Geohydraulics| CE60113

Lecture 1

Overview

Geohydraulics
Applied hydrology

Geohydraulics deal with ⇒
(i) consumption
(ii) Pollution - Contamination - Remediation
(iii) Geotechnical application
(iv) Mining
(v) Hydro-geothermal energy.

Geohydraulics and groundwater hydraulics:-

Geohydraulics → Fluid movement in porous media, soil, rocks and other geological formations. (All geologic materials)
Involves analysis of physical properties ⇒ Porosity/ Permeability how they effect fluid flow.

Groundwater hydraulics:
↳ (Part of fluid mechanics & focus on physical principles).
Specialized field.
Flow through groundwater only.
Water movement ⇒ Sub surface soil and rocks, Contaminant transport etc.

Groundwater hydrology:-

Boarder field than GW hydraulics
groundwater occurance, distribution movement, quality of GW in aquifers

GW hydrologists use mathematical models to simulate the groundwater systems and to predict the impact of human activities on groundwater resources.

Hydrogeology:

Study of properties, distribution and movement of water in subsurface including interaction between SW and GW.

Involves application of geological and hydrological principles of GW systems, aquifers, recharge areas and water quality.
Field obserations, drilling, geophysical surveys, modelling, characterize and quantify groundwater resources

Geohydrology:

Boarder field includes hydrogeology and also includes the study of interaction between water and the geological materials that surround it.
physical and chemical properties of rocks, soils and other subsurface materials are studied to know how they affect GW movement and storage.

Difference between
geohydraulics and geological fluid dynamics:

Geohydraulics ⇒ Fluid movement through porous soil, rock and geological formation.
To study how porosity, permeability affect fluid flow through them.

Geological fluid dynamics:-
More general field,
Fluid behaviour in earth crust, mantle.
fluid like magma, hydrothermal fluids are also included.
It involves study of fluid flow, mixing, transport in geological materials in wide range of scales from individual pores and fractures to entire tectonic plates.
So, GFD includes wide range of fluid processes in the earth's crust and mantle.

☑ How much GW can be extracted from a certain aquifer?

- Aquifer anlaysis
- Catchment delination
- Well head protection
- Sustainability.

Well-head protection:-
⇒ Measures and strategies used to prevent contamination of groundwater at its source, i.e. wellhead.
Includes ⇒ Land use control ⇒ To prohibit certain type of landuse, maintain distance between potential source of contamination and the well head.

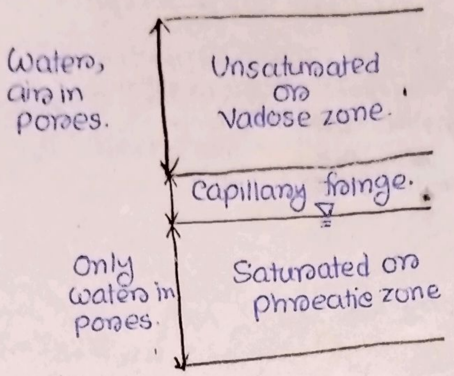
Management Practices ⇒ Minimize use of hazardous materials to properly manage and dispose of waste.

Monitoring activities ⇒ Regular testing of groundwater quality

Lect-1&2

WATER IN SUBSURFACE:

Identify subsurface zones.
To use hydrologic balance equation.
Characterize groundwater hydrograph.



Extinction Depth:-

At this depth, the hydraulic conductivity (K) of the aquifer becomes too low. So, rate of GW flow towards the well is negligible, well can no longer effectively withdraw water from aquifer.

If water pumped too much or too long ⇒ Reaches extinction depth → decline GWL in aquifer and damage to well.

Water injected to aquifer (artificial recharge) to increase hydraulic conductivity and delay the onset of extinction depth.
(সূচনা)

☑ Overland flow:-

Interflow:- Flow of water through vadose/ unsaturated zone.

Baseflow:- Aquifer flow.

Consolidation Problem:-

Occurs when aquifer is overexploited (Pumping rate > Natural recharge rate).

→ water pressure in aquifer pores and fractures decreases. → Land surface subside.

Consequences:- (i) Infrastructure damage.
(ii) Flow direction of GW changes.
(iii) In coastal areas, risk of flooding due to sea level rise.

Management:-

To prevent consolidation ⇒
(i) Reduce water use in dry periods
(ii) Recycle, reuse water
(iii) Alternative source:- Rainwater harvesting, desalination.

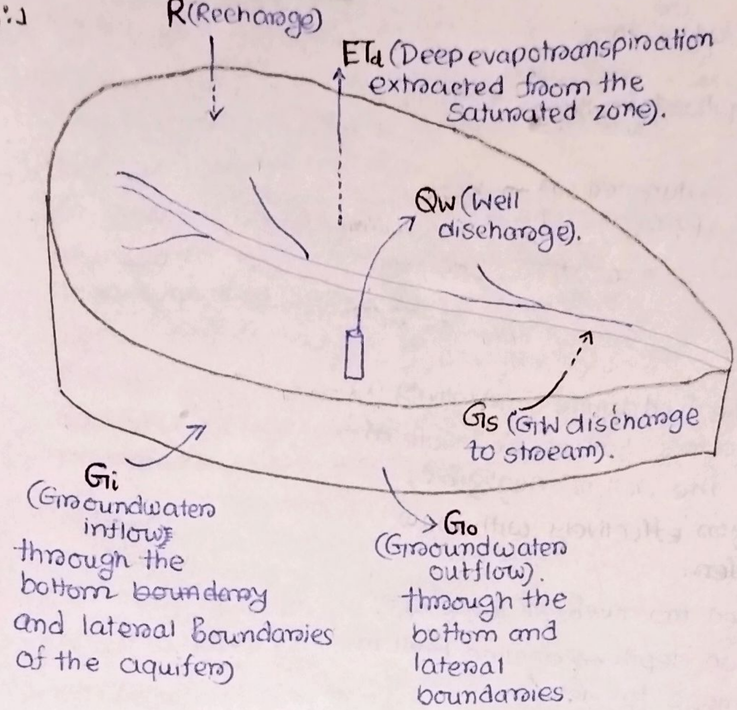
where consolidation already occurred:-

(i) Injecting waters into aquifers to rehydrate soil and rock layers to prevent furthers compaction.

(ii) construct underground storage system

☐ Hydrologic Balance:

Fig: 1



$$R + G_i - G_o - G_S - ET_d - Q_W = \frac{dV}{dt} \quad \ldots\ldots (1)$$

General eqn for hydrologic balance for the piece of aquifer.

☑ For long time span → steady-state balance ⇒ $\frac{dV}{dt} = 0$.

$Q_W = 0$, describes the balance. → # When basin has been operating in a rough steady-state for many years without a pumping well. When, well installed, system thrown into imbalance.

$Q_W > 0$, $Q_W = -\frac{dV}{dt}$

When pumping out, storage is decreasing. $\therefore \frac{dV}{dt}$ is negative.

☑ Groundwater Balance:

Change in GW system storage,

$$\Delta S = \underbrace{R_r + R_s + R_i + R_t}_{\text{Recharge from rainfall, canal seepage, field irrigation, tanks.}} + S_i + I_g - E_t - T_p - B_f - O_g \quad \ldots\ldots (2)$$

S_i → Influent seepage from rivers

I_g → Inflow from other basins

T_p → Draft from GW.

E_t → Draft from Evapotranspiration from groundwater.

Draft ⇒ Amount of GW withdrawn using wells.

B_f = Baseflow, part of GW inflows to rivers.

O_g = Outflow to other basins.

Data requirement:-

(i) Rainfall data # Recharge from rainfall. (I.T)

(ii) Land use data # Evapotranspiration, Recharge etc

(iii) Cropping patterns # Groundwater draft.

(iv) River stage and discharge data. # Baseflow, Seepage from rivers.
River cross sections.

(v) Canal discharge and distributaries data, seepage from canal.

(vi) Tanks depth, capacity, area and seepage data. # Recharge from tanks.

(vii) Water table data.
I.T. This data is necessary to know if GW will flow (inflow/outflow) from aquifer to river, one basin to other basin etc. Because, flow of GW is controlled by its head.

(vii) Groundwater draft.
(No. of wells operational in this area, their corresponding running hours each month and their discharge)

(viii) Aquifer parameters
Porosity, Specific yeild, Specific retention, coff of permeability, Transmissibility, Specific storage, Storage coefficient.

Water balance equation in confined aquifer:-

$$W_{se,i} = W_{se,i-1} + W_{per} - W_{pe} \quad \text{.......(3)}$$

$W_{se,i}$ = Water stored in confined aquifer on day i (mm).

$W_{se,i-1}$ = Water stored in " " on day i-1 (mm).

W_{per} = Amount of water percolating from unconfined aquifer into the confined aquifer on day i (mm).

There is no recharge term (impermeable layer at the top), no base flow contribution for confined aquifer.

? Why reverse process does not occur?

IT ⇒ confined aquifer always stays below the unconfined aquifer. (See fig. P-88).

W_{pe} = Amount of water removed from the confined aquifer by pumping on day i (mm)

Water balance in unconfined aquifer:-

$$W_{su,i} = W_{su,i-1} + \underset{\text{Recharge}}{R_r} - \underset{\text{Base flow}}{B_f} - W_{sd} - \underset{\text{Water percolating from unconfined to confined aquifer.}}{W_{per}} - \underset{\text{Removed by pumping.}}{W_{pu}} \quad \text{........(4)}$$

$W_{su,i}$ = Water stored in unconfined aquifer on day i (mm)

$W_{su,i-1}$ = Water stored in unconfined aquifer on day i-1 (mm).

W_{sd} = Water moving to soil zone in response to water deficiency on day i (mm).

Water Balance in Unsaturated Zone:

$$\Delta S_{unsat} = W_{inf} + W_{cap} + I_g - W_{per} - E_t - B_f \; \ldots\ldots (5)$$

Storage change for reflected time interval. (mm)

Recharge entering the zone by infiltration (mm).

Recharge entering the zone by capillary rise (mm).

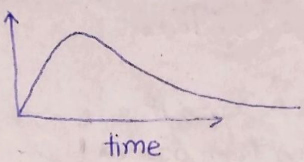
Inflow from other basin.

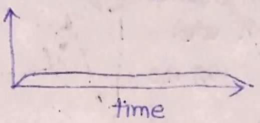
Percolation.

Evapotranspiration

Baseflow.

☒ Infiltration

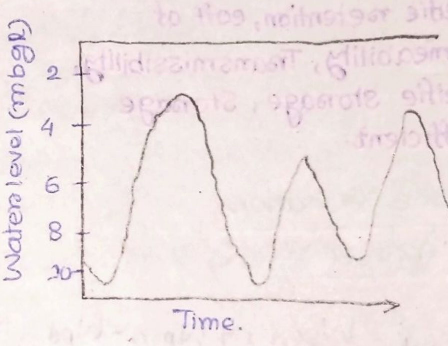




Puddle storage → Storage of water in small depressions or "puddles" on the surface of the ground, which can then infiltrate into the underlaying soil and recharge the groundwater.

Hydrograph:

(GWL hydrograph?!).



(i) Monsoon recharge
⇓
Rising limb generated.

(ii) Based on rainfall pattern and amount, and aquifer character peak takes its shape.

(iii) Probability of getting crest shape in peninsular hard rock region:-
Pointed > Flat > Rounded. Why?

Flat > Rounded.

(iv) Pointed peak may be single-peaked or multi-peaked.

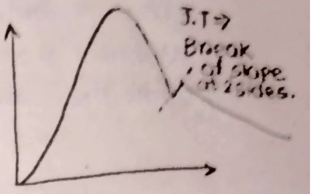
Types of hydrograph curve:

curve based on rising and recession limb pattern:-

(a) V-type
(b) U-type
(c) S-type

<u>V type:-</u> Rising and recession limb has nearly same slope.
Rising limb may be little bit steeper.

<u>U type:-</u> Rising limb → Nearly uniform slope.
Recession limb → Shows a definite break in the slope.



U type

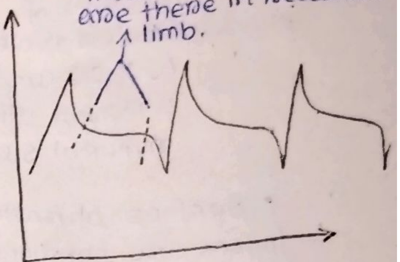
? # Initial discharge is gentle, followed by a fast phase of discharge and then again a slowdown.

<u>S type:-</u> Sharp initial and ultimate slope at recession limb and in between very gentle slope.

∴ Two breaks in actual slope of recession limb.

S curve represents low seasonal fluctuation. So, not very common.

These two breaks are there in recession limb.



☐ Stream- Aquifer Interaction:-
- Losing stream, Gaining stream.

Lec-3
Physical Properties:

☑ Properties of water:-

(i) Mass = 1.94 slugs/ft³
Weight density = 62.4 lb/ft³
compressibility = 4.5×10^{-10} m²/N.
Dynamic viscosity = 1.4×10^{-3} N-s/m².

$\sigma = E\varepsilon$
$\Rightarrow dP = K\left(\frac{dV_w}{V_w}\right)$
$\Rightarrow K = -\frac{V_w}{dV_w} \cdot dP$
$\Rightarrow \beta = -\frac{dV_w}{V_w} \times \frac{1}{dP}$ $\left[\because \beta = \frac{1}{K}\right]$
$\Rightarrow \beta . dP = -\frac{dV_w}{V_w}$

compressibility $(\beta) = \frac{1}{\text{Bulk density}(K)}$

(ii) <u>Viscosity:</u>
- Results from molecule to molecule attractions.

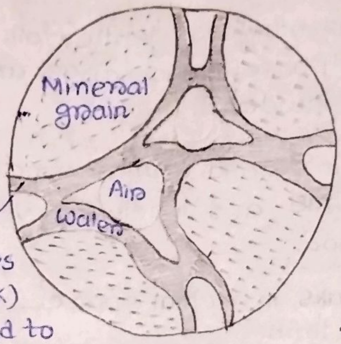
$$F = A\mu\frac{dv}{dz}$$

- Water encounters greater viscous resistance flowing through materials with smaller pores.
- Dynamic viscosity of water → $\mu = 1.79 \times 10^{-3}$ N-s/m² (at 0°C)
$\mu = 1.01 \times 10^{-3}$ N.s/m² (at 20°C).

(iii) Surface tension and Capillarity:

- Polar water molecules - Internal cohesion.
- Water drops in air forms sphere while falling.
- Water wets mineral surface.

Central parts of the pores are for the air.



- Surface attraction forces are stronger in clays. (Due to charged nature of clay mineral surfaces).

- Attraction of water to the mineral surface → Pull and spread the water in mineral surface.

Less amount of water ⇒ More such attraction ⇒ Decrease of pressure within the water.

Attraction of water to mineral surface in partly saturated soil is called **capillarity**.

Capillary allows water to wet pore spaces above the water table.

when both soil have same porosity → Capillary forces are more in fine-grained soils, due to greater amount of mineral surface area.

☑ Properties of air:

Atmospheric pressure $= 1.013 \times 10^5$ N/m^2

Atmospheric density $= 1.2$ kg/m^3.

☑ Properties of porous media:

(i) Soil classification:-

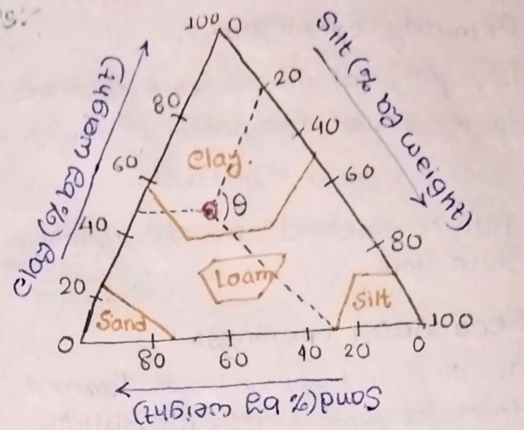
ASTM:- American society for testing and materials.

USDA:- United states department of agriculture Soil classification system.

ISSCS: Indian standard soil Classification system.

(ii) Traingle of soil textures:-

For the red dotted point $\begin{cases} \text{Clay} = 50\% \\ \text{Sand} = 30\% \\ \text{Silt} = 20\% \end{cases}$



USDA Soil textural triangle.

I.T ⇒ θ = 120° ?

(iii) Properties of Porous media:-

?

β - phase

σ - phase

Volume fraction.

Areal fraction.

Volume average factor operator $\langle\ \rangle_\alpha$

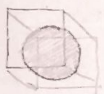
Intrinsic volume average factor operator $\langle\ \rangle$.

Mass average operator $\bar{\ }^{\alpha}$

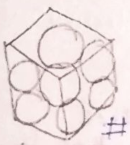
(iv) Porosity and void ratio:

$$\eta = \frac{V_v}{V_t}$$

$$e = \frac{V_v}{V_s}$$

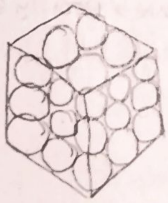


$V_s = 1 \times \frac{\pi}{6} \times d^3$, $V = d^3$ $\therefore \eta = \frac{V_v}{V} = \frac{V - V_s}{V} = \frac{d^3 - \frac{\pi}{6} d^3}{d^3} = 0.4764.$



$V_s = 2^3 \times \frac{\pi}{6} d^3$ $V = (2d)^3$ $\therefore \eta = \frac{(2d)^3 - 2^3 \times \frac{\pi}{6} d^3}{(2d)^3} = 1 - \frac{\pi}{6} = 0.4764.$

2 spheres along length, width and height of the cube.



$V_s = 3^3 \times \frac{\pi}{6} \times d^3$ $V = (3d)^3$ $\therefore \eta = \frac{(3d)^3 - 3^3 \cdot \frac{\pi}{6} d^3}{(3d)^3} = 1 - \frac{\pi}{6} = 0.4764.$

Material	Porosity, η (%)
Narrowly (Poorly!) graded sand, silt, gravel ⇒	30-50
Widely (Well!) graded sand, silt, gravel ⇒	20-35
clay, clay-silt ⇒	35-60.
Limestone, Dolomite ⇒	0-40.

(v) Primary and secondary openings in porous media:

Primary openings:-

- Larger, continuous void formed during initial formation of materials.
- Size range ⇒ mm to cm.
- Interconnected openings, allow fluid flow.

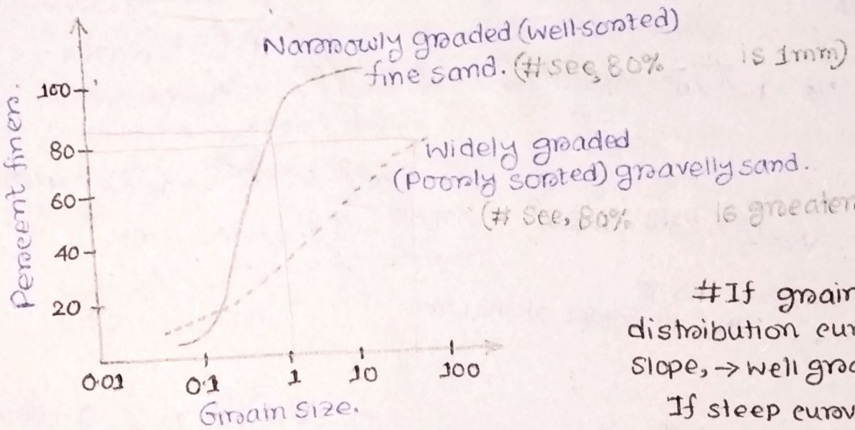
Secondary openings:

- Smaller, irregular voids formed later by weathering, dissolution, fracturing etc.
- Size range ⇒ microns - mm.
  Less inter connected openings
- Affects storage capacity, effective porosity, chemical and mechanical properties

(vi) Grain size:-

US department of Agriculture grain size definition:- (USDA):-

Clay	<0.002 mm
Silt	0.002-0.05 mm
Sand	0.05-2.0 mm
Gravel	>2.0 mm.



#If grain size distribution curve has less slope, → well graded (widely graded)
If steep curve → Poorly graded (Narrowly graded)

(vii) Index properties:

- Volumetric water content, $\theta_v = \frac{V_w}{V}$ ..... (6.1).
- Porosity, $\eta = \frac{V_v}{V}$.
- under saturated condition, $\theta_v = \eta$, Otherwise $\theta_v < \eta$
- Degree of saturation of water, $S_w = \frac{V_w}{V_v} = \frac{\theta_v}{\eta}$ .... (6.2)
- Degree " " " air, $S_a = \frac{V_a}{V_v}$ ..... (6.3)
- $S_w + S_a = 1$.

• Volumetric water content θ_v can be expressed as,

$$\theta_v = \frac{V_w}{V} = \frac{V_w}{V_v} \cdot \frac{V_v}{V} = S_w \times n \quad \ldots (6.4)$$

Note that, in θ_v, volume of water was divided by total volume. But, in case of θ_w, weight of water was divided by weight of solid only.

• Gravimetric water content, $\theta_w = \frac{W_w}{W_s}$

W_s = weight of solid or dry soil. ....(6.5)

• Relation between volumetric water content and Gravimetric water content (θ_v and θ_w):-

$$\theta_v = \theta_w \cdot \frac{\rho_b}{\rho_w} \quad \ldots (6.6)$$

$$\theta_v = \frac{V_w}{V}$$

$$\theta_w = \frac{W_w}{W_s}$$

$$\frac{\theta_w}{\theta_w + 1} = \frac{W_w}{W_w + W_s}$$

$$\Rightarrow \frac{\theta_w}{1+\theta_w} = \frac{W_w}{W}$$

$$\Rightarrow \frac{\theta_w}{1+\theta_w} = \frac{\rho_w g V_w}{\rho_t g V}$$

(ρ_t = Total Bulk density)

$$\Rightarrow \frac{\rho_t}{\rho_w} \cdot \theta_w = (1+\theta_w)\frac{V_w}{V} = (1+\theta_w)\theta_v$$

$$\Rightarrow \boxed{\frac{\rho_t}{\rho_w} \cdot \theta_w = (1+\theta_w) \cdot \theta_v}$$

This is representation by total bulk density (ρ_t)

But, if we represent it by ρ_b,

$$\theta_w = \frac{\rho_w g V_w}{\rho_b g V}$$

$$\Rightarrow \boxed{\theta_w = \frac{\rho_w}{\rho_b} \cdot \theta_v}$$

• **Bulk density: (ρ_b)**

#(Our conventional bulk density here known as "Total bulk density". Here, bulk density has been defined in some other way).

$$\rho_b = \frac{\text{Mass of solid}}{\text{Total volume}} = \frac{m_s}{V} \quad \ldots (6.7)$$

• **Wet or total Bulk density: (ρ_t)**

$$\rho_t = \frac{\text{Mass of solid + Mass of water}}{\text{Total volume}} = \frac{m_s + m_w}{V} \quad \ldots (6.8)$$

• **Density of solids: (ρ_s)**

#(Obviously, ρ_b should be less than ρ_s)

$$\rho_s = \frac{\text{Mass of solid}}{\text{Volume of solid}} = \frac{m_s}{V_s} \quad \ldots (6.9)$$

(viii) Effective stress:

stress experienced by the soil particles.
This stress transferred by grain to grain contact.

Compression and rearrangement of sand grains to form closely packed configuration.

Compression of water in the pores

$$\sigma_T = \sigma_e + P \quad \ldots (6.10)$$

$$\Rightarrow d\sigma_T = d\sigma_e + dP \Rightarrow 0 = d\sigma_e + dP \Rightarrow d\sigma_e = -dP \quad \ldots (6.11)$$

$(\because d\sigma_T = 0)$

(ix) Compressibility:

$$\alpha = \frac{\text{Volumetric Strain}}{\text{Effective stress}} = \frac{-dV_T/V_T}{d\sigma_e} \quad \ldots (6.13)$$

V_T = Total volume of solid mass $= V_s + V_v$.

$\Rightarrow dV_T = dV_s + dV_v$

[illegible] grains are incompressible. $\therefore dV_s = 0$.

$\therefore dV_T = dV_V$ ...... (6.14)

Notes on Total stress, effective stress and pore water pressure:-

Total stress does not change because it is the function of soil weight and applied loads, which do not change unless new loads are added or soil is removed. $\therefore d\sigma_T = 0$ (eqn 6.11).

But, effective stress may change as pore water pressure changes.

Pore water pressure affects the ability of soil skeleton to transmit forces (Grain to grain transfer) and deform, which affects the soil's strength and stiffness. like shear stress, consolidation, settlement.

But, if external load is applied water tries to squeeze out. Then pore water pressure increases. Total stress should increase also. (As external force is applied)?!

Pore water may increase. But, similarly, pressure to the soil grain will decrease. Ultimately, no change to the total stress.

Rising of GWT:

Rising of water table leads to fully saturated soil pores:

- Increase pore water pressure
- Reduce effective stress.
- Total stress remains constant as increase in pore water pressure offset by decrease in effective stress.

(But, saturated bulk unit wt. and unsaturated bulk unit wt. are not same. Then, how total stress can be unchanged?!)

Putting values from (6.11) to (6.13),

$$\alpha = \frac{1}{V_T}\frac{dV_T}{dP} \quad \text{......} \quad (6.15)$$

Types of water:

(i) Magmatic water / Juvenile water:- (New water)

This water is released during the crystalization of magma or cooling of pluotonic rocks.

'Pluotonic' rocks → Formed by solidification of magma deep within the earth's crust.

'Volcanic' rocks → Formed by solidification of lava or pyroclastic material on earth's surface

(ii) Metamorphic water:-

water released from the rocks during metamorphism.

(Metamorphism is a process of changing of minerological, chemical and structural property of rock due to high temperature, pressure and fluid activity).

Metamorphic water comes from different sources, like:-
- Pore fluids
- Dehydration reactions (Water from crystal structures of clay, mica etc during metamorphism).
- Fluid infiltration (Eg. from groundwater, hydrothermal fluid etc).

(iii) Meteoric Water:-
Water originates from atmosphere i.e. precipitation such as rain or snow.

(iv) Connate Water:-
- Water trapped within sedimentary rocks at the time of their formation.
- Isolated from earth's surface since the time of its entrapment.
- Also called 'Fossil water'.
- It may be found in pore spaces of sediment grains or within crystal structures
- Connate water plays an important role in exploration of hydrocarbon reservoirs, after gives valuable information about reservoir's history and fluid migration pathways.

(V) Ocenic water.

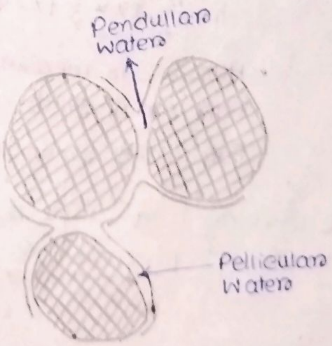
☑ Types of soil water:-

(i) Mobile Water:
- Free to move through the soil.
- Found in larger pores and interstices between soil particles and not bound to soil matrices.
- Can move downward due to gravity / sideways due to pressure gradient or capillary action.

(ii) Adsorbed Water:-
Force of attraction between water molecules and solid mineral surface.

(iii) Capillary Water:-
Water is under negative pressure or suction

(iv) Pendular Water:
- Residual immobile water around grain to grain contact points.
- Disconnected in the hydrodynamic sense.



Lec-5

Physical properties:

To calculate hydraulic head.

To differentiate between aquifer, aquitard, aquiclude.

☑ Energy and Hydraulic head:

$$E = pV + mgz + \frac{1}{2}mv^2 \quad \ldots\ldots (7.1)$$

- E = Mechanical energy
- pV = Elastic potential Energy (Gained by compressing water).
- mgz = Gravitational potential energy. (Achieved by lifting water to higher elevation)
- $\frac{1}{2}mv^2$ = Kinetic Energy (from velocity of water).

• **Alternate definition:-**

Mechanical energy can be thought as work required to compress, elevate a 'm' mass of water to its current state from reference state (i.e $p=0, z=0, v=0$).

• **Hubbert's fluid potential (φ):-**

Energy per unit mass of water.

$$\varphi = \frac{E}{m} = \frac{p}{\rho_w} + gz + \frac{v^2}{2} \text{ (\#From 7.1)} \quad \ldots (7.2)$$

$$\frac{pV}{m} = \frac{pV}{\rho_w V} = \left(\frac{p}{\rho_w}\right)$$

• **Hydraulic head:**

Energy per unit weight of water.

$$h = \frac{E}{mg} = \frac{pV + mgz + \frac{1}{2}mv^2}{mg}$$

$$= \frac{pV}{\rho_w V g} + z + \frac{v^2}{2g}$$

$$= \frac{p}{\rho_w g} + z + \frac{v^2}{2g} \quad \ldots\ldots (7.3)$$

∴ Pressure head, elevation head and velocity head.

• Water always flow towards region of lower hydraulic head.

* Velocity head contributes insignificant amount to hydraulic head. (∵ velocity of GW is very slow).

Hydraulic head for GW flow,

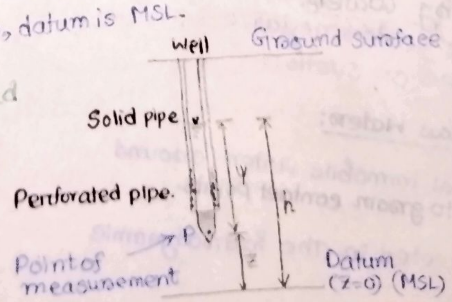
$$h = \frac{p}{\rho_w g} + z = \frac{p}{\gamma} + z = \psi + z \quad \ldots\ldots (7.4)$$

• Here, z is measured wrt datum.

For small study area datum is arbitrary.

For large study area, datum is MSL.

Measuring hydraulic head with wells & piezometers:-



- Hydraulic head in the pore water of saturated zone is measured directly using **hydrostatic** principle.
- Upper end of pipe → open to atmosphere
- Near pipe bottom → hole and slots → allow water to move into pipe from surrounding saturated soil or rock.
- **Piezometers:** Small diameter pipes
- **Wells:** Larger diameter.

- Constant hydraulic head (everywhere in 2 dimensional domain) → no flow → hydrostatic condition

Hydrostatic condition ⇒ No flow
condition ⇒ velocity head = 0

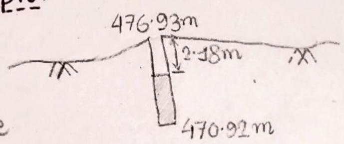
!?

- For a point on surface, if we go down from the surface, hydraulic head **remains constant**, while pressure head ($P = \rho_w g(h - z)$) increases and at same rate z decreases.

See fig of P-84

Pressure head = $\frac{P}{\rho_w g}$.

Example:-



Hydraulic head at piezometer, $h = (476.93 - 2.18) = 474.75$

Water pressure at the bottom of the piezometer $= \rho_w g\,(474.75 - 470.92)$

$= 1000 \times 9.81 \times 3.83$

$= 37572\ N/m^2$.

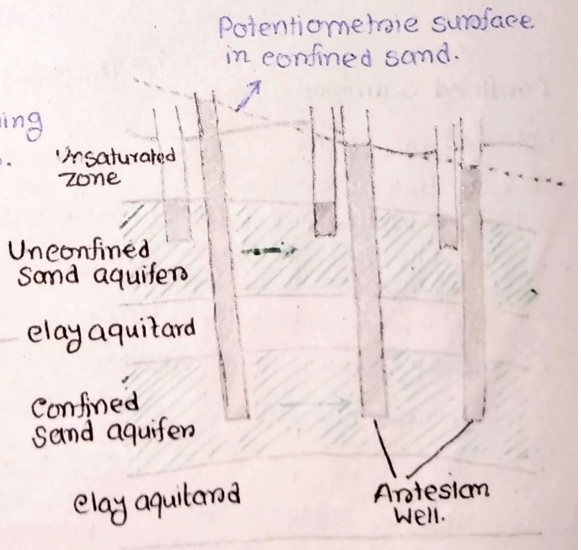
Aquifers and Confining layers:

Preclude ⇒ বর্জন করা
Retard ⇒ স্তিমিত করা

- Aquifers → Tapped by supply wells.
- **Confining layers (aquitards):-**
Retards flow and transmit very less water. # Transmit, can not store water
- **Aquiclude:**
This word no longer used much. Means an extremely low K confining layer → virtually precludes flow.
Stores water, but cannot transmit.

Water stored. But not effective. So, we can't extract it. So, ultimately, no value of this water. So, we will call 'clay' also as aquitard.

So, clay layers should be aquiclude?

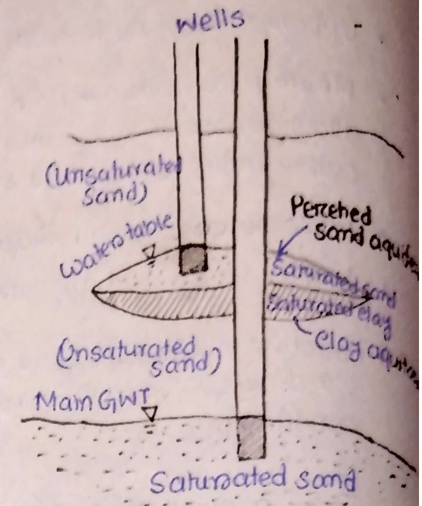


Perched Sand Aquifer:

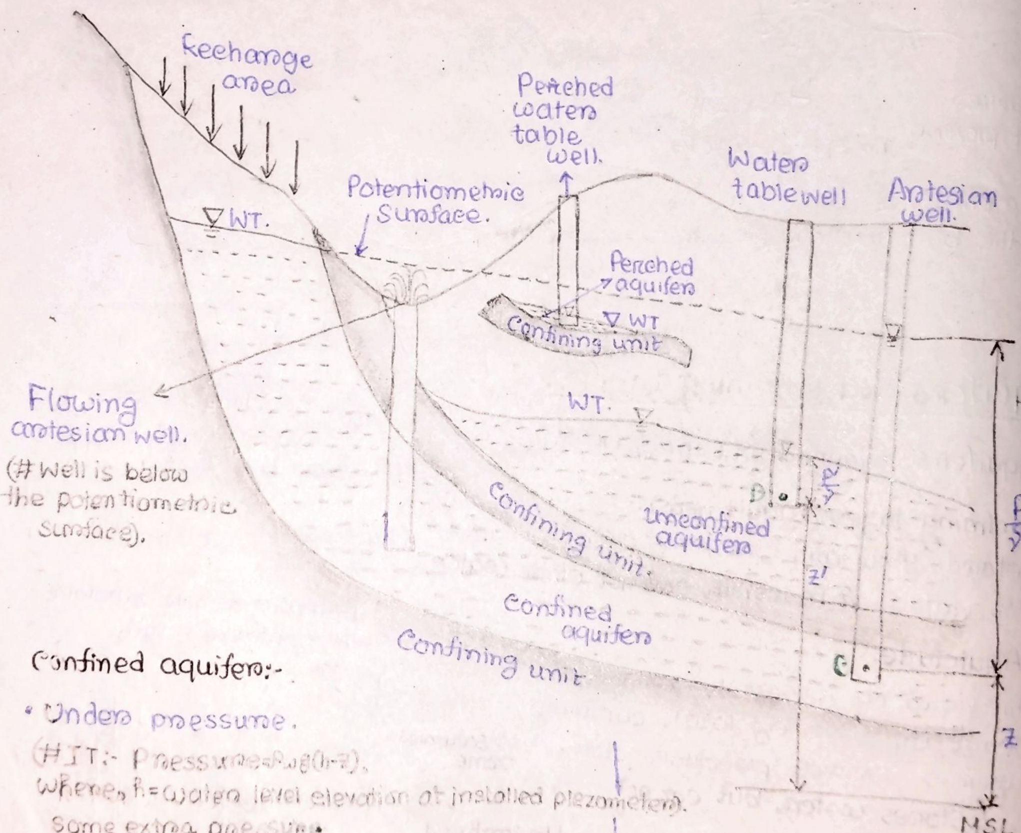
Located above the main water table seperated by impermeable rock or soil.

These aquifers occurs in areas with irregular topography

Impermeable layer causes localized accumulation of water.



- Water is stored in relatively low amount and isolated from main GWT, it is more vulnerable to contamination or depletion.
- Some cases, it is known as Sand lenses, which are small localized deposit of sand or gravel that can store and transmit water.



Confined aquifer:-

- Under pressure.
  (# IT:- Pressure = $\rho_w g(h-z)$. Where, h = water level elevation at installed piezometer. Some extra pressure is there due to confinement). So, water level rises above aquifer.
- Bounded by impervious layer.
- Hydraulic head at C,

$$h_c = \frac{P}{\gamma} + z$$

(# Applying hydrostatic condition at equation (7.3)).
Because, in groundwater, we always use hydrostatic condition, neglect velocity.

Unconfined aquifer:-

- Phreatic / water table aquifer.
- Bounded by water table.
- Hydraulic head at D,

$$h_D = \frac{P'}{\gamma} + z'$$

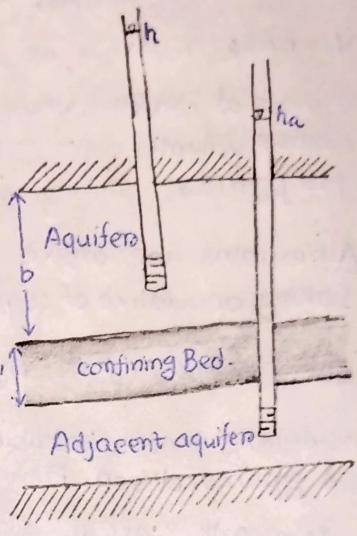
where, $P' = \rho_w g(h_D - z)$.

Lecture 6

Aquifer Properties

To calculate storage coff, specific yeild, specific retention.

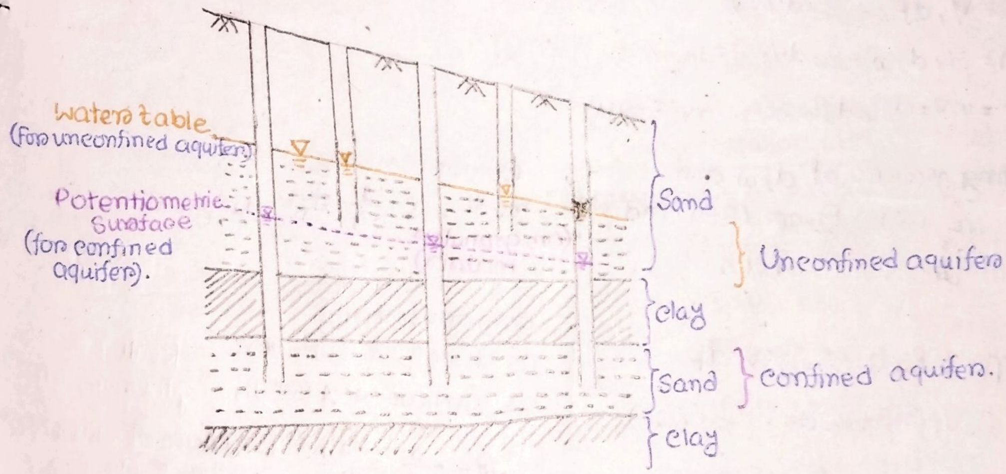
Two confined aquifer with different heads.



- GW tend to flow from top aquifer to bottom aquifer.
- We cann't make any conclusion about horizontal head gradients from this picture.

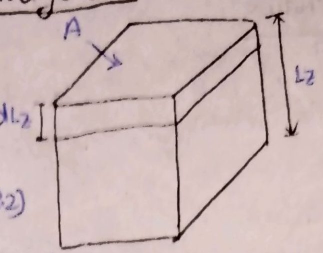
I.T (Because, there is no horizontal flow between top and bottom aquifers. So, $(\frac{h-ha}{L})$ does not make any sense of horizontal head gradient. If two wells were in same aquifer, then it may give horizontal head gradient).

Horizontal and Vertical head gradients:



Specific storage/Storage coff:-

Method 1:



Total volume,

$V_T = AL_z$ ..... (8.1)

Volume of solid, $V_s = (1-\eta) AL_z$ ... (8.2)

$V_v = \eta AL_z$ # Volume of void. ... (8.3)

Total mass of water within the control volume,

$M = \rho_w \eta A L_z$ # Fully saturated (In aquifer)..... (8.4)

If A remains unchanged, taking derivative of (8.4),

$dM = \eta A L_z \, d\rho_w + \rho_w A L_z \, d\eta + \rho_w \eta A \, dL_z$ .... (8.5)

Sometime ρ and sometime ρ_w is written. I think both represents density of water.

Considering incompressible solid grains, $dV_s = 0$. From (8.2),

$(1-\eta) A dL_z - AL_z d\eta = 0$ .... (8.6)

$\Rightarrow L_z d\eta = (1-\eta) dL_z$ .... (8.7)

Putting value obtained from (8.7) to (8.5),

$dM = \eta A L_z d\rho_w + \rho_w A (1-\eta) dL_z + \rho_w \eta A dL_z$

$\Rightarrow dM = \eta A L_z d\rho_w + \rho_w A dL_z$ .... (8.8).

$\Rightarrow dM = V_v d\rho_w + \rho_w d(AL_z)$ [From (8.3), $V_v = \eta AL_z$]

$\Rightarrow dM = V_v d\rho_w + \rho_w dV_T$ [From (8.1), $V_T = AL_z$]

$= \eta V_T d\rho_w + \rho_w dV_T$ .... (8.11)

Compressibility of porous media, $\alpha = \frac{1}{V_T} \cdot \frac{dV_T}{dP}$ (From 6.15) (or, granular matrix) .... (8.9)

Compressibility of water, β (written as similar way (6.13)),

$\beta = -\frac{1}{V_w} \frac{dV_w}{dp}$

$\Rightarrow \beta = \frac{1}{\rho_w} \frac{d\rho_w}{dp}$ .... (8.10)

Really! Then why dM?

Mass of water within control volume remains same,

$\rho_w V_w = \text{const.}$

$\Rightarrow d\rho_w dV_w + V_w d\rho_w = 0$

$\Rightarrow \frac{dV_w}{V_w} = -\frac{d\rho_w}{\rho_w}$

Putting values of $d\rho_w$ and dV_T in (8.11) From (8.9) and (8.10) [# In terms of α and β],

$dM = \eta V_T \beta \rho_w dp + \rho_w \cdot \alpha V_T dp$

$dM = \rho_w V_T (\eta\beta + \alpha) dP$ ..... (8.12)

Putting $dp = \rho g dh$, eq^n (8.12) becomes, (Is this ρ is ρ_w?)

$dM = \rho_w V_T (\eta\beta + \alpha) \rho g dh$ .... (8.13)

$\Rightarrow V_w = \frac{dM}{\rho_w} = V_T (\eta\beta + \alpha) \rho g dh$ ... (8.14)

** $S_s \times (H-\xi) = S_y$ (Unconfined) → P-170
$S_s \times b = S$ (confined) → P-196

Some definitions:-

Storage coff: Volume of water that can be stored in a unit volume of the aquifer per unit change in the hydraulic head. Term is used for unconfined/semi-confined aquifers.

→ Similar definition for confined aquifer is known as Specific Storage. (S_s)

Specific yeild (S_y) ~~Or specific Storage~~: Specific Storage ~~coff~~ multiplied with depth of aquifer is known as specific ~~storage~~ yeild. → It works! See page (170)

It represents fraction of total volume of aquifer that is

~~Some problem here I think!~~

Now, all things are defined as per (8.16.2), (39.4) and (39.4.1) available for release from the storage.

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w Storage coff (S) water volume released per unit area of aquifer per unit change in hydraulic head. $S = S_s \times$ Aquifer thickness.

From definition,

Sp. Storage coff can be defined as,

$$S_s = \frac{1}{V_T}\frac{dV_w}{dh} \cdots\cdots (8.15)$$

• Storage coff also called Storativity.

$$\Rightarrow S_s = (\eta\beta+\alpha)\rho g \text{ [From(8.14)]} \cdots\cdots (8.16)$$

Storage coefficient $S = S_s \times b$ [Where b = Depth of the aquifer. .... (8.15.2)
For our control volume, take, $b = L_z$]

$$= \frac{1}{V_T}\frac{dV_w}{dh}\cdot L_z \quad \text{(From 8.15)}$$

$$= \frac{1}{AL_z}\frac{dV_w}{dh} L_z$$

$$= \frac{1}{A}\frac{dV_w}{dh} \cdots\cdots (8.17).$$

Another way to calculate the ~~specific~~ storage coefficient,

$$S_s = \gamma_w[(1-\eta)\alpha+\eta\beta] \cdots\cdots (8.18)$$

Most popular form.

$(8.16) \rightarrow S_s = (\eta\beta+\alpha)\rho g$
$(8.18) \rightarrow S_s = [(1-\eta)\alpha+\eta\beta]\gamma_w$
} Both will give different values. Applicable for different cases?! (See P-93, Problem-5).

Storage coff derivation:

Method 2 :- (Here, porosity denoted by $\varepsilon = \frac{V_v}{V_T}$).

We know, $\alpha = \frac{1}{V_T}\cdot\frac{dV_T}{dp}$ (From 8.9) ...... (9.1)

and $\beta = \frac{1}{\rho}\frac{d\rho}{dp}$ (From 8.10) .... (9.2)

1st Approach (Expression with C_v)

$$\frac{\partial(\varepsilon\rho)}{\partial t} = \varepsilon\frac{\partial\rho}{\partial t} + \rho\frac{\partial\varepsilon}{\partial t}$$

$$= \varepsilon\frac{\partial\rho}{\partial p}\cdot\frac{\partial p}{\partial t} + \rho\frac{\partial\varepsilon}{\partial t}$$

$$= \varepsilon\frac{d\rho}{dp}\frac{\partial p}{\partial t} + \rho\frac{d\varepsilon}{dp}\frac{\partial p}{\partial t} \cdots (9.3)$$

Now, $\frac{d\varepsilon}{dp} = \frac{d}{dp}\left(\frac{V_v}{V_T}\right)$

$= \frac{dV_v}{V_T\cdot dp}$? Is it properly represented?

$= C_v \cdots\cdots (9.4)$

C_v is related to classical coff of consolidation.

Putting $\frac{d\rho}{dp} = \rho\beta$ (from 9.2) and $\frac{d\varepsilon}{dp} = C_v$ (from 9.4) in the equation (9.3), we have,

$$\frac{\partial(\varepsilon\rho)}{\partial t} = \varepsilon\rho\beta\frac{\partial p}{\partial t} + \rho C_v\frac{\partial p}{\partial t}$$

$$= \rho(\varepsilon\beta+C_v)\frac{\partial p}{\partial t} \cdots\cdots (9.5)$$

In 1st approach, partial derivatives are used. In 2nd approach, ordinary derivatives. But why ?

2nd approach (Representation with α)

Compressibility of porous media (from 9.1),

$$\alpha = \frac{1}{V_T}\frac{dV_T}{dp} \approx \frac{1}{V_T}\cdot\frac{dV_v}{dp} \cdots\cdots (9.6)$$

(Taking $dV_v \approx dV_T$, But why?) ⇒ Because, no volume change for soil grains. [$\because dV_v + dV_s = dV_T$, $dV_s = 0$]

$$\frac{d(\varepsilon\rho)}{dt} = \varepsilon\frac{d\rho}{dt} + \rho\frac{d\varepsilon}{dt}$$

$$= \varepsilon\frac{d\rho}{dp}\cdot\frac{dp}{dt} + \rho\cdot\frac{d\varepsilon}{dp}\cdot\frac{dp}{dt} \cdots (9.7)$$

Now, $d\varepsilon = d\left(\frac{V_v}{V_T}\right) = \frac{V_T dV_v - V_v dV_T}{V_T^2}$

$\approx \frac{(V_T - V_v)dV_v}{V_T^2}$ [$\because V_v \approx V_T$]

$\approx \frac{V_s}{V_T^2}dV_v$ [$\because V_T - V_v = V_s$] ...... (9.8)

$\therefore \frac{d\varepsilon}{dp} = \frac{V_s}{V_T^2}\frac{dV_v}{dp} = \frac{V_s}{V_T}\cdot\left(\frac{1}{V_T}\cdot\frac{dV_v}{dp}\right)$

$= \frac{V_s}{V_T}\cdot\alpha$ (from 9.6).

$= \frac{V_T - V_v}{V_T}\cdot\alpha$

$= \left(1 - \frac{V_v}{V_T}\right)\alpha$

$= (1-\varepsilon)\alpha \cdots\cdots (9.9)$

Putting $\frac{d\rho}{dp}$ and $\frac{d\varepsilon}{dp}$ values from (9.2) and (9.9) to the equation (9.7),

$$\frac{d(\varepsilon\rho)}{dt} = \varepsilon\rho\beta\frac{dp}{dt} + \rho(1-\varepsilon)\alpha\frac{dp}{dt}$$

$$= \rho[\varepsilon\beta + (1-\varepsilon)\alpha]\frac{dp}{dt} \cdots\cdots (9.10)$$

My Calculation:

Eq (9.5) and (9.10) written as,

$$\partial(\varepsilon\rho) = \rho(\varepsilon\beta + C_v)\,\partial P \Rightarrow \frac{\partial(\varepsilon\rho)}{\partial P} = \rho(\varepsilon\beta + C_v) \quad \ldots\ldots (9.11)$$

and

$$d(\varepsilon\rho) = \rho[\varepsilon\beta + (1-\varepsilon)\alpha]\,dP \Rightarrow \frac{d(\varepsilon\rho)}{dP} = \rho[\varepsilon\beta + (1-\varepsilon)\alpha] \quad \ldots (9.12)$$

Now,

$$\frac{d(\varepsilon\rho)}{dP} = \frac{d\left(\frac{V_v}{V_T}\rho\right)}{d(\rho g h)}$$

$$= \frac{\frac{\rho}{V_T}\,dV_v}{\rho g\,dh}$$

$$= \frac{1}{V_T}\frac{dV_v}{dh}$$

$$= \frac{1}{g \cdot V_T}\frac{dV_w}{dh} \quad \left[V_v = V_w \text{, below GWT, all voids are filled with water}\right].$$

$$= \frac{1}{g} \cdot S_s \quad [\text{From } (8.15)] \quad \ldots (9.13)$$

Putting values of (9.13) to (9.11) & (9.12),

$$S_s = \rho g(\varepsilon\beta + C_v) \quad \ldots\ldots (9.14)$$

and

$$S_s = \rho g[\varepsilon\beta + (1-\varepsilon)\alpha] = \gamma_w[\varepsilon\beta + (1-\varepsilon)\alpha] \quad \ldots (9.15)$$

(9.14) and (9.15) both are expressions of storage coff. by two different approach.

(9.15) exactly matches with equation (8.18). So, it is clear that ε and η both are porosity! 😆

Specific yeild:

- In unconfined aquifer, volume of water released from storage per unit surface area per unit decline of water table.

$$S_y = \frac{1}{A}\frac{dV_w}{dh} \quad \ldots\ldots (10.1)$$

(see 8.17)

- Specific yeild can also be defined as the ratio of → Volume of water discharged from the sample considering the gravity & total volume of sample.

$$S_y = \frac{\text{Gravity drainage Volume}}{\text{Bulk volume.}} \quad \ldots\ldots (10.2)$$

For unconfined aquifers,

* Storage coefficient (S) = Sp. yeild (S_y) + Aquifer thickness(b) x specific storage (S_s) ...... (10.3)

why $S = S_s \times b$ in (8.17)? $\left\{\begin{array}{l}\text{Is (8.17) applicable for confined aquifers only?} \\ S_y \text{ in confined aquifer is very less. Hence neglected.}\end{array}\right.$

- In confined aquifers, $S \leq 0.005$

In unconfined aquifers, $0.2 \leq S \leq 0.3$

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☑ confined aquifer has very less Specific yeild compared to unconfined aquifer.

Unit decline in head:-

for unconfined aquifer ⇒ unit Drop of water table.

for confined aquifer ⇒ unit drop of water level in piezometric pipe.

Specific Retention:-

$$\text{Effective porosity } (\eta_e) = \frac{\text{Volume of water able to circulate}}{\text{Volume of rock.}} \quad \text{.... (11.1)}$$

$$\text{Specific retention, } S_r = \eta - \eta_e \quad \text{...... (11.2)}$$

η → Volume Fraction of total water (Because, below GWT, all pores are filled with water). so equals to porosity).

η_e → Volume fraction of water able to circulate.

$$\text{Also, } S_r = \eta - S_y \quad \text{...... (11.3)}$$

IT (comparing 11.2 and 11.3),

Specific yeild(S_y)= Volume fraction of circulatable water(η_e).

Field capacity and Sp. retention:-

#I.T ⇒ Both are same. They represents quantity of water retained by the soil against the pull of gravity.

(?) is it ok?

But, FC is represented by depth/depth ratio.

Sp. retention " " " volume/volume ratio.

Problem-1

Total stress, $\sigma_T = \rho_b g b$

$= 2500 \times 9.81 \times 30$

$= 735750$ Pa.

30m

clayey aquitard

$\rho_b = 2500 \text{ kg/m}^3$

Avg. press head, $p = 40$m.

→ σ_e here = ?

Aquifer

Pore water pressure,

$p = \rho_w g h = 1000 \times 9.81 \times 40$

$= 392400$ Pa.

Effective stress, $\sigma_e = \sigma_T - p = 343350$ Pa.

When hydraulic head decreased by 1m,

$\Delta P = \rho_w g \Delta h = 1000 \times 9.81 \times (-1.5)$

$= -14715$ Pa.

∴ change in effective stress

$\Delta\sigma_e = \Delta\sigma_T - \Delta P$

$= 0 - (-14715)$ # change in total stress = 0

$= 14715 \text{ N/m}^2$

Problem-2:-

$\begin{cases} \text{Porosity} = 0.22 \\ \text{Bulk density} = 1.95\ g/cm^3 \end{cases}$

Particle density, $\rho_s = \frac{1.95\ g/cm^3}{(1-0.22)\ cm^3} = 2.5\ g/cm^3$.

Problem 3:-

$\begin{cases} \text{Aquifer area} = 7km^2 = 7\times10^6\ m^2 \\ \text{Head drop}(\Delta h) = 0.85m. \\ \text{Pumping days} = 8 \text{ years} = 8\times365 \text{ days} \\ \text{Pumping rate} = 5.5\ m^3/day. \end{cases}$

From definition, (eq^n 10.1),

$$S_y = \frac{\Delta V_w}{A\,\Delta h}$$

$$= \frac{(5.5)\frac{m^3}{day} \times (8\times365) day}{(7\times10^6)\ m^2 \times (0.85)\ m}$$

$$= 2.7\times10^{-3}.$$

Problem-4

$\begin{cases} V = \text{Volume of soil sample} = 180\ cm^3 \\ \text{Volume of voids } (V_v) = 67\ cm^3. \\ V_{ev} = \text{Out of vol. of voids, water can move through only} = 45\ cm^3. \end{cases}$

Porosity $= \frac{V_v}{V} = \frac{67}{180} = 0.37$.

From (11.1),

Effective porosity $(\eta_e) = \frac{V_{ev}}{V} = \frac{45}{180} = 0.25$

From (11.2)

Specific retention $(S_r) = \eta - \eta_e$

$= 0.37 - 0.25$

$= 0.12$

From (11.3),

Specific yeild $(S_y) = \eta - S_r$

$= 0.37 - 0.12$

$= 0.25$

Also, we can say, $S_y = \eta_e$.

