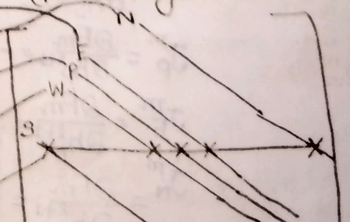
$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = 2\alpha_x h_E^{l+1}$$

$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

$\underset{M_c}{x}$ $\underset{N_c}{y}$ → Number of cells.

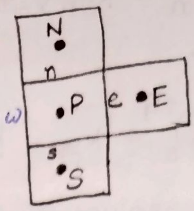
Size of Jacobian matrix, $(M_c N_c \times M_c N_c)$

(Penta diagonal structure)



For Boundary

Left boundary:-



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y}$$

$$\left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y} \quad \text{......} \ (93)$$

Now original eq^n from (88),

$$S_y(h_P^{l+1} - h_P^l)\Delta x \Delta y = \left[\left(K_x h\frac{\partial h}{\partial x}\right)_e^{l+1} \Delta y - \left(K_x h\frac{\partial h}{\partial x}\right)_w^{l+1} \Delta y\right]\Delta t + \left[\left(K_y h\frac{\partial h}{\partial y}\right)_n^{l+1} \Delta x - \left(K_y h\frac{\partial h}{\partial y}\right)_s^{l+1} \Delta x\right]\Delta t$$

$$\Rightarrow S_y \frac{h_P^{l+1} - h_P^l}{\Delta t} = K_x \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x^2} \cdot \frac{h_E^{l+1} + h_P^{l+1}}{2} - K_x h_{BW}^{l+1} \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x^2} + K_y \frac{h_P^{l+1} + h_N^{l+1}}{2} \cdot \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y^2} - K_y \frac{h_P^{l+1} + h_S^{l+1}}{2} \cdot \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y^2}$$

$$\Rightarrow h_P^{l+1} - h_P^l = \frac{K_x \Delta t}{2S_y \Delta x^2}\left\{(h_E^{l+1})^2 - (h_P^{l+1})^2 + \frac{16}{3}(h_{BW}^{l+1})^2 - 6h_{BW}^{l+1} h_P^{l+1} + \frac{2}{3} h_{BW}^{l+1} h_E^{l+1}\right\} + \frac{K_y \Delta t}{2S_y \Delta y^2}\left\{(h_N^{l+1})^2 - 2(h_P^{l+1})^2 + (h_S^{l+1})^2\right\}$$

$$\Rightarrow h_P^{l+1} - h_P^l = \alpha_x\{\text{......}\} + \alpha_y\{\text{.....}\}$$

where, $\frac{K_x \Delta t}{2S_y \Delta x^2} = \alpha_x$ and $\frac{K_y \Delta t}{2S_y \Delta y^2} = \alpha_y$

$$\Rightarrow -h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2 = \alpha_y (h_S^{l+1})^2 + (-\alpha_x - 2\alpha_y)(h_P^{l+1})^2 + (-1 - 6\alpha_x h_{BW}^{l+1}) h_P^{l+1} + \alpha_x (h_E^{l+1})^2 + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1} + \alpha_y (h_N^{l+1})^2$$

$\Rightarrow$ Rearranging,

$$\alpha_y (h_S^{l+1})^2 - (\alpha_x + 2\alpha_y)(h_P^{l+1})^2 - (1 + 6\alpha_x h_{BW}^{l+1}) h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 = -h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2 \quad \text{......} \ (94)$$

In the form of function, (From 94,)

$$F_m(h^{l+1}) = \alpha_y (h_S^{l+1})^2 - \text{........} + \alpha_y (h_N^{l+1})^2 - \left[h_P^l - \frac{16}{3}\alpha_x (h_{BW}^{l+1})^2\right] = 0$$

Elements of Jacobian matrix,

$$J_S^m = \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1}$$

No term for J_W. $\therefore J_W^m = 0$

$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -(1 + 6\alpha_x h_{BW}^{l+1}) - 2(\alpha_x + 2\alpha_y) h_P^{l+1}$$

$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = \frac{2}{3}\alpha_x h_{BW}^{l+1} + 2\alpha_x h_E^{l+1}$$

$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}$$

For Right Boundary:-

$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x}$$

Equation in simplified form,

$$\alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - (\alpha_x + 2\alpha_y)(h_P^{l+1})^2$$
$$- (1 + 6\alpha_x h_{BE}^{l+1}) h_P^{l+1} + \alpha_y (h_N^{l+1})^2 = -h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2$$

In function discretized form,

$$F_m(h^{l+1}) = \alpha_y (h_S^{l+1})^2 + \cdots\cdots + \alpha_y (h_N^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2\right] = 0 \quad \cdots (95)$$

Elements of Jacobian matrix,

$$\left.\begin{aligned}
J_S^m &= \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1} \\
J_W^m &= \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1} + \frac{2}{3}\alpha_x h_{BE}^{l+1} \\
J_P^m &= \frac{\partial F_m}{\partial h_P^{l+1}} = -(1 + 6\alpha_x h_{BE}^{l+1}) - 2(\alpha_x + 2\alpha_y) h_P^{l+1} \\
J_E^m &= 0 \\
J_N^m &= 2\alpha_y h_N^{l+1}
\end{aligned}\right\} \cdots (96)$$

Top Boundary:

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0$$

In simplified form,

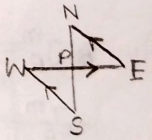
$$\alpha_y (h_S^{l+1})^2 + \alpha_x (h_W^{l+1})^2 - (2\alpha_x + \alpha_y)(h_P^{l+1})^2 - h_P^{l+1} + \alpha_x (h_E^{l+1})^2 = -h_P^l.$$

Function form,

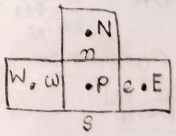
$$F_m(h^{l+1}) = \alpha_y (h_S^{l+1})^2 + \cdots\cdots + \alpha_x (h_E^{l+1})^2 - [-h_P^l] = 0 \quad \cdots\cdots (97)$$

Elements of Jacobian matrix,

$$\left.\begin{aligned}
J_S^m &= 2\alpha_y h_S^{l+1} \\
J_W^m &= 2\alpha_x h_W^{l+1} \\
J_P^m &= -2(2\alpha_x + \alpha_y) h_P^{l+1} - 1 \\
J_E^m &= 2\alpha_x h_E^{l+1}
\end{aligned}\right\} \cdots\cdots (98)$$



Bottom boundary:- (Try this)



Original equation,

$$S_y (h_P^{l+1} - h_P^l)\Delta x \Delta y =$$

$$\int_\Omega \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)\cdot\left(K_x h \frac{\partial h}{\partial x}\hat{i} + K_y h \frac{\partial h}{\partial y}\hat{j}\right) d\Omega$$

$$\int_S \left(K_x h \frac{\partial h}{\partial x}\hat{i} + K_y h \frac{\partial h}{\partial y}\hat{j}\right)\cdot d\vec{S}$$

$$= \left(K_x h \frac{\partial h}{\partial x}\hat{i} + K_y h \frac{\partial h}{\partial y}\hat{j}\right)_e \cdot A_{xe}\hat{i} + (\cdots\cdots)_w A_{xw}(-\hat{i}) + \cdots$$

$$S_y \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial h}{\partial y}\right)$$

Integrate over volume and time

$$\left.\begin{aligned}
&S_y (h_P^{l+1} - h_P^l)\Delta x \Delta y \\
&= \sum K_x h \left(\frac{\partial h}{\partial x}\right)_f A_f
\end{aligned}\right\}$$

We find the derivative on the point pw

Volume to surface

Gauss Divergence

$$(\ldots\ldots)A_{yn}\hat{j} + (\ldots\ldots)A_{ys}(\hat{j})$$

$$=$$

$$\left[K_{xe}h_e\left(\frac{\partial h}{\partial x}\right)_e^{n+1}\Delta y - K_{xw}.h_w\left(\frac{\partial h}{\partial x}\right)_w^{n+1}\Delta y\right]\Delta t + \left[K_{yn}h_n^{l+1}\left(\frac{\partial h}{\partial y}\right)_n^{l+1}\Delta x - K_{ys}\cdot h_s^{l+1}\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\Delta x\right]\Delta t$$

$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x} \qquad \left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y}$$

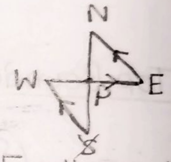
$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x} \quad \text{and} \quad \left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{8h_{BS}^{l+1} - 9h_P^{l+1} + h_N^{l+1}}{3\Delta y} = 0. \text{(Bottom Neumann Boundary).}$$

$$S_y(h_P^{l+1} - h_P^{l})\Delta x\Delta y = K_x\Delta t\Delta y\left[\frac{(h_E^{l+1})^2 - (h_P^{l+1})^2}{2\Delta x} - \frac{(h_P^{l+1})^2 - (h_W^{l+1})^2}{2\Delta x}\right] + K_y\Delta x\Delta t\left[\frac{(h_N^{l+1})^2 - (h_P^{l+1})^2}{2\Delta y} - \cancel{h_{BS}^{l+1}\frac{8h_{BS}-9h_P^{l+1}+h_N^{l+1}}{3\Delta y}}\right] \quad \ldots (98\text{-}1)$$

$$\Rightarrow (h_P^{l+1} - h_P^{l}) = \frac{K_x\Delta t}{2S_y\Delta x^2}\left[(h_E^{l+1})^2 - 2(h_P^{l+1})^2 + (h_W^{l+1})^2\right] + \frac{K_y\Delta t}{2S_y\Delta y^2}\left[(h_N^{l+1})^2 - (h_P^{l+1})^2 - \cancel{h_{BS}^{l+1}\left(\tfrac{16}{3}h_{BS}^{l+1} - 6h_P^{l+1} + \tfrac{2}{3}h_N^{l+1}\right)}\right]$$

$$\Rightarrow (h_P^{l+1} - h_P^{l}) = \alpha_x(h_W^{l+1})^2 + (-2\alpha_x - \alpha_y)(h_P^{l+1})^2 + \cancel{6h_{BS}^{l+1}(h_P^{l+1})} + \alpha_x(h_E^{l+1})^2 + \alpha_y(h_N^{l+1})^2 - \cancel{\tfrac{2}{3}h_{BS}^{l+1}(h_N^{l+1})} - \cancel{\tfrac{16}{3}(h_{BS}^{l+1})^2}$$

$$\Rightarrow \alpha_x(h_W^{l+1})^2 - [(2\alpha_x + \alpha_y)](h_P^{l+1})^2 + \cancel{6h_{BS}^{l+1}.h_P^{l+1}} - h_P^{l+1} + \alpha_x(h_E^{l+1})^2 + \alpha_y(h_N^{l+1})^2 - \cancel{\tfrac{2}{3}h_{BS}^{l+1}.h_N^{l+1}} = -h_P^{l} + \cancel{\tfrac{16}{3}(h_{BS}^{l+1})^2}$$



For the north face, we have

In function form,

$$F_m(h^{l+1}) = \alpha_x(h_W^{l+1})^2 - (2\alpha_x + \alpha_y)(h_P^{l+1})^2 - h_P^{l+1} + \alpha_x(h_E^{l+1})^2 + \alpha_y(h_N^{l+1})^2 - (-h_P^{l}) = 0 \quad \ldots (99)$$

Elements of Jacobian matrix,

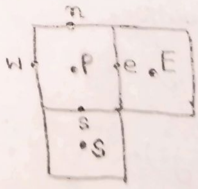
$J_S^m = 0,$

$$\left.\begin{aligned}
J_W^m &= \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1} \\
J_P^m &= \frac{\partial F_m}{\partial h_P^{l+1}} = -1 - 2(2\alpha_x + \alpha_y)h_P^{l+1} \\
J_E^m &= \frac{\partial F_m}{\partial h_E^{l+1}} = 2\alpha_x h_E^{l+1} \\
J_N^m &= \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}.
\end{aligned}\right\} \quad \ldots (100)$$

North-West Corner:-

$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x}.$$

$$\left(\frac{\partial h}{\partial x}\right)_w = \frac{-8h_{BW}^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$



$$\left(\frac{\partial h}{\partial y}\right)_n = \frac{8h_{BN}^{l+1} - 9h_P^{l+1} + h_S^{l+1}}{3\Delta y} = 0 \text{ (Neumann Boundary)}$$

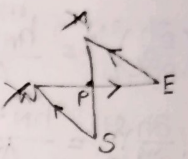
$$\left(\frac{\partial h}{\partial y}\right)_s = \frac{h_P^{l+1} - h_S^{l+1}}{\Delta y}$$

Putting these values equation 88, (Follow 98.1) → simplified form

$$S_y(h_P^{l+1}-h_P^{l})\Delta x\Delta y = K_x\Delta t\Delta y\left[h_e^{l+1}\left(\frac{\partial h}{\partial x}\right)_e^{l+1} - h_w^{l+1}\left(\frac{\partial h}{\partial x}\right)_w^{l+1}\right]$$
$$+ K_y\Delta t\Delta x\left[h_n^{l+1}\cancel{\left(\frac{\partial h}{\partial y}\right)_n^{l+1}}^{\,0} - h_s^{l+1}\left(\frac{\partial h}{\partial y}\right)_s^{l+1}\right]$$

$$\Rightarrow S_y(h_P^{l+1}-h_P^l)\Delta x\Delta y = K_x\Delta y\Delta t\left[\frac{(h_E^{l+1})^2-(h_P^{l+1})^2}{2\Delta x} - h_{BW}^{l+1}\cdot\left(\frac{-8h_{BW}^{l+1}+9h_P^{l+1}-h_E^{l+1}}{3\Delta x}\right)\right]$$
$$- K_y\Delta x\Delta t\left[\frac{(h_P^{l+1})^2-(h_S^{l+1})^2}{2\Delta y}\right]$$

$$\Rightarrow h_P^{l+1}-h_P^l = \overbrace{\frac{K_x\Delta t}{2S_y\Delta x^2}}^{\alpha_x}\left[(h_E^{l+1})^2-(h_P^{l+1})^2 - h_{BW}\left(-\frac{16}{3}h_{BW}^{l+1}+6h_P^{l+1}-\frac{2}{3}h_E^{l+1}\right)\right]$$
$$- \overbrace{\frac{K_y\Delta t}{2S_y\Delta y^2}}^{\alpha_y}\left[(h_P^{l+1})^2-(h_S^{l+1})^2\right]$$



$$\Rightarrow h_P^{l+1}-h_P^l = \alpha_y(h_S^{l+1})^2 + (-\alpha_y-\alpha_x)(h_P^{l+1})^2 +$$
$$(-1-6\alpha_x h_{BW}).h_P^{l+1} + \alpha_x(h_E^{l+1})^2 + \alpha_x(-h_{BW}^{l+1})\left(-\frac{2}{3}\right)h_E^{l+1} + \alpha_x(h_{BW}^{l+1})^2\frac{16}{3}.$$

$$\Rightarrow \alpha_y(h_S^{l+1})^2-(\alpha_x+\alpha_y)(h_P^{l+1})^2-(1+6\alpha_x h_{BW})h_P^{l+1}+\alpha_x(h_E^{l+1})^2+\frac{2}{3}\alpha_x h_{BW}^{l+1}.h_E^{l+1} = -h_P^l$$
$$-\frac{16}{3}\alpha_x(h_{BW}^{l+1})^2.$$

...... (101).

In function form, we have,

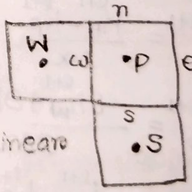
$$F_m(h^{l+1}) = \alpha_y(h_S^{l+1})^2 - \cdots\cdots + \frac{2}{3}\alpha_x h_{BW}^{l+1}h_E^{l+1} - \left(-h_P^l-\frac{16}{3}\alpha_x(h_{BW}^{l+1})^2\right) = 0$$

Elements of Jacobian matrix,

$$\left.\begin{aligned}
\frac{\partial F_m}{\partial h_S^{l+1}} &= 2\alpha_y h_S^{l+1}\\
\frac{\partial F_m}{\partial h_P^{l+1}} &= -(1+6\alpha_x h_{BW}^{l+1})-2(\alpha_x+\alpha_y)h_P^{l+1}\\
\frac{\partial F_m}{\partial h_E^{l+1}} &= 2\alpha_x h_E^{l+1}+\frac{2}{3}\alpha_x h_{BW}^{l+1}
\end{aligned}\right\} \quad \text{1.........(102)}$$

North-east corner:

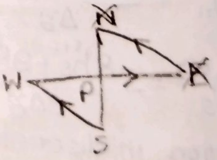
There will be h_{BE} in the function but, no h_{BN} would be there.
$\because h_{BE}(8h_{BE}-9h_P+h_W)$,
$\therefore h_W$ and h_P have both linear and square terms.



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1}-9h_P^{l+1}+h_W^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1}-h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{8h_{BN}^{l+1}-9h_P^{l+1}+h_S^{l+1}}{3\Delta y} = 0$$



$$\left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{h_P^{n+1}-h_S^{n+1}}{\Delta y}.$$

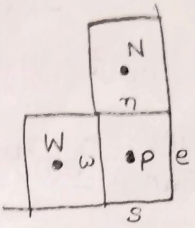
In function form, we have,

$$F_m(h^{l+1}) = \alpha_y(h_S^{l+1})^2+\alpha_x(h_W^{l+1})^2-(\alpha_x+\alpha_y)(h_P^{l+1})^2-(1+6\alpha_x h_{BE}^{l+1})h_P^{l+1}$$
$$+\frac{2}{3}\alpha_x h_{BE}^{l+1}h_W^{l+1}-\left[-h_P^l-\frac{16}{3}\alpha_x(h_{BE}^{l+1})^2\right]=0 \quad \text{.......(103)}$$

Elements of Jacobian matrix,

$$\left.\begin{aligned} J_S^m &= \frac{\partial F_m}{\partial h_S^{l+1}} = 2\alpha_y h_S^{l+1} \\ J_W^m &= \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1} + \frac{2}{3}\alpha_x h_{BE}^{l+1} \\ J_P^m &= \frac{\partial F_m}{\partial h_P^{l+1}} = -2(\alpha_x+\alpha_y) h_P - (1+6\alpha_x h_{BE}^{l+1}) \end{aligned}\right\} \quad \ldots\ldots (104)$$

South East Corner:



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{8h_{BE}^{l+1} - 9h_P^{l+1} + h_W^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{h_P^{l+1} - h_W^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y}$$

$$\left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_S^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

In function form of discretized equation:-

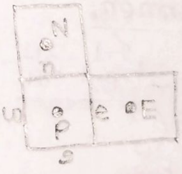
$$F_m(h^{l+1}) = \alpha_x (h_W^{l+1})^2 + \frac{2}{3}\alpha_x h_{BE}^{l+1} h_W^{l+1} - (\alpha_x+\alpha_y)(h_P^{l+1})^2 - (1+6\alpha_x h_{BE}^{l+1}) h_P^{l+1} + \alpha_y (h_N^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BE}^{l+1})^2\right] = 0 \quad \ldots\ldots (105)$$

Elements of Jacobian matrix,

$$\left.\begin{aligned} J_W^m &= \frac{\partial F_m}{\partial h_W^{l+1}} = 2\alpha_x h_W^{l+1} \\ J_P^m &= \frac{\partial F_m}{\partial h_P^{l+1}} = -(1+6\alpha_x h_{BE}^{l+1}) - 2(\alpha_x+\alpha_y) h_P^{l+1} \\ J_N^m &= \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1} \end{aligned}\right\} \quad \ldots\ldots\ldots (106)$$

$$J_E^m = J_S^m = 0$$

South-West corner:



$$\left(\frac{\partial h}{\partial x}\right)_e^{l+1} = \frac{h_E^{l+1} - h_P^{l+1}}{\Delta x}$$

$$\left(\frac{\partial h}{\partial x}\right)_w^{l+1} = \frac{-8h_W^{l+1} + 9h_P^{l+1} - h_E^{l+1}}{3\Delta x}$$

$$\left(\frac{\partial h}{\partial y}\right)_n^{l+1} = \frac{h_N^{l+1} - h_P^{l+1}}{\Delta y}$$

$$\left(\frac{\partial h}{\partial y}\right)_s^{l+1} = \frac{-8h_S^{l+1} + 9h_P^{l+1} - h_N^{l+1}}{3\Delta y} = 0$$

Function in discretized form:-

$$F_m(h^{l+1}) = -(\alpha_x+\alpha_y)(h_P^{l+1})^2 - (1+6\alpha_x h_{BW}^{l+1}) h_P^{l+1} + \frac{2}{3}\alpha_x h_{BW}^{l+1} h_E^{l+1} + \alpha_x (h_E^{l+1})^2 + \alpha_y (h_N^{l+1})^2 - \left[-h_P^l - \frac{16}{3}\alpha_x (h_{BW})^2\right] = 0 \quad \ldots\ldots (107)$$

Elements of Jacobian matrix,

Element of Jacobian matrix for east face.

$$J_P^m = \frac{\partial F_m}{\partial h_P^{l+1}} = -2(\alpha_x+\alpha_y)h_P^{l+1} - (1+6\alpha_x h_{BW}^{l+1})$$

$$J_E^m = \frac{\partial F_m}{\partial h_E^{l+1}} = \frac{2}{3}\alpha_x h_{BW}^{l+1} + 2\alpha_x h_E^{l+1}$$

$$J_N^m = \frac{\partial F_m}{\partial h_N^{l+1}} = 2\alpha_y h_N^{l+1}.$$

$$J_S^m = J_W^m = 0. \quad \ldots\ldots\ldots (108)$$

In general form, GE including boundary condition can be written as,

$$J_S^m \Delta h_S^{l+1} + J_W^m \Delta h_W^{l+1} + J_P^m \Delta h_P^{l+1} + J_E^m \Delta h_E^{l+1} + J_N^m \Delta h_N^{l+1} = -F_m(h^{l+1}) \ldots\ldots (109)$$

From R1-Eq(45),

$$\{\vec{F}_m(h^{l+1}+\Delta h^{l+1})\} = \{\vec{F}_m(h^{l+1})\} + [J(h^{l+1})]\{\vec{\Delta h}^{l+1}\}$$

This is applicable for a particular cell.

Consider this value as zero (Assuming, $h+\Delta h$ as the solution)

$$[J(h^{l+1})]\{\Delta h^{l+1}\} = -\{F_m(h^{l+1})\}$$ ⇒ This is actually eqn (109).

(109) can be written as:-

$$J_S^m\left[h_S^{l+1}\big|^{(P)} - h_S^{l+1}\big|^{(P-1)}\right] + J_W^m\left[h_W^{l+1}\big|^{(P)} - h_W^{l+1}\big|^{(P-1)}\right] + J_P^m\left[h_P^{l+1}\big|^{(P)} - h_P^{l+1}\big|^{(P-1)}\right]$$
$$+ J_E^m\left[h_E^{l+1}\big|^{(P)} - h_E^{l+1}\big|^{(P-1)}\right] + J_N^m\left[h_N^{l+1}\big|^{(P)} - h_N^{l+1}\big|^{(P-1)}\right] = -F_m\left(h_S^{l+1}\big|^{(P)}, h_W^{l+1}\big|^{(P)}, h_P^{l+1}\big|^{(P-1)}, h_E^{l+1}\big|^{(P-1)}, h_N^{l+1}\big|^{(P-1)}\right) \ldots\ldots (110)$$

Iteration goes from → This way.

So, The values for S cell and W cell are already available.

RHS means F_m is the function of these variables. (In function form).

Final iterative form can be written as, (from 110),

$$J_P^m \cdot h_P^{l+1}\big|^{(P)} = J_P^m\, h_P^{l+1}\big|^{(P-1)} + Res$$

$$\Rightarrow \boxed{h_P^{l+1}\big|^{(P)} = h_P^{l+1}\big|^{(P-1)} + \frac{Res}{J_P^m}} \ldots\ldots (111)$$

Here,

$$Res = -F_m - \Big[\underbrace{J_S^m\, \Delta h_S^{l+1}\big|^{(P)} + J_W^m\, \Delta h_W^{l+1}\big|^{(P)} + J_E^m\, \Delta h_E^{l+1}\big|^{(P-1)} + J_N^m\, \Delta h_N^{l+1}\big|^{(P-1)}}_{}\Big] \ldots\ldots (111)$$

In this case, residual excludes the central term.

We can start the iteration and get converged value of h_P^{l+1}. After getting convergence through this iterative format, we can transfer this value to h_P^l. $h_P^l \leftarrow h_P^{l+1}$.

Time loop

Space loop

Space loop can be solved using the iterative form of eqn (111) and we can

$n \leftarrow n+1$ update time. and for updated time, we can transfer this (n+1) level value to nth level directly).

NPTEL: Lecture-37

Module-4, unit-3

Surface Water hydraulics,
Steady-Channel flow (Single/Series)

steady channel flow problem (single or series) using implicit method.

Although, this implicit approach is not related to time. We will talk about 'steady channel flow'. → Solution of non-linear discretized equation will be done.

GVF:- $\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F_r^2}$ .....(112) Initial condition:- $y(x=0) = y_0$

We get, $y(x)$
↓
classify GVF type.
↓
But for classification, we need CDL and NDL (y_c and y_n)

Previously, We have solved one IVP, condition specified at one end. Value at other end was unknown.

For this problem, we would solve, continuity and momentum equations simultaneously. But for special cases, we can continue with one ~~base~~ equation to get the solution.

Multiple channels in series and are directly connected.

We consider, At the junction point, depth at the left side and right side are equal. If there is a hydraulic structure at the junction point, different condition.

We would solve the problem as BVP.

Boundary value problem:

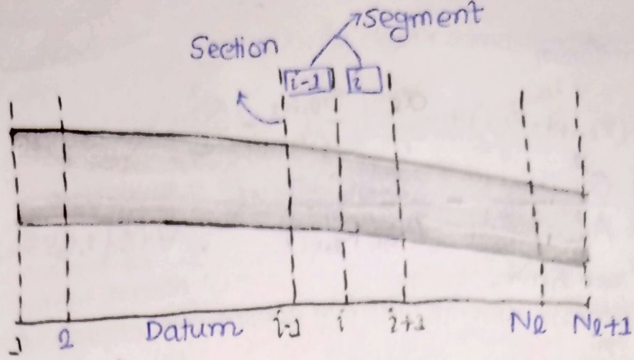
Continuity: $\frac{dQ}{dx} = 0$

Momentum eqn: $\frac{dE}{dx} = -S_f$

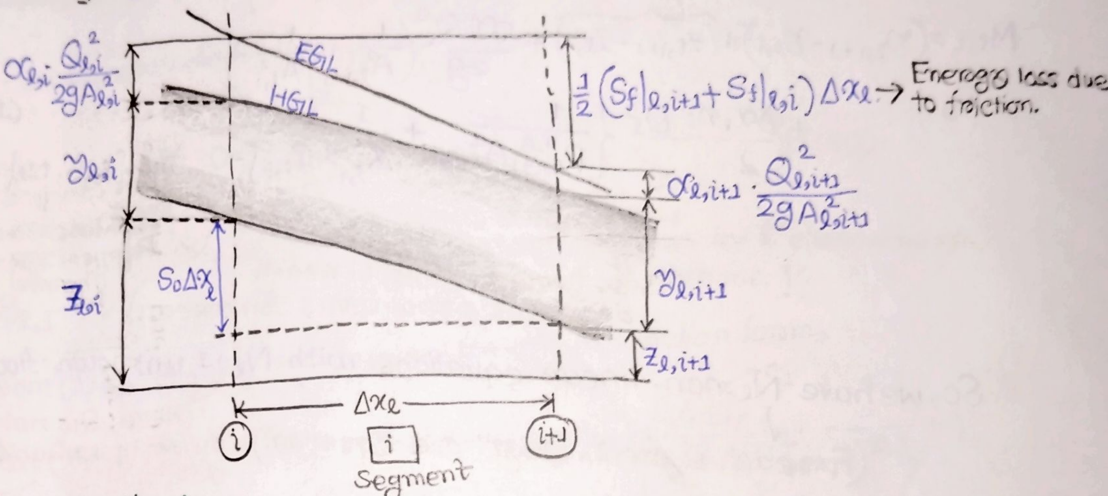
where, $E = y + z + \frac{\alpha Q^2}{2gA^2}$ (α = Momentum correction factor)

S_f = Friction slope $= \frac{n^2 Q^2}{R^{4/3} A^2}$

z = Elevation of channel bottom wrt to datum



Number of segments = N_l
" " sections = N_l+1



Discretization:-

Continuity equation:-

for ith segment of lth channel, (i = Any segment of the channel)

$$\left.\frac{dQ}{dx}\right|_i = 0$$

$$\Rightarrow \frac{Q_{l,i+1}-Q_{l,i}}{\Delta x_l} = 0 \quad \text{------------} (113)$$

$$\Rightarrow Q_{l,i+1} = Q_{l,i}$$

For single channel, $Q_{l,1} = Q_{l,2} = \cdots\cdots = Q_{l,N_l+1} = Q_l$.

Where, N_l = Number of segments for lth channel reach.

Discretization: Momentum equation:

For ith segment (connected to i and i+1th section) of lth channel reach, discretized form,

$$\frac{dE}{dx} = -S_f$$

$$Q = \frac{A}{n}R^{2/3}S^{1/2}$$

$$\Rightarrow \frac{E_{l,i+1}-E_{l,i}}{\Delta x_l} = -\frac{1}{2}\left(S_f|_{l,i+1} + S_f|_{l,i}\right) \quad \text{----------} (114)$$

In expanded form,

$$\frac{\left(y+z+\frac{\alpha Q^2}{2gA^2}\right)_{l,i+1} - \left(y+z+\frac{\alpha Q^2}{2gA^2}\right)_{l,i}}{\Delta x_l} = -\frac{1}{2}\left[\left(\frac{n^2Q^2}{A^2R^{4/3}}\right)_{l,i+1} + \left(\frac{n^2Q^2}{A^2R^{4/3}}\right)_{l,i}\right] \quad \text{----} (115)$$

Here, unknown variable is flow depth(y) only.

Because, for single channel or channel in series Q is constant. Also, steady, so no change with time.

z, Q, n are constant.

y, A(y), R(y) ⇒ Functions of y.

From eqn (115), function form

Momentum function for l^{th} channel reach and i^{th} segment.

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q_{l,i+1}^2}{A_{l,i+1}^2} - \frac{Q_{l,i}^2}{A_{l,i}^2}\right) + \frac{\Delta x_l \cdot n_l^2}{2}\left\{\frac{Q_{l,i+1}^2}{A_{l,i+1}^2 R_{l,i+1}^{4/3}} - \frac{Q_{l,i}^2}{A_{l,i}^2 R_{l,i}^{4/3}}\right\} = 0, \quad \forall i \in \{1,2,\ldots,N_l\}$$

In reduced form, we have,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + \underbrace{(z_{l,i+1} - z_{l,i})}_{\text{constant terms for two consecutive channel sections.}} + \frac{\alpha_l Q_l^2}{2g}\left(\frac{1}{A_{l,i+1}^2} - \frac{1}{A_{l,i}^2}\right) + \frac{\Delta x_l n_l^2 Q_l^2}{2}\left\{\frac{1}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{1}{R_{l,i}^{4/3} A_{l,i}^2}\right\} = 0 \quad \forall i \in \{1,\ldots N_l\} \quad \text{......(116)}$$

N_l = Total number of segment for l^{th} channel reach.

These two are channel reach dependent parameters for our case.

So, we have N_l non-linear equations with N_l+1 unknown flow depth

(For each segment, we get one equation.)

(Here, Q is not, y is our only variables or unknown for each section).

☑ Backwater effect:

obstruction or change in slope.

When there is obstruction, frictional resistance > Gravity (Driving force of water) force ⇒ kinetic energy decreases and potential energy increases in form of increase in the water level.

☑ Subcritical flow → Specified head → Downstream end

Supercritical flow → " → u/s end. why?

For subcritical flow, flow is affected by backwater effects, Upstream depth is not critical because flow is not affected by backwater effects in this region.

In supercritical flow, u/s depth is specified because it is affected by formation of hydraulic jump, but d/s is not affected. In hydraulic jump, energy is lost and u/s water depth increases.

! (The only way to decrease the energy is to increase the depth of flow? Because, the discharge in the channel is constant.

$$E = y + \frac{Q^2}{2gA^2} = y + \frac{Q^2}{2gB^2y^2}$$)

For subcritical flow, (we specify depth conditions at the downstream end)

$$y_{l,N_l+1} = y_d$$

$$\underbrace{DB_{l,N_l+1}}_{\substack{\text{Downstream Boundary condition}\\ \text{for } l\text{th channel reach}\\ \text{and } (l+1)^{th} \text{ section.}}} = y_{l,N_l+1} - y_d = 0 \quad \ldots\ldots\ldots (116.1)$$

→ সাধারণত segment no. কে Represent করা হয়: $M_{l,i}$, ith segment?? কিন্তু $(l+1)^{th}$ segment তো নেই!

→ u/s, d/s ছাড়া section numbering এ N_l+1 কেন?

→ Not a function like $M_{l,i}$? → See matrix (0.25) But not $(N_l+1)^{th}$ segment there!!

For supercritical flows, (Depth specified at u/s end).

$$y_{l,1} = y_u$$

$$UB_{l,N_l+1} = y_{l,1} - y_u = 0 \quad \ldots\ldots\ldots (116.2)$$

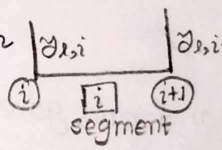
Value specified at 1st section of l^{th} channel. so, it should be $UB_{l,1}$??

→ I think, it represents U/s boundary condition for l^{th} channel reach. having N_l+1 no. of sections. !

From (116), discretized momentum eq^n, in function form,

$$\underbrace{M_{l,i}}_{\substack{\text{Momentum function for}\\ l^{th}\text{ channel reach}\\ \text{and } i^{th}\text{ segment.}}}(y_{l,i+1}, y_{l,i}) = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \overbrace{\frac{\alpha_l Q_l^2}{2g}}^{c_1}\left(\frac{1}{A^2_{l,i+1}} - \frac{1}{A^2_{l,i}}\right) + \underbrace{\frac{Q_l^2 n_l^2 \Delta x_l}{2}}_{c_2}\left[\frac{1}{R^{4/3}_{l,i+1} A^2_{l,i+1}} + \frac{1}{R^{4/3}_{l,i} A^2_{l,i}}\right] = 0 \quad \ldots\ldots (117)$$

For each equation, we have two unknown variables i.e two flow depths



In compact form, (117) written as:-

$$M_{l,i}(y_{l,i}\, y_{l,i+1}) = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + c_1\left(\frac{1}{A^2_{l,i+1}} - \frac{1}{A^2_{l,i}}\right) + c_2\left[\frac{1}{R^{4/3}_{l,i+1} A^2_{l,i+1}} + \frac{1}{R^{4/3}_{l,i} A^2_{l,i}}\right] \quad \ldots\ldots\ldots (118)$$

Elements of Jacobian matrix,

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} = -1 + c_1 \times \frac{2}{A^3_{l,i}} \left.\frac{dA}{dy}\right|_{l,i} + c_2\left(\frac{1}{R^{4/3}_{l,i}}\right) \cdot \frac{(-2)}{A^3_{l,i}} \cdot \left.\frac{dA}{dy}\right|_{l,i} + c_2 \times \left(\frac{1}{A^2_{l,i}}\right) \times \frac{(-4/3)}{R^{7/3}_{l,i}} \times \left.\frac{dR}{dy}\right|_{l,i}$$

$$\Rightarrow \frac{\partial M_{l,i}}{\partial y_{l,i}} = -1 + c_1 \cdot \frac{2}{A^3_{l,i}} \left.\frac{dA}{dy}\right|_{l,i} - c_2\left[\frac{2}{A^3_{l,i} \cdot R^{4/3}_{l,i}} \cdot \left.\frac{dA}{dy}\right|_{l,i} + \frac{4}{3A^2_{l,i} R^{7/3}_{l,i}} \left.\frac{dR}{dy}\right|_{l,i}\right] \quad \ldots (118)$$

and

$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}} = 1 + c_1 \cdot \frac{(-2)}{A^3_{l,i+1}} \cdot \left.\frac{dA}{dy}\right|_{l,i+1} + c_2 \times \left[\frac{(-4/3)}{R^{7/3}_{l,i+1}} \times \frac{1}{A^2_{l,i+1}} \left.\frac{dR}{dy}\right|_{l,i+1} + \frac{(-2)}{A^3_{l,i+1} R^{4/3}_{l,i+1}} \left.\frac{dA}{dy}\right|_{l,i+1}\right]$$

$$= 1 - c_1 \frac{2}{A^3_{l,i+1}} \left.\frac{dA}{dy}\right|_{l,i+1} - c_2\left[\frac{2}{A^3_{l,i+1} R^{4/3}_{l,i+1}} \left.\frac{dA}{dy}\right|_{l,i+1} + \frac{4}{3A^2_{l,i+1} R^{7/3}_{l,i+1}} \left.\frac{dR}{dy}\right|_{l,i+1}\right] \quad \ldots\ldots (119)$$

For general channel cross section,

$$\frac{dR}{dy} = \frac{T}{P} - \frac{R}{P}\frac{dP}{dy} \quad \text{.............. (120).}$$

$$R = \frac{A}{P}$$
$$= \frac{Ty}{P}$$
$$\Rightarrow \frac{dR}{dy} = \frac{T}{P} + y\frac{d}{dy}\left(\frac{T}{P}\right)$$
$$= \frac{T}{P} + yT\frac{d}{dy}\left(\frac{1}{P}\right)$$
$$= \frac{T}{P} - yT\frac{1}{P^2}\frac{dP}{dy}$$
$$= \frac{T}{P} - \frac{A}{P^2}\frac{dP}{dy}$$
$$= \left(\frac{T}{P} - \frac{R}{P}\frac{dP}{dy}\right)$$

(C/S area = Top width × Hydraulic mean depth = T×y)

T is not a func. of y, but P is.

Algebraic Form:

Boundary Conditions:

Elements of Jacobian matrix.

For subcritical flow,

$$\frac{\partial}{\partial y_{l,N_l}}(DB_{l,N_l+1}) = \frac{\partial}{\partial y_{l,N_l}}(y_{l,N_l+1} - y_d) = 0-0=0.$$

and $$\frac{\partial}{\partial y_{l,N_l+1}}(DB_{l,N_l+1}) = \frac{\partial}{\partial y_{l,N_l+1}}(y_{l,N_l+1} - y_d) = 1-0=1$$

From (116.1) ...... (121)

For supercritical flow condition,

$$\frac{\partial}{\partial y_{l,1}}(UB_{l,N_l+1}) = \frac{\partial}{\partial y_{l,1}}(y_{l,1} - y_u) = 1-0=1$$

$$\frac{\partial}{\partial y_{l,2}}(UB_{l,N_l+1}) = \frac{\partial}{\partial y_{l,2}}(y_{l,1} - y_u) = 0-0=0$$

From (116.2) ...... (122)

In general form, governing equation including boundary condition,

$$\frac{\partial M_{l,i}}{\partial y_{l,i}}\cdot\Delta y_{l,i} + \frac{\partial M_{l,i}}{\partial y_{l,i+1}}\Delta y_{l,i+1} = -M_{l,i} \quad \forall i \in \{1,\dots N_l\} \quad \text{.... (123)}$$

Similar to equation (109).

From R_1 - Eq(45), we have,

$$\{F(\varphi+\Delta\varphi)\} = \{F(\varphi)\} + [J(\varphi)]\{\Delta\varphi\}$$

For this case, $\{\cancel{M(y+\Delta y)}^{0}\} = \{M(y)\} + [J(y)]\{\Delta y\}$ # where $J(y)$ has $\frac{\partial M}{\partial y}$ like terms.

Assuming $(y+\Delta y)$ as solution.

$$\Rightarrow [J(y)]\{\Delta y\} = -\{M(y)\} \quad \text{.................. (123.1)}$$

For i^{th} segment of l^{th} channel reach. (2 sections:- i, $i+1$)

$$\begin{bmatrix} \cdots & \frac{\partial M_{l,i}}{\partial y_{l,i}} & \frac{\partial M_{l,i}}{\partial y_{l,i+1}} & \cdots \end{bmatrix} \begin{Bmatrix} \Delta y_{l,i} \\ \Delta y_{l,i+1} \end{Bmatrix} = \begin{Bmatrix} -M_{l,i}(y_{l,i+1}, y_{l,i}) \end{Bmatrix}$$

This gives equation (123).

For subcritical flow, we have,

$$\Delta y_{l,N_l+1} = (y_d - y_{l,N_l+1})$$
$$= -DB_{l,N_l+1} \quad \text{(From eq}^n\text{(116.1))} \quad \text{....... (124)}$$

?? = { But, how can we get Δy for $(N_l+1)^{th}$ segment? Only N_l number of segments are there.....

I think,

$$\Delta y_{l,N_l+1} = y_{l,N_l+1}\Big|^{(p)} - y_{l,N_l+1}\Big|^{(p-1)}$$

This is not difference between two consecutive section. But, diff. for two iterations.

So, after getting solution through iteration, at every iteration we need to update the values.

From (123.1),

$$\{\Delta y\} = -[J(y)]^{-1}\{M(y)\}$$

$$\Rightarrow \{y^{(P)}\} - \{y^{(P-1)}\} = -[J(y)]^{-1}\{M(y)\}$$

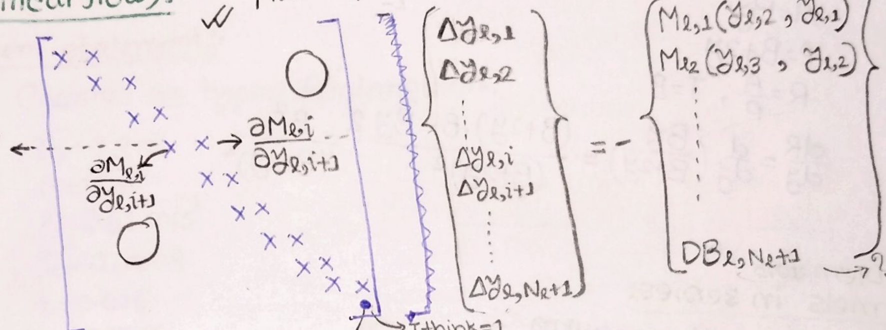
$$\Rightarrow \{y^{(P)}\} = \{y^{(P-1)}\} - [J(y^{(P-1)})]^{-1}\{M(y^{(P-1)})\}, \quad (\text{where } P \geq 1)$$

For this particular l^{th} channel reach, we can simply write,

$$y_l^{(P)} = y_l^{(P-1)} + \Delta y_l^{(P-1)}.$$

Jacobian matrix structure → (when flow depth is only variable)
(Sub-critical flow):-

From eqn (123.1),



→ I think = 1 ↳ yes! from (121)

This is for our boundary condition for subcritical flow.

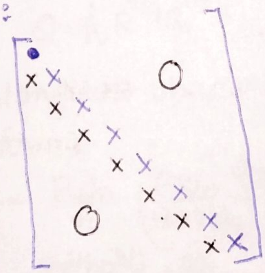
→ ? = (see page-21)

↳ I think, this is the form.

For supercritical flow,
GE is same as eqn (123).
for supercritical flow, the first condition is available at the 1st section.
So, we have,

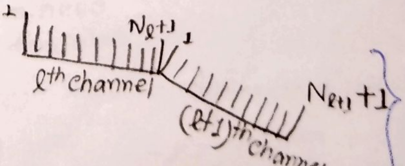
$$\Delta y_{l,1} = -UB_{l,N_l+1}.$$
$$= (y_u - y_{l,1}) \quad \dots (\text{From } 116.2).$$

Jacobian Matrix:



Channel in Series:

Junction conditions:



Continuity:-

$$Q_{l,N_l+1} = Q_{l+1,1}$$

Energy:- Neglecting losses,

$$y_{l,N_l+1} + z_{l,N_l+1} = y_{l+1,1} + z_{l+1,1}. \quad \dots (125)$$

Problem for
Single channel:

Channel c/s type:- Rectangular.

$B=15m$

$g=9.81\ m/s^2$

$S_o=0.0008$

$n=0.015$

$Lx=200m.$

$Q=20\ m^3/s$

$y_d=0.60m.$ (depth at d/s Boundary).

Estimated the flow depth across the channel reach.

$A=By$

$P=B+2y$

$R=\frac{A}{P},\ T=B$

$$\frac{dR}{dy}=\frac{d}{dy}\left(\frac{By}{B+2y}\right)=\frac{(B+2y)\cdot B-By\cdot 2}{(B+2y)^2}=\frac{B^2}{(B+2y)^2}$$

Problem for
Channels in series:

C/s type ⇒ Rectangular

$B=15m$

$S_{o1}=0.0004$

$S_{o2}=0.0008$

$n_1=0.01$

$n_2=0.015$

$Lx_1=100\ m$

$Lx_2=100m$

$Q=20m^3/s$

$y_d=0.6m$

Estimated the flow depths across the channels in series.

NPTEL Lecture 38

Surface Water Hydraulics

GVF - Implicit Approach

$y(x)$ GE :- $\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F_r^2}$ $\Rightarrow$ IVP.

We don't need boundary conditions

Problem definition:

$$\frac{dy}{dx} = \psi(x, y)$$

$$\psi(x, y) = \frac{S_0 - S_f}{1 - F_r^2} = \frac{S_0 - \frac{n^2 Q^2}{R^{4/3} A^2}}{1 - \frac{Q^2 T}{g A^3}} \quad \text{------ (126)}$$

IC :- $y|_{x=0} = y_0$

Problem statement:-

Channel c/s type:- Rectangular.

$y_0 = 0.8 m$
$B = 15 m$
$g = 9.81 m/s^2$
$S_0 = 0.0008$
$n = 0.015$
$L_x = 200 m.$
$Q = 20 m^3/s.$

Identify the GVF profile type.
Plot the GVF profile.

Solution:- $A = By$
$P = B + 2y$
$R = \frac{A}{P}$, $T = B.$

Critical depth, $Cr = 1 \Rightarrow \sqrt{\frac{Q^2 T}{g A^3}} = 1$

$$y_c = \left(\frac{Q^2}{g B^2}\right)^{1/3}$$

Calculation for normal depth:

Normal depth, obtained from Manning's equation.

$$Q = \frac{1}{n} R^{2/3} S_0^{1/2} A$$

Rectangular channel, $A = B y_n$ and $P = B + 2y_n$.

In function form,

$$G(y_n) = \frac{B y_n}{n} \left(\frac{B y_n}{B + 2y_n}\right)^{2/3} S_0^{1/2} - Q$$

$$\Rightarrow G(y_n) = \frac{S_0^{1/2} B^{5/3}}{n} \cdot \left(\frac{y_n}{B + 2y_n}\right)^{2/3} y_n - Q = 0 \quad \text{------ (127)}$$

non-linear equation. So, we need Newton-Raphson technique to solve this equation.

$$y_n|^{(p)} = y_n|^{(p-1)} - \frac{G(y_n|^{(p-1)})}{G'(y_n|^{(p-1)})} \quad \text{------ (128)}$$

Where, $G'(y_n) = \frac{d}{dy_n}\left[\frac{S_0^{1/2} B^{5/3}}{n} \cdot \frac{y_n^{5/3}}{(B + 2y_n)^{2/3}} - Q\right]$

$$= \frac{S_0^{1/2} B^{5/3}}{n} \times \frac{(B + 2y_n)^{2/3} \times \frac{5}{3} \times y_n^{2/3} - \frac{2}{3} \times (B + 2y_n)^{-1/3} \times 2 \times y_n^{5/3}}{(B + 2y_n)^{4/3}} - \frac{dQ}{dy_n}$$

$$= \frac{S_0^{1/2} B^{5/3}}{n} \times \frac{\frac{5}{3} y_n^{2/3}(B+2y_n) - \frac{4}{3} y_n^{5/3}}{(B+2y_n)^{5/3}} - \frac{dQ}{dy_n}$$

$$= \frac{S_0^{1/2} B^{5/3}}{3n} \times \frac{y_n^{2/3}(5(B+2y_n) - 4y_n)}{(B+2y_n)^{5/3}} - \cancel{\frac{dQ}{dy_n}} \to 0$$

$\Rightarrow$ I think, this is zero. Discharge for this channel is constant.

$$G'(y_n) = \frac{S_0^{1/2} B^{5/3}}{3n} \cdot \frac{y_n^{2/3}(5B+6y_n)}{(B+2y_n)^{5/3}} \quad \text{------------} \quad (129)$$

Using (128), our 1st iteration, (i.e, Normal depth after 1st iteration)

$$y_n^{(1)} = y_n^{(0)} - \frac{G(y_n^{(0)})}{G'(y_n^{(0)})}$$

We can start our 1st iteration, using, $y_n^{(0)} = y_c$, Because we already got our Critical depth

Implicit Runge-kutta Method:

Defined as weighted assembly of increments by:-

$$y_{n+1} = y_n + \sum_{j=1}^{m} W_j K_j \quad \cdots\cdots (130)$$

For explicit approach, this value was going upto $(i-1)$.

$$\text{with } \boxed{K_i = \Delta x\, \psi\left(x_n + C_i^x \Delta x,\; y_n + \sum_{j=1}^{i} C_{ij}^y K_j\right)} \quad \text{----------} \quad (131)$$

m = No. of points or number of increments required for calculation of weight.

Butcher Tableau expressed as:-

C_1^x	$C_{1,1}^y$	C_{12}^y	-------	C_{1m}^y
C_2^x	C_{21}^y	C_{22}^y	-------	C_{2m}^y
⋮	⋮	⋮		⋮
C_m^x	C_{m1}^y	C_{m2}^y	-------	C_{mm}^y
	W_1	W_2	-------	W_m.

From this, using (131), we have,

$$K_1 = \Delta x\, \psi(x_n + C_1^x \Delta x,\; y_n + C_{11}^y K_1)$$

$$K_2 = \Delta x \cdot \psi(x_n + C_2^x \Delta x,\; y_n + C_{21}^y K_1 + C_{22}^y K_2)$$

$$K_3 = \Delta x\, \psi(x_n + C_3^x \Delta x,\; y_n + C_{31}^y K_1 + C_{32}^y K_2 + C_{33}^y K_3) \text{ etc.}$$

Backward Euler method:

Butcher Tableau,

1	1
	1

$C_1^x = 1$	$C_{11}^y = 1$
	$W_1 = 1$.

From (131), $K_1 = \Delta x\, \psi(x_n + C_1^x \Delta x,\; y_n + C_{11}^y K_1)$

$$= \Delta x\, \psi(x_n + \Delta x,\; y_n + K_1)$$

$$= \Delta x\, \psi(x_{n+1}, y_{n+1}) \quad \text{------} \quad (132)$$

From (130), (132),

$$y_{n+1} = y_n + W_1 K_1$$

$$\Rightarrow y_{n+1} = y_n + \Delta x\, \psi(x_{n+1}, y_{n+1}) \quad \cdots\cdots (133)$$

Order of Backward Euler, $O(\Delta x)$

In function form,

$$F(y_{n+1}) = y_{n+1} - \underbrace{y_n}_{\text{Known}} - \Delta x\, \psi(x_{n+1}, \underbrace{y_{n+1}}_{\text{unknown}}) = 0 \quad \cdots\cdots (134)$$

This is a non-linear equation.

Implicit $\Rightarrow$ Unseperable unknown terms. Here, $F(y_{n+1})$ is a non-linear function of variable y_{n+1}. We should go for iterative approach to find y_{n+1}. (i.e $y_n + \Delta y$)

From Newton-Raphson's method,

I.T., we are finding channel depth at $(x_i+\Delta x)$ location to get GVF profile which is y_{n+1} or $y_i+\Delta y$

$$y_{n+1}\Big|^{(p)} = y_{n+1}\Big|^{(p-1)} - \frac{F(y_{n+1})\big|^{(p-1)}}{F'(y_{n+1})\big|^{(p-1)}} \quad \cdots\cdots (135)$$

where, $F'(y_{n+1}) = \frac{dF}{dy_{n+1}} = \frac{d}{dy_{n+1}}\left[y_{n+1} - \Delta x\,\psi(x_{n+1}, y_{n+1}) - y_n\right]$

$$= 1 - \Delta x \frac{d\psi(x_{n+1}, y_{n+1})}{dy_{n+1}} - 0 \quad \Rightarrow \psi' = -\frac{\{F'(y_{n+1}) - 1\}}{\Delta x} \quad \cdots (135.1)$$

Now, we know, $\psi(x,y) = \frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} = \frac{S_0 - \frac{n^2Q^2}{R^{4/3}A^2}}{1 - \frac{Q^2T}{gA^3}}$

$$= 1 - \Delta x \cdot \frac{d}{dy_{n+1}}\left\{\frac{S_0 - n^2Q^2/R^{4/3}A^2}{1 - Q^2T/gA^3}\right\}$$

$$= 1 - \Delta x\left[\frac{d}{dy_{n+1}}\left\{\frac{S_0 - \frac{n^2Q^2}{\left(\frac{By_{n+1}}{B+2y_{n+1}}\right)^{4/3}(By_{n+1})^2}}{1 - \frac{Q^2B}{g\cdot(By_{n+1})^3}}\right\}\right] \quad \cdots\cdots (136)$$

Expanding this we get a big term for $F'(y_{n+1})$

Eq^n (131) can be written as:-

$$k_i = \Delta x\,\psi\Big(x_n + \underbrace{c_i^x \Delta x}_{\delta x},\; y_n + \underbrace{\sum_{j=1}^{i-1} c_{ij}^y k_j}_{\delta y} + c_{ii}^y k_i\Big) \quad \cdots\cdots (136.1)$$

$$= \Delta x\,\psi\Big(\underbrace{x_n + \delta x}_{\text{Explicit}},\; \underbrace{y_n + \delta y}_{\text{Explicit}} + \underbrace{c_{ii}^y \cdot k_i}_{\text{Implicit}}\Big)$$

The multivariate function $\psi()$ can be expanded as:-

$$\psi(x_n+\delta x, y_n+\delta y + c_{ii}^y\cdot k_i) = \psi(x_n+\delta x, y_n+\delta y) + c_{ii}^y\cdot k_i\,\psi'(x_n+\delta x, y_n+\delta y) \ldots (137)$$

#using 1st order Taylor series expansion

$$\therefore k_i = \Delta x\left[\psi(x_n+\delta x, y_n+\delta y) + c_{ii}^y k_i \psi'(x_n+\delta x, y_n+\delta y)\right]$$

$$\Rightarrow k_i\left[1 - c_{ii}^y \Delta x \psi'\right] = \Delta x \psi \qquad [\text{where, } \psi = \psi(x_n+\delta x, y_n+\delta y)]$$

$$\Rightarrow k_i = \Delta x\left[1 - c_{ii}^y \Delta x \psi'\right]^{-1}\psi \quad \text{(In compact form)}$$

$$\Rightarrow k_i = \Delta x\left[1 - c_{ii}^y \Delta x \psi'(x_n+\delta x, y_n+\delta y)\right]^{-1}\psi(x_n+\delta x, y_n+\delta y) \quad \cdots\cdots (138)$$

→ This is semi-implicit equation.

Second order RK method:

From (131),

1/2	1/2
	1

$$k_1 = \Delta x\,\psi(x_n + \tfrac{1}{2}\Delta x, y_n + \tfrac{1}{2}k_1) \quad \cdots\cdots (139.1)$$

and $y_{n+1} = y_n + k_1 \quad \cdots\cdots (139.2)$

order of RK2 method $= O(\Delta x^2)$.

Semi-implicit form of (139.1) obtained from (138),

$k_i = k_1$, $c_{ii}^y = 1/2$, $\delta x = \frac{1}{2}\Delta x$, $\delta y = 0$

$$\therefore k_1 = \Delta x\left[1 - \tfrac{1}{2}\Delta x \psi'(x_n + \tfrac{1}{2}\Delta x, y_n)\right]^{-1}\psi(x_n+\tfrac{1}{2}\Delta x, y_n) \quad \cdots\cdots (139.3)$$

অথবা বলা যায়, বিকল্পভাবে,

using Taylor's series, we have, (upto 1st derivative term),

$$k_1 = \Delta x\left[\psi(x_n+\tfrac{1}{2}\Delta x, y_n) + \tfrac{1}{2}k_1\,\psi'(x_n+\tfrac{1}{2}\Delta x, y_n)\right]$$

$$\Rightarrow k_1\left[1 - \Delta x\,\tfrac{1}{2}k_1\psi'\right] = \Delta x \psi$$

$$\Rightarrow k_1 = \Delta x\left[1 - \Delta x\cdot\tfrac{1}{2}\psi'(x_n+\tfrac{1}{2}\Delta x, y_n)\right]^{-1}\psi(x_n+\tfrac{1}{2}\Delta x, y_n) \rightarrow \text{same as } (139.3)$$

RK4 method:

Considering Butcher Tableau,

$$
\begin{array}{c|cc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

RK4, from eqn (131),

$$\sum_{j=1}^{i} c_{ij}^{y} k_j$$ → এখানে, $i=1$. তাহলে, k_2 কেন হলো?

$$k_1 = \Delta x\, \psi(x_n + c_1^x \Delta x,\ y_n + \underbrace{c_{11}^y k_1 + c_{12}^y k_2}) \quad \cdots\cdots (140.1)$$

$$k_2 = \Delta x\, \psi(x_n + c_2^x \Delta x,\ y_n + c_{21}^y k_1 + c_{22}^y k_2) \quad \cdots\cdots (140.2)$$

Order: $O(\Delta x^4)$. $\quad y_{n+1} = y_n + \frac{1}{2}k_1 + \frac{1}{2}k_2 \quad \cdots\cdots (140.3)$

Expanding (140.1) and (140.2) using Taylor's series,

$$k_1 = \Delta x\left[\psi(x_n + c_1^x \Delta x, y_n) + (c_{11}^y k_1 + c_{12}^y k_2)\,\psi'(x_n + c_1^x \Delta x, y_n)\right]$$

$$k_2 = \Delta x\left[\psi(x_n + c_2^x \Delta x, y_n) + (c_{21}^y k_1 + c_{22}^y k_2)\,\psi'(x_n + c_2^x \Delta x, y_n)\right]$$

$$\{1 - \Delta x\, c_{11}^y \psi'(x_n + c_1^x \Delta x, y_n)\} k_1 - \Delta x\, c_{12}^y \psi'(x_n + c_1^x \Delta x, y_n) \cdot k_2 = \Delta x\, \psi(x_n + c_1^x \Delta x, y_n) \quad \cdots\cdots (141.1)$$

$$-\Delta x\, c_{21}^y \psi'(x_n + c_2^x \Delta x, y_n)\, k_1 + \{1 - \Delta x\, c_{22}^y \psi'(x_n + c_2^x \Delta x, y_n)\} k_2 = \Delta x\, \psi(x_n + c_2^x \Delta x, y_n) \quad \cdots\cdots (141.2)$$

Expressing (141.1) and (141.2) in matrix form,

$$
\begin{bmatrix}
1 - \Delta x\, c_{11}^y \psi'(x_n + c_1^x \Delta x, y_n) & -\Delta x\, c_{12}^y \psi'(x_n + c_1^x \Delta x, y_n) \\
-\Delta x\, c_{21}^y \psi'(x_n + c_2^x \Delta x, y_n) & 1 - \Delta x\, c_{22}^y \psi'(x_n + c_2^x \Delta x, y_n)
\end{bmatrix}
\begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix}
=
\begin{bmatrix}
\Delta x\, \psi(x_n + c_1^x \Delta x, y_n) \\
\Delta x\, \psi(x_n + c_2^x \Delta x, y_n)
\end{bmatrix}
$$

NPTEL-39

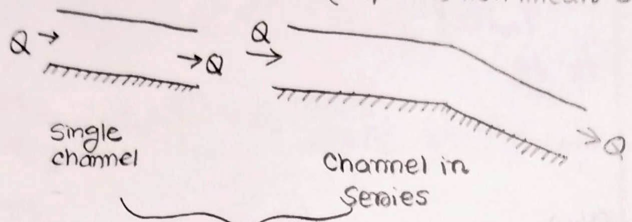
Module-4, Unit-4

Surface Water Hydraulics

Steady channel flow- Channel Network:

To solve steady channel flow for Channel network problem using implicit method.

↳(Implicit, non-linear equation. But, not time implicit)



In these cases, discharge was same everywhere.
Our only variable was depth(y).

Problem definition:

GE of channel flow can be written as boundary value problem:- ??

CE:- $\frac{dQ}{dx}=0$ ⇒ In previous lectures, this equation was not utilized.

ME:- $\frac{dE}{dx}=-S_f$

$E=y+z+\frac{\alpha Q^2}{2gA^2}$.

But now, at junction there will be multiple channels. So, we cannot estimate the discharge value at the junction without solving these equations. So, we will discretize both equations and solve them simultaneously to get y and Q.

Number of sections (N_e+1)

$\therefore (N_e+1) \rightarrow Q$

$(N_e+1) \rightarrow y$

$\therefore$ Total number of variables ⇒ $2(N_e+1)$

Continuity, momentum equations are applicable for segments only.

So, we get ⇒ N_e no. CE + N_e no. ME → Total $2N_e$ equations.

But, we need $2(N_e+1)$ equations to solve to find the solutions. i.e 2 extra eq^{n}s are needed.

Diagram of channel flow ⇒ Same as page 19.

Discretization-CE:

CE for i^{th} segment of l^{th} channel reach,

$$\frac{dQ}{dx}=0$$

$$\frac{Q_{l,i+1}-Q_{l,i}}{\Delta x_l}=0 \quad \text{.........} (142)$$

$\Rightarrow Q_{l,i+1}=Q_{l,i}$ → Now, will not consider a constant value corresponding to this. We will consider these two terms as variables.

In functional form, $C_{l,i}=Q_{l,i+1}-Q_{l,i}=0, \ \forall i \in \{1,\dots N_l\}$ ...... (143)

↳Now, we can use N-R method.
Though, eq^n is linear, we need to get the Jacobian matrix.

$y_{l,i}$ $\qquad\qquad\qquad\qquad$ $y_{l,i+1}$
$Q_{l,i}$ $\qquad\qquad\qquad\qquad$ $Q_{l,i+1}$

(i) $\qquad$ $\boxed{i}$ $\qquad$ (i+1)

Section i $\qquad$ Segment $\qquad$ Section i+1

For any segment, we have ~~two~~ 4 variables. Two depths, 2 discharge.

Now, we should get derivative in terms of these variables.

So from (143),

Coff of Jacobian matrix for Continuity equation.

$$\left\{\begin{aligned} &\frac{\partial C_{l,i}}{\partial y_{l,i}} = 0 \\ &\frac{\partial C_{l,i}}{\partial Q_{l,i}} = -1 \\ &\frac{\partial C_{l,i}}{\partial y_{l,i+1}} = 0 \\ &\frac{\partial C_{l,i}}{\partial Q_{l,i+1}} = 1. \end{aligned}\right\} \quad \cdots\cdots (144)$$

Discretization: Momentum equation:-

For i^{th} segment of lth channel reach,

$$\frac{dE}{dx} = -S_f$$

$$\Rightarrow \frac{E_{l,i+1} - E_{l,i}}{\Delta x_l} = -\frac{1}{2}\left(S_f|_{l,i+1} + S_f|_{l,i}\right)$$

In expanded form,

$$\frac{\left(y+z+\frac{\alpha Q^2}{2gA^2}\right)_{l,i+1} - \left(y+z+\frac{\alpha Q^2}{2gA^2}\right)_{l,i}}{\Delta x_l} = -\frac{1}{2}\left[\left(\frac{n^2 Q^2}{R^{4/3}A^2}\right)_{l,i+1} + \left(\frac{n^2Q^2}{R^{4/3}A^2}\right)_{l,i}\right]$$

In functional form,

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q_{l,i+1}^2}{A_{l,i+1}^2} - \frac{Q_{l,i}^2}{A_{l,i}^2}\right) + \frac{\Delta x_l \cdot n_l^2}{2}\left[\frac{Q_{l,i+1}^2}{R_{l,i+1}^{4/3} A_{l,i+1}^2} + \frac{Q_{l,i}^2}{R_{l,i}^{4/3}A_{l,i}^2}\right] \quad \forall i \in \{1, \ldots N_l\} \quad \cdots\cdots (145)$$

(Direct variables are y and Q. A, R are functions of y).

Considering reverse flow situation,

[Same eq^n as (145), only, Q^2 is written as $Q|Q|$]

ie,
$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q_{l,i+1}|Q_{l,i+1}|}{A_{l,i+1}^2} + \frac{Q_{l,i}|Q_{l,i}|}{A_{l,i}^2}\right) + \frac{n_l^2 \cdot \Delta x_l}{2}\left[\frac{Q_{l,i+1}|Q_{l,i+1}|}{R_{l,i+1}^{4/3}A_{l,i+1}^2} + \frac{Q_{l,i}|Q_{l,i}|}{R_{l,i}^{4/3}A_{l,i}^2}\right] \quad \forall i \in \{1, \ldots N_l\} \quad \cdots\cdots (146)$$

If flow is in forward direction, flow is $+Q$, backward direction, $-Q$

Using square term $\Rightarrow$ loss of information.

Here, we have $2N_l$ non-linear equations with $2(N_l+1)$ unknowns.

$\left\{\begin{array}{l} N_l \text{ number of } C_{l,i} \text{ equation } \forall i \in \{1, \ldots N_l\} \\ N_l \text{ number of } M_{l,i} \text{ " } \quad \forall i \in \{1, \ldots N_l\} \end{array}\right\}$ $\qquad$ $\left[\begin{array}{l} y_1, y_2 \cdots y_{N_l+1} \\ \text{and } Q_1, Q_2, \cdots Q_{N_l+1} \end{array}\right.$

Considering, $D_1 = \frac{\alpha_l}{2g}$ and $D_2 = \frac{1}{2} n_l^2 \Delta x_l$. → For single channel or channel in series, these terms were $c_1 = \frac{\alpha_l Q_l^2}{2g}$ and $c_2 = \frac{n_l^2 \Delta x_l Q_l^2}{2}$, which are different. Because, Q_l was constant.

Elements of Jacobian matrix :

#Absolute value form is not differentiable.
So, we use (145) form, not (146).

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} = -1 + D_1 \cdot \frac{2Q_{l,i}^2}{A_{l,i}^3} \left.\frac{dA}{dy}\right|_{l,i} - D_2 \left\{ \frac{4Q_{l,i}^2}{3A_{l,i}^2 R_{l,i}^{7/3}} \left.\frac{dR}{dy}\right|_{l,i} + \frac{2Q_{l,i}^2}{A_{l,i}^3 R_{l,i}^{4/3}} \left.\frac{dA}{dy}\right|_{l,i} \right\} \quad \text{....(147.1)}$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i}} = -D_1 \cdot \frac{2Q_{l,i}}{A_{l,i}^2} + D_2 \cdot \frac{2Q_{l,i}}{R_{l,i}^{4/3} A_{l,i}^2} \quad \text{.....(147.2)}$$

$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}} = 1 - D_1 \frac{2Q^2}{A^3} \frac{dA}{dy} - D_2 \left\{ \frac{2Q_{l,i+1}^2}{A_{l,i+1}^3 R_{l,i+1}^{4/3}} \left.\frac{dA}{dy}\right|_{l,i+1} + \frac{4Q_{l,i+1}^2}{3A_{l,i+1}^2 R_{l,i+1}^{7/3}} \left.\frac{dR}{dy}\right|_{l,i+1} \right\} \quad \text{.....(147.3)}$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i+1}} = D_1 \cdot \frac{2Q_{l,i+1}}{A_{l,i+1}^2} + D_2 \cdot \frac{2Q_{l,i+1}}{A_{l,i+1}^2 R_{l,i+1}^{4/3}} \quad \text{.....(147.4)}$$

~~notes এ $A^3_{l,i+1}$ আছে~~!! ভুল ছিল

Rough,

$$\left\{-D_1 Q^2 \frac{(-2)}{A^3}\right\}$$

$$D_2 Q^2 \left\{ -\frac{4}{3} \frac{1}{R^{7/3} A^2} + \frac{1}{R^{4/3}} \cdot \frac{(-2)}{A^3} \right\}$$

$$-D_1 \cdot \frac{2Q}{A^2} + D_2 \cdot \frac{2Q}{R^{4/3} A^2}$$

$$1 + D_1 Q^2 \left(-\frac{2}{A^3}\right) \cancel{\frac{dA}{dy}} + D_2 Q^2 \left\{ -\frac{4}{3R^{7/3} A^2} \frac{dR}{dy} - \frac{2}{A^3 R^{4/3}} \frac{dA}{dy} \right\}$$

$$D_1 \frac{2Q}{A^2} + D_2 \times \frac{2Q}{R^{4/3} A^2}$$

These (147.1) and (147.3) are similar to case of channels (single/series).
But, we need (147.2) and (147.4), because we have 2 extra variable per segment.

#D_1, D_2 are constant terms, ~~dep~~ corresponding to l^{th} channel only. Because, $\alpha_l, n_l, \Delta x_l$ ~~depen~~ parameters are for channel reach.

For general channel c/s,

$$\frac{dR}{dy} = \frac{T}{P} - \frac{R}{P} \cdot \frac{dP}{dy} \quad \text{(See Page-22).} \quad \text{....(147.5)}$$

Discretization-Boundary condition:

#We need two more equations.
Because we have $2N_l$ eqns, $2(N_l+1)$ variables.

For subcritical flow,

$$y_{l,N_l+1} = y_d.$$

$$DB_{l,N_l+1} = y_{l,N_l+1} - y_d \quad \text{......148}$$

We can specify discharge B/c at the u/s end to get one more equation.

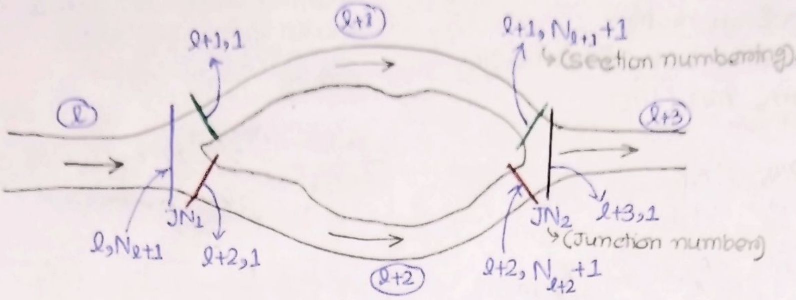
$Q_{1,1} = Q^u$ → This is a specified value.

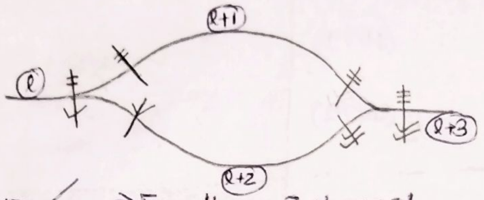
1 ——— Channel-1

Elements of Jacobian matrix,

$$\left.\begin{aligned}
&\frac{\partial}{\partial y_{l,N_l}} (DB_{l,N_l+1}) = 0 \\
&\frac{\partial}{\partial Q_{l,N_l}} (DB_{l,N_l+1}) = 0 \\
&\frac{\partial}{\partial y_{l,N_l+1}} (DB_{l,N_l+1}) = 1 \\
&\frac{\partial}{\partial Q_{l,N_l+1}} (DB_{l,N_l+1}) = 0
\end{aligned}\right\} \quad \text{.....(149)}$$

$\sum Q_I = \sum Q_O$ Channel network:-





⇒ We get 6 eq^ns from continuity and energy for the 2 junctions.

⇒ For these 3 channel cases, we have 1 continuity and 2 momentum equations for each junction.
Momentum or energy equation is applied between (l^{th} and $(l+1)^{th}$ channel) and (l^{th} and $(l+2)^{th}$ channel).

Internal Boundary Condition:-

Junction conditions can be written as:-

☒ Mass conservation:-

$$\underbrace{\sum Q_I}_{\text{at inflow branch}} = \underbrace{\sum Q_O}_{\text{at outflow branch.}}$$

☒ Energy conservation:-

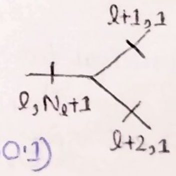
for 1st Junction (JN1),

$$y_{l,N_l+1} + Z_{l,N_l+1} = y_{l+1,1} + Z_{l+1,1} = y_{l+2,1} + Z_{l+2,1}$$

If we have different channel elevation, we can use.
Otherwise, we can ~~use~~ directly consider flow depth conditions.

Junction-1:

$$\underbrace{JC}_{\text{Junction condition}}{}_{\underbrace{JN_1}_{\text{Junction-1.}},\underbrace{1}_{\text{1st equation?}}} = Q_{l,N_l+1} - Q_{l+1,1} - Q_{l+2,1} = 0 \quad \cdots\cdots (150.1)$$



$$JC_{JN_1,2} = y_{l,N_l+1} - y_{l+1,1} + Z_{l,N_l+1} - Z_{l+1,1} = 0 \cdots\cdots\cdots (150.2)$$

$$JC_{JN_1,3} = y_{l,N_l+1} - y_{l+2,1} + Z_{l,N_l+1} - Z_{l+2,1} = 0 \cdots\cdots (150.3)$$

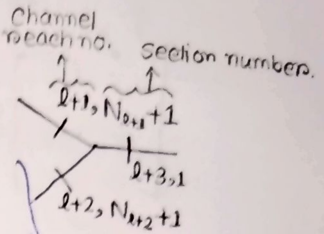
3+2+2=7 entries for Jacobian matrix.

Eq^n (150.1) → 3 variables
Eq^n (150.2) → 2 "
" (150.3) → 2 "

Elements for Jacobian matrix:-

$$\frac{\partial JC_{JN_1,1}}{\partial Q_{l,N_l+1}} = 1 \; ; \; \frac{\partial JC_{JN_1,1}}{\partial Q_{l+1,1}} = -1 \; ; \; \frac{\partial JC_{JN_1,1}}{\partial Q_{l+2,1}} = -1$$

$$\left.\begin{array}{ll} \dfrac{\partial JC_{JN_1,2}}{\partial y_{l,N_l+1}} = 1 & \dfrac{\partial JC_{JN_1,2}}{\partial y_{l+1,1}} = -1 \\[2ex] \dfrac{\partial JC_{JN_1,3}}{\partial y_{l,N_l+1}} = 1 & \dfrac{\partial JC_{JN_1,3}}{\partial y_{l+2,1}} = -1 \end{array}\right\} \quad \cdots\cdots \quad (151)$$



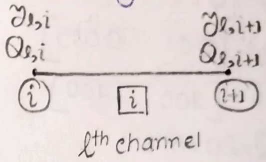
Junction 2:

$$\left.\begin{array}{l} JC_{JN_2,1} = Q_{l+3,1} - Q_{l+1,N_{l+1}+1} - Q_{l+2,N_{l+2}+1} = 0 \\ JC_{JN_2,2} = y_{l+3,1} - y_{l+1,N_{l+1}+1} + Z_{l+3,1} - Z_{l+1,N_{l+1}+1} = 0 \\ JC_{JN_2,3} = y_{l+3,1} - y_{l+2,N_{l+2}+1} + Z_{l+3,1} - Z_{l+2,N_{l+2}+1} = 0 \end{array}\right\} \cdots\cdots (152).$$

I think, this portion is for a channel, not network

Similar Jacobian matrix elements like eq^n(151) will be there for (152).

In general form, Continuity and momentum equation for i^{th} segment,



N_l no of C.E for N_l no of segments

$$\frac{\partial C_{l,i}}{\partial y_{l,i}}\Delta y_{l,i} + \frac{\partial C_{l,i}}{\partial Q_{l,i}}\Delta Q_{l,i} + \frac{\partial C_{l,i}}{\partial y_{l,i+1}}\Delta y_{l,i+1} + \frac{\partial C_{l,i}}{\partial Q_{l,i+1}}\Delta Q_{l,i+1} = -C_{l,i} \quad \cdots\cdots (153.1)$$

N_l no of ME for N_l no. of segment

$$\frac{\partial M_{l,i}}{\partial y_{l,i}}\Delta y_{l,i} + \frac{\partial M_{l,i}}{\partial Q_{l,i}}\Delta Q_{l,i} + \frac{\partial M_{l,i}}{\partial y_{l,i+1}}\Delta y_{l,i+1} + \frac{\partial M_{l,i}}{\partial Q_{l,i+1}}\Delta Q_{l,i+1} = -M_{l,i} \cdots (153.2)$$

$$\forall i \in \{1, \dots N_l\}.$$

$\therefore$ Total number of equations $= 2\times N_l$.

For subcritical flow,

what is the expanded form of this condition?

we are considering l^{th} reach, how 1 came? ?

$$\left.\begin{array}{l} \Delta Q_{1,1} = -UB_{l,N_l+1} \\ \Delta y_{l,N_l+1} = -DB_{l,N_l+1} \end{array}\right\} \cdots (154)$$

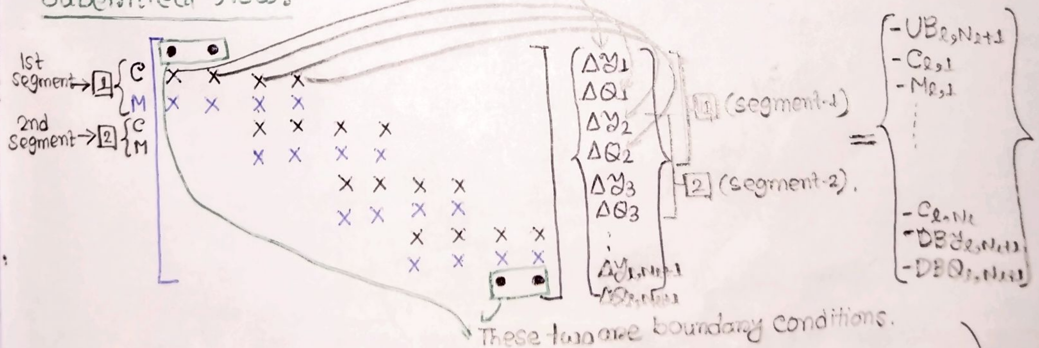
$= y_d - y_{l,N_l+1}$ (from 116.1)?

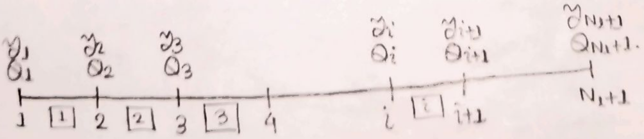
with these 2 BC's, our no. of equations becomes $= 2N_l+2$. So, now we can get solution of the system

↳ But, we have a channel network. Why are we solving for a particular channel reach?

Jacobian Matrix Structure:

Subcritical flow:





? Is it okay??

(See page 39, for more).

$$Q_1 - Q^u \Rightarrow UB_{\ell,N_\ell+1} = Q_1 - Q^u$$

is it $Q_{\ell,1}$?

Jacobian entries. $\begin{cases} A(1,1) = 0 \\ A(1,2) = 1 \end{cases}$

$$\frac{\partial UB}{\partial y_{\ell,1}} = 0$$

$$\frac{\partial UB}{\partial Q_{\ell,1}} = 1$$

$$\begin{bmatrix} 0 & 1 & \\ & & \\ & & \end{bmatrix} \begin{Bmatrix} \Delta y_1 \\ \Delta Q_1 \\ \\ \end{Bmatrix} = \begin{Bmatrix} -UB_{\ell,N_\ell+1} \\ \\ \\ \end{Bmatrix}$$

$$\Delta Q_1 = -UB_{\ell,N_\ell+1} \text{ ?}$$

Data for code

2 channels in series

Rectangular

$B = 15 m$

$g = 9.81 m/s^2$

$S_0 = [0.0004 \quad 0.0008]$

$n = [0.01 \quad 0.015]$

$Lx = [100 \quad 400]$

$Q = 20 m^3/s$

$y_d = 0.60 m$

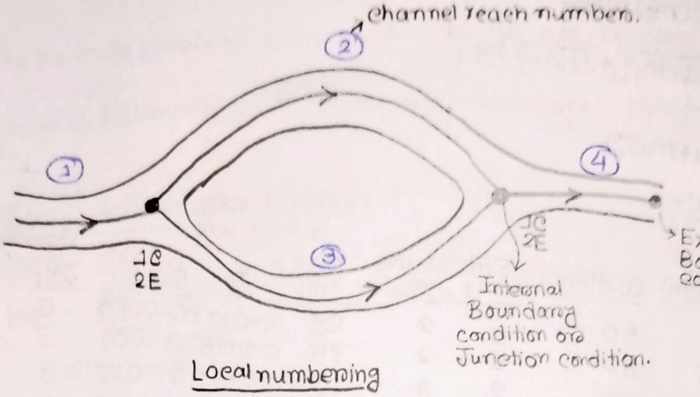
Estimate the flow depths across the channels in series.

NPTEL:- 40

Module-4, Unit-5

Steady Channel flow: Channel Network Without Reverse Flow

↳ To slove using implicit approach.



External Boundary condition (We are considering this as a junction, because one channel is joining at this point) ?

U/S এ joint কেন নয়? ?

Channel reach no	Local numbering Section numbers from	to
1	1	N_1+1
2	1	N_2+1
3	1	N_3+1
4	1	N_4+1

∴ Total sections $=(N_1+N_2+N_3+N_4+4)$

Total no. of variables $= 2(N_1+N_2+N_3+N_4+4)$

To solve we have,

$\left.\begin{matrix}1C\\1M\end{matrix}\right\}$ for $(N_1+N_2+N_3+N_4)$ no. of segments each.

∴ We have $= 2(N_1+N_2+N_3+N_4)$ no. of equations.

∴ We need 8 more equations.

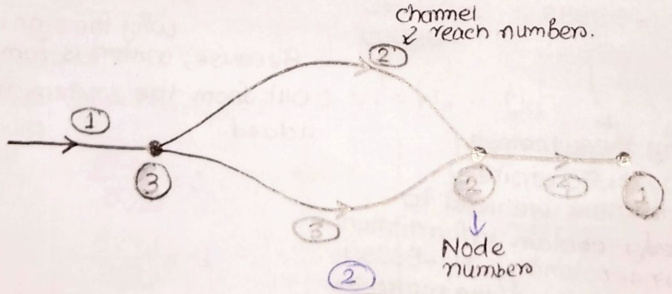
From each internal junction → 1C, 2 Energy equations (see 150, 152)

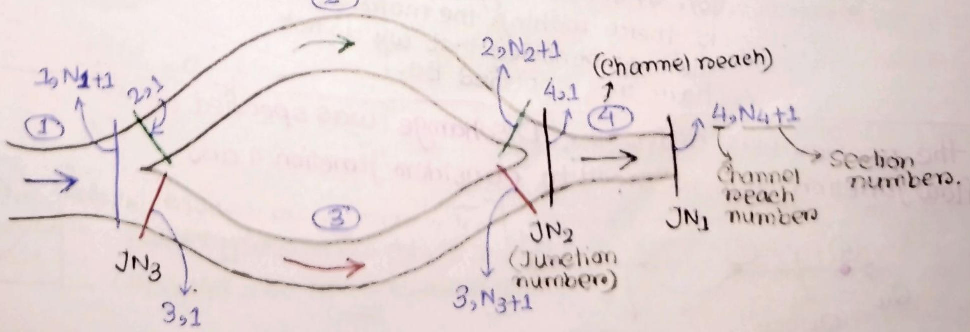
⇒ We get 6 eqⁿs more

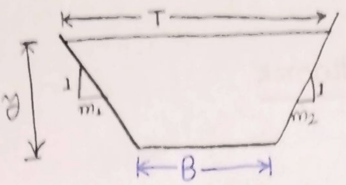
Still we need two eqⁿs.

⇒ May be discharge (Q_d) and depth (y_d) specified at downstream.

Or, discharge specified at u/s (Q_u) and depth specified at d/s (y_d).







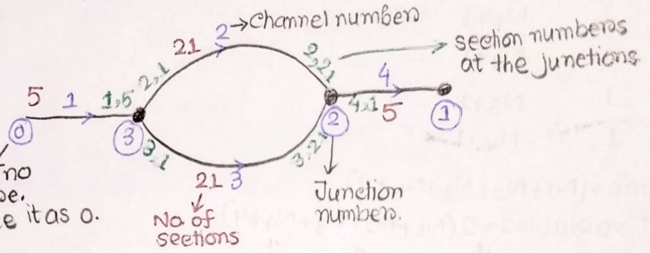
$$A = By + \frac{1}{2}(m_1+m_2)y^2$$
$$P = B + \left(\sqrt{1+m_1^2} + \sqrt{1+m_2^2}\right)y$$
$$R = A/P.$$
$$T = B + (m_1+m_2)y. \quad \Bigg\} \dots\dots (155)$$

Problem statement:-

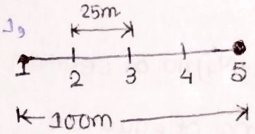
Channel data:

Length of each segment.

Channel	length (m)	width (m)	Side slope m_1	m_2	reach (m)	n	S_0	Connectivity JN_1	JN_2
1	100	50	2	2	25	0.012	0.0005	0	3
2	1500	30	2	2	75	0.0125	0.0004	3	2
3	500	20	2	2	25	0.013	0.0012	3	2
4	100	20	2	2	25	0.0135	0.0005	2	1.



For channel 1,



$\therefore N_1$ = number of segments = 4

No of sections = $N_1+1 = 5$.

Channel 2: $N_2 = \frac{1500}{75} = 20$, $N_2+1 = 20+1 = 21$ sections.

" 3: $N_3 = \frac{500}{25} = 20$, $N_3+1 = 21$ sections

" 4: $N_4 = \frac{100}{25} = 4$ $N_4+1 = 5$ sections

Next, we need the d/s boundary conditions. $y_d = 3$

Junction number	Depth (m)	Discharge (m^3/s)
1	3	-250
2	-99999	-99999
3	-99999	-99999

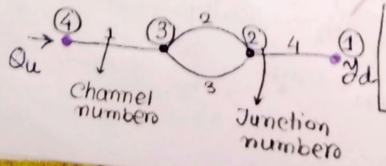
Why these terms?
Because, To write a programme we need to specify certain information for understanding. If -99999 is there within the matrix, then we consider that we do not have any specified BC.

?

But, if I consider this channel reach, water is flowing towards junct hence, +ve Q??

Why the sign is -ve.
Because, water is coming out from the system, not added.

* If the problem was different. Discharge was specified at inflow junction. Then we need to consider junction 4 also.



[That means, we consider junction if there is a boundary value specified ?!]

?

Then,

Junction number	Depth	discharge
1	y_d	-99999
2	-99999	-99999
3	-99999	-99999
4	-99999	$+Q_u$

But, in our present case, we have no u/s junction

discontinued....

Required:- Estimate flow depth and discharge across the channels.

Problem definition:

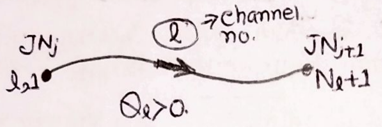
CE:- $\frac{dQ}{dx} = 0$

ME:- $\frac{dE}{dx} = -S_f$

with $E = y + z + \frac{\alpha Q^2}{2gA^2}$

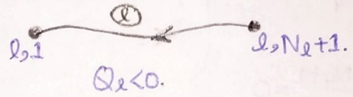
Channel flow diagram: same as Page-19.

Channel flow convention:

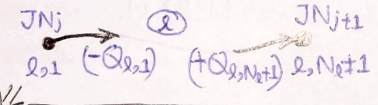


If flow is from section 1 to section N_l+1, then flow is called as +ve and junctions are JN_j and JN_{j+1}.

Reverse flow:-



Positive flow means we extracting some water from junction JN_j and adding some water at junction JN_{j+1}.



⇒ This concept is used for discretizing the governing equation.

For the previous channel, discharge is positive at this point?

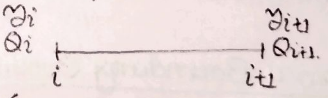
$Q_{l-1,N_{l-1}+1}$ → JN_j $l-1, N_{l-1}+1$. ⇒ (Is it okay)?

→ I think, yes! See eqn (153).

Algebraic form:-

Discretization of continuity equation: $C_{l,i} = Q_{l,i+1} - Q_{l,i} = 0 \quad \forall i \in \{1, \dots N_l\}$. ⇒ Steady state Continuity equation.

$$\left.\begin{aligned} \frac{\partial C_{l,i}}{\partial y_{l,i}} &= 0 \\ \frac{\partial C_{l,i}}{\partial Q_{l,i}} &= -1 \\ \frac{\partial C_{l,i}}{\partial y_{l,i+1}} &= 0 \\ \frac{\partial C_{l,i}}{\partial Q_{l,i+1}} &= 1. \end{aligned}\right\} \quad \dots\dots (156).$$



(Four variable. Jacobian ⇒ 4 entries).

Discretization of momentum equation:- Exactly same as page 30 and 31.

(145, 147 are useful equations).

Trapezoidal section:

$$\frac{dA}{dy} = \frac{d}{dy}\left\{By + \frac{1}{2}(m_1+m_2)y^2\right\} \quad (\text{From } 155)$$

$$= B + (m_1+m_2)y \quad \cdots\cdots\cdots\cdots (157.1)$$

Also, $$\frac{dR}{dy} = \frac{T}{P} - \frac{R}{P}\frac{dP}{dy} \quad (\text{From } 147.5)$$

$$\frac{dP}{dy} = \frac{d}{dy}\left[B + \left(\sqrt{1+m_1^2} + \sqrt{1+m_2^2}\right)y\right]$$

$$= \left(\sqrt{1+m_1^2} + \sqrt{1+m_2^2}\right) \quad - - - - - - (157.2)$$

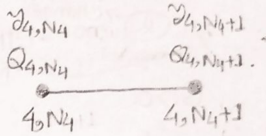
$\frac{dA}{dy}$ and $\frac{dR}{dy}$ are required for elements of Jacobian matrix generated from momentum equations. (See eq^{n}s 147).

Discretization: Boundary Condition

For downstream flow depth condition at Junction 1,

$$\left.\begin{aligned} & y_{4,N_4+1} = y_d. \\ & \therefore DBy_{4,N_4+1} = y_{4,N_4+1} - y_d = 0. \end{aligned}\right\} \cdots\cdots (158)$$

Elements of Jacobian matrix,



$$\left.\begin{aligned} & \frac{\partial(DBy_{4,N_4+1})}{\partial y_{4,N_4}} = 0 \\ & \frac{\partial(DBy_{4,N_4+1})}{\partial Q_{4,N_4}} = 0 \\ & \frac{\partial(DBy_{4,N_4+1})}{\partial y_{4,N_4+1}} = 1 \\ & \frac{\partial(DBy_{4,N_4+1})}{\partial Q_{4,N_4+1}} = 0 \end{aligned}\right\} \cdots - - - (158.1)$$

!? Depth-specified d/s boundary condition for $(N_4+1)^{th}$ section of 4^{th} channel

For d/s discharge condition at Junction 1,

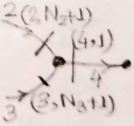
$Q_{4,N_4+1} = -Q_d$. (Negative discharge, because, flow is going out of the system).

$$DBQ_{4,N_4+1} = Q_{4,N_4+1} + Q_d = 0 \cdots\cdots (159)$$

Elements of Jacobian matrix, $\Rightarrow$ 0, 0, 0, 1 (differentiating wrt the variables) $\cdots\cdots (160)$

Internal Boundary Conditions

Junction 2:-



$$\left.\begin{aligned} & JC_{JN2,1} = Q_{2,N_2+1} + Q_{3,N_3+1} - Q_{4,1} = 0 \\ & JC_{JN_2,2} = y_{4,1} - y_{2,N_2+1} + z_{4,1} - z_{2,N_2+1} = 0 \\ & JC_{JN_2,3} = y_{4,1} - y_{3,N_3+1} + z_{4,1} - z_{3,N_3+1} = 0 \end{aligned}\right\} \cdots (161)$$

(Junction condition) for (Junction 2) (2nd equation)

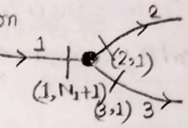
Elements of Jacobian matrix, can be written as,

$$\left.\begin{array}{lll} \dfrac{\partial JC_{JN_2,1}}{\partial Q_{2,N_2+1}} = 1 & \dfrac{\partial JC_{JN_2,1}}{\partial Q_{3,N_3+1}} = 1 & \dfrac{\partial JC_{JN_2,1}}{\partial Q_{4,1}} = -1 \\ \dfrac{\partial JC_{JN_2,2}}{\partial y_{2,N_2+1}} = -1 & \dfrac{\partial JC_{JN_2,2}}{\partial y_{4,1}} = 1. & \\ \dfrac{\partial JC_{JN_2,3}}{\partial y_{3,N_3+1}} = -1 & \dfrac{\partial JC_{JN_2,3}}{\partial y_{4,1}} = 1. & \end{array}\right\} \quad \cdots (162)$$

Junction 3:- (Discharge entering to the junction at 1st channel reach) (+ve)

(Discharge is going out from junction at 2nd and 3rd channel reaches) (-ve)

$$\left.\begin{array}{l} JC_{JN_3,1} = \underline{Q_{1,N_1+1}} - \underline{Q_{2,1}} - \underline{Q_{3,1}} = 0 \\ JC_{JN_3,2} = \underline{y_{1,N_1+1}} - \underline{y_{2,1}} + Z_{1,N_1+1} - Z_{2,1} = 0 \\ JC_{JN_3,3} = \underline{y_{1,N_1+1}} - \underline{y_{3,1}} + Z_{1,N_1+1} - Z_{3,1} = 0 \end{array}\right\} \cdots (163).$$



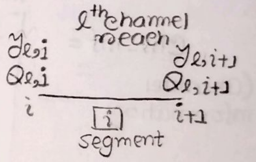
\# ~~There would be no problem if $JC_{JN_3,1} = -Q_{1,N_1+1} + Q_{2,1} + Q_{3,1}$ instead of this. Entries of Jacobian matrix would also have opposite signs. So, ultimately, their effect would be cancelled during matrix solution.~~

☹ Wrong Idea! Because, we need to preserve the sign convention of discharge.

Elements of Jacobian matrix are:-

$$\left.\begin{array}{l} 1, -1, -1. \\ 1, -1. \\ 1, -1. \end{array}\right\} \cdots (163.1)$$

General form of Continuity and momentum equation:-



For lth channel reach, where, $l \in \{1,2,3,4\}$,

$$\frac{\partial C_{l,i}}{\partial y_{l,i}} \Delta y_{l,i} + \frac{\partial C_{l,i}}{\partial Q_{l,i}} \Delta Q_{l,i} + \frac{\partial C_{l,i}}{\partial y_{l,i+1}} \Delta y_{l,i+1} + \frac{\partial C_{l,i}}{\partial Q_{l,i+1}} \Delta Q_{l,i+1} = -C_{l,i} \cdots (164)$$

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} \Delta y_{l,i} + \cdots\cdots + \frac{\partial M_{l,i}}{\partial Q_{l,i+1}} \Delta Q_{l,i+1} = -M_{l,i} \cdots (165)$$

$\forall i \in \{1, \ldots N_l\}$

For channel 1, $i \in \{1, \ldots N_1\}$
Channel 2, $i \in \{1, \ldots N_2\}$
Channel 3, $i \in \{1, \ldots N_3\}$
channel 4, $i \in \{1, \ldots N_4\}$

Here, i denotes each segments

Eqn (164) and (165) are applicable for all segments of 1st, 2nd, 3rd and 4th channel reaches.

At Junction 1, (Downstream Boundary),

$$\left.\begin{array}{l} \dfrac{\partial DBy_{4,N_4+1}}{\partial y_{4,N_4+1}} \Delta y_{4,N_4+1} = -DBy_{4,N_4+1} \\ \dfrac{\partial DBQ_{4,N_4+1}}{\partial Q_{4,N_4+1}} \Delta Q_{4,N_4+1} = -DBQ_{4,N_4+1}. \end{array}\right\} \cdots (166)$$

$$\Rightarrow \begin{bmatrix} & & & \\ & & & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vdots \\ \Delta y_{4,N_4+1} \\ \Delta Q_{4,N_4+1} \end{Bmatrix} = \begin{Bmatrix} \vdots \\ -DBy_{4,N_4+1} \\ -DBQ_{4,N_4+1} \end{Bmatrix}$$

These are only non-zero jacobian entries from (158.1) and (160)

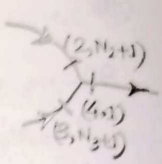
$\dfrac{\partial DBy_{4,N_4+1}}{\partial y_{4,N_4+1}}$ (Obtained from 158.1) $\dfrac{\partial DBQ_{4,N_4+1}}{\partial Q_{4,N_4+1}}$ (Obtained from 160)

At junction 2, (Internal B.C),

$$\frac{\partial JC_{JN_2,1}}{\partial Q_{2,N_2+1}}\Delta Q_{2,N_2+1} + \frac{\partial JC_{JN_2,1}}{\partial Q_{3,N_3+1}}\Delta Q_{3,N_3+1} + \frac{\partial JC_{JN_2,1}}{\partial Q_{4,1}}\Delta Q_{4,1} = -JC_{JN_2,1}$$

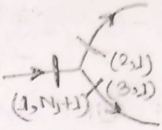
$$\frac{\partial JC_{JN_2,2}}{\partial y_{2,N_2+1}}\Delta y_{2,N_2+1} + \frac{\partial JC_{JN_2,2}}{\partial y_{4,1}}\Delta y_{4,1} = -JC_{JN_2,2}$$

$$\frac{\partial JC_{JN_2,3}}{\partial y_{3,N_3+1}}\Delta y_{3,N_3+1} + \frac{\partial JC_{JN_2,3}}{\partial y_{4,1}}\Delta y_{4,1} = -JC_{JN_2,3}. \quad \text{.....} (167)$$



At junction 3,

এখানে সব '+' চিহ্ন দিয়ে লেখা হচ্ছে, যখন Jacobian element এর value বসানো হবে, তখন Automatic -1 হয়ে যাবে।



$$\frac{\partial JC_{JN_3,1}}{\partial Q_{1,N_1+1}}\Delta Q_{1,N_1+1} + \frac{\partial JC_{JN_3,1}}{\partial Q_{2,1}}\Delta Q_{2,1} + \frac{\partial JC_{JN_3,1}}{\partial Q_{3,1}}\Delta Q_{3,1} = -JC_{JN_3,1}$$

$$\frac{\partial JC_{JN_3,2}}{\partial y_{1,N_1+1}}\Delta y_{1,N_1+1} + \frac{\partial JC_{JN_3,2}}{\partial y_{2,1}}\Delta y_{2,1} = -JC_{JN_3,2}$$

$$\frac{\partial JC_{JN_3,3}}{\partial y_{1,N_1+1}}\Delta y_{1,N_1+1} + \frac{\partial JC_{JN_3,3}}{\partial y_{3,1}}\Delta y_{3,1} = -JC_{JN_3,3} \quad \text{.....} (168)$$

chn_inf = (channel information)

channel number	length	width	side slopes		segment length	Mannings (n)	slope	Junction connectivity	
1	100	50	2	2	25	0.012	0.0005	0	3
2	1500	30	2	2	75	0.0125	0.0004	3	2
3	500	20	2	2	25	0.013	0.0012	3	2
4	100	40	2	2	25	0.0135	0.0005	2	1

যেদিকে Channel এর 1st section

যেদিকে Channel এর $(N+1)^{th}$ বা last section.

$$\text{jun\_inf} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 3 & -250 \\ -99999 & -99999 \\ -99999 & -99999 \end{bmatrix}$$

(Junction information)

(See Page-36).

$$\text{jun\_con} = \begin{array}{c} \text{Junction} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \\ \begin{bmatrix} 1 & -4 & 0 & 0 \\ 3 & 4 & -3 & -2 \\ 3 & -1 & 2 & 3 \end{bmatrix}\end{array}$$

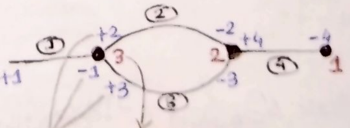
(Junction connectivity).

No. of sections connected to that junction

এই section number গুলো আসলে পড়ে লেখার কোনো sequence আছে?

?