Problem-5

b = Thickness of saturated layer = 8.2m.

Porosity of the soil $(\eta) = 0.28$

? E_s = I think bulk modulus $(K_s) = 5.3\times10^7\ N/m^2$

in notes!

$E_w = K_w = 2.1\times10^9\ N/m^2$

From equation (8.17),

Storage coff, $S = S_s \cdot b$

(From equation 10.3, S_y is neglected)!

From (8.16), $S = S_s \cdot b = (\eta\beta + \alpha)\rho g b$.

From (8.18) or (9.15)

$S = S_s b = \rho g b\,[(1-\eta)\alpha + \eta\beta]$

Now, compressibility $= \frac{1}{\text{Bulk modulus}}$

$\alpha = \frac{1}{5.3\times10^{7}} = 1.89\times10^{-8}$

$\beta = \frac{1}{2.1\times10^{9}} = 4.76\times10^{-10}$

$S = (\eta\beta+\alpha)\rho g b$

$= (0.28\times4.76\times10^{-10} + 1.89\times10^{-8})\times1000\times9.8\times8.2$

$= 1.53\times10^{-3}$

or, $S = [(1-\eta)\alpha+\eta\beta]\rho g b$

$= [(1-0.28)\times1.89\times10^{-8}+0.28\times4.76\times10^{-10}]\times9810\times8.2$

$= 1.105\times10^{-3}$.

which value of storage coff (S) is correct?

Problem-6

For confined aquifer, $\eta = 0.42$.
$b = 78\,m$.
$\beta = 4.76\times10^{-10}$
$\alpha = 6.3\times10^{-8}$

Specific storage $(S_s) = (\eta\beta+\alpha)\rho g$

$= (0.42\times4.76\times10^{-10}+6.3\times10^{-8}).9810$

$= 6.2\times10^{-4}$.

Storativity or storage capacity

$S = S_s\times b = 6.2\times10^{-4}\times78 = 4.836\times10^{-2}$.

Head drop = 100 m
Aquifer area = $1200\,km^2 = 1200\times10^{6}\,m^2$

Volume of water released for this much head drop:- (from 8.17).

$dV_w = SA\,dh$

$\Rightarrow \Delta V_w = S.A.\Delta h$

$= 4.836\times10^{-2}\times1200\times10^{6}\times100$

$= 5.803\times10^{9}\,m^3$.

Problem-7

Soil sample volume, $V = 280\,cm^3$.
$V_w = 65\,cm^3$
$V_v = 115\,cm^3$

Porosity $= \frac{V_v}{V} = \frac{115}{280} = 0.41$.

Volumetric water content (from 6.1),

$\theta_v = \frac{V_w}{V} = \frac{65}{280} = 0.23$

Degree of Saturation, $S_w = \frac{V_w}{V_v} = \frac{65}{115} = 0.56$.

Lecture-7

Principles of flow:

// To apply Darcy's Law.

Darcy's law:-

- I.T ⇒ This setup and equation (12.1) is for steady flow with constant pressure head.

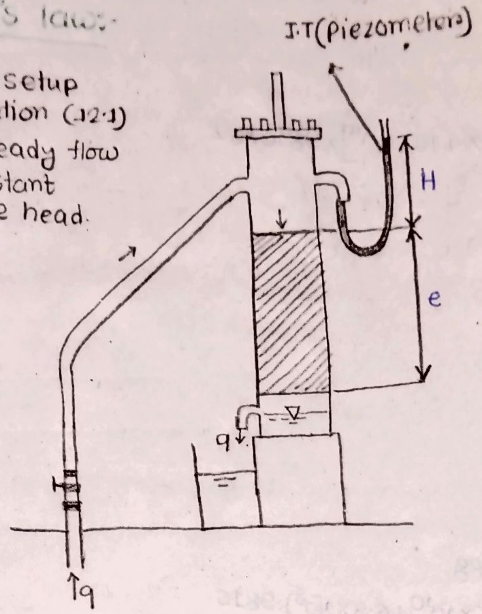


fig: 7.1

Pressure head, Elevation head

Inflow head = H+e

Outflow head = 0 [∵ Atmospheric pressure]

$$q = Ks\frac{H+e}{e} \quad \ldots\ldots (12.1)$$

q = Percolation flux $[L^3T^{-1}]$

K = Permeability factor $[LT^{-1}]$

S = Surface area of sand filter cylinder

e = Length of sand column

Hydraulic gradient = $\frac{(H+e)-0}{e}$ $[\because \frac{\Delta h}{L} = i]$

$= (H+e)/e$.

- Non-steady flow equation with falling pressure head:-

$$q_t = q_0\, e^{-(kt/e)} \quad \ldots\ldots\ldots (12.2).$$

where, t = time

q_0 = Percolation flux at t=0.

Observations of Darcy's Experiment:

$Q \propto \Delta h$, $Q \propto \frac{1}{\Delta S}$, $Q \propto A$

Combining these, Darcy's law for 1D flow,

$$Q = -Ks\frac{\Delta h}{\Delta S}A \ldots\ldots (12.3)$$

Minus sign on the RHS of Darcy's equation is necessary because head decrease in the direction of flow.

i.e. S increases, h decreases.

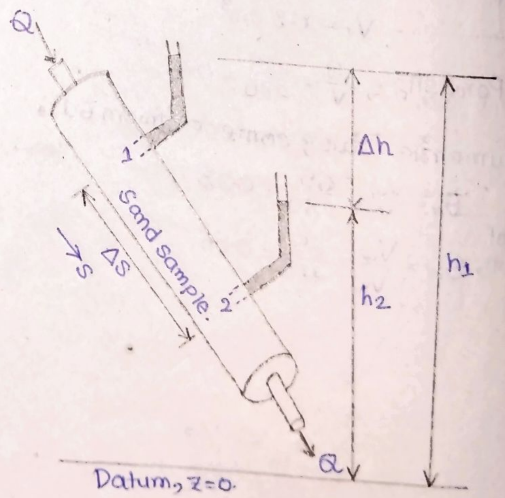


Figure: 7.2

Specific discharge or Darcy's velocity,

$$q_s = \frac{Q_s}{A} = -K_s \frac{dh}{ds} \;\cdots\cdots (12.4)$$

(Unit of Specific discharge $q_s = m/s$).

Limitations of Darcy's law:

- Darcy's velocity (q_s) is fictitious velocity.
  - Assumes flow occurs through entire c/s of the soil sample. Actual it occurs through interconnected ~~flow~~ pore channels.
  - Flow path assumed as linear.

∴ Actual or seepage velocity

$$\bar{v}_s = \frac{q_s}{n_e} = \frac{Q_s}{A n_e} \;\cdots\cdots (12.5)$$

Discharge can be written as,

$$Q = A \cdot q_s = A_v \cdot \bar{v}_s \;\cdots\cdots (12.6).$$

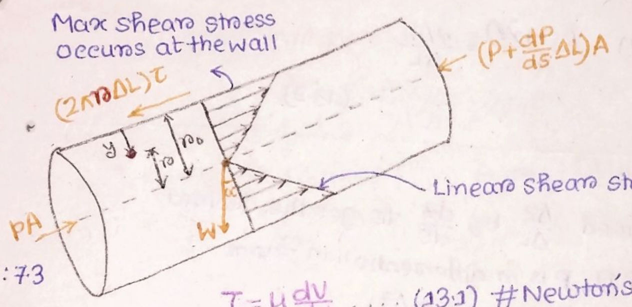
Total area × Darcy's velocity

Pore area × Seepage velocity.

#[Note that, in (12.5); To get seepage velocity, darcy's velocity is divided by effective porosity (n_e), not by total porosity (n)]

Hagen Poiseuille Flow:

Laminar flow in round tube is called Hagen Poiseuille flow.



One mistake done by me. It is not necessary that control volume should consider the entire cross-section. That's why Shear stress ($2\pi r \Delta L \tau$) contains 'r', not 'r_0'. A seperate figure should be drawn for force diagram (see fig 7.4).

Figure: 7.3

$$\tau = \mu \frac{dv}{dy} \;\cdots (13.1)$$ #Newton's law of viscosity.

Where, y = distance from the pipe wall.

#J.T ⇒ We know, in pipe, velocity distribution is parabolic wrt y.

Therefore, $\frac{dv}{dy}$ is a linear quantity.

∴ Shear stress distribution ($\tau = \mu\frac{dv}{dy}$) is linear.

Consider, $y = r_0 - r$ (see fig-7.3) $\cdots\cdots$ (13.2)

$$\therefore \tau = \mu \frac{dv}{dy}$$

$$= \mu \frac{dv}{dr} \cdot \frac{dr}{dy}$$

$$= -\mu \frac{dv}{dr} \quad \left[\because \frac{dr}{dy} = -1, \text{from (13.2)}\right] \cdots\cdots (13.3)$$

Considering force balance in control volume,

$$PA - (p + \frac{dp}{ds}\Delta L)A - W\sin\alpha - (2\pi r \Delta L)\tau = 0$$

$\cdots\cdots$ (13.4)

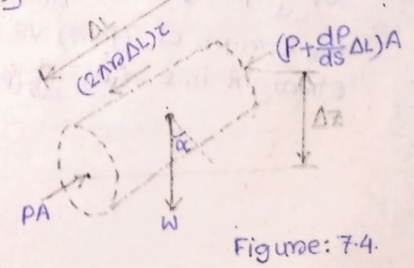


Figure: 7.4.

Where,

$W =$ Weight of c.v $= \gamma A \Delta L$ . . . . (13.5)

and $\sin\alpha = \dfrac{\Delta z}{\Delta L}$ . . . . (13.6)

From (13.4),

$$-\frac{dp}{ds}\Delta L\, A - W\sin\alpha - (2\pi r \Delta L)\tau = 0$$

$$\Rightarrow -\frac{dp}{ds}\Delta L . A - \gamma A \Delta L . \frac{\Delta z}{\Delta L} - (2\pi r \Delta L)\tau = 0 \;\ldots\ldots\; (13.6.1)$$

$$\Rightarrow -A\Delta L . \frac{dp}{ds} - A\Delta L\, \gamma \frac{dz}{ds} - 2 . \frac{\pi r^2}{r} . \Delta L . \tau = 0$$ [$\sin\alpha$ can be written as $\frac{dz}{ds}$ also]

$$\Rightarrow -A\Delta L\left[\frac{dp}{ds} + \frac{d}{ds}(\gamma z) + \frac{2}{r} . \tau\right] = 0$$ [$\because \pi r^2 = A$]

$$\Rightarrow \frac{d}{ds}(P+\gamma z) = -\tau . \frac{2}{r}$$

$$\Rightarrow \boxed{\tau = -\frac{r}{2}\frac{d}{ds}(P+\gamma z)} \;\ldots\ldots\ldots\; (13.7).$$

From, equation (13.1),

$$\tau = \mu\frac{dv}{dy}$$

$$\Rightarrow -\frac{r}{2}\frac{d}{ds}(P+\gamma z) = \mu\frac{dv}{dy}$$

$$= \mu\frac{dv}{dr} . \frac{dr}{dy}$$

$$= -\mu\frac{dv}{dr}$$ [$\because$ From (13.2), $\frac{dr}{dy} = -1$]

$$\Rightarrow \boxed{\left(\frac{2\mu}{r}\right)\frac{dv}{dr} = \frac{d}{ds}(p+\gamma z).} \;\ldots\ldots\; (13.8)$$

↳ Another form of shear stress distribution.

Equation (13.8) can be true if,

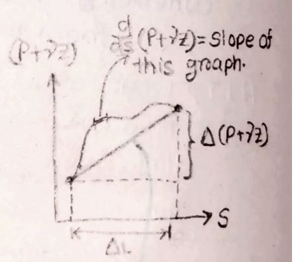
$$\text{Cosntant} = \frac{d}{ds}(p+\gamma z) = \frac{\Delta(P+\gamma z)}{\Delta L} = \frac{\gamma \Delta h}{\Delta L}$$ # Δh = Piezometric head!

. . . . (13.9)

Eq(13.9):- My POV:-

In (13.6.1), we replaced $\frac{\Delta z}{\Delta L}$ by $\frac{dz}{ds}$ to get the desired form. Now, in (13.6.1), p is in differentiation form (i.e. $\frac{dp}{ds}$) and z is in delta form (i.e $\frac{\Delta z}{\Delta L}$).

So, $\frac{d}{ds}(P+\gamma z)$ and $\frac{\Delta(P+\gamma z)}{\Delta L}$ would be same, if these are a constant value.



Because, ~~if slope of (P+γz)~~

$\frac{d}{ds}(P+\gamma z)$ represents slope at a small portion of (P+γz) vs 'S' graph.

On the other hand, $\frac{\Delta(P+\gamma z)}{\Delta L}$ represents slope of the line which connects first and last points of the graph.

So, $\frac{d}{ds}(P+\gamma z)$ and $\frac{\Delta(P+\gamma z)}{\Delta L}$ should be same, if the graph of '(P+γz)' vs 'S' is ~~linear~~ straight line. i.e, $\frac{d}{ds}(P+\gamma z) = \frac{\Delta(P+\gamma z)}{\Delta L}$ = constant.

$\frac{\Delta(P+\gamma z)}{\Delta L}$ = Slope of this line.

So, for both become equal, Graph should be linear. So, curved line is not allowed for making both the quantities same.

Putting value from (13.9) to (13.8),

$$\frac{dv}{dr} = \left(\frac{r}{2\mu}\right) \times \left(\frac{\gamma \Delta h}{\Delta L}\right)$$

Integrating,

$$\int dv = \frac{1}{2\mu} \frac{\gamma \Delta h}{\Delta L} \int r\,dr$$

$$\Rightarrow V = \left(\frac{r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) + C \quad \cdots\cdots (13.10)$$

No-slip condition:-

velocity of fluid at wall is zero.

$$V_{r=r_0} = 0 \quad \cdots\cdots (13.11)$$

$\Rightarrow$ Applying (13.11) boundary condition in equation (13.10),

$$C = -\left(\frac{r_0^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) \quad \cdots\cdots (13.12)$$

Putting value of C in (13.10),

$$\boxed{V(r) = -\left(\frac{r_0^2 - r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)} \quad \cdots\cdots (13.13).$$

$\therefore$ Max. velocity (at $r=0$),

$$V_{max} = -\left(\frac{r_0^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) \quad \cdots\cdots (13.14).$$

$$\Rightarrow \frac{V_{max}}{r_0^2} = -\frac{1}{4\mu}\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

Putting this value in (13.13),

$$V(r) = (r_0^2 - r^2) \cdot \frac{V_{max}}{r_0^2}$$

$$\Rightarrow V(r) = V_{max}\left(1 - \frac{r^2}{r_0^2}\right) \quad \cdots\cdots (13.15).$$

Average Velocity:-

$$V_{avg} = \frac{1}{A} \iint V\,dA$$

$$= \frac{1}{\pi r_0^2} \int_0^{r_0} -\left(\frac{r_0^2 - r^2}{4\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) 2\pi r\,dr$$

[Putting V(r) from (13.13)].

$$= -\frac{1}{r_0^2 \times 2\mu}\left(\frac{\gamma \Delta h}{\Delta L}\right) \int_0^{r_0} (r_0^2 - r^2) r\,dr$$

$$\left[\frac{r_0^2 r^2}{2} - \frac{r^4}{4}\right]_0^{r_0} = \left(\frac{r_0^4}{2} - \frac{r_0^4}{4}\right) = \frac{r_0^4}{4}$$

$$= -\frac{1}{2\mu r_0^2}\left(\frac{\gamma \Delta h}{\Delta L}\right) \times \frac{r_0^4}{4}$$

$$= -\left(\frac{r_0^2}{8\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) \quad \cdots\cdots (13.16)$$

Comparing V_{avg} and V_{max} (13.16 and 13.14),

$$V_{avg} = \frac{V_{max}}{2} \quad \cdots\cdots (13.17).$$

Darcy's law and General fluid flow:-

For Hagen-Poiseuille flow, (from 13.16),

$$V_{avg} = -\left(\frac{r_0^2}{8\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right)$$

Again, $V_{avg} = \frac{q}{n}$ [q = Specific discharge or Darcy's velocity, # Note that, V_{avg} = Seepage velocity]. $\cdots$ (13.18)

$$\Rightarrow q = n \cdot V_{avg} = -n\left(\frac{r_0^2}{8\mu}\right)\left(\frac{\gamma \Delta h}{\Delta L}\right) \quad \cdots\cdots (13.19)$$

If we compare (13.19) with Darcy's law, i.e. $q = -Ki = -K\frac{\Delta h}{\Delta L}$,

$$q = -\left(\frac{\gamma}{\mu}\cdot\frac{n r_0^2}{8}\right)\frac{\Delta h}{\Delta L} \quad \ldots\ldots (13.20)$$

$\therefore$ We can say,

Hydraulic conductivity, $K = \frac{\gamma}{\mu}\cdot\frac{n r_0^2}{8}$ .. (13.21)

Also, we know, $K = \frac{\gamma}{\mu} k$ ..... (13.22)

where, k = Intrinsic permeability.

Comparing (13.21) & (13.22), we get,

$$k = \frac{n r_0^2}{8} \quad \ldots\ldots\ldots (13.23).$$

Validity of Darcy's law:

- Darcy's velocity or specific discharge is a fictitious quantity. This law assumes that flow occurs through entire c/s, but only through pores it occurs.
- This law assumes flow paths are linear. But, actually it is not.
- Darcy's law ignores kinetic energy. (GW has low velocity). (If you look into figure 7.2 of page 94, only pressure and elevation head are considered for hydraulic gradient calculation)
- Applicable for laminar flow. Reynold's number, $R_e = \frac{\rho q d_{10}}{\mu}$ Should be less than 1.

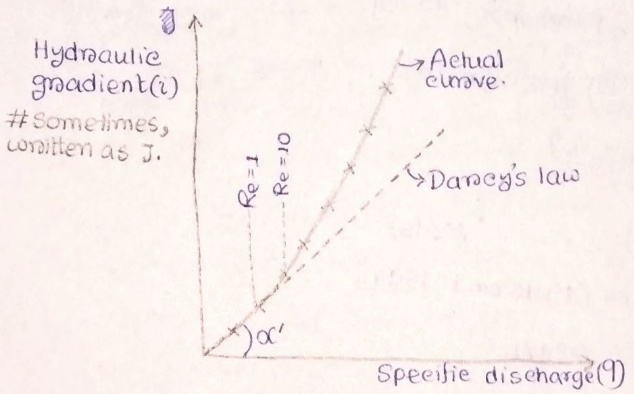


Figure: 7.5

$q = Ki$

$\Rightarrow \frac{i}{q} = \frac{1}{K} = \tan\alpha'$

Lecture-8

Principles of flow:

To estimate hydraulic conductivity.

Relation between
Permeability (Hydraulic conductivity)
and intrinsic permeability:

$$K = k \frac{\rho g}{\mu} \quad \ldots\ldots\ldots \quad (14.1)$$

k = Intrinsic permeability ⇒ Porous media
Property (See 13.23).

ρ, μ = Fluid properties.

☐ Values of hydraulic conductivity:-

Unweathered marine clay < Silt, loess < Clean sand < Gravel.

- Unit of intrinsic permeability,
  1 darcy = 9.87×10^{-9} cm^2.

Darcy's law in Three Dimensions

Flow components:

$$\left.\begin{aligned} q_x &= -K_x \frac{\partial h}{\partial x} \\ q_y &= -K_y \frac{\partial h}{\partial y} \\ q_z &= -K_z \frac{\partial h}{\partial z} \end{aligned}\right\} \quad \ldots\ldots \quad (14.2)$$

$$q = q_x \hat{i} + q_y \hat{j} + q_z \hat{k} \quad \ldots\ldots \quad (14.3)$$

Magnitude of specific discharge vector,

$$|q| = \sqrt{q_x^2 + q_y^2 + q_z^2} \quad \ldots\ldots\ldots \quad (14.4).$$

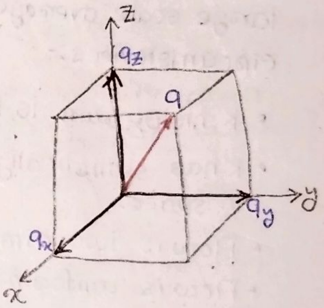


Figure: 8.1

Heterogenity and Anisotropy of Hydraulic conductivity:-

- Deterministic Approach

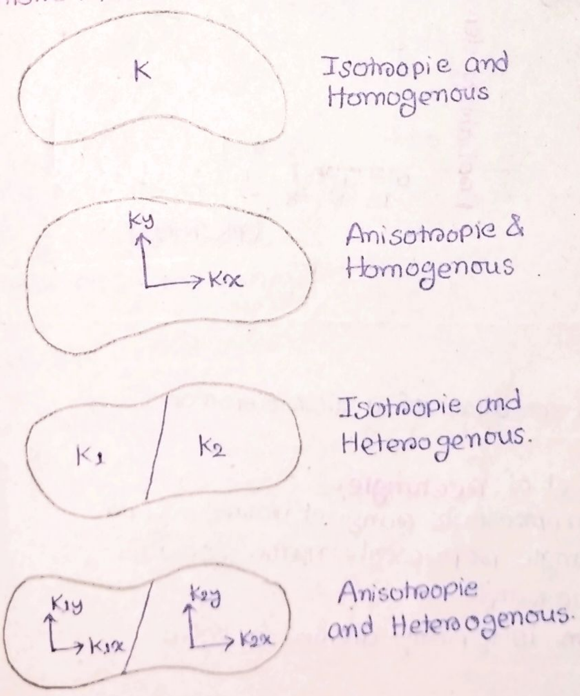


Figure: 8.2

• Stochastic approach:-

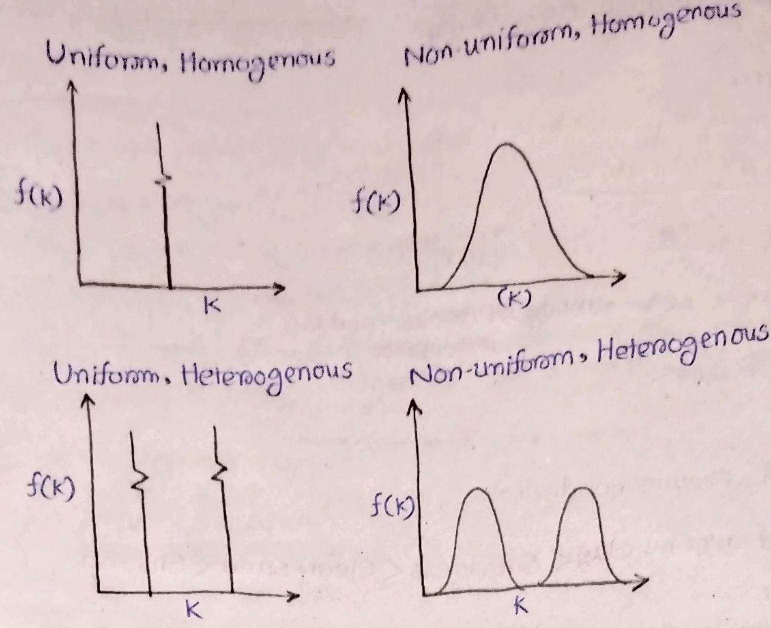


Figure: 8.3

Probabilistic Analysis:-

Matheron (1967) determined that the geometric mean of small-scale K measurements give.s the appropriate large scale average K under the following circumstances:-

- K histogram is log-normal.
- K has statistically isotropic distribution in space.
- Flow is two-dimensional.
- Flow is uniform (1-D in large scale).
  (Flow is 2D, but 1D in large scale. Mean?!).
- The GM of n numbers of K measurements:-

$$K_g = (k_1 k_2 k_3 \ldots\ldots k_n)^{1/n} \quad \ldots\ldots (15)$$

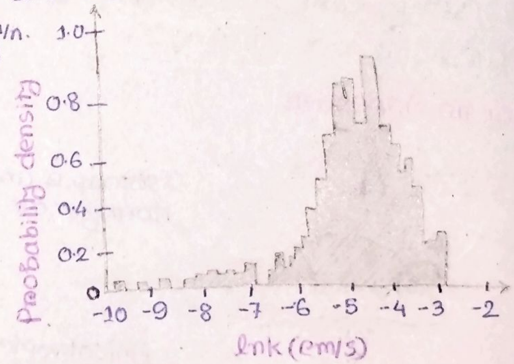


Figure: 8.4.

Histogram:-

- Graphical representation of a distribution of numerical data.
- Consists of a set of rectangle.
  Each rectangle represents range of values in data.
  Height of rectangle represents number of data Points within the range.
- Entire histogram is typically divided in equal

intervals or beans bins along x-axis.

- Histogram helps to provide a quick and intuitive (বোধমূলক) overview of data.

Small scale and large scale measurement:

Relative size or magnification of the object being measured.

S.S.M ⇒ Length of cell, size of molecule. (Requires specialized expertise, techniques and equipments).

L.S.M ⇒ Height of buildings, distance between two cities.

Specific Discharge vectors at an interface:-

At the boundary between two materials with different K values, flow paths are bent in a manner similar to optical refraction.

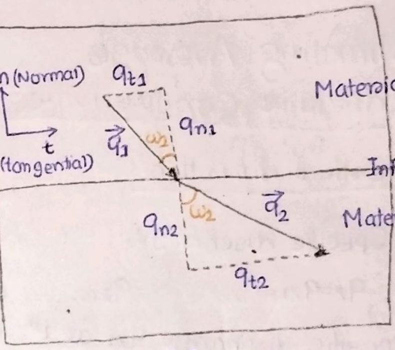


Figure: 8.5

☑ Two conditions must be satisfied:-

(i) Specific discharge component normal to the interface is same on both side to preserve continuity of flow.

$$q_{n1} = q_{n2} \quad \ldots\ldots (16.1)$$

(ii) Pressure is continuous in a fluid.

∴ Head must be continuous across the interface.

$$\left(\frac{\partial h}{\partial t}\right)_1 = \left(\frac{\partial h}{\partial t}\right)_2 \quad \ldots\ldots (16.2).$$

- ω_1, ω_2 angles in figure 8.5:-

$$\tan\omega_1 = \frac{q_{t1}}{q_{n1}} \text{ and } \tan\omega_2 = \frac{q_{t2}}{q_{n2}} \quad \ldots\ldots (16.3)$$

- From (16.3), we have,

$$\frac{\tan\omega_1}{\tan\omega_2} = \frac{q_{t1}}{q_{n1}} \times \frac{q_{n2}}{q_{t2}}$$

$$= \frac{q_{t1}}{q_{t2}} \quad [\because \text{From (16.1), } q_{n1} = q_{n2}]$$

$$= \frac{K_{t1} \cdot i_{t1}}{K_{t2} \cdot i_{t2}}$$

$$= \frac{K_{t1}}{K_{t2}} \quad \ldots\ldots\ldots (16.4).$$

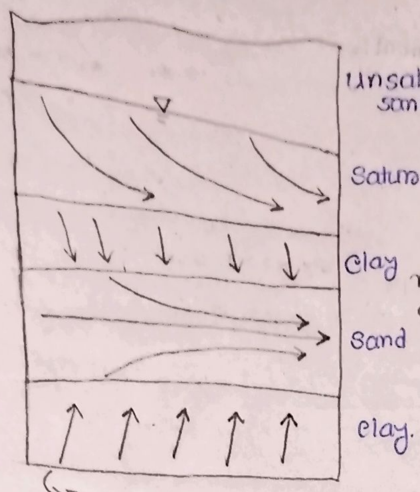
? (# Hydraulic gradient is same in tangential direction for two materials! But, how we knew that ?!)

- Special case:- (From 16.4)

when $K_{t1} \ll K_{t2}$, $\tan\omega_1 \to 0$ and $\tan\omega_2 \to \infty$

i.e $\omega_1 \to 0$ and $\omega_2 \to \frac{\pi}{2}$.

Specific discharge vectors at an interface (contd.)



(Sand layer has more horizontal flow component value than clay).

I.T (Some hard rock region is there, flow is neglected)?!

Fig: 8.6

☑ Estimating Average Hydraulic Conductivities :-

(i) In Vertical directions:

Specific discharge,

$$q_z = q_{z1} = q_{z2} = \cdots = q_{zn} \quad \cdots \cdots (17.1)$$

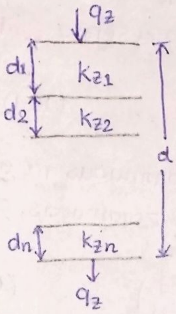


Figure: 8.7

Specific discharge q_{zi} at i^{th} layer,

$$q_{zi} = -K_{zi} \frac{\Delta h_i}{d_i} \quad \cdots\cdots\cdots (17.2)$$

Total head loss,

$$\Delta h = \Delta h_1 + \Delta h_2 + \cdots\cdots + \Delta h_n \quad \cdots\cdots (17.3)$$

Putting the value of Δh_i from (17.2) to (17.3),

$$\frac{d q_z}{K_{ze}} = \frac{d_1 q_{z1}}{K_{z1}} + \frac{d_2 q_{z2}}{K_{z2}} + \cdots\cdots + \frac{d_n q_{zn}}{K_{zn}} \quad \cdots\cdots (17.4)$$

$$\Rightarrow \frac{d}{K_{ze}} = \frac{d_1}{K_{z1}} + \frac{d_2}{K_{z2}} + \cdots + \frac{d_n}{K_{zn}} \quad \left[\because \text{From } (17.1),\ q_z = q_{z1} = q_{z2} = \cdots q_{zn}\right]$$

$$\Rightarrow K_{ze} = \frac{d}{\frac{d_1}{K_{z1}} + \frac{d_2}{K_{z2}} + \cdots + \frac{d_n}{K_{zn}}}$$

#Here, Specific discharges for layers are same.

$$\Rightarrow K_{ze} = \frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} \frac{d_i}{K_{zi}}} \quad \cdots\cdots\cdots (17.5).$$

(ii) In horizontal direction:-

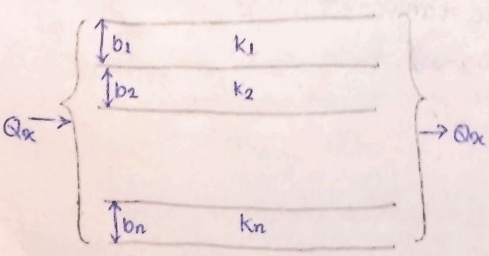


Figure 8.8.

From fig 8.8,
Discharge Q_{xi} for i^{th} layer is given by,

$$Q_{xi} = -k_{xi} \frac{\partial h}{\partial x} d_i \quad \cdots\cdots (18.1)$$

For vertical direction, specific discharge or Darcy's velocity (q_z) was considered. Because, flow area was considered as unit area for each layer.

But in horizontal flow, total discharge (Q_x) was considered. Because, flow area was not same for all layers i.e($b_i \times 1$)

Total discharge, $Q_x = Q_{x1} + Q_{x2} + Q_{x3} + \cdots\cdots + Q_{xn} \quad \cdots (18.2)$

$$\Rightarrow k_{xe} \frac{\partial h}{\partial x} d = k_{x1} \frac{\partial h}{\partial x} d_1 + k_{x2} \frac{\partial h}{\partial x} d_2 + \cdots + k_{xn} \frac{\partial h}{\partial x} d_n \quad \cdots (18.3).$$

$$\Rightarrow k_{xe} \cdot d = k_{x1} d_1 + k_{x2} d_2 + \cdots + k_{xn} d_n$$

Here, hydraulic gradients for layers are same

$$\Rightarrow k_{xe} = \frac{k_{x1} d_1 + k_{x2} d_2 + \cdots + k_{xn} d_n}{d}$$

$$\Rightarrow k_{xe} = \frac{\sum_{i=1}^{n} k_{xi} d_i}{\sum_{i=1}^{n} d_i} \quad \cdots\cdots\cdots (18.4).$$

- Horizontal hydraulic conductivity for alluviam
#(পলিমাটি, loose, unconsolidated deposit by rivers, flood etc. Mixture of clay, sand, silt and gravel).
is greater than that in vertical direction.

- Ratio of k_H/k_V is 2-10 for alluvium, but may be more than 100 when clay layer is present.

J.T ⇒ (When clay layer is present, vertical flow is restricted. So, k_H/k_V ratio increased for other layers.
But, for fig 8.6, why specific discharge vectors of clay are shown vertical for clay if they resist vertical flow) (?)

Transmissivity:

- For a layer, if hydraulic conductivity (K) assumed constant over the thickness(b), transmissivity, $T = kb$

- Definition:- Transmissivity is the amount of water that moves horizontally through the unit width of saturated aquifer as a result of unit change in gradient.

$T = q_x \cdot A$ [q_x = Darcy's velocity, A = Total c/s area.]

$= (k_x \cdot i) \cdot (b \cdot 1)$

$= k_x \cdot b$ [Putting hydraulic gradient and width as 1].

observation:- Transmissivity is a discharge quantity in a sense.

- If a layer is composed of m strata of thickness b_i and hydraulic conductivity k_i, then total transmissivity of that layers is

layers is sum of transmissivity of all stratum.

$$T = \sum_{i=1}^{m} T_i = \sum_{i=1}^{m} K_i b_i$$

- In an unconfined aquifer, transmissivity is not well defined as confined aquifer.

Measuring Hydraulic Conductivity:

- Hazen's (1911) empirical relations, (Sand sample).

$$k = C(d_{10})^2 \quad \ldots\ldots\ldots (19.1).$$

($k \to cm/s$, $C \to (cm.sec)^{-1}$, $d_{10} \to cm$).

C varies from 40 to 150.

Lower end of C values → fine, widely graded Sand.

Upper end of C values → Coarse narrowly graded sand.

- Kozeny karman equation:

$$k = \left(\frac{\rho_w g}{\mu}\right)\cdot\left(\frac{n^3}{(1-n)^2}\right)\cdot\left(\frac{d_{50}^2}{180}\right) \quad \ldots\ldots\ldots\ldots (19.2)$$

This eqn is dimensionally consistent.

Laboratory Measurement:

(i) Constant head Permeameter:

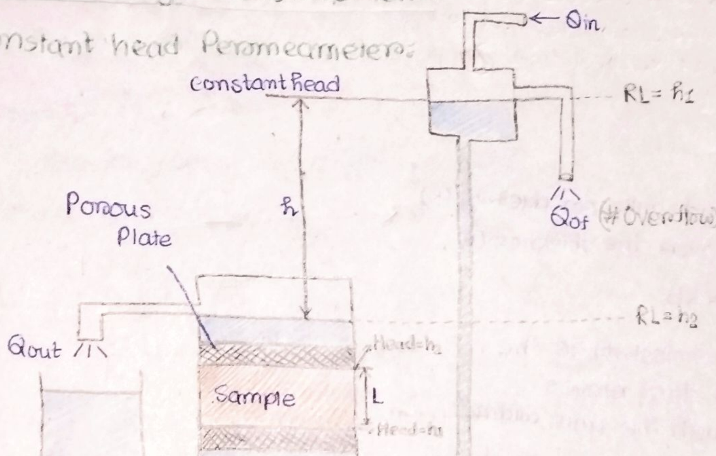


Figure: 8.9

Here, $h = h_1 - h_2$

$$Q = -KiA$$
$$= -K \cdot \frac{h_2 - h_1}{L} A$$
$$= +KA \cdot \frac{h}{L} \quad [\because h_1 - h_2 = h]$$
$$\Rightarrow K = \frac{QL}{Ah} \quad \ldots\ldots\ldots\ldots (20)$$

Where, $h = -\Delta h$
$= -(h_2 - h_1)$
(i.e. h = constant head difference)

Meaning of h_1 and h_2 are different for these cases

(ii) Falling head Permeameter:

Inflow discharge from tube to sample, (# From continuity)

$$Q_{in} = -A_t \frac{dh}{dt} \quad \ldots\ldots (21.1)$$ # (minus, because head decreasing)

Outflow discharge from the sample, (# From Darcy's law).

$$Q_{out} = kA_c \cdot \frac{h}{L} \quad \ldots\ldots (21.2)$$

Where, h = Difference of head (Δh)
$= h_1$ (at the beginning, i.e, $t = t_1$)
$= h_2$ (at the end, i.e $t = t_2$)

For steady-state condition,

$$Q_{in} = Q_{out} \quad \ldots\ldots (21.3)$$

So, from (21.1) and (21.2),

$$-A_t \frac{dh}{dt} = K A_c \frac{h}{L}$$

$$\Rightarrow \int_{h_1}^{h_2} \frac{dh}{h} = -\frac{K A_c}{L A_t} \int_{t_1}^{t_2} dt$$

$$\Rightarrow \ln\left(\frac{h_2}{h_1}\right) = -\frac{K A_c}{L A_t}(t_2 - t_1)$$

$$\Rightarrow K = \frac{A_t \cdot L}{A_c(t_2 - t_1)} \ln\left(\frac{h_1}{h_2}\right) \quad \ldots\ldots (23.4)$$

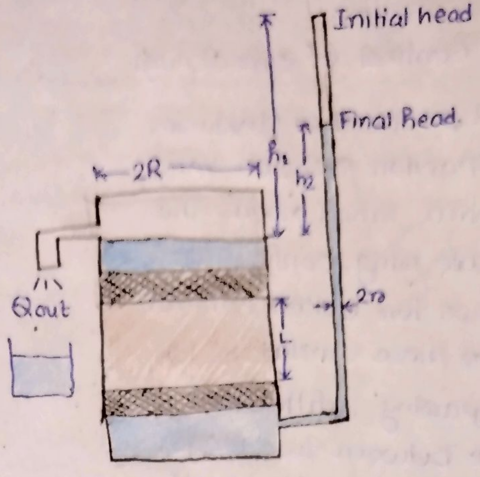


Figure: 8.10.

Lecture-9
Principles of Flow

To estimate field level saturated hydraulic conductivity (H.C).

To determine directional H.C.

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Field Measurement of H.C.

Double ring Infiltrometer:-

- Inner and outer rings are inserted in the ground.
- Same water height in outer and inner cylinders should be maintained.

Otherwise, due to horizontal hydraulic gradient, horizontal movement of water will occur and rate of infiltration (in vertically downward) will decrease.

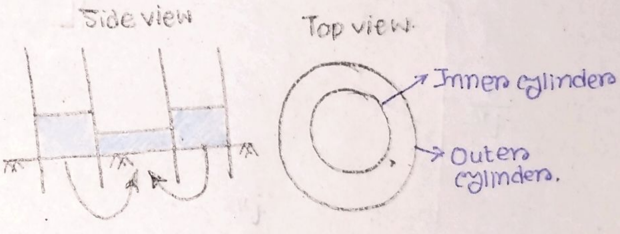


Figure: 9.1

Two rings are chosen for several reason:-

(i) To minimize edge effect:-

Outer ring serves as buffer zone to minimize potential influence of sidewall effects. It helps prevent water from flowing laterally or escaping around the edges of the inner ring, ensuring water infiltrates vertically into the soil.

Outer ring helps isolate the measurement area from surrounding soil condition.

In short, we can say, Outer cylinder reduces 3D radial flow from inner cylinder.

(iii) Control of variability:

Soil properties (texture, compaction etc) can vary within a small area. The double ring configuration allows for better control over these variations. By comparing infiltration rate between two rings, researchers can identify the influence of soil heterogenity on infiltration behaviour.

• Cumulative infiltration:-

$$I = St^{1/2} + At \;\cdots\cdots \quad (24.1)$$

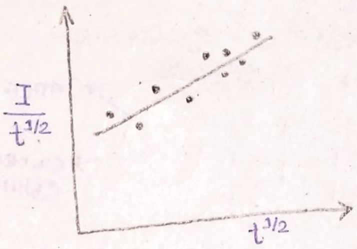
where, S = Soil sorptivity $(LT^{-1/2})$

A = Constant (LT^{-1})

How? $\left\{\begin{array}{l} K = A \text{ as } t \to \infty \\ \text{For shorter time, } A = 0.5K,\ A = \frac{2}{3}K \end{array}\right\}$ These are the observations from Double ring infiltrometer test.

In $y = mx + c$ form, eqn (24.1) can be written as,

$$\frac{I}{t^{1/2}} = At^{1/2} + S \;\cdots \quad (24.2)$$



Darcy's law in 3D:

$$\underset{\sim}{q} = -\underset{\sim}{K} \cdot \nabla h \;\cdots \quad (25.1)$$

$$\underbrace{\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}}_{q = q_x\hat{i} + q_y\hat{j} + q_z\hat{k}} = -\underbrace{\begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}}_{\text{coff matrix (i.e. stiffness matrix)}} \begin{Bmatrix} \frac{\partial h}{\partial x} \\ \partial h/\partial y \\ \partial h/\partial z \end{Bmatrix}$$

$$\nabla h = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)h = \left(\frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j} + \frac{\partial h}{\partial z}\hat{k}\right)$$

In general form,

• Darcy's law can be written as,

$$\left.\begin{array}{l} q_x = -K_{xx}\frac{\partial h}{\partial x} - K_{xy}\frac{\partial h}{\partial y} - K_{xz}\frac{\partial h}{\partial z} \\ q_y = -K_{yx}\frac{\partial h}{\partial x} - K_{yy}\frac{\partial h}{\partial y} - K_{yz}\frac{\partial h}{\partial z} \\ q_z = -K_{zx}\frac{\partial h}{\partial x} - K_{zy}\frac{\partial h}{\partial y} - K_{zz}\frac{\partial h}{\partial z} \end{array}\right\} \cdots\cdots \quad (25.2)$$

(See. Page 143)

After P-106,

$$\underset{\sim}{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

Now, $|K-\lambda I| = \begin{bmatrix} K_{11}-\lambda & K_{12} & K_{13} \\ K_{21} & K_{22}-\lambda & K_{23} \\ K_{31} & K_{32} & K_{33}-\lambda \end{bmatrix}$

Characteristic equation,

$$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0 \cdots\cdots (25.3)$$

Where, $I_1 = tr(\underset{\sim}{K})$

$I_2 = \frac{1}{2}\left[tr(\underset{\sim}{K}^2) - [tr(\underset{\sim}{K})]^2\right]$

$I_3 = \det(\underset{\sim}{K})$ $\cdots\cdots (25.4)$

$tr(\underset{\sim}{K})$ Means trace of the matrix. $\underset{\sim}{K}$. i.e sum of diagonal terms.

?!

$$K' = \begin{bmatrix} K^{(1)} & 0 & 0 \\ 0 & K^{(2)} & 0 \\ 0 & 0 & K^{(3)} \end{bmatrix}$$

$$\left.\begin{aligned} q_1 &= -K^{(1)} \frac{dh}{dx_1} \\ q_2 &= -K^{(2)} \frac{dh}{dx_2} \\ q_3 &= -K^{(3)} \frac{dh}{dx_3} \end{aligned}\right\} \cdots\cdots (25.5)$$

I.T $\Rightarrow$ $\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = -\begin{bmatrix} K^{(1)} & 0 & 0 \\ 0 & K^{(2)} & 0 \\ 0 & 0 & K^{(3)} \end{bmatrix} \begin{Bmatrix} \frac{dh}{dx_1} \\ \frac{dh}{dx_2} \\ \frac{dh}{dx_3} \end{Bmatrix}$

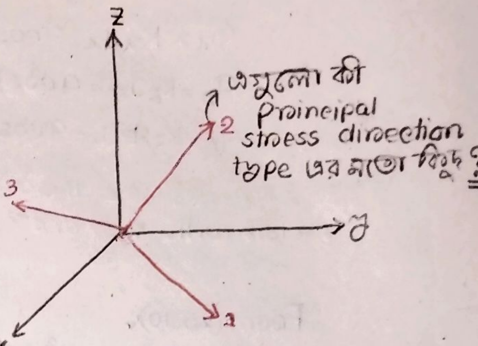


Figure: 9.2

Isotropic condition:- (Occurs when, $K_x = K_y = K_z$) $\Rightarrow$ I.T

Here, $K^{(1)} = K^{(2)} = K^{(3)} = K$.

$$K' = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix} = K \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots\cdots (25.6)$$

$$\underset{\sim}{q}\cdot\underset{\sim}{J} = qJ\cos\theta \cdots\cdots (25.7)$$

$$\cos\theta = \frac{\underset{\sim}{q}\cdot\underset{\sim}{J}}{qJ} = \frac{(q_x\hat{i} + q_y\hat{j} + q_z\hat{k})\cdot(J_x\hat{i} + J_y\hat{j} + J_z\hat{k})}{qJ}$$

$$= \frac{(K_xJ_x\hat{i} + K_yJ_y\hat{j} + K_zJ_z\hat{k})\cdot(J_x\hat{i} + J_y\hat{j} + J_z\hat{k})}{qJ}$$

$$= \frac{K_xJ_x^2 + K_yJ_y^2 + K_zJ_z^2}{qJ} \cdots\cdots (25.8)$$

I.T $\Rightarrow$
$\underset{\sim}{q}$ = Specific discharge vector
$\underset{\sim}{J}$ = Hydraulic gradient vector.

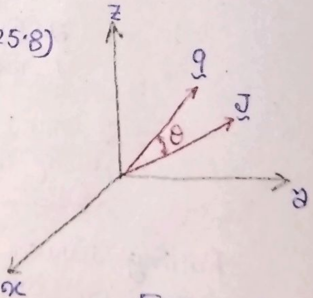


Figure: 9.3

Directional Hydraulic Conductivity

In the Direction of the flow,

Sp. discharge = q
Hydraulic gradient = $J\cos\theta$
Hydraulic conductivity = K_q.

We can write,

$$q = K_q J\cos\theta$$

$$\Rightarrow K_q = \frac{q}{J\cos\theta}$$

$$\Rightarrow K_q = \frac{q * q}{q * J\cos\theta}$$

$$\Rightarrow k_q = \frac{q^2}{q \cdot J}$$

$$\Rightarrow k_q = \frac{q \cdot q}{q \cdot J} \quad [\because q \cdot q = q^2 \cos 0^\circ = q^2]$$

$$\Rightarrow k_q = \frac{(k_x J_x \hat{i} + k_y J_y \hat{j} + k_z J_z \hat{k}) \cdot (k_x J_x \hat{i} + k_y J_y \hat{j} + k_z J_z \hat{k})}{q \cdot J}$$

$$\Rightarrow k_q = \frac{k_x^2 J_x^2 + k_y^2 J_y^2 + k_z^2 J_z^2}{k_x J_x^2 + k_y J_y^2 + k_z J_z^2}$$ (Putting $q \cdot J$ value from equation 25.8).

...... (25.9)

J.T $\Rightarrow$ $q = k \cdot J$

$$(q_x \hat{i} + q_y \hat{j} + q_z \hat{k}) = (k_x \hat{i} + k_y \hat{j} + k_z \hat{k}) \cdot (J_x \hat{i} + J_y \hat{j} + J_z \hat{k})$$

$$\left.\begin{aligned} q_x &= k_x J_x = q \cos \beta_1 \\ q_y &= k_y J_y = q \cos \beta_2 \\ q_z &= k_z J_z = q \cos \beta_3 \end{aligned}\right\} \quad \ldots (25.10)$$

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{Bmatrix} J_x \\ J_y \\ J_z \end{Bmatrix}$$

এরকম হবে কী? তাহলে LHS and RHS এ vector form Maintain হবে।

J.T ($\beta_1, \beta_2, \beta_3$ are the angles of q vector with x, y and z axes).

From (25.10),

$$\left.\begin{aligned} & k_x^2 J_x^2 = q^2 \cos^2 \beta_1 \\ & \Rightarrow \frac{\cos^2 \beta_1}{k_x} = \frac{k_x J_x^2}{q^2} \\ & \text{Similarly, } \frac{\cos^2 \beta_2}{k_y} = \frac{k_y J_y^2}{q^2} \\ & \frac{\cos^2 \beta_3}{k_z} = \frac{k_z J_z^2}{q^2} \end{aligned}\right\} \ldots (25.11)$$

Adding (25.11's),

$$\frac{\cos^2 \beta_1}{k_x} + \frac{\cos^2 \beta_2}{k_y} + \frac{\cos^2 \beta_3}{k_z} = \frac{k_x J_x^2 + k_y J_y^2 + k_z J_z^2}{q^2}$$

$$= \frac{k_x J_x^2 + k_y J_y^2 + k_z J_z^2}{k_x^2 J_x^2 + k_y^2 J_y^2 + k_z^2 J_z^2}$$

$$= \frac{1}{k_q} \quad \text{(from 25.9)}$$

.... (25.12)

Now,

$$\left.\begin{aligned} x &= r \cos \beta_1 \\ y &= r \cos \beta_2 \\ z &= r \cos \beta_3 \end{aligned}\right\} \ldots (25.13)$$

(?) r is the directional vector along the direction of specific discharge?!

Putting values of $\cos\beta_1$, $\cos\beta_2$, $\cos\beta_3$ from eq^n (25.13) to (25.12),

$$\frac{x^2}{k_x} + \frac{y^2}{k_y} + \frac{z^2}{k_z} = \frac{r^2}{k_q} \quad \ldots (25.14)$$

$$r = (k_q)^{1/2} \quad \ldots (25.15).$$

$\hookrightarrow$ How?

Because, then only it would give the equation of ellipsoid.

Directional Hydraulic Conductivity:

In the direction of gradient:

From figure 9.3, Sp. discharge = $q\cos\theta$ (# Component of sp. discharge)
Hydraulic conductivity = K_J (# Assumed quantity Need to be determined)
and hydraulic gradient = J (# We are finding in this direction. So, total value).

$$q\cos\theta = K_J * J$$

$$\Rightarrow K_J = \frac{q\cos\theta}{J}$$

$$= \frac{qJ\cos\theta}{J^2}$$

$$= \frac{\vec{q}\cdot\vec{J}}{J^2} \quad \ldots\ldots (26.1)$$

$$= \frac{(K_x J_x \hat{i} + K_y J_y \hat{j} + K_z J_z \hat{k}) \cdot (J_x \hat{i} + J_y \hat{j} + J_z \hat{k})}{J^2}$$

$$= \frac{K_x J_x^2 + K_y J_y^2 + K_z J_z^2}{J^2} \quad \ldots\ldots (26.2)$$

Now,
$$\left.\begin{aligned} J_x &= J\cos\alpha_1 \\ J_y &= J\cos\alpha_2 \\ J_z &= J\cos\alpha_3 \end{aligned}\right\} \ldots\ldots (26.3)$$

(# $\alpha_1, \alpha_2, \alpha_3$ are the angles made by hydraulic gradient vector (J) with x, y, z axes. (Gives direction cosines))

$$\left.\begin{aligned} x &= r\cos\alpha_1 \\ y &= r\cos\alpha_2 \\ z &= r\cos\alpha_3 \end{aligned}\right\} \ldots\ldots (26.4).$$

$$K_J = \frac{K_x J^2\cos^2\alpha_1 + K_y J^2\cos^2\alpha_2 + K_z J^2\cos^2\alpha_3}{J^2}$$ (Putting J_x, J_y, J_z values from 26.3 to 26.2)

$$= \frac{K_x \cdot x^2 + K_y \cdot y^2 + K_z \cdot z^2}{r^2}$$ (Putting values of $\cos\alpha_1, \cos\alpha_2, \cos\alpha_3$ from 26.4)

$$\Rightarrow \frac{r^2}{1/K_J} = \frac{x^2}{1/K_x} + \frac{y^2}{1/K_y} + \frac{z^2}{1/K_z} \quad \ldots\ldots (26.5)$$

(?) $\Rightarrow r^2 = (K_J)^{-1/2}$ ⟶ But, how?! ...... (26.6) # Then (26.5) becomes eqn of ellipsoid.

Combining (26.5) and (26.6), we have,

$$\frac{x^2}{W_x} + \frac{y^2}{W_y} + \frac{z^2}{W_z} = 1 \quad \ldots\ldots (26.7)$$

where, $W_x = \frac{1}{K_x}$, $W_y = \frac{1}{K_y}$, $W_z = \frac{1}{K_z}$.

GW flow velocity calculation in 2D:

$$h(x,y) = a + bx + cy \quad \ldots\ldots (27.1)$$

$$\left.\begin{aligned} h(x_1,y_1) &= a + bx_1 + cy_1 \\ h(x_2,y_2) &= a + bx_2 + cy_2 \\ h(x_3,y_3) &= a + bx_3 + cy_3 \end{aligned}\right\} \ldots\ldots (27.2)$$

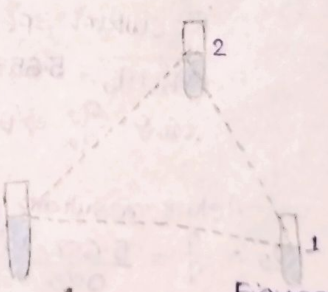


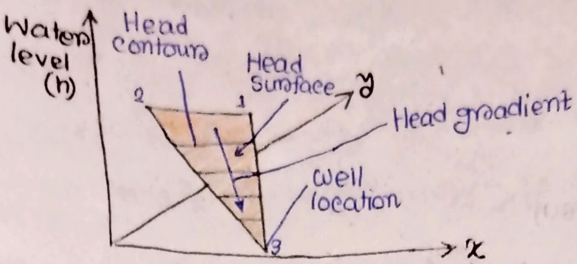
Figure: (9.4)

(?) কেন 2D Plane এ flow.এর কথা বলা হচ্ছে, তাহলে কি wells গুলোর Water level connect করছে?

So, we have three equations and three unknowns a, b, c

$$b = \frac{(h_1-h_2)(y_2-y_3)-(h_2-h_3)(y_1-y_2)}{(x_1-x_2)(y_2-y_3)-(x_2-x_3)(y_1-y_2)} \quad \ldots\ldots (27.3)$$

$$c = \frac{(h_1-h_2)(x_2-x_3)-(h_2-h_3)(x_1-x_2)}{(y_1-y_2)(x_2-x_3)-(y_2-y_3)(x_1-x_2)} \quad \ldots\ldots (27.4)$$

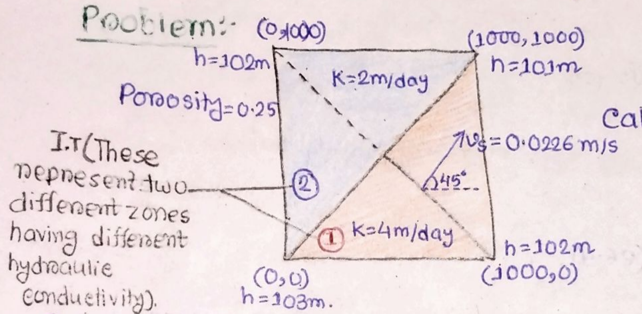


Groundwater gradient is given by:-

$$\nabla h = \frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j} \quad \ldots\ldots (27.5)$$

From 27.1, $\left.\begin{array}{l}\frac{\partial h}{\partial x} = b \\ \frac{\partial h}{\partial y} = c\end{array}\right\} \ldots\ldots (27.6)$

Problem:-



Calculate groundwater velocity.

Solution:

I.T ⇒ (These zones have isotropic media. So, given K values = $K_{xx} = K_{yy}$).