For channel 1, section numbers goes from 1 to N_1+1. Now, we will represent this information using positive and negative numbers



For this matrix,
no. of rows = no. of junctions
no. of columns = 1 + maximum no. of section connected to a junction

To write one additional information i.e no. of sections connected to the junctions.

Discretization techniques:

$(x_0 \longrightarrow y_0) \Rightarrow$ Boundary value

$x_0+\Delta x \longrightarrow (y_0+\Delta y) = ?$ Get $y_1, y_2 \ldots y_n$ values

Backward and forward

Euler, Euler-cauchy, Modified Euler, RK methods

(for foroward, backward difference).

↳ These methods are of two types:
Implicit and Explicit.
(If we want to find $y(i+1)$ explicitly, we know $y(i)$. We get K_1 from $x(i), y(i)$; both are known. K_2 obtained from K_1 etc. So, we can get values directly.) ~~But, if implicit, K_1 value depends on K_1~~ at RHS. So, we need ~~N-R~~ method to get $y(i+1)$).

↳ There are semi-implicit forms

System of algebric equation:
Algebric equation (Multiple variables)

$a_1x+b_1y+c_1z=d_1$
$a_2x+b_2y+c_2z=d_2$
$a_3x+b_3y+c_3z=d_3$

$x=?$
$y=?$
$z=?$

- Gauss elimination
- LU-Decomposition
- TDM method

Iterative
- Jacobi
- Gauss-seidal
- Newton-Raphson

Nonlinear equation with one variable
- Bisection method
- Secant method
- Newton-Raphson method

In GVF codes, normal depth (y_n) need to be calculated using N-R method iteratively, as eq^n is non-linear.
But, calculation of water depth ~~...~~ y_0 for GVF profile done directly for explicit ~~appro~~ RK, euler etc method.

But, for implicit RK, Euler etc method,
y_n and y_0 ~~both need~~ to be calculated using N-R iterative approach.

↓
There is some semi-implicit form, ~~vs~~ ~~form~~

Implicit-RK4
Due (38)

Pickard
Meratli

NPTEL-42,43

Module 4, Unit 06

Surface Water Hydraulics

Steady channel flow-

Channel Network with reverse flow

↳ Solving using implicit approach.

For the previous problem, we knew the flow direction from left to right as the direction of the channel bed. So, we got all flow values (Q) positive as we know the flow direction.

But if we considers flow in right to left direction, then we get -ve Q value.

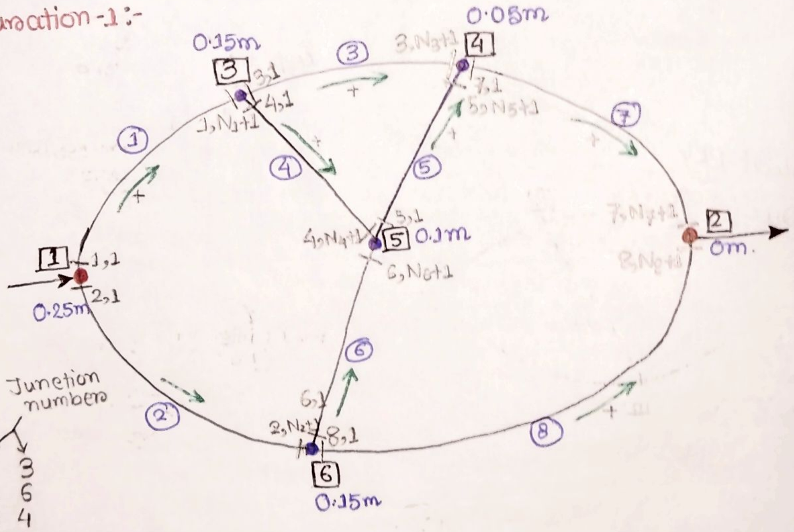
Conceptualization wise GE's are same, but implementation wise we need to consider some points, so that we can include '+' and '-' sign of Q values.

But for y, it is always positive.

Solution procedure:

Finite difference
↓
Non-linear equation
↓
Iterative approach (N-R method).

Configuration-1:-



channel no.	Junction numbers	
1	1	3
2	1	6
3	3	4
4	3	5
5	5	4
6	6	5
7	4	2
8	6	2

↳ These junction numbering follows positive direction of flow.

All the flow directions are taken as per the direction of bed elevation. So, ⟶ indicates +ve direction of flow.

It is a looped network and Junction [5] is the main connector of this loop.

Here, we will consider a rectangular channel.

#Always remember, 1st section and last sections are always defined using flow direction. ⇐ Junction connected to 1st section of the channel.

Junction connected to the last section of the channel

Channel Data:-

Channel no.	length(m)	Width	Side slope m_1	m_2	Reach (m)	n	S_0	Connectivity JN_1	JN_2
1	200	30	0	0	50	0.013	0.0005	1	3
2	200	40	0	0	50	0.013	"	1	6
3	200	20	0	0	50	0.012	"	3	4
4	100	20	0	0	25	0.014	"	3	5
5	100	20	0	0	25	0.013	"	5	4
6	100	25	0	0	25	0.013	"	6	5
7	100	30	0	0	25	0.014	"	4	2
8	300	30	0	0	75	0.014	"	6	2

Total number of sections
$= \left(\frac{200}{50}+1\right)+5+5+5$
$+5+5+5+5 = 40$ sections

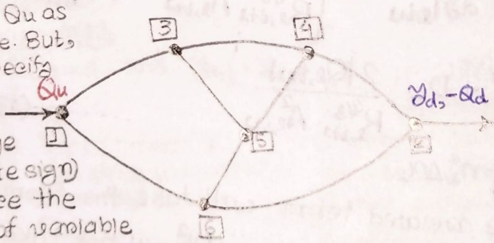
* Total number of unknowns:-
$= 40+40 = 80$ ($\because y$ and Q).

[Again, sub-critical flow, downstream y_d and $-Q_d$ is specified. But, Q_u is unknown.
$\therefore$ Total number of unknown $= 80+1 = 81$].

Junction Data:

Junction number	Depth (m)	Discharge (m^3/s)	Bed elevation (m)
1	-99999	250	0.25
2	5	-250	0
3	-99999	-99999	0.15
4	"	"	0.05
5	"	"	0.10
6	"	"	0.15

#Though, we consider Q_u as a variable. But, we can specify this value (Same as d/s discharge but, opposite sign) and reduce the number of variable to 80.)



Number of equations:-

(i) $4 \times 2 \times 8 = 64$ (for segment)
4 → No. of segments per channel.
2 → One continuity and one momentum equation per segment
8 → No. of Channels.

(ii) For internal Junctions:-
$4 \times 3 = 12$
4 → No. of internal nodes. (3, 4, 5, 6)
3 → One continuity and two energy equations.

(iii) downstream conditions
$1 + 1 + 1 = 3$
Specified depth (y_d)
Specified discharge ($-Q_d$)
Energy continuity equation (Not considered in previous lecture). ?

?

(iv) Upstream Conditions:-
$1 + 1 = 2$
Discharge Continuity at node number 1
Energy continuity for channel reach 1 and 2.

$\therefore$ Total number of equations available
$= (64 + 12 + 3 + 2) = 81.$
Segment; Internal Junctions; Downstream condition; U/s condition.

☑ Estimate Q and y across the channels.

CE: $\frac{dQ}{dx} = 0$

ME:- $\frac{dE}{dx} = -S_f$, $E = y + z + \frac{\alpha Q^2}{2gA^2}$ ...... (169)

Channel flow Convention:-

See Page-37.

Algebraic form:-

Continuity equation:- $C_{l,i} = Q_{l,i+1} - Q_{l,i} = 0 \quad \forall i \in \{1, 2, \ldots N_l\}$ ... (170)

Equations are for i^{th} segment.

Elements of Jacobian matrix, $= 0, -1, 0, 1$. (See P-37).

Momentum equation:-

Somewhat different from previous one.

$$M_{l,i} = (y_{l,i+1} - y_{l,i}) + (z_{l,i+1} - z_{l,i}) + \frac{\alpha_l}{2g}\left(\frac{Q^2_{l,i+1}}{A^2_{l,i+1}} - \frac{Q^2_{l,i}}{A^2_{l,i+1}}\right)$$
$$+ \frac{n_l^2 \Delta x_l}{2}\left[\frac{Q_{l,i+1}|Q_{l,i+1}|}{R^{4/3}_{l,i+1} \cdot A^2_{l,i+1}} + \frac{Q_{l,i}|Q_{l,i}|}{R^{4/3}_{l,i} A^2_{l,i}}\right] \quad \forall i \in \{1, \ldots N\} \ldots (171)$$

This modulus is used so that, we can get ~~modulus value of~~ the direction of the friction slope.

[This equation came from, $\frac{E_{l,i+1} - E_{l,i}}{\Delta x_l} = -\frac{1}{2}(S_{f_{l,i+1}} + S_{f_{l,i}})$

(See P-30).

Elements of Jacobian matrix:-

$$\frac{\partial M_{l,i}}{\partial y_{l,i}} = -1 + D_1 \cdot \frac{2Q^2_{l,i}}{A^3_{l,i}} \frac{dA}{dy}\Big|_{l,i} - D_2\left[\frac{2Q_{l,i}|Q_{l,i}|}{A^3_{l,i} R^{4/3}_{l,i}} + \frac{4Q_{l,i}|Q_{l,i}|}{3 R^{7/3}_{l,i} \cdot A^2_{l,i}} \cdot \frac{dR}{dy}\Big|_{l,i}\right]$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i}} = -D_1 \cdot \frac{2Q_{l,i}}{A^2_{l,i+1}} + D_2 \cdot \frac{2|Q_{l,i}|}{A^2_{l,i} R^{4/3}_{l,i+1}}$$

$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}} = 1 + D_1 \cdot \frac{Q^2_{l,i+1} \cdot 2}{A^3_{l,i+1}} \frac{dA}{dy}\Big|_{l,i+1} - D_2\left[\frac{2Q_{l,i+1}|Q_{l,i+1}|}{R^{4/3}_{l,i+1} A^3_{l,i+1}} + \frac{4 Q_{l,i+1}|Q_{l,i+1}|}{3 R^{7/3}_{l,i+1} A^2_{l,i+1}}\right]$$

$$\frac{\partial M_{l,i}}{\partial Q_{l,i+1}} = D_1 \cdot \frac{2Q_{l,i+1}}{A^2_{l,i+1}} + D_2 \cdot \frac{2|Q_{l,i+1}|}{R^{4/3}_{l,i+1} \cdot A^2_{l,i+1}} \quad \ldots\ldots (172.5)$$

$D_1 = \frac{\alpha_l}{2g}$, $D_2 = \frac{1}{2} n_l^2 \Delta x_l$.

Only friction slope related term will take the form $Q^2 = Q|Q|$, for kinetic energy term $\frac{\alpha V^2}{2g}$ will not take this form. (Because KE is always positive).

Trapezoidal section:-

$\frac{dA}{dy} = B + (m_1 + m_2)y$

$\frac{dR}{dy} = \frac{T}{P} - \frac{R}{P}\frac{dP}{dy}$.

$T = B + (m_1 y + m_2 y)$

$P = B + (\sqrt{1+m_1^2} + \sqrt{1+m_2^2})y$

$\Rightarrow \frac{dP}{dy} = (\sqrt{1+m_1^2} + \sqrt{1+m_2^2})$ .... (173)

$A = \frac{1}{2}(2B + m_1 y + m_2 y) \times y$

$\Rightarrow \frac{dA}{dy} = \frac{1}{2}(m_1 + m_2)y + \frac{1}{2}(2B + m_1 y + m_2 y)$

$\Rightarrow \frac{dA}{dy} = \frac{1}{2} \times 2(B + m_1 y + m_2 y) = B + (m_1 + m_2)y$.

$R = \frac{A}{P} = \frac{Ty}{P}$

$\frac{dR}{dy} = \frac{d}{dy}\left(\frac{Ty}{P}\right) = \frac{T}{P} - \frac{1}{P^2} \cdot Ty \frac{dP}{dy} = \frac{T}{P} - \frac{A}{P^2}\frac{dP}{dy}$

$= \frac{T}{P} - \frac{R}{P}\frac{dP}{dy}$

In general form, continuity and momentum eq^n, (see P-22),

$$\frac{\partial M_{l,i}}{\partial y_{l,i}}\Delta y_{l,i}+\frac{\partial M_{l,i}}{\partial Q_{l,i}}\Delta Q_{l,i}$$

$$\frac{\partial M_{l,i}}{\partial y_{l,i+1}}\Delta y_{l,i+1}+\frac{\partial M_{l,i}}{\partial Q_{l,i+1}}\Delta Q_{l,i+1}=-M_{l,i}\quad\dots(174)$$

$y_{l,i+1}$: $(i+1)^{th}$ section of l^{th} channel.

$M_{l,i}$: (Momentum function for l^{th} channel and i^{th} segment).

$$\frac{\partial C_{l,i}}{\partial y_{l,i}}\Delta y_{l,i}+\frac{\partial C_{l,i}}{\partial Q_{l,i}}\Delta Q_{l,i}$$

$$+\frac{\partial C_{l,i}}{\partial y_{l,i+1}}\Delta y_{l,i+1}+\frac{\partial C_{l,i}}{\partial Q_{l,i+1}}\Delta Q_{l,i+1}=-C_{l,i}$$

$\forall i\in\{1,\dots\dots N_l\}$. .... (175)

$$M(y+\Delta y, Q+\Delta Q)\overset{\nearrow 0}{=}M(y,Q)+J_M(y,Q)\begin{Bmatrix}\Delta y\\ \Delta Q\end{Bmatrix}$$

$$\Rightarrow J_M(y,Q)\begin{Bmatrix}\Delta y\\ \Delta Q\end{Bmatrix}=-M(y,Q)$$

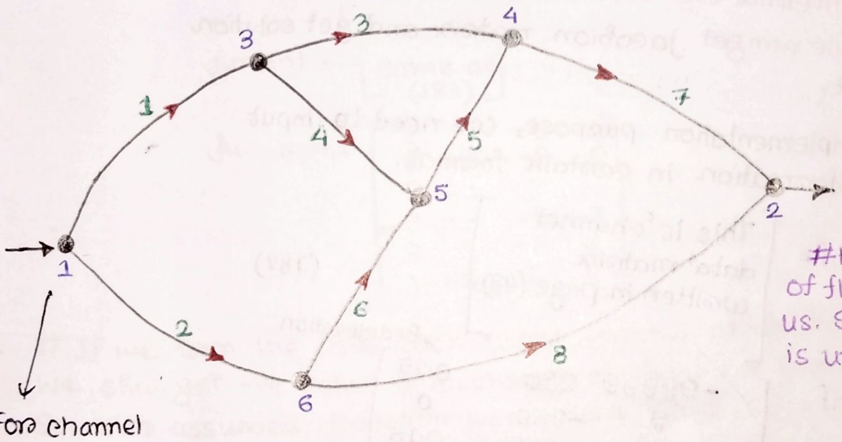
Similarly,

$$J_C(y,Q)\begin{Bmatrix}\Delta y\\ \Delta Q\end{Bmatrix}=-C(y,Q)$$

[where, J_C and J_M are Jacobian matrixes for continuity and momentum functions].

[Note:- Continuity function is actually not a function of depth y. Thats why, $\frac{\partial C_{l,i}}{\partial y_{l,i}}=\frac{\partial C_{l,i}}{\partial y_{l,i+1}}=0$]

Now, we need to talk about discretization of individual internal junctions and our external junction boundary conditions. We have total six junctions.



#Here, direction of flow is assumed by us. Suppose, bed elevation is unknown to us.

For channel reach number 1, (i.e, Junction 1),

$$Q_u-Q_{1,1}-Q_{2,1}=0\quad\text{and}\quad y_{1,1}=y_{2,1}\quad\dots\dots(176)$$

Q_u: Q_u amount is added to the system. So, +ve.

$Q_{1,1}, Q_{2,1}$: These amounts are being extracted from the system. So, -ve.

That means 1^{st} section depth at channel 1 and channel 2 should be same provided we have same elevation for two channel section.

(Section (1,1) and (2,1) both are at 1^{st} junction. জলের সমোচ্চশীলতা ধর্মের জন্য, free surface of water would be at same level.

$\therefore y_{1,1}+z_{1,1}=y_{2,1}+z_{2,1}$).

For Junction-2:

$$Q_{7,N_7+1}+Q_{8,N_8+1}-Q_d=0$$

$$y_{7,N_7+1}=y_{8,N_8+1}$$

$y_{7,N_7+1}=y_d$. (#Because, we have specified depth d/s BC). ..... (177)

Junction 3:

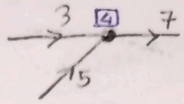
$$Q_{1,N_1+1}-Q_{3,1}-Q_{4,1}=0$$

$$y_{1,N_1+1}=y_{3,1}$$

$$y_{1,N_1+1}=y_{4,1}\quad\dots\dots(178)$$

Note, that, we are not considering any loss at the junctions.

Junction 4:

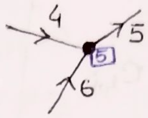


$$Q_{3,N_3+1} + Q_{5,N_5+1} - Q_{7,1} = 0$$

$$y_{3,N_3+1} = y_{7,1}$$
$$y_{3,N_3+1} = y_{5,N_5+1} \quad \cdots\cdots (179)$$

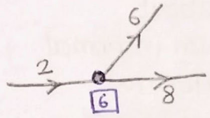
Because, y_{3,N_3+1}, $y_{7,1}$ and y_{5,N_5+1} all these three sections are at same point i.e junction no. 4.

Junction node 5:



$$Q_{4,N_4+1} + Q_{6,N_6+1} - Q_{5,1} = 0$$
$$y_{4,N_4+1} = y_{5,1}$$
$$y_{4,N_4+1} = y_{6,N_6+1} \quad \cdots\cdots (180)$$

Junction 6:



$$Q_{2,N_2+1} - Q_{6,1} - Q_{8,1} = 0$$
$$y_{2,N_2+1} = y_{6,1}$$
$$y_{2,N_2+1} = y_{8,1} \quad \cdots\cdots (181)$$

Now, we can have the elements of Jacobian matrices for each internal and external junctions also. (see p-38 and p-39)
So, now we can get jacobian matrix and get solution out of that.

But, for implementation purpose, we need to input certain information in certain format.

chl inf = [This is 'channel data' matrix written in page (43).] ...... (182)

jun_inf = (Junction Information)

$$\begin{bmatrix} -99999 & 250 & 0.25 \\ 5 & -250 & 0 \\ -99999 & -99999 & 0.15 \\ -99999 & -99999 & 0.05 \\ -99999 & -99999 & 0.10 \\ -99999 & -99999 & 0.15 \end{bmatrix} \quad \cdots\cdots (183)$$

Bed elevation

$$\text{jun\_con} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 2 & 1 & 2 & 0 \\ 2 & -7 & -8 & 0 \\ 3 & -1 & 3 & 4 \\ 3 & -3 & -5 & 7 \\ 3 & -4 & -6 & 5 \\ 3 & -2 & 6 & 8 \end{bmatrix} \quad \cdots\cdots (184)$$

In this matrix, no. of rows = no. of junction and no. of columns = (Max. no. of junctions connected to a junction + 1)

I.T ⇒ There is no problem if we write this in "-4 5 -6" this sequence.

Junction number.

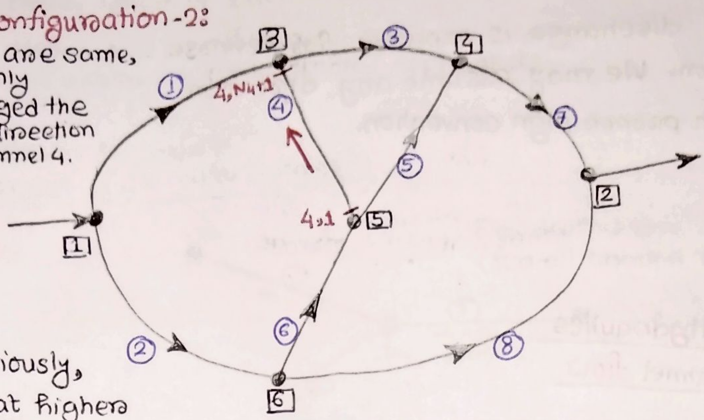
Number of junctions connected to that channel.

These are the channel numbers connected to the junction.
Positive sign represents that 1st section of channel is connected and -ve sign means last section of that channel is connected to that particular junction.

Note that, 1st and last sections are defined by flow direction (see P-37).

Configuration-2:

All are same, we only changed the flow direction at channel 4.



Obviously, 3 is at higher elevation than 5. But, let's go with that and see what happens!

Now, what changed?!

(for this reversing of flow direction).

(i) Section numbering of channel 4 has been changed. For other sections, numbering remains same.

(ii) Changed channel information:-

$$
chl\_inf = \begin{bmatrix} \cdots \\ 4 & 100 & 20 & 0 & 0 & 25 & 0.013 & 0.0005 & 5 & 3 \\ \cdots \end{bmatrix}
$$

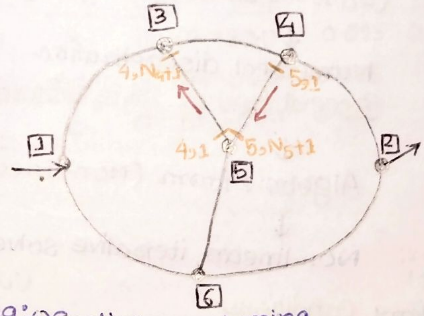
jun_inf = [same as (183)]

$$
jun\_con = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 2 & -7 & -8 & 0 \\ 3 & -1 & 3 & -4 \\ 3 & -3 & -5 & 7 \\ 3 & 4 & -6 & 5 \\ 3 & -2 & 6 & 8 \end{bmatrix}
$$

If we run the code with these updated information, we shall get -ve value of discharge at channel 4 (here, it is $-40.65\ m^3/s$). So, our assumed direction was wrong. (Q at other channels are not affected)

Configuration 3:

When flow direction at channel 4 and 5 both are reversed.



Changed thing: (i) Section numbering

(ii)
$$
chl\_inf = \begin{bmatrix} \cdots \\ 4 & 100 & 20 & 0 & 0 & 25 & 0.014 & 0.0005 & 5 & 3 \\ 5 & 100 & 20 & 0 & 0 & 25 & 0.013 & 0.0005 & 4 & 5 \\ \cdots \end{bmatrix}
$$

(iii) Jun_inf = [Same]

(iii) Jun_con =
$$
\begin{bmatrix} 2 & 1 & 2 & 0 \\ 2 & -7 & -8 & 0 \\ 3 & -1 & 3 & -4 \\ 3 & -3 & +5 & 7 \\ 3 & 4 & -6 & -5 \\ 3 & -2 & 6 & 8 \end{bmatrix}
$$

Here, also, we get -ve discharge at channel 4 and 5.

∴ -ve value of dischange is expected for reverse flow condition. We may assume any direction of flow with proper sign convention.

NPTEL-44
M-4, U-7
Surface Water Hydraulics
Unsteady 1D channel flow

To solve unsteady channel network problem by implicit approach.

For previous steady channel flow, we considered $Q(x)$, $y(x)$
But here, ~~$Q(x,t)$~~, $Q(x,t)$, $y(x,t)$. ⇒ Unsteady flow problem.

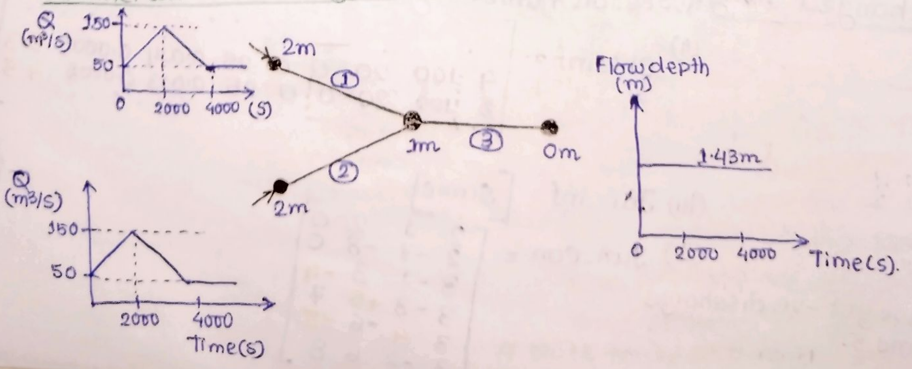
Steady-state problem was a BVP.
But, for unsteady state problem, we need initial condition. Here, BC may be fixed or time varying

Problem Definition to solution:

Hydraulic system (channel network)
↓
Governing equation (Two equations needed for two quantities y and Q).

IC
BC (May be time varying or fixed).
↓
Domain discretization
(Uniform or non-uniform grid).
↓
Numerical discretization
(Special type of finite difference).
↓
Algebraic form (Non-linear equations)
↓
Non-linear iterative solver (N-R method).

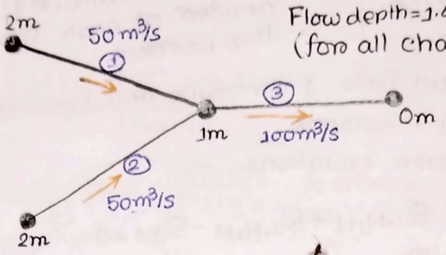
Problem statement:
Channel with Boundary Conditions:



Here, length of each channel = 5000 m.

Here, two specified discharge conditions at upstream and specified depth at d/s.

Problem statement
channel with initial conditions:



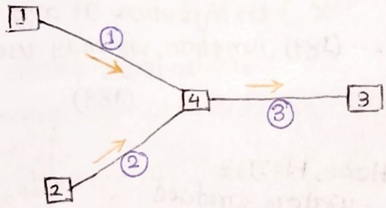
Flow depth = 1.43 m.
(for all channel reaches)

Here, we need to specify initial condition by satisfying the continuity condition. –

In case of steady-state problem, from any arbitrary IC we can get the final result. So, initial condition for steady state problem is nothing but initial guess.

But, for unsteady state, IC is important. Because, it should satisfy the physical equation. Here, for internal junction, both discharge continuity $(50+50=100)$ and energy continuity $(y_{1,N_1+1} = y_{2,N_2+1} = y_{3,1} = 1.43m)$ is satisfied for initial condition.

Problem statement:
Configuration 1:

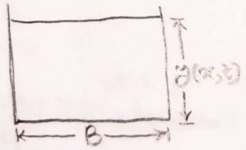


☑ Channel Data:-

Channel	length (m)	Width (m)	Slope (m_1)	(m_2)	Reach (m)	n	S_0	Connectivity JN_1	JN_2
1	5000	50	0	0	$\Delta x_1 = 500$	0.025	0.0002	1	4
2	5000	50	0	0	$\Delta x_2 = 500$	0.025	"	2	4
3	5000	100	0	0	$\Delta x_3 = 500$	0.025	"	4	3.

.... (185)

Rectangular channel:-

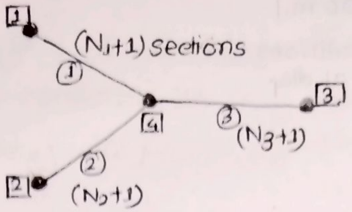


☑ Junction data:-

For unsteady state, we have time-varying boundary condition. So, we can't directly specify the values in a single matrix. like steady-state problem. So, we will write flow depth as equivalent to '1' and discharge as equivalent to '2'.

Junction Data Number	Depth (m)	Discharge (m³/s)	Bed Elevation (m)
1	-99999	2	2
2	-99999	2.	2
3	1	-99999	0
4	-99999	-99999	1

...... (186)



For each section 2 unknowns → y, Q.

∴ For a particular time step, we have total $2(N_1+N_2+N_3+3)$ no. of unknowns.

↳ So, this number of equations are needed at each time-step to solve this problem.

For each segment, we have 1 continuity and 1 momentum equation.

∴ $2(N_1+N_2+N_3)$ equations.

Still we need 6 more equations.

Internal junction:- (JN $\boxed{4}$),

$$Q_{1,N_1+1} + Q_{2,N_2+1} - Q_{3,1} = 0$$
$$y_{1,N_1+1} = y_{3,1}$$
$$y_{2,N_2+1} = y_{3,1}.$$

For junction $\boxed{1}$ and $\boxed{2}$, we have specified discharge and at JN $\boxed{3}$ we have depth boundary condition.

Here, we got more 3 eqns.

Output required for this problem:-

Plot the discharge and depth hydrograph at $x = 4000$ m from internal junction node in channel 3.

If Q-t, or y-t graphs are not given at the boundary.

(Can we solve this problem as BVP/IVP?)

? Governing Equation - IBVP: →

Unsteady 1D channel flow, (St. venant equations), IBVP,

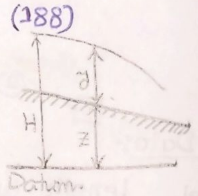
continuity equation, $\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q$ ← # If there is some extraction or injection at any particular ... (187) junction, we may use this term.

Momentum "

$$\frac{\partial}{\partial t}\left(\frac{Q}{A}\right) + \frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{2A^2}\right) + g\frac{\partial H}{\partial x} + g S_f = 0 \quad \text{.... (188)}$$

"কোনো unsteady problem-এ both to and future time level গুলোতে Boundary value দেওয়া থাকে, তাকে IBVP problem বলে।" এটা কী বলা যায়?

Here, $H = y + z$ - water surface elevation wrt datum.



Channel flow diagram

See Page 19.

Channel flow convention.

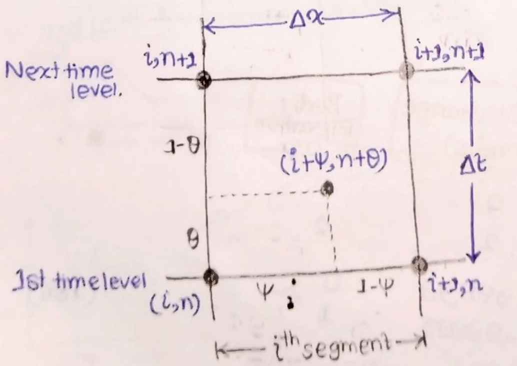
See page 37.

? যদি Future time step এ Boundary value specify না করে পর্যাপ্ত সংখ্যক Equation মেলে যায়, তাহলে IBVP হবে?

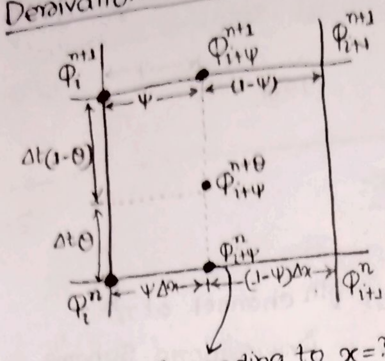
DISCRETIZATION Procedure by Preissmann Scheme:

(General variable)

? Why this scheme, FD is not?!



Derivation of Preissmann Scheme:



According to $x = \frac{m_2 x_1 + m_1 x_2}{m_1 + m_2}$ formula: [Formula $= \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$]

First evaluate $\varphi^n_{i+\psi}$ and $\varphi^{n+1}_{i+\psi}$.

Now, interpolating these two values get $\varphi^{n+\theta}_{i+\psi}$.

$$\varphi^n_{i+\psi} = \frac{\psi.\varphi^n_i + (1-\psi).\varphi^n_{i+1}}{\psi + (1-\psi)} = \psi\varphi^n_i + (1-\psi)\varphi^n_{i+1}$$

$$\varphi^{n+1}_{i+\psi} = \psi.\varphi^{n+1}_i + (1-\psi)\varphi^{n+1}_{i+1}.$$

Now, $\varphi = \varphi^{n+\theta}_{i+\psi} = \frac{(1-\theta).\varphi^{n+1}_{i+\psi} + \theta.\varphi^n_{i+\psi}}{(1-\theta)+\theta} =$

$$\varphi^n_{i+\psi} = \psi.\varphi^n_{i+1} + (1-\psi).\varphi^n_i \quad \text{------ (188.1)}$$

$$\varphi^{n+1}_{i+\psi} = \psi.\varphi^{n+1}_{i+1} + (1-\psi).\varphi^{n+1}_i \quad \text{------ (188.2)}$$

Now, $\varphi = \varphi^{n+\theta}_{i+\psi} = \frac{\Delta t\theta.\varphi^{n+1}_{i+\psi} + (1-\theta)\Delta t\varphi^n_{i+\psi}}{\theta.\Delta t + (1-\theta)\Delta t} = \theta\left[\psi.\varphi^{n+1}_{i+1} + (1-\psi)\varphi^{n+1}_i\right] + (1-\theta)\left[\psi.\varphi^n_{i+1} + (1-\psi)\varphi^n_i\right]$

From the above derivation,

For any general variable φ, Preissmann Scheme can be written as,

$$\boxed{\varphi = \theta\left[\psi.\varphi^{n+1}_{i+1} + (1-\psi)\varphi^{n+1}_i\right] + (1-\theta)\left[\psi\varphi^n_{i+1} + (1-\psi)\varphi^n_i\right]} \quad \text{...... (189)}$$

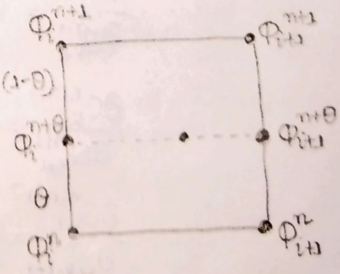
$$\frac{\partial\varphi}{\partial t} = \frac{\partial\varphi^{n+\theta}_{i+\varphi}}{\partial t} = \frac{\varphi^{n+1}_{i+\psi} - \varphi^n_{i+\psi}}{\Delta t}$$

$$= \frac{\{\psi.\varphi^{n+1}_{i+1} + (1-\psi)\varphi^{n+1}_i\} - \{\psi\varphi^n_{i+1} + (1-\psi)\varphi^n_i\}}{\Delta t} \quad \text{(From 188.1 and 188.2)}$$

$$= \frac{\psi(\varphi^{n+1}_{i+1} - \psi^n_{i+1}) + (1-\psi).(\varphi^{n+1}_i - \varphi^n_i)}{\Delta t}$$

$$\Rightarrow \frac{\partial\varphi}{\partial t} = \psi.\frac{\varphi^{n+1}_{i+1} - \varphi^n_{i+1}}{\Delta t} + (1-\psi).\frac{\varphi^{n+1}_i - \varphi^n_i}{\Delta t} \quad \text{.......... (190).}$$

Now, to get the derivative of φ (i.e. $\varphi^{n+\theta}_{i+\psi}$) wrt x, we need to find the values of $\varphi^{n+\theta}_i$ and $\varphi^{n+\theta}_{i+1}$ first.



$$\varphi^{n+\theta}_i = \frac{\theta\varphi^{n+1}_i + (1-\theta)\varphi^n_i}{\theta + (1-\theta)} = \theta\varphi^{n+1}_i + (1-\theta)\varphi^n_i \quad \text{.... (191)}$$

$$\varphi^{n+\theta}_{i+1} = \theta\varphi^{n+1}_{i+1} + (1-\theta).\varphi^n_{i+1} \quad \text{........ (192)}$$

$$\therefore \frac{\partial\varphi}{\partial x} = \left.\frac{\partial\varphi}{\partial x}\right|^{n+\theta}_{i+\psi} = \frac{\varphi^{n+\theta}_{i+1} - \varphi^{n+\theta}_i}{\Delta x} = \frac{\{\theta\varphi^{n+1}_{i+1} + (1-\theta)\varphi^n_{i+1}\} - \{\theta\varphi^{n+1}_i + (1-\theta)\varphi^n_i\}}{\Delta x}$$

IVP: Differential equation problem where value of function and its derivatives are given at specific intial point. Example:-

$y' = 2y$, $y(0) = 1$

BVP: $y'' + y = 0$, $y(0) = 0$, $y(\pi/2) = 1$

IBVP: $y'' + y = 0$, $y(0) = 0$, $y'(\pi) = 1$.

$$= \theta \cdot \frac{\varphi_{i+1}^{n+1} - \varphi_i^{n+1}}{\Delta x} + (1-\theta)\frac{\varphi_{i+1}^{n} - \varphi_i^{n}}{\Delta x}.$$

$$\Rightarrow \frac{\partial \varphi}{\partial x} = \theta \frac{\varphi_{i+1}^{n+1} - \varphi_i^{n+1}}{\Delta x} + (1-\theta)\frac{\varphi_{i+1}^{n} - \varphi_i^{n}}{\Delta x} \quad \ldots\ldots (193).$$

Discretization of continuity equation:-

CE:- $\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} - q = 0 \rightarrow$ (From 187)

The continuity equation for i^{th} segment of l^{th} channel at n^{th} time step can be descretized by four point Preissmann Scheme,

$\frac{\psi}{\Delta t}$

$$C_{l,i}^{n,n+1} = \underbrace{\frac{\psi}{\Delta t}(A_{l,i+1}^{n+1} - A_{l,i+1}^{n})}_{\text{Part1}} + \underbrace{\frac{(1-\psi)}{\Delta t}(A_{l,i}^{n+1} - A_{l,i}^{n})}_{\text{Part2}} + \quad \rightarrow \text{From (190)}$$

$$+ \underbrace{\frac{\theta}{\Delta x_l}(Q_{l,i+1}^{n+1} - Q_{l,i}^{n+1})}_{\text{Part3}} + \underbrace{\frac{(1-\theta)}{\Delta x_l}(Q_{l,i+1}^{n} - Q_{l,i}^{n})}_{\text{Part-4}} \rightarrow \text{From (193)}$$

$$- \Big\{ \underbrace{\theta\left[\psi . q_{l,i+1}^{n+1} + (1-\psi). q_{l,i}^{n+1}\right]}_{\text{Part-5}} - \underbrace{(1-\theta)\left[\psi . q_{l,i+1}^{n} + (1-\psi). q_{l,i}^{n}\right]}_{\text{Part-6}} = 0 \quad \rightarrow \text{From (189)}$$

$$\ldots\ldots (194)$$

I.T. Because, it is evaluated for time step $(n+\theta)$ which is in-between n and $(n+1)$. ?!

Although equation (194) is linear in nature, but problem is totally dependent on θ and ψ values.

Note:- Explicit or implicit consideration for a equation depends on time level of spatial derivative. (Why not on time level of temporal derivative.

?

For example, $\frac{\psi}{\Delta t}(A_{l,i+1}^{n+1} - A_{l,i+1}^{n})$

This term is also unknown?!

If $\theta = 0$, Part-3 of eq (194) is zero. So, explicit case.

If $\theta = 1$, Part-4 of eq^n (194) is zero. So, implicit case.

Now, to implement our Newton - Raphson's method, (To find elements of Jacobian matrix) we need to take derivative of $C_{l,i}^{n,n+1}$ w.r.t. four variables of i^{th} segment. Those variables are:-

$y_{l,i}^{n+1}$, $Q_{l,i}^{n+1}$, $y_{l,i+1}^{n+1}$, $Q_{l,i+1}^{n+1}$

∴ Elements of Jacobian matrix,

$$\frac{\partial C_{l,i}^{n,n+1}}{\partial y_{l,i}^{n+1}} = \frac{1-\psi}{\Delta t} \left.\frac{dA}{dy}\right|_{l,i}^{n+1} \rightarrow \text{Area is function of } y \text{ only, not } Q \text{ here.}$$

$$\frac{\partial C_{l,i}^{n,n+1}}{\partial Q_{l,i}^{n+1}} = -\frac{\theta}{\Delta x_l}$$

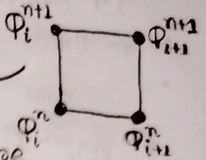
$$\frac{\partial C_{l,i}^{n,n+1}}{\partial y_{l,i+1}^{n+1}} = \frac{\psi}{\Delta t} \left.\frac{dA}{dy}\right|_{l,i+1}^{n+1}.$$

$$\frac{\partial C_{l,i}^{n,n+1}}{\partial Q_{l,i+1}^{n+1}} = \frac{\theta}{\Delta x_l} \quad \ldots\ldots (195)$$

These derivatives will have values particularly for same node number and same time steps. For other cases derivative will be zero.

Discretization of Momentum Equation:

Momentum equation from i^{th} segment of l^{th} channel at n^{th} time step:- (Discretized with four point Preissmann scheme)



Because, values are represented with these four point values.

ME:-

$$\frac{\partial}{\partial t}\left(\frac{Q}{A}\right)+\frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{2A^2}\right)+g\frac{\partial H}{\partial x}+g.S_f=0 \ldots \text{ From (188).}$$

$$M_{l,i}^{n,n+1} = \frac{\psi}{\Delta t}\left(\frac{Q_{l,i+1}^{n+1}}{A_{l,i+1}^{n+1}}-\frac{Q_{l,i+1}^{n}}{A_{l,i+1}^{n}}\right)+\frac{1-\psi}{\Delta t}\left(\frac{Q_{l,i}^{n+1}}{A_{l,i}^{n+1}}-\frac{Q_{l,i}^{n}}{A_{l,i}^{n}}\right) \rightarrow \text{From (190)}$$

Momentum correction factor ($\alpha_{l,i}$) varies with different channel and sections. Not with time!

$$+\frac{\theta}{\Delta x_l}\left\{\frac{\alpha_{l,i+1}}{2}\left(\frac{Q_{l,i+1}^{n+1}}{A_{l,i+1}^{n+1}}\right)^2-\frac{\alpha_{l,i}}{2}\left(\frac{Q_{l,i}^{n+1}}{A_{l,i}^{n+1}}\right)^2\right\}+\frac{1-\theta}{\Delta x_l}\left[\frac{\alpha_{l,i+1}}{2}\left(\frac{Q_{l,i+1}^{n}}{A_{l,i+1}^{n}}\right)^2-\frac{\alpha_{l,i}}{2}\left(\frac{Q_{l,i}^{n}}{A_{l,i}^{n}}\right)^2\right]$$

From (193)

$$+\frac{\theta g}{\Delta x_l}\left\{\left(y_{l,i+1}^{n+1}+z_{l,i+1}\right)-\left(y_{l,i}^{n+1}+z_{l,i}\right)\right\}+\frac{(1-\theta)g}{\Delta x_l}\left\{\left(y_{l,i+1}^{n}+z_{l,i+1}\right)-\left(y_{l,i}^{n}+z_{l,i}\right)\right\}$$

Here, $H=y+z$, $z=$ Channel bed elevation wrt datum. Does not vary with time (rigid bed channel).

$$+g\theta\left[\psi.S_f\Big|_{l,i+1}^{n+1}+(1-\psi)S_f\Big|_{l,i}^{n+1}\right]+g(1-\theta)\left[\psi S_f\Big|_{l,i+1}^{n}+(1-\psi)S_f\Big|_{l,i}^{n}\right]=0 \quad \ldots\ldots (196)$$

where, $S_f=\dfrac{n_m^2 Q^2}{R^{4/3}A^2}$ (I.T. subscript 'm' refers to 'Manning') ?

If we take the reverse flow consideration the discretized equation (196) would be same.

But only S_f written as $=\dfrac{n_m^2 Q|Q|}{R^{4/3}A^2}$ $\Rightarrow Q^2=Q|Q|$ is written only for friction related terms. Not for other terms.

☐ Elements of Jacobian matrix

$$\frac{\partial M_{l,i}^{n,n+1}}{\partial y_{l,i}^{n+1}} = -\frac{1-\psi}{\Delta t}\left(\frac{Q}{A^2}\frac{dA}{dy}\right)_{l,i}^{n+1}+\frac{\theta.\alpha_{l,i}}{\Delta x_l}\left(\frac{Q^2}{A^3}\frac{dA}{dy}\right)_{l,i}^{n+1}-\frac{\theta g}{\Delta x_l}-g\theta(1-\psi)\underbrace{n_{m,l}^2}_{\text{(Manning's n for } l^{th}\text{ channel)}}\left\{\frac{2Q|Q|}{R^{4/3}A^3}\frac{dA}{dy}+\frac{4Q|Q|}{3R^{7/3}A^2}\frac{dR}{dy}\right\}_{l,i}^{n+1}$$

$$\frac{\partial M_{l,i}^{n,n+1}}{\partial Q_{l,i}} = \frac{1-\psi}{\Delta t}\cdot\frac{1}{A_{l,i}^{n+1}}-\frac{\theta\alpha_{l,i}}{\Delta x_l}\left(\frac{Q}{A^2}\right)_{l,i}^{n+1}+2\theta(1-\psi).g.n_{m,l}^2\left[\frac{|Q|}{R^{4/3}A^2}\right]_{l,i}^{n+1}$$

$$-\frac{1-\psi}{\Delta t}\frac{Q}{A^2}\frac{dA}{dy}+\frac{\theta\alpha Q^2}{2\Delta x}\frac{2}{A^3}\frac{dA}{dy}-\frac{\theta g}{\Delta x}+\ldots$$

$$g\theta(1-\psi)\frac{\partial}{\partial y}\left\{\frac{n^2Q^2}{R^{4/3}A^2}\right\}$$

$$+g\theta(1-\psi)n^2Q|Q|\left[-\frac{2}{R^{4/3}A^3}\frac{dA}{dy}-\frac{4}{3R^{7/3}A^2}\frac{dR}{dy}\right]$$

$$\frac{1-\psi}{\Delta t}\frac{1}{A}-\frac{\theta\alpha}{2\Delta x}\frac{2Q}{A^2}+g\theta(1-\psi)\cdot\frac{2n^2|Q|}{R^{4/3}A^2}$$

$$\frac{\partial M_{l,i}^{n+1}}{\partial y_{l,i+1}^{n+1}} = \ldots\ldots$$

$$\frac{\partial M_{l,i}^{n+1}}{\partial Q_{l,i+1}^{n+1}} = \ldots\ldots\ldots\ldots \quad (197)$$

Now, find $\frac{dA}{dy}$ and $\frac{dR}{dy}$ (From Page 44) and put those values in elements of Jacobian matrix.

Algebraic form:

CE and ME from i^{th} segment can be written in Newton Raphson's method,

$$\frac{\partial C^{n,n+1}_{l,i}}{\partial y^{n+1}_{l,i}} \underbrace{\Delta y^{n+1}_{l,i}}_{\text{Increment value}} + \frac{\partial C^{n,n+1}_{l,i}}{\partial Q^{n+1}_{l,i}} \cdot \Delta Q^{n+1}_{l,i} + \frac{\partial C^{n,n+1}_{l,i}}{\partial y^{n+1}_{l,i}} \Delta y^{n+1}_{l,i+1} + \frac{\partial C^{n,n+1}_{l,i}}{\partial Q^{n+1}_{l,i+1}} = \overbrace{-C^{n,n+1}_{l,i}}^{\text{Negative of residual.}} \quad \ldots (198)$$

$$\frac{\partial M^{n,n+1}_{l,i}}{\partial y^{n+1}_{l,i}} \Delta y^{n+1}_{l,i} + \frac{\partial M^{n,n+1}_{l,i}}{\partial Q^{n+1}_{l,i}} \Delta Q^{n+1}_{l,i} + \frac{\partial M^{n,n+1}_{l,i}}{\partial y^{n+1}_{l,i+1}} \Delta y^{n+1}_{l,i+1} + \frac{\partial M^{n,n+1}_{l,i}}{\partial Q^{n+1}_{l,i+1}} \cdot \Delta Q^{n+1}_{1,i+1} = -M^{n,n+1}_{l,i} \quad \ldots (199)$$

$\forall i \in \{1, \ldots, N_l\}$

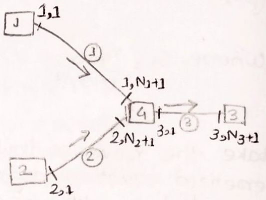
First we can start with a guess value

After getting increment values we can add it with previous level iteration value.

$$y^{n+1\,(P)}_{l,i} = y^{n+1\,(P-1)}_{l,i} + \Delta y^{n+1}_{l,i}.$$

and $$\Delta Q^{n+1\,(P)}_{l,i} = Q^{n+1\,(P-1)}_{l,i} + \Delta Q^{n+1}_{l,i}.$$

Config. 1:



chl_inf and jun_inf matrix → Same as eq^n (185) and (186).

jun-number ↑ ↗ Channel connected to Junc.

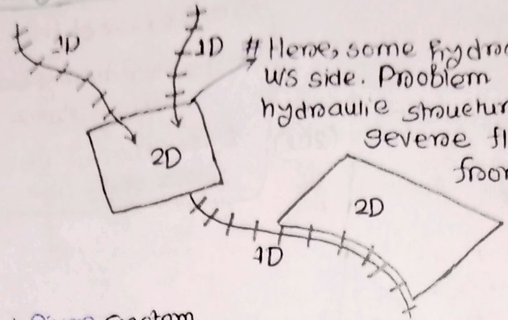
$$\text{jun\_con} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 3 & 3 & -1 & -2 \end{bmatrix}$$

(See P-46 for details).

NPTEL-46
M-4:- Surface Water Hydraulics
U-8:- Unsteady 2D surface flow:

To solve unsteady 2D shallow water flow (free surface) using explicit approach.

1D-2D integrated System:



Here, some hydraulics structures are there at the U/S side. Problem is like water is supplied from the hydraulic structure for irrigation or problem with severe flooding situation due to u/s release from dams and barrages.

+++ River system

☐ → Irrigation command system.

Problem definition to solution:

2D hydraulic system
↓
We need 2D governing equations.
unsteady problem: We need time evaluation of Surface water flooding.
IC + GE + BC
↓
Domain, discretization; and Numerical discretization: Using FVM with a rectangular co-ordinate system.
↓
Algebric form:-
Resulting equations are non-linear in nature
We will reduce the problem to pseudo-linear problem and solve it explicitly. Explicit approach is straight-forward. we donot require any iterative method. Only time stepping is required for forward marching.
↓
Solution.

Governing Equations:-

Conservative form

Depth integrated mass and momentum conservation equations for Surface water flow,

U, E, G, S → Meaning?

$$\frac{\partial \underset{\sim}{U}}{\partial t} + \frac{\partial \underset{\sim}{E}}{\partial x} + \frac{\partial \underset{\sim}{G}}{\partial y} = \underset{\sim}{S} \quad \ldots . \ (200)$$

Infiltration from the bottom of ground surface (going out of the system).

Where,

$$\underset{\sim}{U} = \begin{Bmatrix} h \\ hu \\ hv \end{Bmatrix} \quad \underset{\sim}{E} = \begin{Bmatrix} hu \\ hu^2 + \frac{gh^2}{2} \\ huv \end{Bmatrix} \quad \underset{\sim}{G} = \begin{Bmatrix} hv \\ huv \\ hv^2 + \frac{gh^2}{2} \end{Bmatrix} \quad S = \begin{Bmatrix} -q_s \\ gh(S_{ox} - S_{fx}) \\ gh(S_{oy} - S_{fy}) \end{Bmatrix}$$

...... (201.1 - 201.4)

1st row → continuity eqn

2nd and 3rd row → Momentum equation in x and y direction. We are not considering any variation in z direction. Then we need to consider the full scale Navier-Stokes' equation. But, this is depth-integrated equation, thats why we are not considering z direction.

$\frac{\partial}{\partial t}(hu) + gh\frac{\partial h}{\partial x} + \frac{\partial}{\partial x}(hu^2) + \frac{\partial}{\partial y}(huv) = gh(S_{ox} - S_{fx})$

CE $\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = -q_s$

ME(x) $\Rightarrow \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(\frac{hu^2}{h} + \frac{gh^2}{2}\right) + \frac{\partial}{\partial y}(huv) = gh(S_{ox} - S_{fx})$

ME(y) $\Rightarrow \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}\left(hv^2 + \frac{gh^2}{2}\right) = gh(S_{oy} - S_{fy})$

$gh\frac{\partial h}{\partial x} \Rightarrow$ Gravitational force on water

$\frac{\partial h}{\partial x} \Rightarrow$ Slope of water surface

$gh \Rightarrow$ Gravitational force on water column

$hu \Rightarrow$ horizontal momentum

$\frac{\partial}{\partial x}(hu^2) \Rightarrow$ Flux of horizontal momentum in x direction

$\frac{\partial}{\partial y}(huv) \Rightarrow$ Flux of H.M in y direction

$S_o - S_f \Rightarrow$ Flow driving slope (S_o) - Flow resistance slope (S_f)

$gh(S_o - S_f) \Rightarrow$ Net driving force on the flow

Governing Equations:-

Non-Conservative form:

$$\frac{\partial \underset{\sim}{U}}{\partial t} + \frac{\partial \underset{\sim}{F}}{\partial \underset{\sim}{U}} \cdot (\nabla \cdot \underset{\sim}{U}) = 0 \quad \cdots\cdots (202)$$

where, $\underset{\sim}{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \equiv \begin{Bmatrix} h \\ hu \\ hv \end{Bmatrix}$

So, U, G, E matrixes are exactly same as conservative form. Just difference is elements of E and G vectors are represented using elements of U vector. Because, in Jacobian calculation, we need derivative w.r.t U terms.

$$\underset{\sim}{E} = \begin{bmatrix} U_2 \\ \frac{U_2^2}{U_1} + \frac{1}{2}gU_1^2 \\ \frac{U_2 U_3}{U_1} \end{bmatrix} = \begin{Bmatrix} hu \\ hu^2 + \frac{gh^2}{2} \\ huv \end{Bmatrix}$$

$\frac{U_2^2}{U_1} + \frac{1}{2}gU_1^2 = \frac{h^2u^2}{h} + \frac{1}{2}gh^2 = hu^2 + \frac{1}{2}gh^2$

$\frac{U_2 U_3}{U_1} = \frac{hu.hv}{h} = huv$

$$\underset{\sim}{G} = \begin{bmatrix} U_3 \\ \frac{U_2 U_3}{U_1} \\ \frac{U_3^2}{U_1} + \frac{1}{2}gU_1^2 \end{bmatrix} = \begin{Bmatrix} hv \\ huv \\ hv^2 + \frac{gh^2}{2} \end{Bmatrix} \quad \cdots\cdots (203)$$

$\frac{U_2 U_3}{U_1} = \frac{hu.hv}{h} = huv$

$\frac{U_3^2}{U_1} + \frac{1}{2}gU_1^2 = \frac{h^2v^2}{h} + \frac{1}{2}gh^2 = hv^2 + \frac{1}{2}gh^2$

Now, let us try to derive this ~~eo~~ non-conservative form using equation (200). (i.e. conservative form).

Our conservative eq^n:- $\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial G}{\partial y} = 0$ [Source/sink term ignored!]

Let, $\underset{\sim}{F} = \underset{\sim}{E}\hat{i} + \underset{\sim}{G}\hat{j} \quad \cdots\cdots (204)$

$\therefore$ Jacobian may be calculated as,

$$\underset{\sim}{J} = \frac{\partial \underset{\sim}{F}}{\partial \underset{\sim}{U}} = \frac{\partial \underset{\sim}{E}}{\partial \underset{\sim}{U}}\hat{i} + \frac{\partial \underset{\sim}{G}}{\partial \underset{\sim}{U}}\hat{j}$$

Also, $\nabla \cdot \underset{\sim}{U} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)\underset{\sim}{U} = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j}$

$$\therefore \frac{\partial F}{\partial \underset{\sim}{U}} \cdot (\nabla \cdot \underset{\sim}{U}) = \left(\frac{\partial E}{\partial U}\hat{i} + \frac{\partial G}{\partial U}\hat{j}\right) \cdot \left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j}\right)$$

$$= \frac{\partial E}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial G}{\partial U} \cdot \frac{\partial U}{\partial y}$$

$$\Rightarrow \frac{\partial F}{\partial U} \cdot (\nabla \cdot U) = \frac{\partial E}{\partial x} + \frac{\partial G}{\partial y}$$

Replacing this value in conservative form (in 200), we get,

$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial U} \cdot (\nabla \cdot U) = 0$, which is non-conservative form.

Jacobian of equation (202),

$$\underset{\sim}{J} = \frac{\partial \underset{\sim}{E}}{\partial \underset{\sim}{U}} = \frac{\partial \underset{\sim}{E}}{\partial \underset{\sim}{U}}\hat{i} + \frac{\partial \underset{\sim}{G}}{\partial \underset{\sim}{U}}\hat{j}$$

Termwise Jacobian matrix is calculated as, $\frac{\partial \underset{\sim}{E}}{\partial \underset{\sim}{U}} = \begin{bmatrix} \frac{\partial E_1}{\partial U_1} & \frac{\partial E_1}{\partial U_2} & \frac{\partial E_1}{\partial U_3} \\ \frac{\partial E_2}{\partial U_1} & \frac{\partial E_2}{\partial U_2} & \frac{\partial E_2}{\partial U_3} \\ \frac{\partial E_3}{\partial U_1} & \frac{\partial E_3}{\partial U_2} & \frac{\partial E_3}{\partial U_3} \end{bmatrix}$

$$= \begin{bmatrix} \frac{\partial}{\partial U_1}(U_2) & \frac{\partial}{\partial U_2}(U_2) & \frac{\partial}{\partial U_3}(U_2) \\ \frac{\partial}{\partial U_1}\left(\frac{U_2^2}{U_1}+\frac{1}{2}gU_1^2\right) & \frac{\partial}{\partial U_2}\left(\frac{U_2^2}{U_1}+\frac{1}{2}gU_1^2\right) & \frac{\partial}{\partial U_3}\left(\frac{U_2^2}{U_1}+\frac{1}{2}gU_1^2\right) \\ \frac{\partial}{\partial U_1}\left(\frac{U_2U_3}{U_1}\right) & \frac{\partial}{\partial U_2}\left(\frac{U_2U_3}{U_1}\right) & \frac{\partial}{\partial U_3}\left(\frac{U_2U_3}{U_1}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{U_2^2}{U_1^2}+gU_1 & \frac{2U_2}{U_1}+0 & 0 \\ -\frac{U_2U_3}{U_1^2} & \frac{U_3}{U_1} & \frac{U_2}{U_1} \end{bmatrix}$$

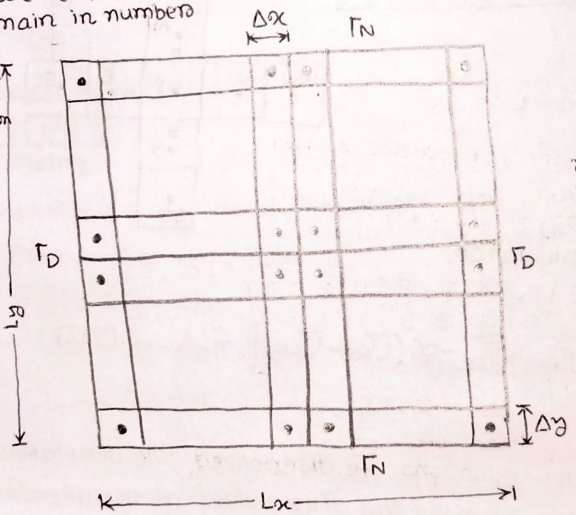
$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{h^2u^2}{h^2}+gh & \frac{2hu}{h} & 0 \\ -\frac{hu.hv}{h^2} & \frac{hv}{h} & \frac{hu}{h} \end{bmatrix}$$ (Putting these values from equation 203).

$$= \begin{bmatrix} 0 & 1 & 0 \\ -u^2+gh & 2u & 0 \\ -uv & v & u \end{bmatrix} \quad \ldots\ldots\ldots\ldots (205)$$

Similarly, we can calculate $\frac{\partial G}{\partial U} = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2+gh & 0 & 2v \end{bmatrix}$ ...... (206)

Domain Discretization:

For FVM, we need to divide our domain in number of cells. m cells in x and n cells in y direction



In some cases, we may have zero Neumann boundary or closed boundary. But, we need to see individual components in that case.

Discretization of governing equation:-

In FVM, the GE is integrated over element volume (in space) and time interval to form the discretized equation at node point P.

From (208),

$$\int_t^{t+\Delta t}\left[\int_{\Omega^P}\frac{\partial U}{\partial t}\,d\Omega\right]dt + \int_t^{t+\Delta t}\left[\int_{\Omega^P}\nabla.F\,d\Omega\right]dt = \int_t^{t+\Delta t}\left[\int_{\Omega^P} S\,d\Omega\right]dt$$

'Omega'

..... (209)

From equation (204),

$F = E\hat{i}+G\hat{j}$

$\therefore \nabla.F = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}\right)(E\hat{i}+G\hat{j})$

$= \frac{\partial E}{\partial x}+\frac{\partial G}{\partial y}$ .... (207)

$\therefore$ Conservative form can be written as (from 207)

$\frac{\partial U}{\partial t}+\nabla.\underset{\sim}{F} = \underset{\sim}{S}$ .... (208)

Discretization:- Temporal Term:-

$$\int_t^{t+\Delta t}\left[\int_{\Omega_P}\frac{\partial U}{\partial t}\,d\Omega\right]dt$$

Similar to the groundwater equations

$$= \int_t^{t+\Delta t}\frac{\partial}{\partial t}\left(\int_{\Omega_P} U\,d\Omega\right)dt$$

$$= \int_t^{t+\Delta t}\frac{\partial}{\partial t}\left(U_P\,\Delta\Omega_P\right)dt$$

I.T., U_P → Cell averaged value of U. $\Delta\Omega_P$ → Volume of the central cell

$$= \left[U_P\,\Delta\Omega_P\right]_t^{t+\Delta t}$$

$$= \left(U_P^{l+1} - U_P^{l}\right)\Delta\Omega_P$$

U_P^{l+1} = Value of U at cell P at $(t+\Delta t)$ time level.

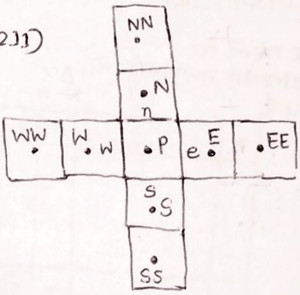
$$= \left(U_P^{l+1} - U_P^{l}\right)\Delta x\,\Delta y \quad \text{.......} \quad (210)$$

Discretization of Spatial Term:-

$$\int_t^{t+\Delta t}\int_{\Omega_P}\nabla\cdot\underline{F}\,d\Omega\,dt = \int_t^{t+\Delta t}\nabla\cdot\left(\frac{\partial E}{\partial x}\hat{i}+\frac{\partial G}{\partial y}\hat{j}\right)d\Omega\,dt$$

$$= \int_t^{t+\Delta t}\int_S \vec{F}\cdot\vec{ds}\,dt \quad \text{(Gauss-Divergence theorem)}$$

$$= \left[\sum_{f=e,w,n,s}\left(\vec{F}_f\cdot\hat{n}_f\right)A_f\right]\Delta t \quad \text{.....} \quad (211)$$



Numerical flux Calculation;

East face:

$$\underline{F}_e\cdot\hat{n}_e = \frac{1}{2}\left[\vec{F}_{Re}+\vec{F}_{Le}-\alpha\left(\vec{U}_{Re}-\vec{U}_{Le}\right)\right]\cdot\hat{n}_e \quad \text{.....} \quad (212)$$

In ground water equations we discretized the derivatives directly. But, here, we need the flux values at the interface.

$\vec{F}_{Re} = f(\vec{U}_{Re})$ = Flux computed using information from the right side of cell face. ..(212.1)

$\vec{F}_{Le} = f(\vec{U}_{Le})$ = " " " " " left side of cell face... (212.2)

Why so complex?!
Face value কেন অন্য 2টা cell centred value average করলে হয় না?

where, $U_{Le} = U_P + \frac{1}{2}\delta U_P$

$U_{Re} = U_E - \frac{1}{2}\delta U_E$

and $\delta U_P = \text{minmod}(\underbrace{U_E - U_P}_{a},\ \underbrace{U_P - U_W}_{b})$

$\delta U_E = \text{minmod}\left(U_E - U_P,\ U_{EE} - U_E\right).$

(↳ See logic at next page)

Now, how to find the values of δU_P and δU_E and evaluate U_{Le} and U_{Re}:-

If we calculate U_e value from left side,

Case I: If $|U_E - U_P| \leq |U_P - U_W|$,

the $\delta U_P = U_E - U_P$.

and, $U_{Le} = U_P + \frac{1}{2}(U_E - U_P)$

$= \frac{U_P + U_E}{2}$

Case II: If $|U_P - U_W| < |U_E - U_P|$,

then, $\delta U_P = U_P - U_W$

and $U_{Le} = U_P + \frac{1}{2}(U_P - U_W)$

$= \frac{3}{2}U_P - \frac{1}{2}U_W$.

The minmod limiter is defined as:-

$$\text{minmod}(a,b) = \begin{cases} a, & \text{if } |a|<|b| \text{ and } ab>0. \\ b, & \text{if } |b|<|a| \text{ and } ab>0 \\ 0, & \text{if } ab<0. \end{cases}$$

The positive coff α is determined by using the maximum value (for all grid points) of the largest eigenvalue of the Jacobian matrix.

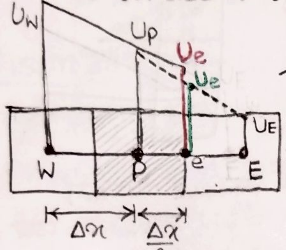
$$\alpha \geq \max|\lambda_P| \quad \forall P \in \Omega$$

with, $\lambda_P = V_P + \sqrt{g h_P}$.

with V_P = Resultant velocity.

h_P = depth of flow in case of surface flooding.

From the left side of east face,



→ Here, Red U_e value is more accurate, as per the logic given below.

Here, $U_e = U_{Le}$. Because, for calculating U_e, we are taking informations from the left cells mainly.

$$U_P = \frac{U_W \times \frac{\Delta x}{2} + U_{Le}\,\Delta x}{\frac{\Delta x}{2} + \Delta x}$$

I.T. #Note: Previously, we used average of previous and next cell centred values to get value at face (U_e). Now, we are evaluating two face values. i.e. U_{Le} and U_{Re}.

$$\Rightarrow U_P = \frac{U_W + 2U_{Le}}{3}$$

$$\Rightarrow U_{Le} = \frac{3U_P - U_W}{2}$$

Using interpolation formula,

$$y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

If we calculate U_e value from the right side:-

Case I: when, $|U_E - U_P| < |U_{EE} - U_E|$

$\delta U_E = (U_E - U_P)$

$\therefore U_{Re} = U_E - \frac{1}{2}(U_E - U_P)$ ⇒

$= \frac{U_E + U_P}{2}$

e is the midpoint between P and E.

$\therefore U_e = \frac{U_E + U_P}{2}$

I.T. # Here, to compute U_e, some quantity (ie $\frac{1}{2}\delta U_E$) is subtracted from U_E. The δU_E quantity will be minimum of $|U_P - U_E|$ and $|U_{EE} - U_E|$. Here, green coloured U_e value is more correct.

Because, if variation of U value is less, then interpolated value would be more accurate at the face.

Case-II: when $|U_{EE} - U_E| < |U_E - U_P|$

$\delta U_E = U_{EE} - U_E$

$\therefore U_{Re} = U_E - \frac{1}{2}(U_{EE} - U_E)$

$= \frac{3}{2}U_E - \frac{1}{2}U_{EE}$

⇓

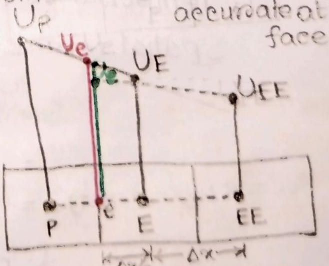
From linear interpolation,

$$\frac{U_e \times \Delta x + U_{EE} \times \frac{\Delta x}{2}}{\Delta x + \frac{\Delta x}{2}} = U_E$$

$$\Rightarrow \frac{2U_e + U_{EE}}{3} = U_E$$

$$\Rightarrow U_e = \frac{3U_E - U_{EE}}{2}$$

$$= \frac{3}{2}U_E - \frac{1}{2}U_{EE}$$



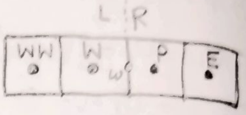
Here, $U_e = U_{Re}$.

Because, we are taking informations from the right cells mainly.

Similarly,
Numerical flux calculations for other faces:-

<u>West face:-</u>

$$F_w \cdot \hat{n}_w = \frac{1}{2}\left[F_{Rw} + F_{Lw} - \alpha(U_{Rw} - U_{Lw})\right] \cdot \hat{n}_w \quad \cdots (213)$$



$$U_{Lw} = U_W + \frac{1}{2}\delta U_W$$

$$U_{Rw} = U_P - \frac{1}{2}\delta U_P$$

$$\delta U_P = \text{minmod}(U_P - U_W, U_E - U_P)$$

$$\delta U_W = \text{minmod}(U_P - U_W, U_W - U_{WW}).$$

<u>North face:</u>

Unit vector perpendicular to north face of cell 'P'.

$$F_n \cdot \hat{n}_n = \frac{1}{2}\left[F_{Tn} + F_{Bn} - \alpha(\underbrace{U_{Tn}}_{\text{U value at north face, evaluated by top informations.}} - U_{Bn})\right] \cdot \hat{n}_n \quad \cdots (214)$$

$$U_{Tn} = U_N - \frac{1}{2}\delta U_N$$

$$U_{Bn} = U_P + \frac{1}{2}\delta U_P.$$

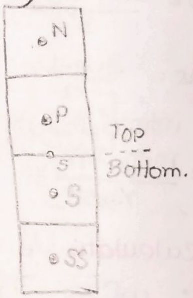
$$\delta U_N = \text{minmod}(U_N - U_P, U_{NN} - U_N)$$

$$\delta U_P = \text{minmod}(U_N - U_P, U_P - U_S)$$

যে face এর জন্য value বের করতে হবে, সেই দিকে একটা extra cell নিতে হবে। (যেমন WW, NN, EE; WW cell এর কথা বলছি)।

<u>South face:-</u>

$$F_s \cdot \hat{n}_s = \frac{1}{2}\left[F_{Ts} + F_{Bs} - \alpha(U_{Ts} - U_{Bs})\right] \cdot \hat{n}_s \quad \cdots (215)$$



$$U_{Ts} = U_P - \frac{1}{2}\delta U_P.$$

$$U_{Bs} = U_S + \frac{1}{2}\delta U_S$$

Because, we are evaluating for 's' face; which is in between P and S node.

$$\delta U_P = \text{minmod}(U_N - U_P, U_P - U_S)$$

$$\delta U_S = \text{minmod}(U_P - U_S, U_S - U_{SS}).$$

Discretized form of GE:

#In FVM, integrated form ⇒ Eq (209)

#Discretization of temporal and spatial terms are mentioned in equation (210) and (211).

$$\text{Temporal term} := (U_P^{l+1} - U_P^{l})\Delta\Omega_P \quad \cdots (216)$$

$$\text{Spatial term} = \left[\sum_{f=e,w,n,s} (\vec{F}_f \cdot \hat{n}_f) A_f\right] dt$$

$$= F_e$$

See next page. → (I.T) →

$$= \left[F_e(\hat{i}) \cdot A_{xe}(\hat{i}) + F_w(\hat{i}) \cdot A_{xw}(-\hat{i}) + F_n(\hat{j}) A_{yn}(\hat{j}) + F_s(\hat{j}) A_{ys}(-\hat{j})\right]$$

$$= \left[F_e^l A_{xe} - F_w^l A_{xw} + F_n^l A_{yn} - F_s^l A_{ys}\right]\Delta t \quad \cdots (217)$$

I.T # All these are evaluated at lth time level. Because, we want to use explicit scheme.

Source sink term $= \int_t^{t+\Delta t} \left[\iint_{\Omega p} S\, d\Omega \right] dt$

$= \int_t^{t+\Delta t} (S_P \, \Delta\Omega_P)\, dt$

S_P → cell averaged Value of source/sink term. evaluated at l^{th} time level.

$\Delta\Omega_P$ → Volume of the cell.

$= \Delta t \, S_P^{l} \, \Delta\Omega_P$ ...... (218)

Volume of the cell $\Delta\Omega_P$ and area of the cell faces ($A_{xe}, A_{xw}, A_{yn}, A_{ys}$) does not varies with time. No time superscript is used.

Putting values from (216), (217) and (218) to GE (209),

$$(U_P^{l+1} - U_P^{l})\Delta\Omega_P + \left[F_e^{l} A_{xe} - F_w^{l} A_{xw} + F_n^{l} A_{yn} - F_s^{l} A_{ys}\right]\Delta t = \Delta t \, S_P^{l} \, \Delta\Omega_P$$

$$\Rightarrow U_P^{l+1} = U_P^{l} - \frac{\Delta t}{\Delta\Omega_P}\left[F_e^{l} A_{xe} - F_w^{l} A_{xw} + F_n^{l} A_{yn} - F_s^{l} A_{ys}\right] + \Delta t \, S_P^{l} \quad \text{..... (219)}$$

↳ Final descretized form of the governing equation.

Discretization:

Predictor - Corrector Approach:

Predictor step, (using equation 219)

$$U_P^{*} = U_P^{l} - \frac{\Delta t}{\Delta\Omega_P}(\cdots\cdots\cdots) + \Delta t \, S_P^{l} \quad \text{..... (220)}$$

For uniform grid system,

$\Omega_P = \Delta x \Delta y$

$A_{xe} = A_{xw} = \Delta y$

$A_{yn} = A_{ys} = \Delta x$.

∴ In simplified form, (From 221),

$$U_P^{*} = U_P^{l} - \frac{\Delta t}{\Delta\Omega_P}\left[E_e^{l}\Delta y - E_w^{l}\Delta y + G_n^{l}\Delta x - G_s^{l}\Delta x\right] + \Delta t \, S_P^{l}$$

$\Delta\Omega_P = \Delta x \Delta y$

$$\Rightarrow U_P^{*} = U_P^{l} - \frac{\Delta t}{\Delta x}\left[E_e^{l} - E_w^{l}\right] - \frac{\Delta t}{\Delta y}\left[G_n^{l} - G_s^{l}\right] + \Delta t \, S_P^{l} \quad \text{...... (222)}$$

Spatial term can be discretized as:- (From 209)

$\int_{\Omega p} \nabla \cdot \vec{F} \, d\Omega$

$= \int_S \vec{F} \cdot d\vec{s}$

$= (E\hat{i} + G\hat{j}) \cdot \left[A_{xe}\hat{i} + A_{xw}(\hat{-i}) + A_{yn}\hat{j} + A_{ys}(-\hat{j})\right]$

$= E_e A_{xe} - E_w A_{xw} + G_n A_{yn} - G_s A_{ys}$

$= E_e \Delta y - E_w \Delta y + G_n \Delta x - G_s \Delta x$

$\therefore \int_t^{t+\Delta t} \left[\iint_{\Omega p} \nabla \vec{F} \, d\Omega \right] dt$

$= (E_e^{l}\Delta y - E_w^{l}\Delta y + G_n^{l}\Delta x - G_s^{l}\Delta x)\Delta t$ ...... (221)

↳ E and G values are evaluated at l^{th} level. Because, explicit scheme.

Corrector step:

Directly using simplified form, (222),

$$U_P^{**} = U_P^{*} - \frac{\Delta t}{\Delta x}\left[E_e^{*} - E_w^{*}\right] - \frac{\Delta t}{\Delta y}\left[G_n^{*} - G_s^{*}\right] + \Delta t \, S_P^{*} \quad \text{..... (223)}$$

∴ Value at future time level is evaluated as:-

$$U_P^{l+1} = \frac{1}{2}(U_P^{*} + U_P^{**}) \quad \text{...... (224)}$$

Actual variables:

U was an assumed variable. (I.T ⇒ formed to get Jacobian matrix).

Remember, U, E, G, S, F these are column vectors. Not scaler terms

From eqn (203),

$$U_P = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}_P = \begin{Bmatrix} U_{P,1} \\ U_{P,2} \\ U_{P,3} \end{Bmatrix} = \begin{Bmatrix} h \\ hu \\ hv \end{Bmatrix}_P = \begin{Bmatrix} h_P \\ h_P \cdot u_P \\ h_P \cdot v_P \end{Bmatrix} \quad \text{.... (225)}$$

Actual variables are:- (# Variables are for $(l+1)^{th}$ time level. Because, l^{th} level values are known).

$h_P^{l+1} = U_{P,1}^{l+1}$ ...(226.1)

$h_P^{l+1} \cdot u_P^{l+1} = U_{P,2}^{l+1}$

$\Rightarrow u_P^{l+1} = \dfrac{U_{P,2}^{l+1}}{U_{P,1}^{l+1}}$ .....(226.2)

$h_P^{l+1} \cdot v_P^{l+1} = U_{P,3}^{l+1}$

$\Rightarrow v_P^{l+1} = \dfrac{U_{P,3}^{l+1}}{U_{P,1}^{l+1}}$ ......(226.3).

For predictor step and corrector step we can use same $\delta U_P, \delta U_E, \cdots$ values because that will lead to numerical stability of the scheme.

$\delta U_P = U_P - U_W, U_E - U_P$

But, I think these values would change in each timestep

$$U^*_{Le} = U^*_P + \tfrac{1}{2}\delta U_P$$

This term considering Predictor time step value

This term is using current time step value

would it give results like,

$$U^*_{Le} = \left(\frac{U^*_P + U^*_E}{2}\right) \text{ or } \left(\tfrac{3}{2}U^*_P - \tfrac{1}{2}U^*_W\right) \ ?$$

$$U^*_{Re} = U^*_E - \tfrac{1}{2}\delta U_E$$

$$U^*_{LW} = U^*_W + \tfrac{1}{2}\delta U_W$$

$$U^*_{RW} = U^*_P - \tfrac{1}{2}\delta U_P$$

$$U^*_{Bn} = U^*_P + \tfrac{1}{2}\delta U_P.$$

$$U^*_{Tn} = U^*_N - \tfrac{1}{2}\delta U_N. \quad \ldots\ldots \ (227)$$

No flow Boundary:

$U_{2,e} = (uh)_e = 0$ → Means, (2nd element of U vector at east face)

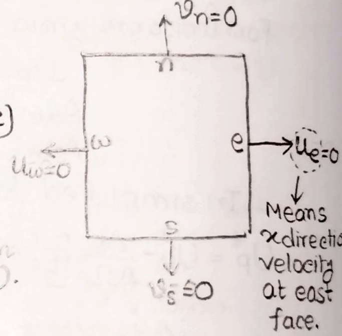
$U_{2,w} = (uh)_w = 0$

$U_{3,n} = (vh)_n = 0$

$U_{3,s} = (vh)_s = 0$

where, $U = \begin{Bmatrix} h \\ hu \\ hv \end{Bmatrix}$ --- (From 201.3).

$\ldots\ldots (228)$



East Boundary:

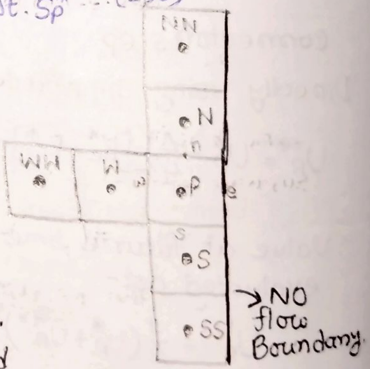
? other places there are 'ℓ'. why 'n' here?! in notes.

$$U^*_P = U^n_P - \frac{\Delta t}{\Delta x}\left[E^\ell_e - E^\ell_w\right] - \frac{\Delta t}{\Delta y}\left[G^\ell_n - G^\ell_s\right] + \Delta t\, S^\ell_P \ \ldots\ (229)$$

$$U^{**}_P = U^*_P - \frac{\Delta t}{\Delta x}\left[E^*_e - E^*_w\right] - \frac{\Delta t}{\Delta y}\left[G^*_n - G^*_s\right] + \Delta t . S^*_P \ \ldots (230)$$

where, (From equation 201.2 and 201.3),

$$E = \begin{Bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{Bmatrix}, \quad G = \begin{Bmatrix} hv \\ huv \\ hv^2 + \frac{gh^2}{2} \end{Bmatrix} \quad \ldots\ (231)$$



These two equations are applicable for west, north and south boundary. i.e E_w, G_n and G_s can be calculated using these two matrix.

But east face is a non flow boundary or zero Neumann Boundary. ∴ u=0 for this case.

$$\therefore \underset{\frown}{E_e} = \begin{Bmatrix} h.0 \\ h.0^2 + \frac{1}{2}gh^2 \\ h.0.v \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{1}{2}gh^2 \\ 0 \end{Bmatrix} \quad \ldots\ (232)$$

West Boundary:

Everything is same like equation (229), (230), (231).
Only change is here is that, west face is a zero Neumann boundary.

$\therefore$ Here, $F_w = \begin{bmatrix} 0 \\ gh^2/2 \\ 0 \end{bmatrix}$ .... (233)

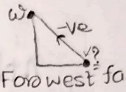
// Here, to calculate the value of 'h' for the west face, we can use the cell centered value here, i.e. $h_w = h_P$.
But, this is approximate specification of boundary conditions. We need characteristic curve or method of characteristics to specify the boundary condition for the explicit cases.

North Boundary:

Everything will be same.

only, $G_n = \begin{bmatrix} h.0 \\ hu.0 \\ h.0^2 + \frac{gh^2}{2} \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ gh^2/2 \end{Bmatrix}$ .... (234)

?



For west face, we have one -ve characteristics equation and one Neumann BC. But, we have to find our variable U. (Because, you can see from 202, E, G, F column vectors can be represented by U.) Then to find only one variable, why two equations will be needed?

$U = \begin{Bmatrix} h \\ hu \\ hv \end{Bmatrix}$

Equation (202),

$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial U}(\nabla \cdot U) = 0$

function — Jacobian — Increment of variables

South Boundary:

$G_s = \begin{Bmatrix} 0 \\ 0 \\ gh^2/2 \end{Bmatrix}$ .... (235)

(l+1) time level West face এর data কোথা থেকে আসবে? previous time step এর cell centred value নেবে? না face value নেবে?

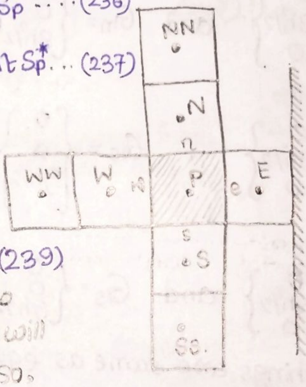
Next to east boundary:

$U_P^* = U_P^n - \frac{\Delta t}{\Delta x}\left[E_e^l - E_w^l\right] - \frac{\Delta t}{\Delta y}\left[G_n^l - G_s^l\right] + \Delta t S_P^l$ .... (236)

$U_P^{**} = U_P^* - \frac{\Delta t}{\Delta x}\left[E_e^* - E_w^*\right] - \frac{\Delta t}{\Delta y}\left[G_n^* - G_s^*\right] + \Delta t S_P^*$ ... (237)

and

$E = \begin{Bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{Bmatrix}$ ...(238) $\quad G = \begin{Bmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{Bmatrix}$ ...(239)



I.I # Here, h, u, v all are non-zero values at face 'e'. So, no term will be zero like boundary cases. So, form of 'E_e' matrix is preserved.
But, it is to be mentioned that, there is no 'EE' cell here.
So, we cannot calculate 'U_{Re}' case-II (i.e. $U_{Re} = \frac{3}{2}U_E - \frac{1}{2}U_{EE}$) here.

Problem is $\Rightarrow$ (E and G are functions of U_{Re}) $\rightarrow$ why??

Here is the answer:- $\underline{F} = \underline{E}\hat{i} + \underline{G}\hat{j}$ (From 204)

$\Rightarrow$ Also, $F_e = f(F_{Re}, F_{Le}, U_{Re}, U_{Le})$ ... (From 212)

$\Rightarrow E\hat{i} + G\hat{j} = f(g(U_{Re}), g(U_{Le}), U_{Re}, U_{Le})$

(From 212.1 and 212.2)

Similarly, equation can be discretized for cells next to west, north and south boundaries. But, to know what is the difference with interior cells, you need to remember the above mentioned logic.

North east corner:-

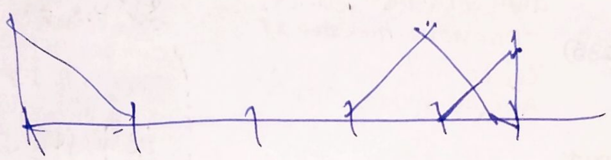
$$E_e = \begin{Bmatrix} 0 \\ gh^2/2 \\ 0 \end{Bmatrix}_e \quad \text{and} \quad G_n = \begin{Bmatrix} 0 \\ 0 \\ gh^2/2 \end{Bmatrix}_n \quad \ldots\ldots (238)$$

Discretized equations would be same as (229) and (230)

North west corner:-

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt}\,dt = \int_{t_n}^{t_{n+1}} \psi(\phi)\,dt$$

$$\phi^{n+1} - \phi^{n} = \int_{t_n}^{t_{n+1}} \psi(\phi)\,dt$$



$$E_w = \begin{Bmatrix} 0 \\ gh^2/2 \\ 0 \end{Bmatrix} \quad \text{and} \quad G_n = \begin{Bmatrix} 0 \\ 0 \\ gh^2/2 \end{Bmatrix} \quad \ldots\ldots (239)$$

South east corner:-

$$E_e = \begin{Bmatrix} 0 \\ gh^2/2 \\ 0 \end{Bmatrix} \quad G_s = \begin{Bmatrix} 0 \\ 0 \\ gh^2/2 \end{Bmatrix} \quad \ldots\ldots (240)$$

South-west corner:-

$$E_w = \begin{Bmatrix} 0 \\ gh^2/2 \\ 0 \end{Bmatrix} \quad \text{and} \quad G_s = \begin{Bmatrix} 0 \\ 0 \\ gh^2/2 \end{Bmatrix} \quad \ldots\ldots (241)$$

Other things are same as equation (229) and (230).

Zero Inertia Model:

For our surface water flow specification of boundary condition and more or less the the solution is difficult due to non-linearity present in the equation. ← (C.T. ⇒ For these types of non-linear terms.) $E = \begin{Bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{Bmatrix}$, $G = \begin{Bmatrix} hv \\ huv \\ hv^2 + \frac{gh^2}{2} \end{Bmatrix}$

So, for surface flooding or simplified modelling, we can reduce our shallow water equations and we can drop some acceleration terms and we can solve the equations for simple flooding situations.

The full shallow water equations:-

$$\frac{\partial h}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = R \quad \ldots\ldots (242.1)$$

$$\frac{\partial q_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q_x^2}{h}+\frac{1}{2}gh^2\right)+\frac{\partial}{\partial y}\left(\frac{q_x q_y}{h}\right)=gh(S_{ox}-S_{fx}) \quad \ldots (242.2)$$

$$\frac{\partial q_y}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q_x q_y}{h}\right)+\frac{\partial q_x q_y}{\partial y}\left(\frac{q_y^2}{h}+\frac{1}{2}gh^2\right)=gh(S_{oy}-S_{fy}) \quad \ldots (242.3)$$

These are exactly same equations mentioned in page number 56. (CE and ME's)

where, $q_x = uh$, $q_y = vh$. ...... (243)

$$\underbrace{S_{ox} = -\frac{\partial z}{\partial x} \quad S_{oy} = -\frac{\partial z}{\partial y}}_{\text{Rate of change in bed elevation in } x \text{ and } y \text{ directions.}} \quad \ldots (244)$$

$$S_{fx}=\frac{n^2 u\sqrt{u^2+v^2}}{h^{4/3}} \text{ and } S_{fy}=\frac{n^2 v\sqrt{u^2+v^2}}{h^{4/3}}. \quad \ldots (245)$$

How these two terms came?!

Considering the flow directions and using Manning's formula,

$$V=\frac{1}{n}R^{2/3}S^{1/2}$$

$$\Rightarrow S=\frac{V^2n^2}{h^{4/3}}$$

[For rectangular channel, R=h= flow depth]

Writing in vector form and considering flow direction, we have,

$$(S_{fx}\hat{i}+S_{fy}\hat{j})=\frac{V|V|n^2}{h^{4/3}}$$

$$=\frac{(u\hat{i}+v\hat{j})|V|n^2}{h^{4/3}}$$

$$\Rightarrow \begin{Bmatrix} S_{fx} \\ S_{fy} \end{Bmatrix}=\begin{Bmatrix} u \\ v \end{Bmatrix}\frac{\sqrt{u^2+v^2}\,n^2}{h^{4/3}}$$

So, we get,

$S_{fx}=(u\sqrt{u^2+v^2}\,n^2)/h^{4/3}$

and $S_{fy}=(v\sqrt{u^2+v^2}\,n^2)/h^{4/3}$.

By neglecting acceleration terms in shallow water equations, zero-inertia system can be expressed as,

Which acceleration related term is neglected here?

$$\frac{\partial h}{\partial t}+\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}=R \quad \text{(CE, Same as 242.1)}$$

$$\left.\begin{aligned}\frac{\partial h}{\partial x}&=S_{ox}-S_{fx}\\ \text{and } \frac{\partial h}{\partial y}&=S_{oy}-S_{fy}\end{aligned}\right\} \quad \ldots (246)$$

Defining water surface slope in x and y directions,

$$S=(S_x, S_y)=-\left[\frac{\partial(h+z)}{\partial x}, \frac{\partial(h+z)}{\partial y}\right] \quad \ldots (247).$$

$$S_x=-\frac{\partial h}{\partial x}-\frac{\partial z}{\partial x} \quad \text{(From 247)}$$

$=-(S_{ox}-S_{fx})+S_{ox}$ (Putting values from (244) and (246));

$= S_{fx}$ ...... (248)

Similarly, $S_y=-\frac{\partial h}{\partial y}-\frac{\partial z}{\partial y}$

$=-(S_{oy}-S_{fy})+S_{oy}$

$=S_{fy}$. ...... (249)

Thus, $S_x = S_{fx}$ and $S_y = S_{fy}$, i.e. $\begin{Bmatrix} S_x \\ S_y \end{Bmatrix}=\begin{Bmatrix} S_{fx} \\ S_{fy} \end{Bmatrix} \Rightarrow \underset{\sim}{S}=\underset{\sim}{S_f}$ ...... (250)

Unit discharge values at x and y direction,

$$q_x=\left(\frac{h^{5/3}n}{\sqrt{|S|}}S_x\right)$$

$$\text{and } q_y=\left(\frac{h^{5/3}\cdot n}{\sqrt{|S|}}\cdot S_y\right) \quad \ldots (251)$$

Manning's 'n' should be at denominator!

$$v=\frac{1}{n}h^{2/3}S_f^{1/2}$$

$$\Rightarrow q=\frac{1}{n}h^{5/3}S^{1/2} \quad [\because S=S_f]$$

$$\Rightarrow q=\frac{1}{n}h^{5/3}\frac{S}{\sqrt{|S|}} \quad \text{(Considering direction of friction slope)}$$

In vector form,

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix}=\frac{h^{5/3}}{n\sqrt{|S|}}\begin{Bmatrix} S_x \\ S_y \end{Bmatrix}$$

$$\Rightarrow q_x=\frac{h^{5/3}}{n\sqrt{|S|}}\times S_x$$

$$\text{and } q_y=\frac{h^{5/3}}{n\sqrt{|S|}}\times S_y$$

By combining Discharge and continuity equations, final form of zero inertia equation can be written as:-

$$\boxed{\frac{\partial h}{\partial t}+\nabla\cdot q=R.} \quad \text{where, } q=(q_x, q_y). \quad \ldots (252)$$

$$\frac{\partial h}{\partial t}+\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}\right)(q_x\hat{i}+q_y\hat{j})=R$$

$$\Rightarrow \frac{\partial h}{\partial t}+\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}=R. \rightarrow \text{This is CE. Same as eq}^n\text{ (242.1).}$$

Equation (252) is of parabolic in nature

I.T(Eigenvalues are zero, that's why).

This equation is similar to the depth-integrated mass conservation equation of groundwater flow. $\Rightarrow S\frac{\partial h}{\partial t} + \nabla . q = f$.

If we consider $\alpha(h)$ as function of h only, zero intertia condition equation can be further simplified.

From equation (251),

$$\text{Let, } \alpha(h) = \frac{h^{5/3} . n}{\sqrt{|S|}} \quad \text{.........} (253)$$

∴ We can write by combining (251.1) and (251.2),

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \alpha(h) . \begin{Bmatrix} S_x \\ S_y \end{Bmatrix}$$

$$\Rightarrow \underset{\sim}{q} = \alpha(h) . \underset{\sim}{S} \quad \text{.......} (254)$$

$$\Rightarrow \underset{\sim}{q} = \alpha(h) \begin{Bmatrix} -\frac{\partial}{\partial x}(h+z) \\ -\frac{\partial}{\partial y}(h+z) \end{Bmatrix}$$

$$\Rightarrow q_x \hat{i} + q_y \hat{j} = - \alpha(h) \left\{ \frac{\partial}{\partial x}(h+z)\hat{i} + \frac{\partial}{\partial y}(h+z)\hat{j} \right\}$$

$$\Rightarrow q_x \hat{i} + q_y \hat{j} = - \alpha(h) \left\{ \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} \right)(h+z) \right\}$$

$$\Rightarrow q_x \hat{i} + q_y \hat{j} = - \alpha(h) \, \nabla(h+z)$$

$$\Rightarrow \underset{\sim}{q} = - \alpha(h) \, \nabla(h+z). \quad \text{.....} (255)$$

Putting value of $\underset{\sim}{q}$ from (255) to zero-intertia equation (252), we get zero-inertia equation in terms of h,

$$\frac{\partial h}{\partial t} = \nabla . \left[\alpha(h) \, \nabla(h+z) \right] + R. \quad \text{.......} (256)$$

Usualy, in surface water flow equations, we need to solve h, u, v but with a zero-inertia model we can reduce our problem to one variable(h) and we can solve that problem.

⇒ So, now we will discretize the zero inertia model equation (256), using FVM again,

$$\int_t^{t+\Delta t} \left[\int_{\Omega_p} \frac{\partial h}{\partial t} d\Omega \right] dt = \int_t^{t+\Delta t} \left[\int_{\Omega_p} \nabla . \left[\alpha(h) . \nabla(h+z) \right] d\Omega \right] dt + \int_t^{t+\Delta t} \left[\int_{\Omega_p} R \, d\Omega \right] dt$$

Here, we are neglecting acceleration related terms. So, obviously, we are considering slow movement of water.

So, example is:- Slow flooding or irrigation of water through canal system.

NPTEL-1

Introduction to CH

ODE, PDEs represent conservation laws.

Solution methods of ODE, PDEs:-

- Analytical (Closed form)
- Semi-Analytical
- Numerical (Approximate form).

Experiments	Simulations
(i) Information of physical phenomenon on representative spatio-temporal observation points	(i) Prediction of physical phenomenon on discretized nodes
(ii) Measurement errors	(ii) Conceptualization and numerical errors.

NPTEL-2

Problem definition and governing Equations:

ODE:- DE with one independent variable

PDE:- Differential equation with two or more independent variable.

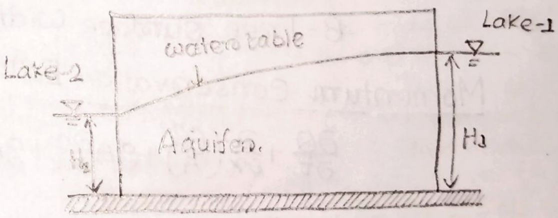
Incompressible fluid flow:-

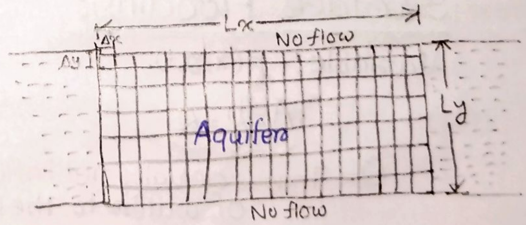
Mass conservation:- $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ .... (257.1)

Momentum ":-

$$x \text{ dir}:- \underbrace{\frac{\partial u}{\partial t}}_{\text{Temporal term}}+\underbrace{\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}}_{\text{Advective term}}=\underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{Pressure term}}+\underbrace{g_x}_{\text{Gravity term}}+\underbrace{\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}_{\text{Diffusion term}} \quad ...(257.2)$$

Groundwater movements in aquifers:





Depth integrated mass conservation equation:-

$S\frac{\partial h}{\partial t}+\nabla.\underset{\sim}{q}=f$ ..... (258) → Source/sink term.

S → Storativity

- Specific yeild (Unconfined aquifer)
- Storage coefficient (Confined)

$\underset{\sim}{q}=q_x\hat{i}+q_y\hat{j}$ = flux.

Momentum CE:- (Darcy's law):-

$$\left.\begin{aligned} q&=-T\nabla h\\ T&=K[\min(h,Z_u)-Z_b]\end{aligned}\right\} \quad ......(259)$$

$[\min(h,Z_u)-Z_b]$ → Means Saturated thickness portion.

T = Aquifer transmissivity $[L^2/T]$

$k=k(x,y)$

Z_u, Z_b = Top and bottom aquifer elevation.

$S\frac{\partial h}{\partial t}+\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}\right).\left(-T\left(\frac{\partial h}{\partial x}\hat{i}+\frac{\partial h}{\partial y}\hat{j}\right)\right)=f$

(From 258 and 259)

$\Rightarrow S\frac{\partial h}{\partial t}-T\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)=f$

(Considering source sink term $f=0$),

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t}$$

(Same equation at page 2).

Steady 1D groundwater flow:

Mass conservation Equation: $\frac{dq_x}{dx} = f$ ..... (260.1)

Momentum " " : $q_x = -T\frac{dh(x)}{dx}$ ..... (260.2)

$\Rightarrow v\cancel{h} = -k\cancel{h}\cdot i$

Contaminant transport: $\Rightarrow v = -ki \rightarrow$ which is Darcy's law.

Mass CE.: $\frac{\partial}{\partial x}(k_{xx}\frac{dh}{dx}) + \frac{\partial}{\partial y}(k_{yy}\frac{\partial h}{\partial x}) + q_s = S_s\frac{\partial h}{\partial t}$ ...(261.1)

h = potentiometric head.

q_s = Volumetric flux per unit volume. (1/T)

S_s = Specific storage (1/L) (This 'specific storage' is applicable for confined aquifer system).

This is hydraulic head related equation.

Concentration equation (Scaler Transport equation):

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x}\left(\eta D_{xx}\frac{\partial C}{\partial x} + \eta D_{xy}\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial y}\left(\eta D_{yx}\frac{\partial C}{\partial x} + \eta D_{yy}\frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial x}(\eta v_x C) - \frac{\partial}{\partial y}(\eta v_y C) + q_s C_s \quad \text{...... (261.2)}$$

η = Effective porosity

D_{xx}, D_{yy} = Hydraulic dispersion coff along x and y axes $[L^2/T]$.

D_{xy}, D_{yx} = " " " " cross directions "

C_s = Source concentration $[M/L^3]$.

v_x, v_y = velocity along x and y axes # Calculated from Darcy's velocity.

Channel Networks:

Depth integrated mass C.E:-

$$\frac{\partial h}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = 0 \quad \text{(\#1-D space and time both considered). --- (262.1)}$$

h = h(x,t) = channel water depth.

B = Free surface width.

Momentum Conservation Eqn:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h}{\partial x} + gA(S_0 - S_f) = 0 \quad \text{........ (262.2)}$$

Surface Flooding:

Variable :- h(x,y,t).

$A\frac{dh}{dt} = Q$ .... (263)

Q = That is actually net inflow of water to the plot. Inflow can be from canal section or from rainfall to the surface area.

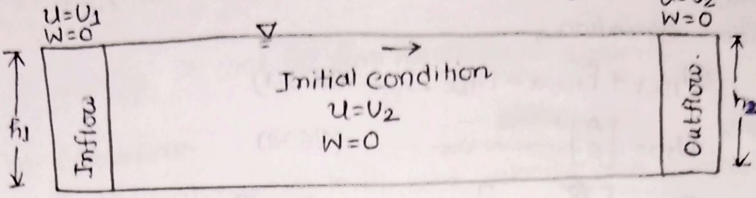
A = Cross sectional area of the plot (I.T: Remember, not the plan area).

h = Depth of water.

Open channel flow:
Hydraulic Jump:

With time, if we change the inlet velocity, we can get desired hydraulic jump simulation.

Variables:- $u(x,z,t)$, $w(x,z,t)$



Initial condition of hydraulic Jump.