Zone 1:

$$\frac{dh}{dx} = \frac{102-103}{1000-0} = -\frac{1}{1000}$$

$$\frac{dh}{dy} = \frac{101-102}{1000-0} = -\frac{1}{1000}$$

$$q_x = -K_{xx}\frac{dh}{dx} = -4\times\frac{-1}{1000} = 0.004$$

$$q_y = -K_{yy}\frac{dh}{dy} = -4\times\left(-\frac{1}{1000}\right) = 0.004$$

These are sp. discharge or Darcy's velocity.

∴ Resultant specific discharge (q)

$$= \sqrt{q_x^2+q_y^2} = 5.657\times10^{-3}\ m/s.$$

$$\tan\theta = \frac{q_y}{q_x} \Rightarrow \theta = \tan^{-1}(1) = 45°$$

Actual resultant seepage velocity

$$v_s = \frac{q}{\eta} = \frac{5.657\times10^{-3}}{0.25} = 0.0226\,m/s.$$

Lecture 10

Groundwater flow equations

To derive the governing equation for saturated confined aquifer.

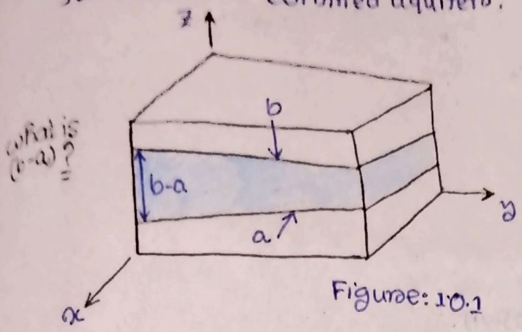


Figure: 10.1

3D saturated flow in confined aquifer:

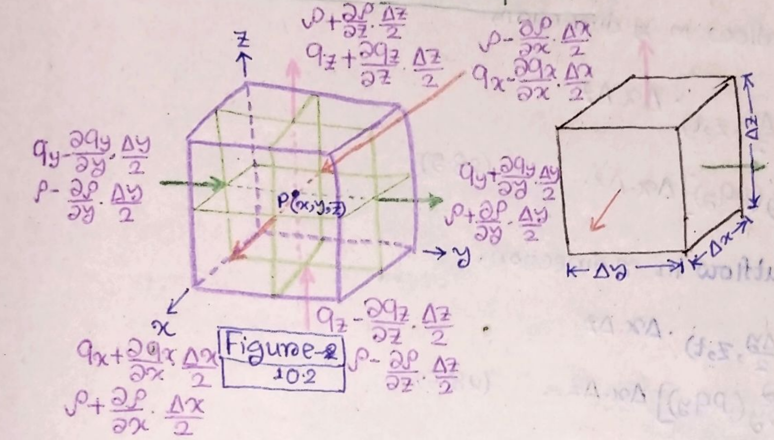
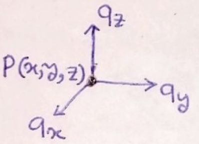


Figure-2 10.2



$P\,(q_x, q_y, q_z, \rho, \eta)$

J.T ⇒ (P is the centrepoint of the cuboid of sides $\Delta x, \Delta y, \Delta z$. $q_x, q_y, q_z, \rho, \eta$ are the values of parameters at point P. From these, we evaluated the face values of the parameters).

- Mass inflow rate − Mass outflow rate = Rate of change of mass storage with time.

- Fig-2; Elementary control volume of soil.
  Volume $= \Delta x \Delta y \Delta z$ .... (28.1)

  Mass of groundwater in this C.V,
  $M = \rho \eta \Delta x \Delta y \Delta z$....... (28.2).

J.T # ρ is the average density in the control volume.
It is saturated flow. So, volume of water = Porosity × Total volume.

Mass of Ground water inflow in x direction,

$$= \rho\Big|_{(x-\frac{\Delta x}{2}, y, z, t)} \cdot q_x\Big|_{(x-\frac{\Delta x}{2}, y, z, t)} \cdot \Delta y \Delta z$$

$$\approx \left(\rho - \frac{\partial \rho}{\partial x}\cdot\frac{\Delta x}{2}\right)\left(q_x - \frac{\partial q_x}{\partial x}\cdot\frac{\Delta x}{2}\right)\cdot \Delta y \Delta z \qquad \text{[\# See fig (10.2)]}$$

$$\approx \left[\rho q_x - \frac{\Delta x}{2}\left(\rho\frac{\partial q_x}{\partial x} + q_x\frac{\partial \rho}{\partial x}\right) + \left(\frac{\Delta x}{2}\right)^2 \frac{\partial \rho}{\partial x}\cdot\frac{\partial q_x}{\partial x}\right]\Delta y \Delta z \qquad \text{[\# only consider 1st order term]}$$

$$\approx \left[\rho q_x - \frac{\Delta x}{2}\frac{\partial}{\partial x}(\rho q_x)\right]\Delta y \Delta z \,.\,.\,.\,.\,.\,.\,.\; (28.3).$$

Similarly, if we put density and specific discharge values for different faces from fig (10.2), we get,

- Mass of GW outflow in x direction,

$$\left[\rho q_x\right]_{(x+\frac{\Delta x}{2},\, y,\, z,\, t)} \cdot \Delta y \Delta z$$

$$\approx \left(\rho+\frac{\partial \rho}{\partial x}\cdot\frac{\Delta x}{2}\right)\cdot\left(q_x+\frac{\partial q_x}{\partial x}\cdot\frac{\Delta x}{2}\right)\cdot\Delta y\cdot\Delta z$$

$$\approx\left[\rho q_x+\frac{\Delta x}{2}\cdot\frac{\partial}{\partial x}(\rho q_x)\right]\Delta y\Delta z \ldots\ldots (28.4).$$

Similarly,

- Mass of GW inflow in y direction,

$$\left[\rho q_y\right]_{(x,\, y-\frac{\Delta y}{2},\, z,\, t)} \cdot \Delta x \Delta z$$

$$\approx\left[\rho q_y-\frac{\Delta y}{2}\frac{\partial}{\partial y}(\rho q_y)\right]\Delta x . \Delta z \ldots\ldots (28.5)$$

- Mass of GW outflow in y direction,

$$\left[\rho q_y\right]_{(x,\, y+\frac{\Delta y}{2},\, z,\, t)} \cdot \Delta x . \Delta z$$

$$\approx\left[\rho q_y+\frac{\Delta y}{2}\cdot\frac{\partial}{\partial y}(\rho q_y)\right]\Delta x . \Delta z \ldots\ldots (28.6).$$

- Mass of GW inflow in z direction,

$$\left[\rho q_z\right]_{(x,\, y,\, z-\frac{\Delta z}{2},\, t)} \cdot \Delta x \Delta y$$

$$\approx\left[\rho q_z-\frac{\Delta z}{2}\cdot\frac{\partial}{\partial z}(\rho q_z)\right]\Delta x . \Delta y \ldots\ldots (28.7)$$

- Mass of GW outflow in z direction,

$$\left[\rho q_z\right]_{(x,\, y,\, z+\frac{\Delta z}{2},\, t)} \cdot \Delta x \Delta y$$

$$=\left[\rho q_z+\frac{\Delta z}{2}\frac{\partial}{\partial z}(\rho q_z)\right] . \Delta x \Delta y \ldots\ldots (28.8)$$

Now, we know,

$$\frac{\partial M}{\partial t}=\text{Inflow}-\text{Outflow} \ldots\ldots (28.9)$$

If outflow is more, mass of the system will decrease. so, $\frac{\partial M}{\partial t}$ is negative.

considering equations (28.3) to (28.8), (inflow - outflow) of (28.9) can be written as,

$$\boxed{\frac{\partial M}{\partial t}=-\left[\frac{\partial}{\partial x}(\rho q_x)+\frac{\partial}{\partial y}(\rho q_y)+\frac{\partial}{\partial z}(\rho q_z)\right]\Delta x\Delta y\Delta z} \ldots\ldots (28.10)$$

From (28.2), we can write,

$$\frac{\partial M}{\partial t}=\frac{\partial}{\partial t}(\rho\eta\Delta x\Delta y\Delta z)$$

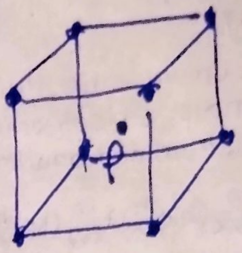
$$=\frac{\partial}{\partial t}(\rho\eta A L_z) \ldots\ldots (28.11)$$

I.T #($\Delta x\Delta y = A$ and $\Delta z = L_z$) ?

• change in mass can be written as:-

$$dM = \rho V_T(n\beta+\alpha)dP \cdots\cdots (28.12)$$

$$\left.\begin{array}{l} \text{with } V_T = \Delta x \Delta y \Delta z = AL_z \\ \text{and } dP = \rho g dh \end{array}\right\} \cdots\cdots (28.13)$$



(28.12) can be written as,

$$dM = \rho V_T \underline{(n\beta+\alpha)\rho g} dh$$

$$= \rho V_T S_s\, dh \quad [\text{where, } S_s = \text{Specific storage}].$$

Specific storage is water volume stored defined for unit volume of aquifer and unit decline in piezometric head. So, if we multiply it with $V_T.dh$, we get dV_w (i.e. change of water volume). Multiply it with ρ gives change in mass.

Or, you can see (8.14) to get dM (Page-88).

From (28.12),

$$dM = \rho V_T(n\beta+\alpha)\rho g dh$$

$$= \rho^2 g V_T (n\beta+\alpha)\, dh$$

$$\Rightarrow \frac{\partial M}{\partial t} = \underline{\rho^2 g V_T(n\beta+\alpha)} \cdot \frac{\partial h}{\partial t} \cdots\cdots (28.14).$$

From (28.3) to (28.8), we can see, ρ changes with space. In (28.14), we can see, ρ does not change with time. Why? → (average value of density - ρ in CV)

As, we defined ρ as $\rho(x,y,z,t)$

From (8.16), $S_s = \rho g(n\beta+\alpha)$.
Putting this value in (28.14),

$$\frac{\partial M}{\partial t} = \rho S_s . V_T . \frac{\partial h}{\partial t} \cdots\cdots (28.15)$$

Comparing (28.10) and (28.15),
Conservation eq^n can be written as,

$$\rho . S_s . \frac{\partial h}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z)\right] \cdots\cdots (28.16)$$

$[\# \because V_T = \Delta x \Delta y \Delta z]$

Under incompressible condition (i.e ρ does not change with x,y,z,t)
(28.16) becomes,

$$S_s \frac{\partial h}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) \cdots\cdots (28.17) \rightarrow \text{Also, written as, } S_s \frac{\partial h}{\partial t} = -\nabla . q$$

In terms of piezometric head,

$$q = -\underset{\sim}{K} . \nabla h \cdots\cdots (28.18)$$

If x,y,z are principal flow directions, then,

$$\left.\begin{aligned} q_x &= -k_x\frac{\partial h}{\partial x} \\ q_y &= -k_y\frac{\partial h}{\partial y} \\ q_z &= -k_z\frac{\partial h}{\partial z} \end{aligned}\right\} \cdots\cdots (28.19).$$

Putting values from (28.19) to (28.17):-

- # (For incompressible fluid with heterogenous, anisotropic, saturated, confined aquifer),

$$S_s\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(k_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial h}{\partial z}\right)\cdots\cdots (28.20)$$

Isotropic: • $k_x=k_y=k_z$ at a particular point (Let, K).
• But, point to point K may vary. i.e changable with space (x,y,z).

Homogenous:
• For all points in a domain:-
k_x values are same.
k_y values are same.
k_z values are same. → i.e. unchangable with space (x,y,z).
• But, $k_x \neq k_y \neq k_z$ at a point.

• GW flow eqⁿ for 3D homogenous, anisotropic, saturated confined aquifer,

$$S_s\frac{\partial h}{\partial t} = k_x\frac{\partial^2 h}{\partial x^2}+k_y\frac{\partial^2 h}{\partial y^2}+k_z\frac{\partial^2 h}{\partial z^2}\cdots (28.21)$$

As homogenous, k_x, k_y, k_z values are unchangable with space (i.e x,y,z).

• GW Flow eqⁿ for 3D heterogenous isotropic, saturated confined aquifer,

$$S_s\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right)\cdots\cdots (28.22).$$

• GW flow eqⁿ for 3D homogenous, isotropic, saturated confined aquifer,

$$S_s\frac{\partial h}{\partial t} = K\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}\right)$$

$$\Rightarrow S_s\frac{\partial h}{\partial t} = K\nabla^2 h\cdots\cdots (28.23)$$

28.20 to 28.23 were derived from 28.17. So, these equations are applicable for incompressible fluids only.

Darcy's Law:-

(i) Divergence in Cartesian Co-ordinate system:

?

Also, it may be 'gradient' of a scaler function (·) and e_x, e_y, e_z are unit vectors. But, we are interested in specific discharge q, which is a vector. So, possibly, all these things are 'divergence', not gradient, I think.

$$\nabla(\cdot) = \frac{\partial}{\partial x}(\cdot)e_x+\frac{\partial}{\partial y}(\cdot)e_y+\frac{\partial}{\partial z}(\cdot)e_z\cdots\cdots (29.1)$$

(I.T (it should be written as:- $\nabla(e) = \frac{\partial}{\partial x}(e_x)+\frac{\partial}{\partial x}(e_y)+\frac{\partial}{\partial z}(e_z)$.

$$\nabla^2(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot) + \frac{\partial}{\partial y^2}(\cdot) + \frac{\partial}{\partial z^2}(\cdot) \quad \ldots\ldots\ldots (29.2)$$

From (29.1), for specific discharge vector ($\underset{\sim}{q}$),

$$\nabla \cdot \underset{\sim}{q} = \frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z) \quad \ldots\ldots (29.3)$$

- If x, y, z are the principal flow directions,

$$\left.\begin{aligned} q_x &= -K_x \frac{\partial h}{\partial x} \\ q_y &= -K_y \frac{\partial h}{\partial y} \\ q_z &= -K_z \frac{\partial h}{\partial z} \end{aligned}\right\} \quad \ldots\ldots (29.4)$$

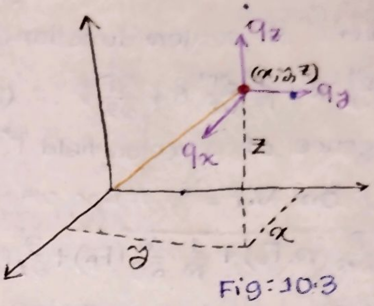


Fig: 10.3

(ii) Cylindrical co-ordinate system:

?
$$\nabla(\cdot) = \frac{\partial}{\partial r}(\cdot)e_r + \frac{1}{r}\frac{\partial}{\partial \theta}(\cdot)e_\theta + \frac{\partial}{\partial z}(\cdot)e_z \quad \ldots\ldots (30.1)$$

I think, It should be,

$$\nabla(\underset{\sim}{e}) = \frac{1}{r}\frac{\partial}{\partial r}(r \cdot e_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(e_\theta) + \frac{\partial}{\partial z}(e_z) \quad \ldots (30.1)$$

$$\nabla^2(\cdot) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\cdot)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\cdot) + \frac{\partial^2}{\partial z^2}(\cdot) \quad \ldots (30.2)$$

From (30.1), for specific discharge vector, $\underset{\sim}{q}$,

$$\nabla \cdot \underset{\sim}{q} = \frac{1}{r}\frac{\partial}{\partial r}(r q_r) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}(q_\theta) + \frac{\partial}{\partial z}(q_z) \quad \ldots\ldots (30.3)$$

- If r, θ, z are principal flow directions,

$$\left.\begin{aligned} q_r &= -K_r \frac{\partial h}{\partial r} \\ q_\theta &= -\frac{K_\theta}{r}\frac{\partial h}{\partial \theta} \\ q_z &= -K_z \frac{\partial h}{\partial z} \end{aligned}\right\} \quad \ldots\ldots (30.4)$$

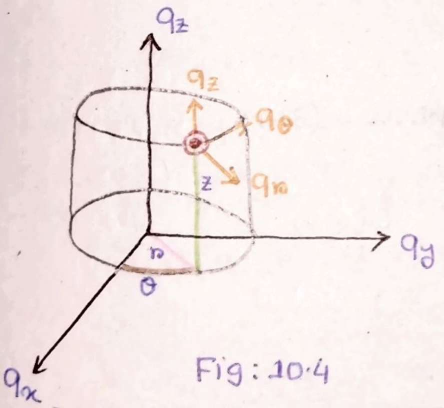


Fig: 10.4

> In Cartesian co-ordinate system, we represent as (x, y, z). Sp. discharge vector q has components q_x, q_y, q_z.
>
> In cylindrical c.s, we write co-ordinate as (r, θ, z), so, specific discharge vector q has components q_r, q_θ, q_z.

Groundwater flow equation.

Notes on Gradient and Divergence in cylindrical c/s: (from chatgpt).

- Gradient of scaler function, F'

$$\nabla F' = \frac{\partial F'}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F'}{\partial \theta}\hat{\theta} + \frac{\partial F'}{\partial z}\hat{z} \quad \dots (31.1)$$

- Divergence of a vector field $\vec{F}$, (I.T, $\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{z}$)

div(F) or' $\nabla . F$ = #Dot product.

$$\frac{1}{r}\frac{\partial}{\partial r}(r.F_r) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(F_z) \quad \dots (31.2)$$

#Note: Gradient is a vector quantity and divergence is a scaler quantity.

#If divergence is +ve, vector field is diverging or spreading away from that point. In that case, vector field can be thought as a source.

#If divergence is -ve, vector field → converging or contracting i.e sink.

when divergence=0, ⇒ No flow of the vector field at that point.

Groundwater flow eq^n for 3D heterogenous, anisotropic, saturated confined aquifer,

$$S_s\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right) - \underline{W} \quad \dots\dots (32.1)$$

#Almost same as (28.20), but what is $\underline{W}$? $\underline{Q}$? Source-sink term?!

Or, $S_s\frac{\partial h}{\partial t} = \nabla.(K.\nabla h) - \underline{Q}$ .... (32.2)

Where, $h = h(x, y, z, t)$ ..... (32.3).

#I.T { $\nabla . \underline{(K.\nabla h)}$

This should be a vector, because we need to do dot product again.

$$\nabla . \left[\begin{pmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{pmatrix} . \begin{pmatrix} \partial h/\partial x \\ \partial h/\partial y \\ \partial h/\partial z \end{pmatrix}\right]$$

$$= \left(\frac{\partial}{\partial x} \;\; \frac{\partial}{\partial y} \;\; \frac{\partial}{\partial z}\right) . \begin{pmatrix} K_x . \partial h/\partial x \\ K_y \, \partial h/\partial y \\ K_z . \partial h/\partial z \end{pmatrix}$$

$$= \frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)$$

↳ This is finally a dot product. Scaler quantity. }

In cylindrical c/s, Same equation written as,

(As our mother equations are 28.17 and 28.19, let's try to derive from those)

Now, we need to apply same transformation for eq^n (32.1), which was applied for (29.3)→(30.3)

$$\therefore \frac{\partial}{\partial x}\left(k_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial h}{\partial z}\right)$$

can be written as,

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r.\,k_r\frac{\partial h}{\partial r}\right)+\frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(k_\theta\cdot\frac{1}{r}\cdot\frac{\partial h}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial h}{\partial z}\right)$$

From (29.3) and (30.3),

$$\frac{\partial}{\partial x}(q_x)+\frac{\partial}{\partial y}(q_y)+\frac{\partial}{\partial z}(q_z)$$
$$=\frac{1}{r}\frac{\partial}{\partial r}(rq_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta)+\frac{\partial}{\partial z}(q_z)$$

See, in place of '∂y' we always replace with $r\partial\theta$.

$$S_s\frac{\partial h}{\partial t}=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r.\,k_r\frac{\partial h}{\partial r}\right)+\frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{k_\theta}{r}\cdot\frac{\partial h}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial h}{\partial z}\right)-W\;\cdots\cdots(32.2)$$

where, $h=h(r,\theta,z,t)$.

Reduction in Dimensionality:

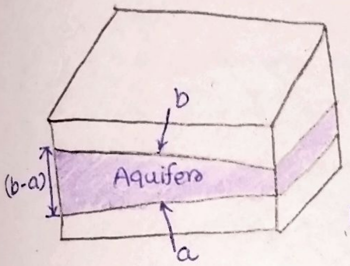


Fig: 10.5.

? Adding a source/sink term W, Flow equation (28.17) can be written as,

$$S_s\frac{\partial h}{\partial t}=-\nabla.q-W\;\cdots\cdots(33.1)$$

vertical integration of the flow equation, (33.1) gives,

$$\boxed{\int_{a(x,y,t)}^{b(x,y,t)}\left(\nabla.q+S_s\frac{\partial h}{\partial t}+W\right)dz=0\;\cdots\cdots(33.2)}$$

Using Leibniz Integral rule, 1st term of (33.2) written as,

$$\int_{a(x,y,t)}^{b(x,y,t)}\nabla.\underset{\sim}{q}\,dz=\int_a^b\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)dz$$

$$=\frac{\partial}{\partial x}\int_a^b q_x\,dz-q_x\Big|_{z=b}\cdot\frac{\partial b}{\partial x}+q_x\Big|_{z=a}\cdot\frac{\partial a}{\partial x}$$

$$+\frac{\partial}{\partial y}\int_a^b q_y\,dz-q_y\Big|_{z=b}\frac{\partial b}{\partial y}+q_y\Big|_{z=a}\cdot\frac{\partial a}{\partial x}$$

$$+q_z\Big|_{z=b}-q_z\Big|_{z=a}\;\cdots\cdots\cdots\cdots(33.3).$$

In generalized form, (33.3) can be written as,

$$\int_{a(x,y,t)}^{b(x,y,t)} \nabla \cdot \underset{\sim}{q}\, dz = \nabla_{xy} \cdot \int_{a(x,y,t)}^{b(x,y,t)} \underset{\sim}{q}_{xy}\, dz + \underset{\sim}{q}\Big|_{z=b} \nabla(z-b)$$

$$- \underset{\sim}{q}\Big|_{z=a} \nabla(z-a) \quad \ldots\ldots (33.4)$$

where, $\nabla_{xy}(\cdot) = \frac{\partial}{\partial x}(\cdot)\, e_x + \frac{\partial}{\partial y}(\cdot)\, e_y \quad \ldots\ldots (33.5)$

? Can we write this as:-

$$\nabla_{xy}(\underset{\sim}{e}) = \frac{\partial}{\partial x}(e_x) + \frac{\partial}{\partial y}(e_y)$$

To compare (33.3) and (33.4):-

$$\nabla_{xy} \cdot \int_{a(x,y,t)}^{b(x,y,t)} \underset{\sim}{q}_{xy}\, dz$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \cdot \left(\int_a^b (q_x \hat{i} + q_y \hat{j})\, dz\right)$$

$$= \frac{\partial}{\partial x}\int_a^b q_x\, dz + \frac{\partial}{\partial y}\int_a^b q_y\, dz \quad \ldots\ldots (33.6)$$

$$\underset{\sim}{q}\Big|_{z=b} \cdot \nabla(z-b)$$

$$= \left(q_x\Big|_{z=b}\hat{i} + q_y\Big|_{z=b}\hat{j} + q_z\Big|_{z=b}\hat{k}\right) \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(z-b).$$

$$= \left(q_x\hat{i} + q_y\hat{j} + q_z\hat{k}\right)_{z=b} \cdot \left(\frac{\partial}{\partial x}(z-b)\,\hat{i} + \frac{\partial}{\partial y}(z-b)\,\hat{j} + \frac{\partial}{\partial z}(z-b)\,\hat{k}\right)$$

$$= q_x\Big|_{z=b}\frac{\partial}{\partial x}(z-b) + q_y\Big|_{(z-b)}\frac{\partial}{\partial y}(z-b) + q_z\Big|_{z=b}\frac{\partial}{\partial z}(z-b)$$

⇒ Seperating and rearranging,

$$= \left(q_x\Big|_{z=b}\frac{\partial z}{\partial x} + q_y\Big|_{z=b}\frac{\partial z}{\partial y}\right) + \left(-q_x\Big|_{z=b}\frac{\partial b}{\partial x} - q_y\Big|_{z=b}\frac{\partial b}{\partial y} + q_z\right)_{z=b} \quad \ldots\ldots (33.7)$$

$\frac{\partial b}{\partial z} = 0$. As, b is not a function of z. $b(x,y,t)$.

These terms will be not there.
Derivative of one co-ordinate wrt other doesn't give any significant value.

Similarly,

$$-\underset{\sim}{q}\Big|_{z=a} \cdot \nabla(z-a)$$

$(z-a)$ → Represents the plane situated at elevation 'a'.

$$= \left(-q_x\Big|_{z=a}\frac{\partial z}{\partial x} - q_y\Big|_{z=a}\frac{\partial z}{\partial y}\right) + \left(q_x\Big|_{z=a}\frac{\partial a}{\partial x} + q_y\Big|_{z=a}\frac{\partial a}{\partial y} - q_z\Big|_{z=a}\right) \quad \ldots\ldots (33.8)$$

Study 3D co-ordinate geometry and vector calculus for more.

Adding $\{(33.6) + (33.7) + (33.8)\}$,
we have,

$$\nabla_{xy} \cdot \int_{a(x,y,t)}^{b(x,y,t)} \underset{\sim}{q}_{xy}\, dz + \underset{\sim}{q}\Big|_{z=b} \cdot \nabla(z-b) - \underset{\sim}{q}\Big|_{z=a} \cdot \nabla(z-a)$$

$$= \frac{\partial}{\partial x}\int_a^b q_x\, dz + \frac{\partial}{\partial y}\int_a^b q_y\, dz + q_x\Big|_{z=a}\frac{\partial a}{\partial x} + q_y\Big|_{z=a}\frac{\partial a}{\partial y} - q_z\Big|_{z=a}$$

$$- q_x\Big|_{z=b}\frac{\partial b}{\partial x} - q_y\Big|_{z=b}\frac{\partial b}{\partial y} + q_z\Big|_{z=b} + \underbrace{\left(q_x\Big|_{z=b}\frac{\partial z}{\partial x} + q_y\Big|_{z=b}\frac{\partial z}{\partial y} - q_x\Big|_{z=a}\frac{\partial z}{\partial x} - q_y\Big|_{z=a}\frac{\partial z}{\partial y}\right)} \quad \ldots (33.9)$$

If we compare with (33.3), this portion is coming extra.

If $q_x|_{z=b} = q_x|_{z=a}$ and $q_y|_{z=a} = q_y|_{z=b}$, then it will get cancelled.

• In terms of piezometric head,

$$\underset{\sim}{q} = -\underset{\sim}{K} \cdot \nabla h \quad \cdots (33.6)$$

J.T # Piezometric head is a scaler quantity. So, we take gradient ~~to get~~ of that to get vector form.

Piezometric head = Pressure head + Datum head.

Pressure of a fluid element is a scaler quantity, as it does not require any direction to specify, it always acts perpendicular to the surface. So, it is a zero-order tensor.

Eg. At some depth of the ocean, pressure acts on a body in all directions (Perpendicular to every point on the surface of the body) - it is directionless.

J.T → Datum head is also measured from a reference surface. So, vertical length (i.e magnitude) is sufficient to define it. It is also a scalar quantity.

Scaler: Only magnitude (zero-order tensor)

Vector: Magnitude and a particular direction. (1^{st} order tensor). → 3 components (x, y, z)

Tensor:-

(i) Dyad: Magnitude and 2 directions (9 components) #Tensor of rank 2

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

(ii) Triad: Magnitude and 3 directions. #Tensor rank 3. (27 components).

#Tensor is a physical quantity which has no specified direction but have different values in different directions.

Eg. Moment of inertia.

Tensor is in some sense, collection of vectors.

Tensors do not follow the law of vector addition. Eg. Electric current.

#Tensor is a quantity which requires, direction, magnitude and plane to define.

Three dimensional stress, strain are known as tensors.

Integrating (33.6) over depth,(l)

$$\int_{a(x,y,t)}^{b(x,y,t)} \underset{\sim}{q}_{xy}\, dz = -\underset{\sim}{\overline{K}} \cdot \int_a^b \nabla_{xy} h \, dz \quad \cdots (33.7)$$

What is this? Average Hydraulic conductivity!

? Why divergence is defined for x and y here? i.e. $\nabla_{xy} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$

($h = h(x, y, t)$ here.

In the domain, for a point on x-y plane, at any elevation Piezometric head (h) is same. So, h does not depend on z.) → J.T. Not sure.

Applying Libehitz rule at (33.7),

$$= -\bar{K}\left[\nabla_{xy}\int_a^b h\,dz - h|_{z=b}\nabla_{xy}b + h|_{z=a}\cdot\nabla_{xy}a\right] \quad \dots (33.8)$$

Now, Time derivative term in equation (33.2),

$$\int_{a(x,y,t)}^{b(x,y,t)} S_s\frac{\partial h}{\partial t}dz = \bar{S}_s\int_a^b \frac{\partial h}{\partial t}dz$$

I.T ⇒ $\bar{S}_s$ = Average specific storage.

$$= \bar{S}_s\left[\frac{\partial}{\partial t}\int_a^b h\,dz - h|_{z=b}\frac{\partial b}{\partial t} + h|_{z=a}\frac{\partial a}{\partial t}\right] \quad \dots (33.9)$$

- # Now, we will combine all the terms to get GW flow equation,

From (33.2),

$$\int_a^b \nabla.\underset{\sim}{q}\,dz + \int_a^b S_s\frac{\partial h}{\partial t}dz + \int_a^b W\,dz = 0 \quad \dots (33.10)$$

#Putting value of $\int_a^b \nabla.\underset{\sim}{q}\,dz$ from (33.4),

$$\Rightarrow \left[\nabla_{xy}\cdot\left(\int_a^b q_{xy}dz\right) + \underset{\sim}{q}|_{z=b}\cdot\nabla(z-b) - \underset{\sim}{q}|_{z=a}\cdot\nabla(z-a)\right] + \int_a^b S_s\frac{\partial h}{\partial t}dz + \int_a^b W\,dz = 0$$

Putting value of $\int_a^b q_{xy}dz$ from (33.8), and $\int_a^b S_s\frac{\partial h}{\partial t}dz$ from (33.9),

$$\nabla_{xy}\cdot\left[-\bar{K}\left(\nabla_{xy}\int_a^b h\,dz - h|_{z=b}\nabla_{xy}b + h|_{z=a}\nabla_{xy}a\right)\right] + \underset{\sim}{q}|_{z=b}\cdot\nabla(z-b) - \underset{\sim}{q}|_{z=a}\cdot\nabla(z-a) + \bar{S}_s\left[\frac{\partial}{\partial t}\int_a^b h\,dz - h|_{z=b}\frac{\partial b}{\partial t} + h|_{z=a}\frac{\partial a}{\partial t}\right] + \int_a^b W\,dz = 0 \quad \dots (33.11)$$

- # Now, consider vertical average values as:-

$$\bar{h} \equiv \frac{1}{(b-a)}\int_a^b h\,dz = \frac{1}{l}\int_a^b h\,dz \quad \dots\dots (33.12)$$

$$\bar{W} = \frac{1}{(b-a)}\int_a^b W\,dz = \frac{1}{l}\int_a^b W\,dz \quad \dots\dots (33.13)$$

- Putting values from (33.12) and (33.13) in (33.11), we have,

$$-\nabla_{xy}\cdot\left[\underset{\sim}{K}\cdot(\nabla_{xy}l\bar{h}-\bar{h}\nabla_{xy}l)\right]+\underset{\sim}{q}\big|_{z=b}\cdot\nabla(z-b)$$
$$-\underset{\sim}{q}\big|_{z=a}\cdot\nabla(z-a)+\bar{S}_s\left[\frac{\partial}{\partial t}(l\bar{h})-\bar{h}\frac{\partial l}{\partial t}\right]+l\bar{W}=0 \quad \cdots\cdots (33.17)$$

Here,
$h|_{z=b}\nabla_{xy}b-h|_{z=a}\nabla_{xy}a$
can be written as,
I.T (Here $h|_{z=b}$ and $h|_{z=a}$ both are replaced by average value $\bar{h}$)

$$=\bar{h}\nabla_{xy}(b-a)$$
$$=\bar{h}\nabla_{xy}l \quad \cdots\cdots (33.14)$$

Also, From (33.12),
$$\int_a^b h\,dz=l\bar{h} \quad \cdots\cdots (33.15)$$

Again,
$$h|_{z=b}\frac{\partial b}{\partial t}-h|_{z=a}\frac{\partial a}{\partial t}$$
$$=\bar{h}\frac{\partial}{\partial t}(b-a)$$
$$=\bar{h}\frac{\partial l}{\partial t} \quad \cdots\cdots (33.16)$$

6 # Putting values from (33.12), (33.14), (33.15), (33.16) and (33.13) respectively.

Now, taking $l=$ constant,
Putting $\nabla_{xy}l=0$ and $\frac{\partial l}{\partial t}=0$ in equation (33.17), we get,

$$-\nabla_{xy}\cdot\left[l\underset{\sim}{K}\cdot\nabla_{xy}\bar{h}\right]+\underset{\sim}{q}\big|_{z=b}\cdot\nabla(z-b)$$
$$-\underset{\sim}{q}\big|_{z=a}\cdot\nabla(z-a)+l\bar{S}_s\frac{\partial\bar{h}}{\partial t}+l\bar{W}=0 \quad \cdots\cdots (33.18)$$

Remember $\bar{h}$ is the depth-integrated value. So, $\bar{h}$ is not a fixed value for the domain. It would change with (x,y)
So, $\nabla_{xy}(\bar{h})\neq 0$.

I.T # (So, we reduced the dimensionality in z direction incorporating depth-integrated piezometric head value, $\bar{h}$). So, we will convert
3D eq^n $\rightarrow$ 2D eq^n.

Two Dimensional Saturated flow in confined Aquifer: (This is continuation, not a seperate topic)

Storage coefficient, $S=l\bar{S}_s$ ..... ~~(34.1)~~ (33.19)

? # But, I thought, storage coff defined for unconfined aquifer and specific storage for confined aquifer. But, both have same definition. (See Page-86).

Transmissivity tensor,
$$\underset{\sim}{T}=l\underset{\sim}{\bar{K}} \quad \cdots\cdots (34.2?).\ (33.20)$$

In 2D, governing equation can be written as, (Putting values from 33.19 and 33.20 to (33.18)),

$$-\nabla_{xy}\cdot\left[\underset{\sim}{T}\cdot\nabla_{xy}\bar{h}\right]+\underset{\sim}{q}\big|_{z=b}\cdot\nabla(z-b)$$
$$-\underset{\sim}{q}\big|_{z=a}\cdot\nabla(z-a)+S\frac{\partial\bar{h}}{\partial t}+l\bar{W}=0$$

$$\Rightarrow \boxed{S\frac{\partial\bar{h}}{\partial t}+q_T+q_B+q_{ext}=\nabla_{xy}\cdot\left[\underset{\sim}{T}\cdot\nabla_{xy}\bar{h}\right]} \quad \cdots\cdots (33.21)$$

where,
$$q_T=\underset{\sim}{q}\big|_{z=b}\cdot\nabla(z-b) \quad \cdots\cdots (33.22)$$
$$q_B=-\underset{\sim}{q}\big|_{z=a}\cdot\nabla(z-a) \quad \cdots\cdots (33.23)$$
$$q_{ext}=l\bar{W} \quad \cdots\cdots (33.24).$$

What does these mean?

? I.T $\Rightarrow$ q_T = Inflow to the aquifer from top.
q_B = Outflow from the " through bottom (As -ve sign)
q_{ext} = Inflow/outflow from some external sources?

• GW flow equation for 2D heterogenous, anisotropic, Saturated confined aquifer,

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right) \quad .... (34.1)$$

Equation (33.21), excluding q terms,

$$S\frac{\partial \bar{h}}{\partial t} = \nabla_{xy} \cdot \left[\underset{\sim}{T} \cdot \nabla_{xy}\bar{h}\right]$$

$$\Rightarrow S\frac{\partial h}{\partial t} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \cdot \left[\begin{bmatrix} T_x & 0 \\ 0 & T_y \end{bmatrix} \cdot \begin{Bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{Bmatrix}\right]$$

$$\Rightarrow S\frac{\partial h}{\partial t} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} T_x \frac{\partial h}{\partial x} \\ T_y \frac{\partial h}{\partial y} \end{bmatrix}$$

$$\Rightarrow S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right).$$

• GW flow eq^n for 2D homogenous, anisotropic, saturated, confined aquifer,

$$S\frac{\partial h}{\partial t} = \cancel{\frac{\partial}{\partial x}} T_x\frac{\partial^2 h}{\partial x^2} + T_y\frac{\partial^2 h}{\partial y^2} \quad ......(34.2)$$

T_x, T_y values donot varies in the domain.

• For heterogenous, isotropic, saturated confined aquifer,

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right) \quad ..... (34.3)$$

$T_x = T_y = T$ at a point. But, T may vary at different points.

• For homogenous, isotropic, saturated confined aquifer,

$$S\frac{\partial h}{\partial t} = T\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) \quad ..... (34.4).$$

Lecture-11

General Flow Equations

To derive governing equation for Saturated Unconfined Aquifer

☑ Dupit-Forchheimer Assumptions:-

(1) • For small inclinations of the free surface of a gravity flow system, streamlines can be taken as horizontal. (So, no flow in vertical direction. i.e, no vertical hydraulic gradient, i.e)

(2) • The velocities associated with these streamlines are proportional to the slope of the free surface, but independent of the depth.

Head is independent of depth. i.e,

$h(x,y,z,t) \Longrightarrow h(x,y,t)$. ...... (34.5)

Discharge is proportional to the slope of the water table.

$h(x,y,t) \rightarrow H(x,y,t)$. .... (34.6).

J.T ⇒ [For Darcy's law,
Discharge was proportional to hydraulic gradient.

$q \propto \frac{dh}{dx}$

But, now, $q \propto \frac{dH}{dx}$

H = Elevation of free water surface. (from Datum!)
and $\frac{dH}{dx}$ = Slope of the free surface).]

• Aquifer flux per unit width

$U_x = -k_x (H-\xi) \frac{\partial H}{\partial x}$ ...... (35.1)

$U_y = -k_y (H-\xi) \frac{\partial H}{\partial y}$ ...... (35.2).

Where, ξ = elevation of aquifer base.

$\underbrace{U_x}_{\text{Discharge per unit width.}} = -K_x \underbrace{\frac{\partial H}{\partial x}}_{\text{Hydraulic gradient}} \cdot \underbrace{(H-\xi).1}_{\text{Aquifer depth} \times \text{width} = \text{cross sectional area.}}$

From 33.12,
$U_x = \int q_x dz$, q_x = Sp. discharge

• No vertical flow occurs.

J.T ⇒ No hydraulic gradient in vertical direction. $\frac{\partial H}{\partial z} = 0$

Two Dimensional saturated flow in unconfined Aquifer

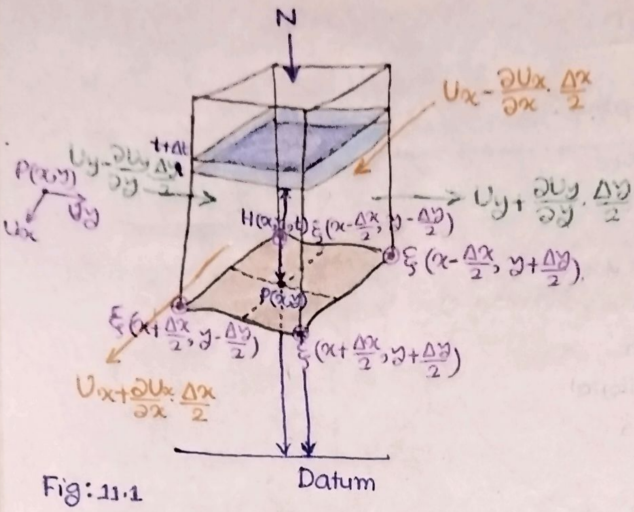


Fig: 11.1

This surface represents the base of the aquifer.

\# $\xi(x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2})$ represents elevation of the aquifer base at point $(x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2})$.

\# $H(x,y,t)$ represents elevation of the top water surface at point $P=(x,y)$ at time t.

\# There is no vertical flow. So, only U_x and U_y are considered.

I.T (But, there is some amount of recharge N due to some amount of infiltration through unsaturated zone). [N = Recharge per unit area. See-(36.6)]!

- Inflow and outflow is calculated along x and y directions along each side of the element.

- GW inflow in x direction,

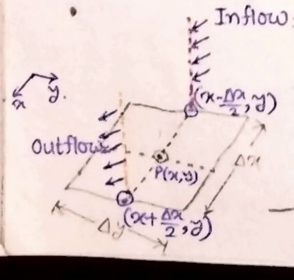
$$U_x(x-\frac{\Delta x}{2}, y, t)\Delta y \approx \left[U(x,y,t) - \frac{\partial U_x}{\partial x}\cdot\frac{\Delta x}{2}\right]\Delta y, \quad \ldots (36.1)$$

Upto 1st order. I.T ⇒ (36.1 measures inflow through unit depth of aquifer). ie $U_x(x-\frac{\Delta x}{2}, y, t)\Delta y \cdot 1$

- If we use Taylor's series expansion,

$$U_x(x-\frac{\Delta x}{2}, y, t)$$
$$= U_x(x,y,t) - \frac{\Delta x}{2}\frac{\partial U_x}{\partial x} + \left(\frac{\Delta x}{2}\right)^2\frac{\partial^2 U_x}{\partial x^2} - \ldots\ldots$$

So, from this equation, neglecting higher order terms, we get (36.1).



\# Along the red vertical line, for red point, ~~Darcy's Velocity~~* is $U_x(x-\frac{\Delta x}{2}, y, t)$. and for orange one it is $U_x(x+\frac{\Delta x}{2}, y, t)$.

*Aquifer flux per unit width in x direction.

This is as per 2nd assumption as stated in page (159).

\# As velocity independent of depth

$$\begin{matrix} U_x(x,y,z,t) \\ U_y(x,y,z,t) \end{matrix} \Rightarrow \begin{matrix} U_x(x,y,t) \\ U_y(x,y,t) \end{matrix}$$

• GW outflow in x direction,

$$U_x(x+\frac{\Delta x}{2}, y, t)\,\Delta y \approx \left(U_x + \frac{\Delta x}{2}\frac{\partial U_x}{\partial x}\right)\Delta y$$
$$= U_x \Delta y + \frac{\partial U_x}{\partial x}\frac{\Delta x \Delta y}{2} \quad \cdots\cdots (36.2).$$

$U_x(x,y,t)$ in short written as, U_x.

• GW inflow in y direction,

$$U_y(x, y-\frac{\Delta y}{2}, t)\,\Delta x \approx \left(U_y - \frac{\Delta y}{2}\frac{\partial U_y}{\partial y}\right)\Delta x$$
$$= U_y \Delta x - \frac{\partial U_y}{\partial y}\cdot\frac{\Delta y \Delta x}{2} \quad \cdots\cdots (36.3).$$

• GW outflow in y direction,

$$U_y(x, y+\frac{\Delta y}{2}, t) \approx \left(U_y + \frac{\Delta y}{2}\frac{\partial U_y}{\partial y}\right)\Delta x$$
$$= U_y \Delta x + \frac{\partial U_y}{\partial y}\cdot\frac{\Delta y \Delta x}{2} \quad \cdots\cdots (36.4).$$

• Now,
Rate of change of storage volume with time = Volume inflow rate − Volume outflow rate ......(36.5)

$$\Rightarrow \frac{\partial V}{\partial t} = -\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y}\right)\Delta x \Delta y + N \Delta x \Delta y.$$

Putting value from (36.1), (36.2), (36.3), (36.4) to eq^n (36.5).

IT ⇒ Here, N = Recharge rate per unit Area ($m^3/s/m^2$).

$$\Rightarrow \frac{\partial V}{\partial t} = -\left[\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} - N\right]\Delta x \Delta y \quad \cdots\cdots (36.6)$$

$$V = S_y V_T = S_y\left[H(x,y,t) - \xi(x,y)\right]\Delta x \Delta y \quad \cdots\cdots (36.7).$$

IT ⇒ Specific yeild gives the volumetric fraction of available or flowable water.
So, $V = S_y . V_T$ gives total water volume in c.v.
$(H-\xi)$ gives thickness of aquifer.
Water level may change with time ⇒ $H(x,y,t)$
Aquifer base elevation does not change with time. So, $\xi(x,y)$.

From (36.7), rate of change in storage,

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left[S_y\{H(x,y,t) - \xi(x,y)\}\Delta x \Delta y\right]$$
$$= S_y \frac{\partial H}{\partial t}\Delta x \Delta y \quad \cdots\cdots\cdots (36.8)$$

Because, elevation of water level changes with time only among these quantities.

Comparing (36·6) and (36·8), we can write, the conservation equation as,

$$S_y \frac{\partial H}{\partial t} = -\left[\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} - N\right] \dots\dots (36{\cdot}9)$$

Putting values of U_x and U_y from (35·1) and (35·2), → (36·9) becomes,

$$\boxed{S_y \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[K_x (H-\xi)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y (H-\xi)\frac{\partial H}{\partial y}\right] + N} \dots\dots (36{\cdot}10)$$

- (36·10) gives GW flow equation for heterogenous, anisotropic, saturated unconfined aquifer.

- For homogenous, anisotropic, saturated unconfined aquifer,

$$S_y \frac{\partial H}{\partial t} = K_x \frac{\partial}{\partial x}\left[(H-\xi)\frac{\partial H}{\partial x}\right] + K_y \frac{\partial}{\partial y}\left[(H-\xi)\frac{\partial H}{\partial y}\right] + N \dots\dots (36{\cdot}11)$$

- For heterogenous, isotropic, saturated unconfined aquifer,

$$S_y \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[K(H-\xi)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[K(H-\xi)\frac{\partial H}{\partial y}\right] + N \dots\dots (36{\cdot}12).$$

- For homogenous, isotropic, saturated unconfined aquifer,

$$S_y \frac{\partial H}{\partial t} = K\left[\frac{\partial}{\partial x}\left\{(H-\xi)\frac{\partial H}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{(H-\xi)\frac{\partial H}{\partial y}\right\}\right] + N \dots\dots (36{\cdot}13)$$

- Now, GW flow flow equation for homogenous, isotropic, saturated, unconfined aquifer with bed level elevation, ($\xi = 0$) can be written as,

$$S_y \frac{\partial H}{\partial t} = K \frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right) + K \frac{\partial}{\partial y}\left(H\frac{\partial H}{\partial y}\right) + N \dots\dots (36{\cdot}14)$$

[We know,
$$\frac{\partial^2 H^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial H^2}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(H^2)\right] = \frac{\partial}{\partial x}\left[2H\frac{\partial H}{\partial x}\right] = 2\frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right)\Big] \dots (36{\cdot}15)$$

Similarly, $\frac{\partial^2 H^2}{\partial y^2} = 2\frac{\partial}{\partial y}\left(H\frac{\partial H}{\partial y}\right) \dots\dots (36{\cdot}16)$

Substituting value of $\frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(H\frac{\partial H}{\partial y}\right)$ from (36·15) and (36·16) to equation (36·14), we can write,

$$S_y \frac{\partial H}{\partial t} = \frac{K}{2}\frac{\partial^2 H^2}{\partial x^2} + \frac{K}{2}\frac{\partial^2 H^2}{\partial y^2} + N \dots\dots (36{\cdot}16{\cdot}2)$$

$$\Rightarrow \frac{2S_y}{K}\frac{\partial H}{\partial t} = \nabla^2 H^2 + \frac{2N}{K} \dots\dots (36{\cdot}17).$$

Equation (36·17) can be called as Boussinesq Equation.

J.T ⇒ Equation (36.17) can also be written as $\frac{2S_0}{k}\frac{\partial H}{\partial t} = \nabla^2_{xy} H^2 + \frac{2N}{k}$

Because, there is no z component in ∇ operator.
For unconfined aquifer, Piezometric head (h) is equivalent to GW surface elevation (H). And head does not change in vertical direction (As per assumption, see P-159)
So. $\frac{\partial^2}{\partial z^2}(H^2) = 0$. So, we can write ∇^2_{xy} instead of ∇^2.

⊙ For 2D steady flow (i.e $\frac{\partial H}{\partial t} = 0$) in homogenous, isotropic, saturated unconfined aquifer with bed $\xi = 0$, (Put $\frac{\partial H}{\partial t} = 0$ at 36.16.2),

Are we considering the aquifer bed as a plane surface. so that $\xi = 0$ everywhere?

$$0 = \frac{k}{2}\frac{\partial^2 H^2}{\partial x^2} + \frac{k}{2}\frac{\partial^2 H^2}{\partial y^2} + N \quad \text{.....} \quad (36.18)$$

⊙ 1D steady flow for homogenous, isotropic, saturated unconfined aquifer with no bed level variation ($\xi = 0$),

$$0 = \frac{k}{2}\frac{\partial^2 H^2}{\partial x^2} + N \quad \text{......} \quad (36.19)$$

⊙ Dupit's equation for radial flow,
Obtained from equation (30.2), neglecting vertical z direction.

$$\nabla^2(\cdot) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\cdot)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\cdot) \quad \text{.....} \quad (36.20)$$

$$\approx \frac{d^2}{dr^2}(\cdot) + \frac{1}{r}\frac{d}{dr}(\cdot) \quad \text{.......} \quad (36.21)$$

↳ How this is obtained?

(I tried with formula in (P-47)
i.e $\frac{dH}{dr} = \frac{\partial H}{\partial r} + \frac{\partial H}{\partial \theta}\cdot\frac{d\theta}{dr}$ .... (36.21.2)
But result not coming)

$\frac{k}{2}\nabla^2(H^2) + N = 0$

$\Rightarrow \frac{k}{2}\left(\frac{d^2H^2}{dr^2} + \frac{1}{r}\frac{dH^2}{dr}\right) + N = 0$

$\Rightarrow \frac{k}{2}\left[\frac{d}{dr}\left(\frac{dH^2}{dr}\right) + \frac{1}{r}\frac{dH^2}{dr}\right] + N = 0$

$\Rightarrow \frac{k}{2}\cdot\frac{1}{r}\left[r\cdot\frac{d}{dr}\left(\frac{dH^2}{dr}\right) + \frac{dH^2}{dr}\right] + N = 0$

$\Rightarrow \frac{k}{2}\cdot\frac{1}{r}\left[r\cdot\frac{d}{dr}\left(\frac{dH^2}{dr}\right) + \frac{dH^2}{dr}\cdot\frac{d}{dr}(r)\right] + N = 0$

$\Rightarrow \frac{k}{2}\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot\frac{dH^2}{dr}\right) + N = 0$ ..... (36.22)

⊙ Replacing N by '-W' in equation (36.22)
(What does 'W' means? Is it 'withdrawl', as 'N means 'recharge'?)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH^2}{dr}\right)=\frac{2W}{K}\ldots\ldots(36.23).$$

$$\Rightarrow \int d\left(r\frac{dH^2}{dr}\right)=\frac{2W}{K}\int r\,dr$$

$$\Rightarrow r\frac{dH^2}{dr}=\frac{2W}{K}\frac{r^2}{2}+A \quad [A=\text{Integral const.}]$$

$$\Rightarrow \int dH^2=\frac{W}{K}\int r\,dr+A\int\frac{dr}{r}$$

$$\Rightarrow H^2=\frac{Wr^2}{2K}+A\ln r+B\ldots\ldots(36.24).$$

Agricultural Drain:

⊙ The Governing equation can be written as, (Same as 36.19),

$$\frac{\partial^2 H^2}{\partial x^2}+\frac{2R}{K}=0\ldots\ldots(37.1)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial H^2}{\partial x}\right)=-\frac{2R}{K}$$

$$\Rightarrow \int\partial\left(\frac{\partial H^2}{\partial x}\right)=-\frac{2R}{K}\int\partial x$$

$$\Rightarrow \frac{\partial H^2}{\partial x}=-\frac{2R}{K}\cdot x+A$$

$$\Rightarrow \partial H^2=-\frac{2R}{K}\int x\,\partial x+A\int\partial x$$

$$\Rightarrow H^2=-\frac{2R}{K}\cdot\frac{x^2}{2}+Ax+B$$

[where, A, B = constant]

$$\Rightarrow \boxed{H^2+\frac{Rx^2}{K}=Ax+B}\ldots\ldots(37.2)$$

(37.2) is the solution of differential equation for (37.1).

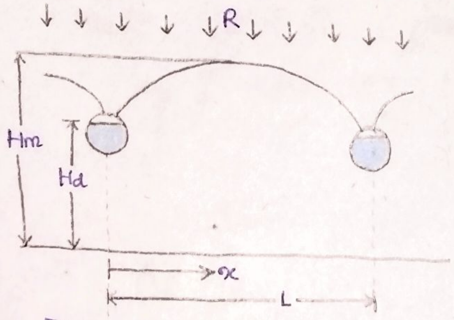


Figure: 11.2.

⊙ Boundary conditions are:-

$$H^2(0)=H_d^2\;;\;H^2(L)=H_d^2\ldots\ldots(37.3)$$

or

$$H^2(0)=H_d^2\;;\;\left.\frac{dH^2}{dx}\right|_{x=\frac{L}{2}}=0\ldots\ldots(37.4)$$

⊙ For Boundary condition set I: (from 37.3), Applying to equation (37.2),

$$H_d^2 + 0 = 0 + B$$

$$\Rightarrow H_d^2 = B \quad \ldots\ldots\ldots \quad (37.5)$$

and, $H^2|_{x=L} = H_d^2$

$$\Rightarrow H_d^2 + \frac{RL^2}{K} = AL + B \quad \ldots\ldots \quad (37.6)$$

Putting $B = H_d^2$ from (37.5), (37.6) becomes,

$$A = \frac{RL}{K} \quad \ldots\ldots\ldots \quad (37.7).$$

For set I Boundary conditions,
Putting values of A and B in (37.2),

$$H^2 + \frac{Rx^2}{K} = \frac{RL}{K}x + H_d^2$$

$$\Rightarrow \boxed{H^2 = H_d^2 + \frac{Rx}{K}(L-x)} \quad \ldots\ldots \quad (37.8)$$

Equation (37.8) is the formula of water table elevation.

- For boundary condition set II:-

$$H^2|_{x=0} = H_d^2$$

Putting in (37.2),

$$H_d^2 + 0 = 0 + B$$

$$\Rightarrow B = H_d^2 \quad \ldots\ldots\ldots \quad (37.9)$$

Also, $\left.\frac{dH^2}{dx}\right|_{x=\frac{L}{2}} = 0$

Now, differentiating (37.2) with x,

$$\frac{d}{dx}(H^2) + \frac{R.2x}{K} = A$$

#To apply BC, put $\frac{dH^2}{dx} = 0$ and $x = \frac{L}{2}$.

$$0 + \frac{R.2.L}{2K} = A$$

$$\Rightarrow A = \frac{RL}{K} \quad \ldots\ldots\ldots \quad (37.10).$$

Putting A and B values from boundary condition set II in eq^n (37.2),

$$H^2 + \frac{Rx^2}{K} = \frac{RL}{K}x + H_d^2$$

$$\Rightarrow \boxed{H^2 = H_d^2 + \frac{Rx}{K}(L-x)} \quad \ldots\ldots\ldots \quad (37.11)$$

Observation:-

Applying 2 sets of boundary conditions, (i.e. 37.3 and 37.4) we obtained the same equations (i.e 37.8 and 37.11)

- For $x = \frac{L}{2}$, we get maximum saturated thickness i.e $H = H_m$.