Mass conservation:-

$$\nabla \cdot \underset{\sim}{u} = 0 \quad \cdots\cdots (264)$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Here, $\underset{\sim}{u} = u\hat{i} + \omega\hat{j}$

$$\nabla \cdot \underset{\sim}{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(u\hat{i} + \omega\hat{k}) = \left(\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z}\right)$$

Momentum Conservation Eqn:

$$\boxed{\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho}\nabla P + \underset{\sim}{g} + \nu \nabla^2 \underset{\sim}{u} + \frac{1}{\rho}\nabla\cdot\tau} \quad \cdots\cdots (264.2)$$

$\frac{D\underset{\sim}{u}}{Dt}$ = Total derivative → (It considers the advective term plus temporal change in the velocity.)

P = Fluid pressure

$\frac{\tau}{\rho}$ = Sub-particle scale tensor.

ν = kinematic viscosity

Pressurized Conduits:

Variables $H(x,t)$, $Q(x,t)$

In pressurized conduits left and right boundary fixed head is maintained. Closure time of the valve should be known to specify the conditions. It is IBVP problem.

Depth integrated Mass CE:-

$$\frac{\partial H}{\partial t} + \frac{c^2}{gA}\frac{\partial Q}{\partial x} = 0 \ldots (265)$$

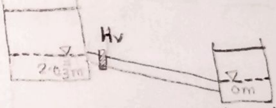
Here, it is 1-D flow system (only x direction).

$H = H(x,t)$ = Piezometric head.

c = wave speed.

Momentum C.E.:-

$$\frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + J_s + J_u = 0 \cdots\cdots (266)$$



I.T (Initially, system is at steady-state condition. Discharge and head at valve are Q_0 and H_0). That is IC. Then valve is closed, transient starts. Two boundary values at u/s and d/s reservoir are also specified. So, problem is IBVP).

A = c/s area

J_s = steady friction loss $= \frac{f_s Q|Q|}{2DA} \cdots\cdots (266.1)$

J_u = Unsteady " " $= \frac{k}{2}\left(Q_t + c\Phi_A|Q_x|\right) \cdots\cdots (266.2)$

where, $\Phi_A = \text{sgn}(Q)$.

K = Brunone friction coefficient [L]

Pressurized Conduits Network:

Mass C.E:- $\frac{\partial q_j}{\partial t} + \frac{A_j}{\rho}\frac{\partial P_j}{\partial x} + \tau_j(q_j) = 0$, $x \in [0, l_j]$, $j \in \Lambda \cdots\cdots (267)$

Along the pipeline j, P_j = Pressure,

q_j = Flow

c_j = Fluid line wave speed.

A_j = C/S area l_j = length of pipe j.

Lambda → Λ = Set of fluid lines.

Momentum Conservation eqn: $\frac{\partial P_j}{\partial t} + \frac{\rho c_j^2}{A_j}\frac{\partial q_j}{\partial x} + \tau_j(q_j) = 0$, $x \in [0, l_j]$, $j \in \Lambda \cdots (268)$

τ_j = Cross sectional frictional resistance.

1D-2D integrated System: (1D governing Eq^ns).

variables:- $h_c(x,t)$, $Q_c(x,t)$, $h_f(x,y,t)$, $v_f(x,y,$ $v_f(x,y,t)$.

Depth integrated mass and momentum Conservation equation,

$$U_{1D,t} + F_{1D,x} = H_{1D} \quad \text{..... (269.1)}$$

where,

$$U_{1D} = \begin{bmatrix} A \\ Q \end{bmatrix} \begin{matrix} \rightarrow \text{Mass} \\ \rightarrow \text{Momentum} \end{matrix} \quad \text{..... (269.2)}$$

$$F_{1D} = \begin{bmatrix} Q \\ Q^2/A + gI_1 \end{bmatrix} \quad \text{..... (269.3)}$$

$$H_{1D} = \begin{bmatrix} q_l \\ gI_2 + gA(S_0 - S_f) \end{bmatrix} \quad \text{..... (269.4)}$$

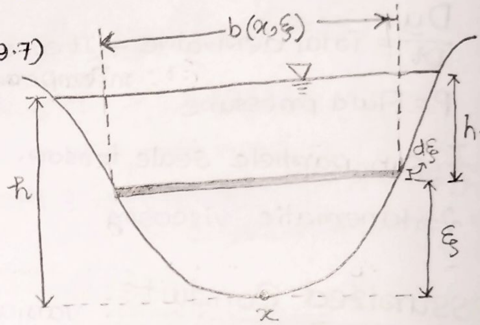
q_l is lateral discharge (I.T, l is subscript here).

Expanding equation (269.1) in algebraic form,

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_l \quad \text{....... (269.5)}$$

and $$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + gI_1\right) = gI_2 + gA(S_0 - S_f). \quad \text{..... (269.6)}$$

$$I_1 = \int_0^h (h-\xi)\, b(x,\xi)\, d\xi \quad \text{.. (269.7)}$$



I.T (# I_1 looks like the moment of the area strip w.r.t water surface).

$$I_2 = \int_0^h (h-\xi) \frac{\partial b(x,\xi)}{\partial x} d\xi. \quad \text{..... (269.8).}$$

1D-2D integrated system (2D governing equations):-

$$U_{2D,t} + \nabla F_{2D} = H_{2D} \quad \text{...... (270)}$$

$$U_{2D} = \begin{Bmatrix} h \\ hu \\ hv \end{Bmatrix} \quad \text{..... (270.1)}$$

$$F_{2D} = \begin{bmatrix} hu & hv \\ hu^2 + \frac{gh^2}{2} & huv \\ huv & hv^2 + \frac{gh^2}{2} \end{bmatrix} \quad \text{.... (270.2)}$$

$$H_{2D} = \begin{bmatrix} i_R \rightarrow \text{Rainfall} \\ gh(S_{0x} - S_{fx}) \\ gh(S_{0y} - S_{fy}) \end{bmatrix} \quad \text{..... (270.3)}$$

Expanding equation (270) in ~~algebraic for~~ algebraic form, we can get the equations 242's.

NPTEL-3

Classification of problems based on IC and BC

To identify IC and BC for hydraulic systems.

Initial condition:-

Initial water water level and velocity in a channel network.

Initial GWL in an aquifer system for time dependent problem.

Types of BC (Location Based):-

External Boundary condition:-

(i) u/s and d/s locations of a river. (# Like we can define inflow discharge, or depth at d/s section).

(ii) River boundary for an aquifer region.
(# constant or varying head boundary)

Internal Boundary Condition:-

(i) Operating conditions for hydraulic structures within channel network.
(# Also called junction condition)

(ii) Constant water level maintained in a pond of an aquifer region.

Types of BC (based on physical nature):

Dirichlet/specified:- Discharge specified at inlet/outlet of a channel network.

Neumann/Flux boundary:- No flow boundary
Near impermeable region of the aquifer system. (# Like rock boundary).

Robin/mixed boundary:-
Weighted Combination of Dirichlet and Neumann conditions.

Classification of Differential equation:

(i) If ODE is given, the problem are:-

IVP ⇒ (GE+IC)

BVP ⇒ (GE+BC).

So, Initial value problem is only defined for ODE. !

(ii) If governing equation is PDE, then.

BVP ⇒ (GE+BC)

IBVP ⇒ GE+IC+BC.

ODE-IVP

GVF in open channel:

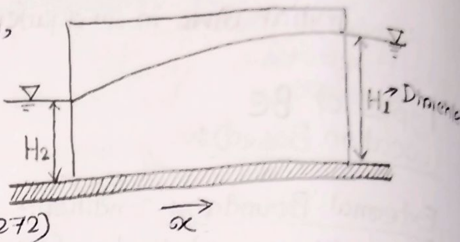
GE: $\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F_r^2}$

IC:- $y|_{x=0} = y_0$ ..... (271).

ODE-BVP

Steady 1D groundwater flow in unconfined aquifer:-

Variable: $h(x)$



Here h is function of x only. (So, 1D problem. Here, we have constant levels of H_1 and H_2 with time. So, it is BVP for ODE.

GE:-

$$-\frac{d}{dx}\left(T\frac{dh(x)}{dx}\right) = f \quad \ldots\ldots (272)$$

It is combined mass and momentum equation. See (260.1) and (260.2).

BC:- $h|_{x=0} = H_2$

and $h|_{x=L_x} = H_1$.

I think → Check it later!

BVP for PDE:-

$$\begin{cases} a\frac{\partial\varphi}{\partial x} + b\frac{\partial\varphi}{\partial y} = c \\ \varphi|_{x=0} = \varphi_1 \quad \varphi_{y=0} = \varphi_3 \\ \varphi|_{x=L_x} = \varphi_2 \quad \varphi_{y=L_y} = \varphi_4 \end{cases}$$

IBVP for PDE

?!
$$\begin{cases} a\frac{\partial\varphi}{\partial t} + b\frac{\partial\varphi}{\partial x} = c \\ \text{at } t=0,\ \varphi_{x=0} = \varphi_5 \\ \text{at } t=5,\ \varphi_{x=0} = \varphi_6 \end{cases}$$

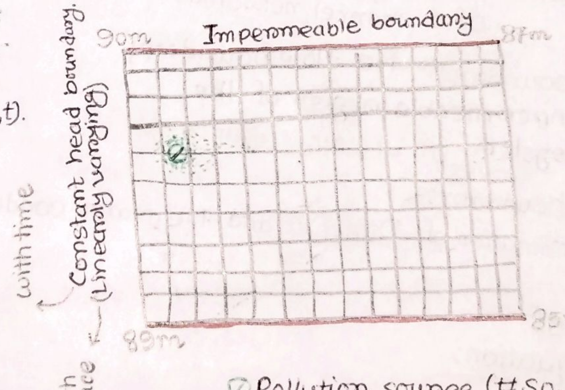
Groundwater movement in aquifers:

Variable: $h(x, y, t)$.

Refer to figure at Page 67. Only difference is we are considering that head is varying in y direction also. (That means, head is not H_2 everywhere in left boundary and not H_1 everywhere in right boundary).

Contaminant transport:-

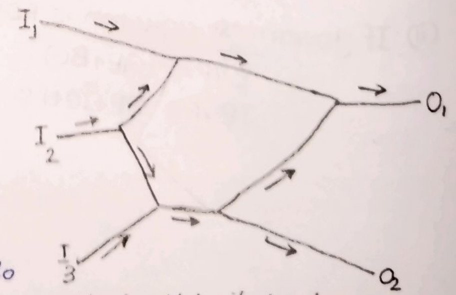
Variables:- $h(x,y,t)$, $C(x,y,t)$.



⊙ Pollution source. (# So, time varying injection is there. So, with time there would be variation in head values).

So, it is IBVP for PDE.

Channel Network:



(1) Internal BC on Junction Conditions:-

(a) Mass conservation:-

$\sum Q_i = \sum Q_o$

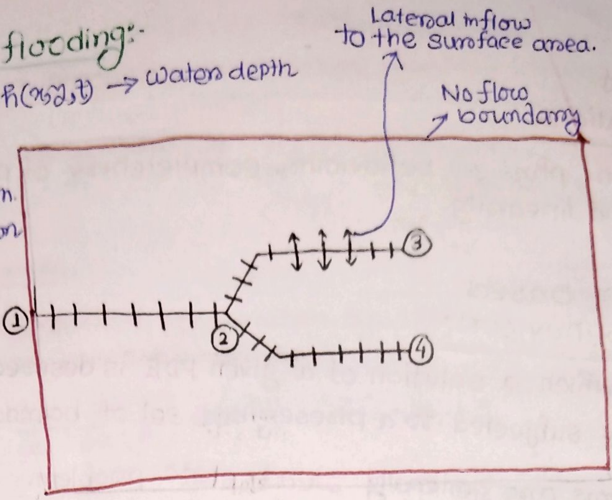
(b) Energy conservation:- $h_i + Z_i = h_o + Z_o$

↳ # without considering velocity head.

Surface flooding:-

Variable $h(x,y,t) \rightarrow$ water depth

① ⇒ Inlet condition.
③,④ ⇒ Outlet condition
② ⇒ Junction ".



Open channel flow: Hydraulic Jump

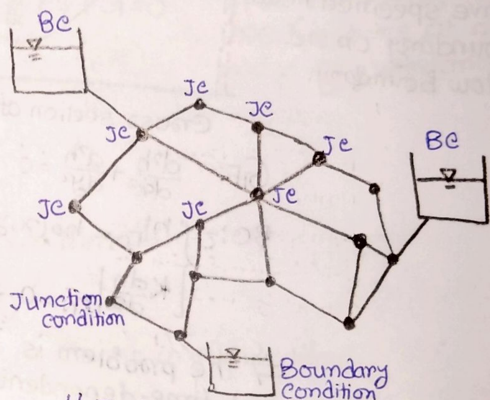
See page 69.

Pressurized Conduits:

Single: see page 69.

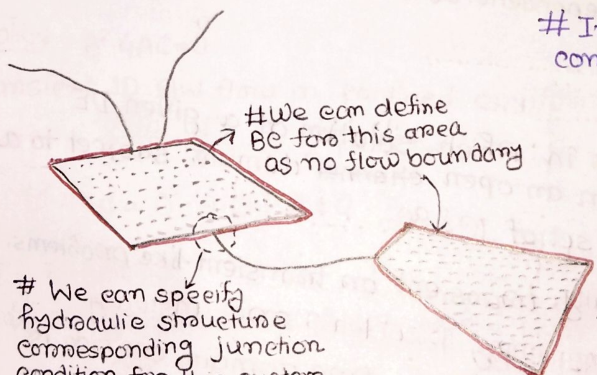
Network:-

Here, we have two variables $P(x,t)$, $q(x,t)$
We have mass and momentum conservation equations. We have internal or junction condition and boundary conditions.



Initial condition is steady state condition.

1D-2D integrated system:



We can define BC for this area as no flow boundary

We can specify hydraulic structure corresponding junction condition for this system.

Depending upon physical situations within hydraulic system, we can change BC's and the combined form along with governing eq^{n}s, and try to solve the problem.

NPTEL-4

Classification of Differential equations:

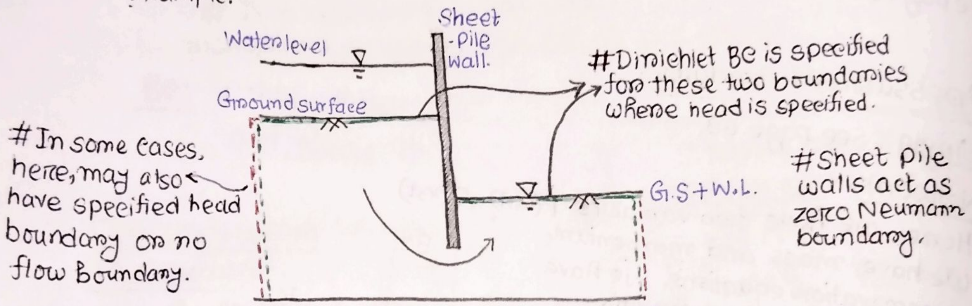
↳ # Based on physical behaviour, completeness of problem definition and linearity.

Classification based on physical behaviour:

(i) Equilibrium Problems:- Problems in which a solution of a given PDE is desired in a closed domain subjected to a prescribed set of boundary conditions.

These problems are generally steady state problem.
From all sides boundaries are defined.
Solution is always smooth even there is disturbances.

Example:-



Cross section of a foundation pit.

GE:- $\frac{d^2h}{dx^2} + \frac{d^2h}{dy^2} = 0$

BC:- $\begin{cases} h|_{\Gamma_D} = h_0(x,y) \rightarrow \text{DBC.} \\ K\frac{dh}{dx}\Big|_{\Gamma_N} = 0 \rightarrow \text{NBC.} \end{cases}$

The problem is space dependent, not time-dependent. So it is steady-state problem.

(ii) Marching Problem:- Problems in which solution of a given DE is desired in an open ~~channel~~ domain subject to a prescribed set of IC & BC.

☐ Generally, transient or transient-like problems.
Not all marching problems are unsteady

Initial condition means, in time domain, one side is defined, but other side is open. So, it is open domain problem.

Classification based on Completeness of Problem definition:

(i) Well posed problem:-

Unique solution of the problem exists.
Solution continuously depends on data and parameters.

(ii) Ill posed problem:- Not well-posed.

Classification based on linearity:-

(i) Linear:- Groundwater equation for confined aquifer.

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2} \quad \ldots (273)$$

Because, there is no multiplication of dependent variables.

(ii) Non-linear:

Momentum conservation equation for surface water.

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)+gA\frac{\partial h}{\partial x}+gA(S_f-S_o)=0 \quad \ldots (274)$$

In our hydraulic systems, in most of the cases we consider linear 2nd order PDE.

Classification of 2nd Order PDE: (Based on B^2-4AC value)

For 2 independent variables). General variable $=\varphi$ (dependent variable)

If we consider 2nd order PDE with 2 independent variable x and y, and dependent variable $=\varphi$, i.e $\varphi=\varphi(x,y)$

$$\underbrace{A\frac{\partial^2\varphi}{\partial x^2}+B\frac{\partial^2\varphi}{\partial x\partial y}+C\frac{\partial^2\varphi}{\partial y^2}}+D\frac{\partial\varphi}{\partial x}+E\frac{\partial\varphi}{\partial y}+F\varphi+G=0 \quad \ldots (275)$$

where, A, B, ... F. are functions of x and y or constants.

Highest partial derivatives determine the nature of the equations. These are those terms.

The characteristics equation can be written as:-

How? $\leftarrow$ $A\left(\frac{dy}{dx}\right)^2-B\left(\frac{dy}{dx}\right)+C=0 \quad \ldots (276)$

$\hookrightarrow$ Depending the sign of (B^2-4AC) equations are classified.

(i) Parabolic: $B^2-4AC=0$

Transient 1D GW flow in confined aquifer:-

$$\frac{S}{T}\frac{\partial h}{\partial t}-\frac{\partial^2 h}{\partial x^2}=0, \quad h=h(t,x)$$

$$B^2-4AC=0^2-4.0.(-1)=0$$

(ii) Elliptic: $B^2-4AC<0$

Steady 2D groundwater flow equation in confined aquifer,

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0$$

$$B^2-4AC=0^2-4.1.1=-4$$

(iii) Hyperbolic: $B^2-4AC>0$

$u(t,x)$ One dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0$$

$$B^2-4AC=0^2-4.1(-1)=4.$$

Classification of 2nd order PDE based on Eigenvalues (with N independent variables)

A general second order PDE with N independent variables, $(x_1, x_2, \dots x_N)$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial \varphi}{\partial x_i} + c\varphi + d = 0 \quad \dots (277)$$

$a_{ij}, b_i, c, d \Rightarrow$ Functions of $x_1, x_2, \dots x_N$.

Assumptions:-

$$\frac{\partial^2\varphi}{\partial x_i \partial x_j} = \frac{\partial^2\varphi}{\partial x_j \partial x_i} \Rightarrow \text{Cross second order derivatives are same.} \quad \dots (278)$$

$A_\lambda = [a_{ij}]$ is symmetric $\Rightarrow$ Because, coefficients of cross terms are also same

If $\hat{\lambda}$ are the eigenvalues of A_λ, then it should satisfy the equation,

$|A_\lambda - \lambda I| = 0 \Rightarrow$ i.e determinant of $[A - \lambda I]$ matrix should be zero.

☑ The equation can be classified based on sign of eigenvalues $(\lambda_1, \lambda_2, \dots \lambda_N)$ of matrix A_λ:-

Parabolic equation:

One or more zero eigenvalues $(\lambda_i = 0)$.

Elliptic equation:

All eigenvalues are non-zero having same sign.

(i.e $\lambda_i > 0, \forall i$ or $\lambda_i < 0, \forall i$)

Hyperbolic equation:

All eigenvalues are non-zero but one of them has different sign.

$\lambda_i > 0, \; i \in \{1, 2, \dots N\} \setminus \{j\}$

$\lambda_j < 0.$

or $\lambda_i < 0, \; i \in \{1, 2, \dots N\} \setminus \{j\}$

$\lambda_j > 0.$

Example: Parabolic: $\frac{S}{T}\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0$

$$A_\lambda = \begin{bmatrix} a_{tt} & a_{tx} \\ a_{xt} & a_{xx} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

a_{tt} = coff of $\frac{\partial^2 h}{\partial t^2}$

a_{tx} = coff of $\frac{\partial^2 h}{\partial t \partial x}$

a_{xt} = coff of $\frac{\partial^2 h}{\partial x \partial t}$

a_{xx} = coff of $\frac{\partial^2 h}{\partial x^2}$

$\Rightarrow |A_\lambda - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(1+\lambda) = 0$

$\Rightarrow \lambda_1 = 0, \lambda_2 = -1.$

$\therefore$ One or more zero eigenvalue condition satisfied.

$\therefore$ Parabolic.

Similarly, do for elliptic and hyperbolic eq^{n}s, given in Page 75.

Now,

General form of differential equation:

ϕ = Dependent variable = $\phi(x,y,z,t)$

Lambda, Epsilon, Gamma

$$\underbrace{\frac{\partial(\Lambda_\phi \phi)}{\partial t}}_{\#\text{Temporal}} + \underbrace{\nabla\cdot(\Upsilon_\phi\, \phi u)}_{\#\text{Advection}} = \underbrace{\nabla\cdot(\Gamma_\phi \nabla\phi)}_{\#\text{Diffusion}} + F_{\phi_0} + S_\phi \quad \cdots\cdots (279)$$

→ It defines our all hydraulic system related problems.

ϕ = General variable

$\Lambda_\phi, \Upsilon_\phi$ = Problem dependent parameters.

Γ_ϕ = Tensor (Dispersion coff tensor) = Diffusion coff.

F_{ϕ_0} = other forces

S_ϕ = Source/Sink term.

Different equations represented by General form:

(i) Mass conservation;
(In incompressible fluid flow):
compare (257.1) and (279),

$$\Lambda_\phi = 1,\ \phi = \rho = \text{constant},\ \Upsilon_\phi = 1,\ \Gamma_\phi = \underline{0},\ F_{\phi_0} = 0,\ S_\phi = 0 \quad \cdots\cdots (280.1)$$

↳ General variable is taken constant?

$$\frac{\partial(1.\rho)}{\partial t} + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(1.\rho.(u\hat{i}+v\hat{j}+w\hat{k})) = \nabla.(\underline{0}.\nabla\phi) + 0 + 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial y}(\rho w) = 0$$

Now, ρ is constant for in-compressible fluid flow.

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad \cdots\cdots (280.2)$$

(ii) Momentum Conservation
(Incompressible flow)

$$\Lambda_\phi = \rho,\ \phi = u,\ \Upsilon_\phi = \rho,\ \Gamma_\phi = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix},\ F_{\phi_0} = -\frac{\partial P}{\partial x} + \rho g_x,\ S_\phi = 0. \quad \cdots\cdots (281.1)$$

$$\frac{\partial}{\partial t}(\rho u) + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).\left[\rho u(u\hat{i}+v\hat{j}+w\hat{k})\right] = \nabla. \left[\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \times \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial u/\partial z \end{Bmatrix}\right] - \frac{\partial P}{\partial x} + \rho g_x + 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho u) + \rho\left\{\frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial x} + \frac{\partial(uw)}{\partial x}\right\} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right).\left(\mu\frac{\partial u}{\partial x}\hat{i} + \mu\frac{\partial u}{\partial y}\hat{j} + \mu\frac{\partial u}{\partial z}\hat{k}\right) - \frac{dP}{dx} + \rho g_x$$

$$= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{dP}{dx} + \rho g_x.$$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial x} + \frac{\partial(uw)}{\partial x} = -\frac{1}{\rho}\frac{dP}{dx} + g_x + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \cdots (281.2)$$

(iii) Contaminant transport
Concentration equation:- # Two dimensional contaminant transport is considered in plane.

$$\Lambda_\phi = 1,\ \phi = \eta c,\ \Upsilon_\phi = 1,\ \Gamma_\phi = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix},\ F_{\phi_0} = 0,\ S_\phi = q_s c_s. \quad \cdots (282.1)$$

(See page 72)

$u = u_x\hat{i} + u_y\hat{j}$

$$\frac{\partial}{\partial t}(1.\eta c) + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).(1.\eta c.(u_x\hat{i}+u_y\hat{j})) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).\left[\begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix} \times \begin{Bmatrix} \frac{\partial}{\partial x}(\eta c) \\ \frac{\partial}{\partial y}(\eta c) \end{Bmatrix}\right] + 0 + q_s c_s$$

$$\Rightarrow \frac{\partial}{\partial t}(\eta c) + \frac{\partial}{\partial x}(\eta c u_x) + \frac{\partial}{\partial y}(\eta c u_y) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).\begin{Bmatrix} D_{xx}\frac{\partial}{\partial x}(\eta c) + D_{xy}\frac{\partial}{\partial y}(\eta c) \\ D_{yx}\frac{\partial}{\partial x}(\eta c) + D_{yy}\frac{\partial}{\partial y}(\eta c) \end{Bmatrix} + q_s c_s.$$

$$\begin{Bmatrix} a \\ b \end{Bmatrix} \equiv a\hat{i} + b\hat{j}$$

$$\Rightarrow \frac{\partial}{\partial t}(\eta c) = \underbrace{\frac{\partial}{\partial x}\left\{D_{xx}\frac{\partial}{\partial x}(\eta c)+D_{xy}\frac{\partial}{\partial y}(\eta c)\right\} + \frac{\partial}{\partial y}\left\{D_{yx}\frac{\partial}{\partial x}(\eta c)+D_{yy}\frac{\partial}{\partial y}(\eta c)\right\}}_{\text{Advection terms}}$$

$$\underbrace{-\frac{\partial}{\partial x}(\eta u_x c)-\frac{\partial}{\partial y}(\eta u_y c)}_{\text{Advection}} + q_s c_s \ldots\ldots \quad (282.2)$$

(This is called Scalar transport equation). → Meaning?

NPTEL-5

Numerical methods

Connection between GE's and numerical discretizations

Exact solution is not available for complex system.

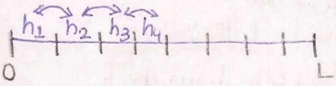
But, exact solution is continuous and defined for the whole system. But, numerical solution is defined for set of points/elements.

Continuous domain can be divided into parts and sub-parts for discrete representation.

Numerical discretization defines the mathematical relation between parts and sub-parts in terms of field variables.

☑ Lets have an example.

For 1D unsteady GW flow, $h(x,t)$.
↳ This is field variable.



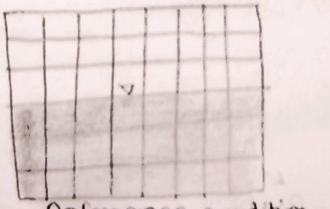
1D space domain divided into parts for solution.

By the help of numerical discretization, we can define the relationship between field variable values like (h_1 & h_2) or (h_2 & h_3)... etc.

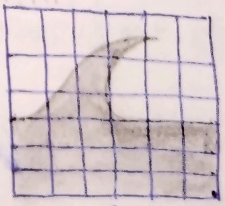
Different length & time scales and corresponding computational methods:-

Computational methods	Length scale (m)	Time scale. (s)
(i) Atomistic models (Classic MD, Ab initio molecular dynamics (M.D), Kinetic MC)	$10^{-8}-10^{-7}$	$10^{-11}-10^{-10}$
(ii) Mesoscopic models (DPD LBM = Lattice Bolzmann)	$10^{-6}-10^{-5}$	$10^{-9}-10^{-8}$
(iii) Continuum models	$10^{-4}-10^{-3}$ m or higher	10^{-3} s or higher.

Eulerian description:



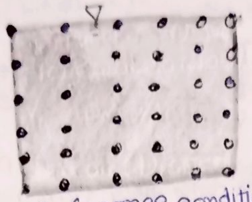
Reference condition



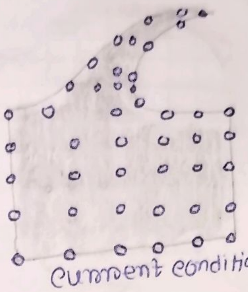
Current condition

Change of water level we are tracking with available grids. Grids are not changing.

Lagrangian Description:



Reference condition



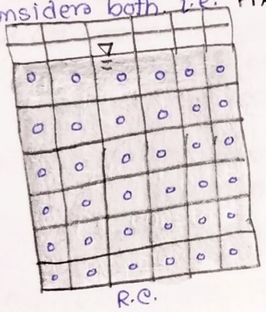
Current condition

Here, we divide the domain into numbers of particles and track these particles with time. Each particles represent element volume and for each particle, we can get information about fill variance.

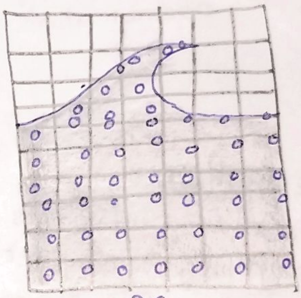
In terms of particles, we define our current position or condition.

Eulerian Lagrangian System:

We consider both i.e. Fixed grid system and moving particle system.



R.C.



C.C

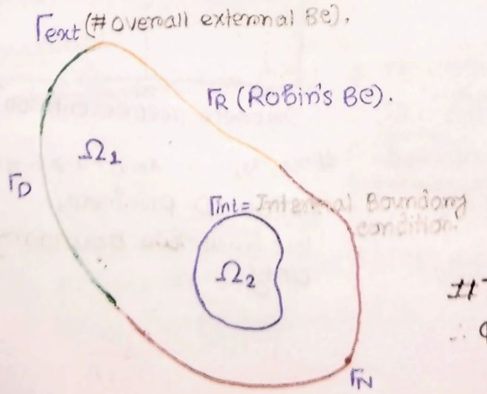
There will be movement of particles from grid to grid. Using the information, we can get field variable values based on grids. Also, we can move the particles base-d on the velocity calculated with the help of the values of support domain.

Physical Domain:

For these space-time points, we can get the field variables(Φ).

$\Phi(x, y, z, t) \longrightarrow \Phi(x_i, y_j, z_k, t_l)$.

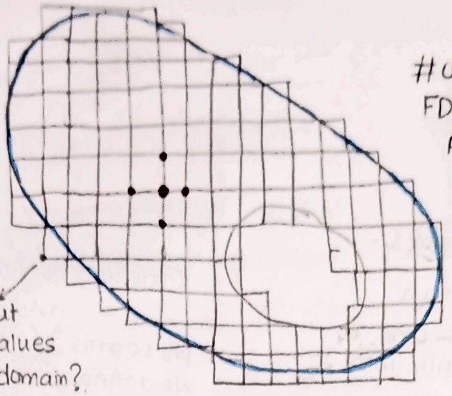
Continuous domain (Closed form solution) — Discrete ~~domain~~ values for Numerical discretization.



Though, this is 2D case.
$\therefore \Phi = \Phi(x_i, y_j, t_l)$, no z terms.

If we consider flow and other kind of problem in Ω_1 domain excluding Ω_2 domain, we need to divide domain into parts or sub-parts.

Finite difference method:



When we choose FD, we choose intersection points to assign single point value.
For these grid points only field variables are defined.

What's about these nodal values outside the domain?
(If some node-points are not at boundary then how to apply BC ?!)

Finite volume method:

It may be structured or unstructured mesh. Can divide it in triangular parts.

Mess free method:

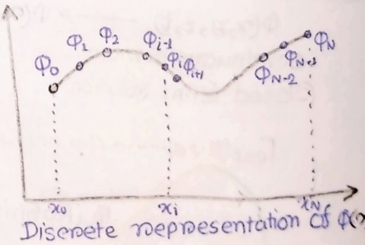
Domain is divided into points.
From those points, we can get information about field variable.

Our numerical discretization should define the relationship between field variable at that point and surrounding points.

NPTEL-6

Finite Difference:

$\phi \rightarrow$ Single valued, finite and continuous function of x.



Discrete representation of $\phi(x)$

Forward difference:

$$\delta_+\phi_i = \phi_i' \approx \frac{\phi(x_i+\Delta x)-\phi(x_i)}{\Delta x}$$ (From limit concept, $\lim_{\Delta x\to 0}$. So, the value is approximated).

$$\approx \frac{\phi_{i+1}-\phi_i}{\Delta x}$$

ϕ_N' cannot be calculated

$x_0, x_1, \ldots\ldots x_{N-1}, x_N \rightarrow$ Grid points.

For 1D problem, we have two boundary values only.

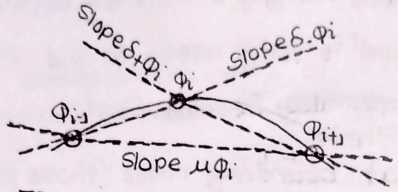
Backward difference:

$$\phi'_N \cong \frac{\phi(x_N - \Delta x) - \phi(x_N)}{\Delta x}$$

$$\phi'(x)|_{BD} = \delta_- \phi_i = \frac{\phi_i - \phi_{i-1}}{\Delta x}$$

Center difference:

$$\phi'(x)|_{CD} = \mu\phi_i = \frac{1}{2}(\delta_+ + \delta_-)\phi_i = \frac{\phi_{i+1} - \phi_{i-1}}{x_{i+1} - x_{i-1}} = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} \quad \text{.... (283)}$$



For FD, BD and CD, results should converge to the same value as $\Delta x \to 0$. This is called consistency of the discretization.

FD, BD, CD approximations are consistent. However, they will not produce same value for finite Δx due to associated truncation errors.

Associated Computational Errors:-

Round-off error:- Computer related errors, as they can store only finite number of decimal places.

Truncation error:- Human error due to approximation being made.

Taylor's Series expansion:-

If the function is infinitely differentiable, then Taylor series expansion about point x_i evaluated at $x_i + \Delta x$,

$$\phi(x_i + \Delta x) = \phi(x_i) + \sum_{m=1}^{\infty} \frac{(\Delta x)^m}{m!} \phi^{(m)}(x_i) \quad \text{...... (284)}$$

Similarly, $\phi(x_i - \Delta x) = \text{.........} (-1)^m \text{..........}$

Approximation by Taylor's series:

(i) FD approximation: See (R1, P-134).

(ii) BD " :- See (R1, P-134).

(iii) CD " :- See (R1, P-135).

Observations:

FD approximation for $\phi'(x)$ $\Rightarrow$ TE $\sim O(\Delta x)$ $\Rightarrow$ 1st order discretization.

BD " " " $\Rightarrow$ TE $\sim O(\Delta x)$ $\Rightarrow$ " " "

CD " " " $\Rightarrow$ TE $\sim O(\Delta x^2)$ $\Rightarrow$ 2nd order " .

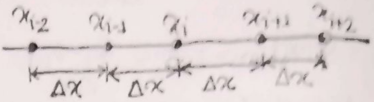
If we reduce the grid size from Δx to $\frac{\Delta x}{2}$,

In FD and BD $\to$ Error halved $(\Delta x \to \frac{\Delta x}{2})$

In CD $\to$ Error become $\frac{1}{4}$th $((\Delta x)^2 \to (\frac{\Delta x}{2})^2)$

Higher order derivative discretization for 1st order derivative:-

See (R1 - Page 135), [for derivation]



We use left sided three point Stencil/ Backward 3 point stencil,

$$\varphi_i' = \frac{\varphi_{i-2} - 4\varphi_{i-1} + 3\varphi_i}{2\Delta x} + O(\Delta x^2) \quad \ldots\ldots (285)$$

Just using one sided stencil, we get 2nd order accuracy (i.e TE ~ $O(\Delta x^2)$) here. This method is very useful to find 1st order derivative for boundary nodes. Because, Center difference can not be applied there.

We can apply FD or BD at boundary nodes (those are 2 point stencils). But, using FD and BD, we get 1st order accuracy only; here we get 2nd order accuracy.

Second Order Derivative:

(i) Using Symmetric Stencil:-

For derivation, see (R1, Page-135)

$$\varphi_i''\big|_{FD} = \varphi_i''\big|_{BD} = \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{\Delta x^2} \quad \ldots (286)$$

$$= \varphi_i'' + 2\sum_{m=1}^{\infty} \frac{(\Delta x)^{2m}}{2(m+1)!}\varphi^{2(m+1)}(x_i)$$

$$= \underbrace{\varphi_i''}_{\text{\#Exact value}} + O(\Delta x^2)$$

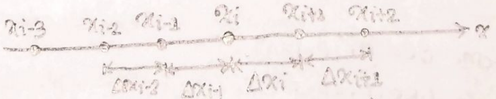
(ii) One-sided three point Second order derivative:-

Backward 3 point stencil, $\varphi_i'' = \dfrac{\varphi_{i-2} - 2\varphi_{i-1} + \varphi_i}{\Delta x^2} + O(\Delta x) \ldots (287)$

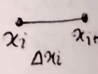
Forward 3 point stencil, $\varphi_i'' = \dfrac{\varphi_i - 2\varphi_{i+1} + \varphi_{i+2}}{\Delta x^2} + O(\Delta x) \ldots (288)$

But, order of accuracy reduced than symmetric stencil

(iii) For non-uniform grid:-



→ Grid size is named by left sided node.



$$\varphi''(x_i) = \alpha_{i-1}\varphi_{i-1} + \alpha_i\varphi_i + \alpha_{i+1}\varphi_{i+1}$$

$$= \alpha_{i-1}\varphi(x_i - \Delta x_{i-1}) + \alpha_i\varphi(x_i) + \alpha_{i+1}\varphi(x_i + \Delta x_i)$$

Now, use Taylor's series expansion and comparing coff of derivative terms in LHS and RHS, form 3 equations, find the values of $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$ solving those equations.

Observations:

(i) One sided m point stencils provides:-

m-1 order accurate 1^{st} order derivative. (For m=3, we get $O(\Delta x^2)$)

m-2 " " 2^{nd} order " (For m=3, we get $O(\Delta x)$)

(ii) To approximate n^{th} order derivative, at least (n+1) neighbouring points are required.

For 1st order derivative, we need atleast 2 points (Eg. FD & BD)

For 2^{nd} " " " " 3 points. (Eg eq^n 286, 287, 288)

(iii) If order of accuracy of discretization of differential equation & BC is more, our accuracy of solution will be more.

NPTEL-7

ODE-IVP

☑ ODE with IC can be solved as IVP with time / time-like discretization.

#Because, ODE may not be time-dependent always. May be space dependent.

☑ ODE can be solved by finite difference approach.

→# Because, only one independent variable is there (∵ODE). It seems to be a 1D problem

☑ Accuracy of the solution only depends on discretization of ODE.

(# In IVP, initial condition is like specified or Dirichlet kind of boundary condition)

I.T⇒(If it was Neumann type of BC, then we need to do discretization and truncation error may come that case)

General structure of IVP:-

1^{st} order ODE with general variable φ,

$$\frac{d\varphi}{dt} = \psi(t, \varphi) \quad \text{.......} \quad (289.1)$$

Subjected to initial condition,

$$\varphi(t_0) = \varphi_0$$

#In a particular problem, we may have multiple dependent variables. So, we will have multiple ODE's and there should be initial condition for each D.E.

Numerical Discretization:-

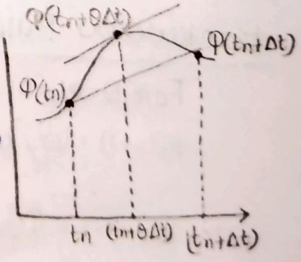
$$\int_{t_n}^{t_{n+1}} \frac{d\varphi}{dt} = \int_{t_n}^{t_{n+1}} \psi(t, \varphi) \quad \text{....} \quad (289.2)$$

Using Mean value theorem to evaluate the RHS,

$$\frac{\varphi^{n+1} - \varphi^n}{\Delta t} = \psi(t_n + \theta\Delta t, \varphi(t_n + \theta\Delta t))$$

$$\Rightarrow \varphi^{n+1} = \varphi^n + \Delta t\, \psi(t_n + \theta\Delta t, \varphi(t_n + \theta\Delta t)) \text{...} (290)$$

where $0 \le \theta \le 1$.



(Mean Value Theorem)

Different values of θ and evaluation of $\psi(t_n + \theta\Delta t, \varphi(t_n + \theta\Delta t))$ yeilds different numerical methods.

Truncation errors analysis:

The function Φ at t_{n+1} can be expanded as,

$$\underbrace{\Phi(t_{n+1})=\Phi(t_n)}_{\text{1st part}}+\underbrace{\Delta t\Phi'(t_n)+\dots+\frac{\Delta t^P}{P!}\varphi^{(P)}(t_n)}_{\substack{\text{2nd part} \\ \Delta t\Psi(t_n,\Phi(t_n),\Delta t)}}+\underbrace{\frac{(\Delta t)^{P+1}}{(P+1)!}\varphi^{(P+1)}(t_n+\theta\Delta t)}_{\substack{\text{3rd part} \\ \text{Truncation error}}} \quad \dots (291)$$

Ψ represents $\frac{d\varphi}{dt}$ or φ'. But, here, how whole thing is φ'?

Why associated derivative is going upto $\varphi^{(P+1)}$, not infinity?

$$\Rightarrow \Phi(t_{n+1})=\Phi(t_n)+\Delta t\,\Psi(t_n,\Phi(t_n),\Delta t)+TE. \quad \dots (292)$$

Euler's method:

? For $\theta=0$, we can write, (and also $P=0$) $\to$ I.T. (Then it is possible).

$$\varphi^{n+1}=\varphi^n+\Delta t\,\Psi(t_n,\varphi^n). \quad \dots (293)$$

(# At the starting point, we are evaluating the function) !?

For $P=1$, $\theta=0$,

$$\Phi(t_{n+1})=\Phi(t_n)+\frac{(\Delta t)^1}{1!}\varphi^{(1)}(t_n)+\frac{(\Delta t)^{1+1}}{(1+1)!}\varphi^{(1+1)}(t_n+0.\Delta t).$$

($\because P=1$, so second term itself is $\frac{\Delta t^P}{P!}\varphi^{(P)}(t_n)$)

$$=\Phi(t_n)+\Delta t\,\Phi'(t_n)+\frac{\Delta t^2}{2!}\Phi''(t_n).$$

$$\Rightarrow \Phi(t_{n+1})=\Phi(t_n)+\Delta t\,\varphi'(t_n)+O(\Delta t^2) \quad \dots (294)$$

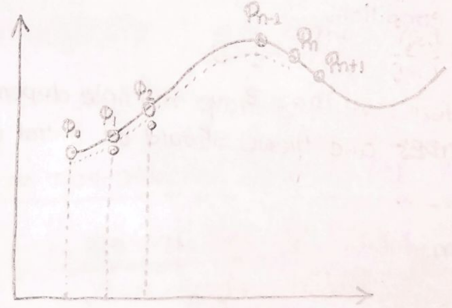
I think!

$$\Rightarrow \frac{\Phi(t_{n+1})-\Phi(t_n)}{\Delta t}=\varphi'(t_n)+\frac{O(\Delta t^2)}{\Delta t}$$

$$\Rightarrow \underbrace{\varphi'^{n}\big|_{FD}}_{\substack{\text{Discretized} \\ \text{derivative}}}=\underbrace{\Phi'(t_n)}_{\substack{\text{Exact} \\ \text{value}}}+\underbrace{O(\Delta t)}_{\substack{\text{Truncation} \\ \text{error.}}}$$

$\therefore$ Order of Euler's method is actually $\to O(\Delta t)$.

?



Backward Euler's method:

For $\theta=1$, (and $P=0$)?, [From (291)],

$$\varphi(t_{n+1})=\frac{(\Delta t)^0}{0!}\varphi^{(0)}(t_n)+\frac{(\Delta t)^{0+1}}{(0+1)!}\varphi^{(0+1)}(t_n+1.\Delta t)$$

[If $P=0$, it will not take any derivative term from 2nd part of equation (291)].

$$\Rightarrow \Phi(t_{n+1})=\Phi(t_n)+\Delta t\,\varphi'(t_n+\Delta t)$$

$$\Rightarrow \varphi^{n+1}=\varphi^n+\Delta t\,\Psi(t_{n+1},\varphi^{n+1}) \quad \dots (295)$$

** (# From equation 289,

$$\varphi'(t)=\frac{d\varphi}{dt}=\Psi(t,\varphi) \Rightarrow \varphi'(t_n+\Delta t)\equiv\varphi'(t_{n+1}) = \Psi(t_{n+1},\Phi(t_{n+1})) \equiv \Psi(t_{n+1},\varphi^{n+1}))$$

In our Euler's method, we have used t_n and φ^n (Equation 294). In this case, (equation 295) we are using future time level value.

? For $\theta = 0$ and $P = 1$, (From equation 297),

$$\varphi(t_n) = \varphi(t_n+\Delta t) + \frac{(-\Delta t)^1}{1!}\varphi^{(1)}(t_n+\Delta t) + \frac{(-\Delta t)^{1+1}}{(1+1)!}\varphi^{1+1}(t_n+\Delta t + 0\cdot(-\Delta t))$$

$$\Rightarrow \varphi(t_n) = \varphi(t_n+\Delta t) - \Delta t\,\varphi'(t_n+\Delta t) + \frac{\Delta t^2}{2!}\varphi''(t_n+\Delta t)$$

$$\Rightarrow \varphi(t_n) = \varphi(t_{n+1}) - \Delta t\,\varphi'(t_{n+1}) + \underbrace{\frac{\Delta t^2}{2!}\varphi''(t_{n+1})}_{\text{Leading error.}} \quad \ldots\ldots (298)$$

Or, $$\varphi(t_{n+1}) = \varphi(t_n) + \Delta t\,\varphi'(t_{n+1}) + O(\Delta t^2) \quad \ldots\ldots (299)$$

It looks like 2nd order accuracy, but it is not.

Eq^n (291) can be written as

$$\varphi(t_n+\Delta t) = \varphi(t_n) + \Delta t\,\varphi'(t_n) + \ldots + \frac{\Delta t^P}{P!}\varphi^{(P)}(t_n) + \frac{(\Delta t)^{P+1}}{(P+1)!}\varphi^{(P+1)}(t_n+\theta\Delta t)$$

If we replace, $t_n \to t_n+\Delta t$ and increment as $\Delta t \to -\Delta t$, we can write (291) as,

$$\varphi(t_n+\Delta t-\Delta t) = \varphi(t_n+\Delta t) - \Delta t\,\varphi'(t_n+\Delta t) + \ldots + \frac{(-\Delta t)^P}{P!}\varphi^{(P)}(t_n+\Delta t) + \frac{(-\Delta t)^{P+1}}{(P+1)!}\varphi^{(P+1)}(t_n+\Delta t-\theta\Delta t) \quad \ldots\ldots (297)$$

Observe the difference between (294) and (299)!

$$\Rightarrow \frac{\varphi(t_{n+1})-\varphi(t_n)}{\Delta t} = \varphi'(t_{n+1}) + \frac{O(\Delta t^2)}{\Delta t}$$

$$\Rightarrow \varphi'(t_n)\big|_{FD} = \underbrace{\varphi'(t_{n+1})}_{\text{Exact}} + O(\Delta t).$$

$\therefore$ Actual order of backward Euler's method:- $O(\Delta t)$.

Now, if we used any intermediate values then what will be the situation?!

Modified Euler's method:-

For $\theta = \frac{1}{2}$, we can write, (take $P=0$) (at eq^n 291).

$$\varphi(t_{n+1}) = \frac{(\Delta t)^0}{0!}\varphi^0(t_n) + \frac{(\Delta t)^{0+1}}{(0+1)!}\varphi^{(0+1)}(t_n+\tfrac{1}{2}\Delta t)$$

$$\Rightarrow \varphi^{n+1} = \varphi^n + \Delta t\,\varphi'(t_n+\tfrac{\Delta t}{2})$$

$$\Rightarrow \boxed{\varphi^{n+1} = \varphi^n + \Delta t\,\psi\left[t_n+\tfrac{\Delta t}{2}, \varphi(t_n+\tfrac{\Delta t}{2})\right]} \quad \ldots\ldots (300)$$

But $(t_n+\frac{\Delta t}{2})$ is not a nodal point. So, we need to figure out how to approximate this term.

First approach:

Using Euler's method, equation (294),

$$\varphi(t_n+\tfrac{\Delta t}{2}) = \varphi(t_n) + \frac{\Delta t}{2}\psi(t_n, \varphi^n) \ldots\ldots (301)$$

Putting $\varphi(t_n+\frac{\Delta t}{2})$ value from eq^n (301) to equation (300),

? $$\underbrace{\varphi^{n+1} = \varphi^n + \Delta t\cdot\psi\left[t_n+\tfrac{\Delta t}{2}, \varphi(t_n) + \tfrac{\Delta t}{2}\underbrace{\psi(t_n,\varphi^n)}_{K_1}\right]}_{K_2} \ldots (302)$$

In simplified form,

$$\varphi^{n+1} = \varphi^n + K_2 + O(\Delta t^3), \quad \text{where, } K_1 = \Delta t\psi^n,\ K_2 = \Delta t\,\psi(t_n+\tfrac{\Delta t}{2}, \varphi^n+\tfrac{1}{2}K_1). \quad \ldots (303)$$

How?! Check it later!

However, actual order of accuracy is 2nd order.

Second - approach:

Using averaging approach, $\phi'(t_n+\frac{\Delta t}{2}) = \frac{1}{2}\left[\phi'(t_n)+\phi'(t_n+\Delta t)\right]$ .... (304)

From equation (305), with Euler approximation, (Put value from 305 to 304),

$$\phi'(t_n+\tfrac{\Delta t}{2}) = \tfrac{1}{2}\left[\psi(t_n,\phi^n)+\psi(t_{n+1},\phi^n+\Delta t\psi^n)\right] \quad \text{.....(306)}$$

$\phi'(t_n+\Delta t)$
$= \psi(t_n+\Delta t, \phi(t_n+\Delta t))$
$= \psi(t_{n+1}, \phi(t_n)+\Delta t\phi'(t_n))$
$= \psi(t_{n+1}, \phi^n+\Delta t\psi^n)$ .... (305)
Using (293)

From equation (300),

$$\phi^{n+1} = \phi^n+\Delta t\,\psi\left[t_n+\tfrac{\Delta t}{2}, \phi(t_n+\tfrac{\Delta t}{2})\right]$$

$$= \phi^n+\Delta t \cdot \phi'(t_n+\tfrac{\Delta t}{2}) \quad \text{.....} \left[\because \phi'(t)=\psi(t,\phi)\right] \text{... From (289.1)}$$

$$= \phi^n+\tfrac{\Delta t}{2}\left[\psi(t_n,\phi^n)+\psi(t_{n+1},\phi^n+\Delta t\psi^n)\right] \text{.... Putting value from (306). ..... (307)}$$

$$= \phi^n+\tfrac{\Delta t}{2}\left[\psi^n+\psi(t_{n+1},\phi^n+\Delta t\psi^n)\right]$$

$$= \phi^n+\tfrac{\cancel{\Delta t}\,1}{2}\left[\Delta t\psi^n+\Delta t\,\psi(t_{n+1},\phi^n+\Delta t\psi^n)\right]$$

$$= \phi^n+\tfrac{1}{2}\left[K_1+K_2\right]+O(\Delta t^3) \quad \text{....... (308)}$$

where, $K_1 = \Delta t\psi^n$
and $K_2 = \Delta t\,\psi(t_{n+1},\phi^n+K_1)$. $\}$ ...... (309)

It is 3rd order accurate, but resulting thing is 2nd order accurate.