So, from (37.11),

$$H_m^2 = H_d^2 + \frac{RL}{2K}\left(L - \frac{L}{2}\right)$$

$$\Rightarrow H_m^2 = H_d^2 + \frac{RL^2}{4K}$$

$$\Rightarrow H_m = \sqrt{H_d^2 + \frac{RL^2}{4K}} \quad \ldots\ldots\ldots \quad (37.12).$$

☑ Dupuit Forchheimer Discharge Formula:

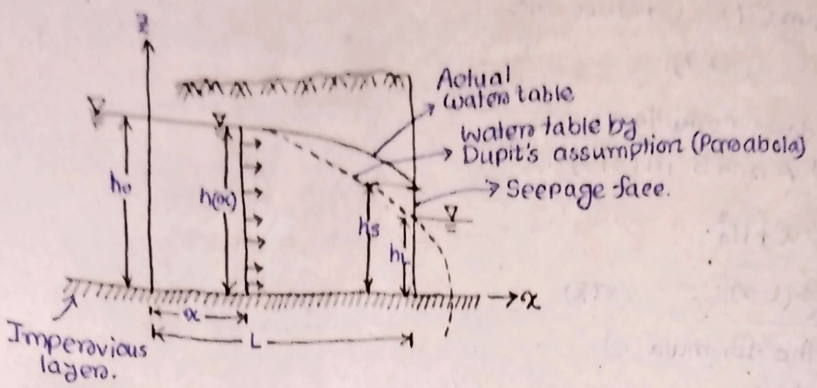


Fig: 11.3.

Paradoxes in Dupit-Forchheimer Seepage theory:-

- Two Dupit's assumption (I.T ⇒ Mentioned in page 159) on which it rests are 'entirely contradictory to the implications of Darcy's law'.
  ⓠ (How? Because, in unconfined aquifer free surface gives the piezometric head. So, I think h→H transformation is good. I think, the problem is with horizontal streamline. As free surface slope is very less, can't we neglect that?) / [Also, this assumption doesn't consider vertical flow, that may be a problem].
- Many problem solutions based on Dupit's assumption gives compareable favo results with other rigorous method.
- Here, we get correct solution for the discharge, but not the free surface.

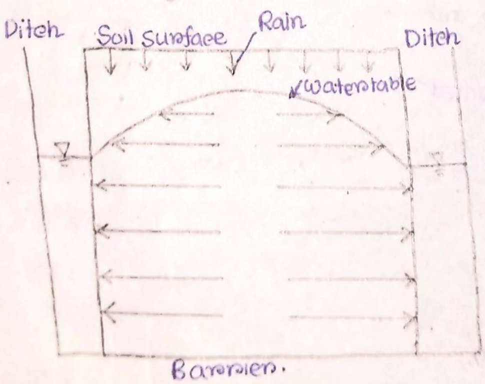
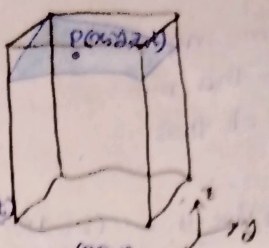


Fig: 11.4 ⇒ Horizontal flow lines as (improperly) convinced for Dupit-Forchheimer flow.

Free Surface condition:



⊙ At every point within the considered domain and on its boundary,

Piezometric head $h(x,y,z,t) = z + \frac{p(x,y,z,t)}{\gamma}$ ..... (38.1).

⊙ On the phreatic surface, atmospheric pressure $\Rightarrow p = 0$.

$\therefore h(x,y,z,t) = z$ ..... (38.2).

$\therefore$ Free surface function,

$F_1(x,y,z,t) = z - h(x,y,z,t) = 0$ ..... (38.3)

⊙ Free surface elevation at a given (x,y) location,

$z = H(x,y,t)$ ...... (38.4)

$\therefore$ Free surface function,

$F_2(x,y,z,t) = z - H(x,y,t) = 0$ ...... (38.5)

W # Two Free surface function F_1 & F_2 is there. F_1 is related to 'piezometric head' and F_2 is related to 'elevation'. But, F_1 and F_2 represent the same free surface.

☑ ⊙ F_1 does not vary on the surface as it move.

What?! As $F_1 = z - h = 0$, so always zero on the surface?!

$$\frac{DF_1}{Dt} = \frac{D}{Dt}\left[z - h(x,y,z,t)\right]$$

$$= \frac{\partial}{\partial t}\left[z - h(x,y,z,t)\right] + \underset{\sim}{V}_{wt} \cdot \nabla(z - h(x,y,z,t)) \text{ ..... (38.6)}$$

I.T $\Rightarrow$ For (38.6), we applied the formula,

$$\frac{DF_1}{Dt} = \frac{\partial F_1}{\partial t} + \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) \cdot \left(\frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_1}{\partial y}\hat{j} + \frac{\partial F_1}{\partial z}\hat{k}\right) \text{ ... (38.7)}$$

I.T $\Rightarrow$ Here, V_{wt} represents some velocity. $\rightarrow$ See the next page.

I.T $\Rightarrow$ (38.7) is vector equivalent to the formula mentioned in (P-47, eqn 18.1).

In 1D, $\frac{DF_1}{Dt} = \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial x} \cdot \frac{dx}{dt}$ this is the equivalent formula, I think.

$$= -\frac{\partial h}{\partial t} - V_{wtx}\frac{\partial h}{\partial x} - V_{wty}\frac{\partial h}{\partial y} + V_{wtz}\left(1 - \frac{\partial h}{\partial z}\right) \text{ ..... (38.8)}$$

$= 0$ $\left[\because F_1 = 0 \text{ from 38.3, so, } \frac{DF_1}{Dt} = 0 \text{ as well}\right]$.

** # In (38.8), $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$, $\frac{\partial z}{\partial z} = 1$ ...... (38.9).

Equation (38.8) can be written as,

$$-n_e\frac{\partial h}{\partial t} - n_e V_{wtx}\frac{\partial h}{\partial x} - n_e V_{wty}\frac{\partial h}{\partial y} + n_e V_{wtz}\left(1 - \frac{\partial h}{\partial z}\right) = 0 \text{ ..... (38.10).}$$

⊙ The BC on the moving surface is obtained from the mass balance condition at that surface.

$$(\underset{\sim}{q}-\underset{\sim}{N})\cdot n = \eta_e \underset{\sim}{V_{wt}}\cdot n \quad \ldots\ldots (38.11)$$ → (For more see p:210).

I.T ⇒ q ⇒ Darcy velocity or sp. discharge, from CV.
$\underset{\sim}{N}$ ⇒ Recharge rate per unit area.
? $\underset{\sim}{V_{wt}}$ ⇒ Actual velocity of water from CV.
η_e ⇒ Effective porosity.
$\hat{n}$ ⇒ unit vector perpendicular to the plane.
If A = Total c/s area, then,

$A(\underset{\sim}{q}-\underset{\sim}{N})\cdot n$ → Discharge through Area A

$A\eta_e \underset{\sim}{V_{wt}}\, n = \underbrace{A_v \cdot V_{wt}\cdot\hat{n}}$

Area of void × Actual velocity that also give discharge through that area.

where η = Drainable porosity or effective porosity.

In (38.11), $\underset{\sim}{N} = -Ne_z \quad \ldots\ldots (38.12)$.

I think, recharge is from z direction only. So, only that component exist. But, why minus sign?
(?)

⊙ In terms of piezometric head,

$$\underset{\sim}{q} = -\underset{\sim}{K}\cdot\nabla h \quad \ldots\ldots (38.12)$$

If x, y, z are in the principal flow directions,

$$\left.\begin{aligned} q_x &= -K_x\frac{\partial h}{\partial x}\\ q_y &= -K_y\frac{\partial h}{\partial y}\\ q_z &= -K_z\frac{\partial h}{\partial z}\end{aligned}\right\} \quad \ldots\ldots (38.13)$$

You can see Page-106.

⊙ From (38.10),

$$-\eta_e\frac{\partial h}{\partial t} - \eta_e V_{wtx}\frac{\partial h}{\partial x} - \eta_e V_{wty}\frac{\partial h}{\partial y} + \eta_e V_{wtz}\left(1-\frac{\partial h}{\partial z}\right) = 0$$

$$\Rightarrow -\eta_e\frac{\partial h}{\partial t} - q_x\frac{\partial h}{\partial x} - q_y\frac{\partial h}{\partial y} + (q_z+N)\left(1-\frac{\partial h}{\partial z}\right) = 0 \quad \ldots (38.14)$$

I.T ⇒ From (38.11),

$$(\underset{\sim}{q}-\underset{\sim}{N})\cdot n = \eta_e \underset{\sim}{V_{wt}}\cdot n$$

$$\Rightarrow \begin{pmatrix} q_x\\ q_y\\ q_z\end{pmatrix} - \begin{pmatrix} 0\\ 0\\ -N\end{pmatrix} = \eta_e\begin{pmatrix} V_{wtx}\\ V_{wty}\\ V_{wtz}\end{pmatrix}$$

Value of N obtained from equation (38.12)

$$\Rightarrow \begin{pmatrix} q_x\\ q_y\\ q_z+N\end{pmatrix} = \begin{pmatrix} \eta_e\cdot V_{wtx}\\ \eta_e\cdot V_{wty}\\ \eta_e\cdot V_{wtz}\end{pmatrix} \quad \ldots\ldots (38.14.2)$$

⇒ These values were put to get equation (38.14).

Putting values from (38.13) to equation (38.14), (i.e q_x, q_y, q_z values),

$$-\eta_e \frac{\partial h}{\partial t} + k_x\left(\frac{\partial h}{\partial x}\right)^2 + k_y\left(\frac{\partial h}{\partial y}\right)^2 + \left(-k_z\frac{\partial h}{\partial z} + N\right)\left(1 - \frac{\partial h}{\partial z}\right) = 0$$

Rearranging,

$$\eta_e \frac{\partial h}{\partial t} = k_x\left(\frac{\partial h}{\partial x}\right)^2 + k_y\left(\frac{\partial h}{\partial y}\right)^2 + k_z\left(\frac{\partial h}{\partial z}\right)^2 - (k_z + N)\frac{\partial h}{\partial z} + N$$

...... (38.15)

⊙ Again, Similarly, as F_1, F_2 also does not vary on the surface as it moves.

$$\therefore \frac{DF_2}{Dt} = \frac{D}{Dt}\left[z - H(x,y,t)\right] \quad \text{\# From (38.5)}$$

$$= \frac{\partial F_2}{\partial t} - \underset{\sim}{V}_{wt} \cdot \nabla F_2$$

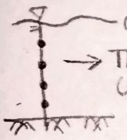
$$= \frac{\partial}{\partial t}\left[z - H(x,y,t)\right] + \underset{\sim}{V}_{wt} \cdot \nabla\left[z - H(x,y,t)\right]$$

$$= -\frac{\partial H}{\partial t} - V_{twx}\frac{\partial H}{\partial x} - V_{wty}\frac{\partial H}{\partial y} + V_{wtz}\left(1 - \cancelto{0}{\frac{\partial H}{\partial z}}\right) \ldots \; (38.16)$$

\# Here, $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$, $\frac{\partial z}{\partial z} = 1$. (See P-167)

Here, $\frac{\partial H}{\partial z} = 0$ also. Because, $H = H(x,y,t)$

We took $h = h(x,y,z,t)$. Because, pizometric head may vary in vertical direction (Though, as per Dupit's assumption, it does not vary). But, water surface elevation H is function of x, y, t only, not z.



→ Think, would you get different water surface level for different points in this vertical line? No. That mean H is not function of z.

$$= -\frac{\partial H}{\partial t} - V_{twx}\frac{\partial H}{\partial x} - V_{wty}\frac{\partial H}{\partial y} + V_{wtz}$$

$$= 0$$

So, this equation can be written as,

$$-\eta_e \frac{\partial H}{\partial t} - \eta_e V_{wtx}\frac{\partial H}{\partial x} - \eta_e V_{wty}\frac{\partial H}{\partial y} + \eta_e V_{wtz} = 0$$

-------- (38.17)

Putting values from (38.14.2),

$$-\eta_e \frac{\partial H}{\partial t} - q_x\frac{\partial H}{\partial x} - q_y\frac{\partial H}{\partial y} + (q_z + N) = 0 \; \ldots\ldots \; (38.18)$$

Putting values from (38.13) and rearranging,

$$\eta_e \frac{\partial H}{\partial t} = K_x\frac{\partial h}{\partial x}\frac{\partial H}{\partial x} + K_y\frac{\partial h}{\partial y}\frac{\partial H}{\partial y} - K_y\frac{\partial h}{\partial z} + N \ldots\ldots (38.19)$$

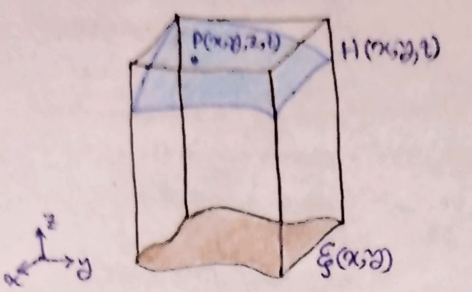
⊙ For very small slope of the preatic surface, from 38.19,

$$\eta_e \frac{\partial H}{\partial t} = -K_z\frac{\partial h}{\partial z} + N \ldots\ldots\ldots (38.20).$$

I.T ⇒ for flat surface, we considered $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y} = 0$

But, more importantly, From Dupit's assumption, no head variation along z direction. so, why $\frac{\partial h}{\partial z}$ is not zero?

☑ Reduction in Dimensionality:



Vertical integration of the flow equation,

$$\int_{\xi(x,y)}^{H(x,y,t)} \left(\nabla\cdot \underset{\sim}{q} + S_s\frac{\partial h}{\partial t}\right)dz = 0 \quad \ldots\ldots (39.1).$$

---

* How we got (39.1) from general flow equation, (36.10):-

Ans:- From (36.10),

$$S_y\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[k_x(H-\xi)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y(H-\xi)\frac{\partial H}{\partial y}\right] + N \ldots (39.2)$$

[# For unconfined aquifer,
Head = water surface elevation.
So, let replace 'H' with 'h'.]

$$S_y\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[k_x(H-\xi)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y(H-\xi)\frac{\partial h}{\partial y}\right] + N \ldots (39.3)$$

[# From Dupit's assumption, streamlines are horizontal and no head variation along z direction. So, no vertical flow occurs. $\frac{\partial h}{\partial z} = 0$
I.T ⇒ Also, N may increase water level, but it also does not introduce any q_z.]

$$\frac{S_y}{(H-\xi)}\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(k_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial h}{\partial y}\right) \ldots (39.4)$$

[# [Neglecting, recharge term, N.
$H-\xi$ = Aquifer thickness. We know,
(from P-88), S_y = Aquifer thickness × S_s.
Also, $q_x = -k_x\frac{\partial h}{\partial x}$ & $q_y = -k_y\frac{\partial h}{\partial y}$]

? But, why, $(H-\xi)$ would come out of the derivative. Both are functions of x and y!?]

∴ We get,

$$S_s\frac{\partial h}{\partial t} = -\left(\frac{\partial}{\partial x}q_x + \frac{\partial}{\partial y}q_y\right) \ldots\ldots (39.4.1)$$

$$\Rightarrow \nabla\cdot \underset{\sim}{q} + S_s\frac{\partial h}{\partial t} = 0$$

This is eqⁿ (39.1).

---

⑥ Using Leibniz rule for 1st term of equation (39.1),

$$\int_{\xi(x,y)}^{H(x,y,t)} \nabla\cdot \underset{\sim}{q}\, dz = \int_{\xi}^{H}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)dz$$

$$= \frac{\partial}{\partial x}\int_{\xi}^{H} q_x dz - q_x|_{z=H}\frac{\partial H}{\partial x} + q_x|_{x=\xi}\frac{\partial \xi}{\partial x}$$
$$+\frac{\partial}{\partial y}\int_{\xi}^{H} q_y dz - q_y|_{z=H}\frac{\partial H}{\partial y} + q_y|_{z=\xi}\frac{\partial \xi}{\partial y}$$
$$+\frac{\partial}{\partial z}\int_{\xi}^{H} q_z dz - q_z|_{z=H}\frac{\partial H}{\partial z} + q_z|_{z=\xi}\frac{\partial \xi}{\partial z} \quad \cdots (39.5)$$

$$= \frac{\partial}{\partial x}\int_{\xi}^{H} q_x dz - q_x|_{z=H}\frac{\partial H}{\partial x} + q_x|_{x=\xi}\frac{\partial \xi}{\partial x}$$
$$+\frac{\partial}{\partial y}\int_{\xi}^{H} q_y dz - q_y|_{z=H}\frac{\partial H}{\partial y} + q_y|_{z=\xi}\frac{\partial \xi}{\partial y}$$
$$+ q_z|_{z=H} - q_z|_{z=\xi} \quad \cdots\cdots (39.6)$$

For 'z' related part of 39.5,

$q_z = 0$, As there is no vertical flow.

Here, $H = H(x,y,t)$

and $\xi = \xi(x,y)$

So, $\frac{\partial H}{\partial z}$ and $\frac{\partial \xi}{\partial z}$ should be zero also. why not?

I think, here these are taken as 1.

(39.6) can be written in general form,

$$\int_{\xi(x,y)}^{H(x,y,t)} \nabla\cdot q\, dz = \nabla_{xy}\cdot\int_{\xi(x,y)}^{H(x,y,t)} q_{xy}\, dz + q|_{z=H}\cdot\nabla(z-H)$$
$$- q_{z=\xi}\cdot\nabla(z-\xi) \quad \cdots\cdots (39.7)$$

#(See page 154 for more).

- Free surface function,

$$F_2(x,y,z,t) = z - H(x,y,t) = 0 \quad \cdots\cdots (39.8)$$

(# see 38.5)

F_2 does not vary on the surface as it moves.

$$\frac{DF_2}{Dt} = \frac{D}{Dt}[z - H(x,y,t)] = 0 \qquad (39.9)$$

Using formula from (38.7),

$$\Rightarrow \frac{\partial}{\partial t}(z-H) + V_{wt}\cdot\nabla(z-H) = 0 \quad \text{\# From (38.6)}$$

⇒ Multiplying both sides with effective porosity, n_e,

$$\Rightarrow n_e\frac{\partial}{\partial t}(z-H) + n_e V_{wt}\cdot\nabla(z-H) = 0$$
$$\Rightarrow n_e\frac{\partial}{\partial t}(z-H) + (q - N)\cdot\nabla(z-H) = 0 \quad \cdots\cdots (39.10)$$

(From 38.11).

As, we are considering (39.10) for free surface,

$$q|_{z=H}\cdot\nabla(z-H) = n_e\frac{\partial H}{\partial t} - N \quad \cdots\cdots (39.11)$$

J.T ⇒ $F_2 = z - H$ represents a plane passing through elevation H.
So, $\frac{\partial H}{\partial t}$ is important. $\frac{\partial z}{\partial t}$ should not be considered.

(39.11) can be written as,

$$n_e\left(\frac{\partial \cancel{z}}{\partial t}-\frac{\partial H}{\partial t}\right)+\underset{\sim}{q}\cdot\nabla(z-H)-\underset{\sim}{N}\cdot\nabla(z-H)=0$$

$$\Rightarrow n_e\frac{\partial H}{\partial t}+\underset{\sim}{q}\cdot\nabla(z-H)-(0\hat{i}+0\hat{j}-N\hat{k})\cdot\left(\frac{\partial(z-H)}{\partial x}\hat{i}+\frac{\partial(z-H)}{\partial y}\hat{j}+\frac{\partial(z-H)}{\partial z}\hat{k}\right)=0$$

From (38.12)

$$\Rightarrow -n_e\frac{\partial H}{\partial t}+\underset{\sim}{q}\cdot\nabla(z-H)+N(1-0)=0$$

$\left[\because \frac{\partial z}{\partial z}=1 \text{ and } \frac{\partial H}{\partial z}=0, \text{ because } H(x,y,t)\right]$

$$\Rightarrow \underset{\sim}{q}\cdot\nabla(z-H)=n_e\frac{\partial H}{\partial t}-N$$

This is equation (39.11).

⊙ Now, let us consider the vertical average values,

$$\bar{q}_x=\frac{1}{(H-\xi)}\int_\xi^H q_x dz=\frac{1}{l}\int_\xi^H q_x dz=\frac{U_x}{l}\quad\ldots(39.12)$$

$$q_y=\frac{1}{(H-\xi)}\int_\xi^H q_y dz=\frac{1}{l}\int_\xi^H q_y dz=\frac{U_y}{l}\quad\ldots(39.13)$$

$$\bar{h}=\frac{1}{(H-\xi)}\int_\xi^H h\,dz=\frac{1}{l}\int_\xi^H h\,dz\quad\ldots\ldots(39.14)$$

⊙ Time derivative term of flow equation (39.1),

$$\int_{\xi(x,y)}^{H(x,y,t)} S_s\frac{\partial h}{\partial t}dz=\bar{S}_s\int_\xi^H\frac{\partial h}{\partial t}dz$$

Applying Libniz rule,

$$=\bar{S}_s\left[\frac{\partial}{\partial t}\int_\xi^H h\,dz-h|_{z=H}\frac{\partial H}{\partial t}+h|_{z=\xi}\frac{\partial \xi}{\partial t}\right]$$

Putting values from (39.12),

$$=\bar{S}_s\left[\frac{\partial}{\partial t}(l\bar{h})-h|_{z=H}\frac{\partial H}{\partial t}+0\right]$$

As $\frac{\partial \xi}{\partial t}=0$. Because, $\xi=\xi(x,y)$ not a function of time

$$=\bar{S}_s\left[l\frac{\partial \bar{h}}{\partial t}+\bar{h}\frac{\partial l}{\partial t}-h|_{z=H}\frac{\partial H}{\partial t}\right]$$

(In case of confined aquifer, (see 33.12), $l=b-a$ ~~which is constant wrt time.~~) X → No, it was not constant previously. see at (33.9), it was $a(x,y,t)$, $b(x,y,t)$. But before getting (33.13), it was taken as constant. But, I think, $(b-a)$ i.e thickness of confined aquifer should be invariant wrt time.

But for unconfined aquifer, $l=H(x,y,t)-\xi(x,y)$.

$$=\bar{S}_s\left[l\frac{\partial \bar{h}}{\partial t}+\bar{h}\frac{\partial(H-\xi)}{\partial t}-h|_{z=H}\frac{\partial H}{\partial t}\right]$$

$$=S_s\left[l\frac{\partial \bar{h}}{\partial t}+\cancel{\bar{h}\frac{\partial H}{\partial t}}-\bar{h}\overset{\nearrow 0}{\frac{\partial \xi}{\partial t}}-\cancel{h|_{z=H}\frac{\partial H}{\partial t}}\right]$$

[As per Dupit's Assumption h does not vary in vertical direction. So, at any depth $h=\bar{h}$. Here, $h|_{z=H}=\bar{h}$ as well]

Also, $\xi=\xi(x,y)$, $\frac{\partial \xi}{\partial t}=0$

$= \bar{S}_s l \frac{\partial \bar{h}}{\partial t}$ .... (39.15)

Putting values from (39.7) and (39.15) in the flow equation, i.e (39.1), we get,

$$\nabla_{xy}\cdot\int_{\xi(x,y)}^{H(x,y,t)} \underset{\sim}{q}_{xy}\,dz + \underset{\sim}{q}\big|_{z=H}\cdot\nabla(z-H) - \underset{\sim}{q}\big|_{z=\xi}\cdot\nabla(z-\xi) + \bar{S}_s l\frac{\partial\bar h}{\partial t} = 0 \quad \cdots\cdots (39.16)$$

$$\Rightarrow \nabla_{xy}\cdot\underset{\sim}{U} + \underset{\sim}{q}\big|_{z=H}\cdot\nabla(z-H) - \underset{\sim}{q}\big|_{z=\xi}\cdot\nabla(z-\xi) + \bar{S}_s l\frac{\partial\bar h}{\partial t} = 0 \quad \cdots\cdots (39.17)$$

[Aquifer flux per unit width $\underset{\sim}{U} = \int_{\xi}^{H} \underset{\sim}{q}_{xy}\,dz$, where, $\underset{\sim}{q}_{xy}$ = specific discharge vector].

- Non-leaky condition, in (39.17),

$$q_B = -\underset{\sim}{q}\big|_{z=\xi}\cdot\nabla(z-\xi) = 0 \quad \cdots\cdots (39.18)$$

I.T ⇒ No specific discharge through the bottom surface.

- Free surface condition, in (39.17),

$$q_T = \underset{\sim}{q}\big|_{z=H}\cdot\nabla(z-H) = n_e\frac{\partial H}{\partial t} - N \quad \cdots (39.19)$$

This is from (39.11).

Now, if we consider free surface condition in a non-leaky aquifer,
(Put values of q_B and q_T in from 39.18 and 39.19 to equation 39.17),

$$\nabla_{xy}\cdot\underset{\sim}{U} + n_e\frac{\partial H}{\partial t} - N + 0 + \bar{S}_s l\frac{\partial\bar h}{\partial t} = 0$$

⇒ Putting $\bar{h} = H$ we get,

$$\Rightarrow (n_e + S_s\cdot l)\frac{\partial H}{\partial t} = -\nabla_{xy}\cdot\underset{\sim}{U} + N \quad \cdots\cdots (39.20)$$

⇒ as $n_e \gg S_s\cdot l$, so we can write, ⟶ But, why so?

$$n_e\frac{\partial H}{\partial t} = -\nabla_{xy}\cdot\underset{\sim}{U} + N \quad \cdots\cdots (39.21)$$

considering, $n_e = S_y$, (Effective porosity is volumetric fraction of mobile water, as flow is saturated flow).

$$S_y\frac{\partial H}{\partial t} = -\nabla_{xy}\cdot\underset{\sim}{U} + N \quad \cdots\cdots (39.22).$$

Lecture-12

Parameter Estimation:

\# To derive G.E for Saturated Leaky confined aquifer

\# " " " " " " unconfined "

\# To estimate aquifer parameters for steady confined flow condition.

Basic Aquifer types:

Confined

Unconfined

Semi confined/Leaky aquifer.

Confined Aquifer

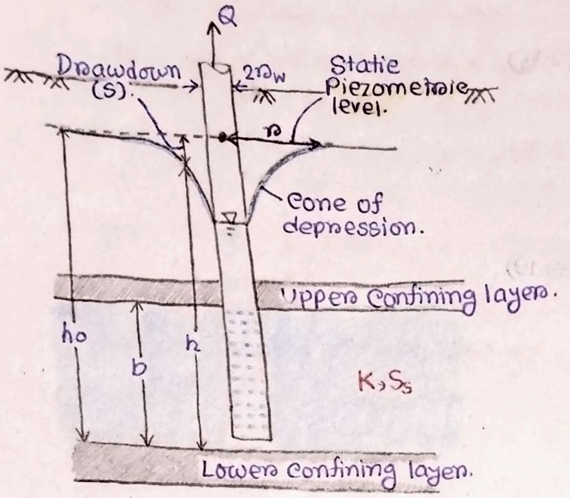


Fig: 12.1

Unconfined Aquifer:

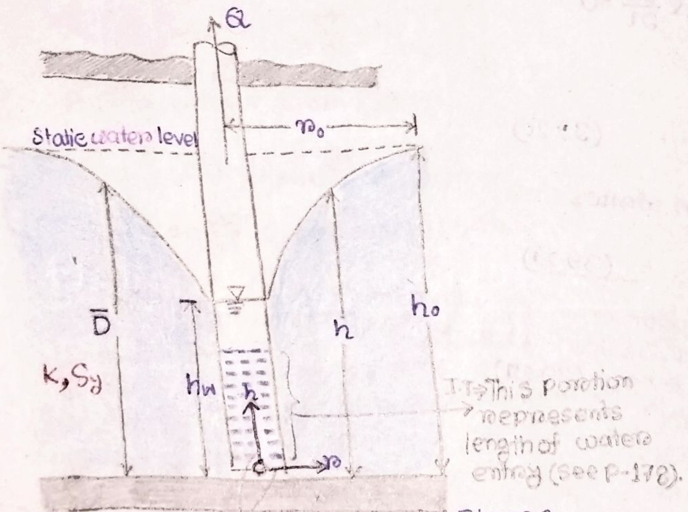


Fig: 12.2

I.T ⇒ These two represents h and r axis (The origin was taken as bottom of the upper confining layer for confined aquifer. See fig-12.10)!?

(?) What is $\bar{D}$. (Average water depth in the zone of influence)?

\# Note that, for confined Aquifer, upper surface of cone is called 'static piezometric level'.

For unconfined aquifer, it is static water level.

Semi-confined Aquifer:

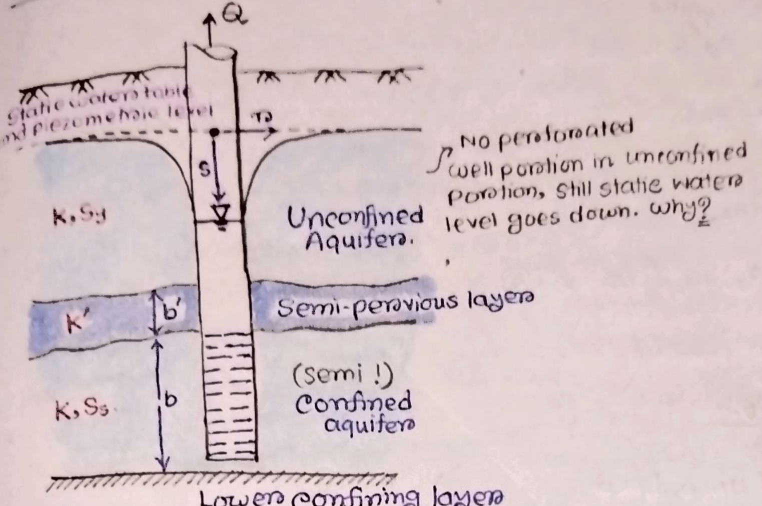


Fig: 12.3.

Flow in a Leaky Phreatic Aquifer:

- The governing equation for Phreatic Aquifer,

$$S_y \frac{\partial H}{\partial t} = -\nabla_{xy} \cdot \underset{\sim}{U} + N - q_B \cdots (40.1).$$

Using (39.18) and (39.19), (39.17) can be written as,

$$\nabla_{xy} \cdot \underset{\sim}{U} + q_T + q_B + \bar{S}_s l \frac{\partial \bar{h}}{\partial t} = 0 \ldots\ldots (40.2)$$

Using (39.19), we replace q_T from (40.2),

$$\Rightarrow \nabla_{xy} \cdot \underset{\sim}{U} + \eta_e \frac{\partial H}{\partial t} - N + q_B + \bar{S}_s l \frac{\partial \bar{h}}{\partial t} = 0$$

Now, putting $\eta_e = S_y$ (see 39.22) and $\bar{h} = H$ (See P-172), we can write this as,

$$\nabla_{xy} \underset{\sim}{U} + (S_y + \bar{S}_s l)\frac{\partial H}{\partial t} - N + q_B = 0$$

Here, $S_y \gg \bar{S}_s l$, so, it becomes,

$$S_y \frac{\partial H}{\partial t} = -\nabla_{xy} \cdot \underset{\sim}{U} - q_B + N$$

This is equation (40.1)

Or, $$S_y \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[K_x (H - \xi)\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y (H - \xi)\frac{\partial H}{\partial y}\right] - q_B + N \ldots\ldots (40.3)$$

\# Putting values of U_x and U_y from (35.1) and (35.2).

- Leaky Condition:-

If we assume steady flow across the horizontal leaky confining layer ($\nabla \xi = 0$), from (39.18),

$$q_B = -\underset{\sim}{q}\big|_{z=\xi} \cdot \nabla (z - \xi)$$

$$= -q_z\big|_{z=\xi} \ldots\ldots (40.4)$$

$$= k' \frac{H - h_{cA}}{b'} \ldots\ldots (40.5)$$

$$-\underset{\sim}{q}\big|_{z=\xi} \cdot \nabla (z - \xi) = -(q_x \hat{i} + q_y \hat{j} + q_z \hat{k})\big|_{z=\xi} \cdot \left(\frac{\partial}{\partial x}(z - \xi)\hat{i} + \frac{\partial}{\partial y}(z - \xi)\hat{j} + \frac{\partial}{\partial z}(z - \xi)\hat{k}\right) \ldots \quad (40.6)$$

Here, $\frac{\partial z}{\partial x}=0$, $\frac{\partial z}{\partial y}=0$

We have considered horizontal leaky, confining layers. So ξ is not $\xi(x,y)$ anymore. $\therefore \frac{\partial \xi}{\partial x}=\frac{\partial \xi}{\partial y}=0$ now.

$\frac{\partial z}{\partial z}=1$, $\frac{\partial \xi}{\partial z}=0$

$\therefore$ (40.6) can be written as,

$$-\underset{\sim}{q}\big|_{z=\xi}\cdot\nabla(z-\xi) = -(q_x\hat{i}+q_y\hat{j}+q_z\hat{k})\big|_{z=\xi}\cdot(0\hat{i}+0\hat{j}+\hat{k})$$

$$= -q_z\big|_{z=\xi}$$

This is equation (40.4).

For equation (40.5).

Head at the bottom of unconfined aquifer = Head at the top of leaky layer = H. # H = elevation of static water table

Head at the bottom of leaky layer = Head at the top of confined aquifer = h_{CA}

$$\therefore -q_z\big|_{z=\xi} = -[-K' i]$$

$$= K'\frac{h_{CA}-H}{b'} \Rightarrow \text{sign Not matching with (40.5).} \; ?$$

I.T ⇒ # As discharge is from upper unconfined to confined aquifer,

$$i = \frac{\Delta h}{L} = \frac{h_2-h_1}{L} = \frac{h_{CA}-H}{b'}$$

By applying Leaky condition from (40.5) in equation (40.3),

$$S_y\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[K_x(H-\xi)\frac{\partial H}{\partial x}\right]+\frac{\partial}{\partial y}\left[K_y(H-\xi)\frac{\partial H}{\partial y}\right] - K'\frac{H-h_{CA}}{b'}+N \quad \cdots\cdots (40.7)$$

The term $\left(\frac{K'}{b'}\right)$ is called 'leakance'.

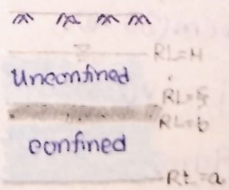
Flow in a Leaky CONFINED aquifer

2D governing equation for confined aquifer, (From 33.21),

$$\left.\begin{aligned} S\frac{\partial h}{\partial t}+q_T+q_B+q_{ext} &= \nabla_{xy}\cdot[\underset{\sim}{T}\cdot\nabla_{xy}h] \\ \text{where, } q_B &= -\underset{\sim}{q}\big|_{z=a}\cdot\nabla(z-a)=0 \\ q_T &= -\underset{\sim}{q}\big|_{z=b}\cdot\nabla(z-b) \end{aligned}\right\} \cdots (41.1).$$

? # Why $q_B=0$ but, q_T not?

I Think, at the bottom, there is some impermeable layer, But, at the top, there is some leaky confined layer.



Leaky Condition:- If we assume steady flow across the horizontal leaky confining layer, $(\nabla b=0)$,

$\nabla b=0$ means, $\frac{\partial b}{\partial x}=0$ and $\frac{\partial b}{\partial y}=0$. So, b is constant (horizontal surface) and nomore function of x and y.

$$q_T = \underset{\sim}{q}\Big|_{z=b} \cdot \nabla(z-b)$$

$$= q_z\Big|_{z=b} \quad \text{\#(See below 40·4 for more)} \quad \ldots (41.2)$$

$$= -K'\frac{H_{UA}-h}{b'} \quad \ldots\ldots\ldots (41.3).$$

(?) I have some confusion regarding flow direction of (40·4) and (41·3)

Putting, q_T value from (41·3), $q_B=0$ in equation (41·1), GIE for leaky confined aquifer becomes,

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right) - q_T - \cancelto{0}{q_B} - q_{ext}$$

$$= \frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right) + K'\frac{H_{UA}-h}{b'} - q_{ext} \quad \ldots (42.4)$$

Here also, *same as leaky phreatic aquifer, $\frac{K'}{b'}$ is called as leakance.

I think, →

$$\nabla_{xy}\cdot[T\cdot\nabla_{xy}h]$$
$$= \nabla_{xy}\cdot\left[\begin{pmatrix} T_x & 0 \\ 0 & T_y \end{pmatrix}\cdot\begin{pmatrix} \partial h/\partial x \\ \partial h/\partial y \end{pmatrix}\right]$$
$$= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}\cdot\begin{bmatrix} T_x\frac{\partial h}{\partial x} \\ T_y\frac{\partial h}{\partial y} \end{bmatrix}$$
$$= \frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)$$

Parameter Estimation:

- Mathematical method:
  - Analytical method
  - Numerical method.

- Aquifer test:-

Controlled stress-response experiment at field level for determination of hydraulic conductivity, transmissivity, storage coefficient.

(i) Pumping test:-
Aquifer is stressed using controlled extraction or injection from a pumping well.

(ii) Slug test:-
Aquifer is stressed using sudden change in water level in the control well.

(iii) Constant head test:-
Aquifer is stressed using constant head (no change in head) in the control well.

- Ground water flow equation, (See 32·1),

$$S_s\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right) - W \quad \ldots (43.1)$$

I.T ⇒ (43·1) is applicable for confined aquifer. (Yes! See 43·3).

(43·1) represented in cylindrical co-ordinate system, (See P-153),

$$S_s\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(K_r\cdot r\frac{\partial h}{\partial r}\right) + \frac{1}{r\cdot\partial\theta}\ \frac{\partial}{\partial\theta}\left(\frac{\partial h}{r\,\partial\theta}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right) - W \quad \ldots (43.2)$$

⊙ Parameters to be determined:-

- hydraulic conductivity tensor, $\underline{K}$
- Specific storage, S_s.

$\}$ .... (43.3)

For unconfined aquifer, it is called Storage coefficient. (see P-88).

⊙ Initial condition

⊙ Boundary Conditions

- Dirichlet, also called fixed head.
- Neumann, also called fixed flux.
- Robin's, also called induced flux.

(?) Did not understand what is in slide-11.

☑ Partially Penetrating Wells

(From DK Todd, Page-149),

⊙ A well whose length of water entry is less than aquifer is called partially penetrating well.

⊙ For PPW, average length of a flow line is more than FPW. So, it encounters greater resistance to flow.

⊙ If PPW and FPW in same aquifer has same discharge (i.e $Q_P = Q$), PPW will have more drawdown. In figure 12.4, drawdown at well face,

$$S_p = S + \Delta S \text{ .... } (44.1).$$

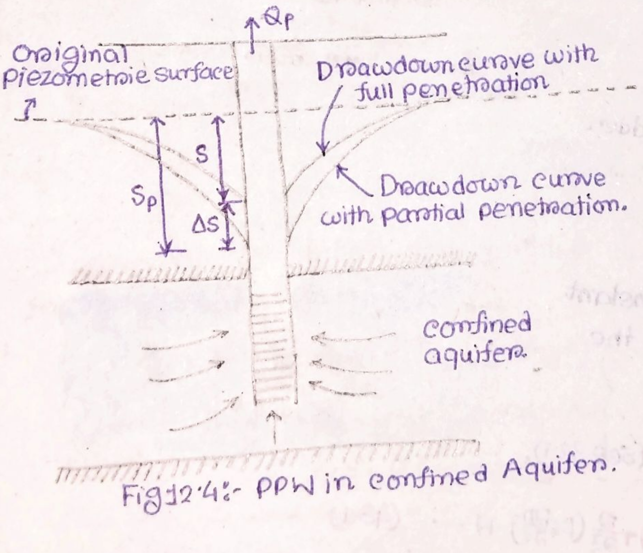


Fig 12.4:- PPW in confined Aquifer.

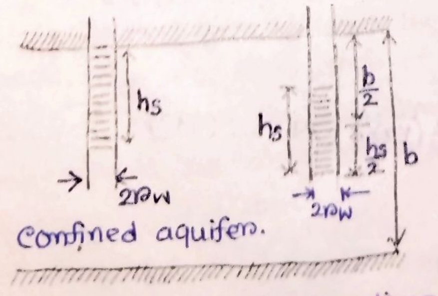


Fig: 12.5: Two configurations of PPW.

@ The, effect of partial penetration is negligible on flow pattern and drawdown for:-

radial distance > 0.5 to 2 times the saturated thickness b ...... (44.2)

The multiplying factor varies from 0.5 to 2, that depends on amount of well penetration.

@ Penetration factor, $p = \frac{h_s}{b}$ ...... (44.3)

@ If penetration factor $p > 0.2$, for steady-state condition in typical condition in confined aquifer,

$$\Delta S = \frac{Q_p}{2\pi T}\,\frac{1-p}{p}\,\ln\frac{(1-p)h_s}{r_w} \quad ......(44.4).$$

(See figure 12.4)

For the case, when well screen centered at the aquifer (see fig-12.5),

$$\Delta S = \frac{Q_p}{2\pi T}\cdot\frac{1-p}{p}\cdot\ln\frac{(1-p)h_s}{2r_w} \quad ...... (44.5)$$

@ Equation, (44.4) can be modified in case of unconfined aquifer,

$$\Delta S = \frac{Q_p}{2\pi K h_w}\cdot\frac{1-p}{p}\ln\frac{(1-p)h_s}{r_w} \quad ..... (44.6).$$

Where, h_w = Saturated thickness of the well with full penetration.

h_s = Length of well screen or perforated zone or length of water entry. (see 44.5)

p = penetration factor (see 44.3).

☑ Parameter Estimation (Continued)

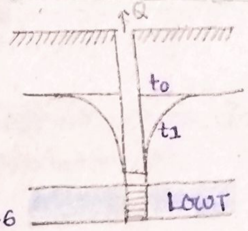


fig: 12.6

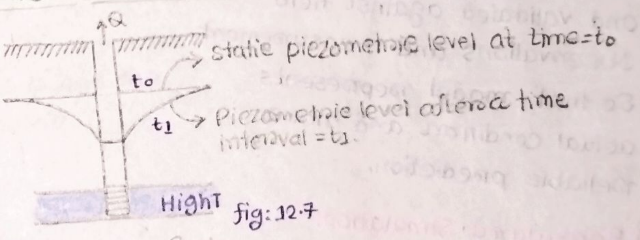


fig: 12.7

I think, if transmissivity of aquifer is high, when water is extracted from the well, water from surrounding places can easily fill that gap. So, less drawdown will occur at well face.

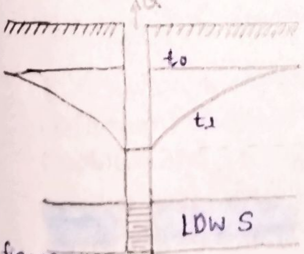


fig: 12.8

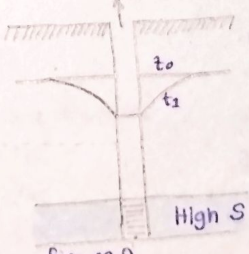


fig: 12.9.

From page 88, in broad sense, Specific Storage of confined aquifer means → water volume that can be stored per unit decline of piezometric head.

As, fig 12.9 has higher S value, so to create same discharge as fig 12.8, its declination of head should be less.

Inverse Problem:

(a) Backward/Retrospective problem

Here, initial conditions need to be found.

A.I.→ One example is process of the movement of pollutants in GW from a specific location back to their original source or starting point.

Steps to be followed-

(i) Data collection:- Data such as hydraulic conductivity, aquifer properties, water levels, contaminant measurement at different locations etc.

(ii) Conceptual model development: Identifying potential sources, understanding hydrogeological conditions, mapping the flow pathway.

(iii) Numerical modelling: Models use mathematical eqns to represent the behavior of GW and simulate movement of pollutants based on available data.

(iv) Calibration and Validation: Model need to be calibrated and validated against field observations and measurement. So that model represents actual condition and make reliable prediction.

(v) Backward Simulation:- This involves starting from observation points where the contaminents were detected and tracing the movement of GW backward in time to determine the likely source/sources.

Although, the problem is challanging due to uncertainties, incomplete knowledge of data, incomplete knowledge of subsurface conditions, and complex nature of GW flow.

(b) Coefficient Inverse problem:

- Classical parameters estimation problem, where constant multipliers of a governing equation is to be found.

- Example:- Estimating hydraulic conductivity or other aquifer properties of a subsurface system based on observed data such as groundwater level or flow rates.

- Steps:-

(i) Data collection:- Gathering data such as hydraulic head measurements, pumping tests, tracer experiments etc.

(ii) Forward modelling:- Develop a groundwater flow model that simulate behaviour of GW in aquifer. The model should have proper GE, IC, and BC's. The model requires an initial estimate of hydraulic conductivity distribution. (I.T ⇒ For initialization).

(iii) Calibration:- Adjust the hydraulic conductivity values in numerical model to match the observed data. Run multiple model simulations and compare the model results with observed values. Calibration techniques like parameter estimation algorithms or optimization methods can be employed to find the best fit values for hydraulic conductivities.

(iv) Inverse problem formulation:- This involves defining an Objective function that quantifies the difference between observed data and model prediction. We need to minimize this function.

(v) Solution methods:- Different method to solve inverse problem and get hydraulic conductivity distribution are:- least square methods, Bayesian techniques, Regularization techniques. These methods helps to find the best estimate and balance smoothness of the parameter distribution.

(vi) Uncertainty analysis:- It helps to understand the reliability of estimated parameters and quantify the potential errors or variation in the results.

(c) Boundary Inverse Problem:-

① Some missing information at the boundary of a domain is to be found.

* In the context of groundwater, the boundaries may represent between different geological formations, groundwater recharge or discharge areas or the extent of contaminant plume.

Methodology is same as coefficient inverse problem.

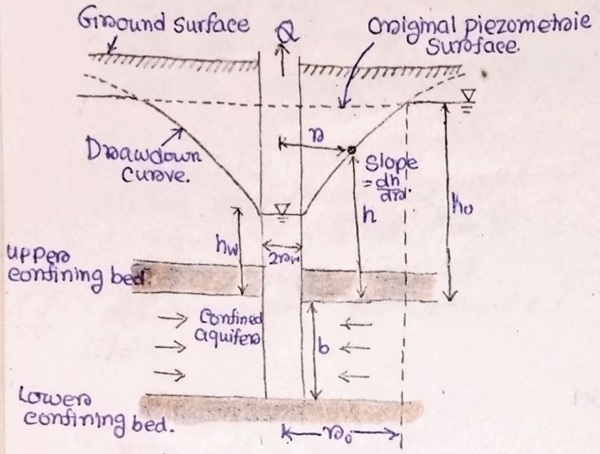


Fig: 12.10

☑ Steady Radial flow in a Confined Aquifer (Fig-12.10)

For confined aquifer, well discharge can be written as,

$$Q = Aq_r = (2\pi rb)\left(-K_r \frac{\partial h}{\partial r}\right)$$
$$= -2\pi rbK_r \frac{\partial h}{\partial r} \cdots\cdots (45.1).$$

Head difference between radius of influence and well face is:-

$$h_0 - h_w = \frac{Q}{2\pi bK_r} \ln\left(\frac{r_0}{r_w}\right) \cdots\cdots (45.2)$$

From (45.2), discharge can be written as,

$$Q = \frac{2\pi bK_r (h_0 - h_w)}{\ln(r_0/r_w)} \cdots\cdots (45.3).$$

negative sign neglected (Todd- Page-116).

Neglecting the minus sign in equation (45.1).

[#(If minus sign is not neglected, and we integrate that then, $h_0 - h_w$ in (45.2) would have negative value. Which is not acceptable.)] → See 45.7 and 46.2.

$$Q = 2\pi r b K_r \frac{\partial h}{\partial r}$$

$$\Rightarrow \int_{r_w}^{r_o} \frac{\partial r}{r} = \frac{2\pi b K_r}{Q} \int_{h_w}^{h_o} \partial h$$

$$\Rightarrow [\ln r]_{r_w}^{r_o} = \frac{2\pi b K_r}{Q} [h]_{h_w}^{h_o}$$

$$\Rightarrow \ln\left(\frac{r_o}{r_w}\right) = \frac{2\pi b K_r}{Q} (h_o - h_w)$$

$$\Rightarrow Q = 2\pi b K_r \frac{(h_o - h_w)}{\ln(r_o / r_w)}$$

This gives equation (45.3)

Now, groundwater flow equation for confined aquifer in radial co-ordinate system, (from 32.4),

$$\cancel{S_s \frac{\partial h}{\partial t}}^{0} = \frac{1}{r}\frac{\partial}{\partial r}\left(K_r \cdot r \frac{\partial h}{\partial r}\right) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{K_\theta}{r}\frac{\partial h}{\partial \theta}\right)}^{0} + \cancel{\frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right)}^{0} - \cancel{W}^{0} \quad \ldots\ldots (45.4)$$

I.T $\Rightarrow$ [$S_s \frac{\partial h}{\partial t} = 0$, Because steady-state is considered.

$\frac{\partial h}{\partial \theta} = 0$, Because, no change of head in radial direction. In 3D, water surface profile is cone shaped. (Radially symmetric)

$\frac{\partial h}{\partial z} = 0$, It is as per Dupit's assumption. (P-152).

? (But, this was applicable for unconfined aquifer there)

W=0, No recharge was considered.]

So,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(K_r \cdot r \frac{\partial h}{\partial r}\right) = 0 \quad \ldots\ldots (45.5).$$

$$\Rightarrow \frac{\partial}{\partial r}\left(K_r \cdot r \frac{\partial h}{\partial r}\right) = 0$$

$$\Rightarrow \partial\left(K_r \cdot r \frac{\partial h}{\partial r}\right) = 0 \quad [\because K_r \text{ is constant term}]$$

$$\Rightarrow K_r \cdot r \frac{\partial h}{\partial r} = A \quad \ldots\ldots (45.5.2).$$

$$\Rightarrow K_r \int \partial h = A \int \frac{dr}{r}$$

$$\Rightarrow h(r) = A \ln(r) + B \ldots\ldots (45.6)$$

Where, A, B are constant of integration.

Again, $Q = -A \cdot q_r(r) \ldots\ldots (45.7)$

[#Negative sign to represent inward flow]

$$= -2\pi r b \cdot q_r(r)$$

$$= -(2\pi r b) \cdot \left(-K_r \cdot \frac{\partial h}{\partial r}\right)$$

$$= 2\pi r b \cdot K_r \frac{\partial h}{\partial r}$$

$$= 2\pi \cdot K_r b \cdot r \frac{\partial h}{\partial r}$$

$$= 2\pi T A \ldots\ldots (45.8)$$

[#In (45.8), T = Transmissvity. But, A is not area, it is the constant of integration obtained from (45.5.2).]

? In (45.1), we considered, $Q = A q_r$, in (45.7), $Q = -A q_r$. But, if we considered $Q = -A q_r$ in (45.1), considering inward flow, we need not neglect negative sign to derive (45.3)!

• Now, if we consider two piezometric observation wells located at a radial distance r_1 and r_2 from the well-centerline, and their piezometric heads after drawdown are h_1 and h_2,

then in similar fashion of eqⁿ (45.2),

We can write Thiem equation,

$$Q = 2\pi b k_r \frac{h - h_w}{\ln\left(\frac{r}{r_w}\right)} \quad \ldots\ldots (45.9)$$

#where, h = Piezometric head at any radial distance r from the well (r, h)

Eqⁿ (45.9) is general formula.

For observation well 1 & 2, (45.9) can be written as:-

$$h_2 - h_1 = \frac{Q}{2\pi T} \ln\left(\frac{r_2}{r_1}\right) \quad \ldots\ldots (45.10)$$

$[T = \text{Transmissvity} = k_r \cdot b]$

$$(H - S_2) - (H - S_1) = \frac{Q}{2\pi T} \ln\left(\frac{r_2}{r_1}\right)$$

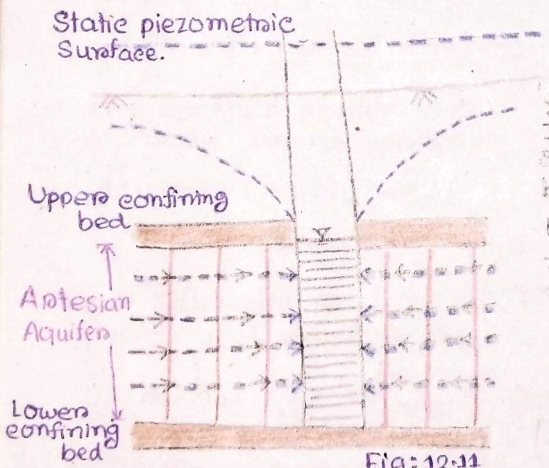
[H = Elevation of static piezometric level. S_1, S_2 are drawdowns at observation wells].

$$\Rightarrow S_1 - S_2 = \frac{Q}{2\pi T} \ln\left(\frac{r_2}{r_1}\right) \quad \ldots\ldots (45.11)$$

$$\Rightarrow T = b k_r = \frac{Q}{2\pi (S_1 - S_2)} \ln\left(\frac{r_2}{r_1}\right) \quad \ldots (45.12).$$

Confined Aquifer:

Flow lines & Equipotential lines:



I think, ground surface should be below the static piezometric surface for artesian aquifer!!

Fig: 12.11

--→ Flow lines

—— Equipotential lines.

I think, horizontal flow lines are considered as per Dupit's assumption. (But, it is applicable for unconfined aquifers only).

AI ⑥ Why Dupit's Assumption is generally not valid for confined aquifer?

It assumes, aquifer is unconfined and vertical hydraulic gradient is very less compared to horizontal hydraulic gradient (i.e vertical flow is negligible). In other words, (all the points have same head along a vertical line (∵ No vertical hydraulic gradient is there).

But, confined aquifer is under significant overlaying pressure, they have distinct vertical hydraulic gradient. In confined aquifer, confining layers are there to restrict vertical flow and leads to a different flow regime. As flow in confined aquifer is quite complicated, we use Darcy's law, numerical modelling (FEM, FDM) for confined aquifers flow.

AI ⑦ Difference between Dupuit's assumptions and Dupuit-Forchheimer's assumption?

(i) Dupuit's assumption:
- For unconfined aquifers.
- Flow is considered as two-dimensional (in the horizontal plane) and vertical component is neglected.

(ii) Dupuit-Forchheimer's assumption:-
- Extension of Dupuit's assumption, incorporates additional considerations.
- Analyzes GW flow in unconfined aquifers with partially penetrating boundaries, such as partially penetrating wells or drains.
- It takes into account the effect of vertical leakage/flow across the partially penetrating boundaries. (For PPW, see P-178).

→ Another source tells both are same! ?

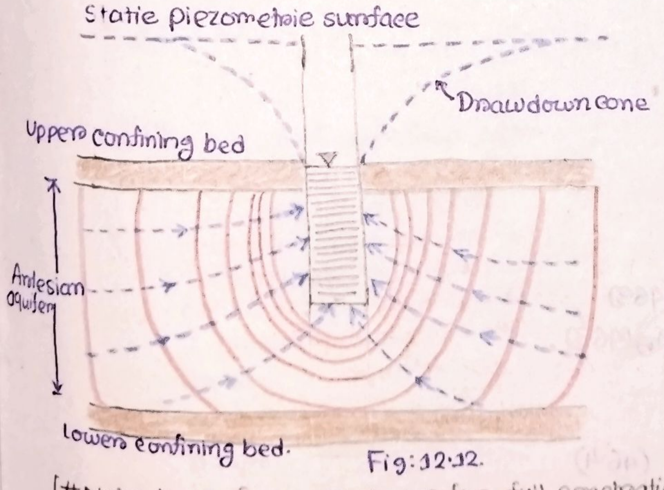


Fig: 12.12.

[#Note that, figure 12.11 was for full penetration, but fig 12.12 is for partial penetration of well.]

Lecture-13

Parameter Estimation:

Aquifer Parameter for steady, unconfined flow condition.

Aquifer parameter for steady, leaky, confined flow condition.

Steady Radial flow in Unconfined Aquifer:-

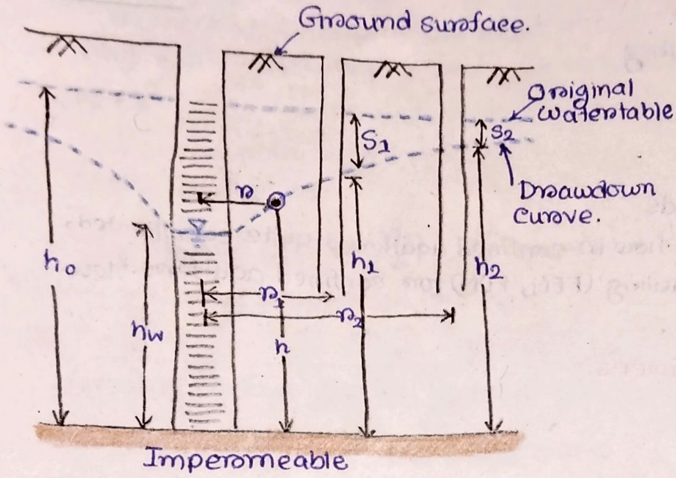


Fig: 13.1

Well discharge can be written as,

$$Q = A q_r = (2\pi r H)\left(-K_r \frac{\partial H}{\partial r}\right)$$

$$= -2\pi r K_r H \frac{\partial H}{\partial r} \quad \ldots (46.1)$$

For unconfined aquifer/water table aquifer, piezometric head at any point (h) = Water table elevation at that point. So, we replace h_0, h_w, h_1, h_2, h by H_0, H_w, H_1, H_2, H.

Neglecting minus sign in (46.1)

$$Q = 2\pi r K_r H \frac{\partial H}{\partial r} \quad \ldots (46.2)$$

(# Actually, it should be, $Q = -A \cdot q_r$ (See 46.1 & 46.2)
Because, inward flow to well should be negative. So, ultimately,
$Q = 2\pi r K_r H \frac{\partial H}{\partial r}$ is true without any assumption) → (See 45.7 and 45.3)

$$\Rightarrow \frac{Q}{2\pi K_r} \int_{r_w}^{r_0} \frac{\partial r}{r} = \int_{H_w}^{H_0} H \partial H$$

$$\Rightarrow \frac{Q}{2\pi K_r} \ln\left(\frac{r_0}{r_w}\right) = \frac{H_0^2 - H_w^2}{2}$$

$$\Rightarrow H_0^2 - H_w^2 = \frac{Q}{\pi K_r} \ln\left(\frac{r_0}{r_w}\right) \quad \ldots\ldots (46.3)$$

Head difference represented by (46.3).

⊙ From 46.3, discharge,

$$Q = \pi K_r \cdot \frac{H_0^2 - H_w^2}{\ln\left(\frac{r_0}{r_w}\right)} \quad \ldots (46.4)$$

② If we consider two observation wells, similarly,

$$K_r = \frac{Q}{\pi(H_2^2 - H_1^2)} \ln\left(\frac{r_2}{r_1}\right) \dots (46.5)$$

Governing equation for unconfined aquifer, (I.T ⇒ Free surface condition for non leaky aquifer).

$$\frac{2S_y}{K}\cancelto{0}{\frac{\partial H^2}{\partial t}} = \nabla^2 H^2 + \cancelto{0}{\frac{2N}{K}} \dots (46.6).$$

From (39.22),

$$S_y \frac{\partial H}{\partial t} = -\nabla_{xy} \cdot \underset{\sim}{U} + N$$

$$\Rightarrow S_y \frac{\partial H}{\partial t} = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \cdot \left(-K_x(H-\xi)\frac{\partial H}{\partial x}\hat{i} - K_y(H-\xi)\frac{\partial H}{\partial y}\right) + N \dots (46.7).$$

Considering isotropic condition i.e $K_x = K_y$, horizontal bottom bed. i.e. $\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial y} = 0$, we have,

$$S_y \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(KH\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(KH\frac{\partial H}{\partial y}\right) + N \dots (46.8)$$

considering homogenous media,

$$S_y \frac{\partial H}{\partial t} = K\left[\frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial H}{\partial y}\right)\right] + N \dots (46.9)$$

Now, $\frac{\partial}{\partial x}(H^2) = 2H\frac{\partial H}{\partial x}$

$$\Rightarrow \frac{1}{2}\frac{\partial H^2}{\partial x} = H\frac{\partial H}{\partial x} \dots (46.10)$$

Similarly, $\frac{1}{2}\frac{\partial H^2}{\partial y} = H\frac{\partial H}{\partial y} \dots (46.11)$

Putting values from (46.10) and (46.11) to eq^n (46.9), we get,

$$S_y \frac{\partial H}{\partial t} = \frac{K}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial H^2}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial H^2}{\partial y}\right)\right] + N$$

$$= \frac{K}{2}(\nabla^2 H^2) + N$$

$$\Rightarrow \frac{2S_y}{K} = \nabla^2 H^2 + \frac{2N}{K} \rightarrow \text{This is } (46.6)$$

Now, $\frac{\partial H}{\partial t} = 0 \Rightarrow$ Steady-state condition

$\frac{2N}{K} = 0 \Rightarrow$ Neglecting recharge!

So, (46.6) becomes,

$$\nabla^2 H^2 = 0 \dots (46.12)$$

In cylindrical co-ordinate system,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(K_r \cdot r\frac{\partial H^2}{\partial r}\right) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{K_\theta}{r}\frac{\partial H}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial H}{\partial z}\right) = 0 \dots (46.13)$$

Now, $\frac{\partial H}{\partial \theta} = 0$, water surface after drawdown is cone-shaped. At a particular radial distance from the well, water surface elevation is same in all direction. It is a circle parallel to the horizontal aquifer bed.

$\frac{\partial H}{\partial z} = 0$, Property of unconfined aquifer $\left(\frac{\partial h}{\partial z} = \frac{\partial H}{\partial z} = 0\right)$, from Dupuit assumption.

So, 46.13 becomes, $\frac{1}{r}\frac{\partial}{\partial r}\left(K_r \cdot r\frac{\partial H^2}{\partial r}\right) = 0 \dots (46.14)$

$$\Rightarrow \frac{\partial}{\partial r}\left(K_r \cdot r \frac{\partial H^2}{\partial r}\right) = 0$$

Now, we already assumed homogenous and isotropic aquifer. So,

$$K \cdot \frac{\partial}{\partial r}\left(r \frac{\partial H^2}{\partial r}\right) = 0$$

$$\Rightarrow \int \partial\left(r \frac{\partial H^2}{\partial r}\right) = 0$$

$$\Rightarrow r \frac{\partial H^2}{\partial r} = A \; \ldots\ldots \; (46.14.2)$$

$$\Rightarrow \int \partial H^2 = A \int \frac{\partial r}{r}$$

$$\Rightarrow H^2(r) = A \ln(r) + B \ldots \; (46.15)$$

Radius of influence:

$H = H_0$ at $r = r_0$.

∴ From 46.15,

$$H_0^2(r) = A \ln(r_0) + B \; \ldots \; (46.16)$$

I.T ⇒ (we put a constant value H_0, so it is not a function of r anymore.)

Also, $Q = -A \cdot q_r(r) \; \ldots\ldots \; (46.16.2)$

$$= -2\pi r H \cdot \left(-K_r \frac{\partial H}{\partial r}\right)$$

$$= 2\pi K_r H \left(r \frac{\partial H}{\partial r}\right)$$

$$= 2\pi K_r \cdot r \cdot \left(H \frac{\partial H}{\partial r}\right)$$

Now, applying formula,

$H \frac{\partial H}{\partial r} = \frac{1}{2} \frac{\partial H^2}{\partial r}$ (see-46.10), we get,

$$Q = 2\pi K_r \cdot r \cdot \frac{1}{2} \frac{\partial H^2}{\partial r}$$

$$= \pi K_r \cdot \left(r \frac{\partial H^2}{\partial r}\right)$$

Putting value from (46.14.2),

$$Q = \pi K_r A \; \ldots\ldots \; (46.17)$$

[# Here, A is not area. Its an constant of integration.]

$$\Rightarrow A = \frac{Q}{\pi K_r} \; \ldots\ldots \; (46.18)$$

Putting values of A from (46.18) in (46.15) and (46.16),

$$H^2(r) = \frac{Q}{\pi K_r} \ln(r) + B \; \ldots \; (46.19)$$

$$H_0^2 = \frac{Q}{\pi K_r} \ln(r_0) + B \; \ldots \; (46.20)$$

Subtracting (46.20) to (46.19),

$$H_0^2 - H^2(r) = \frac{Q}{\pi K_r} \ln\left(\frac{r_0}{r}\right)$$

$$\Rightarrow H^2(r) = H_0^2 - \frac{Q}{\pi K_r} \ln\left(\frac{r_0}{r}\right) \; \ldots \; (46.21)$$

- If we consider two observation wells at radial distances r_1 and r_2,

$$\left.\begin{aligned} S_1 &= H_0 - H_1 \Rightarrow H_1 = H_0 - S_1 \\ S_2 &= H_0 - H_2 \Rightarrow H_2 = H_0 - S_2 \end{aligned}\right\} \; \ldots \; (46.22)$$

Putting these values in (46.21),

$$(H_0-S_1)^2 = \frac{Q}{\pi k_{ro}} \ln(r_{01}) + B$$

$$(H_0-S_2)^2 = \frac{Q}{\pi k_{ro}} \ln(r_{02}) + B$$

Subtracting these expressions,

$$\frac{Q}{\pi k_{ro}} \ln\left(\frac{r_{02}}{r_{01}}\right) = (H_0-S_2)^2 - (H_0-S_1)^2$$

$$= \cancel{H_0^2} - 2H_0S_2 + S_2^2 - \cancel{H_0^2} + 2H_0S_1 - S_1^2$$

$$= (S_2^2 - 2H_0S_2) - (S_1^2 - 2H_0S_1)$$

$$= \frac{2H_0}{\cancel{2H_0}}\left[\left(\frac{S_2^2}{2H_0} - S_2\right) - \left(\frac{S_1^2}{2H_0} - S_1\right)\right] \ldots (46.23)$$

Transmissivity for the full thickness,

$$T = k_{ro}H_0 = \frac{Q\ln(r_{02}/r_{01})}{2\pi\left[\left(S_1 - \frac{S_1^2}{2H_0}\right) - \left(S_2 - \frac{S_2^2}{2H_0}\right)\right]} \ldots (46.24)$$

obtain by rearranging (46.23).

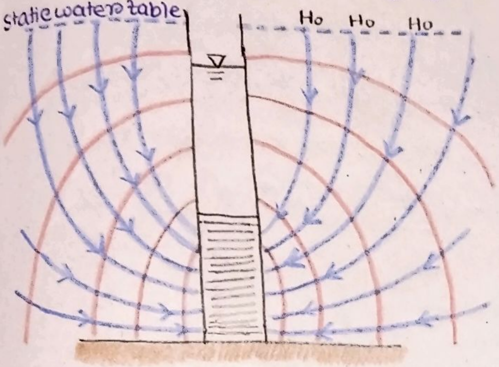


Fig: 13.2 ⇒ Intial stage of pumping a free (unconfined) aquifer. Most water follows a path with high vertical component from the water table to the screen.

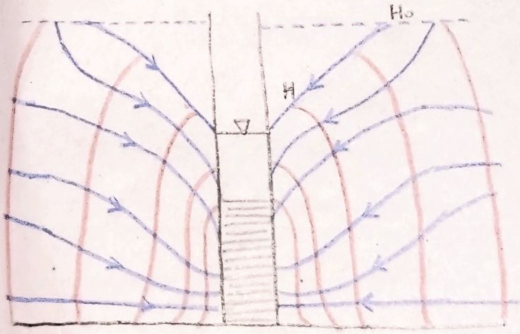


Fig 13.3:- Intermediate state in pumping a free aquifer. Radial component of flow becomes more pronounced. But, contribution from drawdown cone in immediate vicinity of well is still important. (see figure. Some vertical component of flow is still there for water at nearby locations of the well).

My observation:- At initial stage, at near or far distance, everywhere head = Ho. But nearby water need to travel less distance to reach well screen. So, high vertical hydraulic gradient. So water flows vertically.

But, after sometime, due to drawdown, hydraulic gradient of nearby water decreases. So, water comes from far distance.

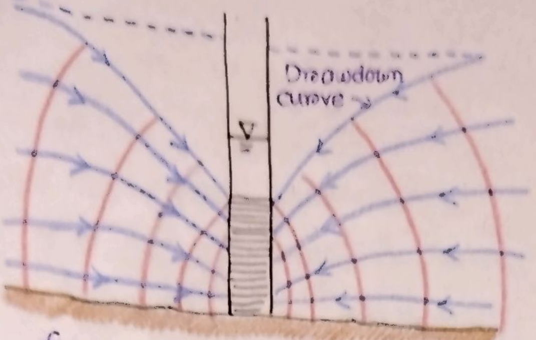


fig: 13.4:- Approximate steady-state in pumping a free aquifer. Profile of cone of depression is established. Total zone which are originating water (i.e zone within radius of influence) is stable. Primarily radial flow pattern established, minimum vertical flow.

Concept of image well:-

When a well is pumped near an aquifer boundary, the assumption that the aquifer is of infinite areal extent no longer holds.

Then, we introduce image wells, use principle of superposition, transform aquifer of finite extent into an infinite aquifer to apply previously described solution method.

Example: For well near a perennial stream.

An imaginary well (recharge well) is placed directly opposite and at the same distance from the stream as the real well. The image well is operated simultaneously at the same rate as real well so that no change of water level occurs along the line of the stream.

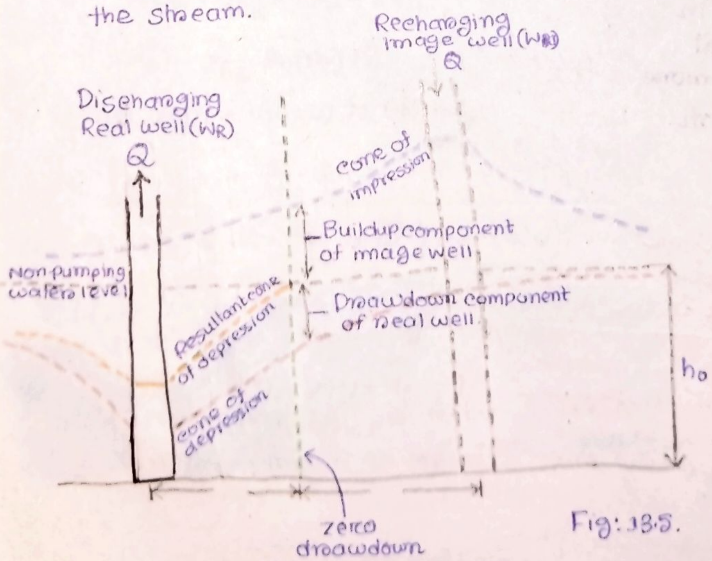


Fig: 13.5.

• River or surface water body (constant head):

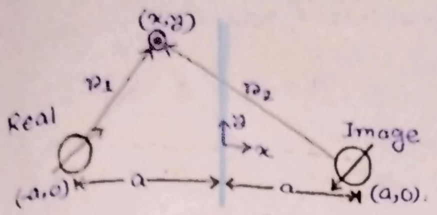


Figure 13.6.

11→ Resultant discharge at point (x,y)

$$Q = Q_R + Q_I$$
$$= (-Aq_R) + (Aq_I)$$

Real well is a drawdown well.
So, inward flow. Hence, negative
But, image well is a buildup well.
So, outward flow. Hence, positive.

$$\boxed{\begin{aligned} &= \cancel{-2\pi r H\left(-k_r \frac{\partial H}{\partial r}\right)} + \cancel{2\pi r H\cdot\left(-k_r\frac{\partial H}{\partial r}\right)} \\ &= \cancel{2\pi k_r\left(rH\frac{\partial H}{\partial r} - rH\frac{\partial H}{\partial r}\right)} \end{aligned}}$$

Considering real well,

$$Q_R = -2\pi r H\cdot\left(-k_r\frac{\partial H}{\partial r}\right)$$

$$\Rightarrow \frac{Q_R}{2\pi k_r}\int_{r_1}^{r_0}\frac{\partial r}{r} = \int_{H_1}^{H_0} H\,\partial H$$

$$\Rightarrow \frac{Q_R}{2\pi k_r}\ln\left(\frac{r_0}{r_1}\right) = \frac{1}{2}\left(H_0^2 - H_1^2\right) \quad \text{.....(47.1)}$$

Considering image well,

$$Q_I = Aq_I$$
$$= 2\pi r H\cdot\left(-k_r\frac{\partial H}{\partial r}\right)$$

$$\Rightarrow \frac{Q_I}{2\pi k_r}\int_{r_2}^{r_0}\frac{\partial r}{r} = -\int_{H_2}^{H_0} H\,\partial H$$

$$\Rightarrow \frac{Q_I}{2\pi k_r}\ln\left(\frac{r_0}{r_2}\right) = -\frac{1}{2}\left(H_0^2 - H_2^2\right) \quad \text{.....(47.2)}$$

Combining (47.1) and (47.2),

$$Q = Q_R + Q_I = \frac{\pi k_r\left(H_0^2 - H_1^2\right)}{\ln\left(\frac{r_0}{r_1}\right)} - \frac{\pi k_r\left(H_0^2 - H_2^2\right)}{\ln\left(\frac{r_0}{r_2}\right)}$$

$$\Rightarrow \frac{Q}{\pi k_r} = \frac{H_0^2 - H_1^2}{\ln\left(\frac{r_0}{r_1}\right)} - \frac{H_0^2 - H_2^2}{\ln\left(\frac{r_0}{r_2}\right)}$$

proper form not coming! (as 48.1)

Equation for figure 13.6,

$$H_0^2 - H^2(r) = \frac{Q}{\pi k_r}\ln\left(\frac{r_0}{r_1}\right) - \frac{Q}{\pi k_r}\ln\left(\frac{r_0}{r_2}\right) \quad \text{.....(48.1)}$$

$$= \frac{Q}{\pi k_r}\ln\left(\frac{r_2}{r_1}\right) \quad \text{.....(48.2)}$$

$$\left.\begin{aligned} r_1 &= \sqrt{(x+a)^2 + (y-0)^2} = \sqrt{(x+a)^2 + y^2} \\ r_2 &= \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x-a)^2 + y^2} \end{aligned}\right\} \quad \text{.....(48.3)}$$

Along the y axis, i.e along the length of the stream $r_1 = r_2$

So, from (48.2),

$$H_0^2 - H^2(r) = 0$$

$$\Rightarrow H^2 = H_0^2 \cdots\cdots (48.4)$$

⊙ Impermeable faults (No flow boundary):

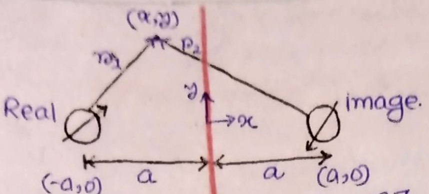


Fig: 13.7

$$\left.\begin{aligned} r_1 &= \sqrt{(x+a)^2 + y^2} \\ r_2 &= \sqrt{(x-a)^2 + y^2} \end{aligned}\right\} \cdots\cdots (48.5)$$

$$H^2(r) = \frac{Q}{\pi K_r} \ln(r_1) + \frac{Q}{\pi K_r} \ln(r_2) + B$$

How derived?

$$\Rightarrow H^2(r) = \frac{Q}{\pi K_r} \ln(r_1 r_2) + B \cdots\cdots (49.1)$$

⊙ x component of flow,

$$U_x = H q_x = -KH \frac{\partial H}{\partial x}$$

$$= -\frac{K}{2} \frac{\partial H^2}{\partial x} \quad \text{(From 46.10)} \cdots (49.2)$$

Putting value of H^2 from (49.1),

$$U_x = -\frac{K}{2} \frac{\partial}{\partial x} \left[\frac{Q}{\pi K_r} \ln(r_1 r_2) + B \right]$$

$$= -\frac{K}{2} \times \frac{Q}{\pi K_r} \frac{\partial}{\partial x} \left[\ln r_1 + \ln r_2 \right]$$

#I.T ⇒ B is some constant of integration.
$K_r = K$, as we already considered, homogenous and isotropic media.

$$= -\frac{Q}{2\pi} \left[\frac{\partial}{\partial r_1} (\ln r_1) \frac{\partial r_1}{\partial x} + \frac{\partial}{\partial r_2} (\ln(r_2)) \cdot \frac{\partial r_2}{\partial x} \right]$$

$$= -\frac{Q}{2\pi} \left[\frac{1}{r_1} \cdot \frac{x+a}{r_1} + \frac{1}{r_2} \cdot \frac{x-a}{r_2} \right] \cdots\cdots (49.3)$$

$$\frac{\partial r_1}{\partial x} = \frac{\partial}{\partial x} \left[\sqrt{(x+a)^2 + y^2} \right]$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{(x+a)^2 + y^2}} \times \frac{\partial}{\partial x} \left[(x+a)^2 + y^2 \right]$$

$$= \frac{2(x+a)}{2\sqrt{(x+a)^2 + y^2}} = \frac{x+a}{r_1}$$

Similarly, $\frac{\partial r_2}{\partial x} = \frac{x-a}{r_2}$

These values were put in (49.3).

Putting values of r_1 and r_2 from (48.5), (49.3) becomes,

$$U_x = -\frac{Q}{2\pi} \left[\frac{x+a}{(x+a)^2 + y^2} + \frac{x-a}{(x-a)^2 + y^2} \right] \cdots\cdots (49.4)$$

∴ Along the no-flow boundary, (i.e $x = 0$), aquifer flux,

$$U_x = -\frac{Q}{2\pi} \left[\frac{0+a}{(0+a)^2 + y^2} + \frac{0-a}{(0-a)^2 + y^2} \right]$$

$$= -\frac{Q}{2\pi} \left[\frac{a}{a^2 + y^2} - \frac{a}{a^2 + y^2} \right] = 0$$

12. Steady radial flow in Leaky confined Aquifer:

(I think this is also called semiconfined aquifer.)

Governing equation, (see similarity of fig 12.3 and 13.8)

$$S\frac{\partial \bar{h}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(Tr\frac{\partial \bar{h}}{\partial r}\right) - \frac{K'}{b'}(\bar{h}-\bar{h}_a) \quad \ldots\ldots (50.1)$$

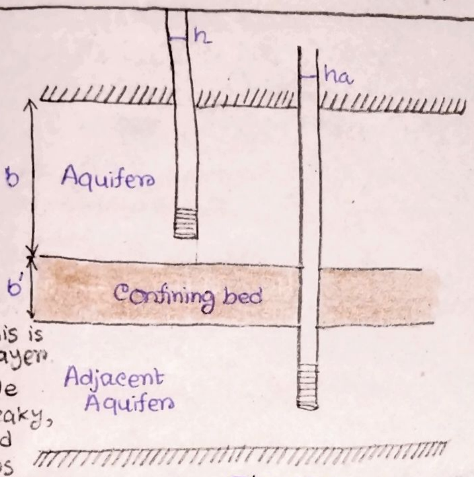
From (42.1), GE for leaky, confined aquifer,

$$S\frac{\partial \bar{h}}{\partial t} = \frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right) + \frac{K'}{b'}(H_{UA}-h) - q_{ext}^{\,0} \quad \ldots (50.2)$$

Modyfying this equation in cylindrical co-ordinate system (only considering radial flow) and as per figure 13.8,

$$S\frac{\partial \bar{h}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(T\,r\,\frac{\partial \bar{h}}{\partial r}\right) - \frac{K'}{b'}(h-h_a) = 0$$

I expected '+' here!?



I.T→ This is leaky layer. can we tell leaky, confined aquifers as semi-confined aquifers?

Fig: 13.8.

Semi confined aquifer under Steady-state, radial flow and homogenous media, (50.1) becomes,

$$T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) - \frac{K'}{b'}(\bar{h}-\bar{h}_a) = 0 \ldots. (50.3).$$

From (50.3),

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = \frac{T}{r}\left(\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right) = T\frac{\partial^2 h}{\partial r^2} + \frac{T}{r}\frac{\partial h}{\partial r}$$

In original eqn (50.3), no bar sign was there in h and ha. So, its okay now. I.T→ We can't replace h with $\bar{h}$ in confined aquifer.

Putting this value in 50.3,

$$T\frac{\partial^2 h}{\partial r^2} + \frac{T}{r}\frac{\partial h}{\partial r} - \frac{K'}{b'}h = -\frac{K'}{b'}h_a$$

$$\Rightarrow \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{h}{\left(\frac{Tb'}{K'}\right)} = -\frac{h_a}{\left(\frac{Tb'}{K'}\right)}$$

#Dividing both sides by T.

Now, replacing, $\frac{Tb'}{K'} = B^2$, we get, ... (50.4)

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{h}{B^2} = -\frac{h_a}{B^2} \quad \ldots\ldots (50.5)$$

⊙ Bessel's modified differential equation of order zero,

$$Q = -A.q_r(r) = -2\pi r b.\left(-K_r.\frac{\partial h}{\partial r}\right)$$

$$= 2\pi K_r b.\, r.\frac{\partial h}{\partial r} \quad \ldots (50.6)$$

Solution:

$$S(r) = h_a - h(r)$$

$$= \frac{Q}{2\pi T} K_0\left(\frac{r}{B}\right) \cdots\cdots (50.7)$$

$K_0\left(\frac{r}{B}\right)$ is called modified Bessel function of second kind of order zero. → (?) ha and h(r) are for two different aquifers. How their relation established?

- From (50.7), we get, Transmissivity,

$$T = \frac{Q}{2\pi S} K_0\left(\frac{r}{B}\right) = \frac{Q}{2\pi S} K_0\left(\frac{r}{\sqrt{\frac{Tb'}{K'}}}\right) \cdots\cdots (50.8)$$

Putting B values from (50.4)

- From (50.8), non-linear function can be written as,

$$F(T) = T - \frac{Q}{2\pi S} K_0\left(\frac{r}{\sqrt{\frac{Tb'}{K'}}}\right) = 0 \cdots\cdots (50.9)$$

Now, differentiating,

$$\frac{dF}{dT} = 1 - \frac{Q}{2\pi S} K_1\left(\frac{r}{\sqrt{\frac{Tb'}{K'}}}\right) \cdot r \cdot \frac{d}{dT}\left(\cancel{\sqrt{}}\frac{Tb'}{K'}\right)^{-1/2}$$

(# Applying chain rule. I.T ⇒ K_0 is function, K_1 is 1st derivative.)

$$= 1 - \frac{Q}{2\pi S} \cdot K_1\left(\frac{r}{\sqrt{\frac{Tb'}{K'}}}\right) r \left(-\frac{1}{2}\right)\left(\frac{Tb'}{K'}\right)^{-3/2} \cdot \left(\frac{b'}{K'}\right)$$

$$= 1 + \frac{Q}{4\pi S} \cdot \frac{(rb'/K')}{(Tb'/K')^{3/2}} K_1\left(\frac{r}{\sqrt{\frac{Tb'}{K'}}}\right) \cdots\cdots (50.10)$$

Lecture এ − আছে!
+ হওয়ার কথা!

- For p^{th} iteration,

$$T^{(p)} = T^{(p-1)} - \frac{F(T^{(p-1)})}{\left(\frac{dF}{dT}\right)^{(p-1)}} \cdots\cdots (50.11)$$

Function (50.9) and (50.10) are used here to get new iteration value.

Lecture-14:

Parameter Estimation:

Aquifer parameters under unsteady, confined flow condition.
(Previous cases were steady)

Unsteady Radial flow:

- Confined aquifer:

Assumptions:-
- Aquifer ⇒ Homogenous, isotropic, infinite extent. .... (51.1)
- Initial piezometric surface ⇒ Horizontal ... (51.2)
- Pumping at well ⇒ constant rate. ..... (51.3)
- well ⇒ Fully penetrating .... (51.4)
- Flow ⇒ Horizontal. .... (51.5)
- Well diameter ⇒ Infinitesimal or very small (i.e storage can be neglected in well!?) .... (51.6)
- Aquifer storage release is instantaneous. .... (51.7).

- Unconfined aquifer
- Semi-confined / Leaky aquifer

} Later lectures!

Unsteady Radial flow in confined aquifer:

Plan:

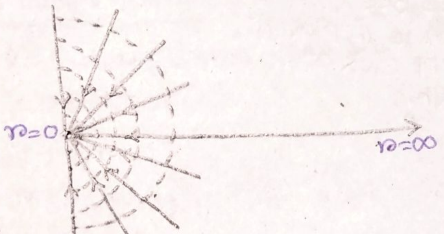


Fig: 14.1

Section:

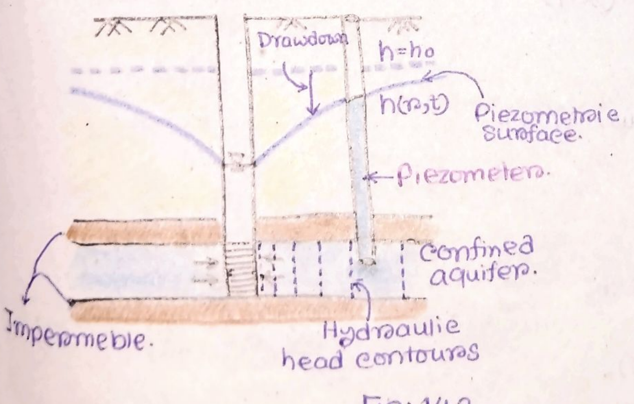


Fig: 14.2.

Governing eq^n (See 32.1 and 43.2),

$$S_s \frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(K_r \cdot r \cdot \frac{\partial h}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\cancel{\left(\frac{K_\theta \partial h}{r\,\partial \theta}\right)}^{0} + \frac{\partial}{\partial z}\cancel{\left(K_z \frac{\partial h}{\partial z}\right)} - \cancel{W}^{0} \quad \dots (52.1).$$

(Why zero? See P-183)

$$\Rightarrow S_s \frac{\partial h}{\partial t} = \frac{K_r}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h}{\partial r}\right)$$

$$\Rightarrow S_s\frac{\partial h}{\partial t} = \frac{k_r}{r}\left[\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right]$$

$$\Rightarrow S_s\frac{\partial h}{\partial t} = \frac{k_r}{r}\frac{\partial h}{\partial r} + k_r\frac{\partial^2 h}{\partial r^2}$$

$$\Rightarrow \frac{S_s}{k_r} = \frac{1}{r}\cdot\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}$$

$$\Rightarrow \frac{S_s.b}{k_r.b} = \frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}$$

$$\Rightarrow \boxed{\frac{S}{T} = \frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}} \quad \ldots\ldots (52.2)$$

[From page 89,
Storage coff, S = Sp. storage (S) × Confined aquifer depth (b).
Transmissivity, T = $k_r \times b$.]

Eq^n (52.2) subject to:-

IC:- $h(r,0) = h_0$ ...... (52.3)
At t=0, head = h_0 everywhere.

BC's:- $h(r\to\infty, t) = h_0$ ...... (52.4)

$$Q = \lim_{r\to 0}\left[2\pi r b(-q_r)\right]$$

I.T ⇒ Negative sign is for inward flow.

$$= \lim_{r\to 0} 2\pi r b.\,K_r\frac{\partial h}{\partial r} \quad \# q_r = -k_r\frac{\partial h}{\partial r}$$

$$\Rightarrow Q = \lim_{r\to 0}\left[2\pi r\left(T\frac{\partial h}{\partial r}\right)\right]$$

$$\Rightarrow \lim_{r\to 0}\left(r\frac{\partial h}{\partial r}\right) = \frac{Q}{2\pi T} \quad \ldots\ldots (52.5)$$

• Theis solution:- [# Applying these BC's means that well is replaced by a mathematical sink of constant strength.]

Applying BC's of (52.3), (52.4) and (52.5), Solution of (52.2) can be obtained as:-

$$h_0 - h(r,t) = s(r,t) = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-w}}{w}\,dw \ldots\ldots (52.6).$$

$$= \frac{Q}{4\pi T} W(u) \ldots\ldots (52.7)$$

where, $u = \frac{r^2 S}{4Tt}$ .... (52.8)

and Theis Well function, $W(u) = \int_u^\infty \frac{e^{-w}}{w}\,dw$ ... (52.9).

where, s = drawdown.
S = Storage coff.

We know, incomplete gamma function,

$$\Gamma(a,u) = \int_u^\infty w^{a-1} e^{-w}\,dw \ldots\ldots (52.10).$$

If we put, a=0 in (52.10), the we get the well function (i.e 52.9).

$\therefore W(u) = \Gamma(0,u)$ ..... (52.11).

∴ Well function is an incomplete gamma function.

- **Analysis of well function:**

Well function in terms of infinite series,

$$W(u) = -\gamma - \ln(u) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} u^k}{k\,k!} \quad \cdots\cdots (52.11)$$

Complete and incomplete gamma function:

(i) Complete gamma function.

Complete gamma function extend the concept of factorial to complex number.

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \cdot e^{-t}\,dt$$

- It is defined for all complex numbers except non-positive integers
- For positive integers, Γ function reduces to factorial function.

(ii) Incomplete gamma function:

A generalization of complete gamma function. $\Gamma(z,x)$ has two arguments z and x.

$$\Gamma(z,x) = \int_x^{\infty} t^{z-1} e^{-t}\,dt$$

In summary, complete gamma f^n is extension of factorial function to complex numbers.

Incomplete $[\Gamma(z,x)]$ gamma f^n represents the integral of gamma probability density function from a specific lower bound (x) to infinity.

In 52.11, γ = Euler-gamma constant
$= 0.5772156649$.

In expanded form,

$$W(u) = \left[-0.5772 - \ln u + \frac{u}{1.1!} - \frac{u^2}{2.2!} + \frac{u^3}{3.3!} - \cdots\cdots\right] \quad \cdots\cdots (52.12)$$

If time, $t \to \infty$,
$u \to 0$ (see 52.8).

$\therefore W(u) \to 0$

[But, $W(u) \to 0$, written in lecture, how?]

Now, $$\frac{\partial s}{\partial t} = \frac{\partial}{\partial t}\left[\frac{Q}{4\pi T}\int_u^{\infty} \frac{e^{-\omega}}{\omega}\,d\omega\right] = \frac{Q}{4\pi T}\left(-\frac{1}{u}e^{-u}\right)\frac{\partial u}{\partial t}$$

$$= \frac{Q}{4\pi T}\left(\frac{1}{t}e^{-u}\right) \cdots\cdots (52.13).$$

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial t}\left[\frac{Q}{4\pi T}\int_u^{\infty}\frac{e^{-\omega}}{\omega}\,d\omega\right] = \frac{Q}{4\pi T}\frac{\partial}{\partial t}\left[\int_u^{\infty}\frac{e^{-\omega}}{\omega}\,d\omega\right]$$

Applying Leibniz rule,

$$= \frac{Q}{4\pi T}\left[\int_u^{\infty}\underbrace{\frac{\partial}{\partial t}\left(\frac{e^{-\omega}}{\omega}\right)}_{0}d\omega + \cancel{\frac{e^{-\infty}}{\infty}\frac{\partial \infty}{\partial x}}^{\,0} - \frac{e^{-u}}{u}\frac{\partial u}{\partial t}\right]$$

? (how? $\frac{\partial}{\partial t}\left(\frac{e^{-\omega}}{\omega}\right) = 0$??)

$$= \frac{Q}{4\pi T}\left[-\frac{e^{-u}}{u}\frac{\partial}{\partial t}\left(\frac{r^2 S}{4Tt}\right)\right] \quad \text{\# value of u from 52.8}$$

$$= \frac{Q}{4\pi T}\left[-\frac{e^{-u}}{u}\cdot\left(-\frac{r^2 S}{4Tt^2}\right)\right]$$

Leibniz Rule - Differentiation formula for Definite Integral

This rule provides a way to differentiate a definite integral wrt one of its parameters:-

Let's consider one definite integral:-

$$F(x) = \int_{a(x)}^{b(x)} f(t)\,dt$$

where, $a(x)$ and $b(x)$ are differentiable functions of x and $f(t)$ is a function continuous over the integration interval.

From Leibniz rule,

$$F'(x) = \frac{d}{dx} F(x)$$

$$= \frac{d}{dx} \int_{a(x)}^{b(x)} f(t)\,dt$$

$$= \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[f(t)]\,dt + f(b)\cdot\frac{db}{dx} - f(a)\cdot\frac{da}{dx}$$

Here, no need to write $\frac{\partial b}{\partial x}$ or $\frac{\partial a}{\partial x}$ in place of $\frac{db}{dx}$ and $\frac{da}{dx}$. Because, a and b are functions of x only.

$$= \frac{Q}{4\pi T}\left[\frac{e^{-u}}{u}\cdot\frac{\left(\frac{r^2 S}{4Tt}\right)}{t}\right]$$

$$= \frac{Q}{4\pi T}\left[\frac{e^{-u}}{u}\cdot\frac{u}{t}\right]$$

$$= \frac{Q}{4\pi T}\cdot\frac{e^{-u}}{t}$$

This is the equation (52.13)

As, $u = \frac{r^2 S}{4Tt}$

So, for large t or small r,

$u \to 0$

$\therefore e^{-u} \to 1.$

$\therefore$ Equation (52.13) becomes,

$$\frac{\partial s}{\partial t} = \frac{Q}{4\pi T}\left(\frac{1}{t}\right) \quad \cdots\cdots (52.14).$$

- Well function:-

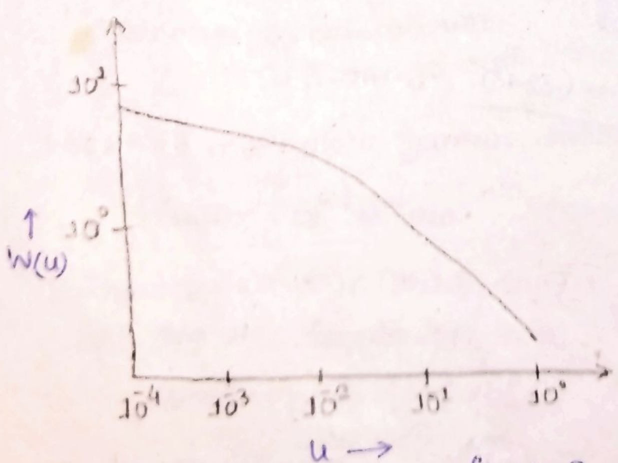


fig: 14.3

• Pumping test data:-

$Q = 2500\ m^3/day$.

t, min	S, m	r^2/t, m^2/min
0	0	∞
1	0.20	3600
1.5	0.27	2400
2	0.30	⋮
2.5	0.34	⋮
3	⋮	⋮
4	⋮	⋮
5	⋮	⋮
6	⋮	⋮
8	⋮	⋮
10	⋮	⋮
⋮	⋮	⋮
210	1.10	17
240	1.12	15

Question:- A well penetrating a confined aquifer pumped at a uniform rate of $2500\ m^3/d$. Drawdowns during a pumping period was measured in a observation well 60 m away. Using Theis method, determine T and S for this confined aquifer.

• Numerical solution:- what are N_T and N_l?

$$\text{Min } F(S,T) = \sum_{l=1}^{N_T} \sum_{i=1}^{N_l} \left[s_i^l - \frac{Q}{4\pi T} W\left(\frac{r_i^2 S}{4Tt_l}\right)\right]^2 \cdots (53.1)$$

why function was not directly taken as 0, instedd min? !

clearly F is the function related to drawdown (s). From (52.7), you can see, $s = \frac{Q}{4\pi T} W(u)$.

I think → We need to find the min value of F so that it goes closer to zero and we get the solution. And it was squared to get positive value. !

First derivatives of F:-

$$\left.\begin{aligned} F_1(S,T) &= \frac{\partial F}{\partial S} = 0 \\ F_2(S,T) &= \frac{\partial F}{\partial T} = 0. \end{aligned}\right\} \cdots\cdots (53.2).$$

I.T ⇒ Derivatives zero, as we need minimum value ! ?

Taylor Series:-

~~$F_1(S+\Delta S)$ =~~

$$F_1(S+\Delta S, T+\Delta T) = F_1(S,T) + \frac{\partial F_1}{\partial S}\Delta S + \frac{\partial F_1}{\partial T}\Delta T + O(\Delta S^2, \Delta T^2) \ldots\ldots (53.3)$$

$$F_2(S+\Delta S, T+\Delta T) = F_2(S,T) + \frac{\partial F_2}{\partial S}\Delta S + \frac{\partial F_2}{\partial T}\Delta T + O(\Delta S^2, \Delta T^2) \ldots (53.4).$$

(53.3) and (53.4) in compact form,

$$\begin{Bmatrix} F_1(S+\Delta S, T+\Delta T) \\ F_2(S+\Delta S, T+\Delta T) \end{Bmatrix} = \begin{Bmatrix} F_1(S,T) \\ F_2(S,T) \end{Bmatrix} + \begin{bmatrix} \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial T} \\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial T} \end{bmatrix} \begin{Bmatrix} \Delta S \\ \Delta T \end{Bmatrix} \ldots\ldots (53.5)$$

In (53.5), $\begin{bmatrix} \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial T} \\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial T} \end{bmatrix}$ called Jacobian i.e. $\underset{\sim}{J}(S,T)$.

• From (53.1), it is clear that F is a nonlinear function of storage coff (S) and Transmissvity (T). So, we need to use Newton-Raphson's method to get the solution.

$$\begin{Bmatrix} S^{(P)} \\ T^{(P)} \end{Bmatrix} = \begin{Bmatrix} S^{(P-1)} \\ T^{(P-1)} \end{Bmatrix} - J(S^{(P-1)}, T^{(P-1)})^{-1} \begin{Bmatrix} F_1(S^{(P-1)}, T^{(P-1)}) \\ F_2(S^{(P-1)}, T^{(P-1)}) \end{Bmatrix} \cdots\cdots (53.6).$$

Where P = iteration number ≥ 1

NR method for one variable $x^P = x^{(P-1)} - \frac{f(x)}{f'(x)}$, Here, $f'(x) \rightarrow J^{-1}$ replaced.

Graphical Solution: Theis Method.

Case I:-

$$\frac{r^2}{t} = \frac{4T}{S} u \quad \ldots\ldots (54)$$

$$S(r,t) = \frac{Q}{4\pi T} W(u) \ldots (54.2)$$

Case-II:

$$t = \frac{r^2 S}{4T} \frac{1}{u} \quad \ldots\ldots (55)$$

$$S(r,t) = \frac{Q}{4\pi T} W(u) \quad \ldots\ldots (55.2)$$

Case III:

$$\frac{t}{r^2} = \frac{S}{4T} \cdot \frac{1}{u} \quad \ldots\ldots (56)$$

$$S(r,t) = \frac{Q}{4\pi T} W(u). \quad \ldots\ldots (56.2)$$

Case-I:

Taking logarith on both sides of (54) and (54.2)

$$\log\left(\frac{r^2}{t}\right) = \log\left(\frac{4T}{S}\right) + \log(u) \ldots\ldots (57.1)$$

$$\log(s) = \log\left(\frac{Q}{4\pi T}\right) + \log(W(u)) \ldots (57.2)$$

Observations:

- From (57.1) and (57.2), it is obvious that relation between log(s) and $\log\left(\frac{r^2}{t}\right)$ has the same form as the relation between log(W(u)) and log(u). — This is the fundamental basis of Theis curve fitting method.

- Two relations differ by a constant factor. .... (57.2.2)

What is that factor?

I.T ⇒ Not sure ⇒

$$\left\{\log s - \log\left(\frac{r^2}{t}\right)\right\} - \left\{\log(W(u)) - \log(u)\right\}$$
$$= \log\left(\frac{Q}{4\pi T}\right) - \log\left(\frac{4T}{S}\right) \ldots (57.3)$$

// At RHS of (57.3), all quantities are constant for an aquifer during pumping test

- A plot of log(s) vs $\log\left(\frac{r^2}{t}\right)$ should look the same as a plot of log(W(u)) vs log(u). ... (57.3.2)

- Valid for multiple well case. (But, I think, as RHS of 57.3 are aquifer parameters, so, to be constant, all wells should be within an aquifer).

Steps for case-I:-

From (57.1) and (57.2) ⇒

- Plot the function log(W(u)) vs log(u) on log-log paper. This is known as type curve.

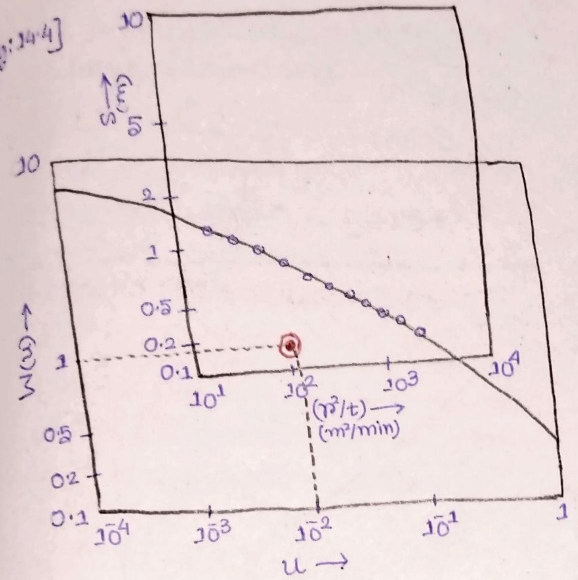
• Plot the measured $\log(S)$ and $\log(\frac{r^2}{t})$ on same size and scale log-log paper. This plot is known as data curve.

• Match the data curve with type curve.

Considering the statements (57.2.2) and (57.3.2), just shifting the axes of S vs $\frac{r^2}{t}$ curve, it should exactly match with $W(u)$ vs u curve.

(For more, see. Georesearch international you tube channel).

[fig: 14.4]



• For a pair of arbitrary points, $[(\frac{r^2}{t})^+, S^+]$ and $[u^+, W(u)^+]$ calculate

$$T = \frac{Q}{4\pi}\frac{W(u)^+}{S^+} \quad \ldots\ldots (57.4)$$

$$S = 4T\frac{u^+}{(r^2/t)^+} \quad \ldots\ldots (57.5)$$

Pair of points is told because, point indicated by red dot gives two point in two curves.

For that point,

$[(\frac{r^2}{t})^+, S^+] = [1.5\times10^2, 0.18]$

$[u^+, W(u)^+] = [10^{-2}, 1]$.

Now, put these values in (57.4) and and (57.5) to get desired value of transmissivity and storage coff.

I think, similar curve matching should be done for case II and case III also.

Lecture-15

Parameter Estimation:

To estimate aquifer parameters under unsteady confined flow with recovery test.

To estimate aquifer parameters under unsteady unconfined flow condition.

☑ Unsteady Radial flow in confined Aquifer:-

• Recovery test → Pumping well is shut off.

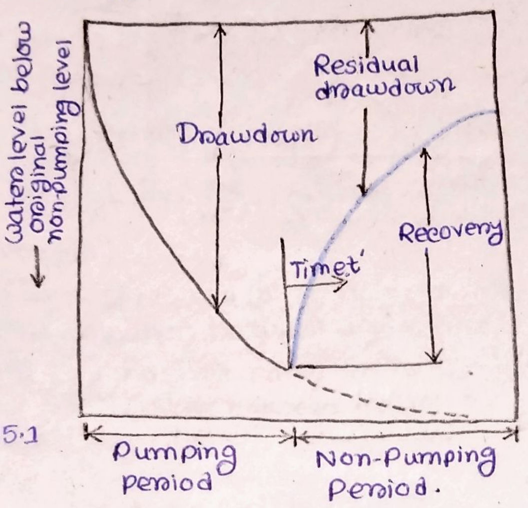


Fig: 15.1

YouTube; Georesearch international

Cooper Jacob straight line method for analysis of unsteady pumping test in confined aquifer

Cooper Jacob method is a simplification of Theis method.

From (52.12),

$$s=\frac{Q}{4\pi T}W(u)=\frac{Q}{4\pi T}\left[-0.5772-\ln u+u-\frac{u^2}{2.2!}+\frac{u^3}{3.3!}-\frac{u^4}{4.4!}+\cdots\right]\cdots(58.1)$$

Where, $u=\frac{r^2 S}{4Tt}$ ...... (58.2)

What Jacob said:-

u becomes small when - The distance r is small.

- The time t becomes large.

Then, we can write (58.1) as,

$$s=\frac{Q}{4\pi T}\left[-0.5772-\ln u\right]\ldots\ldots(58.3)$$

Remember $\left(u-\frac{u^2}{2.2!}+\frac{u^3}{3.3!}-\frac{u^4}{4.4!}+\cdots\right)$ this part can be neglected when, $u<0.01$

⊙ for u smaller than 0.03 0.05 0.1 0.15

an error less than 1% 2% 5% 10%

Most of the hydrologists accepted that 2% error can be acceptable. (I.T ⇒ Error to determine W(u))

The cooper-Jacob equation:-

$$s=\frac{Q}{4\pi T}\left(-0.5772-\ln\frac{r^2S}{4Tt}\right)=\frac{Q}{4\pi T}\left(-\ln 1.78-\ln\frac{r^2S}{4Tt}\right)=\frac{Q}{4\pi T}\ln\left(\frac{4Tt}{1.78r^2S}\right)$$

$$=\frac{2.303Q}{4\pi T}\log_{10}\left(\frac{2.25Tt}{r^2S}\right)\cdots\cdots(58.4)$$

where s = drawdown at distance r at time t after the start of pumping

key assumption: $u<0.05$

($\because$ error < 2% is acceptable).

• There are two types of Cooper - Jacob method.

(i) Time-drawdown method

(where, r = constant)

$$T=\frac{2.303Q}{4\pi\Delta s}\cdots\cdots(58.5)$$

$$S=\frac{2.25Tt_0}{r^2}\cdots\cdots(58.6).$$

Derivation of (58.5) and (58.6):-

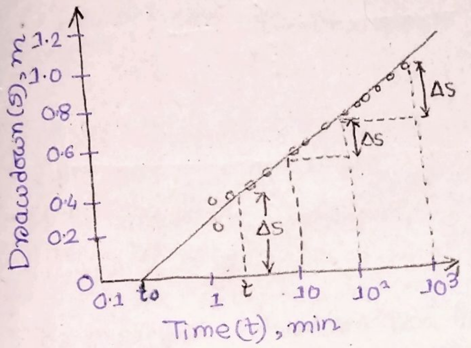


Fig: 15.2.

For one log cycle, $\log\left(\frac{t}{t_0}\right)=\log(10)=1\cdots(58.6.2)$

and Drawdown difference per log cycle = Δs.

So, applying cooper-Jacob eq^n for time t_0 and t,

$$s_0=\frac{Q}{4\pi T}\left(-0.5772-\ln\frac{r^2S}{4Tt_0}\right)\cdots\cdots(58.7)$$

$$s=\frac{Q}{4\pi T}\left(-0.5772-\ln\frac{r^2S}{4Tt}\right)\cdots\cdots(58.8).$$

Subtracting {(58.8) - (58.7)},

$$s-s_0=\frac{Q}{4\pi T}\left(+\ln\frac{1}{t_0}-\ln\frac{1}{t}\right)$$

(# other terms Q, r, S, T are constant).

$$\Rightarrow \Delta s=\frac{Q}{4\pi T}\ln\left(\frac{t}{t_0}\right)$$

$$\Rightarrow \Delta s=\frac{Q}{4\pi T}\ln(10) \quad \text{\# From 58.6.2}$$

$$\Rightarrow \Delta s=\frac{2.303Q}{4\pi T}$$

$$\Rightarrow T=\frac{2.303Q}{4\pi\Delta s} \Rightarrow \text{(This is 58.5)}$$

From figure 15.2, at $t=t_0$, $s=0$.

(# Because, t_0 is initial state of pumping, i.e. undisturbed state of the aquifer).

Putting this BC is 58.4,

$$0=\frac{2.303Q}{4\pi T}\log\left(\frac{2.25Tt_0}{r^2S}\right)$$

$\Rightarrow S = \frac{2.25 T t_0}{r^2} \Rightarrow$ This is (58·6)

(ii) Distance -Drawdown method:
(where time t = constant).

$$T = \frac{2.303 Q}{2\pi \Delta s} \quad \ldots\ldots (58.9)$$

and $$S = \frac{2.25 T t}{r_0^2} \quad \ldots\ldots (58.10).$$

Derivations of (58·9) and (58·10)

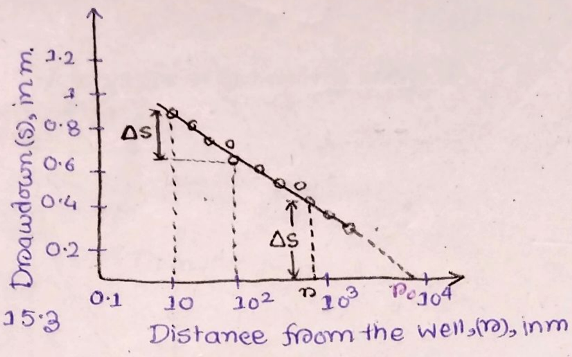


Figure: 15·3

For one log cycle,
change of drawdown, $= \Delta s$
and $\log\left(\frac{r_0}{r}\right) = \log(10) = 1 \ldots\ldots (58.11)$

I.T $\Rightarrow r_0$ = Radius of influence
Because, there drawdown = 0.