Explicit RK methods:

(See Page-8) Or, (R1-Page-106)

NPTEL-8

ODE-BVP

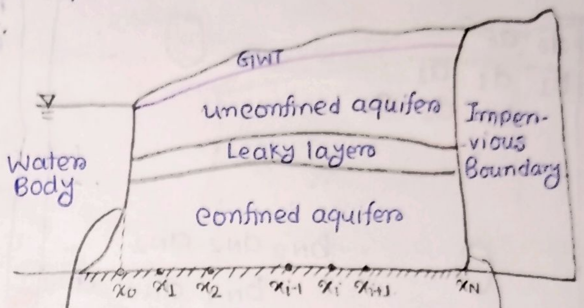
ODE with

Space discretization = BVP

Time/Time like ,, = IVP.

ODE can be solved by Finite difference

Accuracy of the solution depends on both discretization of ODE and BC(s). for BVP.



This is specified head boundary. So, no discretization is required, no error.

This is Neumann Boundary $\left(\frac{\partial u}{\partial x}\right) = 0$. It may have error in discretization.

Problem definition:

$h = h(x)$. = head.

$$\frac{d^2h}{dx^2} = \frac{C_{conf}}{T}(h - h_{wt}) \quad \ldots\ldots (310).$$

$C_{conf} = \left(\frac{\text{Hydraulic conductivity}}{\text{Thickness}}\right)$ ratio of confining layer.

h_{wt} = Overlaying water table elevation. $= C_0 + C_1x + C_2x^2$.

BC(s): $h(x=0) = h_s$ (Left)

$\left.\frac{dh}{dx}\right|_L = 0$ (Right).

Numerical discretization:

Governing equation:

$$\frac{h_{i-1} - 2h_i + h_{i+1}}{\Delta x^2} + O(\Delta x^2) = \frac{C_{conf}}{T}\left[h_i - h_{wt}(x_i)\right]$$

$$\Rightarrow \frac{1}{\Delta x^2}h_{i-1} - \left(\frac{C_{conf}}{T} + \frac{2}{\Delta x^2}\right)h_i + \frac{1}{\Delta x^2}h_{i+1} = -\frac{C_{conf}}{T}h_{wt}(x_i) \quad \ldots\ldots (311)$$

Applicable for interior points only. $i = 1, 2, \ldots N-1$.

Equation (311) can be further simplified as:-

$$b_i h_{i-1} + d_i h_i + a_i h_{i+1} = r_i \quad \ldots\ldots (312)$$

where, $b_i = \frac{1}{\Delta x^2}$, $d_i = -\left(\frac{C_{conf}}{T} + \frac{2}{\Delta x^2}\right)$, $a_i = \frac{1}{\Delta x^2}$

and $r_i = -\frac{C_{conf}}{T}h_{wt}(x_i)$. .... (312.1)

d_i is coeff of diagonal term. Coeff of head (h_i) for the point for which eq^n is written

Boundary condition:-

(i) Left boundary: $h(x = x_0) = h_s$

$\Rightarrow h_0 = h_s$.

in eq^n (312), $b_0 = 0$, $d_0 = 1$, $a_0 = 0$, $r_0 = h_s$

Dirichlet boundary, no truncation error.

(ii) Right Boundary: $\frac{h_N - h_{N-1}}{\Delta x} + O(\Delta x) = 0$

$\Rightarrow h_N - h_{N-1} = 0$

$$\Rightarrow \frac{1}{\Delta x}(-h_{N-1} + h_N + 0 \cdot h_{N+1}) = 0 \quad \ldots (313)$$

$\therefore b_N = -\frac{1}{\Delta x}$ #N represents node number for which equation has been written.

$d_N = \frac{1}{\Delta x}$

$a_N = 0$

$r_N = 0$

Governing equations for interior points and BC's for boundary points.

$$\left[\begin{array}{ccccccc} d_0 & a_0=0 & & & & & \\ b_1 & d_1 & a_1 & & 0 & & \\ & b_2 & d_2 & a_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & 0 & & b_{N-2} & d_{N-2} & a_{N-2} & \\ & & & & b_{N-1} & d_{N-1} & a_{N-1} \\ & & & & & b_N & d_N \; a_N \end{array}\right] \begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{N-2} \\ h_{N-1} \\ h_N \end{Bmatrix} = \begin{Bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{Bmatrix}$$

This portion is for interior points.

$A_{(N+1)\times(N+1)}$ $\quad h_{(N+1)\times 1}$ $\quad r_{(N+1)\times 1}$.

This is called the sparse matrix structure.

Problem is accuracy.

Because, the governing equation has second order accuracy. But, ~~RBC has~~ right hand boundary has 1st order accuracy. So, overall accuracy of the sol. problem is 1st order.

Here, we have minimum storage requirement when we can store 3 matrices on column vectors. i.e $\{a\}, \{b\}, \{d\}$. Here, b_0 and a_N terms are always zero.

$\therefore$ Total storage requirement $= (N+1)\times 3$.

$$[b] = \begin{Bmatrix} b_0 \to 0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N-2} \\ b_{N-1} \\ b_N \end{Bmatrix} \quad [d] = \begin{Bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{Bmatrix} \quad [a] = \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \to 0 \end{Bmatrix}$$

More accuracy for Right BC's :-

If we consider 2nd order discretization for Right BC,

$$\frac{3h_N - 4h_{N-1} + h_{N-2}}{2\Delta x} + O(\Delta x^2) = 0 \quad \cdots\cdots (3.14)$$

(#This condition is for N^{th} node. comparing with equation (3.12), $b_i h_{i-1} + d_i h_i + a_i h_{i+1} = r_i$, $b_N h_{N-1} + d_N h_N + a_N h_{N+1} = r_N$)

$b_N = -\frac{4}{2\Delta x}$, $d_N = \frac{3}{2\Delta x}$, $a_N = 0$ and $r_N = 0$

But, here, we have an extra term (corresponding to h_{i-2} i.e for N^{th} cell h_{N-2} take the coff for h_{N-2} as e_N).

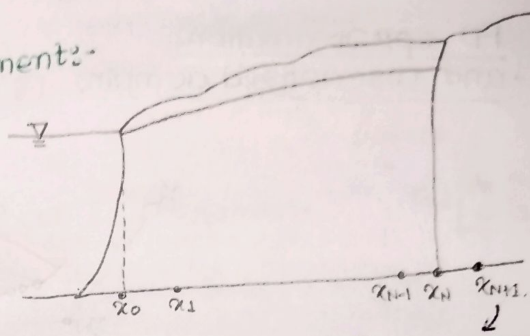
$e_N = \frac{1}{2\Delta x}$.

But, problem is here, e_N breaking tridiagonal structure. So, it is

$$\left[\begin{array}{cccccc} d_0 & a_0 & & & & \\ b_1 & d_1 & a_1 & & & \\ & b_2 & d_2 & a_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & b_{N-2} & d_{N-2} & a_{N-2} & \\ & & & b_{N-1} & d_{N-1} & a_{N-1} \\ & & & e_N & b_N & d_N \end{array}\right]$$

increasing the storage requirement. However, we are getting 2nd order accuracy. If we want to use direct inversion approach (i.e $h = A^{-1} r$), we need to store the full matrix.

Impermeable boundary treatment:-



We have created one fictitious point N+1.

considering $(N+1)^{th}$ point, zero Neumann BC can be written as,

$$\frac{h_{N+1} - h_{N-1}}{2\Delta x} + O(\Delta x^2) = 0 \quad \ldots\ldots (315)$$

$$\Rightarrow h_{N+1} = h_{N-1} \quad \ldots\ldots (316)$$

Fictitious Point method:

Writing the discretized governing equation at $i = N$,

$$b_N h_{N-1} + d_N h_N + a_N h_{N+1} = r_N \quad \ldots\ldots (317)$$

Using boundary condition, (Equation 316), (317) can be written as:-

$$b_N h_{N-1} + d_N h_N + a_N h_{N-1} = r_N.$$

$$\Rightarrow (b_N + a_N) h_{N-1} + d_N h_N = r_N \quad \ldots\ldots (318)$$

coff a_N, b_N, d_N, r_N are same as (312.1).

In matrix form, equation (318) can be included as:-

$$0 = b_0 \quad \begin{bmatrix} d_0 & a_0 & & & & \\ b_1 & d_1 & a_1 & & & \\ & b_2 & d_2 & a_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{N-1} & d_{N-1} & a_{N-1} \\ & & & & b_N + a_N & d_N \end{bmatrix} \begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{N-2} \\ h_{N-1} \\ h_N \end{Bmatrix} = \begin{Bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{Bmatrix}$$

$a_N \to 0$

Now to get solution, we can use direct inversion, but not necessary to do that. Because, tri-diagonal form is preserved here. So, no need to store the full matrix.

So, ultimate advantage is:- (i) Tri-diagonal form preserved

(ii) 2nd order accuracy achieved.

NPTEL-9

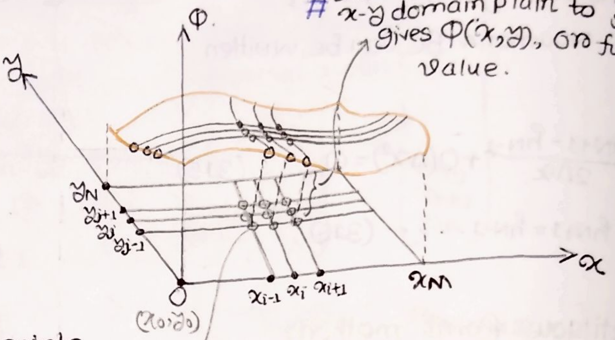
PDE-BVP

To discretize single-valued multi-dimensional functions using finite difference approximation.

To derive the algebraic form using discretized PDEs and BCs.

FD approximation and discretized domain:

Distance from rectangular x-y domain plain to surface gives $\Phi(x,y)$, or function value.



We have informations for discretes points only, not for intermediate points.

For these internal general points, we define our partial derivatives and correspond Finite difference approximations.

Taylor series for functions with two independent variables:

$$\Phi(x+\Delta x, y+\Delta y) = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty} \frac{\Delta x^{n_x}\,\Delta y^{n_y}}{n_x!\,n_y!}\cdot\frac{\partial^{n_x+n_y}\Phi(x,y)}{\partial x^{n_x}\,\partial y^{n_y}} \quad \cdots (319)$$

$$= \underbrace{\frac{(\Delta x)^0(\Delta y)^0}{0!\,0!}\frac{\partial^0\Phi}{\partial x^0\partial y^0}}_{n_x=0,\,n_y=0} + \underbrace{\frac{\Delta x}{1!}\frac{\partial\Phi}{\partial x}}_{n_x=1,\,n_y=0} + \underbrace{\frac{\Delta y}{1!}\cdot\frac{\partial\Phi}{\partial y}}_{n_x=0,\,n_y=1}$$

$$+ \underbrace{\frac{\Delta x^2}{2!}\frac{\partial^2\Phi}{\partial x^2}}_{n_x=2,\,n_y=0} + \underbrace{\frac{\Delta x\,\Delta y}{1!\,1!}\frac{\partial^2\Phi}{\partial x\partial y}}_{n_x=1,\,n_y=1} + \underbrace{\frac{\Delta y^2}{2!}\frac{\partial^2\Phi}{\partial y^2}}_{n_x=0,\,n_y=2} + \cdots\cdots$$

So, this expression need to be expanded for all possible combinations of n_x and n_y.

$$= \Phi(x,y) + \Delta x\frac{\partial\Phi}{\partial x} + \Delta y\frac{\partial\Phi}{\partial y} + \frac{1}{2!}\left[\Delta x^2\frac{\partial^2\Phi}{\partial x^2} + 2\Delta x\Delta y\frac{\partial^2\Phi}{\partial x\partial y} + \Delta y^2\frac{\partial^2\Phi}{\partial y^2}\right] + \cdots$$

$$\cdots (320)$$

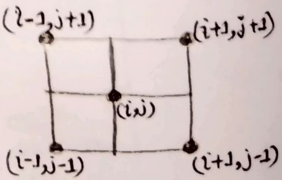
Some finite difference approximations:

Second order CD wrt x:-

$$\left.\frac{\partial^2\Phi}{\partial x^2}\right|_{i,j} = \frac{\Phi_{i-1,j} - 2\Phi_{i,j} + \Phi_{i+1,j}}{\Delta x^2} + O(\Delta x^2) \quad \cdots\cdots (321)$$

Second order center mix difference wrt y:

$$\left.\frac{\partial^2\Phi}{\partial x\partial y}\right|_{i,j} = \frac{\Phi_{i+1,j+1} + \Phi_{i-1,j-1} - \Phi_{i+1,j-1} - \Phi_{i-1,j+1}}{4\Delta x\Delta y} + O(\Delta x^2, \Delta y^2)$$



$$\cdots (322)$$

Form of differential equations with a general variable φ,

$$\frac{\partial}{\partial t}(\cancelto{0}{\Lambda_\varphi \varphi}) + \nabla\cdot(\cancelto{0}{\gamma_\varphi \varphi \underline{u}}) = \nabla\cdot(\Gamma_\varphi \cdot \nabla\varphi) + \cancelto{0}{F_{\varphi_u}} + S_\varphi \quad \text{From (Equation 279)} \quad \ldots (323).$$

In case of BVP, temporal, advective, and other force terms should be zero.

In this case del operator is 2D only.
i.e. $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$.

Problem definition:-

From (323), two dimensional BVP can be written as:-

$$\Omega: \Gamma_x \frac{\partial^2 \varphi}{\partial x^2} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2} + S_\varphi(x,y) = 0 \ldots (324)$$

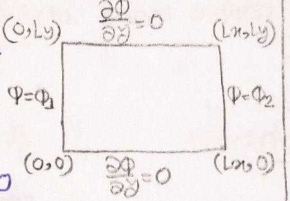
Subject to-
Boundary condition ⇒

Γ_D^1 :- $\varphi(0,y) = \varphi_1$

Γ_D^2 :- $\varphi(L_x, y) = \varphi_2$

Γ_N^3 :- $\left.\frac{\partial \varphi}{\partial y}\right|_{(x,0)} = 0$

Γ_N^4 : $\left.\frac{\partial \varphi}{\partial y}\right|_{(x,L_y)} = 0$



Gamma ← $\Gamma_\varphi = \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix}$

$$\nabla\cdot\left(\begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix}\begin{bmatrix} \partial\varphi/\partial x \\ \partial\varphi/\partial y \end{bmatrix}\right) + S_\varphi = 0$$

$$\Rightarrow \nabla\cdot\begin{Bmatrix} \Gamma_x \frac{\partial \varphi}{\partial x} \\ \Gamma_y \frac{\partial \varphi}{\partial y} \end{Bmatrix} + S_\varphi = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)\cdot\left(\Gamma_x \frac{\partial \varphi}{\partial x}\hat{i} + \Gamma_y \frac{\partial \varphi}{\partial y}\hat{j}\right) + S_\varphi = 0$$

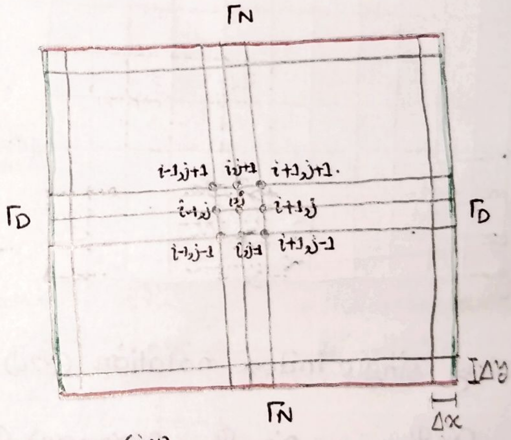
$$\Rightarrow \Gamma_x \frac{\partial^2 \varphi}{\partial x^2} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2} + S_\varphi = 0$$

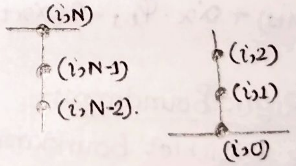
Here, Γ_x and Γ_y are taken as constant. Not varies with x and y.

Domain Discretization:-

Corner points are here considered Dirichlet boundary. Though, it can be both.

We can define the top and bottom Neumann boundary condition using two or three points based on desired accuracy.





Numerical discretization:-

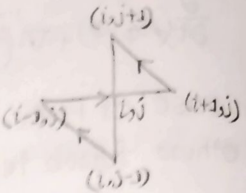
Governing equation (324) can be discretized as:-

$$\Gamma_x \frac{\varphi_{i-1,j} - 2\varphi_{i,j} + \varphi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\varphi_{i,j-1} - 2\varphi_{i,j} + \varphi_{i,j+1}}{\Delta y^2} + O(\Delta x^2, \Delta y^2) = -S_\varphi|_{i,j} \quad \ldots (325)$$

GE for nodal point (i,j).

We have $(M+1)\times(N+1)$ total number of nodes.
So, we must have this number of equations.
We have $(M-1)\times(N-1)$ number of governing equations for interior nodes. $2(M+N)$ no. of equations corresponding to BC (Dirichlet and Neumann). ∴ Total no. of eq^{n}s = $(M-1)(N-1) + 2(M+N) = (M+1)\times(N+1)$ = No. of nodes.

Equation can be rearranged as:-



$$\frac{\Gamma_y}{\Delta y^2}\cdot\phi_{i,j-1}+\frac{\Gamma_x}{\Delta x^2}\cdot\phi_{i-1,j}-2\left(\frac{\Gamma_x}{\Delta x^2}+\frac{\Gamma_y}{\Delta y^2}\right)\phi_{i,j}$$

$$+\frac{\Gamma_x}{\Delta x^2}\phi_{i+1,j}+\frac{\Gamma_y}{\Delta y^2}\phi_{i,j+1}=-S_\phi|_{i,j}$$

⇒ In simplified form,

$$\alpha_y\phi_{i,j-1}+\cancel{\Gamma_x}\,\alpha_x\phi_{i-1,j}-2(\alpha_x+\alpha_y)\phi_{i,j}+\alpha_x\phi_{i+1,j}+\alpha_y\phi_{i,j+1}=-S_\phi|_{i,j} \quad \dots\dots(326)$$

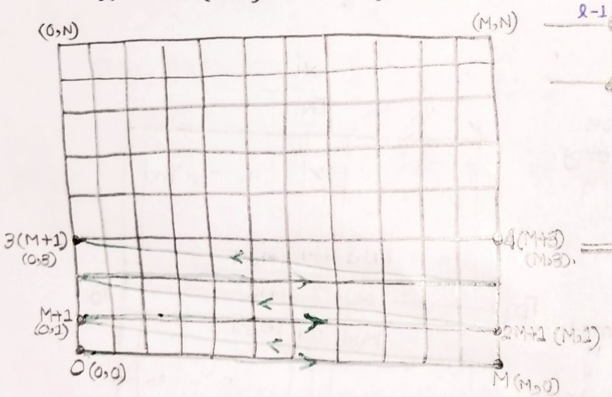
But, the problem with (326) is, we cannot form the algebraic matrix, because we have double index ~~ma~~ notation.

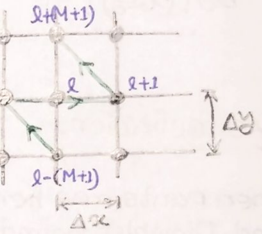
একটা node point এর governing equation এর coff গুলো লেখার জন্য matrix এর একটা row বরাদ্দ আছে। Double index notation নিজেই হল 2D, তাই 2D matrix এর একটা row তে double index notation দ্বারা coff লেখা সম্ভব নয়।

Single Index Notation:-

(i) GE's for Interior nodes:

$$l=i+j\times(M+1) \quad \dots(327)$$





Using single index notation (326) written as,

$$\alpha_y\phi_{l-(M+1)}+\alpha_x\cdot\phi_{l-1}-2(\alpha_x+\alpha_y)\cdot\phi_l+\alpha_x\phi_{l+1}+\alpha_y\cdot\phi_{l+(M+1)}=-S_\phi|_l \quad \dots(327)$$

(ii) Left and Right Boundary:-

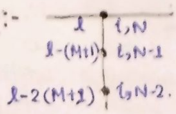
These are Dirichlet boundaries, values are specified. So, clearly we can apply BC's without any errors.

(iii) Top and bottom boundary:

Neumann boundary.

Consider second order discretization to get 2nd order accuracy.

Top:-



$$\frac{3\phi_{i,N}-4\phi_{i,N-1}+\phi_{i,N-2}}{2\Delta y}+O(\Delta y^2)=0 \quad \dots(328)$$

Equation (328) in single index notation format,

$$\frac{3\varphi_l - 4\varphi_{l-(M+1)} + \varphi_{l-2(M+1)}}{2\Delta y} = 0.$$

Bottom:

$$\frac{-3\varphi_{i,0} + 4\varphi_{i,1} - \varphi_{i,2}}{2\Delta y} + O(\Delta y^2) = 0 \quad \ldots\ldots (329.1)$$

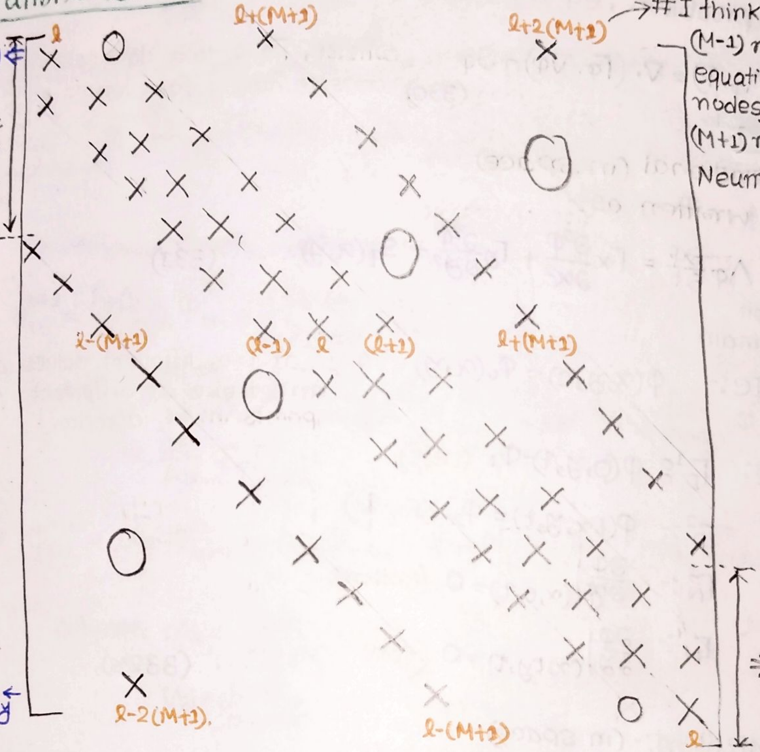
In single index,

$$\frac{-3\varphi_l + 4\varphi_{l+(M+1)} - \varphi_{l+2(M+1)}}{2\Delta y} = 0. \quad \ldots\ldots (329.2)$$

Matrix form:-

Bottom boundary

I.T, these are for bottom boundary nodes.

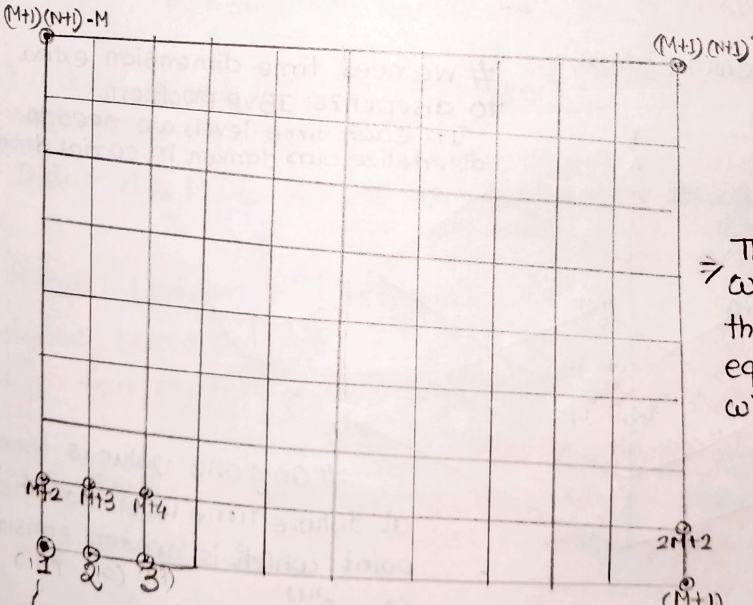


#I think, there should be (M-1) numbers of this kind of equations (excluding 2 corner nodes). Because, there are (M+1) no. of nodes at botom Neuman boundary.

I.T. These are for top boundary

Top boundary

Matrix size ⇒ $\{(M+1)(N+1)\} \times \{(M+1)(N+1)\}$



এই node এর coff গুলো Last row তে যাবে।

This figure shows at which row of matrix A, the coefficients of governing equation of that node will go.

এই node এর coff গুলো 1 no. row তে যাবে।

Solution can be obtained by,

#Defined for single index notation.

$$A\{\varphi_\ell\}=\{b_\ell\}$$

$$\Rightarrow \{\varphi_\ell\}=A^{-1}\{b_\ell\}$$

NPTEL-10
PDE-IBVP

To descretize the spatial and temporal derivatives of single-valued multi-dimensional functions using FD approximations

#Derive algebraic form using PDE, IC, BCs.

For IBVP problem, eq^n (279) becomes,

$$\frac{\partial}{\partial t}(\Lambda_\varphi.\varphi)=\nabla.(\Gamma_\varphi.\nabla\varphi)+S_\varphi \quad \ldots (330)$$

where, Γ_φ is two-dimensional tensor having four terms $=\begin{bmatrix}\Gamma_x & 0\\ 0 & \Gamma_y\end{bmatrix}$

Problem Definition:

A two dimensional (in space) IBVP can be written as:-

$$\Omega: \Lambda_\varphi\frac{\partial\varphi}{\partial t}=\Gamma_x\frac{\partial^2\varphi}{\partial x^2}+\Gamma_y\frac{\partial^2\varphi}{\partial y^2}+S_\varphi(x,y) \ldots (331)$$

Defined for interior domain

Subject to $\Rightarrow$ IC:- $\varphi(x,y,0)=\varphi_0(x,y)$

#So, at $t=0$, different values of φ are there at different points in the domain.

BC's: Γ_D^1:- $\varphi(0,y,t)=\varphi_1$ (Left)

Γ_D^2:- $\varphi(L_x,y,t)=\varphi_2$ (Right)

Γ_N^3:- $\left.\frac{\partial\varphi}{\partial y}\right|_{(x,0,t)}=0$ (Bottom)

Γ_N^4:- $\left.\frac{\partial\varphi}{\partial y}\right|_{(x,L_y,t)}=0$ (Top). ..... (332's).

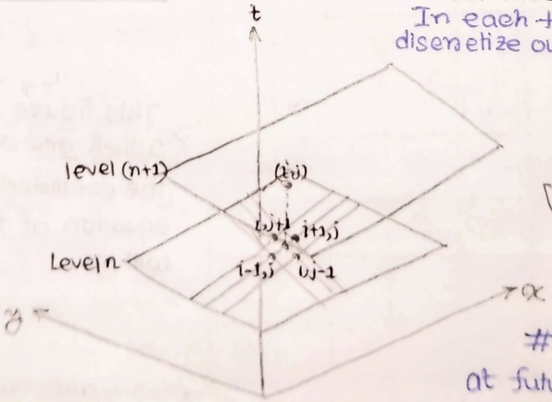
Domain discretization:- (in space)

same as page 91.

Space-time discretization
(Explicit Scheme)

#We need time-dimension extra to discretize IBVP problem.

In each time level, we need to discretize our domain in spatial direction



#Only one value is there at future time level for the point which is under consideration ie. $\varphi_{i,j}^{n+1}$

GE discretization by Explicit Scheme:

$$\Lambda_\varphi \frac{\partial \varphi}{\partial t}\Big|_{i,j}^{n} = \Gamma_x \frac{\partial^2 \varphi}{\partial x^2}\Big|_{i,j}^{n} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2}\Big|_{i,j}^{n} + S_\varphi(x,y)\Big|_{i,j}^{n} \cdots\cdots (333)$$

Time discretization: $$\frac{\partial \varphi}{\partial t}\Big|_{i,j}^{n} = \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}}{\Delta t} + O(\Delta t) \cdots\cdots (334.1)$$

Space discretization:- In explicit scheme, space derivatives are discretized at present time level (n).

$$\frac{\partial^2 \varphi}{\partial x^2}\Big|_{i,j}^{n} = \frac{\varphi_{i-1,j}^{n} - 2\varphi_{i,j}^{n} + \varphi_{i+1,j}^{n}}{\Delta x^2} + O(\Delta x^2). \cdots\cdots (334.2)$$

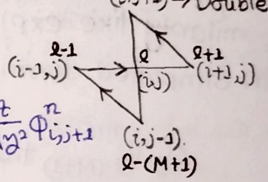
and $$\frac{\partial^2 \varphi}{\partial y^2}\Big|_{i,j}^{n} = \frac{\varphi_{i,j-1}^{n} - 2\varphi_{i,j}^{n} + \varphi_{i,j+1}^{n}}{\Delta y^2} + O(\Delta y^2). \cdots\cdots (334.3).$$

∴ Discretized form of the governing equation,

$$\Lambda_\varphi . \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\varphi_{i-1,j}^{n} - 2\varphi_{i,j}^{n} + \varphi_{i+1,j}^{n}}{\Delta x^2} + \Gamma_y . \frac{\varphi_{i,j-1}^{n} - 2\varphi_{i,j}^{n} + \varphi_{i,j+1}^{n}}{\Delta y^2} + S_\varphi\Big|_{i,j}^{n} + O(\Delta x^2, \Delta y^2, \Delta t) \cdots (335.1)$$

This is the only unknown term.

In compact form, we have,



$$\varphi_{i,j}^{n+1} = \frac{\Gamma_y \Delta t}{\Lambda_\varphi \Delta y^2}\varphi_{i,j-1}^{n} + \frac{\Gamma_x \Delta t}{\Lambda_\varphi \Delta x^2}\varphi_{i-1,j}^{n} + \left(1 - 2\frac{\Gamma_x \Delta t}{\Lambda_\varphi \Delta x^2} - 2\frac{\Gamma_y \Delta t}{\Lambda_\varphi \Delta y^2}\right)\varphi_{i,j}^{n} + \frac{\Gamma_x \Delta t}{\Lambda_\varphi \Delta x^2}\varphi_{i+1,j}^{n} + \frac{\Gamma_y \Delta t}{\Lambda_\varphi \Delta y^2}\varphi_{i,j+1}^{n} + \frac{\Delta t}{\Lambda_\varphi} S_\varphi\Big|_{i,j}^{n}$$

$$\Rightarrow \varphi_{i,j}^{n+1} = \alpha_y \varphi_{i,j-1}^{n} + \alpha_x \varphi_{i-1,j}^{n} + (1 - 2\alpha_x - 2\alpha_y)\varphi_{i,j}^{n} + \alpha_x \varphi_{i+1,j}^{n} + \alpha_y \varphi_{i,j+1}^{n} + \frac{\Delta t}{\Lambda_\varphi} S_\varphi\Big|_{i,j}^{n} \cdots\cdots (335.2)$$

where, $\alpha_x = \dfrac{\Gamma_x \Delta t}{\Lambda_\varphi \Delta x^2}$

$\alpha_y = \dfrac{\Gamma_y \Delta t}{\Lambda_\varphi \Delta y^2}$

Using single index notation, equation can be written as,

$$\varphi_l^{n+1} = \alpha_y \varphi_{l-(M+1)}^{n} + \alpha_x \varphi_{l-1}^{n} + [1 - 2(\alpha_x + \alpha_y)]\varphi_l^{n} + \alpha_x . \varphi_{l+1}^{n} + \alpha_y . \varphi_{l+(M+1)}^{n} + \frac{\Delta t}{\Lambda_\varphi} S_\varphi\Big|_l^{n} \cdots\cdots (335.3).$$

Standard Steps for explicit scheme (Time-stepping Algorithm)

Data:- $\Lambda_\varphi, \Gamma_x, \Gamma_y, S_\varphi, \Delta x, \Delta y, \Delta t, \varphi^n$ at time step n. (We have initial condition available for n=0).

Result:- Updated φ^{n+1} at time-step (n+1)

While t<end time do

For interior points:- $\varphi_{i,j}^{n+1} = \alpha_y \varphi_{l-(M+1)}^{n} + \alpha_x \varphi_{l-1}^{n} + [1 - 2(\alpha_x + \alpha_y)]\varphi_l^{n} + \alpha_x \varphi_{l+1}^{n} + \alpha_y \varphi_{l+(M+1)}^{n} + \frac{\Delta t}{\Lambda_\varphi} S_\varphi\Big|_{i,j}^{n}$

For Boundary points: Use Boundary condition

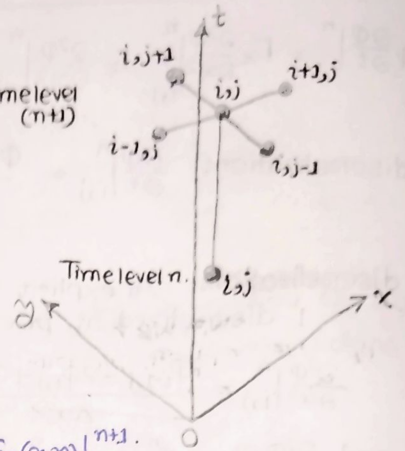
$n \leftarrow n+1.$

end

For explicit scheme, first we need to solve the interior points. Then we need to get the informations about the boundary points.

Implicit Scheme:

In explict scheme, we discretize at n^{th} time level (Eq 333). But, for implicit scheme, we discretize it at $(n+1)^{th}$ time level.



$$\Lambda_\varphi \frac{\partial \varphi}{\partial t}\Big|_{i,j}^{n+1} = \Gamma_x \frac{\partial^2 \varphi}{\partial x^2}\Big|_{i,j}^{n+1} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2}\Big|_{i,j}^{n+1} + S_\varphi(x,y)\Big|_{i,j}^{n+1}.$$

Time discretization:-

$$\frac{\partial \varphi}{\partial t}\Big|_{i,j}^{n+1} = \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}}{\Delta t} + O(\Delta t) \quad \text{\# Backward in time.}$$

Space discretization:

In implicit scheme, derivatives are discretized at future time level $(n+1)$.

Similarly like explicit scheme, (eq^n 335·3), the governing equation in simplified form using single index notation, can be written as,

$$\alpha_y \varphi_{\ell-(M+1)}^{n+1} + \alpha_x \varphi_{\ell-1}^{n+1} - [1+2(\alpha_x+\alpha_y)]\varphi_\ell^{n+1} + \alpha_x \varphi_{\ell+1}^{n+1} + \alpha_y \varphi_{\ell+(M+1)}^{n+1}$$
$$= -\varphi_\ell^n - \frac{\Delta t}{\Lambda_\varphi} S_\varphi\Big|_\ell^{n+1} + O(\Delta x^2, \Delta y^2, t) \quad \ldots (33$$

Here, only known is φ_ℓ^n, $S_\varphi|_\ell^{n+1}$ (Because, source/sink term for a system may be defined)
Unknowns are $\varphi_{\ell-(M+1)}^{n+1}$, $\varphi_{\ell-1}^{n+1}$, φ_ℓ^{n+1}, $\varphi_{\ell+1}^{n+1}$, $\varphi_{\ell+(M+1)}^{n+1}$.

Standard steps for implicit scheme (Time-stepping Algorithm):

Data:- Λ_φ, Γ_x, Γ_y, S_φ, Δx, Δy, Δt, φ^n, at time-step n.

Result:- Updated φ^{n+1} at time step n+1.

While t<end time do,

For interior and boundary points: Solve governing equation and boundary conditions simultaneously in discretized form.

end.

θ SCHEME:

In θ scheme, we consider some intermediate time step. First we define our expliet step (n level) and implicit step (n+1 level).

Explicit step:-

$$\Lambda_\varphi \frac{\partial \varphi}{\partial t}\Big|_{i,j}^{n} = \Gamma_x \frac{\partial^2 \varphi}{\partial x^2}\Big|_{i,j}^{n} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2}\Big|_{i,j}^{n} + S_\varphi(x,y)\Big|_{i,j}^{n} \quad \ldots (337\cdot1)$$

only this value is unknown at nth time level.

$$\Rightarrow \Lambda_\varphi \frac{\varphi_{i,j}^{n+\theta} - \varphi_{i,j}^{n}}{\theta \Delta t} = \Gamma_x \frac{\varphi_{i+1,j}^n - 2\varphi_{i,j}^n + \varphi_{i-1,j}^n}{\Delta x^2} + \Gamma_y \frac{\varphi_{i,j+1}^n - 2\varphi_{i,j}^n + \varphi_{i,j-1}^n}{\Delta y^2} + S_\varphi\Big|_{i,j}^n + O(\Delta x^2, \Delta y^2, \Delta t) \quad (337\cdot2)$$

Implicit step:-

$$\Lambda_\phi \frac{\partial \phi}{\partial t}\Big|_{i,j}^{n+1} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2}\Big|_{i,j}^{n+1} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2}\Big|_{i,j}^{n+1} + S_\phi(x,y)\Big|_{i,j}^{n+1} \quad \ldots\ldots (338.1)$$

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n+\theta}}{(1-\theta)\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{\Delta y^2} + S_\phi\Big|_{i,j}^{n+1}$$
$$+ O(\Delta x^2, \Delta y^2, (1-\theta)\Delta t) \quad \ldots (338.2)$$

In implicit step, we have all unknowns at $(n+1)^{th}$ level. Only known is $\phi_{i,j}^{n+\theta}$ from the explicit 1^{st} step.

Combining explicit and implicit steps Equations (337.2) and (338.2),

(# Multiply 337.2 with θ and 338.2 with $(1-\theta)$ and then add these two),

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t} = \Gamma_x \left[\theta \frac{\phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n}}{\Delta x^2} + (1-\theta) \frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{\Delta x^2}\right]$$
$$+ \Gamma_y \left[\theta \cdot \frac{\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n}}{\Delta y^2} + (1-\theta) \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{\Delta y^2}\right]$$
$$+ \left[\theta S_\phi\Big|_{i,j}^{n} + (1-\theta) S_\phi\Big|_{i,j}^{n+1}\right] + O(\Delta x^2, \Delta y^2, ?) \quad \ldots\ldots (339)$$

#Here, the important point is, what will be the truncation error for time? I.T ⇒ (It will be the minimum of $\theta\Delta t$ and $(1-\theta)\Delta t$)

Truncation error of time discretization in θ scheme:

#We need to get order of time for the term ⇒ $\left(\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t}\right)$ in equation (339) in terms of θ to get the time related truncation error of eq (339).

Explicit step:-

$$\phi_{i,j}^{n} = \phi(x_0, y_0, t_0) = \phi(x_0, y_0, t_0 + \theta\Delta t - \theta\Delta t) \equiv \phi_{i,j}^{(n+\theta)-\theta}$$
$$= \phi_{i,j}^{n+\theta} - \theta\Delta t \frac{\partial \phi}{\partial t}\Big|_{i,j}^{n+\theta} + \frac{(\theta\Delta t)^2}{2!} \frac{\partial^2 \phi}{\partial t^2}\Big|_{i,j}^{n+\theta} - \frac{(\theta\Delta t)^3}{3!} \frac{\partial^3 \phi}{\partial t^3}\Big|_{i,j}^{n+\theta} + \ldots\ldots \quad (340.1)$$

Implicit Step:-

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{(n+\theta)+(1-\theta)} \equiv \phi(x_0, y_0, t_0+\Delta t) = \phi(x_0, y_0, t_0 + \theta\Delta t + (1-\theta)\Delta t)$$
$$= \phi_{i,j}^{n+\theta} + (1-\theta)\Delta t \frac{\partial \phi}{\partial t}\Big|_{i,j}^{n+\theta} + \frac{\{(1-\theta)\Delta t\}^2}{2!} \frac{\partial^2 \phi}{\partial t^2}\Big|_{i,j}^{n+\theta} + \frac{(1-\theta)^3 \Delta t^3}{3!} \frac{\partial^3 \phi}{\partial t^3}\Big|_{i,j}^{n+\theta} + \cdots (340.2)$$

Using (340.1) and (340.2),

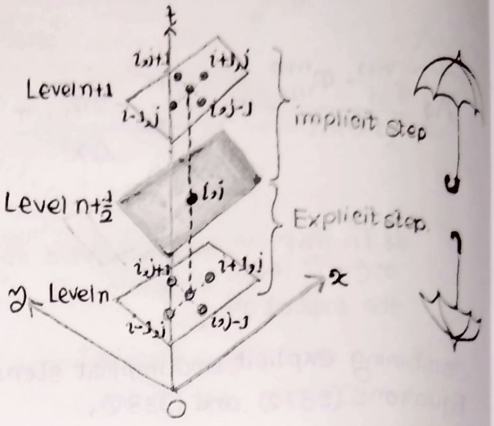
$$\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t} = \frac{\partial \phi}{\partial t}\Big|_{i,j}^{n+\theta} + \underbrace{\frac{[(1-\theta)^2 - \theta^2]\Delta t}{2!} \frac{\partial^2 \phi}{\partial t^2}\Big|_{i,j}^{n+\theta} + \frac{[(1-\theta)^3 - \theta^3]\Delta t^2}{3!} \frac{\partial^3 \phi}{\partial t^3}\Big|_{i,j}^{n+\theta} + \ldots\ldots}_{\text{Truncation error}} \quad \ldots\ldots (341)$$

#If we use $\theta = 1/2$, the 2^{nd} term of RHS gets vanished. That is called Crank-Nicolson Scheme. Thus, we get 2^{nd} order accuracy in time.

Crank-Nicolson Scheme: ($\theta = 0.5$)

Truncation error for C-N scheme $\Rightarrow O(\Delta x^2, \Delta y^2, \Delta t^2)$

It is combination of implicit scheme and explicit scheme.



Crank-Nicolson Scheme can be solved by implicit algorithm. (Because, we have unknowns at both LHS and RHS)?

NPTEL-11

PDE: Numerical stability of IBVP

To analyze the numerical stability of discretized PDE.

Problem Definition:

Governing equation for 2D (in space) IBVP,

$$\Omega :- \Lambda_\varphi \frac{\partial \varphi}{\partial t} = \Gamma_x \frac{\partial^2 \varphi}{\partial x^2} + \Gamma_y \frac{\partial^2 \varphi}{\partial y^2} + S_\varphi(x, y) \quad \cdots\cdots (342).$$

We learned explicit, implicit and C-N scheme. For all these schemes, we need to define $\Delta x, \Delta y, \Delta t$ to satisfy numerical stability.

IC & BC :- Same as eq^n (332's).

ERRORS

(i) Discretization error:

= Analytical solution of PDE − Exact solution of finite difference equation (Obtained from a hypothetical infinite precision computer)

= Truncation error + Error due to treatment of boundary conditions

I.T $\Rightarrow$ (For Governing equation, i.e interior nodes) + I.T $\Rightarrow$ (For boundary nodes, i.e Boundary conditions are applicable) = Error for all nodes in the domain

(ii) Round off error (ε):-

= Numerical solution of finite difference equation (Obtained from finite precision computer) − Exact solution of the FDE (obtained from a hypothetical infinite precision computer).

Numerical Errors

Some points:-

☑ Every algorithm requires repeated operations (eg. $\pm, \times, \div$). There is accumulation of round-off errors.

☑ In time steeping algorithm, accumulated round-off error may magnify / Reduce with every step.

☑ Error may increase exponentially. Called Numerical instability.

☑ Numerical instability / stability is a property of the algorithm and discretization of PDE+BC's. It does not depend on computer used. But, round off error may increase if we use low precision computer.?!

STABILITY ANALYSIS:

In stability analysis of linear PDE, we analyze only one arbitrary Fourier mode

Consider, error can be represented in the form of Fourier series and single arbitrary term is written as,

$$\varepsilon_{i,j}^{n} = A^{n} e^{\sqrt{-1}\, i \omega_x \Delta x + \sqrt{-1}\, j \omega_y \Delta y} \quad \text{....(343).}$$

i, j are corresponding to x and y directions respectively.

$\omega_x, \omega_y \rightarrow$ wave numbers in x and y directions, $A^n \Rightarrow$ Amplitude.
↳ I.T (n represents amplitude for n time step).

Note that, $|\varepsilon_{i,j}^{n}| = |A^n|$

$$\# \left| e^{\sqrt{-1}\, i\omega_x \Delta x + \sqrt{-1}\, j\omega_y \Delta y} \right| = \left| e^{i(i\omega_x \Delta x + j\omega_y \Delta y)} \right|$$

$$= |e^{i\theta}|$$

(where, $\theta = i\omega_x \Delta x + j\omega_y \Delta y$)

'i' inside bracket (represents x direction) and 'i' outside bracket (imaginary number 'iota') are not same.

$$= |\cos\theta + i\sin\theta|$$

$$= \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1.$$

That's why modulus of error depends on amplitude only.

In simplified form, error can be written as, (from 343),

$$\varepsilon_{i,j}^{n} = A^{n} e^{\sqrt{-1}\, i\varphi_x + \sqrt{-1}\, j\varphi_y} \quad \text{........ (344).}$$

where, $\varphi_x (= \omega_x \Delta x)$ and $\varphi_y (= \omega_y \Delta y)$ are phase values in x and y directions respectively.

Von-Neumann Stability condition:-

Amplification factor, $G = \dfrac{A^{n+1}}{A^n}$ .... (345).

'G' governs the growth of Fourier component.

Von-Neumann stability condition is given by $|G| \leq 1$..... (345.1)

$|G| > 1 \Rightarrow$ Error grows (Unstable scheme)

$|G| < 1 \Rightarrow$ " reduces (Stable ")

$|G| = 1 \Rightarrow$ " remains same (Neutrally stable scheme).

Stability: Explicit Scheme

From (342), discretized GE for IBVP with explicit Scheme,

$$\Lambda_\varphi \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\varphi_{i-1,j}^{n} - 2\varphi_{i,j}^{n} + \varphi_{i+1,j}^{n}}{\Delta x^2} + \Gamma_y \frac{\varphi_{i,j-1}^{n} - 2\varphi_{i,j}^{n} + \varphi_{i,j+1}^{n}}{\Delta y^2} + S_\varphi\Big|_{i,j}^{n} \quad \cdots\cdots (346)$$

General variable φ written as,

$$\varphi_{i,j}^{n} = \hat{\varphi}_{i,j}^{n} + \varepsilon_{i,j}^{n} \quad \cdots\cdots (347).$$

$\varphi_{i,j}^{n}$ = Numerical solution obtained from finite precision computer.

$\hat{\varphi}_{i,j}^{n}$ = Exact discrete solution obtained from a hypothetical infinite-precision computer.

$\varepsilon_{i,j}^{n}$ = Accumulated round-off errors at time level n.

Using (347), (346) can be written as:-

$$\Lambda_\varphi \frac{(\hat{\varphi}_{i,j}^{n+1} + \varepsilon_{i,j}^{n+1}) - (\hat{\varphi}_{i,j}^{n} + \varepsilon_{i,j}^{n})}{\Delta t} = \Gamma_x \frac{(\hat{\varphi}_{i-1,j}^{n} + \varepsilon_{i-1,j}^{n}) - 2(\hat{\varphi}_{i,j}^{n} + \varepsilon_{i,j}^{n}) + (\hat{\varphi}_{i+1,j}^{n} + \varepsilon_{i+1,j}^{n})}{\Delta x^2}$$
$$+ \Gamma_y \frac{(\hat{\varphi}_{i,j-1}^{n} + \varepsilon_{i,j-1}^{n}) - 2(\hat{\varphi}_{i,j}^{n} + \varepsilon_{i,j}^{n}) + (\hat{\varphi}_{i,j+1}^{n} + \varepsilon_{i,j+1}^{n})}{\Delta y^2} + S_\varphi\Big|_{i,j}^{n} \quad \cdots\cdots (347).1)$$

By definition $\hat{\varphi}$ is the exact discrete solution of the finite difference equation. So, discretized FDE can be written as:-

$$\Lambda_\varphi \frac{\hat{\varphi}_{i,j}^{n+1} - \hat{\varphi}_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\hat{\varphi}_{i-1,j}^{n} - 2\hat{\varphi}_{i,j}^{n} + \hat{\varphi}_{i+1,j}^{n}}{\Delta x^2} + \Gamma_y \frac{\hat{\varphi}_{i,j-1}^{n} - 2\hat{\varphi}_{i,j}^{n} + \hat{\varphi}_{i,j+1}^{n}}{\Delta y^2} + S_\varphi\Big|_{i,j}^{n} \quad \cdots (348)$$

This equation is ideally satisfied.

I think source/sink term is defined for a particular system. That's why no change in different cases (see P-96).

By {(347)-(348)}, we have, the error equation,

$$\Lambda_\varphi \frac{\varepsilon_{i,j}^{n+1} - \varepsilon_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\varepsilon_{i-1,j}^{n} - 2\varepsilon_{i,j}^{n} + \varepsilon_{i+1,j}^{n}}{\Delta x^2} + \Gamma_y \frac{\varepsilon_{i,j-1} - 2\varepsilon_{i,j} + \varepsilon_{i,j+1}}{\Delta y^2}$$

Error equation is without source/sink term. Because, no error involved there due to discretization.