Applying equation (58·4) for distances r and r_0, we have,

$$s = \frac{Q}{4\pi T}\left(-0.5772 - \ln \frac{r^2 S}{4Tt}\right) \ldots\ldots (58.12)$$

and, $$s_0 = \frac{Q}{4\pi T}\left(-0.5772 - \ln \frac{r_0^2 \cdot S}{4Tt}\right) \ldots (58.13).$$

Subtracting {(58·12) − (58·13)},
(Because, here $s > s_0$, s_0 may be zero here).

$$s - s_0 = \frac{Q}{4\pi T}\left(-\ln r^2 + \ln r_0^2\right)$$

#(Other quantities like Q,T,S,t does not vary with radial distance.)

$$\Rightarrow \Delta s = \frac{Q}{4\pi T} \times 2 \ln\left(\frac{r_0}{r}\right)$$

$$\Rightarrow \Delta s = \frac{Q}{2\pi T} \times 2.303 \log(10) \quad \text{(\# From 58.11)}$$

$$\Rightarrow T = \frac{2.303 Q}{2\pi \Delta s} \Rightarrow \text{This is (58.9)}$$

Now, at $r = r_0$, $s = s_0 = 0$,
So, from (58·4),

$$0 = \frac{2.303 Q}{4\pi T} \log\left(\frac{2.25 T t}{r_0^2 S}\right)$$

$$\Rightarrow S = \frac{2.25 T t}{r_0^2} \Rightarrow \text{This is (58.10)}$$

Why s vs $\log r$ or s vs $\log t$ curves are linear?
This statement can be confirmed from equation (58·4)

$s = \frac{2.303Q}{4\pi T}\log\left(\frac{2.25Tt}{r^2 S}\right)$

$= \frac{2.303Q}{4\pi T}\log\left(\frac{2.25Tt}{S}\right) - \frac{2.303Q}{4\pi T}\times 2\log r \ldots$

$\therefore s = A + B.\log(r)$ form, i.e linear. (Provided, when t = Constant)

Similarly for s versus log t curve. (Provided, r = constant).

But, if we considered Theis equation (58.1), the equation contains u^2, u^3, u^4 etc. ($u = \frac{r^2 S}{4Tt}$). So the equation should be non-linear.

So, we can say,

Cooper-Jacob method transforms non-linear s vs log r and s vs log t relationships to a linear relationships.

Now, Back to lecture,

☑ Unsteady Radial flow in confined Aquifer (contd.):-

⊙ Superposition with transient flow:-

$\nabla^2 h = 0$ .... (59.1)

$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t}$ (From 34.4) ... (59.2)

- Equations are linear in h. (59.1 and 59.2)
- If h_1 and h_2 are solutions, then $h = h_1 + h_2$ .... (59.3) (How?)
- The principle of superposition is not affected by transient condition.

⊙ Aquifer test data:-

$Q = 504\ m^3/day$.

$r = 18.3\ m$.

	t = Time (min)	s = Drawdown (m).	t−Δt	t/(t−Δt)
Pumping test Data (∵ s value increasing)	0.6	0.0366		
	1	0.1067		
	2	0.2377		
	4	⋮		
	8	⋮		
	12	⋮		
	⋮	⋮		
	180	1.7404		
	210	1.8014		
	240	1.8379		
Recovery test Data (s value decreasing)	241	1.7407	→ (241−240) = 1	→ 241/1 = 241
	242	1.6002	→ (242−240) = 2	→ 242/2 = 121
	245	1.3655	→ (245−240) = 5	→ 245/5 = 49
	250	⋮		
	255	⋮		
	270	⋮		
	⋮	⋮		
	400	0.3383	→ (400−240) = 160	→ 400/160 = 2.5
	455	0.2774	→ (455−240) = 215	→ 2.1163
	545	0.2103	→ (545−240) = 305	→ 1.7869

I.T ⇒ Δt = 240 min here. After that recovery started.

Figure: Table 154

Plotting aquifer test data Obtained from table 15.4, We get drawdown(s) vs time (t) graph, (In log-log Scale),

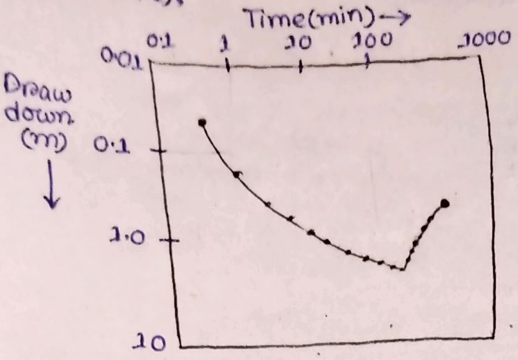


Fig: 15.5

- Jacob's approximation,

$$W(u) = -\gamma - \ln(u) \;\cdots\; (59.3.2)$$

[where γ = Euler-gamma constant, (see P52.11)]

$$= \ln\left(\frac{e^{-\gamma}}{u}\right) \qquad \# \gamma = 0.5772.$$

$$= \ln\left(\frac{0.561}{u}\right)$$

$$= \ln\left(\frac{2.25Tt}{r^2 S}\right) \;\cdots\cdots\; (59.4)$$

Putting value of well function in equation (52.7),

$$s(r,t) = \frac{Q}{4\pi T} W(u)$$

$$= \frac{Q}{4\pi T} \ln\left(\frac{2.25Tt}{r^2 S}\right)$$

$$= \frac{2.303Q}{4\pi T} \log\left(\frac{2.25Tt}{r^2 S}\right) \cdots (59.5)$$

Equation (59.5) would be valid, when $u < 0.01$,

Now, considering two different points in time axis (fig: 15.5):

$$s(r,t) = \frac{Q}{4\pi T}\left[W(u_1) - W(u_2)\right] \cdots (59.6)$$

Where, $u_1 = \frac{r^2 S}{4Tt} \;\cdots\; (59.7)$

and $u_2 = \frac{r^2 S}{4T(t-\Delta t)} \;\cdots\; (59.8)$

For a particular radial distance (only time is varying here),

$$s(r,t) = \frac{Q}{4\pi T}\left[(-\gamma - \ln u_1) - (-\gamma - \ln u_2)\right]$$

~~(?) It should be (Δs)~~ # [Putting values of $W(u_1)$ and $W(u_2)$ from (59.3.2)].

(No, here 's' is used directly in curve)

$$= \frac{Q}{4\pi T} \ln\left(\frac{u_2}{u_1}\right)$$

$$= \frac{Q}{4\pi T} \times 2.303 \log\left(\frac{t}{t-\Delta t}\right) \cdots (59.9)$$

Putting values of u_1 and u_2 from (59.7) and (59.8).

Horner plot:-

Horner plot is a graph of Drawdown(s) vs $t/(t-\Delta t)$

[From table 15.4, it is clear that Δt is the time when pumping test (i.e drawdown) stops and recovery test starts]

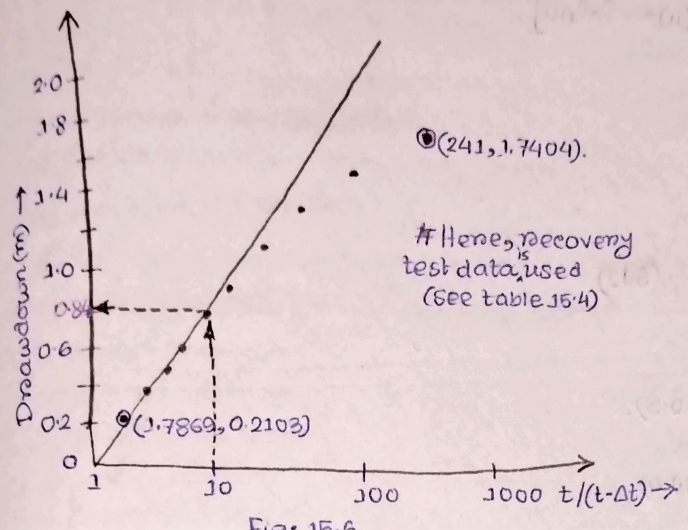


Fig: 15.6.

From this Horner's plot,

For $\frac{t}{t-\Delta t}=10$,

$s=0.84m$,

As per provided data, $Q=504 m^3/day$.

From (59.9), We have, (Putting these values),

$$s=\frac{Q}{4\pi T}.2.303\times\log 10$$

$$\Rightarrow T=\frac{2.303\times Q}{4\pi s} \quad \text{......(59.10)}$$

$$=\frac{2.303\times 504}{4\times\pi\times 0.84}$$

$$=109.96\ m^2/day.$$

Jacob's Method

(Same data used as table 15.4)

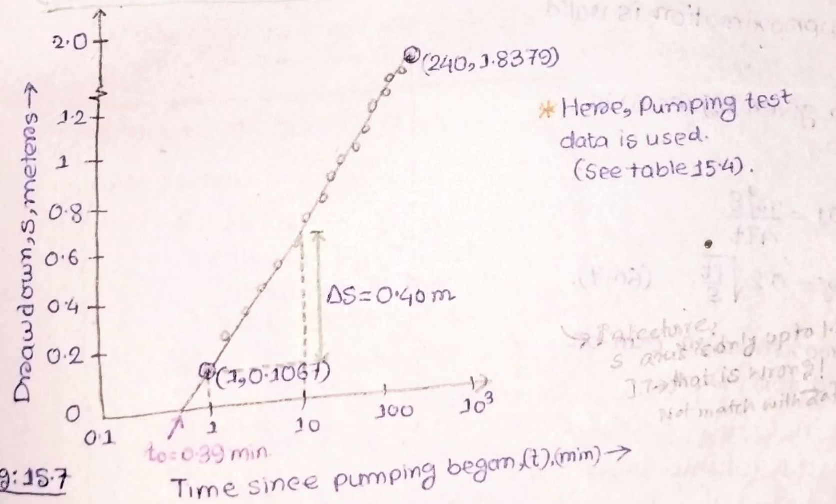


Fig: 15.7

→ In lecture: S axis is only upto 1.2 m. 1.7 → that is wrong! not match with data.

⊙ t_0 is the time corresponding to zero drawdown.

From (52.7), $0=\frac{Q}{4\pi T}\ln\left(\frac{2.25Tt_0}{r^2S}\right)$

$$\Rightarrow \frac{2.25Tt_0}{r^2S}=1 \quad \text{......(60.1)}$$

From Semi-logarithmic plot (figure 15.7),

$$\Delta s = \frac{Q}{4\pi T}[W(u_{10}) - W(u_1)]$$

$$= \frac{Q}{4\pi T}[\{-\gamma - \ln(u_{10})\} - (-\gamma - \ln(u_1))]$$

[From 59.3.2, $W(u) = -\gamma - \ln u$]

$$= \frac{Q}{4\pi T} \ln\left(\frac{u_1}{u_{10}}\right)$$

$$= \frac{Q}{4\pi T} \ln\left\{\frac{\left(\frac{r^2 S}{4Tt_1}\right)}{\left(\frac{r^2 S}{4Tt_{10}}\right)}\right\}$$

$$= \frac{Q}{4\pi T} \ln\left(\frac{t_{10}}{t_1}\right) \text{.....} (60.2)$$

$$= \frac{Q}{4\pi T} \ln(10)$$

$$\Rightarrow \Delta s = \frac{2.303 Q}{4\pi T} \text{.......} (60.3).$$

Or, $T = \frac{2.303 Q}{4\pi \Delta s}$ ..... (60.4)

Now, radius of influence, $r = r_0$.

Theim equation:-

$$s(r,t) = \frac{Q}{4\pi T} \ln\left(\frac{r_0^2}{r^2}\right)$$

(See P-204 for more)

$$= \frac{Q}{2\pi T} \ln\left(\frac{r_0}{r}\right) \text{....} (60.5)$$

(Approximate formula!) for radius of influence,

$$r_0 = 1.5\sqrt{\frac{Tt}{S}} \text{......} (60.6)$$

This formula gives,

$$r_0^2 = 2.25 \times \frac{Tt}{S}$$

$$\Rightarrow u = \frac{r_0^2 S}{4Tt} = \frac{2.25}{4} = 0.5625$$

But, Jacob's approximation is valid for $u < 0.01$ (# See 59.8)

(# So, formula given by (60.6) is not acceptable!)

Now, $0.01 = \frac{r_j^2 S}{4Tt}$

$$\Rightarrow r_j = 0.2\sqrt{\frac{Tt}{S}} \text{....} (60.7).$$

∴ Jacob approximation can be suitably applied for $r < r_j$.

So, when we are applying Jacob's method (s vs t), we should take an observation well data, whose distance from main well should be less than r_j.

Unsteady Radial Flow in Confined Aquifers:-

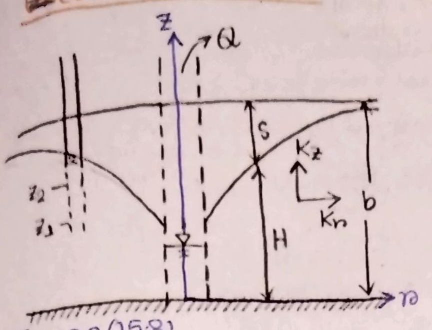


Figure (15.8)

Unconfined or confined? (61.1. 61.10)?

GE for confined aquifer, (from 43.2),

(Radially symmetric flow) (No recharge condition)

$$S_s \frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(K_r . r\frac{\partial h}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\cancel{\frac{K_\theta}{r}\frac{\partial h}{\partial \theta}}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) - \cancel{W} \quad \dots (61.1)$$

$$\Rightarrow S_s \frac{\partial h}{\partial t} = \frac{1}{r} K_r \frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + K_z \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\right).$$

(# Aquifer is not isotropic, because, $K_r \neq K_z$. But, it is homogenous i.e K_r and K_z does not depend on r and z co-ordinates)

(# I.T ⇒ If it was an unconfined aquifer, as per Dupuit's assumption, $\frac{\partial h}{\partial z}$ should be zero. But, here, it is not).

$$\Rightarrow S_s \frac{\partial h}{\partial t} = \frac{K_r}{r}\left(r\frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{\partial r}\right) + K_z \frac{\partial^2 h}{\partial z^2}$$

$$\Rightarrow S_s \frac{\partial h}{\partial t} = \frac{K_r}{r}\frac{\partial h}{\partial r} + K_r \frac{\partial^2 h}{\partial r^2} + K_z \frac{\partial^2 h}{\partial z^2}$$

Dividing both sides with K_r and rearranging,

$$\Rightarrow K_r . \frac{\partial^2 h}{\partial r^2} + \frac{K_r}{r}\frac{\partial h}{\partial r} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t}$$

$$\Rightarrow \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{K_z \partial^2 h}{K_r \partial z^2} = \frac{S_s}{K_r}\frac{\partial h}{\partial t} \quad \dots (61.2)$$

(61.2), in terms of drawdown,

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{K_z}{K_r} . \frac{\partial^2 s}{\partial z^2} = \frac{S_s}{K_r}\frac{\partial s}{\partial t} \quad \dots (61.3)$$

Calculation for (61.3):-

We know,

$$s = H_0 - h \quad \dots (61.4)$$

(where, H_0 = Initial piezometric level). ↳ a constant value.

$$\left.\begin{aligned} \frac{\partial s}{\partial r} &= -\frac{\partial h}{\partial r} \\ \frac{\partial^2 s}{\partial r^2} &= -\frac{\partial^2 h}{\partial r^2} \\ \frac{\partial s}{\partial t} &= -\frac{\partial h}{\partial t} \\ \frac{\partial^2 s}{\partial z^2} &= -\frac{\partial^2 h}{\partial z^2} \end{aligned}\right\} \quad \dots (61.5)$$

Putting these values in (61.2),

$$-\frac{\partial^2 s}{\partial r^2} - \frac{1}{r}\frac{\partial s}{\partial r} - \frac{K_z}{K_r}\frac{\partial^2 s}{\partial z^2} = -\frac{S_s}{K_r}\frac{\partial s}{\partial t}$$

$$\Rightarrow \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{K_z}{K_r}\frac{\partial^2 s}{\partial z^2} = \frac{S_s}{K_r}\frac{\partial s}{\partial t} \Rightarrow \text{This is (61.3).}$$

(6.1.3) subjected to:-

Initial condition, $s(r,z,0) = 0 \cdots$ (6.1.6)

(i.e drawdown = 0 before pumping)

Boundary Conditions (BC's):-

$$s(r \to \infty, z, t) = 0. \quad \cdots (6.1.7)$$

(At any distance after radius of influence, no drawdown is there)

$$\frac{\partial s}{\partial z}(r,0,t) = 0 \quad (?) \cdots (6.1.8)$$

why this limit? (see p-196)

$$\lim_{r \to 0} \int_0^b r \frac{\partial s}{\partial r} dz = \frac{Q}{2\pi K_r} \cdots (6.1.9)$$

For (6.1.9):-

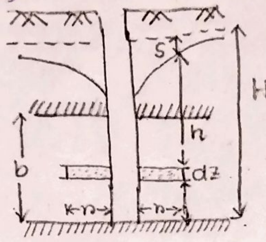


Fig: (15.9)

Applying Darcy's law for that small strip,

$$q = 2\pi r dz \cdot q_r$$

(q = Discharge through the outer surface of the strip, q_r = Sp. discharge in radial direction).

$$q = 2\pi r dz \cdot \left(-K_r \frac{\partial h}{\partial r}\right) \cdots (6.1.10)$$

Now, from (6.1.5), $\frac{\partial h}{\partial r} = -\frac{\partial s}{\partial r}$

$$\therefore q = 2\pi r \cdot dz \cdot K_r \frac{\partial s}{\partial r}$$

$$\Rightarrow Q = 2\pi K_r \int_0^b r \frac{\partial s}{\partial r} dz$$

$$\Rightarrow \frac{Q}{2\pi K_r} = \int_0^b r \frac{\partial s}{\partial r} dz \Rightarrow \text{This is } (6.1.9)$$

- Two boundary conditions must be satisfied simultaneously on FS (Free surface?!)

  - Continuity of energy with assuming zero gauge pressure on FS.

$$h(r,z,t) = H(r,t). \cdots (6.1.10)$$

(Head = elevation of FS) → Looks like unconfined (?!)

  - Continuity of mass, assuming that water and mediums are incompressible.

$$(\underset{\sim}{q} - \underset{\sim}{N}) \cdot \hat{n} = S_y \underset{\sim}{V_s} \cdot \hat{n} \quad (\text{see-}38.11)$$

(See Exercise book 6 for next)

(183) why $\frac{\partial h}{\partial z}=0$, for confined aquifer??

(185) Is Dupit and Dupit-Forchheimer assumptions are same?

(191) Derivation of (48.1) was not ~~correctly~~ done!

(193) Leaky, Confined aquifer & Semi-confined Aquifer হবে (0)!

(194) Drawdown, $s = h_0 - h(r)$
Different aquifers heads!?

One ± related doubt.

(199) what is N_T and N_L?!

(209 and 210) ~~Compare~~ why (61.10) will be applicable for a confined aquifer

3 no Copy

(53) what is $q_s C_s$? → (74.16)
what is Kd? → (74.18)

(56) why $C(0,t) = C_0$, is it a ~~Constant~~ continuous source of pollutant?
$C(x,0)$ Should be $= 0$, for $x>0$ ~~not $x \geq 0$~~ not $x \geq 0$

(58) ADE also comes from mass balance!

R2

(76) → flat > Round.

(91) FC and Sp. retention same!?

→(157)

(163) $\xi = 0$ not $\xi(x,y)$ → Means assuming aquifer bed as plane surface!?

(163) (36.21) eqⁿ

(166) Dupit ↔ Darcy contradictory
Because Dupit does not assume vertical flow??

(168) $N = -Ne_z$, because recharge in -ve z direction?? e_z is unit vector??

(169) why $\frac{\partial h}{\partial z}$ is not zero? (38.20).

(170) $(H-\xi)$ is not function ~~of~~ of spatially terms! (x,y)???

(173) why $n_e >> S_s$.

(178) $q_B = -K' \frac{h_{CA}-H}{b'}$ → Sign not matching!

6 no copy

(1) What is η_0?

(5) why t_s is higher value in time axis than t_p?

(16) Why I_3 is there?!

(21) $S' = \frac{dB}{dt}$

(24) Pendullar / adsoped water preesent for θr or not?

self check

(27) Horton's, Philips

(28) Why horton and philip gives different results at t=0!

(33) Two layer Green-Ampt
(71.1) এর derivation?! not matching!

(36) Significance of tr

(37) what is ρ^*?!

(45) $Mg(HCO_3)_2$ type থেকে To $CaCO_3$ type হওয়া যায়?

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* Continued after R_2

$$(\underset{\sim}{q}-\underset{\sim}{N})\cdot\hat{n} = n_e \underset{\sim}{V_S}\cdot\hat{n} \quad \ldots\ldots (61.11)$$

Specific yeild = Effective porosity (Because, effective porosity gives the fraction of void for which we can extract the water).

$$= n_D\cdot \underset{\sim}{V_S}\,\hat{n} \quad \ldots\ldots (61.12)$$

(what is n_D?)

$$\frac{DF}{Dt} = \frac{D}{Dt}\left[z - H(r,t)\right]$$

('$F = z - H(r,t)$' represents a plane which passes through piezometric surface or free surface ?!)

$$= \frac{\partial}{\partial t}\left[z - H(r,t)\right] + \underset{\sim}{V_S}\cdot\nabla\left[z - H(r,t)\right] = 0 \quad \ldots\ldots (61.13).$$

(For more, see R_1-P-167).

If $\underset{\sim}{N}=0$ and $\underset{\sim}{q} = q_z\hat{e}_z$,

$$q_z = S_y\frac{\partial H}{\partial t} = -K_z\frac{\partial h}{\partial z} \quad \ldots\ldots (61.14)$$

or, $$K_z\frac{\partial s}{\partial z} = -S_y\frac{\partial s}{\partial t} \text{ at } z=H \quad \cdots (61.15)$$

From (38.20),

$$n_e\frac{\partial H}{\partial t} = -K_z\frac{\partial h}{\partial z} + N.$$

$n_e = S_y$ (See-61.11) and here, $N=0$.

$$\therefore S_y\frac{\partial H}{\partial t} = -K_z\frac{\partial h}{\partial z}$$

(Now, for unconfined aquifer, $h=H$.

Also, $s = (H_0 - H)$

$\Rightarrow H = (H_0 - s)$

$$\therefore S_y\frac{\partial}{\partial t}(H_0 - s) = -K_z\frac{\partial}{\partial z}(H_0 - s)$$

$$\Rightarrow -S_y\frac{\partial s}{\partial t} = K_z\frac{\partial s}{\partial z} \Rightarrow \text{This is } (61.15).$$

Assumption → $S_y \gg S$ (why? Is it necessary!)

⊙ Drawdown:-

$$s(r,t) = \frac{Q}{4\pi T}W(u_A, u_B, \beta). \;?\ldots\ldots (62.1)$$

$$\left.\begin{aligned} \text{with } \frac{1}{t_s} &= u_A = \frac{r^2 S}{Tt} \\ \frac{1}{t_y} &= u_B = \frac{r^2 S_y}{Tt} \\ \beta &= \frac{K_z r^2}{K_r b^2} \end{aligned}\right\} \ldots\ldots (62.2)$$

⊙ Solution has two components

- Transition from elastic storage release to drainage associated with vertical flow near water table. $W(u_A, u_B, \beta) = W(u_A, \beta) \ldots (62.3)$

- Transition from drainage associated with vertical flow to drainage associated with primarily horizontal flow when water table decline has slowed down sufficiently.

$$W(u_A, u_B, \beta) = W(u_B, \beta) \ldots (62.4)$$

I.T ⇒ Initial drawdown is from elastic storage (S) and then from specific yeild (S_y). So firstly $W(u_A, \beta)$ and secondly, $W(u_B, \beta)$.

Mechanism:-

(a) Initially, the water supply derives from the elastic storage, S, of the aquifer, analogously to confined aquifer.

(b) A phase dominated by delayed gravity drainage follows, during which the drawdown curve flattens out somewhat;

(c) Once the gravity drainage ends, the drawdown starts to increase again, depending on the value of the specific yeild, S_y.

Difference between Elastic storage and specific yeild:- (From AI)

- When groundwater is pumped from an aquifer, the water pressure within the aquifer decreases causing the aquifer to undergo compression or 'squeeze' as it adjusts to reduced pressure.
This compression of aquifer's pore space results in a decrease in the volume of stored water and is known as elastic storage. When pumping stops and GWL recovers, the aquifer expands or "rebounds" elastically. Stored water volume increases again.
So, elastic storage represents temporary compression or expansion of aquifer due to changes in GWL.
- Specific yield:- Measure of permanent or sustained storage capacity of the aquifer. It represents the proportion of water ...

that an aquifer can yield or release under the influence of gravity after it has been completely drained or dewatered. Specific yeild, which takes into account the porosity of the aquifer, is expressed as the ratio of the volume of water that drains from the aquifer by gravity to the total volume of the aquifer.

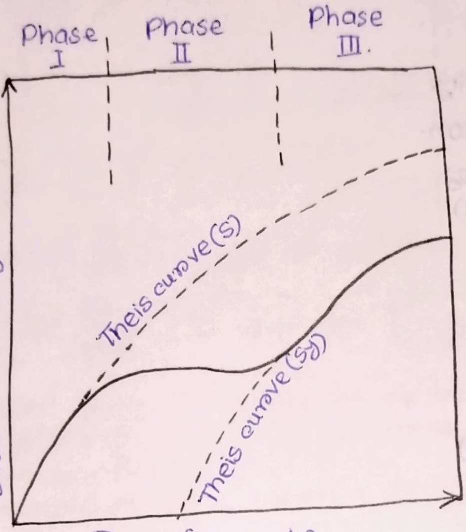


Figure: 15.10

$$\varsigma(r,z,t)=\frac{Q}{4\pi T}\cdot\int_0^\infty 4y J_0(y\beta^{\frac{1}{2}})\cdot\left[W_0(y)+\sum_{n=1}^{\infty} W_n(y)\right]dy \cdots (62.5)$$

where, J_0 is the zero-order Bassel's function of first kind. → [Can we write $J_0(y\beta^{1/2}) = 1-\frac{(y\beta^{1/2})^2}{2!}+\frac{(y\beta^{1/2})^4}{4!}-\frac{(y\beta^{1/2})^6}{6!}+\cdots$ using (62.7)] ?

How to do the integration of (62.5)?
An infinite series?
(See Page-6).

Zero-order Bessel's function of first kind:
It is one of the solutions to Bessel's differential equation of order zero, given by,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2y=0 \cdots (62.6)$$

The zeroth order Bessel's function of the first kind can be defined in terms of a power series expansion

$$J_0(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\cdots \rightarrow (62.7)$$

Alternatively, it can also be expressed in terms of the modified Bassel's function of the first kind

$$J_0(x)=I_0(ix)\cdots (62.8)$$

where, $I_0(x)$ is modified Bassel's function of the first kind of order zero.

Zero-order Bassel's function exhibits oscillatory behaviour for positive values of x, with an infinite number of positive and negative zeros. It arises in problems involving circular or cylindrical symmetry, such as wave propagation, heat conduction.

Bessel's equation:

It is a 2nd order ODE, given by,

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \dots (62.9)$$

Where, $y = y(x)$ is an unknown function

v = constant parameter, known as order of Bessel's equation.

This equation used in problems with circular or cylindrical symmetry.

Bessel's functions are defined for non-negative integer values of v, denoted by $J_v(x)$

Bessel functions have oscillatory behaviours and typically written as a linear combination of two linearly independent solutions, $J_v(x)$ and $Y_v(x)$ (where, $Y_v(x)$ is the Bessel's function of 2nd kind).

Modified Bessel's DE:-

It is a variation of Bessel's equation that includes a modification term.

It is given by:-

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+v^2)y = 0 \dots (62.10)$$

The sol^{n}s of modified Bassel's differential equation are called modified Bassel's function and are denoted as $I_v(x)$.

Modified Bessel's functions are defined for any real or complex value of v.

Modified Bessel's function exhibit oscillatory growth or decay behaviour as x tends to infinity, depending on the value of v. They are commonly encountered in problems involving cylindrical symmetry or exponential growth or decay phenomena, such as heat conduction in a long rod or cylindrical waveguide's wave propagation.

In (62.5),

$$\omega_0(\gamma) = \frac{\{1 - \exp[-t_s\beta(\gamma^2 - \eta_0^2)]\}\cdot\cosh(\eta_0 z_D)}{\left\{\gamma^2 + (1+\sigma)\eta_0^2 - \left[\frac{\gamma^2 - \eta_0^2}{\sigma}\right]\right\}\cdot\cosh(\eta_0)} \dots (62.11)$$

$$\omega_n(\gamma) = \frac{\{1 - \exp[-t_s\beta(\gamma^2 + \eta_n^2)]\}\cdot\cosh(\eta_n z_D)}{\left\{\gamma^2 + (1+\sigma)\eta_n^2 - \left[\frac{(\gamma^2+\eta_n^2)^2}{\sigma}\right]\right\}\cdot\cosh(\eta_n)} \dots (62.12)$$

The terms η_0 and η_n are the solutions of following equations:-

$\sigma\gamma_0 \cdot \sinh(\gamma_0) - (\gamma^2 - \gamma_0^2)\cdot\cosh(\gamma_0) = 0$ with $\gamma_0^2 < \gamma^2$ . . . . (62.13)

$\sigma\gamma_n \cdot \sin(\gamma_n) + (\gamma^2 + \gamma_n^2)\cos(\gamma_n) = 0$ . . . . (62.14)

where, $(2n-1)\frac{\pi}{2} < \gamma_n < n\pi$, $n \geq 1$

#Observing values of w_0 and W_n from (62.11) and (62.12), Drawdowns (from 62.5) can be written as:-

$$s(r_0, z, t) = \frac{Q}{4\pi T} s_D(\sigma, \beta, t_s, z_D) \; \ldots\ldots (62.15)$$

(I.T⇒ γ_0 and γ_n are solutions of two particular equations (i.e 62.13 and 62.14). They are not parameters.)!

From Chatgpt,
t_D = Dimensionless time $= \frac{t - t_0}{t_0}$ from pumping start
Z_D = Dimensionless distance from pumping well $= \frac{r_0}{\sqrt{\frac{4Tt_0}{S}}}$
#But, I am not sure!

where, $\sigma = \frac{S}{S_y} = \frac{\left(\frac{Tt}{r^2 t_s}\right)}{\left(\frac{Tt}{r^2 t_y}\right)}$ #From (62.2)

$$= \frac{t_y}{t_s} \; \ldots\ldots (62.16)$$

$$\beta = \frac{K_z}{K_r} \cdot \frac{r^2}{b^2} \quad (\text{\# From 62.2}) \ldots (62.17)$$

$t_D = t_s = \frac{Tt}{r^2 S}$, for short times

$t_D = t_y = \frac{Tt}{r^2 S_y}$, for long times. . . . (62.18)

But in 62.11 and 62.12 only t_s has been used, no t_y.?

[I.T⇒ Water comes from elastic storage for initial phase or shorter period of time. After that, water only comes from permanent storage i.e specific yeild.]

#Compare (62.15) and (62.5) with $s = \frac{Q}{4\pi T} W(u)$.
You can see 's_D' is somewhat Equivalent to well function $W(u)$.
Am I correct?

(a) Type A curve:-

$s_D = s_D(t_s, \beta)$, which is valid for short periods of time.

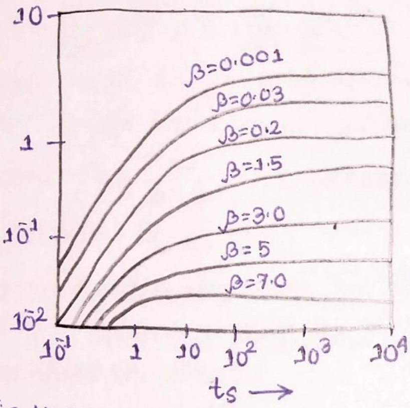


Fig 15.11: Neuman's dimensionless function $s_D(t_s, \beta)$ valid for short times. → But values are higher in t_s axis than t_y axis!? Contradiction!

(b) Type B curve:-

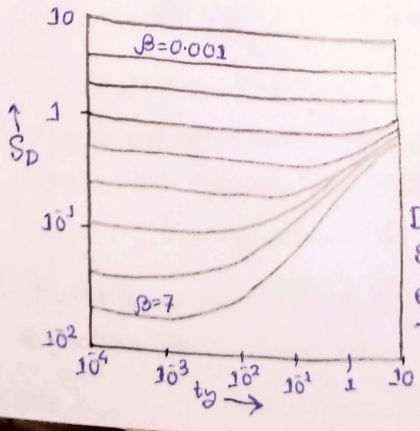


Figure 15.12:- Neumann's Dimensionless function $s_D(t_y, \beta)$ valid for extended period of time.

How to do integration of an infinite series?

(i) Power series:-

Integrate term by term. Integration becomes easy, if series is convergent.

(ii) Taylor's series:

Integrating each term of Taylor's series, if the series can be expanded in Taylor's series form.

(iii) Special functions:-

Some infinite series have well-known closed-form solutions in terms of special function.

Eg. Exponential function, trignometric functions, Bessel's functions, hypergeometric functions.

(iv) Transform techniques:-

In some cases, integral transforms such as Laplace transform, Fourier transform, Mellin transform can be applied. After the transform is applied, the resulting transformed series can be integrated and then transformed back to obtain the integral of the original series.

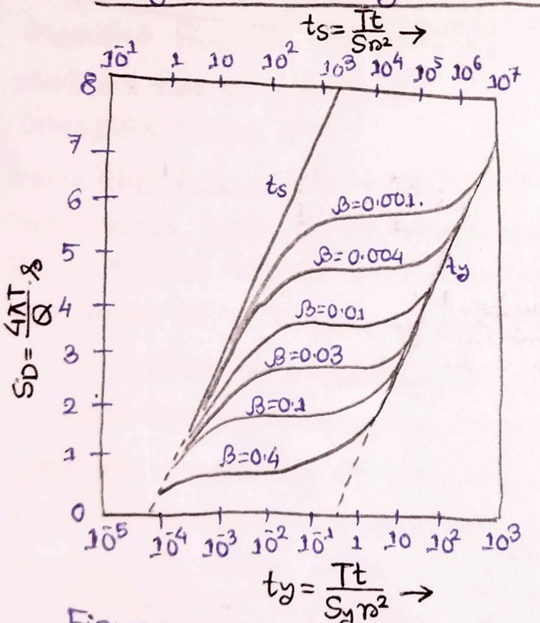


Figure 15.13:- Neumann's dimensionless function $S_D(t_s, t_y, \beta)$ represented for $\sigma = 10^{-4}$.

I think, this curve is combination of figure (15.11) and (15.12). Observe the similarity of this curve with Drawdown vs time (s vs t) curve of figure (15.10).

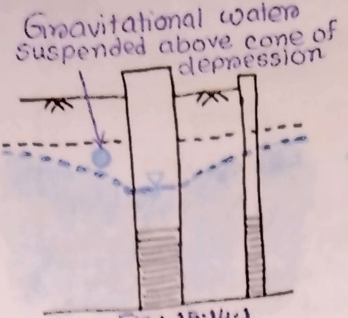


Fig: 15.14.1

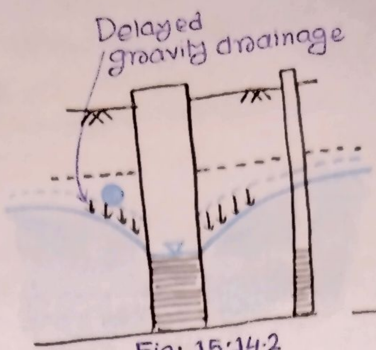


Fig: 15.14.2

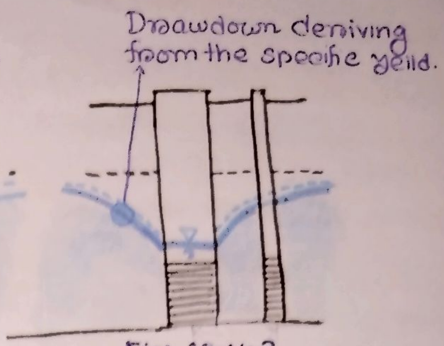


Fig: 15.14.3.

#I think, when drawdown occurs, pore pressure decreases an aquifer gets compressed. So, due to decrease in pore volume, gravitational waters are forced to move downward. (Gravity drainge!) So, initially water comes from elastic storage. After sometimes, drawdown comes from specific yeild.

To find different Aquifer Parameters with type A, type B curves and drawdown vs. time curve

1. Match field data with type B curve and identify β.

2. For an arbitrary point (t_s^*, S_D^*) (#From type B curve) or (t^*, s^*) (#From field data, drawdown curve) calculate,

$$T = \frac{Q}{4\pi}\frac{S_D^*}{s^*} \quad \cdots\cdots (62.19)$$

#[$s = \frac{Q}{4\pi T} \cdot S_D \Rightarrow T = \frac{Q}{4\pi} \cdot \frac{S_D}{s}$.
I Think, Well function, $W(u)$ and Neumann's dimensionless function, S_D both are same]

and $S_y = \frac{Tt^*}{r^2 t_y^*} \quad \cdots\cdots (62.20)$. Or, $S_y = \frac{Tt^*}{r^2 \left(\frac{1}{u_B^*}\right)}$ (# when, S_D vs $\frac{1}{u_B}$ curve is there).

#[Obtained from 62.2]

3. Match field data with type A curve corresponding to β identified in step 1. #Because, $\beta = \frac{K_z}{K_r} \times \left(\frac{r}{b}\right)^2$ which is constant for both typeA and typeB curves

4. Now for this, for a arbitrary point (t_s^+, S_D^+) (#From type A curve) or (t^+, s^+) (#From field data curve) Calculate,

$$S = \frac{Tt^+}{r^2 t_s^+} \quad \cdots\cdots (62.21) \quad \text{Or, } S_y = \frac{Tt^+}{r^2 \cdot \left(\frac{1}{u_A^+}\right)}$$

#(When, S_D vs $\frac{1}{u_A}$ curve curve is provided)

(#From 62.2).

5. Now, using (62.19), calculate horizontal hydraulic conductivity,

$$K_r = \frac{T}{b} \quad \cdots\cdots (62.22)$$

6. Putting K_r value from (62.22) in equation (62.2), calculate vertical

Hydraulic conductivity,

$$k_z = \beta k_r \left(\frac{b}{r}\right)^2 \cdots\cdots (62.23)$$

7. Calculate σ using (62.16),

$$\sigma = \frac{S}{S_y}$$

(S_y and S values are from 62.20 and 62.21).

Problem:

$Q = 4.1\ m^3/min.$

t(min)	s(m)
0.165	0.037
0.25	0.059
0.34	0.078
⋮	⋮
10	0.311
12	0.314
15	0.317
18	0.320
⋮	⋮
2000	0.759
2500	0.789
3000	0.811.

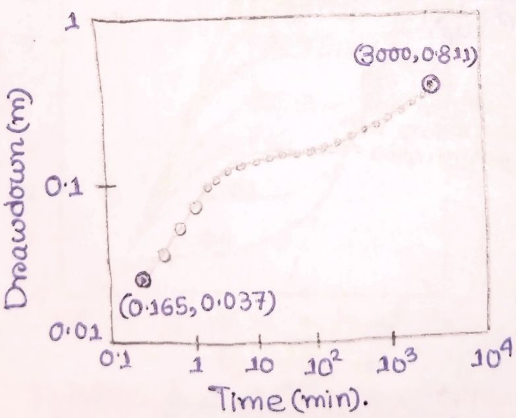
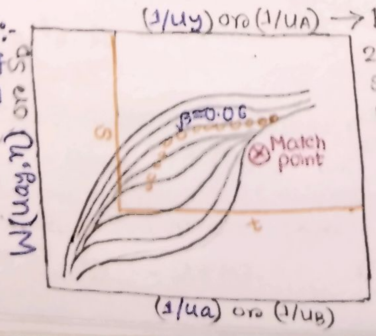


Fig 15.15: Aquifer test data.

Fig: 15.16.1:- Matching of type B and field data curve.



→ Because, I think, 2 different symbols are used in 2 different pages.

Parameters in the axis may look different in Figure 15.13 and 15.16.1

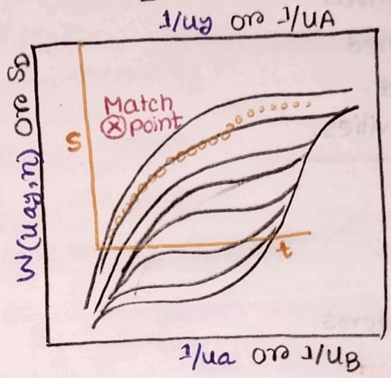
But, actually they are not different.

Because, $t_s = \frac{1}{u_y}$
and $t_y = \frac{1}{u_a}$
(Though, in 62.2, u_y and u_a were named as u_A and u_B.) → Don't be confused!

→ I think, two pages in lectures are from 2 different books. That's why symbols are confusing!

and $S_D = W(u_{ay}, \eta)$ # u_{ay} means both u_a and u_y.
But, why η ?

Figure: 15.16.2:- Matching of type A curve with field data curve maintaining same β value as obtained from 15.16.1.



Recommended drawdown measurement frequency in observation wells in confined and unconfined aquifers:-

Time from start (min)	Measurement interval (min) [confined aquifer]	Measurement interval (min) [Unconfined aquifer]
0-2	→ 0.5	→ 0.25
2-5	→ 1	→ 0.5
5-15	→ 5	→ 1
15-60	→ 5	→ 5
60-120	→ 15	→ 10
120-240	→ 30	→ 30
240-360	→ 60	→ 30
360-720	→ 60	→ 60
720-2880	→ 60	→ 180
>2880	→ 60	→ 480

Why pumping test data is collected ~~to~~ more frequently in unconfined aquifer?

1. It exhibit rapid response to pumping due to direct connection with water table. More frequent measurements are necessary to capture the dynamic changes in water table.
2. Transient behaviour is more during.

Pumping test.

3. Unconfined aquifer has higher K and S_y compared to confined aquifers. Drawdown occurs over a shorter period.

4. Unconfined aquifers → More heterogenous geological formations, leads to spatial variations of hydraulic properties. To capture the change, more data is required.

5. Regulatory agencies required more data where unconfined aquifers are heavily utilized or sensitive to pumping activities.

Lecture-16

Parameter Estimation

To estimate aquifer parameters under unsteady, semi confined flow condition

To utilize unsteady pumping solutions for different boundary conditions.

Semi confined Aquifer:

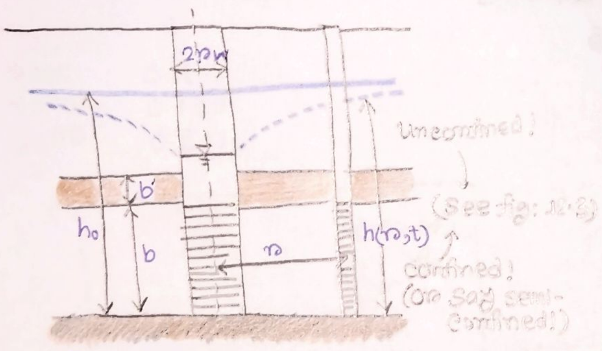


Figure: 16.1

Under unsteady state, radial flow condition in semi-confined aquifer,

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}-\frac{h_0-h}{B^2}=\frac{S}{T}\frac{\partial h}{\partial t} \quad \cdots (63.1)$$

From (42.4),

$$S\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)+\frac{K'}{b'}(H_{UA}-h)-q_{ext} \quad \cdots (63.1.2)$$

$H_{UA}, h \rightarrow h, h_0$ (see 64.1) ? Is it true

Converting it to cylindrical co-ordinate system,

$$S\frac{\partial h}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r T_r\frac{\partial h}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(T_\theta\frac{1}{r}\frac{\partial h}{\partial \theta}\right)+\frac{K'}{b'}(h-h_0)$$

Flow is radially symmetric,

So, $\frac{\partial h}{\partial \theta}=0$.

// ? b' is replaced by B? Both are aquifer thickness? (No, see 63.9)

$$\Rightarrow S\frac{\partial h}{\partial t} = \frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) - \frac{K'}{b'}(h_0-h)$$

$$\Rightarrow S\frac{\partial h}{\partial t} = \frac{T}{r}\left[r\frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{\partial r}\right] - \frac{K'}{b'}(h_0-h)$$

$$\Rightarrow \frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{K'}{Tb'}(h_0-h)$$

$$= \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{K'}{Kb \cdot b'}(h_0-h)$$

$$= \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{(h_0-h)}{B^2}$$ [Putting value of B from equation (63.9)]

But it ~~should be minus~~?

Putting upper unconfined aquifer head = h and below confined aquifer head = h_0, we get the correct sign! Is it okay?

Subject to:-

Intial condition:- $h(r,0)=h_0 \cdots (63.2)$

Boundary conditions:-

$$h(r\to\infty, t)=h_0 \cdots\cdots (63.3)$$

$$\lim_{r\to 0}\left(r\frac{\partial h}{\partial r}\right) = \frac{Q}{2\pi T} \cdots\cdots (63.4)$$

Similar to 61.9. But, why $r\to 0$? → (see Page-196)

Hantush-Jacob solution:-

$$h_0 - h(r,t) = s(r,t) = \frac{Q}{4\pi T}\int_u^\infty \frac{1}{\omega} e^{-\left(\omega+\frac{\beta^2}{4\omega}\right)} d\omega \cdots (63.5)$$

$$= \frac{Q}{4\pi T} W(u,\beta) \cdots\cdots (63.6)$$

Firstly, Well function was defined as,

① $W(u) = \int_u^\infty \frac{e^{-\omega}}{\omega} d\omega$ (For confined aquifer).

Now, for semi-confined aquifer it changed to (63.5) with some β related term

where, $u = \frac{r^2 S}{4Tt} \cdots\cdots (63.7)$

and $\beta = \frac{r}{B} \cdots\cdots (63.8)$

(But, in 62.2, $\beta = \frac{K_z}{K_r}\cdot\left(\frac{r}{B}\right)^2$. If we take $K_r = K_z$, $\beta = \frac{r^2}{B^2}$, not $\frac{r}{B}$.)

$$B = \sqrt{\frac{T}{c}} = \sqrt{\frac{T}{(K'/b')}} = \sqrt{\frac{Kb}{(K'/b')}} = \sqrt{\frac{K}{K'}bb'} \cdots (63.9)$$

What is 'c' here? (c has dimension of velocity)!

Now, consider figure (16.2), Under unsteady state, radial flow condition in semi-confined aquifer (Similar to 63.1).

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{h-h_a}{B^2} = \frac{S}{T}\frac{\partial h}{\partial t} \cdots (64.1).$$

If [In (63.1.2),] we replace H_{UA} with h and h with h_0, (Maintaining similarity with figure 16.2), The problem of ± sign would be solved!)

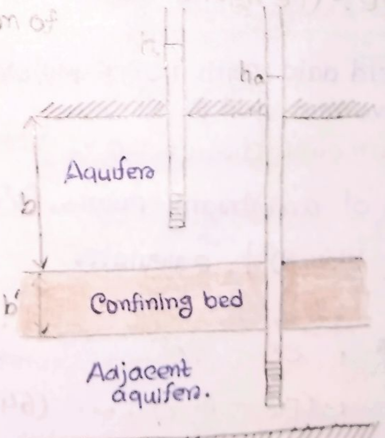


Figure: 16.2.

which one is semi-confined!?

Subject to

Initial condition:-(IC):-

$$h(r,0)=h_a \ldots\ldots (64.2)$$

#(So, IC is applied for below aquifer. So, I think that is semi-confined!)

Boundary Conditions (BCs)

$$h(r\to\infty,t)=h_a \ \ldots (64.3)$$

$$\text{and } \lim_{r\to 0}\left(r\frac{\partial h}{\partial r}\right)=\frac{Q}{2\pi T} \ldots\ldots (64.4)$$

(See R_2-P-196)

Hantush solution,

$$h_a-h(r,t)=s(r,t)=\frac{Q}{4\pi T}\int_u^{\infty}\frac{1}{\omega}e^{-\left(\omega+\frac{\beta^2}{4\omega}\right)}d\omega \ldots (64.5)$$

$$=\frac{Q}{4\pi T}W(u,\beta) \ldots (64.6)$$ → In lecture, $W(u,\beta)$ written I.T wrong!

$$u=\frac{r^2S}{4Tt} \ldots (64.7)$$

(See 63.8)? $\beta=\frac{r}{B} \ldots (64.8)$

$$B=\sqrt{\frac{T}{c}}=\sqrt{\frac{Tb'}{K'}}=\sqrt{\frac{K}{K'}bb'} \ldots (64.9)$$

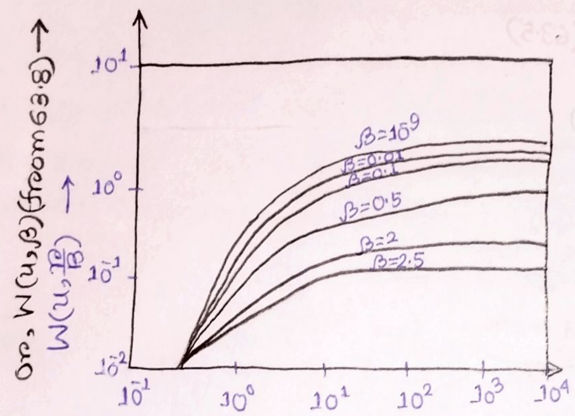


Figure: 16.3.

How to get aquifer parameters using field data and non-equilibrium type curve:- (i.e figure 16.3)

1. Match field data with non equilibrium type curve and find β.

#[Field data means drawdown vs time curve]

2. For a pair of arbitrary points, $[t^+, s^+]$ and $\left[\left(\frac{1}{u}\right)^+, W(u,\beta)^+\right]$, calculate,

$$T=\frac{Q}{4\pi}\cdot\frac{W(u,\beta)^+}{s^+} \quad \text{(From 63.6)} \ldots (64.10)$$

$$B=\frac{r}{\beta} \quad \text{(From 63.8)} \ldots\ldots (64.11)$$

$$c=\frac{T}{B^2} \ldots (64.12)$$

$$S=4T\frac{t^+}{r^2\left(\frac{1}{u}\right)^+} \quad \text{(From 63.7)} \ldots (64.13).$$

Problem:-

Semiconfined aquifer

$Q = 17\ m^3/min$	t (min)	s (m)
$b = 425$ m.	0	0.0
$r = 12.2$ m.	2	1.722
	4	2.121
	⋮	⋮
	25	3.155
	30	3.261
	40	3.395
	⋮	⋮
	110	3.77
	120	3.783
	150	3.868
	⋮	⋮
	300	4.063
	360	4.075
	420	4.087.

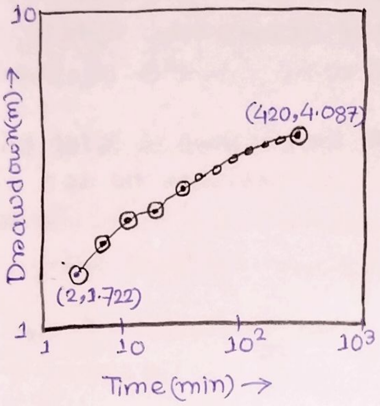


Figure: 16.4: Aquifer Test data.

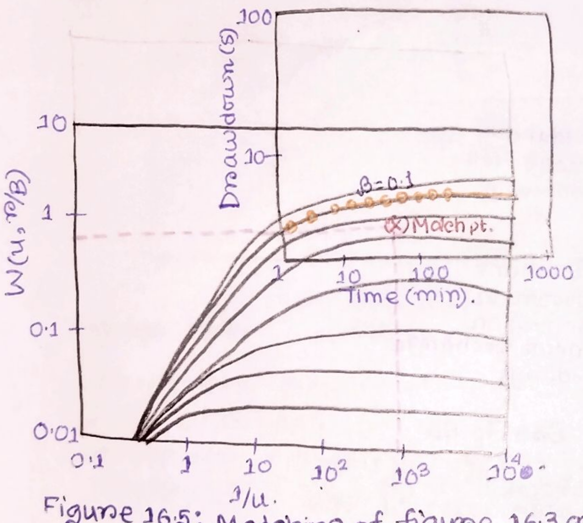


Figure 16.5: Matching of figure 16.3 and 16.4.

Approx. estimate of Aquifer parameters:

Co-ordinate of match point,

take, $\beta = 0.1$, $[t^+, s^+] = [\cancel{900}, [70, 3]$

$\left[\frac{1}{u}^+, W(u, \frac{r}{B})^+\right] = [900, 0.8]$

$\therefore T = \frac{Q}{4\pi} \cdot \frac{W(u,\beta)^+}{s^+} = \frac{17}{4 \times 3.14} \times \frac{0.8}{3} = 0.36\ m^2/min$

$B = \frac{r}{\beta} = \frac{12.2}{0.1} = 122$ m.

$c = \frac{T}{B^2} = \frac{0.36}{122} = 2.95 \times 10^{-3}$. m/min.

$S = 4T \frac{t^+}{r^2 (\frac{1}{u})^+} = \frac{4 \times 0.36 \times 70}{(12.2)^2 \times 900} = 7.52 \times 10^{-4}$

(Storage coff is dimensionless. See R_2-P-89).

Well flow for Special conditions:-

(1) Constant well drawdown.

(2) Varying, cycling and intermittent well discharges.

(3). Sloping aquifers.

(4) Aquifers of variable thickness.

(5) Two layered/Multilayered aquifers.

(6) Anisotropic aquifers.

(7) Aquifer conditions varying with depth.

(8) Large diameter wells.

(In our cases, we considered infinitesimal diameter of well and storage in well was neglected).

(9) Collector wells.

(10) Wells with mutiple sectioned well screens.

(11) Effect of boundary conditions.

Well flow near a stream:-

(See figure 13.5, R_2-P-190)

The drawdown equation can be written as:-

$$S_r = \frac{Q}{4\pi T}\left[\underbrace{W(u_p)}_{\text{For Pumping well}} - \underbrace{W(u_i)}_{\text{For image well.}}\right] \quad \dots (65.1)$$

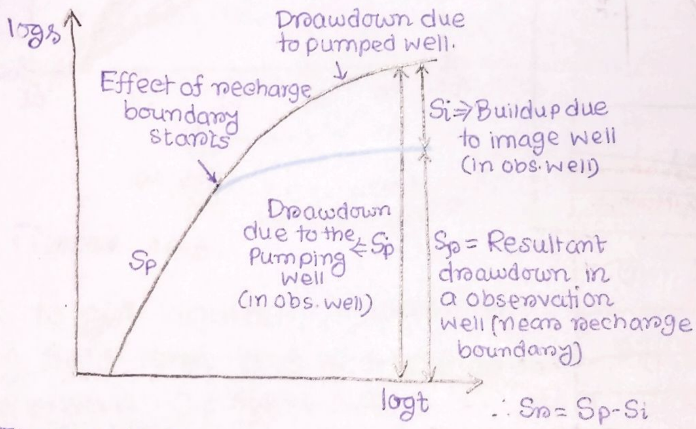


Figure: 16.6

$\therefore S_r = S_p - S_i$

All S_r, S_p, S_i reading are for an observation well near the recharge boundary (i.e. stream).

Well flow near an Impermeable Boundary:

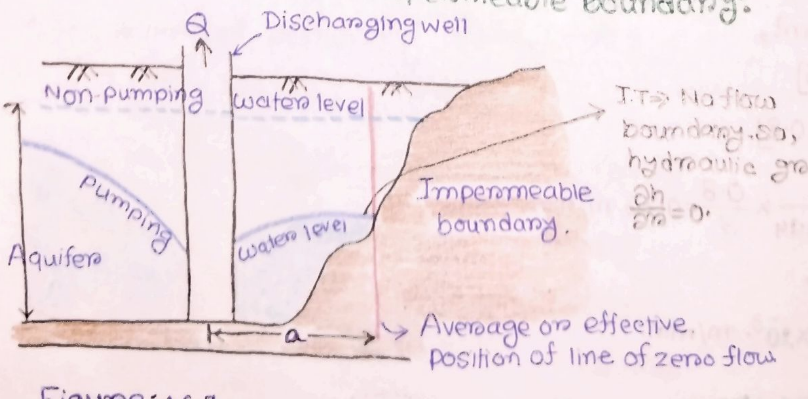


Figure: 16.7.