Note that, all error eq^n, exact eq^n, numerical discretization eq^{n}s are at time level n only.

$$\varepsilon_{i,j}^{n+1} = \alpha_y \varepsilon_{i,j-1}^{n} + \alpha_x \varepsilon_{i-1,j}^{n} + [1 - 2(\alpha_x + \alpha_y)]\, \varepsilon_{i,j}^{n} + \alpha_x \varepsilon_{i+1,j}^{n} + \alpha_y \varepsilon_{i,j+1}^{n} \cdots (349)$$

where, $\alpha_x = \dfrac{\Gamma_x \Delta t}{\Lambda_\varphi \Delta x^2}$ $\quad \alpha_y = \dfrac{\Gamma_y \Delta t}{\Lambda_\varphi \Delta y^2}$

using eq^n (343), we have,

$$\varepsilon_{i,j}^{n+1} = A^{n+1} e^{\sqrt{-1}[i\varphi_x + j\varphi_y]} \quad \cdots\cdots (350.1)$$

$$\varepsilon_{i,j}^{n} = A^n e^{\sqrt{-1}(i\varphi_x + j\varphi_y)} \quad \ldots\ldots (350.2)$$

$$\varepsilon_{i,j-1}^{n} = A^n e^{\sqrt{-1}[i\varphi_x + (j-1)\varphi_y]} \quad \ldots\ldots (350.3)$$

$$\varepsilon_{i-1,j}^{n} = A^n e^{\sqrt{-1}[(i-1)\varphi_x + j\varphi_y]} \quad \ldots\ldots (350.4)$$

$$\varepsilon_{i+1,j}^{n} = A^n e^{\sqrt{-1}[(i+1)\varphi_x + j\varphi_y]} \quad \ldots\ldots (350.5)$$

$$\varepsilon_{i,j+1}^{n} = A^n e^{\sqrt{-1}[i\varphi_x + (j+1)\varphi_y]}. \quad \ldots\ldots (350.6).$$

By doing $\{(350.1) \div (350.2)\}$,

$$\frac{A^{n+1} e^{\sqrt{-1}(i\varphi_x + j\varphi_y)}}{A^n e^{\sqrt{-1}(i\varphi_x + j\varphi_y)}} = \frac{\varepsilon_{i,j}^{n+1}}{\varepsilon_{i,j}^{n}}$$

$$\Rightarrow \frac{A^{n+1}}{A^n} = \frac{\varepsilon_{i,j}^{n+1}}{\varepsilon_{i,j}^{n}}$$

$$\Rightarrow \frac{A^{n+1}}{A^n} = \frac{\alpha_y \varepsilon_{i,j-1}^{n} + \alpha_x \varepsilon_{i-1,j}^{n} + [1-2(\alpha_x+\alpha_y)]\varepsilon_{i,j}^{n} + \alpha_x \varepsilon_{i+1,j}^{n} + \alpha_y \varepsilon_{i,j+1}^{n}}{\varepsilon_{i,j}^{n}}$$

$$\Rightarrow \frac{A^{n+1}}{A^n} = \alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} + [1-2(\alpha_x+\alpha_y)] + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y} \quad \ldots (351)$$

↳ # This is also called as growth factor or Amplification factor (G) (from 345).

Applying formula $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ in equation (351), we get,

$$G = \frac{A^{n+1}}{A^n} = \alpha_y\left(e^{\sqrt{-1}\varphi_y} + e^{-\sqrt{-1}\varphi_y}\right) + \alpha_x\left(e^{\sqrt{-1}\varphi_x} + e^{-\sqrt{-1}\varphi_x}\right) + [1-2(\alpha_x+\alpha_y)]$$

$$= 2\alpha_y \cos\varphi_y + 2\alpha_x \cos\varphi_x + 1 - 2(\alpha_x+\alpha_y)$$

$$= 1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)$$

$$= 1 - 4\alpha_y \sin^2\frac{\varphi_y}{2} - 4\alpha_x \sin^2\frac{\varphi_x}{2}. \quad \ldots\ldots (352).$$

The Von-Neumann Stability Condition,

$$|G| \leq 1 \quad \text{(From 345.1).}$$

$$\Rightarrow \left|1 - 4\alpha_y \sin^2\frac{\varphi_y}{2} - 4\alpha_x \sin^2\frac{\varphi_x}{2}\right| \leq 1$$

$$\Rightarrow -1 \leq 1 - 4\alpha_y \sin^2\frac{\varphi_y}{2} - 4\alpha_x \sin^2\frac{\varphi_x}{2} \leq 1.$$

$$\Rightarrow -2 \leq -4\alpha_y \sin^2\frac{\varphi_y}{2} - 4\alpha_x \sin^2\frac{\varphi_x}{2} \leq 0$$

$$\Rightarrow 0 \leq \alpha_y \sin^2\frac{\varphi_y}{2} + \alpha_x \sin^2\frac{\varphi_x}{2} \leq \frac{1}{2} \quad \ldots (353)$$

For two boundary cases:-

(i) case 1:- $\alpha_y \sin^2\frac{\varphi_y}{2} + \alpha_x \sin^2\frac{\varphi_x}{2} = 0$ (From 353).

$\sin\frac{\varphi_y}{2} = 0$ and $\sin\frac{\varphi_x}{2} = 0$

From (352),

$$G = 1 - 4\alpha_y \cdot 0 - 4\alpha_x \cdot 0 = 1$$

Case 2:-

$$\alpha_y \sin^2\frac{\varphi_y}{2} + \alpha_x \sin^2\frac{\varphi_x}{2} \le \frac{1}{2}$$

If we take max values of $\sin\frac{\varphi_y}{2}$ and $\sin\frac{\varphi_x}{2}$ and the above condition is satisfied, then it will be satisfied for all other values.

$\therefore \sin\frac{\varphi_y}{2} = 1$ and $\sin\frac{\varphi_x}{2} = 1$.

$$\therefore \alpha_x + \alpha_y \le \frac{1}{2} \quad \ldots\ldots\ldots\ldots (354).$$

$$\therefore G = 1 - 4\alpha_y \cdot 1^2 - 4\alpha_x \cdot 1^2$$
$$= 1 - 4(\alpha_x + \alpha_y).$$

From (354), we can say, explicit scheme is unconditionally stable.

These α_x, α_y terms contain $\Delta x, \Delta y, \Delta t$ terms. So, we cannot specify arbitrary values of Δt in explicit scheme. That should be related to Δx or Δy!

Stability: Implicit Scheme

The discretized GE for IBVP with implicit scheme, (Applying (347) to equation (338·2)),

$$\Lambda_\varphi \frac{(\hat{\varphi}_{i,j}^{n+1} + \varepsilon_{i,j}^{n+1}) - (\hat{\varphi}_{i,j}^{n} + \varepsilon_{i,j}^{n})}{\Delta t} = \Gamma_x \frac{(\hat{\varphi}_{i+1,j}^{n+1} + \varepsilon_{i+1,j}^{n+1}) - 2(\hat{\varphi}_{i,j}^{n+1} + \varepsilon_{i,j}^{n+1}) + (\hat{\varphi}_{i-1,j}^{n+1} + \varepsilon_{i-1,j}^{n+1})}{\Delta x^2}$$
$$+ \Gamma_y \frac{(\hat{\varphi}_{i,j+1}^{n+1} + \varepsilon_{i,j+1}^{n+1}) - 2(\hat{\varphi}_{i,j}^{n+1} + \varepsilon_{i,j}^{n+1}) + (\hat{\varphi}_{i,j-1}^{n+1} + \varepsilon_{i,j-1}^{n+1})}{\Delta y^2}$$
$$+ S_\varphi\big|_{i,j}^{n+1} \quad \ldots\ldots (355)$$

Discretized finite difference equation in terms of exact discrete solution ($\hat{\varphi}$),

$$\Lambda_\varphi \frac{\hat{\varphi}_{i,j}^{n+1} - \hat{\varphi}_{i,j}^{n}}{\Delta t} = \Gamma_x \frac{\hat{\varphi}_{i+1,j}^{n+1} - 2\hat{\varphi}_{i,j}^{n+1} + \hat{\varphi}_{i-1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\hat{\varphi}_{i,j+1}^{n+1} - 2\hat{\varphi}_{i,j}^{n+1} + \hat{\varphi}_{i,j-1}^{n+1}}{\Delta y^2} + S_\varphi\big|_{i,j}^{n+1}$$
$$\ldots\ldots (356)$$

By {(355)-(356)}, we get the error equation, (Expressing in simplified form),

$$\alpha_y \varepsilon_{i,j-1}^{n+1} + \alpha_x \varepsilon_{i-1,j}^{n+1} - [1 + 2(\alpha_x + \alpha_y)]\varepsilon_{i,j}^{n+1} + \alpha_x \varepsilon_{i+1,j}^{n+1} + \alpha_y \varepsilon_{i,j+1}^{n+1} = -\varepsilon_{i,j}^{n} \ldots (357)$$

Here, $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\varphi \Delta x^2}$

$\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\varphi \Delta y^2}$

$$\varepsilon_{i,j}^{n+1} = A^{n+1} e^{\sqrt{-1}(i\varphi_x + j\varphi_y)} \ldots (358\cdot1) \quad \varepsilon_{i,j}^{n} = A^{n} e^{\sqrt{-1}(i\varphi_x + j\varphi_y)} \ldots (358\cdot2)$$

$$\varepsilon_{i,j-1}^{n+1} = A^{n+1} e^{\sqrt{-1}(i\varphi_x + (j-1)\varphi_y)} \ldots (358\cdot3)$$

$$\varepsilon_{i-1,j}^{n+1} = A^{n+1} e^{\sqrt{-1}[(i-1)\varphi_x + j\varphi_y]} \ldots (358\cdot4)$$

$$\varepsilon_{i+1,j}^{n+1} = A^{n+1} e^{\sqrt{-1}[(i+1)\varphi_x + j\varphi_y]} \ldots (358\cdot5).$$

$$\varepsilon_{i,j+1}^{n+1} = A^{n+1} e^{\sqrt{-1}[i\varphi_x + (j+1)\varphi_y]} \quad \ldots\ldots (358.6)$$

$$G = \frac{A^{n+1}}{A^n} = \frac{\varepsilon_{i,j}^{n+1}}{\varepsilon_{i,j}^n}$$

$$= \frac{\varepsilon_{i,j}^{n+1}}{-[\alpha_y \varepsilon_{i,j-1}^{n+1} + \alpha_x \varepsilon_{i-1,j}^{n+1} - (1+2\alpha_x+2\alpha_y)\varepsilon_{i,j}^{n+1} + \alpha_x \varepsilon_{i+1,j}^{n+1} + \alpha_y \varepsilon_{i,j+1}^{n+1}]}$$

[from equation 357]

$$= -\frac{1}{\alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} - (1+2\alpha_x+\alpha_y) + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y}}$$

$$= -\frac{1}{\alpha_y \cdot 2\cos\varphi_y + \alpha_x \cdot 2\cos\varphi_x - (1+2\alpha_x+2\alpha_y)}$$

$$= +\frac{1}{2\alpha_y(1-\cos\varphi_y) + 2\alpha_x(1-\cos\varphi_x) + 1}$$

$$G = \frac{1}{4\alpha_y \sin^2\frac{\varphi_y}{2} + 4\alpha_x \sin^2\frac{\varphi_x}{2} + 1} \quad \ldots\ldots (359).$$

Case I:

Von-Neumann stability condition:-

$$|G| = \left| \frac{1}{4\alpha_y \sin^2\frac{\varphi_y}{2} + 4\alpha_x \sin^2\frac{\varphi_x}{2} + 1} \right| \leq 1.$$

Two cases:-

- ☐ $\sin\frac{\varphi_x}{2} = 0, \ \sin\frac{\varphi_y}{2} = 0 \Rightarrow G = 1$
- ☐ $\sin\frac{\varphi_x}{2} = 1, \ \sin\frac{\varphi_y}{2} = 1 \Rightarrow G = \frac{1}{1+4\alpha_x+4\alpha_y} < 1 \ \ldots\ldots (360)$

∴ Implicit scheme is unconditionally stable.
We don't need to think about value of Δt for any Δx value for numerical stability. Though, we use small values of Δt for the problems.

NPTEL-12

Numerical stability of 1D PDE:-

To analyze the numerical stability of discretized 1-D conservation law in terms of PDE.

General equation:-

$$\underbrace{\frac{\partial(\Lambda_\varphi \varphi)}{\partial t}}_{\text{Temporal}} + \underbrace{\nabla\cdot(\Upsilon_\varphi \varphi \underline{u})}_{\text{Advective}} = \underbrace{\nabla\cdot(\Gamma_\varphi \cdot \nabla\varphi)}_{\text{Diffusion}} + F_{\varphi_0} + S_\varphi \text{ (Same as 279)} \quad \ldots\ldots (361)$$

Unlike BVP and IBVP, here we use advective term also. This is somewhat related to velocity term.

1D Scaler Conservation law:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \quad \ldots \quad (362).$$

Here $\phi = \phi(x,t)$.

$\mathcal{F}_\phi$ = flux function
= (Amount of ϕ passes at the abscissa x per unit time due to displacement of ϕ).

$\mathcal{F}_\phi(\phi, x, t)$ does not depend on $\frac{\partial \phi}{\partial x}$ or $\frac{\partial \phi}{\partial t}$.

S_ϕ = Source term
= Amount of ϕ that appears per unit time per unit volume irrespective of amount transported via flux.

For example, $\mathcal{F}_\phi = u\phi \Rightarrow$ Allowed

$\mathcal{F}_\phi = -\Gamma_x \frac{\partial \phi}{\partial x} \Rightarrow$ Not allowed.

How to get equation (362) from General Equation.

I.T.

$\mathcal{F}_\phi = u\phi$, $\Lambda_\phi = 1$, $\gamma_\phi = 1$.

$$\frac{\partial \phi}{\partial t} + \nabla(\phi \underset{\sim}{u}) = 0 + S_\phi$$

$\Downarrow$
(Diffusion term = 0 ?!) why ?

$$\Rightarrow \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi u) = S_\phi \quad [\text{considering 1D, } \underset{\sim}{u} = u]$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi$$

Non-conservative form (of eq^n 362)

Non-conservative forms are written in terms of general variable ϕ, not in terms of flux $\mathcal{F}_\phi$.

$$\frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial x} = \hat{S}_\phi \quad \ldots \quad (363)$$

where, $\lambda = \frac{\partial \mathcal{F}_\phi}{\partial \phi} \quad \ldots \quad (363.1)$

and, $\hat{S}_\phi = S_\phi - \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_{\phi = const}$

From eq (362),

$$\frac{\partial \mathcal{F}_\phi}{\partial x} = \underbrace{\frac{\partial \mathcal{F}_\phi}{\partial \phi}}_{\lambda} \cdot \frac{\partial \phi}{\partial x} + \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_{\phi = constant}$$

$$= \lambda \frac{\partial \phi}{\partial x} + \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_{\phi = constant}$$

Putting this value at (362),

$$\frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial x} + \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_{\phi = constant} = S_\phi$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial x} = \underbrace{S_\phi - \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_{\phi = constant}}$$

$$\frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial x} = \hat{S}_\phi$$

Explicit upwind scheme

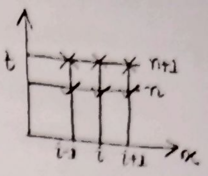
Conservative form:

Governing Equation:- $\left.\frac{\partial \phi}{\partial t}\right|_i^n + \left.\frac{\partial \mathcal{F}_\phi}{\partial x}\right|_i^n = S_\phi|_i^n \quad \ldots \quad (364.1)$

Time discretization:- $\frac{\partial \phi}{\partial t} = \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + O(\Delta t). \quad \ldots \quad (364.2)$

Space discretization:- # $\lambda = \frac{\partial F_\varphi}{\partial \varphi}$

$$\frac{\partial F_\varphi}{\partial x} = \begin{cases} \frac{F_{\varphi_i}^n - F_{\varphi_{i-1}}^n}{\Delta x} + O(\Delta x) & \text{if } \lambda_i^n > 0 \\ \frac{F_{\varphi_{i+1}}^n - F_{\varphi_i}^n}{\Delta x} + O(\Delta x) & \text{if } \lambda_i^n \leq 0 \end{cases}$$ why ?



what is the significance of these two eqns of $\lambda > 0$ or $\lambda < 0$? (364.3)

Source term:- $S_\varphi = S_{\varphi_i}^n$

Final solution can be written as:- (Putting discretized forms in governing equation 364.1),

$$\varphi_i^{n+1} = \begin{cases} \varphi_i^n + \frac{\Delta t}{\Delta x}(F_{\varphi_{i-1}}^n - F_{\varphi_i}^n) + \Delta t\, S_{\varphi_i}^n & \text{if } \lambda_i^n > 0. \\ \varphi_i^n + \frac{\Delta t}{\Delta x}(F_{\varphi_i}^n - F_{\varphi_{i+1}}^n) + \Delta t\, S_{\varphi_i}^n & \text{if } \lambda_i^n \leq 0. \end{cases} \quad \ldots\ldots (365).$$

Stability analysis

☐ Von-Neumann stability analysis can be performed for linear equations.

Let flux function term,

$$F_\varphi = a\varphi. \quad , a = \text{constant} \ldots\ldots (366)$$

$$\therefore \lambda = \frac{\partial F_\varphi}{\partial \varphi} = a \quad \ldots (367)$$ # a is equivalent to λ (see 363.1).

☐ Error is represented as fourier series form.

$$\varepsilon_i^n = A^n e^{\sqrt{-1}\, i\, \omega_x \Delta x}$$, ω_x = wave number in x direction.

In simplified form,

$$\varepsilon_i^n = A^n e^{\sqrt{-1}\, i\, \varphi_x} \quad \ldots (368)$$, φ_x = Phase value corresponding to x direction.

These eqns are comparable with eqns (365). In (365), $\frac{\partial F_\varphi}{\partial \varphi} = \lambda$. Here, $\frac{\partial F_\varphi}{\partial \varphi} = a$. Here in place of $F_\varphi \rightarrow a\varphi$ has been used.

☐ $$\varphi_i^{n+1} = \begin{cases} \varphi_i^n + a\frac{\Delta t}{\Delta x}(\varphi_{i-1}^n - \varphi_i^n) + \Delta t\, S_{\varphi_i}^n & \text{if } a > 0. \\ \varphi_i^n + a\frac{\Delta t}{\Delta x}(\varphi_i^n - \varphi_{i+1}^n) + \Delta t\, S_{\varphi_i}^n & \text{if } a < 0 \end{cases} \quad \ldots\ldots (369)$$

Equations (369) can be written as,

(The discretized GE for IBVP with explicit scheme),

But, here IC is the main thing because only one side is defined. So, it is like IVP. ?

$$\hat{\varphi}_i^{n+1} + \varepsilon_i^{n+1} = \begin{cases} (\hat{\varphi}_i^n + \varepsilon_i^n) + a\frac{\Delta t}{\Delta x}\left[(\hat{\varphi}_{i-1}^n + \varepsilon_{i-1}^n) - (\hat{\varphi}_i^n + \varepsilon_i^n)\right] + \Delta t\, S_{\varphi_i}^n & \text{if } a > 0 \\ (\hat{\varphi}_i^n + \varepsilon_i^n) + a\frac{\Delta t}{\Delta x}\left[(\hat{\varphi}_i^n + \varepsilon_i^n) - (\hat{\varphi}_{i+1}^n + \varepsilon_{i+1}^n)\right] + \Delta t\, S_{\varphi_i}^n & \text{if } a < 0. \end{cases} \quad \ldots\ldots (370).$$

Discretized Finite difference equation for exact solution ($\hat{\varphi}$) (# obtained from infinite precission computer),

$$\hat{\varphi}_i^{n+1} = \begin{cases} \hat{\varphi}_i^n + a\frac{\Delta t}{\Delta x}\left[\hat{\varphi}_{i-1}^n - \hat{\varphi}_i^n\right] + \Delta t\, S_{\varphi_i}^n & \text{if } a > 0 \\ \hat{\varphi}_i^n + a\frac{\Delta t}{\Delta x}\left[\hat{\varphi}_i^n - \hat{\varphi}_{i+1}^n\right] + \Delta t\, S_{\varphi_i}^n & \text{if } a < 0. \end{cases} \quad \ldots\ldots (371)$$

By doing {(370) - (371)}, we get error equation,

$$\varepsilon_i^{n+1} = \begin{cases} \varepsilon_i^n + a\frac{\Delta t}{\Delta x}(\varepsilon_{i-1}^n - \varepsilon_i^n) & \text{if } a > 0 \\ \varepsilon_i^n + a\frac{\Delta t}{\Delta x}(\varepsilon_i^n - \varepsilon_{i+1}^n) & \text{if } a < 0. \end{cases} \quad \ldots\ldots (372)$$

Where, $\varepsilon_i^{n+1} = A^{n+1} e^{\sqrt{-1}\, i\varphi_x}$ .... (373.1)

$\varepsilon_i^n = A^n e^{\sqrt{-1}\, i\varphi_x}$ ..... (373.2)

$\varepsilon_{i-1}^n = A^n e^{\sqrt{-1}(i-1)\varphi_x}$ ..... (373.3)

$\varepsilon_{i+1}^n = A^n e^{\sqrt{-1}(i+1)\varphi_x}$ ..... (373.4)

$\therefore$ Growth factor, $G = \frac{A^{n+1}}{A^n} = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}$ (# From 373.1 and 373.2).

$$= \begin{cases} 1 + a\frac{\Delta t}{\Delta x}\left(\frac{\varepsilon_{i-1}^n}{\varepsilon_i^n} - 1\right) & \text{if } a>0 \\ 1 + a\frac{\Delta t}{\Delta x}\left(1 - \frac{\varepsilon_{i+1}^n}{\varepsilon_i^n}\right) & \text{if } a<0 \end{cases} \Rightarrow \text{From (372)}$$

$$= \begin{cases} 1 + a\frac{\Delta t}{\Delta x}\left(e^{-\sqrt{-1}\varphi_x} - 1\right) & \text{if } a>0 \quad \ldots (374.1) \\ 1 + a\frac{\Delta t}{\Delta x}\left(-e^{\sqrt{-1}\varphi_x} + 1\right) & \text{if } a<0 \quad \ldots (374.2) \end{cases}$$ Putting values from (373s)

Now, we define Courant number as, $C_r = |a|\frac{\Delta t}{\Delta x}$

So, Always C_r is positive. But, for (374.2), a is negative. So, we can write that equation as,

$$G = 1 - |a|\frac{\Delta t}{\Delta x}\left(1 - e^{\sqrt{-1}\varphi_x}\right) \ldots\ldots (374.3).$$

when $a<0$

Now, using the formula, $e^{\sqrt{-1}\theta} = \cos\theta + \sqrt{-1}\sin\theta$, (374.1) and (374.3) can be written as,

$$G = \begin{cases} 1 + C_r\left(\cos\varphi_x - \sqrt{-1}\sin\varphi_x\right) - 1) & \text{, if } a>0 \\ 1 - C_r\left(1 - \cos\varphi_x - \sqrt{-1}\sin\varphi_x\right) & \text{, if } a<0 \end{cases}$$

$$= \begin{cases} [(1-C_r) + C_r\cos\varphi_x] + \sqrt{-1}\,[-C_r\sin\varphi_x] & \text{if } a>0 \\ \underbrace{[(1-C_r) + C_r\cos\varphi_x]}_{\text{Real part } (\mathrm{Re}\,G)} + \sqrt{-1}\,\underbrace{[C_r\sin\varphi_x]}_{\text{Imaginary part } (\mathrm{Im}\,G)} & \text{if } a<0. \end{cases} \ldots\ldots (375\text{'s})$$

Courant-Friedrichs-Lewy condition:

From (375's), we have, the modulus of amplification factor,

$$|G|^2 = [(1-C_r) + C_r.\cos\varphi_x]^2 + [C_r\sin\varphi_x]^2$$

$$= (1-C_r)^2 + 2(1-C_r)C_r\cos\varphi_x + C_r^2\cos^2\varphi_x + C_r^2\sin^2\varphi_x$$

$$= 1 - 2C_r + C_r^2 + 2C_r(1-C_r)\cos\varphi_x + C_r^2$$

$$= 1 - 2C_r + 2C_r^2 + 2C_r(1-C_r)\left(1 - 2\sin^2\frac{\varphi_x}{2}\right)$$

$$= 1 - 2C_r + 2C_r^2 + 2C_r(1-C_r) - 4C_r(1-C_r)\sin^2\frac{\varphi_x}{2}$$

$$= 1 - \cancel{2C_r} + \cancel{2C_r^2} + \cancel{2C_r} - \cancel{2C_r^2} + 4C_r(C_r-1)\sin^2\frac{\varphi_x}{2}$$

$$= 1 + 4C_r(C_r-1)\sin^2\frac{\varphi_x}{2} \quad \ldots\ldots (376).$$

$|G|$ should be less than 1, if $\left[4C_r(C_r-1)\sin^2\frac{\varphi_x}{2}\right]$ is negative.

But here C_r is +ve, $\sin^2\frac{\varphi_x}{2}$ is +ve. (∵ C_r is always +ve, because, it takes $|a|\frac{\Delta t}{\Delta x}$ all positive values)

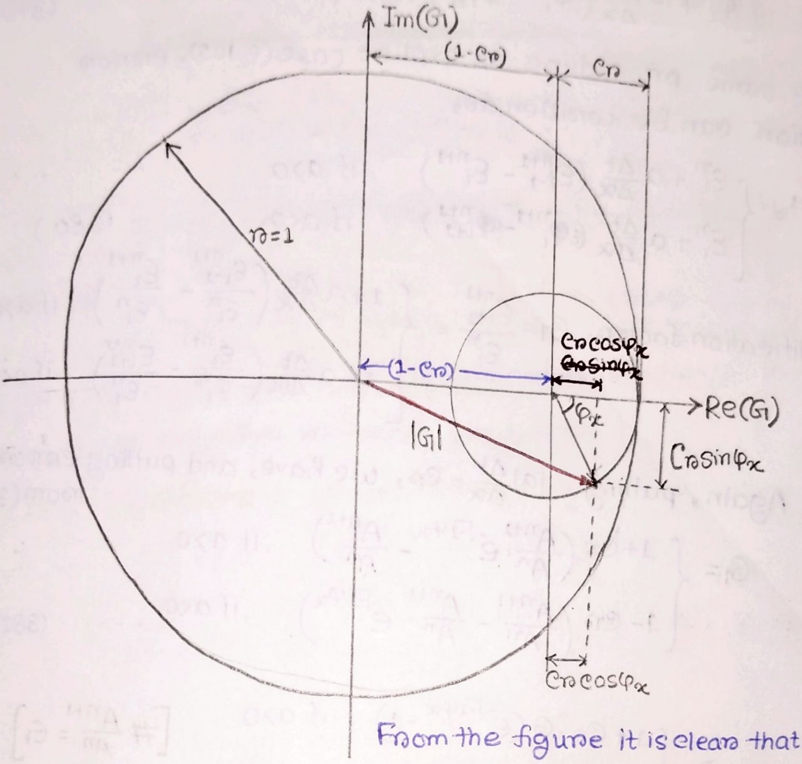
∴ only (C_r-1) can be negative, if,

$$C_r-1\leq 0$$
$$\Rightarrow C_r\leq 1$$

∴ Condition for explicit scheme to be stable:-

$0<C_r\leq 1$, this is called CFL condition.

∴ Explicit scheme is conditionally stable.



From the figure it is clear that,

$$|G|=\sqrt{\left[(1-C_r)+C_r\cos\varphi_x\right]^2+(C_r\sin\varphi_x)^2}$$

The modulus of G should be in the circle of unit radius.

If $0<C_r\leq 1$, then small circle will be within the big circle. ∴ Stability criterion should be satisfied.

☑ IMPLICIT UPWIND SCHEME:

$$\left.\frac{\partial\varphi}{\partial t}\right|_i^{n+1}+\left.\frac{\partial F_\varphi}{\partial x}\right|_i^{n+1}=\left.S_\varphi\right|_i^{n+1} \quad \text{---- (377)}$$

$$\frac{\partial\varphi}{\partial t}=\frac{\varphi_i^{n+1}-\varphi_i^n}{\Delta t}+O(\Delta t)$$

$$\frac{\partial F_\varphi}{\partial x}=\begin{cases}\dfrac{F_{\varphi_i}^{n+1}-F_{\varphi_{i-1}}^{n+1}}{\Delta x}+O(\Delta x) & \text{if } \lambda_i^{n+1}>0\\[2ex] \dfrac{F_{\varphi_{i+1}}^{n+1}-F_{\varphi_i}^{n+1}}{\Delta x}+O(\Delta x) & \text{if } \lambda_i^{n+1}\leq 0.\end{cases}$$

$$S_\varphi=S_{\varphi_i}^{n+1}.$$

If $\lambda_i^{n+1}>0$, Backward difference
if $\lambda_i^{n+1}\leq 0$, Forward difference } why.

For implicit scheme, space discretization is done at $(n+1)$ time level.

Final form:-

$$\phi_i^{n+1} = \begin{cases} \phi_i^n + \frac{\Delta t}{\Delta x}\left(-F_{\phi_i}^{n+1} + F_{\phi_{i-1}}^{n+1}\right) + \Delta t\, S_{\phi_i}^{n+1} & \text{if } \lambda_i^{n+1} > 0 \\ \phi_i^n + \frac{\Delta t}{\Delta x}\left(F_{\phi_i}^{n+1} - F_{\phi_{i+1}}^{n+1}\right) + \Delta t\, S_{\phi_i}^{n+1} & \text{if } \lambda_i^{n+1} < 0. \end{cases} \quad \ldots (378's)$$

If we put $F_\phi = a\phi$ (as, Von-Neuman Stability analysis can be Performed for linear equations). (See page 104 and 105)

$$\phi_i^{n+1} = \begin{cases} \phi_i^n + a\frac{\Delta t}{\Delta x}\left(\phi_{i-1}^{n+1} - \phi_i^{n+1}\right) + \Delta t\, S_{\phi_i}^{n+1}, & \text{if } a>0 \\ \phi_i^n + a\frac{\Delta t}{\Delta x}\left(\phi_i^{n+1} - \phi_{i+1}^{n+1}\right) + \Delta t\, S_{\phi_i}^{n+1}, & \text{if } a<0. \end{cases} \quad \ldots (379's)$$

Using same procedure as explicit case (P-105), error equation can be written as,

$$\varepsilon_i^{n+1} = \begin{cases} \varepsilon_i^n + a\frac{\Delta t}{\Delta x}\left(\varepsilon_{i-1}^{n+1} - \varepsilon_i^{n+1}\right) & \text{if } a>0 \\ \varepsilon_i^n + a\frac{\Delta t}{\Delta x}\left(\varepsilon_i^{n+1} - \varepsilon_{i+1}^{n+1}\right) & \text{if } a<0. \end{cases} \quad \ldots (380)$$

Amplification factor, $$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \begin{cases} 1 + a\frac{\Delta t}{\Delta x}\left(\frac{\varepsilon_{i-1}^{n+1}}{\varepsilon_i^n} - \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}\right) & \text{if } a>0 \\ 1 + a\frac{\Delta t}{\Delta x}\left(\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} - \frac{\varepsilon_{i+1}^{n+1}}{\varepsilon_i^n}\right) & \text{if } a<0. \end{cases} \quad \ldots (381)$$

Again, putting $|a|\frac{\Delta t}{\Delta x} = C_r$, we have, and putting error value from (373's)

$$G = \begin{cases} 1 + C_r\left(\frac{A^{n+1}}{A^n} e^{-\sqrt{-1}\varphi_x} - \frac{A^{n+1}}{A^n}\right), & \text{if } a>0 \\ 1 - C_r\left(\frac{A^{n+1}}{A^n} - \frac{A^{n+1}}{A^n} e^{\sqrt{-1}\varphi_x}\right), & \text{if } a<0 \end{cases} \quad \ldots (382's)$$

$$= \begin{cases} 1 + C_r \cdot G\left(e^{-\sqrt{-1}\varphi_x} - 1\right), & \text{if } a>0 \\ 1 + C_r \cdot G\left(e^{\sqrt{-1}\varphi_x} - 1\right), & \text{if } a<0 \end{cases} \qquad \left[\# \frac{A^{n+1}}{A^n} = G\right]$$

$\ldots (383's)$

From (383.1), $(a>0)$

$$G = 1 + C_r \cdot G\left(e^{-\sqrt{-1}\varphi_x} - 1\right)$$

$$\Rightarrow G\left[1 - C_r\left(e^{-\sqrt{-1}\varphi_x} - 1\right)\right] = 1$$

$$\Rightarrow G = \frac{1}{1 - C_r\left(e^{-\sqrt{-1}\varphi_x} - 1\right)}$$

From (383.2), $(a<0)$

$$G = 1 + C_r \cdot G\left(e^{\sqrt{-1}\varphi_x} - 1\right)$$

$$\Rightarrow G\left(1 - C_r\left(e^{\sqrt{-1}\varphi_x} - 1\right)\right) = 1$$

$$\Rightarrow G = \frac{1}{1 - C_r\left(e^{\sqrt{-1}\varphi_x} - 1\right)}$$

$\therefore$ Final form of amplification factor,

$$G = \begin{cases} \frac{1}{1 - C_r\left(e^{-\sqrt{-1}\varphi_x} - 1\right)} & \text{if } a>0 \\ \frac{1}{1 - C_r\left(e^{\sqrt{-1}\varphi_x} - 1\right)} & \text{if } a<0. \end{cases}$$

Stability Analysis $(a>0)$:-

$$G = \frac{1}{1 - C_r\left(\cos\varphi_x - \sqrt{-1}\sin\varphi_x\right) - 1)}$$

$$= \frac{1}{(1 + C_r - C_r\cos\varphi_x) + \sqrt{-1}\,(C_r\sin\varphi_x)}$$

We know, $|G|^2 = G.G^*$ where, G^* = conjugate of complex number G.

$$\Rightarrow |G|^2 = \frac{1}{[(1+C_r - C_r\cos\varphi_x) + \sqrt{-1}(C_r\sin\varphi_x)]\,[(1+C_r - C_r\cos(\varphi_x) - \sqrt{-1}(C_r\sin\varphi_x)]}$$

$$= \frac{1}{(1+C_r - C_r\cos\varphi_x)^2 + (C_r\sin\varphi_x)^2}$$

$$= \frac{1}{1+C_r^2+C_r^2\cdot\cos^2\varphi_x + 2C_r - 2C_r^2\cos\varphi_x - 2C_r\cdot\cos\varphi_x + C_r^2\sin^2\varphi_x}$$

$$= \frac{1}{1+2C_r+2C_r^2-(2C_r^2+2C_r)\cos\varphi_x}$$

$$= \frac{1}{1+2C_r+2C_r^2-(2C_r^2+2C_r)(1-2\sin^2\frac{\varphi_x}{2})}$$

$$= \frac{1}{1+\cancel{2C_r}+\cancel{2C_r^2}-\cancel{2C_r^2}-\cancel{2C_r}+4C_r^2\sin^2\frac{\varphi_x}{2}+4C_r\sin^2\frac{\varphi_x}{2}}$$

$$|G|^2 = \frac{1}{1+4C_r(C_r+1)\sin^2\frac{\varphi_x}{2}} \quad \ldots\ldots (384)$$

As C_r is always positive. so denominator is always >1.

$\therefore |G| \leq 1$ even for extreme conditions.

$\therefore$ Implicit Scheme is unconditionally stable.

NPTEL-50

SURFACE WATER & GROUNDWATER INTERACTION

To solve unsteady interaction problem between channel flow, surface flow and groundwater flow.

We integrated 1D channel flow, 2D surface flow and 2D ground water flow for this problem.

Problem statement:

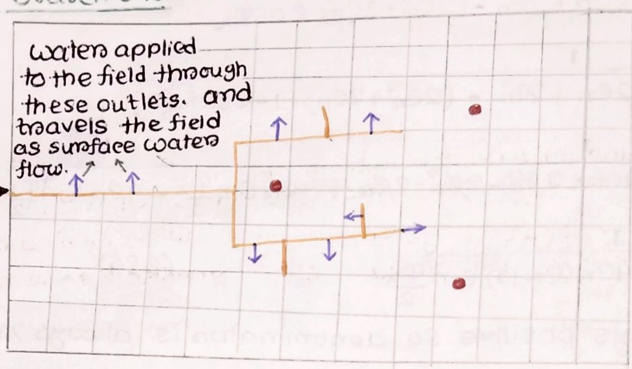


Fig: 50.1

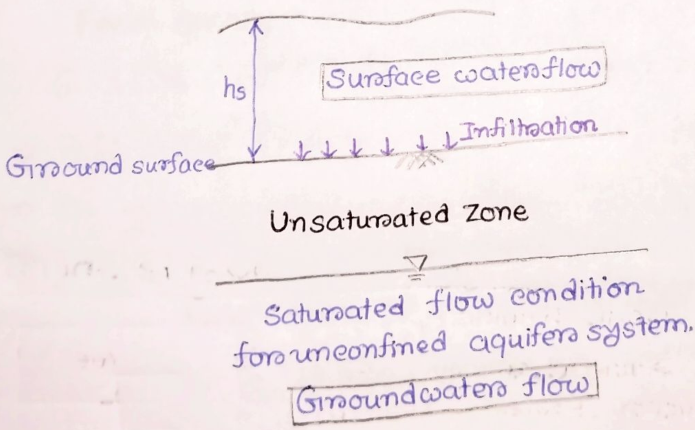
Water is to be supplied to this irrigation command area through 1D channel network.

Required:-

(i) Unsteady channel flow ⇒ $Q_c(x,t)$, $y_c(x,t)$. # 'C' means channel.

(ii) Unsteady free-surface flow (Shallow water): $h_s(x,y,t)$, $u_s(x,y,t)$, $v_s(x,y,t)$.

(iii) Unsteady unconfined aquifer flow: $h_g(x,y,t)$.



Unconfined aquifer is also called as phreatic aquifer or water table aquifer.

There will be recharge through the unsaturated zone towards GWT and after recharge, we can solve GW equation

Problem Definition:

(1) channel flow:-

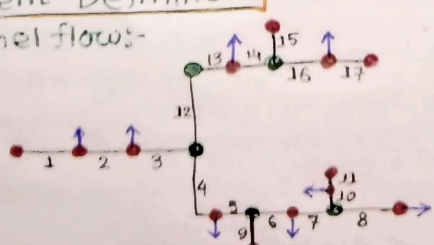


Fig: 50.2

- outflow points are also considered as external junction points.
- Others are internal junction points.

Here, we have total 17 no. of channel reaches.
For i^{th} channel reaches, number of sections $= (N_i+1)$, where, N_i = Number of segments.

∴ Total number of section in this channel network $= \sum_{i=1}^{17}(N_i+1)$

∴ Total number of unknowns $= 2 \times \sum_{i=1}^{17}(N_i+1)$ # Two unknowns y_c and Q_c for each sections.

Depending on upstream inflow condition, there will be changes inflow pattern and discharge within the channel network.
The outlet points may be by direct pipe flow or some hydraulic structures.
↳ So, we need to incorporate the effect of hydraulic structures within the system. But, here, in simplified form, we use junction condition in terms of discharge or energy condition (i.e flow depth) using network flow equation. We can use our generalized code for reverse flow situation.

☑ Governing equation for unsteady 1D channel flow (st. venant eqⁿs), as IBVP,

Continuity Eqⁿ: $\frac{\partial A}{\partial t} + \frac{\partial q_c}{\partial x} = -q_c$ .... (385)

"−" sign Because, it is a outflow component, called "Lateral outflow".

Momentum eqⁿ:- $\frac{\partial}{\partial t}\left(\frac{Q_c}{A}\right) + \frac{\partial}{\partial x}\left(\frac{\alpha Q_c^2}{2A^2}\right) + g\frac{\partial H}{\partial x} + gS_f = 0$ ..... (386).

we used these equations for unsteady channel network problem (P-50).

Where, $S_f = \frac{n^2 Q^2}{R^{4/3}A^2}$, $A = A(y_c)$.

H = Water surface elevation $= y_c + z$.

We need to link the component $-q_c$ from equation (385) with surface water flow. Because, lateral outflow will obviously contribute to 2D surface water flow.

(2) Unsteady free surface flow:
We have information about discharge values at inflow points and outflow junctions for the surface water system. With these informations, we can start our unsteady problem and can solve it with definite boundary.
Obviously, BC's should be for total command area because we have considered the rectangular system as irrigation command.

Required: $h_s(x,y,t)$, $u_s(x,y,t)$, $v_s(x,y,t)$.

Governing Equation: (see P-55).

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial G}{\partial y} = S \quad .... (387)$$

$$\underset{\sim}{U} = \begin{Bmatrix} h_s \\ h_s u_s \\ h_s v_s \end{Bmatrix} \quad \underset{\sim}{E} = \begin{Bmatrix} h u_s \\ h_s u_s^2 + \frac{g h_s^2}{2} \\ h_s u_s v_s \end{Bmatrix} \quad \underset{\sim}{G} = \begin{Bmatrix} h_s v_s \\ h_s u_s v_s \\ h_s v^2 + \frac{g h_s^2}{2} \end{Bmatrix}$$

$$\underset{\sim}{S} = \begin{Bmatrix} R + q_c - q_s \\ g h_s (S_{ox} - S_{fx}) \\ g h_s (S_{oy} - S_{fy}) \end{Bmatrix}$$

→ This gives linkage between channel and surface flow.

R = Rainfall, q_c = Channel outflow, q_s = Infiltration component.

h_s = Water height

u_s, v_s = Surface water velocity at x and y directions.

(3) Unsteady unconfined aquifer flow:-

Required ⇒ $h_g(x, y, t)$.

For every cell, we have information about q_s. q_s is the amount which is coming out from the surface water system and getting recharged into the saturated groundwater flow.

We consider no loss in unsaturated zone.

Governing Equation:-

2D groundwater flow in unconfined aquifers:-

$$S_y \frac{\partial h_g}{\partial t} = \frac{\partial}{\partial x}\left(k_x (h_g - \xi)\frac{\partial h_g}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y (h_g - \xi)\frac{\partial h_g}{\partial y}\right) - W_P + W_I + q_s \quad \cdots (388)$$

W_P = Pumping rate

W_I = Injection rate (For injection well).

q_s = Infiltration component.

ξ = Elevation of aquifer base w.r.t a particular datum = $\xi(x, y)$.

q_s can be calculated using our standard 1-D conceptual models or analytical models like Green-Ampt model.

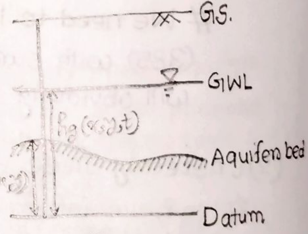
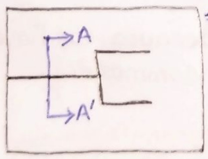


Fig: 50.3

Canal, SW, GW Interaction:



⇒ # This AA' section is shown in figure 50.4.

Canal is on top, because canal bed is always at a higher elevation compared to ground surface.

If canal water level is above the pipe, there will be flow from left to right.

If there is flooding situation at RHS, pipe flow may occur from right to left. So, q_c is not a fixed amount.

Here, we consider canal as a 1D or lined system. So, no rainfall to canal and no infiltration from canal bed to groundwater aquifer is considered. Rainfall and infiltration is considered only for 2D surface flooding system.

Canal - Surface water - GW interaction :-

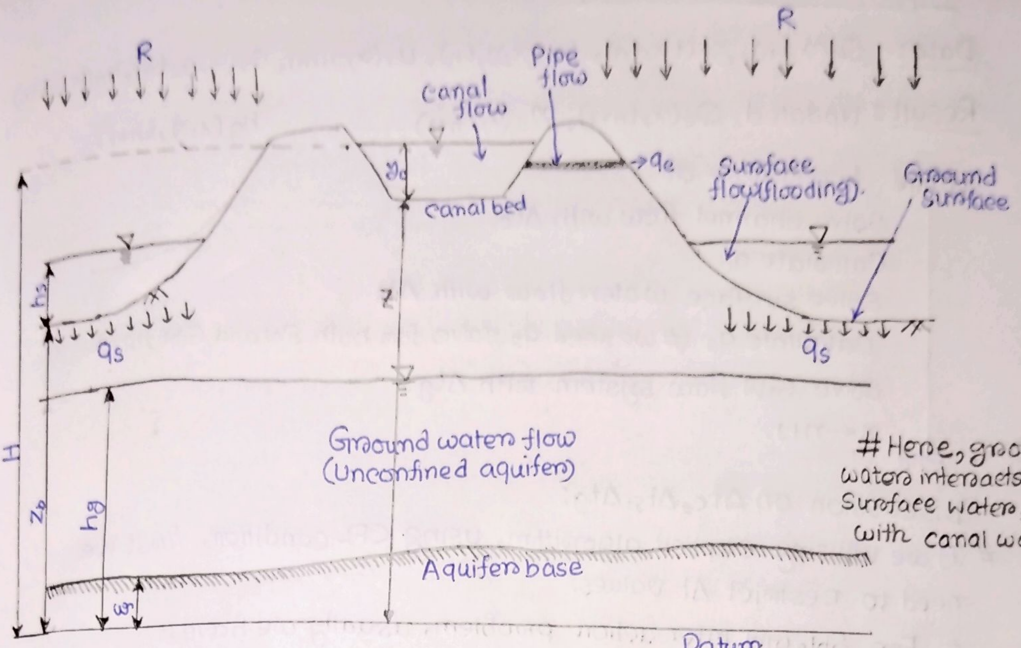


Fig: 50·4

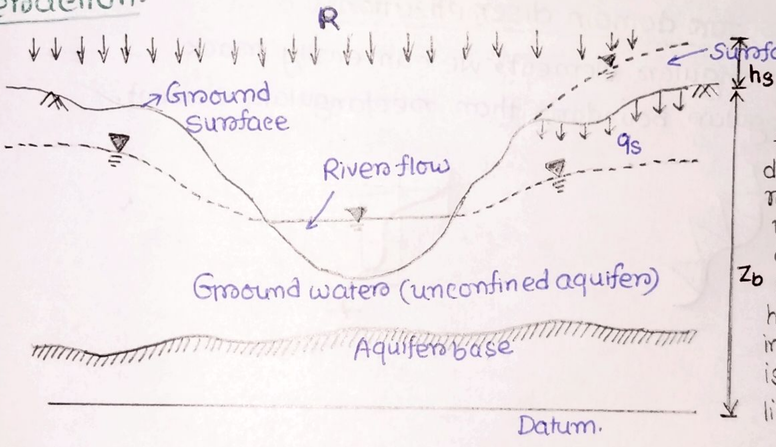
Here, ground water interacts with Surface water, not with canal water.

H = channel water surface elevation $= y_c + z$

Here, we have channel flow, pipe flow, Surface flooding, groundwater flow ⇒ these components.

We can integrate the discretization of different components using governing equations.

River, SW, GW Interaction:



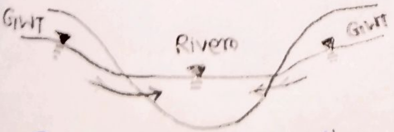
Here, GW is directly linked with river flow situation unlike the previous case.
But, still here may have SW-GW interaction if there is water-logging like situation.

In this case, river is at lower elevation compared to ground surface.

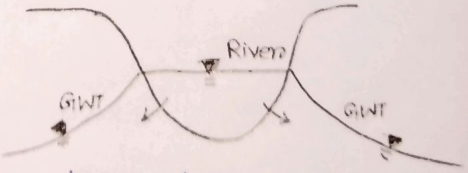
Here, unconfined GWT elevation should coincide with the stage of the river.

We can solve integrated SW-GW-River flow situation using our governing equations

There still have surface water flow due to rainfall.



Gaining stream situation (when GWT > River stage)



Losing stream situation.

Algorithm Structure:

Data:- $Q_c(x,t_n)$, $y_c(x,t_n)$, $h_s(x,y,t_n)$, $u_s(x,y,t_n)$, $v_s(x,y,t_n)$, $h_g(x,y,t_n)$

Result: Updated $Q_c(x,t_{n+1})$, $y_c(x,t_{n+1})$ .... $h_g(x,y,t_{n+1})$

While $t <$ end time do
- Solve channel flow with Δt_c.
- Calculate q_c
- Solve surface water flow with Δt_s
- Calculate q_s (# we need q_s value for both SW and GW flow).
- Solve GW flow system with Δt_g.
- $n \leftarrow n+1$.

end

Discussion on $\Delta t_c, \Delta t_s, \Delta t_g$:

If we utilizing explicit algorithm, using CFL condition, first we need to restrict Δt values.

For SW, GW interaction problem, usually we have:-

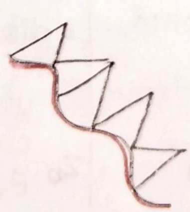
$$\Delta t_c < \Delta t_s < \Delta t_g .$$

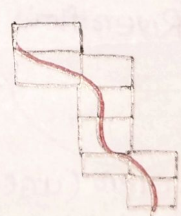
Because, larger time-step is adopted for slower process.

Obviously, for time Δt_g, we need to iterate our surface flow situation for a number of times. Same thing for Δt_s and Δt_c

Rectangular and triangular elements for domain discretization:

Using triangular elements we can easily track the irregular boundary than rectangular element.





NPTEL-47

M-5, U-1

Flows in Pressurized Conduits

Steady Flow in Pipe Network

We previously discussed surface water hydraulics, mainly free surface flow.
There flowdepth, elevation head and in some cases velocity was important.

Here, specifically pressure head component is most important part.

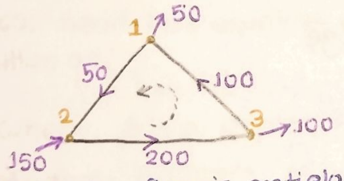
#To simulate steady flow in pipe networks in Hardy-Cross method.

We consider steady discharge condition and we start with some arbitrary values so that they satisfy continuity equation at each junction.