Now, illustration of figure 16.7 using the concept of image well, In figure 16.8,

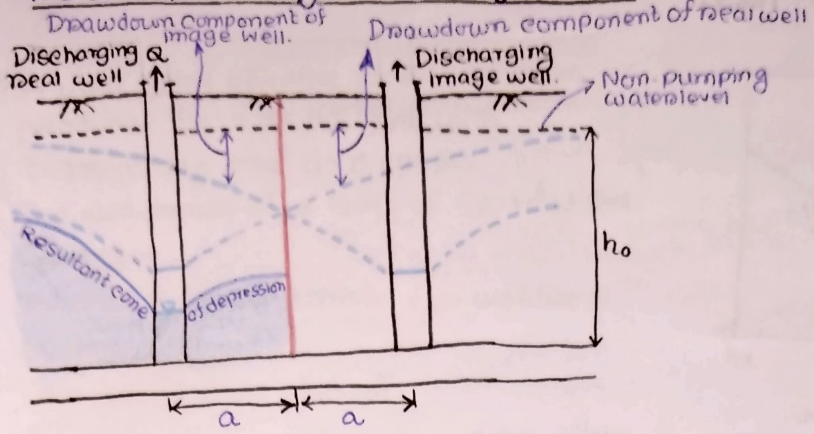


Figure: 16.8

Here, aquifer thickness(h_0) should be very large compared to resultant drawdown $s(r \to 0, t)$ near the real well.

The total drawdown at an observation well can be expressed as, (I.T⇒ observation well should be near the impermeable boundary to capture its effect properly).

$$S_b = \frac{Q}{4\pi T} W(u_p) + \frac{Q}{4\pi T} W(u_i) \ldots\ldots (66.1)$$
$$= S_p + S_i \ldots\ldots (66.2).$$

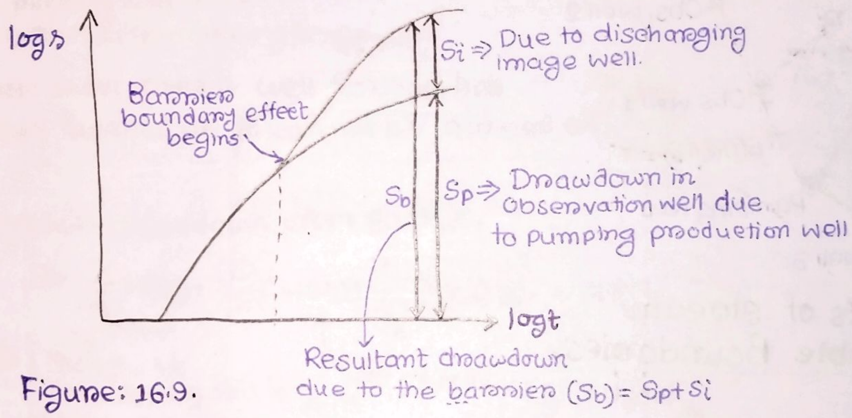


Figure: 16.9.

! [#Note that, barrier effect does not start from begining, after a few times.
I.T⇒ This time is time required for proper development of flow upto the barrier boundary.] .

Now, suppose that we choose drawdowns at time t_p and t_i (# Two different times) such that $s_p = s_i$, ... (66.3)

$$\text{so, } \frac{Q}{4\pi T} W(u_p) = \frac{Q}{4\pi T} W(u_i)$$

$$\Rightarrow W(u_p) = W(u_i) \ldots\ldots\ldots (66.4)$$

$$\Rightarrow u_p = u_i \ldots\ldots\ldots (66.5)$$

$$\Rightarrow \frac{r_p^2 S}{4Tt_p} = \frac{r_i^2 S}{4Tt_i}$$

$$\Rightarrow \frac{r_p^2}{t_p} = \frac{r_i^2}{t_i} \ldots\ldots (66.6)$$

($\because$ T and S are constants i.e aquifer properties).

From (66.6), Distance of a image well from observation well,

$$r_i = r_{rp}\sqrt{\frac{t_i}{t_p}} \;\cdots\cdots \quad (66.7)$$

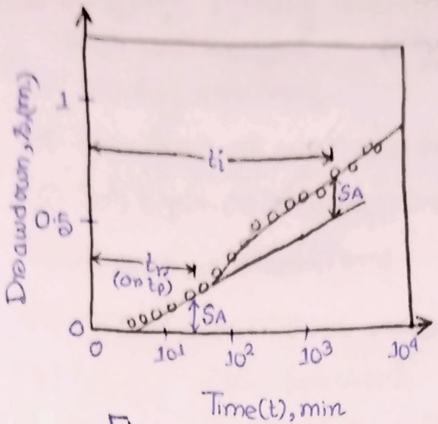


Figure 16.10:- Cooper-Jacob drawdown curve showing effect of an impermeable boundary.

! (I.T ⇒ t_p in (66.7) and t_r in figure(16.10) both are same. 'r' means real well, 'p' means pumping well)

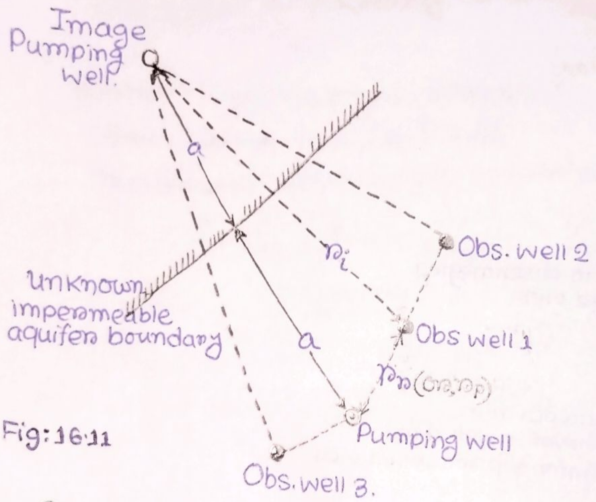


Fig: 16.11

Combined effects of stream and impermeable Boundaries:

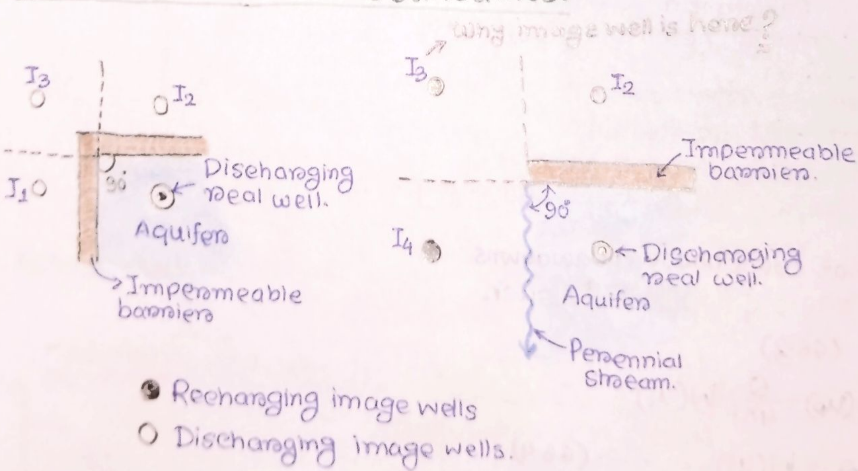


Figure: 16.12.

Superposition with muliple pumping Rates:-

Problem:

A well with effective radius of 0.3m produces from an aquifer with $T = 300\, m^2/day$ and $S = 0.0006$.
The well pumps at a rate of $550\, m^3/d$ for 30 days.
Then at a rate of $800\, m^3/d$ for additional 30 days.
The well is then shut off.
Calculate the residual drawdown after 80 days from the start of pumping (20 days after the well was shut off).

Soln:

	Period1	priod 2	period 3.
Time (days)	0-30	30-60	60-80
$Q(m^3/d)$	550	800	0
$\Delta Q(m^3/d)$	550	250	-800
Duration(t), days	80	50	20

Duration for which the pumping effects the drawdown = (80 - strating time) days.

Basic rule: Once a well function has been "turned on" it can not be trurned off. } why Such Statement?

Residual drawdown after 80 days,

$$S = \underbrace{\frac{Q_1}{4\pi T} W(u_1)} + \frac{\Delta Q_{12}}{4\pi T} W(u_2) + \frac{\Delta Q_{23}}{4\pi T} W(u_3) \cdots (67)$$

[Though, we considering this for 80 days. But, we have ΔQ_{12} and ΔQ_{23} terms to balance this].

Now, $u_1 = \frac{r^2 S}{4Tt_1} = \frac{0.3^2 \times 0.0006}{4 \times 300 \times 80} = 5.625 \times 10^{-10}$

[r = 0.3m, we are calculating drawdown at the well face].

$u_2 = \frac{r^2 S}{4Tt_2} = \frac{0.3^2 \times 0.0006}{4 \times 300 \times 50} = 9 \times 10^{-10}$

$u_3 = \frac{r^2 S}{4Tt_3} = \frac{0.3^2 \times 0.0006}{4 \times 300 \times 20} = 2.25 \times 10^{-9}$.

So, u is less than 0.01 for every cases.
So, we can neglect higher order terms of well functions of Theis equation (52.12).
Using, cooper Jacob's approximation (i.e 58.3).

$\cancel{W(u_1) = \frac{2.303 Q}{4\pi T}[-0.5772 - \ln u_1]}$ (i.e $W(u) = -0.5772 - \ln u$)

=

equation (67) can be written as,

$$s = \frac{Q_1}{4\pi T}(-0.5772 - \ln u_1) + \frac{\Delta Q_{12}}{4\pi T}(-0.5772 - \ln u_2) + \frac{\Delta Q_{23}}{4\pi T}(-0.5772 - \ln u_3)$$

$$\Rightarrow s = \frac{550}{4\pi \times 300}(-0.5772 - \ln 5.625\times10^{-10}) + \frac{250}{4\pi\times 300}(-0.5772 - \ln 9\times10^{-10}) + \frac{(-800)}{4\pi\times300}(-0.5772 - \ln 2.25\times10^{-9})$$

$$= 0.1103\,m.$$

Lecture-17
Unsaturated Flow

- To explain unsaturated flow
- To estimate infiltration.

Unsaturated flow:-

- Partially saturated flow
- Unsaturated flow
- vadose zone flow.

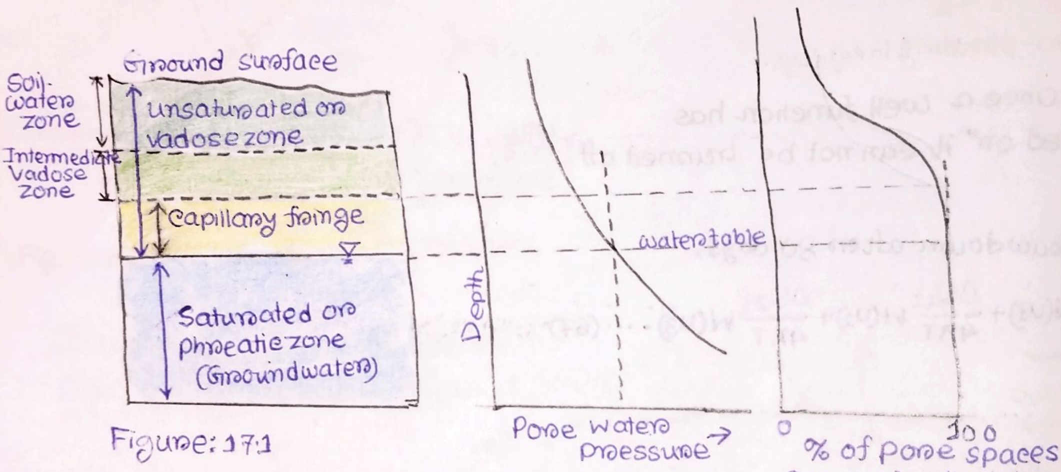


Figure: 17.1

→ [Moisture content (θ) $= \frac{\text{Volume of water}}{\text{Total volume}}$.]

(1) Zone of aeration:- (From Todd).

Also called unsaturated or vadose zone.

'Vadose' water (latin word means shallow) occurs.

Divided into 3 sub-zones:-

(i) Soil water zone:- Extends upto major root zone. Thickness depends on soil and vegetation type. Its thickness depends on soil type and vegetation. The amount of moisture present on soil water zone depends on primarily the recent exposure of the soil to moisture.

(ii) Intermediate vadose zone:- Non-moving vadose water in held in place by hygroscopic and capillary forces. Temporary excess of water migrate downward as gravitational water.

(iii) Cappillary zone:-

capillary rise he derived from equilibrium of surface tension of water and weight of water raised.

$$h_e = \frac{4\sigma\cos\theta}{\rho g d} \quad \ldots\ldots (68.1)$$

Though, cappillary zone is saturated, it is considered to be within the vadose zone.

(2) Saturated or Phreatic zone:-

Greek word Phrear-atos means → 'a well'.

Capillary zone and Phreatic zone has only water in the pores, no air.

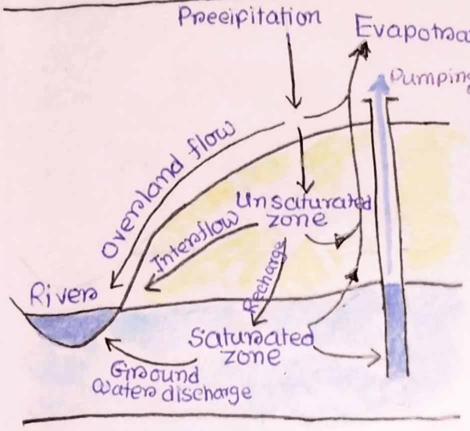


Figure: 17.2

From AI:-

Total head in unsaturated zone,
= Elevation head + Pressure head + capillary head. [In figure 17.3, Total head = Elevation head + capillary (or suction) head.]

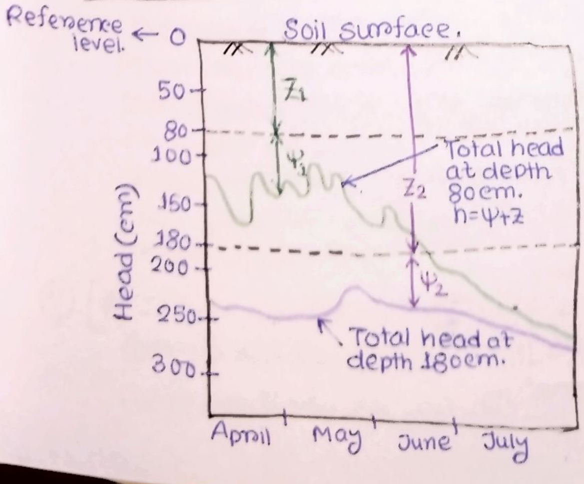
Suction head in unsaturated zone:-

Negative pressure or tension that draws water upward against the force of gravity.

Suction head arises due to capillary action.

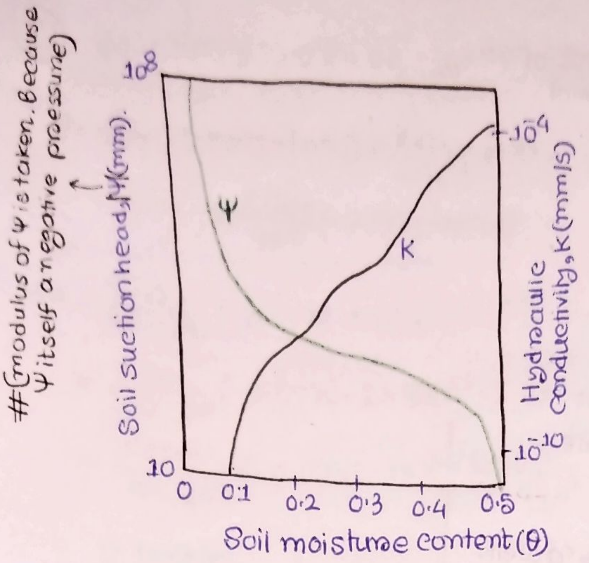
As water content decreases, suction head increases.

Measured by tensiometers, pressure transducers, soil-moisture sensors.



From Page 18, we know, upper portion of zone of aeration has more variation in moisture content with time. and moisture content determines suction head.
so, we can see more variation in suction head at 80 cm than 180 cm.

Fig: 17.3: Profiles of total soil moisture head with time.



ψ is also called 'matric potential' which is a measure of energy required to remove water from the soil.

Fig: 17.4:- Variation of ψ and K with soil moisture content(θ).

Parameters and relationships in unsaturated flow:

- Total head, $h = z + \psi$ .....(68.2)
- Darcy's law,

$$\underset{\sim}{q} = -K(\theta)\nabla h. \quad \cdots(68.3)$$

$K = K(\theta)$. Because, from figure 17.4, it is obvious that, Permeability depends on soil moisture content (θ).

- Relative permeability:-

$$K_r(\theta) = \frac{K(\theta)}{K_s} \quad \cdots(68.4)$$

$$0 \leqslant K_r(\theta) \leqslant 1$$

Permeability has maximum value when the soil is saturated.

- Expanded form of Darcy's law:

~~$q = -K_s K_r$~~

From (68.3), $\underset{\sim}{q} = -K(\theta)\nabla h$

Putting value of $K(\theta)$ from (68.4), and value of h from (68.2), we have,

$$q = -K_s K_r(\theta)\nabla(z+\psi) \quad \cdots\cdots(68.5)$$

$$= -K_s K_r(\theta)\underset{\sim}{\hat{e}_z} - K_s K_r(\theta)\nabla\psi. \cdots\cdots(68.6)$$

$$q = -K_s K_r(\theta)\nabla z - K_s K_r(\theta)\nabla\psi$$

Now, $\nabla z = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)z$

$$= \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} + \frac{\partial z}{\partial z}\hat{k}$$

$$= 0 + 0 + \hat{k} \quad \left[\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0\right]$$

$$\therefore q = -K_s K_r(\theta)\hat{k} - K_s K_r(\theta)\nabla\psi \cdots\cdots(68.7)$$

?! (Comparing (68.6) and (68.7), I think e_z is the unit vector in z direction)

⊙ continuity equation

$$\frac{\partial \theta}{\partial t} + \nabla \cdot q = S' \rightarrow S' = \frac{dB}{dt}? \;\ldots\ldots (68.7)$$

(See- My observation, P-22)

From AI

S' = Source/sink term representing water input or extraction.

⊙ Mixed form (h-θ) based.

Put q value in (68.7) from (68.3) and $S'=0$,

$$\frac{\partial \theta}{\partial t} - \nabla \cdot [K(\theta)\nabla h] = 0 \;\ldots\ldots (68.8)$$

Or, put q value from (68.6),

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [K_s K_r(\theta) e_z + K_s K_r(\theta) \nabla \psi] \;\ldots (68.9)$$

Here, one equation with two unknowns.

(Two unknowns are → Soil moisture content (θ) and suction head (ψ))!?

Chapter 4: Subsurface water (Ven-te-Chow).

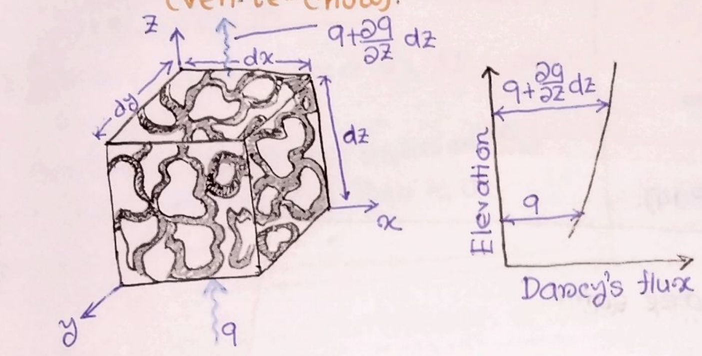


Figure: 17.5

⊙ Continuity Equation:-

From figure (17.5), water contained in the control volume = $\theta\, dx\, dy\, dz$ (# θ = Moisture content). .... (68.10)

Darcy's flux (i.e volumetric flow rate per unit area of soil), $q = Q/A$.

#(Though, q is a vector, but here, horizontal fluxes are assumed to be zero, only vertical component of Darcy's flux is considered).

In this case, for RTT,

B = Mass of soil water.

$$\therefore \beta = \frac{dB}{dm} = 1 \;\ldots\ldots (68.11)$$

$\frac{dB}{dt} = 0$ .... (68.12) (Because, no phase change is occuring in the water. I think, if phase changes occur, then density will also change. So, mass of water within the CV would be changed with time. Because, pore volume is constant within the CV i.e $= \eta\, dx\, dy\, dz$ and water can't take more volume than that).

From R_2-P-38,

Reynold's Transport theorem,

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int \beta \rho\, d\forall + (\beta \rho A V_s)_{out} - (\beta \rho A V_s)_{in} \;\ldots (68.9.2)$$

(Same eqn written here, with a little bit different form).

B = Extensive property

β = Intensive property. (May be the ratio of two extensive property).

$\forall$ = Total volume.

$V_s = \begin{cases} V, \text{ for fixed control volume} \\ V - \omega, \text{ when CV is moving with velocity } \omega. \end{cases}$

(?) [# If you considered $\frac{dB}{dt}$ in (68.13), then (68.7) would look like that. So, comparing (68.7) and (68.14), can we say, $S' = \frac{dB}{dt}$?] ... (68.12.2)

Now, from Reynold's Transport theorem,

$$\frac{dB}{dt} = \frac{d}{dt}\iiint_{CV} \beta \rho_w dV + \iint_{CS} \beta\rho_w \vec{V}.d\vec{A} \; \ldots\ldots (68.13)$$

From (68.11) and (68.12), Putting $\beta = 1$ and $\frac{dB}{dt} = 0$,

$$0 = \frac{d}{dt}\iiint_{CV} \rho_w dV + \iint_{CS} \rho_w \vec{V}.d\vec{A}.$$

$$\Rightarrow 0 = \left[\frac{d}{dt}(\rho_w \theta dx dy dz)\right] + \left[\underbrace{\rho_w\left(q + \frac{\partial q}{\partial z}dz\right)dx dy}_{\text{Outflow}} - \underbrace{\rho_w q dx dy}_{\text{Inflow}}\right] \ldots (68.13.1)$$

Now, see this equation looks like Transient one, i.e (68.9.2)

$$\Rightarrow 0 = \underbrace{\rho_w dx dy dz}\frac{\partial \theta}{\partial t} + \rho_w dx dy dz \frac{\partial q}{\partial z}.$$

ρ_w assumed to be constant and spatial dimensions of control volume (i.e dx, dy, dz) are also fixed.

$$\Rightarrow 0 = \frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} \;\ldots\ldots (68.14).$$

My observation:-

Equation (68.7) and (68.14) are same.

From, (68.7),

$$\frac{\partial \theta}{\partial t} + \nabla . q = S'$$

$$\Rightarrow \frac{\partial \theta}{\partial t} + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(q_x\hat{i} + q_y\hat{j} + q_z\hat{k}) = \frac{dB}{dt} \quad (\#\ \text{I.T, } S' = \frac{dB}{dt})$$

Now, considering only vertical flux, i.e $q_x = q_y = 0$, and $\frac{dB}{dt} = 0$, we have,

$$\Rightarrow \frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} = 0 \ldots \Rightarrow \text{which is } (68.14).$$

- Momentum equation:-

Darcy's law, $q = KS_f \ldots (68.15)$

where, S_f = Head loss per unit length of the medium. $= -K\frac{\partial h}{\partial z} \ldots (68.16)$

[# We are considering vertical flow only.]

Note: For unconfined saturated flow, only gravity and friction forces are involved.

But, for unsaturated flow, suction force binding water to soil particles through surface tension must be included also.

Total head(h) = Suction head(ψ) + gravity head(z) $\ldots\ldots (68.17)$

Substituting h from (68.17) to (68.16),

$$q = -K\frac{\partial}{\partial z}(\psi + z)$$

$= -K\frac{\partial \psi}{\partial z} - K$

$= -K\frac{\partial \psi}{\partial \theta}\cdot\frac{\partial \theta}{\partial z} - K$

[θ = Soil moisture content.

$\cdot\frac{\partial \psi}{\partial \theta}$ because suction head depends on moisture content]

$\Rightarrow q = -(D\frac{\partial \theta}{\partial z} + K)$ .... (68.18)

where, $D = K\frac{\partial \psi}{\partial \theta}$ .... (68.19)

is known as soil-water diffusivity, which has dimensions $[L^2/T]$.

Now, Substituting value of q from (68.18) to continuity equation (68.14), we get,

$$\boxed{\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z}(D\frac{\partial \theta}{\partial z} + K) = 0} \quad \text{.... (68.20)}$$

Equation (68.20) is known as **1D** form of Richard's equation.

This is the governing equation for unsteady unsaturated flow in a porous media.

Governing equation of unsteady, unsaturated flow in porous media,

- Pressure form (ψ based).

$$C(\psi)\frac{\partial \psi}{\partial t} = \nabla.(K(\psi)\nabla\psi) + \frac{\partial K(\psi)}{\partial z} \quad \text{..... (68.21)}$$

where, $C(\psi) = \frac{d\theta}{d\psi}$ ..... (68.22)

$C(\psi)$ is known as specific capacity or capillary capacity.

K can be represented by θ or ψ both. Because, moisture content and suction heads are interdependent.

- Moisture form (θ based):

$$\frac{\partial \theta}{\partial t} = \nabla.(D(\theta)\nabla\theta) + \frac{\partial K(\theta)}{\partial z} \quad \text{.... (68.23)}$$

where, $D(\theta) = K_\theta\frac{\partial \psi}{\partial \theta}$ .... (68.24)

$D(\theta)$ is known as specific diffusivity or capillary (soil-water) diffusivity. (This parameter represents the ability of water to move through the soil matrix.)

How to get moisture form (68.23) and pressure form (68.21):-

(68.23) is just 3D form of equation (68.20).

From (68.21), $\frac{\partial \theta}{\partial t} = \nabla.[D(\theta)*\nabla(\theta)] + \frac{\partial K(\psi\theta)}{\partial z}$

$= (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}).(D\overset{0}{\frac{\partial \theta}{\partial x}}\hat{i} + D\overset{0}{\frac{\partial \theta}{\partial y}}\hat{j} + D\frac{\partial \theta}{\partial z}\hat{k}) + \frac{\partial K}{\partial z}$

Now, assuming moisture content only varies in vertical direction. (i.e $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y} = 0$),

$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}(D\frac{\partial \theta}{\partial z}) + \frac{\partial K}{\partial z}$ $\Rightarrow$ which is (68.20)

Let's get pressure form.

From moisture form, (68·23),

$$\frac{\partial \theta}{\partial t} = \nabla\cdot[D(\theta) * \nabla\theta] + \frac{\partial K(\theta)}{\partial z}$$

Hydraulic conductivity can be represented as $K(\theta)$ or $K(\psi)$.

Let, $\boxed{K(\theta) = K(\psi) = K}$ Because, θ and ψ are functions of each other. (see fig 17·4)

$$\Rightarrow \frac{\partial \theta}{\partial \psi}\cdot\frac{\partial \psi}{\partial t} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[K\frac{\partial \psi}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\hat{i} + \frac{\partial \theta}{\partial y}\hat{j} + \frac{\partial \theta}{\partial z}\hat{k}\right)\right] + \frac{\partial K}{\partial z}$$

Putting value of $D(\theta)$ from (68·24).

$$\Rightarrow C(\psi)\cdot\frac{\partial \psi}{\partial t} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(K\frac{\partial \psi}{\partial x}\hat{i} + K\frac{\partial \psi}{\partial y}\hat{j} + K\frac{\partial \psi}{\partial z}\hat{k}\right) + \frac{\partial K}{\partial z}$$

$$\Rightarrow C(\psi)\frac{\partial \psi}{\partial t} = \nabla\cdot(K\nabla\psi) + \frac{\partial K}{\partial z}$$

$$\Rightarrow C(\psi)\frac{\partial \psi}{\partial t} = \nabla\cdot[K(\psi) * \nabla\psi] + \frac{\partial}{\partial z}K(\psi)$$

↳ This is pressure form i.e equation (68·21)

◎ Van-Genuchten-Mualem model:-

Mathematical model describes flow of water in unsaturated soil. It is an extension of classical Darcy's law, which describes the flow of water in saturated soils.

This model takes into account the capillary forces that affect the movement of water in unsaturated soil.

It combines two key concepts:- i.e Van-Genuchten retention model & Mualem permeability model.

(1) Degree of saturation:-

$$S_e(\theta) = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \begin{cases} \dfrac{1}{(1+|\alpha\psi|^n)^m}, & \psi < 0 \text{ (\# -ve suction means unsaturated)} \\ 1, & \psi \geq 0. \text{ (\# +ve suction means saturated).} \end{cases}$$

.... (68·25)

#[Equation (68·25) can also be called as relation between soil moisture content (θ) and soil suction head (ψ)]

Parameter definitions are from A], θ_r = Residual water content → Pendular water (R_2-P-83) & adsorbed water are also

(when, soil has least water drained or present? or it is completely dry, or water content at low pressure head)

θ_s = Saturated water content

(Max water content that soil can hold or water content at high pressure head).

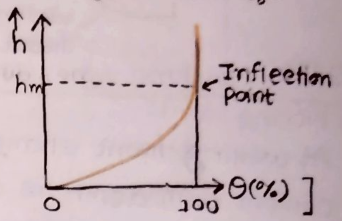
α = Empirical parameter related to inverse of the air entry value of the soil.

n = Empirical parameter controlling the shape of retention curve.
(n is related to pore size distribution)

m = Empirical parameter related to rate of change of retention curve w.r.t water content.
(m is related to pore connectivity).

? = [We can write "$\alpha\psi = \frac{h}{h_m}$" in equation (68·25).
Where, h_m = Pressure head at the inflection point of the (retention!?) curve.]

[I think, inflection point occurs when θ = 100%.
Curve may be like this:-



]

#Retention curve:

Also known as soil-water retention curve or moisture characteristics curve.

It describes how soil retain water at different suctions or pressure head.

Generally, retention curve shows that as pressure head (h) decreases (or suction increases), the soil's water content (θ) decreases.

I.T #(Similar to ψ vs θ curve in figure (17·4), But, with opposite sign. (Because, ψ itself negative, but we consider pressure head as positive)

(2) Hydraulic Conductivity:

$$K(\psi) = \begin{cases} K_s S_e^{0.5}\left[1-(1-S_e^{1/m})^m\right]^2 & , \psi<0 \\ K_s & , \psi \geq 0 \end{cases} \quad \cdots\cdots (68·26).$$

Where, K_s = Saturated hydraulic conductivity (H.C at high pressure head).

S_e = Degree of saturation (from 68·25)

(3) Soil-water capacity:

$$C(\psi) = \frac{d\theta}{d\psi} = \begin{cases} \dfrac{-\alpha mn(\theta_s-\theta_r)|\alpha\psi|^{n-1}}{(1+|\alpha\psi|^n)^{m+1}} & \psi<0 \quad \cdots\cdots (68·27) \\ 0 & \psi \geq 0 \end{cases}$$

$\psi \geq 0$ → This represent saturated condition where, θ is always 100%. So, $\frac{d\theta}{d\psi}$ is zero. Also, for saturated condition, pressure head is predominant, suction head is insignificant.

(4) Soil-water diffusivity:

$$D(\theta) = K(\theta)\frac{d\psi}{d\theta} \cdots\cdots (68·28).$$

$K(\theta) = K(\psi)$, put value from (68·26)

and $\frac{d\psi}{d\theta}$ from (68·27).

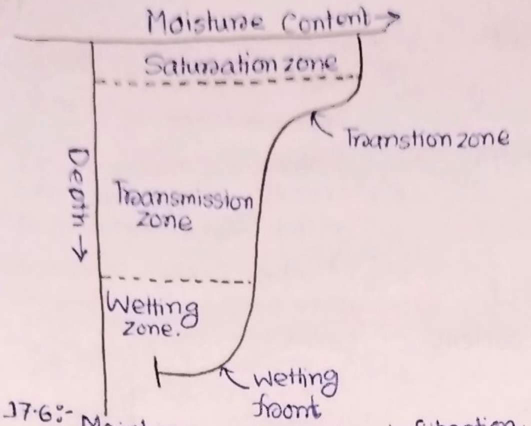


Fig 17.6:- Moisture zones during infiltration.

Wetting front:-

At wetting front change of moisture content with depth is abrupt.
So, there is a sharp discontinuity between wet soil above and dry soil below.
Length of wetting front may depend on amount of infiltration and physical properties of soil.

Potential infiltration rate:-

If water is ponded over the surface the infiltration occurs at the potential infiltration rate.
Most infiltration equation describe this rate.

- Cumulative infiltration depth over time t,

$$F(t) = \int_0^t f(\tau)\,d\tau \quad \cdots\cdots (69.1)$$

where f is the rate of integration.

In converse,

$$f(t) = \frac{dF(t)}{dt} \quad \cdots\cdots (69.2)$$

Here, τ is nothing but a dummy variable.
(Dummy variable is a placeholder or temporary variable in integration. These variables are not essential in integration, can be changed to other symbol without affecting the result of integration)

Eg: $\int_2^4 f(x)\,dx = \int_2^4 f(t)\,dt$ will give same result.

Here, x and t are dummy variable.

☐ Horton's equation:

$$f(t) = f_c + (f_0 - f_c)\,e^{-kt} \quad \cdots\cdots (69.3)$$

where, ~~$f(t)$~~ f_0 and f_c are intial and final infiltration rates respectively.
k is decay constant

- We know, hydraulic conductivity (K) and soil water diffusivity (D) are the functions of moisture content (θ), i.e $K(\theta)$ and $D(\theta)$.

But,
If we assume that they are constant value,

then from Richards eq^n (68.23),

$$\frac{\partial\theta}{\partial t} = \nabla\cdot\left(D(\theta)\nabla\theta\right) + \frac{\partial K(\theta)}{\partial z} \text{ becomes,}$$

$$\Rightarrow \frac{\partial\theta}{\partial t} = D*\nabla\cdot(\nabla\theta) + 0$$

$$\Rightarrow \frac{\partial\theta}{\partial t} = D\nabla^2\theta \quad \ldots\ldots (69.4)$$

Now, if we consider only vertical infiltration, (69.4) becomes, $\frac{\partial\theta}{\partial t} = D\frac{\partial^2\theta}{\partial z^2}$ ...... (69.5)

(69.5) is the standard form of a diffusion equation and may be solved to yield the moisture content (θ) as a function of time and depth.

Horton's equation results from solving for the rate of moisture diffusion ($D\frac{\partial\theta}{\partial z}$) at the soil surface. → How?

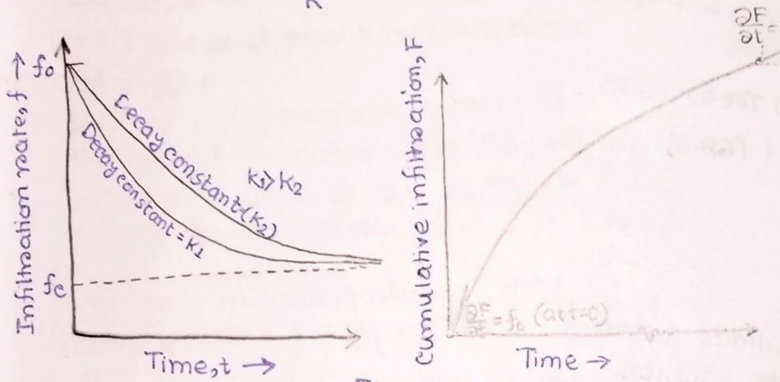
⊙ If we put the value of f(t) from Horton's equation (69.3) to eq^n (69.1), cumulative infiltration becomes,

$$F(t) = \int_0^t \left[f(c) + (f_0 - f_c)e^{-kt}\right]dt$$

$$= \left[f_c.t + (f_0 - f_c)\left(-\frac{1}{k}\right)e^{-kt}\right]_0^t$$

$$= \left[f_c.t + (f_0 - f_c)\left(-\frac{1}{k}\right)e^{-kt}\right] - \left[0 + (f_0 - f_c)\left(-\frac{1}{k}\right)\right]$$

$$= f_c t + \frac{(f_0 - f_c)}{K}(1 - e^{-kt}) \ldots\ldots (69.6)$$



Combining (69.2) and (69.3)

$$\frac{dF}{dt} = f_c + (f_0 - f_c)e^{-kt}$$

at t=0, $\frac{dF}{dt} = f_c + (f_0 - f_c).1 = f_0$

at t=∞, $\frac{dF}{dt} = f_c + (f_0 - f_c).0 = f_c$

Figure 17.7

Philip's Equation:-

Horton solved ~~Philip's~~ Richard's equation taking K and D as constant.

But, Philip solved Richard's equation under less restrictive condition.

He assumed K and D may vary with θ (i.e $K(\theta)$ and $D(\theta)$), but he employed Boltzmann transformation $B(\theta) = zt^{-1/2}$ to convert (68.20) into an ordinary differential equation in B, and solved this equation to yield an infinite series for cumulative infiltration F(t). → How?

Here $F(t)$ is approximated by,

$$F(t) = St^{1/2} + kt \quad \cdots\cdots (69.7)$$

where, S is sorptivity ⇒ function of soil suction potential.

k = Hydraulic conductivity.

Now, differentiating (69.7), we get rate of infiltration,

$$f(t) = \frac{dF(t)}{dt} = \frac{1}{2}St^{-1/2} + k \quad \cdots\cdots (69.8)$$

The two terms in Philip's equation (69.7), represents the effect of soil suction head and gravity head.

If we compare Horton's and Philip's equation, (69.3) and (69.8), for infiltration rate,

$f(t) = f_c + (f_0 - f_c)e^{-kt}$

and $f(t) = \frac{1}{2}St^{-1/2} + K$.

At $t \to \infty$, Horton gives, $f(t) = f_c$ } Both has similarity.
Philip gives, $f(t) = K$ }

But, at $t=0$, Horton gives $f(t) = f_0$ } why?
but Philip gives, $f(t) = \infty$ }

For a horizontal column of soil, Soil suction is the only force drawing water into the column,

#(As we only considered the vertical flow, So, $K = K_z$ in 69.8. So, K_z will not work for horizontal column flow).

∴ Philip's equation (69.7) reduces to

$$F(t) = St^{1/2} \quad \cdots\cdots (69.9)$$

☑ Green-AMPT method:

- Horton's and Philip's equations were developed from approximate solution of Richard's equation. (I need the complete process)!
- Green and Ampt gave an alternative approach to develop a more approximate physical theory that has exact analytical solution.
- They proposed a simplified infiltration picture in figure 17.8. Here, wetting front is a totally sharp boundary (# compare with fig 17.6), having moisture content = η above (all pores are filled with water!) and moisture content = θ_i below.

The wetting front has penetrated to a depth L in time t since infiltration began.

Water is ponded to a small depth h_0 on the soil surface.

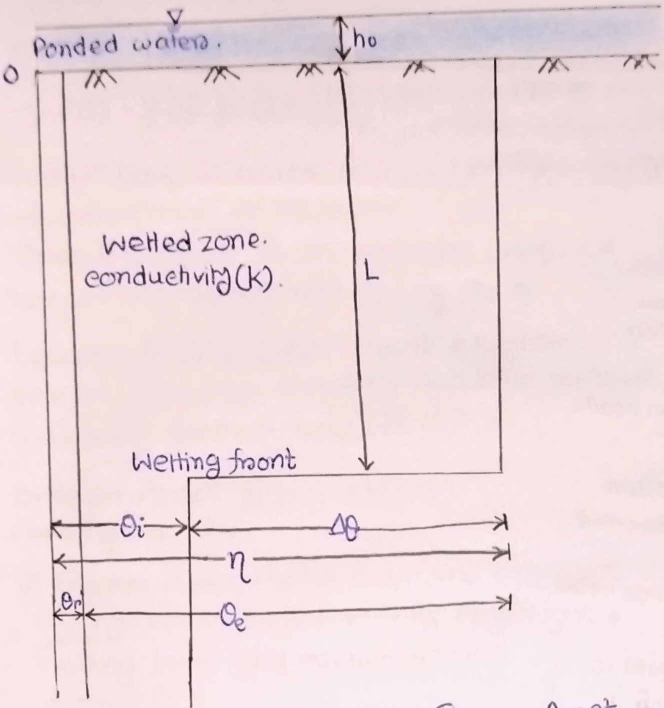


Figure 17.8:- Variables in Green-Ampt model. Vertical axis ⇒ Distance from the soil surface. Horizontal axis ⇒ Moisture content of soil.

Continuity:

Consider a vertical column of soil with unit horizontal cross sectional area and length L

Soil had initial moisture content θ_i throughout its entire depth. Then it increases from θ_i to η as the wetting front passes.

∴ Increase in water stored within the control volume $= L \cdot 1 \cdot (\eta - \theta_i)$ . . . . . . (70.1)

∴ Cumulative depth of water infiltrated into the soil,

$$F(t) = L(\eta - \theta_i)$$

$$= L\Delta\theta \quad \text{. . . . . (70.2)}$$

where, $\Delta\theta = \eta - \theta_i$

Momentum:-

Darcy's law can be expressed as, $q = -K\frac{\partial h}{\partial z}$ . . . . (70.3)

Here, Darcy's flux q (i.e discharge per unit area) is constant throughout the depth and equal to $-f$. (because q is positive upward), f is positive downward).

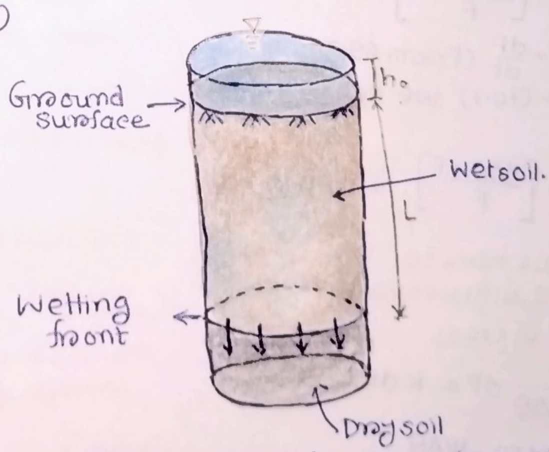


Figure 17.9:- Infiltration into a soil column of unit c/s area.

If point 1 and 2 are located respectively at the ground surface and just on the dry side of the wetting front.

$$f = -q$$
$$= -(-k\frac{\partial h}{\partial z})$$
$$= k\frac{\partial h}{\partial z}$$
$$= K\left(\frac{h_2-h_1}{z_2-z_1}\right)$$

$$\text{Or, } f = K.\left[\frac{h_1-h_2}{z_1-z_2}\right] \quad \text{.......(70.4)}$$

Now, $h_1 = h_0$ and $h_2 = -\psi - L$ } ..... (70.5).

h_1 is near the surface where water is ponded. So, it is saturated.

So, h_1 = Pressure head + Elevation head $= h_0 + 0 = h_0$

But, point 2 is in dry zone (Just after the wetting front). So, it is in unsaturated zone.

$\therefore h_2$ = suction head + Elevation head $= -\psi - L$

($-L$, because, reference level is ground surface and elevation head is -ve downward).

Putting values of h_1 and h_2 in (70.4),

$$f = K.\left[\frac{h_0-(\psi-L)}{L}\right]$$
$$= k\left(\frac{h_0+\psi+L}{L}\right)$$
$$\approx k\left[\frac{\psi+L}{L}\right] \quad \text{........ (70.6)}$$

#(In (70.6), we considered $h_0 \ll \psi$ and L. This assumption is applicable for surface water hydrology problems because it is assumed that ponded water becomes surface runoff.)

Now, From (70.2), wetting front depth, $L = \frac{F}{\Delta\theta}$,

Put this value in (70.6),

$$f = k\left[\frac{\psi\Delta\theta+F}{F}\right] \quad \text{....... (70.7)}$$

Since, $f = \frac{dF}{dt}$ (From 69.8),
so, from (70.7) we can write,

$$\frac{dF}{dt} = K\left[\frac{\psi\Delta\theta+F}{F}\right] \quad \text{.... (70.8)}$$

#Now we need to solve this differential equation of (70.8).

$$\Rightarrow \frac{F}{F+\psi\Delta\theta}dF = Kdt$$

$$\Rightarrow \frac{(F+\psi\Delta\theta)-\psi\Delta\theta}{F+\psi\Delta\theta}dF = Kdt$$

$$\Rightarrow \int_0^{F(t)} \left(1-\frac{\psi\Delta\theta}{F+\psi\Delta\theta}\right) dF = \int_0^t k\,dt$$

$$\Rightarrow \left[F - \psi\Delta\theta \ln(F+\psi\Delta\theta)\right]_0^{F(t)} = [kt]_0^t$$

$$\Rightarrow F(t) - \psi\Delta\theta \ln[F(t)+\psi\Delta\theta] + \psi\Delta\theta \ln(\psi\Delta\theta) = kt$$

$$\Rightarrow F(t) - \psi\Delta\theta \ln\left[\frac{F(t)+\psi\Delta\theta}{\psi\Delta\theta}\right] = kt$$

$$\Rightarrow F(t) - \psi\Delta\theta \ln\left(1+\frac{F(t)}{\psi\Delta\theta}\right) = kt \ldots\ldots (70.9)$$ $\Rightarrow$ when Ponding depth(h_0) is not negligible, Substitute ψ with $(\psi - h_0)$ here.

Equation (70.9) is called Green-Ampt equation for cumulative infiltration.

Once F is found from equation (70.9), we can get infiltration rate, using (70.7).

Equation (70.9) is a non-linear equation, can be solved by Newton's iterative method. (i.e Newton-Raphson method!?)

Green Ampt parameters:
(see figure 17.8).

- Green Ampt model requires estimates of hydraulic conductivity k, Porosity η & wetting front soil suction head ψ.

- Brooks and Corey (1964) expressed ψ as a logarithmic function of effective saturation (S_e).

where, $S_e = \dfrac{\theta - \theta_r}{\eta - \theta_r}$ .... (70.10)

θ_r = Residual moisture content after it was thoroughly drained

$\theta - \theta_r$ = Available moisture content

$\eta - \theta_r$ = Max possible moisture content. it is also called effective porosity, θ_e.... (70.11)

Obviously, $0 \leq S_e \leq 1$ ...... (70.12)

Now, putting $\eta - \theta_r = S_e$ (from 70.11) in (70.10), we get, $\theta - \theta_r = S_e\theta_e$ ...... (70.13).

- For the initial condition where, $\theta = \theta_i$,

We have, $\theta_i - \theta_r = S_e\theta_e$ ...... (70.14)

and the change in moisture content when wetting front passes,

$$\Delta\theta = \eta - \theta_i = \eta - (\theta_r + S_e\theta_e) \quad \text{\# (From 70.14)}$$
$$= (\eta - \theta_r) - S_e\theta_e$$
$$= \theta_e - S_e\theta_e \quad \text{(\# From 70.11)}$$
$$= \theta_e(1 - S_e) \ldots\ldots (70.15)$$

The logarithmic relation between S_e and ψ can be expressed by Brooks-Corey equation,

$$S_e = \left[\frac{\psi_b}{\psi}\right]^{\lambda} \ldots\ldots (70.16)$$

where ψ_b and λ are constants obtained by draining a soil in stages, measuring the values of S_e and ψ at each stage.

Then we put values of S_e and ψ in equation (10.16) and estimate the values of ψ_b and λ.

Soil class	Porosity (n)	Effective Porosity (θ_e)	Wetting front Soil suction head, ψ (cm)	Hydraulic conductivity, K (cm/h).
Sand	0.437 (0.374-0.5)	0.417 (0.354-0.480)	4.95 (0.97-25.36)	11.78
Loam	0.463 (0.375-0.551)	0.434 (0.334-0.534)	8.89 (1.33-59.38)	0.34
Silty clay loam	0.471 (0.418-0.524)	0.432 (0.347-0.517)	27.30 (5.67-131.50)	0.10
clay	0.475 (0.427-0.523)	0.385 (0.269-0.501)	31.63 (6.39-156.5).	0.03

Table 17.10:- Green-Ampt infiltration parameters for various soil classes.

My observations:-

Though clay has more porosity, but its effective porosity is less than sand. Because, clay has more surface area, so more adhesion force between water and clay particle. So, sand has more mobile water

Sand particles are arranged generally in hexagonal close packing. But, in clay, due to higher particle-particle attraction and chain like formation of clay mineral, clay media is porous. So, it has higher porosity.

Lecture-18

Unsaturated Flow

To estimate infiltration:-

Two Layer Green-Ampt Model:

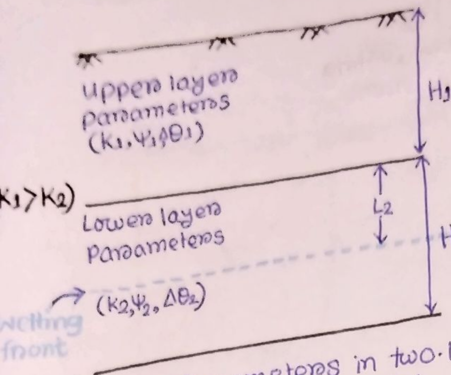


Fig: 18.1: Parameters in two-layered Green-Ampt model.

Here, "$K_1 > K_2$" is required for the "upper layer to remain saturated" while water infiltrates to the lower layer.

It can be shown that, infiltration rate,

$$f = \frac{K_1 K_2}{H_1 K_2 + L_2 K_1}(\psi_2 + H_1 + L_2) \quad \ldots\ldots (71.1).$$

Proof of equation (71.1):

Wetting front penetrates depth L_2 at second layer.

$\therefore$ Total depth of saturated portion $= (H_1 + L_2)$

$\therefore$ Equivalent permeability of this portion

$$K = \frac{H_1 + L_2}{\frac{H_1}{K_1} + \frac{L_2}{K_2}}$$

$$= \frac{K_1 K_2 (H_1 + L_2)}{(H_1 K_2 + L_2 K_1)} \quad \ldots\ldots (71.2).$$

Now, if we consider point at ground surface as point 1 and point just after the wetting front as point 2, then,

$h_1 = h_0$ (# h_0 = Ponding depth)

$z_1 = 0$ (reference level)

and $h_2 = -(H_1 + L_2 + \psi_2)$

$z_2 = -(H_1 + L_2)$.

Now, Rate of infiltration,

$$f = -q \quad \text{(see 70.3)}$$

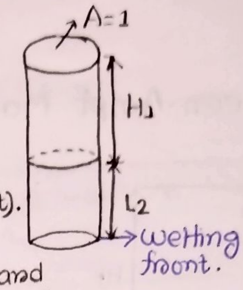
$$= -\left(-K \frac{\partial h}{\partial z}\right)$$

$$= K \frac{h_2 - h_1}{z_2 - z_1}$$

$$= \frac{K_1 K_2 (H_1 + L_2)}{(H_1 K_2 + L_2 K_1)} \times \frac{-(H_1 + L_2 + \psi_2)}{-(H_1 + L_2)} = \frac{K_1 K_2 (\psi_2 + H_1 + L_2)}{(H_1 K_2 + L_2 K_1)} \Rightarrow \text{This is (71.1)}$$

and cumulative infiltration is given by, (considering control volume with unit surface area),

$$F = (H_1 \cdot 1)\cdot \Delta\theta_1 + (L_2 \cdot 1)\Delta\theta_2$$
$$= H_1\Delta\theta_1 + L_2\Delta\theta_2 \quad \ldots (7.1.3)$$



[# Here, value of L_2 depends on time, ie $L_2(t)$. Because, wetting front continues to move downward with time. So, $F = F(t)$ also.]

Now, $$f = \frac{dF}{dt} = \frac{d}{dt}(H_1\Delta\theta_1 + L_2\Delta\theta_2)$$
$$= 0 + \Delta\theta_2 \frac{dL_2}{dt}$$
$$= \Delta\theta_2 \frac{dL_2}{dt} \quad \ldots (7.1.4)$$

Comparing (7.1.1) and (7.1.4), (i.e value of rate of infiltration, f),

$$K_1K_2 \cdot \frac{\psi_2 + H_1 + L_2}{H_1K_2 + L_2K_1} = \Delta\theta_2 \frac{dL_2}{dt}$$

$$\Rightarrow \frac{K_1K_2}{\Delta\theta_2} dt = \frac{H_1K_2 + L_2K_1}{\psi_2 + H_1 + L_2} dL_2$$

$$= H_1K_2 \cdot \frac{dL_2}{\psi_2 + H_1 + L_2} + K_1 \cdot \frac{L_2}{\psi_2 + H_1 + L_2} dL_2$$

$$= \text{"} + K_1 \cdot \frac{(\psi_2 + H_1 + L_2) - (\psi_2 + H_1)}{\psi_2 + H_1 + L_2} dL_2$$

$$\frac{K_1K_2}{\Delta\theta_2}\int dt = H_1K_2\int\frac{dL_2}{\psi_2 + H_1 + L_2} + K_1\int dL_2 - K_1(\psi_2 + H_1)\cdot\int\frac{dL_2}{\psi_2 + H_1 + L_2}$$

$$\Rightarrow \frac{K_1K_2}{\Delta\theta_2} t = H_1K_2 \ln(\psi_2 + H_1 + L_2) + K_1L_2 - K_1(\psi_2 + H_1)\ln(\psi_2 + H_1 + L_2)$$

$$\Rightarrow t = \frac{\Delta\theta_2}{K_1K_2}\left[K_1L_2 + \{H_1K_2 - K_1(\psi_2 + H_1)\}\ln(\psi_2 + H_1 + L_2)\right]$$

$$= L_2\frac{\Delta\theta_2}{K_2} + \frac{1}{K_1K_2}\{\Delta\theta_2H_1K_2 - \Delta\theta_2K_1(\psi_2 + H_1)\}\ln(\psi_2 + H_1 + L_2) \quad \ldots (7.1.5)$$

But, it is not properly matching with book,

$$L_2\frac{\Delta\theta_2}{K_2} + \frac{1}{K_1K_2}\{\Delta\theta_2H_1K_2 - \Delta\theta_2K_1(\psi_2 + H_1)\}\ln\left(1 + \frac{L_2}{\psi_2 + H_1}\right) = t.$$

Remember:-

- Green-Ampt model can be applied when more permeable soil is at upper layer i.e $K_1 > K_2$ (see-P-33).
- This model can be applied when wetting front enters from upper layers to lower layers.

Ponding Time: (t_p)

- t_p is the elapsed time between rainfall begins and time when water begins to pond on the surface.

- Ponding occurs when,
  Rainfall intensity > Soil infiltration capacity. or potential infiltration rate.

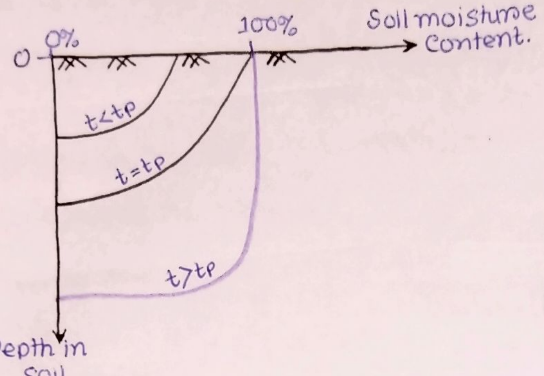


Figure 18.2:- Soil moisture profiles before, during and after ponding occurs.