We have discharge condition Q and is varying with pipe numbers i.e. Q_i.
+ve or -ve direction of flow is important.
(But, this is not same as channel network where we considered sign of discharge as per channel numbering).
In Hardy-Cross method, clockwise direction was considered as +ve direction for a particular loop.

⊙ Let us have an example:- considering mass conservation let us assume a combination of demands at the junctions and discharge values at this following pipe network:-

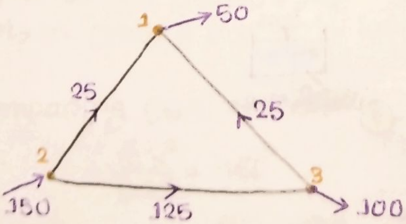
Fig: 47.1



Obviously, flow is anticlockwise for this loop. So negative.

⊙ But, if we consider figure (47.2),

Fig: 47.2



Here, no specific discharge direction is there for entire loop. For pipes connecting junctions 2-3 and 3-1 discharge is negative (as anti-clock wise). For pipes connecting junction 1-2 we have positive discharge condition.

So, for channel network flow,
Our specification of discharge direction depends on the direction of discretization.
I.T ⇒ (i.e if discharge is along the channel numbering, then positive discharge).

But, in case of pipe flow, we have fixed convention. (i.e clockwise positive, anti-clockwise flow is negative).

In this problem, equations will be non-linear. So, for non-linear solver, here, we will use "modified Newton-Raphson method".

☑ Typical pipe Network:-

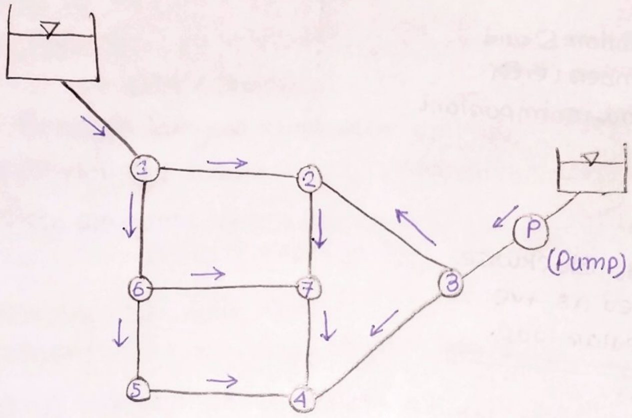


Fig: 47.3

I.T ⇒ (# Note, all directions of flow are targeting junc. 4).

We have 7 nodes, 10 pipes and one extra pump.

So, here, we need to find out 10 discharge values for each pipe and one additional discharge for pump.

☐ Problem Statement:-

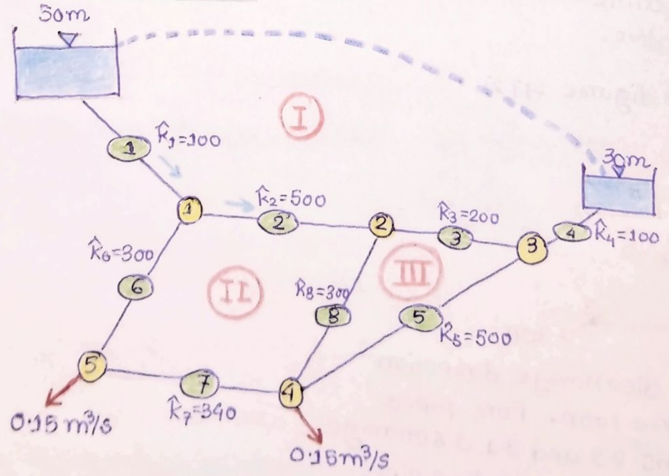


Fig: 47.4

In figure 474,
Difference of elevation of two tanks,
$\Delta H = 50 - 30 = 20m$

There are two demand nodes. i.e 4 and 5.

We can represent our head loss equation as, $h_L = \hat{k} Q^{\beta}$ .... (388.2)
If we consider Darcy-Weisbach equation, k is dependent on area/pipe diameter, transition loss coff (because, at the junction or entrance there would be loss)

K value is a 'physical parameter', because it has different values depending on the nature of the pipe and type of material.

Here, we have 8 pipes.
3 loops are there. Loop (I) is a pseudo loop. Loop II, III are internal loops.
'Pseduo loop' because, we have no direct connections with tanks.

Note that, if we consider direction of discharge in pipe ① or pipe ② along '→'-this direction, for loop I, discharge is negative (as counter-clockwise).

But for same direction of discharge, in loop (II), discharge is positive (as clockwise).

Friction losses in pipe system:-

Friction Head loss equation, $h_L = \frac{fLV^2}{D \cdot 2g}$ ...... (389)

where, V = average velocity.

Also, head loss equation can be written as,

$$h_L = KQ^{\beta} \text{ .... } (390)$$

where, β is a constant exponent.

[Note that, in (388.2), we used $\hat{k}$ but in (390) we used K.
$\hat{k}$ is the coff which considers the loss component in junction entrance or exit points. So, it is total loss thing.
But, in (390) it considers loss in pipe only.]

Comparing (389) and (390),

$$\frac{fLV^2}{D.2g} = KQ^{\beta}$$

$$\Rightarrow KQ^{\beta} = \frac{fL}{2gD} \times \left(\frac{Q}{A}\right)^2$$

$$= \frac{fL}{2gD} \times \frac{Q^2}{\frac{\pi^2}{16} D^4}$$

$$= \left(\frac{8}{\pi^2 g} \frac{fL}{D^5}\right) Q^2 \text{ ...... } (391)$$

Comparing LHS and RHS,

$$\beta = 2 \quad \text{and,} \quad K = \frac{8fL}{\pi^2 g D^5}$$

$\therefore \beta = 2$ for Darcey-Weisbach equation.

☑ Pipes in Series:-

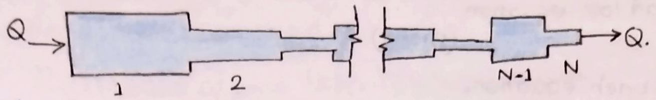


Fig: 47.5:


⇒ Open channel flow


⇒ Same pipe cross section in running-full condition, is called pipe flow.

Energy equation,

The total head loss between two end points (starting H_S, ending H_E), of the pipes connected in series,

[I.T ⇒ $\sum K$ looks like minor loss coefficient i.e $\left(\sum K \cdot \frac{v^2}{2g}\right)$]

$$H_E - H_S = \left(K_1 + \frac{\sum k}{2gA_1^2}\right)Q_1^2 + \cdots + \left(K_N + \frac{\sum k}{2gA_N^2}\right)Q_N^2$$

(I think; $h_L = H_S - H_E$)!?

$$\Rightarrow h_L = \sum_{i=1}^{N}\left(K_i + \frac{\sum k}{2gA_i^2}\right)Q_i^2 \cdots\cdots (392)$$

Continuity eqn,

$$Q_1 = Q_2 = \cdots = Q_i = \cdots = Q_N = Q \cdots (393).$$

☑ Pipes in parallel:-

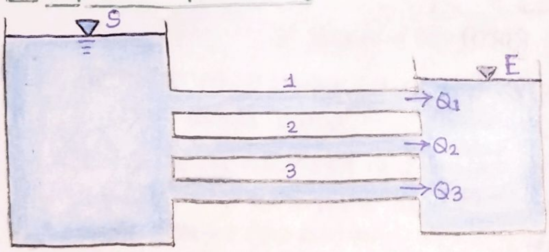


Fig 47.6:

Here different discharge, but same head loss for all pipes.

Energy equation,

$$H_E - H_S = \left(k_i + \frac{\sum k}{2gA_i^2}\right)Q_i^2, \ \forall i \in \{1, 2, \ldots N\} \cdots (394)$$

Continuity equation,

$$Q = \sum_{i=1}^{N} Q_i \cdots\cdots (395).$$

In (394), $\hat{k} = \left(k_i + \frac{\sum k}{2gA_i^2}\right) \cdots\cdots (396).$

(See-388.2)

I.T ⇒ [Here, k_i = for friction loss along pipe.
$\frac{\sum k}{2gA_i^2}$ = for loss due to junction, entrance, exit or other pipe fittings.
$\hat{K}$ = Considering all these above mentioned effects]

* Rember the difference among $\hat{K}$, K and k.

Pipes in Network:

For junction j in a pipe network, conservation of mass should be satisfied.

$$\sum_{i\in J_{in}^{j}} Q_i - \sum_{i\in J_{out}^{j}} Q_i = q_j \;\ldots\ldots (397)$$

where, q_j = External demand (withdrawl) from that node.

J_{in}^{j} = Set of pipes with inflow to the junction.

J_{out}^{j} = Set of pipes with outflow from the junction.

Conservation of energy;

$$H_E - H_S = \sum_{i\in Y} h_{(i)} = \sum_{i\in Y} k_i Q_i |Q_i|^{\beta-1} \ldots (398)$$

where, Y is the set of pipes along a path.

Explanation of (398):-

⊙ Let us take, this is our network and 'orange' coloured line is the path.

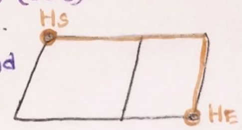


Fig: 47.7

Along this path, there should be difference of head. This head difference is nothing but total friction loss from the pipes.
(Because, in 398, it is k_i, not $\hat{k}$ or $\sum k$).
(see P-118, below eqn 396).

⊙ Let us consider figure 47.8.
If we follow this loop, HE and Hs would be same quantity.

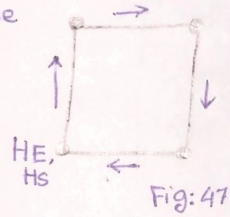


Fig: 47.8

∴ $H_E - H_S = 0$

∴ From (398),

$$0 = \sum_{i\in Y} h_{(i)} = \sum_{i\in Y} k_i Q_i |Q_i|^{\beta-1} \ldots (399)$$

∴ For a particular loop in the network, we should have a total head loss = 0

Pipes in network: Interior loop:

In a closed loop, total head loss, F_ℓ

$$F_\ell(Q) = \sum_{i\in Z^\ell} \hat{k}_i Q_i |Q_i|^{\beta-1} = 0 \ldots\ldots (400)$$

[Its $\hat{k}_i$, not k_i. Because, not only frictional head loss, but total head loss is zero].

#[Interior loop considers only pipes without any reservoir or pumps].

$$\text{In (400), } \hat{k}_i = k_i + \frac{\sum k}{2gA_i^2} \ldots (401).$$

#[In (400), modulus sign was used to get the actual direction of flow.]
[If we not used mod sign, we will not get a zero value in a particular loop].

#[# In 400, Z^ℓ is the set of all pipes in a particular loop 'ℓ'.]

From Taylor series expansion, # (Looks like N-R method for multiple variables. Or, $\left[x_2 = x_1 + \frac{dy}{dx}\Big|_{x_1} \times \Delta x\right]$ → from Calculus)

$$F_\ell(Q^{(P)}) = F_\ell(Q^{(P-1)}) + \sum_{i\in Z^\ell} \left(Q_i^{(P)} - Q_i^{(P-1)}\right) \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}} \cdots \quad (402)$$

I think,

[F_ℓ or total head loss is a function of discharge values in p^{th} iteration. Those discharges are unknown (Because, future iteration values)
∴ F_ℓ is a function of set of variables Z_ℓ. (Z_ℓ = Set of all pipes in a particular loop 'ℓ'). (i.e single function with multiple variables)]

Using (388.2), or (400),

$$F_\ell(Q^{(P)}) = \sum_{i\in Z^\ell} \hat{k}_i \left[Q_i^{(P-1)}\right]^\beta + \sum_{i\in Z^\ell} \left(Q_i^{(P)} - Q_i^{(P-1)}\right) \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}} \cdots \quad (403)$$

[In (402), it will be $\frac{\partial F_\ell}{\partial Q_i}$, not $\frac{dF_\ell}{dQ_i}$. Because, F_ℓ is a multivariate Function. $F_\ell = F_\ell(Q_i)$, where, $i \in Z^\ell$]

In Hardy-Cross method, we have an assumption, that,

$$Q_i^{(P)} - Q_i^{(P-1)} = \Delta Q_\ell \quad \forall i \in Z_\ell \cdots \quad (404)$$

[So, this increment is same for all pipes in a particular loop 'ℓ']

Putting this value is (402),

$$0 = F_\ell(Q^{(P-1)}) + \Delta Q_\ell \sum_{i\in Z^\ell} \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}} = 0$$

[ΔQ_ℓ is same. So, it comes out of summation.
$F_\ell(Q^{(P)}) = 0$, consider $Q^{(P)}$ is the solution set, for which head loss in the loop is zero]

$$\Rightarrow \Delta Q_\ell = -\frac{F_\ell(Q^{(P-1)})}{\sum_{i\in Z^\ell} \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}}} \quad (405)$$

[Remember, ΔQ_ℓ is applicable for a particular loop only]

In (405), derivative can be computed as,

$$\frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}} = \frac{\partial}{\partial Q_i}\left(\hat{k}_i Q_i^\beta\right)$$

$$= \beta \hat{k}_i Q_i^{\beta-1} \cdots \quad (406).$$

$$= \beta k_i |Q_i|^{\beta-1} \cdots$$

Now, Putting the values of F_ℓ from (406), $\sum \frac{\partial F_\ell}{\partial Q_i}$ from (405) into the equation (405),

$$\Delta Q_\ell = - \frac{\sum_{i \in Z^\ell} \hat{K}_i Q_i |Q_i|^{\beta-1}}{\sum_{i \in Z^\ell} \beta \hat{K}_i |Q_i|^{\beta-1}} \quad \cdots\cdots (407)$$

[If this increment is within loop for a particular pipe, we have to add it. If it is comming from another loop, we have to subtract it]

Pipes in Network: Pseudo loop

In pseudo loop, total head-loss considering head difference between two fixed grade nodes,

$$F_\ell(\underset{\sim}{Q}) = \sum_{i \in Z^\ell} \hat{K}_i Q_i |Q_i|^{\beta-1} + \Delta H = 0 \cdots\cdots (408).$$

where, $\hat{K}_i = K_i + \frac{\sum k}{2gA_i^2}$ (See-396)

From Taylor Series expansion,

$$F_\ell(\underset{\sim}{Q}^{(P)}) = F_\ell(\underset{\sim}{Q}^{(P-1)}) + \sum_{i \in Z^\ell} (Q_i^{(P)} - Q_i^{(P-1)}) \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}}$$

$$= \sum_{i \in Z^\ell} \hat{K}_i [Q_i^{(P-1)}]^\beta + \Delta H + \sum_{i \in Z^\ell} (Q_i^{(P)} - Q_i^{(P-1)}) \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}} \cdots\cdots (409)$$

Now, in Hardy-Cross method, we assume that

$$Q_i^{(P)} - Q_i^{(P-1)} = \Delta Q_\ell \quad \forall i \in Z^\ell$$

[Z^ℓ = Set of all pipes in a particular loop].

So, (409), can be written as,

$$0 = F_\ell(\underset{\sim}{Q}^{(P-1)}) + \Delta Q_\ell \sum_{i \in Z^\ell} \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}}$$

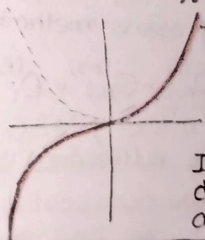
$$\Rightarrow \Delta Q_\ell = - \frac{F_\ell(\underset{\sim}{Q}^{(P-1)})}{\sum_{i \in Z^\ell} \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(P-1)}}} \quad \cdots\cdots (410).$$

From (408),

$$\frac{\partial}{\partial Q_i}(F_\ell)\Big|_{Q_i^{(P-1)}} = \frac{\partial}{\partial Q_i}\left(\sum_{i \in Z^\ell} \hat{K}_i Q_i |Q_i|^{\beta-1} + \Delta H\right)$$

$$= \sum_{i \in Z^\ell} \beta \hat{K}_i |Q_i|^{\beta-1} \quad \cdots\cdots (411).$$

(But, if β value is other than 2, can we take mod as derivative?!)



→ Curve of $x|x|$. Which is always increasing with x. So, I think it's derivative is always positive

why $\frac{\partial}{\partial Q}(Q|Q|^{\beta-1}) = \beta |Q|^{\beta-1}$ → Function may be positive or negative.

→ Derivative is always positive. (Provided $\beta > 1$).

[$\frac{\partial}{\partial Q_i}(\Delta H) = 0$, Because, ΔH is a fixed quantity, which represents difference in elevation of two fixed tanks]

Now, Putting values from (408) and (411) in equation (410), Discharge correction in lth loop can be calculated as,

$$\Delta Q_l = -\frac{\sum_{i\in Z^l} \hat{R}_i Q_i |Q_i|^{\beta-1} + \Delta H}{\sum_{i\in Z^l} \beta \hat{R}_i |Q_i|^{\beta-1}} \quad \cdots (412)$$

Pipes in network: Pseudo loop with pump:-

In pseudo loop, total head loss (F_l) considering head difference between two fixed grade nodes and pump, can be calculated as,

$$F_l(Q, Q_{P,l}) = \sum_{i\in Z^l} \hat{R}_i Q_i |Q_i|^{\beta-1} - (H_P)_l + \Delta H = 0 \cdots (413)$$

[I.T ⇒ $Q_{P,l}$ means Discharge of pump in lth loop.]

Where, $\hat{R}_i = K_i + \frac{\Sigma k}{2gA_i^2}$

and $(H_P)_l = a_0 + a_1 . Q_{P,l} + a_2 . Q_{P,l}^2 \cdots (414)$

[Here, $(H_P)_l$ does not mean head loss. It is head delivered by the pump. If discharge through pump is more, more head is generated.]

From Taylor's series expansion:-

$$F_l(Q^{(P)}, Q_{P,l}^{(P)}) = F_l(Q^{(P-1)}, Q_{P,l}^{(P-1)}) + \underbrace{\sum_{i\in Z^l}(Q_i^{(P)} - Q_i^{(P-1)}) \frac{\partial F_l}{\partial Q_i}\Big|_{Q_i^{(P-1)}}}_{\text{all pipe's terms}} + \underbrace{(Q_{P,l}^{(P)} - Q_{P,l}^{(P-1)}) \frac{\partial F_l}{\partial Q_{P,l}}\Big|_{Q_{P,l}^{(P-1)}}}_{\text{Pump. term}} \cdots (415)$$

[If there are N number of pipes in the set of pipes in the loop, Total number of unknowns/variables to calculate F_l is $= N+1$ [See, in 415, total (N+1) no. 1st order derivatives are there] (∵ 'N' no. of discharge for N pipes, one extra discharge for pump)]

For Hardy-cross method, we assume that,

$$Q_{P,l}^{(P)} - Q_{P,l}^{(P-1)} = Q_i^{(P)} - Q_i^{(P-1)} = \Delta Q_l \quad \forall i \in Z^l \cdots (416)$$

P,l → Pump in loop. ; i = pipe numbering

[i.e. difference of discharges for two consecutive iterations for pump and all the pipes are equal]

So, using (416), eq^n (415) can be written as,

$$0 = F_l(Q^{(P-1)}, Q_{P,l}^{(P-1)}) + \Delta Q_l \sum_{i\in Z^l} \frac{\partial F_l}{\partial Q_i}\Big|_{Q_i^{(P-1)}} + \Delta Q_l \frac{\partial F_l}{\partial Q_{P,l}}\Big|_{Q_{P,l}^{(P-1)}}$$

Pipes are in numbers. So, 'Σ' used. Pump is only one.

$$\Rightarrow \Delta Q_\ell = -\frac{F_\ell\left(Q^{(p-1)}, Q_{p,\ell}^{(p-1)}\right)}{\sum_{i\in Z^\ell} \frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(p-1)}} + \frac{\partial F_\ell}{\partial Q_{p,\ell}}\Big|_{Q_{p,\ell}^{(p-1)}}} \quad \cdots\cdots (417)$$

[Now, we need to calculate the derivative values and put those into (417) to get final form of ΔQ_ℓ]

$$\frac{\partial F_\ell}{\partial Q_i}\Big|_{Q_i^{(p-1)}} = \frac{\partial}{\partial Q_i}\left(\hat{k}_i Q_i^{\beta}\right)$$
$$= \beta \hat{k}_i Q_i^{\beta-1}$$
$$= \beta \hat{k}_i |Q_i|^{\beta-1}. \quad \cdots\cdots (418)$$

(why modulus!? See page 121).

$$\frac{\partial F_\ell}{\partial Q_{p,\ell}}\Big|_{Q_{p,\ell}^{(p-1)}} = -\frac{\partial}{\partial Q_{p,\ell}}\left(a_0 + a_1 Q_{p,\ell} + a_2 Q_{p,\ell}^2\right) \cdots (418.2)$$
$$= -(a_1 + 2a_2 Q_{p,\ell})$$
$$= -(a_1 + 2a_2 |Q_{p,\ell}|) \rightarrow \text{why, here is modulus sign?}$$
$$(419)$$

why $F_\ell = -(a_0 + a_1 Q_{p,\ell} + a_2 Q_{p,\ell}^2)$? (in 418.2).

Ans: I think, For a particular loop, total change in head should be zero.

$\therefore$ Total head loss in the loop + Head generated in the loop = 0

$$\Rightarrow F_\ell + (H_p)_\ell = 0 \quad \cdots\cdots\cdots\cdots (420)$$

From (414), $(H_p)_\ell = a_0 + a_1 \cdot Q_{p,\ell} + a_2 Q_{p,\ell}^2$

$\therefore F_\ell = -(H_p)_\ell$

$= -(a_0 + a_1 Q_{p,\ell} + a_2 Q_{p,\ell}^2)$

This is used to derive equation (413).

? [But, see in (413), $F_\ell = 0$ is taken, which does not match with (420)]

We can evaluate ΔQ_ℓ using (417).

by putting values from (413), (414), (418), (419),

Q এর জায়গায় |Q|
Q^2 এর জায়গায় Q|Q| } কেন?

$$\Delta Q_\ell = -\frac{\sum_{i\in Z^\ell} \hat{k}_i Q_i |Q_i|^{\beta-1} - (a_0 + a_1 |Q_{p,\ell}| + a_2 Q_{p,\ell} |Q_{p,\ell}|) + \Delta H}{\sum_{i\in Z^\ell} \beta \hat{k}_i |Q_i|^{\beta-1} - (a_1 + 2a_2 |Q_{p,\ell}|)} \cdots (420).$$

Hardy-Cross Method

Steps :-

- Assume an intial flow distribution in network that satisfies,

$$\sum_{i\in J_{in}^j} Q_i - \sum_{i\in J_{out}^j} Q_i = q_j \quad \cdots\cdots (421)$$

- If the values of initial estimates are closer, we need fewer iterations.
- Q will decrease for higher $\hat{k}$.

⊙ Determine ΔQ_ℓ for each path or loop using appropriate equations.

[Path term is applicable for Pseudo loops.]

⊙ Adjust the flows in each path element in all loops and paths using the relation,

$$\underbrace{Q_{\ell,i}^{(p)}}_{\text{(Updated value)}} = \underbrace{Q_{\ell,i}^{(p-1)}}_{} + \Delta Q_\ell - \sum_{\forall k \setminus \{\ell\} : i \in Z^k} \Delta Q_k \quad \text{.....(422)}$$

(Updated value)

↓ (Previous iteration value for ℓ^{th} loop only.)

↓ (This term should be subtracted for contribution from the other loop.)

I.T ⇒ [$Q_{\ell,i}$ means ⇒ Discharge for i^{th} pipe of ℓ^{th} loop.

$\forall k \setminus \{\ell\} : i \in Z^k$ ⇒ Means, pipe number 'k' does not belongs to ℓ^{th} loop.

The set of those pipes is represented by Z^k]

- Repeat last two steps untill a desired accuracy is reached.

Configuration 1:

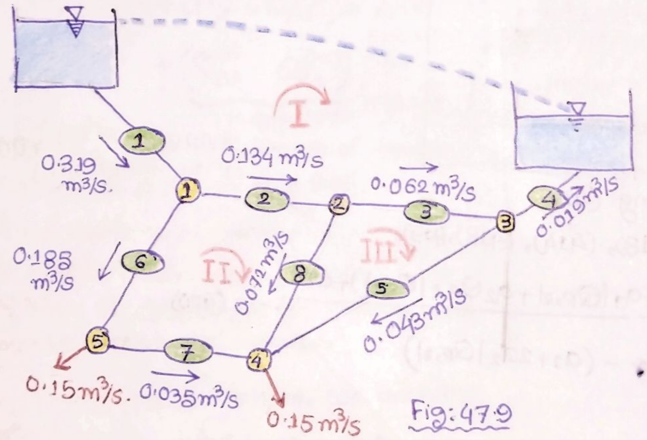


Fig: 47.9

So, we have external demands at 4 and 5 junctions no.

Maintaining continuity equation at each junctions, we need to specify initial discharge values here. These are arbitrary discharge values. You may change the intial pipe discharge values in figure 47.9 and as well as the signs. also. But, finally after iterations, you will get the same result.

Pipe numbers	Loop numbers I	II	III
1	−		
2	−	+	
3	−		+
4	−		
5			+
6		−	
7		−	
8		+	−

[clockwise positive, anti-clockwise negative. We need to impose these positive-negative informations for individual loop-specific calculations.]

Loop Connectivity for configuration 1

$$
loop_{con} = \begin{array}{c|c|cccc}
\text{Loop number} & \text{Number of pipes connected to that loop.} & \multicolumn{4}{c}{\text{Pipe numbers connected to that loop (with +ve/-ve sign of discharge direction)}} \\
1 & 4 & -1 & -2 & -3 & -4 \\
2 & 4 & 2 & 8 & -7 & -6 \\
3 & 3 & 3 & 5 & -8 & 0
\end{array} \quad \ldots (423)
$$

NPTEL-49
M-5: Flows in Pressurized Conduits
U-2: Unsteady Flows in pipes

[In this problem, we will consider 1D space variation. i.e (x,t). We will not consider 2D or radial direction in our calculations]

[We have continuity and momentum equations here and we will discretize it with Finite volume approach or Godunov Scheme. Resulting thing can be solved using Predictor-corrector approach]

Problem statement:

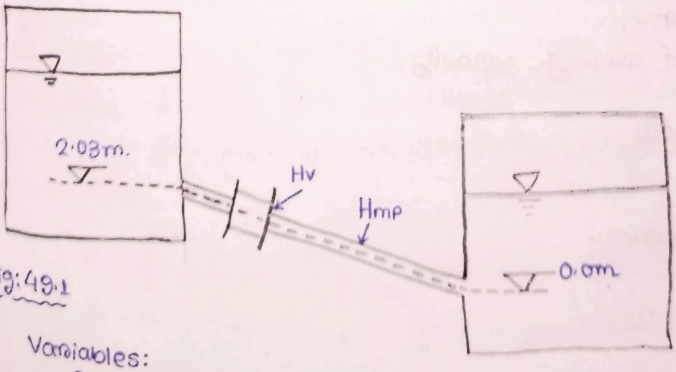


Fig: 49.1

Variables:

Piezometric head, $H(x,t)$

velocity $V(x,t)$.

Governing Equations:-

1-D unsteady flow in pipes can be represented as in terms of following differential eqns:-

$$\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + \frac{a^2}{g}\frac{\partial V}{\partial x} = 0 \quad \cdots\cdots (424)$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} + J = 0 \quad \cdots\cdots (425).$$

$H = H(x,t)$ = Piezometric head.

a = Wave speed.

V = Cross sectional mean velocity.

J = Friction force at the pipe wall per unit mass.

t = time.

[Equations (424) and (425) are called non-conservative form.]

Governing eqns (424) and (425) can be written in non-conservative form as:-

$$\underset{\sim}{U}_{,t} + \underset{\sim}{A}(\underset{\sim}{U})\underset{\sim}{U}_{,x} = \underset{\sim}{S} \quad \cdots\cdots (426)$$

where,

$$\underset{\sim}{U} = \begin{bmatrix} H \\ V \end{bmatrix}, \quad \underset{\sim}{A} = \begin{bmatrix} V & a^2/g \\ g & V \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ -J \end{bmatrix} \quad \cdots\cdots (426.1)$$

In short, (426), can be represented as,

$$\frac{\partial \underset{\sim}{U}}{\partial t} + \underset{\sim}{A}\frac{\partial \underset{\sim}{U}}{\partial x} = \underset{\sim}{S} \quad \cdots\cdots (426.2)$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial H}{\partial t} \\ \frac{\partial V}{\partial t} \end{Bmatrix} + \begin{bmatrix} V & a^2/g \\ g & V \end{bmatrix} \begin{Bmatrix} \partial H/\partial x \\ \partial V/\partial x \end{Bmatrix} = \begin{Bmatrix} 0 \\ -J \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial H}{\partial t} \\ \frac{\partial V}{\partial t} \end{Bmatrix} + \begin{Bmatrix} V\frac{\partial H}{\partial x} + \frac{a^2}{g}\frac{\partial V}{\partial x} \\ g\frac{\partial H}{\partial x} + V\frac{\partial V}{\partial x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -J \end{Bmatrix}$$

From this, get (424) & (425)

The non-conservative form of (426) can be converted to conservative form as,

$$\underset{\sim}{U}_{,t} + \underset{\sim}{F}_{,x} = \underset{\sim}{S}(\underset{\sim}{U}) \quad \cdots\cdots (427)$$

where, $\underset{\sim}{F} = \bar{\underset{\sim}{A}}\,\underset{\sim}{U} \quad \cdots\cdots (428)$

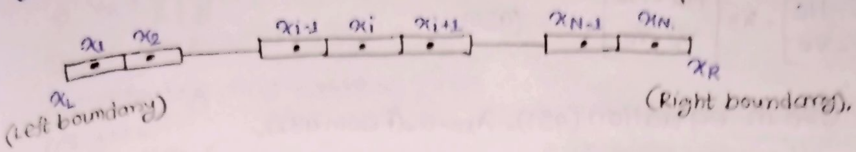
and A is te in terms of average velocity.

$$\bar{A} = \begin{bmatrix} \bar{V} & a^2/g \\ g & \bar{V} \end{bmatrix} \quad \cdots\cdots (429)$$

If in (429), $\bar{V} = 0$, it yields classical water-hammer equation.

Discretization:-

GE is discretized using FVM.



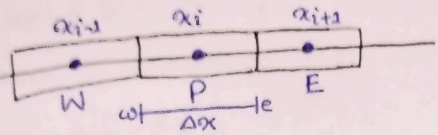


Fig: 49.2

The discretized form can be written as,

$$\frac{d\underset{\sim}{U}}{dt} = \frac{F_w - F_e}{\Delta x} + \frac{1}{\Delta x}\int_{x_w}^{x_e} \underset{\sim}{S}\, dx \quad \cdots\cdots (430).$$

> Eq^n (430):-
>
> From (427),
>
> $$\frac{\partial \underset{\sim}{U}}{\partial t} + \frac{\partial \underset{\sim}{F}}{\partial x} = \underset{\sim}{S}(\underset{\sim}{U})$$
>
> $$\Rightarrow \frac{\partial \underset{\sim}{U}}{\partial t} + \frac{F_e - F_w}{\Delta x} = \frac{1}{\Delta x}\int_{x_w}^{x_e} \underset{\sim}{S}\, dx$$
>
> I.T $\Rightarrow$ RHS represents average value of Source/sink term. $\Rightarrow$ (See 460 and 461)
>
> $$\Rightarrow \frac{\partial \underset{\sim}{U}}{\partial t} = \frac{\underset{\sim}{F_w} - \underset{\sim}{F_e}}{\Delta x} + \frac{1}{\Delta x}\int_{x_w}^{x_e} \underset{\sim}{S}\, dx$$
>
> This is (430).

Riemann Problem:

conservative form:-

Riemann problem for 1D conservation law.

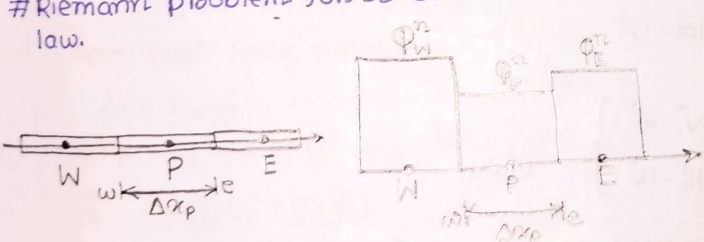


Fig: 49.3

$U_L^n \underset{x_e}{|} U_R^n$ (# From 431)

$$\frac{\partial \underset{\sim}{U}}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad \cdots\cdots (431).$$

$$\underset{\sim}{U}(x,t) = \begin{cases} U_L^n & \text{if } x < x_e \\ U_R^n & \text{if } x > x_e. \end{cases}$$

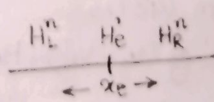
The eigenvalues of matrix $\underset{\sim}{\bar{A}}$ (See-429) are,

$\lambda_1 = \bar{V} - a$ and $\lambda_2 = \bar{V} + a \quad \cdots\cdots (432)$ #(can be obtained from $|\underset{\sim}{\bar{A}} - \lambda I| = 0$)

Applying Rankine-Hugoniot conditions → what is it?

across λ_1,

$$\begin{bmatrix} \bar{V} & \frac{a^2}{g} \\ g & \bar{V} \end{bmatrix} \begin{bmatrix} H_e - H_L^n \\ V_e - V_L^n \end{bmatrix} = \lambda_1 \begin{bmatrix} H_e - H_L^n \\ V_e - V_L^n \end{bmatrix} \quad \cdots (433)$$



Applying Rankine-Hugoniot conditions across λ_2,

$$\begin{bmatrix} \bar{V} & \frac{a^2}{g} \\ g & \bar{V} \end{bmatrix} \begin{bmatrix} H_R^n - He \\ V_R^n - Ve \end{bmatrix} = \lambda_2 \begin{bmatrix} H_R^n - He \\ V_R^n - Ve \end{bmatrix} \quad \dots\dots (434)$$

If we consider $\bar{V}=0$ in equation (433), $\lambda_1 = -a$, (From 432), we have,
(Fast face value=0!?)
(Time=9:19)

$$\begin{bmatrix} 0 & \frac{a^2}{g} \\ g & 0 \end{bmatrix} \begin{Bmatrix} He - H_L^n \\ Ve - V_L^n \end{Bmatrix} = -a \begin{Bmatrix} He - H_L^n \\ Ve - V_L^n \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{a^2}{g}(Ve - V_L^n) \\ g(He - H_L^n) \end{Bmatrix} = -a \begin{Bmatrix} He - H_L^n \\ Ve - V_L^n \end{Bmatrix} \quad \dots (435)$$

From (435), we get,

$$\frac{a^2}{g}(Ve - V_L^n) = -a(He - H_L^n) \Rightarrow \frac{a}{g}(Ve - V_L^n) + (He - H_L^n) = 0 \quad \dots (436.1)$$

and $g(He - H_L^n) = -a(Ve - V_L^n) \Rightarrow \frac{a}{g}(Ve - V_L^n) + (He - H_L^n) = 0 \quad \dots (436.2)$

$\therefore$ Equations (436.1) and (436.2) both are same.

Similarly, using (434), putting $\bar{V}=0$ and $\lambda_2 = +a$, we can obtain,

$$\frac{a}{g}(Ve - V_R^n) - (He - H_R^n) = 0 \quad \dots\dots (437).$$

$\boxed{\text{With this Rankine-Hugunoit condition (433 and 434), we can write our governing equations for a particular face using Rimenn Problem.}}$

For all internal cell P, the following solution can be written,

$$\underset{\sim}{Ve}(t) = \begin{bmatrix} He \\ Ve \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (H_L^n + H_R^n) + \frac{a}{g}(V_L^n - V_R^n) \\ (V_L^n + V_R^n) + \frac{a}{g}(H_L^n - H_R^n) \end{bmatrix} \quad \dots (438)$$

$$= \underset{\sim}{B}\, \underset{\sim}{U}_L^n + \underset{\sim}{C}\, \underset{\sim}{U}_R^n \quad \dots (439)$$

where,
$$\left.\begin{aligned} \underset{\sim}{B} &= \frac{1}{2}\begin{bmatrix} 1 & \frac{a}{g} \\ \frac{g}{a} & 1 \end{bmatrix} \\ \underset{\sim}{C} &= \frac{1}{2}\begin{bmatrix} 1 & -a/g \\ -\frac{g}{a} & 1 \end{bmatrix} \end{aligned}\right\} \dots\dots (440)$$

Derive (438), (439):-

To get He and Ve, we need to solve two equations (436.1) and (437).

From (437),

$$He = \frac{a}{g}(Ve - V_R^n) + H_R^n \quad \dots (441)$$

Putting this value in (436.1),

$$\frac{a}{g}(Ve - V_L^n) + \frac{a}{g}(Ve - V_R^n) + H_R^n - H_L^n = 0$$

$$\Rightarrow \frac{a}{g} \times 2V_e = (H_L^n - H_R^n) + \frac{a}{g}(V_L^n - V_R^n)$$

$$\Rightarrow V_e = \frac{1}{2}\left[\frac{g}{a}(H_L^n - H_R^n) + (V_L^n - V_R^n)\right]$$

This is 2nd row of (438).

Substituating this value in (441), we get,

$$H_e = \frac{1}{2}\left[(H_L^n + H_R^n) + \frac{a}{g}(V_L^n - V_R^n)\right]$$

Now, let us expand (439),

$$\underset{\sim}{B}\underset{\sim}{U}_L^n + \underset{\sim}{C}\underset{\sim}{U}_R^n = \frac{1}{2}\begin{bmatrix} 1 & \frac{a}{g} \\ \frac{g}{a} & 1 \end{bmatrix}\begin{Bmatrix} H_L^n \\ V_L^n \end{Bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -\frac{a}{g} \\ -\frac{g}{a} & 1 \end{bmatrix}\begin{Bmatrix} H_R^n \\ V_R^n \end{Bmatrix}$$ (# From 440 value of B and C)

For 1st row,

$$= \frac{1}{2}(H_L^n + \frac{a}{g} \cdot V_L^n) + \frac{1}{2}(H_R^n - \frac{a}{g}V_R^n)$$

$$= \frac{1}{2}\left[(H_L^n + H_R^n) + \frac{a}{g}(V_L^n - V_R^n)\right]$$

$= H_e$ [# comparing with (438)]

Similarly, for 2nd row,

$$= \frac{1}{2}(\frac{g}{a}H_L^n + V_L^n) + \frac{1}{2}(-\frac{g}{a}H_R^n + V_R^n)$$

$$= \frac{1}{2}\left[(V_L^n + V_R^n) + \frac{g}{a}(H_L^n - H_R^n)\right]$$

$= V_e$ [# comparing with (438)]

∴ We can say,

$$\underset{\sim}{B}\underset{\sim}{U}_L^n + \underset{\sim}{C}\underset{\sim}{U}_R^n = \begin{Bmatrix} H_e \\ V_e \end{Bmatrix} = \underset{\sim}{U}_e \text{ (Proved).}$$

We know, general form of flux,

$$\underset{\sim}{F} = \bar{\underset{\sim}{A}}\underset{\sim}{U} \quad \text{\# [From (428)].}$$

∴ For east face value,

$$\underset{\sim}{F}_e = \bar{\underset{\sim}{A}}_e \underset{\sim}{U}_e \quad \cdots\cdots (442)$$

Putting the value of $\underset{\sim}{U}_e$ from (439),

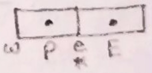
$$\underset{\sim}{F}_e = \bar{\underset{\sim}{A}}_e(\underset{\sim}{B}\underset{\sim}{U}_L^n + \underset{\sim}{C}\underset{\sim}{U}_R^n)$$

$$= \bar{\underset{\sim}{A}}_e\underset{\sim}{B}\underset{\sim}{U}_L^n + \bar{\underset{\sim}{A}}\underset{\sim}{C}\underset{\sim}{U}_R^n \quad \cdots\cdots (443)$$

In (443),

$\bar{\underset{\sim}{A}}_e$ can be calculated by, # $\left\{\bar{A} = \begin{bmatrix} \bar{V} & a^2/g \\ g & \bar{V} \end{bmatrix}, \text{ from (429)}\right\}$

- Approach 1:- By setting $\bar{V} = 0$
- Approach 2:- By setting $\bar{V} = \frac{1}{2}(V_P^n + V_E^n)$ #



- Approach 3: By setting $\bar{V} = \bar{V}_e$ from Reimann solution. (J.T ⇒ From (438)) ..... (444)

☑ First order Godunov approach:

Interior Cells: In first order Godunov approximation,

$$\underset{\sim}{U}_L^n = \underset{\sim}{U}_P^n \text{ and } \underset{\sim}{U}_R^n = \underset{\sim}{U}_E^n \quad \cdots (445)$$

∴ Substituating these values in (443), numerical flux can be calculated

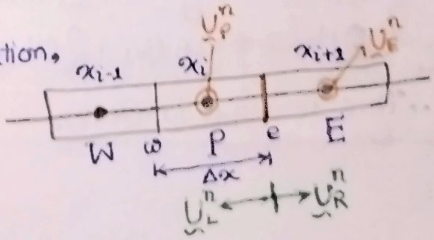
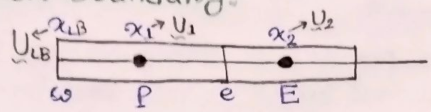


Fig: 49.4

as, at east face as, (use 443).

$$\underset{\sim}{J_e} = \underset{\sim}{\bar{A}_e}\, \underset{\sim}{B}\, \underset{\sim}{U_P^n} + \underset{\sim}{\bar{A}_e}\, \underset{\sim}{C}\, \underset{\sim}{U_E^n} \quad \ldots\ldots (446)$$

Left boundary:-



$$\underset{\sim}{U_{LB}} = \begin{Bmatrix} H_{LB} \\ V_{LB} \end{Bmatrix} \quad \ldots\ldots (447)$$

The Riemann invarient associated with negative characteristics line,

$$H_{LB} - \frac{a}{g} V_{LB} = H_1^n - \frac{a}{g} V_1^n \quad \ldots\ldots (448)$$

How this came?

$$\underset{\sim}{J_{LB}} = \underset{\sim}{\bar{A}_{LB}}\, \underset{\sim}{U_{LB}}$$

$$= \begin{Bmatrix} \bar{V}_{LB} H_{res} + \frac{a^2}{g}\left(V_1^n + \frac{g}{a}(H_{res} - H_1^n)\right) \\ g H_{res} + \bar{V}_{LB}\left(V_1^n + \frac{g}{a}(H_{res} - H_1^n)\right) \end{Bmatrix} \quad \ldots\ldots (449)$$

Equation (449):-

For left boundary condition,

$$H_{LB} = H_{res} \quad (\text{\# Reservoir head}) \ldots (450)$$

Putting this on (448),

$$H_{res} - \frac{a}{g} V_{LB} = H_1^n - \frac{a}{g} V_1^n$$

$$\Rightarrow \frac{a}{g} V_{LB} = (H_{res} - H_1^n) + \frac{a}{g} V_1^n$$

$$\Rightarrow V_{LB} = V_1^n + \frac{g}{a}(H_{res} - H_1^n) \quad \ldots\ldots (451)$$

$$\therefore \underset{\sim}{J_{LB}} = \underset{\sim}{\bar{A}_{LB}}\, \underset{\sim}{U_{LB}}$$

$$= \begin{bmatrix} \bar{V}_{LB} & \frac{a^2}{g} \\ g & \bar{V}_{LB} \end{bmatrix} \begin{Bmatrix} H_{LB} \\ V_{LB} \end{Bmatrix}$$

$$= \begin{bmatrix} \bar{V}_{LB} & \frac{a^2}{g} \\ g & \bar{V}_{LB} \end{bmatrix} \begin{Bmatrix} H_{res} \\ V_{LB} \end{Bmatrix}$$

[# Putting value of $\underset{\sim}{\bar{A}_{LB}}$ from (429) and H_{LB} from (450)]

$$= \begin{Bmatrix} \bar{V}_{LB} \cdot H_{res} + \frac{a^2}{g} V_{LB} \\ g H_{res} + \bar{V}_{LB} \cdot V_{LB} \end{Bmatrix}$$

[Note that, V_{LB} and $\bar{V}_{LB}$ are not same here. Value of V_{LB} has been put from (451) to get the final form (i.e. 449)]

What is the difference between $\bar{V}_{LB}$ and V_{LB}?

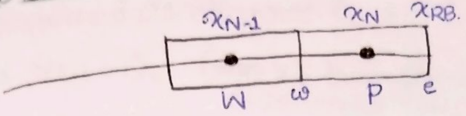
$$= \begin{Bmatrix} \bar{V}_{LB} \cdot H_{res} + \frac{a^2}{g}\left(V_1^n + \frac{g}{a}(H_{res} - H_1^n)\right) \\ g H_{res} + \bar{V}_{LB}\left(V_1^n + \frac{g}{a}(H_{res} - H_1^n)\right) \end{Bmatrix}$$

$\therefore$ This is equation (449).

If advective term is neglected,

?! then $V_{LB}=0$ . . . . (452)

Right Boundary:



$$\underset{\sim}{U}_{RB}=\begin{Bmatrix}H_{RB}\\V_{RB}\end{Bmatrix} \cdots\cdots (453)$$

The Riemann invarient associated with positive characteristics line,

$$H_{RB}+\frac{a}{g}V_{RB}=H_N^n+\frac{a}{g}V_{N}^n \cdots (454) \rightarrow \text{see (448)}!?$$

Now, for fully closed valve, we have,

$$V_{RB}=0 \cdots\cdots (455).$$

Putting, $V_{RB}=0$ in (454),

$$H_{RB}=H_N^n+\frac{a}{g}V_N^n \cdots (456).$$

$$\underset{\sim}{J}_{RB}=\underset{\sim}{\bar{A}}_{RB}\,\underset{\sim}{U}_{RB}$$

$$=\begin{bmatrix}\bar{V}_{RB} & a^2/g\\ g & \bar{V}_{RB}\end{bmatrix}\begin{Bmatrix}H_{RB}\\V_{RB}\end{Bmatrix}$$

$$=\begin{bmatrix}\bar{V}_{RB} & a^2/g\\ g & \bar{V}_{RB}\end{bmatrix}\begin{Bmatrix}H_N^n+\frac{a}{g}V_N^n\\ 0\end{Bmatrix}$$

[# H_{RB} and V_{RB} values are from (455) and (456)]

$$=\begin{Bmatrix}\bar{V}_{RB}(H_N^n+\frac{a}{g}V_N^n)\\ gH_N^n+aV_N^n\end{Bmatrix} \cdots\cdots (457).$$

☐ Numerical Discretization:

⊙ In absence of friction, FVM yeilds,

$$\underset{\sim}{U}_P^{n+1}=\underset{\sim}{U}_P^n-\frac{\Delta t}{\Delta x}\left[\underset{\sim}{\bar{F}}(x_e,t)-\underset{\sim}{\bar{F}}(x_w,t)\right] \cdots (458)$$

? [In (446), (449), (457) all were $\underset{\sim}{F}$. But Here $\underset{\sim}{\bar{F}}$ is used. what is the difference of $\underset{\sim}{F}$ and $\underset{\sim}{\bar{F}}$?]

[Equation (458), is discretized form of equation (431) (i.e, Riemann equation conservative form).]

T.T⇒ [In absence of friction, energy conservation is applicable. So, Riemann conservative form can be used. No source/sink term]

⊙ In presence of friction, two step approach is adopted

First step:- $\underset{\sim}{\bar{U}}_P^{n+1}=\underset{\sim}{U}_P^n-\frac{\Delta t}{\Delta x}\left[\underset{\sim}{\bar{F}}(x_e,t)-\underset{\sim}{\bar{F}}(x_w,t)\right] \cdots (459)$ (# Same form as 458)

Second step:- $\underset{\sim}{\bar{\bar{U}}}_P^{n+1}=\underset{\sim}{\bar{U}}_P^{n+1}+\frac{\Delta t}{2}\underset{\sim}{S}(\underset{\sim}{\bar{U}}_P^{n+1}) \cdots\cdots (460)$ → [I.T⇒ It is same as $\frac{\partial \underset{\sim}{U}}{\partial t}=\frac{\underset{\sim}{S}}{2}$. Here, $\underset{\sim}{S}(\underset{\sim}{U})$ means $\underset{\sim}{S}$ is function of $(\underset{\sim}{U})$, not multiplication (See 427 and below 430)]

Final step:- $\underset{\sim}{U}_P^{n+1}=\underset{\sim}{\bar{U}}_P^{n+1}+\frac{\Delta t}{2}\underset{\sim}{S}(\underset{\sim}{\bar{\bar{U}}}_P^{n+1}) \cdots\cdots (461).$

How?

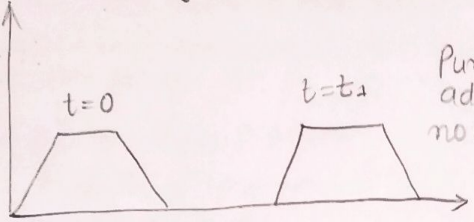
?

Stability criteria:

$$C_r = a\frac{\Delta t}{\Delta x} \leq 1 \quad \cdots\cdots (462)$$

It is expliet, first order approach (Eqns 459, 460, 461 confirms it is explicit and Eqn 427 shows it is 1st order). So, stability condition is required.

Purely advection Problem → without any loss of information!



But if dispersion occurs, then,

This type of shape change may occur.

☑ **FVM: conservation law:-**

If there is change in values

After page-33:

I think, there, 1st changnel was considered as channel 1, not 2,

So, $Q_{1,1} = Q_u$ (for u/s boundary) discharge boundary condition.

$\therefore UB_{1,1} = Q_u - Q_{1,1}$.

$$\frac{\partial(UB_{1,1})}{\partial y_{1,1}} = 0$$

and $\dfrac{\partial(UB_{1,1})}{\partial Q_{1,1}} = -1$

From Netwon's-Raphson's method,

$$[F(\phi)] = -[\Delta\phi]\,[J]$$

$$\Rightarrow UB_{1,1} = -\Delta Q_{1,1} \cdot \frac{\partial(UB_{1,1})}{\partial Q_{1,1}}$$

$$\Rightarrow UB_{1,1} = -\Delta Q_{1,1}.$$

$ Equation (~~15.4~~) (154.1)

I think, this should not be $(UB_{2,Ne+1})$