I.T (# Initially, Just below the surface, soil is unsaturated. So, there is suction pressure. So, ponding is not possible.
Then time go ahead, water infiltrates. Wetting front moves downward and soil gets saturated. So, now, there is no suction. water starts ponding.)

- Mein and Larson (1973) determined ponding time by Green-Ampt equation.

They assumed, rainfall with intensity i started instanteneously and continuing indefinitely.

Three principles involved:-

(i) Prior to time ponding occurs, all the rainfall is infiltrated.

(ii) The potential infiltration rate f is a function of cumulative infiltration rate i.e $f = f(F)$.

(Cumulative infiltration, F varies with time i.e increases. So, potential infiltration rate is not a constant quantity !?)

(iii) Ponding occurs when potential infiltration rate is less than or equal to rainfall intensity. i.e, $f \leq i$

- Relation between f and F (from 70.7),

$$f = K\left(\frac{\psi \Delta\theta}{F} + 1\right) \quad \ldots\ldots (72.1)$$

Cumulative infiltration at the ponding time t_p,

$$\boxed{F_p = i \cdot t_p} \quad \text{[From assumption (i)]} \ldots (72.2)$$

and infiltration rate, $f = i$ [from assumption (iii)] .... (72.3).

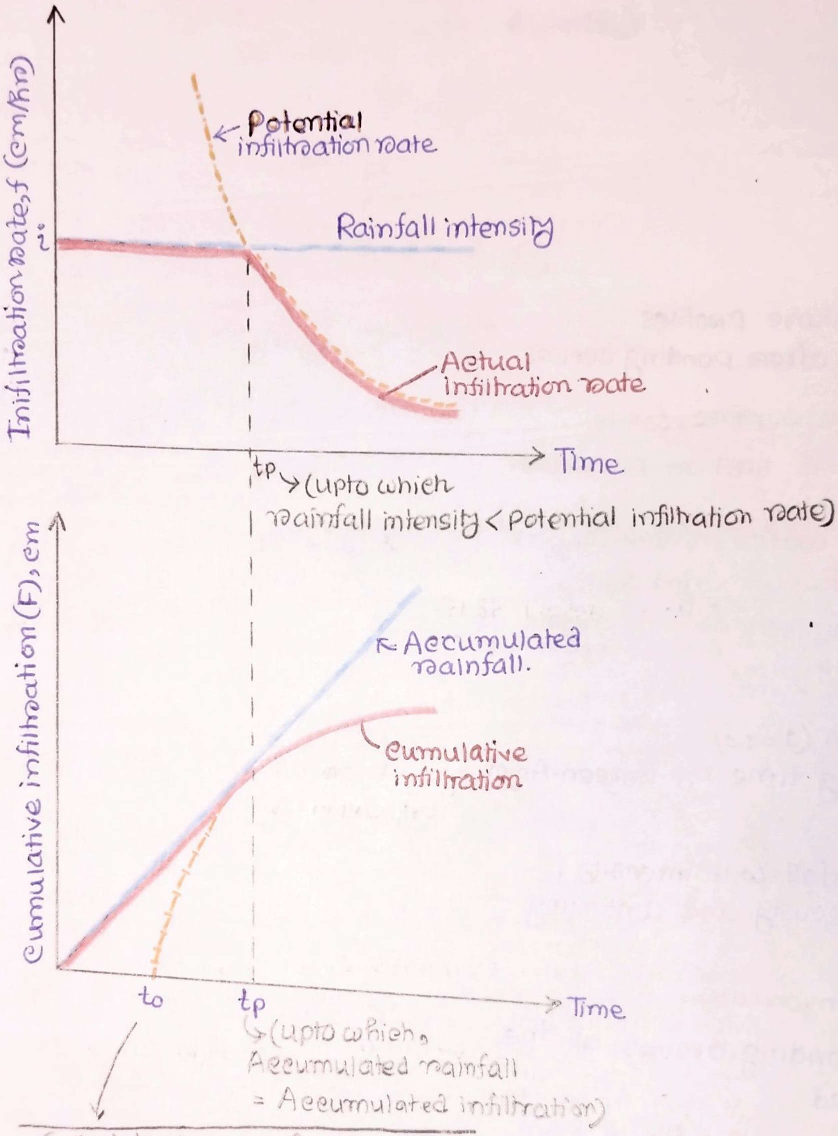
Substituating these two values in (72.1),

$$i = K\left(\frac{\Psi \Delta\theta}{i t_p} + 1\right)$$

$$\Rightarrow \frac{i}{K} - 1 = \frac{\Psi \Delta\theta}{i t_p}$$

$$\Rightarrow t_p = \frac{K \Psi \Delta\theta}{i(i-K)} \quad \ldots\ldots (72.4)$$

So, (72.4) gives ponding time under constant rainfall intensity using Green - Ampt infiltration eq^n.



(What is the significance of t_o? Just to construct the curve of cumulative rainfall as per potential infiltration rate?).

Now, Substituating $t = t_p - t_o$ and $F = F_p$ into Green-Ampt equations (70.9),
#(Looks like t_o is the initial time)!

$$F_p - \Psi\Delta\theta \ln\left(1 + \frac{F_p}{\Psi\Delta\theta}\right) = K(t_p - t_o) \quad \ldots (72.5)$$

for $t > t_p$,

$$F - \Psi\Delta\theta \ln\left(1 + \frac{F}{\Psi\Delta\theta}\right) = K(t - t_o) \quad \ldots (72.6)$$

#[For $t < t_p$, it follows (72.2), i.e., $F = i.t$]

Now, {(72.6)-(72.5)}.

$$F-F_p-\psi\Delta\theta\left[\ln\left(1+\frac{F}{\psi\Delta\theta}\right)-\ln\left(1+\frac{F_p}{\psi\Delta\theta}\right)\right]=K(t-t_p)$$

$$\Rightarrow F-F_p-\psi\Delta\theta\ln\left(\frac{\psi\Delta\theta+F}{\psi\Delta\theta+F_p}\right)=K(t-t_p) \ldots\ldots (72.7)$$

Equation (72.7) can be used to calculate depth of infiltrating after ponding.

[F_p from 72.2, t_p from 72.4 (needed to calculate F_p). $\Delta\theta, \psi \rightarrow$ these values will be given. We need to get F at time = t] !

- Total head, $h=z+\psi$. .... (73.1)

Darcy's law, $\underset{\sim}{q}=-K(\theta)\nabla h$ .... (73.2)

Relative Permeability, $K_r(\theta)=\frac{K(\theta)}{K_s}$ ... (73.3)

$$0\leqslant K_r(\theta)\leqslant 1.$$

Expanded form of Darcy's law,

$$\underset{\sim}{q}=-K_s \cdot K_r(\theta)\nabla(z+\psi)$$

$$=-K_s \cdot K_r(\theta)\underset{\sim}{e}_z-K_s \cdot K_r(\theta) * \nabla\psi \ldots (73.4)$$

(# See P-20)

- Continuity equation:-

* This is not multiplication. This is ρ^*. But what is that? (see 73.13)

$$\frac{\partial(\rho\theta)}{\partial t}+\nabla\cdot(\rho\underset{\sim}{q})=\rho * q$$ → How? (try with 68.13). ....(73.5) (See next page!)

- Temporal term,

$$\frac{\partial(\rho\theta)}{\partial t}=\frac{\partial}{\partial t}(\rho\eta S_w)$$ [I.T ⇒ Moisture content, θ = Porosity (η) × Degree of saturation (S_w)]

$$=\eta S_w\frac{\partial\rho}{\partial t}+\rho S_w\frac{\partial\eta}{\partial t}+\eta\rho\frac{\partial S_w}{\partial t} \ldots (73.6)$$

In equation (73.6),

First term, $\frac{\partial\rho}{\partial t}=\frac{\partial\rho}{\partial p}\cdot\frac{\partial p}{\partial t}+\frac{\partial\rho}{\partial c}\cdot\frac{\partial c}{\partial t}$ [c = soil water capacity = $\frac{d\theta}{d\psi}$ (see 68.27)!?]

$$=\rho\beta\frac{\partial p}{\partial t}+\frac{\partial\rho}{\partial c}\cdot\frac{\partial c}{\partial t}$$ [ρ is function of pressure (p) and soil water capacity (c)]

..... (73.7).

For (73.7),

We know, $K=\frac{\Delta p}{-\frac{\Delta V}{V}}$

ρV = constant

$\Delta(\rho V)=0$

$\Rightarrow \rho\Delta V+V\Delta\rho=0$

$\Rightarrow \frac{\Delta\rho}{\rho}=-\frac{\Delta V}{V}$

$\therefore K=\frac{\Delta p}{\left(\frac{\Delta\rho}{\rho}\right)}$

$\Rightarrow \frac{K}{\rho}=\frac{\Delta p}{\Delta\rho}$

$\Rightarrow \frac{1}{\rho\beta}=\frac{\Delta p}{\Delta\rho} \Rightarrow \rho\beta=\frac{\Delta\rho}{\Delta p}=\frac{\partial\rho}{\partial p}$

[∵ Bulk modulus (K) = $\frac{1}{\text{compressibility of water}(\beta)}$]

$\therefore \frac{\partial\rho}{\partial p}$ is substituted as $\rho\beta$ in (73.7).

* [I think, in equations (73)'s all the pressures used are suction pressures. Because, in (73.10) $\rho g\psi=p$ was written.]
(Doubt! 73.14) See.

For continuity equation (73.5),

$$\frac{\partial}{\partial t}(\rho\theta) + \nabla\cdot(\rho q) = \rho q^*$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\theta) = \rho q^* - \nabla\cdot(\rho q)$$ → Dimiensionally terms are not matching!?

[Rate of change of mass within the volume.]

[Rate at which mass is added or extracted.] This term takes into account any source or sink of mass within the system.

[Inflow or outflow of mass to a control volume]. It is divergence of mass flux vector.

ρq represents amount of mass flowing per unit area per unit time.

Second term of (73.6),

$$\frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial p}\cdot\frac{\partial p}{\partial t} = (1-\eta)\alpha\frac{\partial p}{\partial t} \quad \cdots\cdots (73.8)$$

For (73.8),

From (8.9), compressibility of porous media,

$$\alpha = \frac{1}{V_T}\cdot\frac{dV_T}{dp}$$

$V_T = V_V + V_S$

$\Rightarrow dV_T = dV_V + dV_S$

$\Rightarrow dV_T = dV_V$

[we consider, solid particles are incompressible, $dV_S = 0$

Now, $$\frac{\partial \eta}{\partial p} = \frac{\partial}{\partial p}\left(\frac{V_V}{V_T}\right)$$

$$= \frac{V_T\frac{\partial V_V}{\partial p} - V_V\frac{\partial V_T}{\partial p}}{V_T^2}$$

$$= \frac{1}{V_T}\frac{\partial V_V}{\partial p} - \frac{V_V}{V_T}\cdot\frac{1}{V_T}\frac{\partial V_T}{\partial p}$$

$$= \frac{1}{V_T}\frac{\partial V_T}{\partial p} - \eta\cdot\frac{1}{V_T}\frac{\partial V_T}{\partial p}$$

$= \alpha - \eta\alpha = \alpha(1-\eta) \Rightarrow$ This is used in (73.8).

⊙ Discharge vector,

$$\underset{\sim}{q} = -\underset{\sim}{K}\cdot\nabla h$$

$$= -\underset{\sim}{K}\cdot\nabla(\psi + z) \quad \text{(from 73.1)}$$

Now, if we replace permeability (K) with intrinsic permeability,

$$q = -\frac{k\rho g}{\mu}\cdot\nabla(\psi + z) \quad \cdots\cdots (73.9)$$

$$= -\frac{k}{\mu}\cdot\nabla(\rho g\psi + \rho g z)$$

$$= -\frac{k}{\mu}\cdot(\nabla p + \rho g\nabla z) \quad \cdots (73.10)$$

[ψ is suction head and p is suction pressure]

⊙ Now, continuity equation from (73.5), becomes,

$$\frac{\partial}{\partial t}(\rho\theta) + \nabla\cdot(\rho q) = \rho^* q$$

$$\Rightarrow \eta S_w \frac{\partial \rho}{\partial t} + \rho S_w \frac{\partial \eta}{\partial t} + \eta\rho\frac{\partial S_w}{\partial t} + \nabla\cdot(\rho q) = \rho^* q \quad \text{(\# From 73.6)}$$

$$\Rightarrow \eta S_w\left[\rho\beta\frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial c}\cdot\frac{\partial c}{\partial t}\right] + \rho S_w(1-\eta)\alpha\frac{\partial p}{\partial t} + \eta\rho\frac{\partial S_w}{\partial t} - \nabla\cdot\left[\frac{\rho k}{\mu}\cdot(\nabla p + \rho g\nabla z)\right] = \rho^* q$$

[Putting $\frac{\partial \rho}{\partial t}$ from (73.7), $\frac{\partial \eta}{\partial t}$ from (73.8) and q from (73.10)]

$$\Rightarrow \left[\eta S_w\rho\beta + \rho S_w(1-\eta)\alpha\right]\frac{\partial p}{\partial t} + \eta S_w\frac{\partial \rho}{\partial c}\cdot\frac{\partial c}{\partial t} + \eta\rho\frac{\partial S_w}{\partial t}$$

$$= \nabla\cdot\left[\frac{\rho k}{\mu}\cdot(\nabla p + \rho g\nabla z)\right] + \rho^* q.$$

$$\Rightarrow \left[\theta\rho\beta + \frac{\rho\theta}{\eta}(1-\eta)\alpha\right]\frac{\partial p}{\partial t} + \theta\frac{\partial \rho}{\partial c}\cdot\frac{\partial c}{\partial t} + \rho\eta\frac{\partial S_w}{\partial t}$$

$$= \nabla\cdot\left[\frac{\rho k}{\mu}\cdot(\nabla p + \rho g\nabla z)\right] + \rho^* q.$$

[Here, Porosity (η) × Degree of saturation (S_w) was replaced by moisture content (θ).
In 2nd term, $S_w = \frac{S_w\cdot\eta}{\eta} = \frac{\theta}{\eta}$ was written]

∴ Rearranging,

$$\rho\left[\theta\beta + \frac{\theta}{\eta}(1-\eta)\alpha\right]\frac{\partial p}{\partial t} + \theta\frac{\partial \rho}{\partial c}\cdot\frac{\partial c}{\partial t} + \rho\eta\frac{\partial S_w}{\partial t}$$

$$= \nabla\cdot\left[\frac{\rho k}{\mu}\cdot(\nabla p + \rho g\nabla z)\right] + \rho^* q \quad \ldots\ldots (73.11).$$

⊙ Now,

$p = \rho g \psi$ [From 73.10]

$\Rightarrow dp = \rho g\, d\psi$ ---------- (73.11?)

$\Rightarrow dp = \rho g \frac{d\psi}{d\theta}\cdot d\theta$

$= \rho g \frac{d\psi}{d\theta}\, d(\eta S_w)$ [∵ $\theta = \eta S_w$, from 73.6]

$= \rho g \frac{d\psi}{d\theta}\left[\eta\, dS_w + S_w\, d\eta\right]$

$\approx \rho g \frac{d\psi}{d\theta}\cdot(\eta\, dS_w)$.

[∵ $d\eta \ll dS_w$. I.T ⇒ Porosity of the porous media does not change much.
But, degree of saturation may change drastically depending on moisture availability).

Rearranging,

$\Rightarrow \frac{1}{g}\frac{d\theta}{d\psi}\, dp = \rho\eta\frac{\partial S_w}{\partial t}\, dS_w$

$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{g}\frac{d\theta}{d\psi}\, dp\right) = \rho\eta\frac{\partial}{\partial t}(S_w)$ # $C = \frac{d\theta}{d\psi}$ = soil water capacity. I think this does not change with time.

$\Rightarrow \frac{1}{g}\cdot\frac{d\theta}{d\psi}\frac{\partial p}{\partial t} = \rho\eta\frac{\partial S_w}{\partial t} \quad \ldots\ldots (73.12)$

Now, substituting this value in eq^n (73.11),

$$\rho\left[\theta\beta + \frac{\theta}{\eta}(1-\eta)\alpha\right]\frac{\partial p}{\partial t} + \theta\frac{\partial \rho}{\partial c}\cdot\frac{\partial c}{\partial t} + \frac{1}{g}\frac{d\theta}{d\psi}\frac{\partial p}{\partial t}$$

$$= \nabla\cdot\left[\frac{\rho k}{\mu}\cdot(\nabla p + \rho g\nabla z)\right] + \rho^* q \quad \ldots\ldots (73.13)$$

Reference pressure head,

$$h = \frac{p}{\rho_0 g} \cdots\cdots (73.14)$$

Relationship between pressure head and reference pressure head,

$$\rho_0 dh = \rho d\psi \cdots (73.15)$$

I think, $p = \rho_0 g h$ (From 73.14) $\Rightarrow dp = \rho_0 g dh$

$p = \rho g \psi$

$\Rightarrow dp = \rho g d\psi$ (from 73.11.2)

$\therefore$ Comparing these two,

$\rho_0 g dh = \rho g d\psi$

$\Rightarrow \rho_0 dh = \rho d\psi$ (This is 73.15)

Still doubt.... why ψ is pressure head and h is reference pressure head? What is the difference between ρ and ρ_0?

Using "$p = \rho_0 g h$" $\rightarrow$ (73.14) continuity equation (73.13) can be written as,

$$\rho\left[\theta\beta + \frac{\theta}{\eta}(1-\eta)\alpha\right]\rho_0 g \frac{\partial h}{\partial t} + \theta \frac{\partial \rho}{\partial c}\cdot\frac{\partial c}{\partial t} + \frac{1}{g}\cdot\frac{\rho}{\rho_0}\frac{d\theta}{dh}\rho_0 g\frac{\partial h}{\partial t}$$

$$= \nabla\cdot\left[\frac{\rho k}{\mu}\cdot(\rho_0 g \nabla h + \rho g \nabla z)\right] + \rho^* q \cdots\cdots (73.16)$$

3rd term of equation (73.15),

$$\frac{1}{g}\cdot\frac{d\theta}{d\psi}\cdot\frac{\partial p}{\partial t} = \frac{1}{g}\cdot\frac{d\theta}{dh}\frac{\partial h}{\partial \psi}\cdot\frac{\partial}{\partial t}(\rho_0 g h)$$

$$= \frac{1}{g}\cdot\frac{d\theta}{dh}\cdot\left(\frac{\rho}{\rho_0}\right)\rho_0 g\frac{\partial h}{\partial t}$$

$\left[\frac{\partial h}{\partial \psi} = \frac{\rho}{\rho_0} \text{ is from } (73.15)\right]$

Let us define modified compressibilities (Not actual compressibility, just like terms to simplify expression 73.16), of media and water as,

$$\alpha' = (1-\eta)\rho_0 g\,\alpha \cdots\cdots (73.17)$$

$$\beta' = \beta\rho_0 g \cdots\cdots (73.18)$$

Dividing both sides of (73.16) by ρ_0 and substituating α' and β' values from (73.17) and (73.18), we get,

$$\frac{\rho}{\rho_0}\left[\theta\beta' + \frac{\theta}{\eta}\alpha'\right]\frac{\partial h}{\partial t} + \frac{\theta}{\rho_0}\frac{\partial \rho}{\partial c}\cdot\frac{\partial c}{\partial t} + \frac{\rho}{\rho_0}\frac{d\theta}{dh}\frac{\partial h}{\partial t}$$

$$= \nabla\cdot\left[\frac{\rho k}{\mu\rho_0}\cdot(\rho_0 g\nabla h + \rho g \nabla z)\right] + \frac{\rho^*}{\rho_0}q$$

$$\Rightarrow \frac{\rho}{\rho_0}\left[\theta\beta' + \frac{\theta}{\eta}\alpha' + \frac{d\theta}{dh}\right]\frac{\partial h}{\partial t} + \frac{\theta}{\rho_0}\frac{\partial \rho}{\partial c}\frac{\partial c}{\partial t}$$

$$= \nabla\cdot\left[\frac{\rho g k}{\mu}\left(\nabla h + \frac{\rho}{\rho_0}\nabla z\right)\right] + \frac{\rho^*}{\rho_0}q \cdots\cdots (73.19)$$

In compact form,

$$\frac{\rho}{\rho_0}F\cdot\frac{\partial h}{\partial t} + \frac{\theta}{\rho_0}\frac{\partial \rho}{\partial c}\frac{\partial c}{\partial t} = \nabla\cdot\left[\underline{K}\cdot\left(\nabla h + \frac{\rho}{\rho_0}\nabla z\right)\right] + \frac{\rho^*}{\rho_0}q$$

$$\cdots (73.20)$$

Where, Permeability, $K = \frac{\gamma k}{\mu} = \frac{\rho g k}{\mu} \cdots (73.21)$

where,

$$\frac{\rho}{\rho_0} = a_0 + a_1 c + a_2 c^2 + a_3 c^3 \ldots \quad (73.22)$$

$$\frac{\mu}{\mu_0} = b_0 + b_1 c + b_2 c^2 + b_3 c^3 \ldots \quad (73.23)$$

These ratio's are functions of soil water capacity $(c) = \frac{d\theta}{d\psi}$! why?!
what are ρ, ρ_0, μ, μ_0?!

⊙ In case of seawater intrusion,

$$\frac{\rho}{\rho_0} = 1 + \varepsilon c \ldots \quad (73.24)$$

$$\varepsilon = \frac{\rho_{max}}{\rho_0} - 1 \ldots \quad (73.25).$$

Lecture 19

Groundwater Contamination

To detect GW contamination level.

☑ Sources of groundwater Contamination:

- Disposal of solid wastes.
- Underground petroleum tank leakage.
- Disposal of liquid waste.
- Sewage disposal on land.
- Agricultural activities (infiltration of pesticides and

Lechate spring:
Lechate, a liquid, forms as water interacts with waste materials within a landfill or waste disposal site.
When waste is disposed of in a landfill, rainwater or other sources of water infiltrate the waste mass, extracting soluble masses, forms leachate. They accumulate at bottom of waste layers, forming a pool.
The pressure from the accumulated leachate can cause it to flow or seep out of landfill (formation of "leachate spring") and transport contaminants to groundwater.

fertilizers).

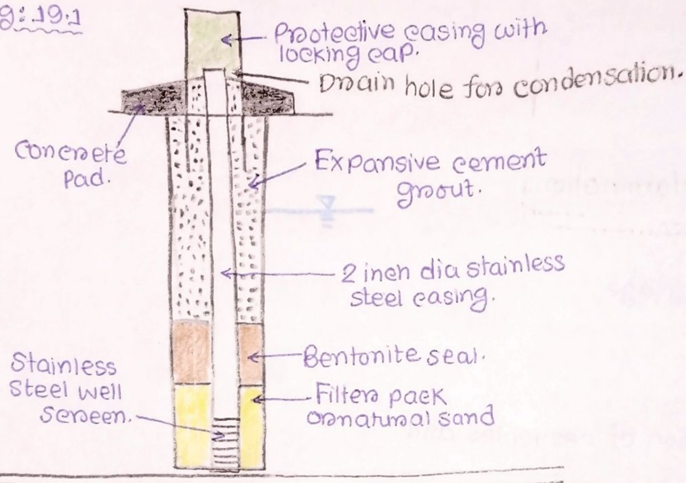
- Infiltration of de-icing chemical from roadways to water table.

De-icing chemicals:-
Lowers freezing point of water, causes melting of existing ice.

- NaCl: Inexpensive and effective. But, causes corrosion to infrastructure, damage vegetation and contaminants waterbodies
- $CaCl_2$:- More effective at lower temperature than NaCl. But, expensive.
- $MgCl_2$:- Less environmental impact than NaCl.
- Potassium acetate (CH_3COOK): Less harmful to environment. Used in airports and environmentally sensitive areas.

Groundwater Monitoring Well:-

Fig: 19.1



Groundwater monitoring well measure and monitor characteristics of GW such as its level, quality, flow direction.

Components:-

(i) Casing:- Pipe that extends from ground surface to the depth of targeted aquifer.
Made of PVC or Stainless steel.
It provides stability and prevents the collapse of well bore.

(ii) Screen: Lower section of casing, Perforated.
Allows GW to enter the well, filters large particles and sediments.

(iii) Gravel Pack: Space between the screen and the casing filled with layers of gravel or sand known as gravel pack.
It helps to stabilize the surrounding formation, prevents fine sediments from entering the well, and faciliates flow into the well.

(iv) Well cap:-
At top of the casing cap or cover.
It protects the well from entering unwanted substances to it.

(v) Monitoring Equipments:-
Are used to collect data.
Eg. Water level loggers, Pressure transducers, temperature sensors, GW sampling devices.

(vi) Well development:-
Techniques such as surging, pumping, jetting are performed to remove fine sediment and improve hydraulic ~~conductivity~~ connectivity between the well and surrounding Aquifers.

(vii) Annular seal (Bentonite):-
Annular seal such as bentonite clay or grout, is placed between the casing and the surrounding formation to prevent the migration of water and contaminents from well to outside aquifer.

Monitoring Network:-

Some common considerations for positioning wells in groundwater monitoring network:-

(i) Well distribution:-
Required to capture spatial variability of groundwater system.

It should consider variations in hydrogeological properties, potential contamination area sources, known or suspected area of concern.

(ii) Hydrogeological setting:-
Presence of aquifers, confining layer, fractures, faults, groundwater flow patterns influence the placement of monitoring wells.

Well should be positioned to capture groundwater at different depths within different hydrogeological units to provide a comprehensive understanding of the groundwater system.

(iii) Well spacing:-
Based on different factors like size of groundwater plume or affected area, flow characteristics of aquifer, regulatory requirements.

(iv) Well depth:-

Should extend below the water table.
The depth should be sufficient to capture relevant hydrogeological units or contaminant migration pathways.

(v) Screen Placement:-

It should be positioned within relevant hydrogeological unit or depth range to allow representative sampling and monitoring.

(vi) Well network design:-

It should consider factors such as location of known or suspected contamination sources, the direction and velocity of groundwater flow. regulatory requirements.

It may involve (i.e well network) a combination of upgradient, down-gradient and lateral wells to capture the flow direction and asses the extent of contamination.

(vii) Transect or line configuration:

Wells are arranged in a line or transect across the target area, typically ⊥r to expected GW flow direction.
(Then how can we get gradient in direction of the flow ?!).
This configuration allows for the characterization of groundwater flow gradients, contamination plumes, or other spatial variations in GW quality.

(viii) Grid configuration:-

It provides a more comprehensive coverage of the monitoring area.
Grid configurations are usally used in large-scale monitoring programs.

(ix) Backward Wells:-

These wells are placed in areas which are considered as representative of backward or reference condition. These wells are typically located away from potential contaminant. Sources and are used to establish baseline groundwater quality data for comparison with other monitoring wells.

(x) Compliance Wells:-

Purpose is to monitor or evaluate GW quality in vicinity of the of the regulated facility or activity. The specific distance and placement may be dictated by regulatory guidelines to ensure that wells capture potential impacts on groundwater.

⊙ Long-term GW monitoring:

- Ambient monitoring:- Regional, annual monitoring for water safety.
- Detection monitoring:- watch a dangerous spot.
- Compliance monitoring: Evaluate the progress of a management policy.
- Research monitoring:- Monitoring for a specific research purpose.

⊙ GW constituents and Contaminents:

(i) Inorganic:-

- Nitrogen:- Main form is NO_3^-. others forms are:- NH_4^+, NH_3, NO_2^-, N_2, N_2O, organic nitrogen.
- Metals:- Cd, Cr, Cu, Hg, Fe, Mn, Zn,
- Non-metals:- Carbon, chlorine, sulpher, nitrogen, fluorides, As, Selenium (Se), Phosphorus (P), Boron (B).

(ii) Organic:-

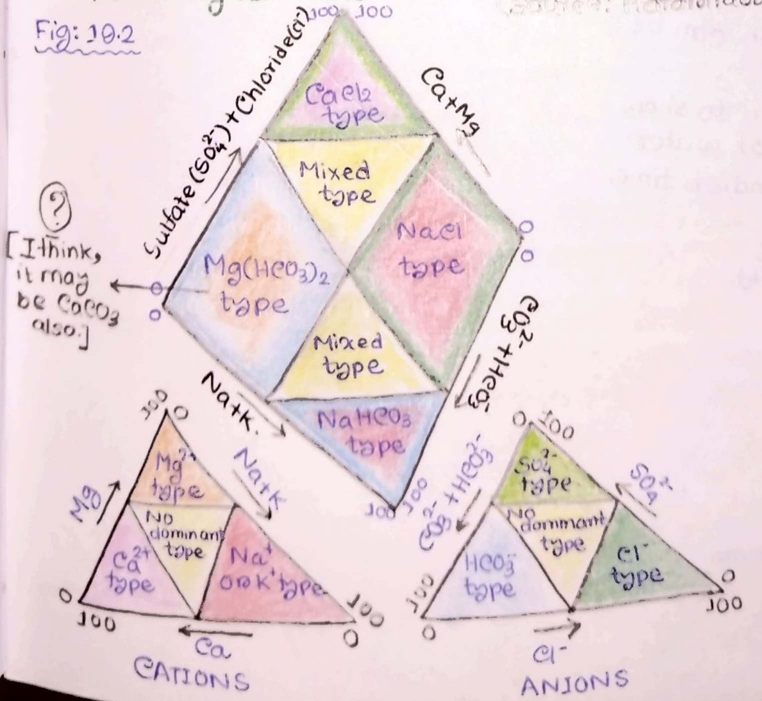
Carbon is the key element in organic compound.
The compositions H_2CO_3, CO_2, HCO_3^-, CO_3^{2-} are some exceptions that are not considered as organic components.

(iii) Dissolved gases:-
CO_2, O_2, N_2.

⊙ Piper diagram for Water chemistry Analysis:

(Source: Hatarilabs.com)

Fig: 10.2

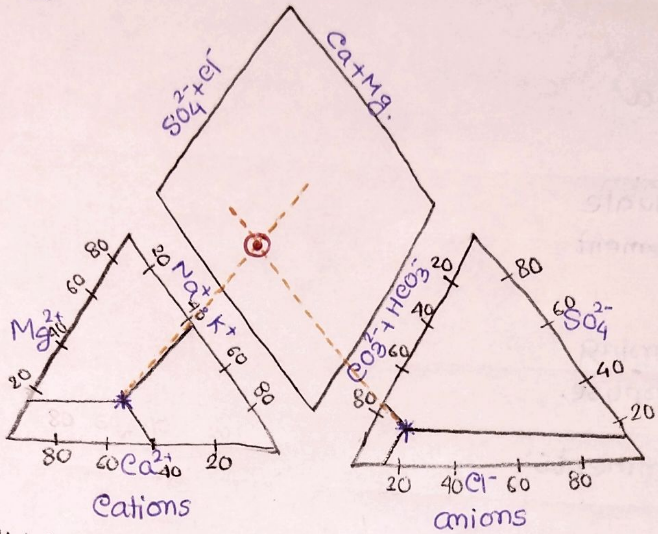


Example:-

Data for a sample is given here

Cations (%meq/L)		Anions	%meq/L
Ca^{2+}	40.4	HCO_3^-	77.8
Mg^{2+}	17	SO_4^{2-}	11.6
$Na^+ + K^+$	42.6	Cl^-	10.6

Fig: 19.3



#[I think, the lines drawn to get '*' are parallel to sides of triangle.]

#So, "⊙" is the position of sample in piper diagram. To know the type of the sample, superimpose the point in figure 19.2. It looks like $Mg(HCO_3)_2$ type.

☑ Stiff Diagram:

Graphical representation to display major ion composition of a water sample. A polygonal shape is created from four parallel horizontal axes ~~and~~ (Though, Fe vs CO_3 is often neglected, it is optional) extending on either side of a vertical zero axis. Cations are plotted in meqv/L on the left side and anions on the right side. Stiff diagram can be used:-

(i) To help visualize ionically related waters from which a flow path can be determined.

(ii) If the flow path is known, to show how the ionic composition of water body changes over space and/or time.

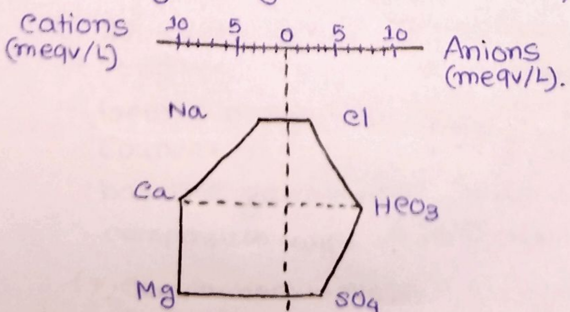


Fig 19.4:- Example of stiff diagram.

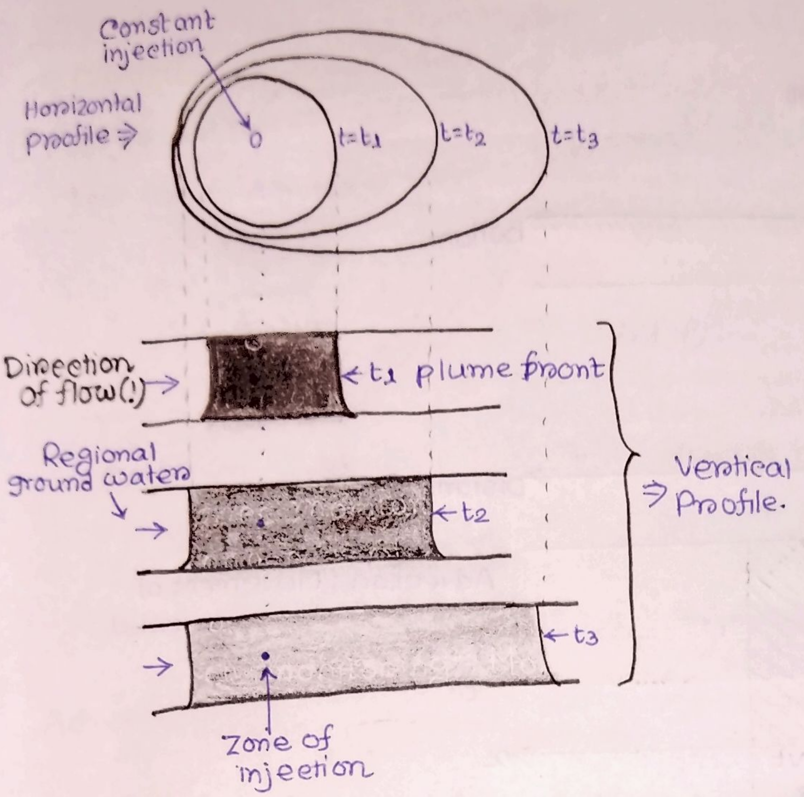
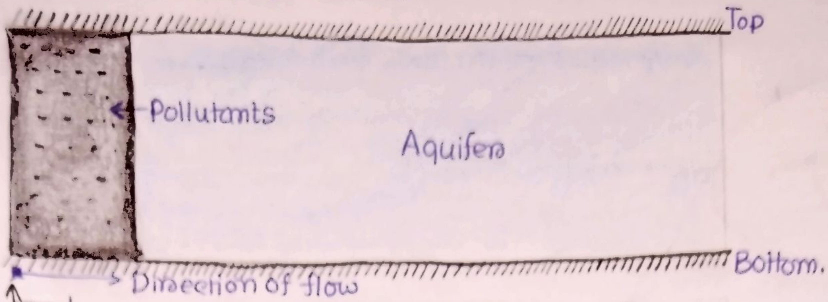
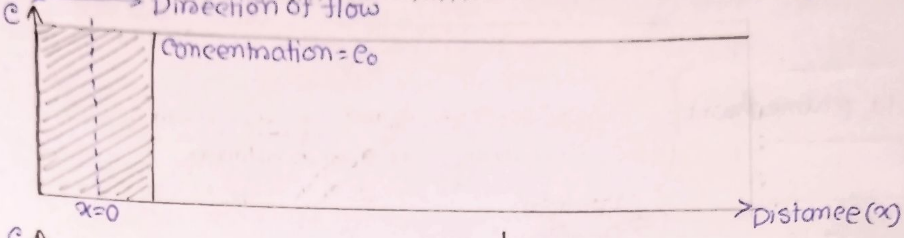


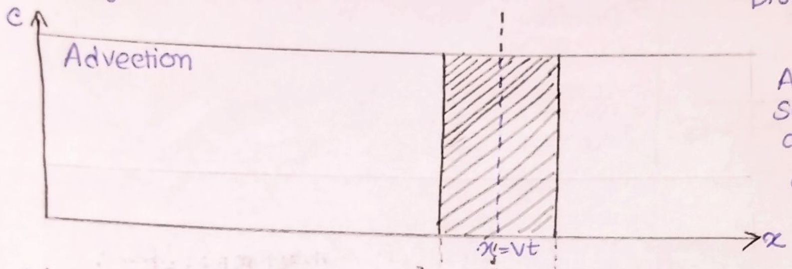
Figure 19.5:- Potentiometric mount caused by waste disposal injection and the expansion of affected zone occupied at times t_1, t_2, t_3.

(P.T.O)

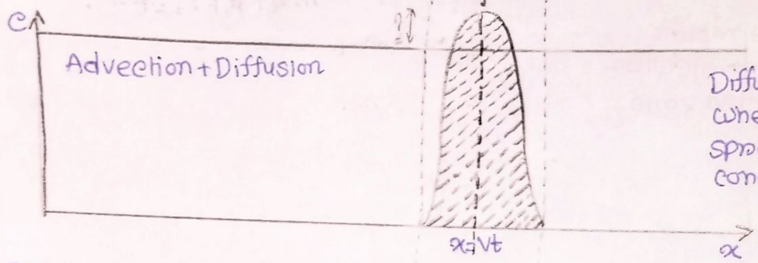
Pollutant distribution at t=0,



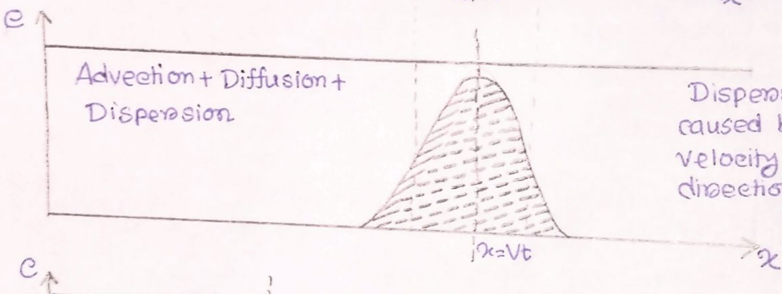




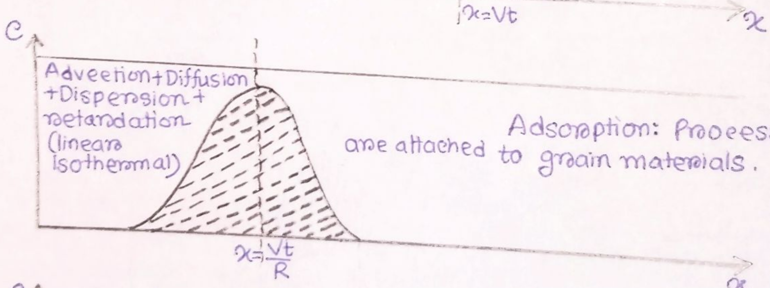
Advection: Movement of solutes that are carried along with flowing groundwater.



Diffusion: Molecular process where constituents are spread due to differences in concentrations.

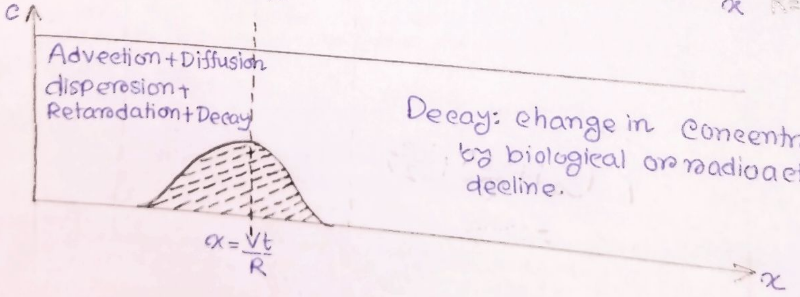


Dispersion:- Mixing process caused by differences in velocity (in magnitude and in direction) of water particles.



Adsorption: Process where certain constituents are attached to grain materials.

R = Retardation factor



Decay: change in concentration by biological or radioactive decline.

Figure:- 19.6 Movement of affected zone with time (x axis) and change of its concentration (y axis) considering various combinations of processes.

? (Are these figures for 1D flow only!?)

AI → * Mass flux vector
It represents flow of mass per unit area per unit time.
$\underline{F} = \rho \underline{v}$, where, $\underline{v}$ = velocity vector.

Lecture-20
Groundwater Contamination:
continued after lecture 19,

⊙ Advection:-
For saturated condition,

$$\left.\begin{aligned} F_{Ax} &= q_x C = \eta v_x C \\ F_{Ay} &= q_y C = \eta v_y C \\ F_{Az} &= q_z C = \eta v_z C \end{aligned}\right\} \ldots (74.1's)$$

At saturated condition, water content = η, otherwise θ.
q_x, q_y, q_z = Specific discharge or Darcy's velocity.
v_x, v_y, v_z = Seepage velocity or actual velocity.

For unsaturated condition,

$$\left.\begin{aligned} F_{Ax} &= q_x C = \theta v_x C \\ F_{Ay} &= q_y C = \theta v_y C \\ F_{Az} &= q_z C = \theta v_z C \end{aligned}\right\} \ldots (74.2's)$$

Advection mass flux vector,

$$\vec{F_A} = \vec{q}\, C \ldots\ldots (74.3).$$

i.e, $(F_{Ax}\hat{i} + F_{Ay}\hat{j} + F_{Az}\hat{k}) = q_x C\hat{i} + q_y C\hat{j} + q_z C\hat{k}$
$= (q_x\hat{i} + q_y\hat{j} + q_z\hat{k})C$

From AI, Advection eqn $\frac{\partial c}{\partial t} + V \cdot \nabla c = 0$
V = velocity vector of GW flow.
∇c = Spatial variation of contaminant conc.

⊙ Molecular Diffusion:
Diffusion is a net transport of molecules from a region of higher concentration to lower concentration by random molecular motion.

Under steady-state condition, Fick's first law describe the flux of a solute as:-

$$\left.\begin{aligned} F_{Diff\,x} &= -\eta D^* \frac{\partial c}{\partial x} = -\eta \tau D_m \frac{\partial c}{\partial x} \\ F_{Diff\,y} &= -\eta D^* \frac{\partial c}{\partial y} = -\eta \tau D_m \frac{\partial c}{\partial y} \\ F_{Diff\,z} &= -\eta D^* \frac{\partial c}{\partial z} = -\eta \tau D_m \frac{\partial c}{\partial z} \end{aligned}\right\} \ldots (74.4)$$

D^* = Effective molecular diffusion coff.
D_m = Molecular diffusion coff in open water
$\tau \Rightarrow$ Tortuosity of the porous medium (<1)
Range 0.01 - 0.5.

Diffusion moss flux vector,

$\vec{F} = \eta D^* \nabla C$ (# For saturated condition) ....(74.5)

$\Rightarrow F_{Diff\,x}\hat{i} + F_{Diff\,y}\hat{j} + F_{Diff\,z}\hat{k} = -\eta D^*\left(\frac{\partial c}{\partial x}\hat{i} + \frac{\partial c}{\partial y}\hat{j} + \frac{\partial c}{\partial z}\hat{k}\right)$

$\vec{F} = -\theta D^* \nabla C$ (# For unsaturated condition) ....(74.6).

⊙ Dispersion or mechanical dispersion:-

$$F_{Disp\,x} = -\eta D_{xx}\frac{\partial c}{\partial x} - \eta D_{xy}\frac{\partial c}{\partial y} - \eta D_{xz}\frac{\partial c}{\partial z} \ldots (74.7)$$

$$F_{Disp\,y} = -\eta D_{yx}\frac{\partial c}{\partial x} - \eta D_{yy}\frac{\partial c}{\partial y} - \eta D_{yz}\frac{\partial c}{\partial z} \ldots (74.8)$$

$$F_{Disp\,z} = -\eta D_{zx}\frac{\partial c}{\partial x} - \eta D_{zy}\frac{\partial c}{\partial y} - \eta D_{zz}\frac{\partial c}{\partial z} \ldots (74.9)$$

Physical mechanisms responsible for "Mechanical dispersion" are:-

- The particles nearest the wall of the pore channel move more slowly than near the channel center.

- The variations of pore dimensions along the pore axes cause the particles to move at different relative speeds. (9 dispersion coff s to cover all possible directions?!).

- Adjacent particles in one channel can flow different streamlines that lead to different channels.

  These particles may later come together in another channel or they may continue to move farther apart.

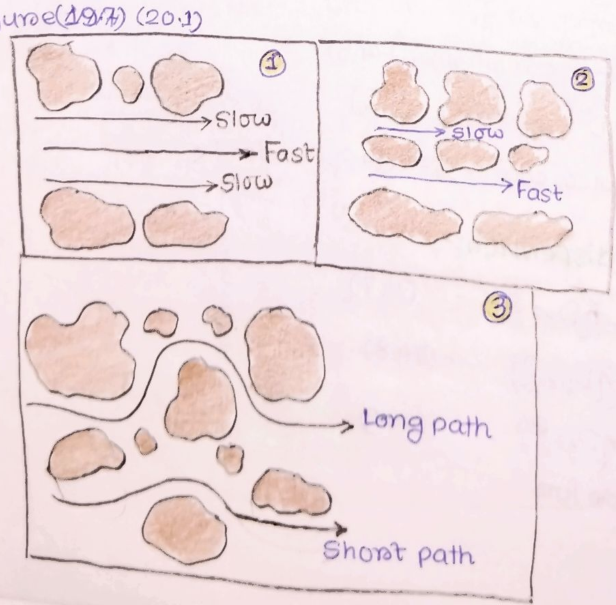
- Solute mole-cules move at different speeds (even when hydraulic gradient is uniform) due to heterogenous hydraulic conductivity field. ...... (74.10)

⊙ Longitudinal dispersion occurs based on the following reasons.

- Fluids move faster through the center of the pores than along the edges. (See fig-19.7.1)
- Some portions of fluid travel in longer pathways than other portions (# So, fuid particles in longer path lags behind). (see fig 19.7.3)
- The movement of fluid through larger pores is faster. (see fig-19.7.2)
- Heterognity of occurs the aquifers cause the groundwater to move faster in some layers and slower in some others. .... (74.11)

So, these four reasons cause "relative speed" of fluid particles in the medium and dispersion occurs. (See figure 19.6).

Figure(19.7) (20.1)



- Similarly, transverse dispersion may may occur if solutes spreads or dispersed in direction perpendicular to flow. [Eg. If flow occurs in x direction, then dispersion in xy or xz plane would be called transverse dispersion].
- Combining (74.7), (74.8) and (74.9),

$$\left.\begin{array}{l}\vec{F}_{Disp} = -\eta \underset{\sim}{D} \nabla c \text{ (saturated condition)} \\ \vec{F}_{Disp} = -\theta \underset{\sim}{D} \nabla c \text{ (unsaturated condition)}\end{array}\right\} \dots (74.12)$$

$$\vec{F}_{Disp} = -\eta \underset{\sim}{D} \nabla c$$

$$\Rightarrow \begin{Bmatrix} F_{Dispx} \\ F_{Dispy} \\ F_{Dispz} \end{Bmatrix} = -\eta \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \begin{Bmatrix} \partial c/\partial x \\ \partial c/\partial y \\ \partial c/\partial z \end{Bmatrix}$$

Now,

$$\left.\begin{array}{l} D_{xx} = \alpha_L \frac{v_x^2}{|v|} + \alpha_{TH} \frac{v_y^2}{|v|} + \alpha_{TV} \frac{v_z^2}{|v|} + D^* \\ D_{yy} = \alpha_L \frac{v_y^2}{|v|} + \alpha_{TV} \frac{v_z^2}{|v|} + \alpha_{TH} \frac{v_x^2}{|v|} + D^* \\ D_{zz} = \alpha_L \frac{v_z^2}{|v|} + \alpha_{TV} \frac{v_x^2}{|v|} + \alpha_{TV} \frac{v_y^2}{|v|} + D^* \\ D_{xy} = D_{yx} = (\alpha_L - \alpha_{TH}) \frac{v_x v_y}{|v|} \\ D_{xz} = D_{zx} = (\alpha_L - \alpha_{TV}) \frac{v_x v_z}{|v|} \\ D_{yz} = D_{zy} = (\alpha_L - \alpha_{TV}) \frac{v_y v_z}{|v|} \end{array}\right\} \dots (74.13\text{'s})$$

J.T.⇒ # x-y plane is horizontal plane. In equations (74.13), So, x and y will take α_{TH} for each other. z will take α_{TV} for both x and y. x, y, z will take α_L for themselves

- Comparison of longitudinal dispersivity (α_L) and transverse dispersivity (α_{TH}, α_{TV})

(i) Direction of dispersion:

$\alpha_L \rightarrow$ characterizes how solutes spread along flow direction.

$\alpha_T \rightarrow$ Characterizes how solutes dispersed perpendicular to flow direction.

(ii) Physical interpretation:-

α_L is associated with factors that affect solute spreading along flow directions i.e, velocity variation, molecular diffusion & heterogeneity in porous media in that direction.

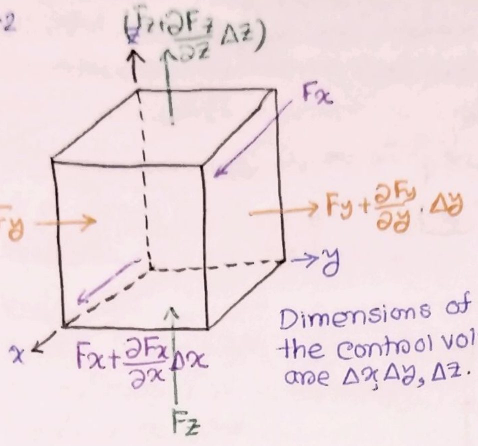
α_L is related to the factors that affect in directions perpendicular to the flow, i.e, velocity gradient (in ⊥r direction!), Preferential flow paths, structural characteristics of porous media.

(iii) Magnitude and behaviour:-

Generally, $\alpha_L > \alpha_T$ in many practical cases, as solutes tends to spread more readily along flow directions due to

the influence of bulk flow.
However, α_T plays a crucial role where solutes encounter obstacles, heterogenity and preferential flow paths perpendicular to flow direction.

Fig: 20.2



Dimensions of the control volume are $\Delta x, \Delta y, \Delta z$.

Mass conservation equation can be written as,

$$\left(F_x + \frac{\partial F_x}{\partial x}\Delta x\right)\Delta y \Delta z - F_x \Delta y \Delta z$$
$$+\left(F_y + \frac{\partial F_y}{\partial y}\Delta y\right)\Delta x \Delta z - F_y \Delta x \Delta z$$
$$+\left(F_z + \frac{\partial F_z}{\partial z}\Delta z\right)\Delta x \Delta y - F_z \Delta x \Delta y$$
$$= -\frac{\partial (\eta c)}{\partial t}\Delta x \Delta y \Delta z \cdots\cdots (74.14)$$

Why is it mass conservation eqn:-
F_x represents mass flux vector → (Here, mass flux for solute, not water, I think).
i.e mass per unit area per unit time. So, $(F_x \Delta y \Delta z)$ i.e (mass flux×area) gives mass flow rate per unit time.

Also, $-\frac{\partial (\eta c)}{\partial t}\Delta x \Delta y \Delta z$

$\equiv -\frac{\partial}{\partial t}(\underbrace{\eta \Delta x \Delta y \Delta z}_{\text{Volume of water (saturated)}} . \underbrace{c}_{\text{conc. of solute.}})$ Concluded from here!

$= -\frac{\partial}{\partial t}(\text{mass of solute})$

So, Both LHS and RHS gives rate of change of mass.

Now, why -ve sign in RHS?:-
Let, us take, outflow is more than inflow from control volume,
So, LHS of (74.14) would be +ve.

But, mass of solute is decreasing in CV (as more outflow). So, conc. will also decrease.

$\therefore \frac{\partial (\eta c)}{\partial t}$ is negative itself.

$\therefore$ To balance the sign with LHS -ve sign is taken at RHS.

Subtracting and Rearranging, from (74.14), we get,

$$\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}=-\frac{\partial(\eta c)}{\partial t}\quad\ldots(74.15)$$

In (74.15),

F_x, F_y, F_z are mass flux of solute for all possible causes. i.e advection, dispersion, adsorption and decay are considered here (See fig.-19.6).

(Diffusion not considered in (74.16) why?) → (Though in P-57, it has been told that D represents both diffusion and dispersion)!

Now, $F = F_A + F_{Disp} + F_{adsorption} + F_{decay}$.

Putting advection flux values from (74.1) and dispersion flux from (74.7), (74.8) and (74.9); (74.15) becomes,

$$\frac{\partial}{\partial x}\left(q_x c-\eta D_{xx}\frac{\partial c}{\partial x}-\eta D_{xy}\frac{\partial c}{\partial y}-\eta D_{xz}\frac{\partial c}{\partial z}\right)$$
$$+\frac{\partial}{\partial y}\left(q_y c-\eta D_{yx}\frac{\partial c}{\partial x}-\eta D_{yy}\frac{\partial c}{\partial y}-\eta D_{yz}\frac{\partial c}{\partial z}\right)$$
$$+\frac{\partial}{\partial z}\left(q_z c-\eta D_{zx}\frac{\partial c}{\partial x}-\eta D_{zy}\frac{\partial c}{\partial y}-\eta D_{zz}\frac{\partial c}{\partial z}\right)$$
$$-q_s c_s-\sum_{n=1}^{N}R_n=-\frac{\partial(\eta c)}{\partial t}\quad\ldots(74.16)$$

Not sure! May be wrong! Verify later!

From AJ,

$-q_s c_s$ ⇒ Represents source/sink term that accounts for any external input or removal of solute.

~~I think, for this problem, this term is adsorption related. As mass is decreasing due to adsorption, so -ve sign is there.~~

But what q_s, c_s individually means?

? → See (74.22)

$-\sum_{n=1}^{N}R_n$ ⇒ Represents the reaction term which incorporates any chemical reactions or transformation of solute. For this problem, decay of mass of solute is occurring, hence, -ve sign

→ Because, R_n contains both retardation (Adsorption) and decay. Then what is "$q_s c_s$"?

Rearranging (74.16),

$$\frac{\partial}{\partial t}(\eta c)=\frac{\partial}{\partial x}\left(\eta D_{xx}\frac{\partial c}{\partial x}+\eta D_{xy}\frac{\partial c}{\partial y}+\eta D_{xz}\frac{\partial c}{\partial z}\right)-\frac{\partial}{\partial x}(q_x c)$$
$$+\frac{\partial}{\partial y}\left(\eta D_{yx}\frac{\partial c}{\partial x}+\eta D_{yy}\frac{\partial c}{\partial y}+\eta D_{yz}\frac{\partial c}{\partial z}\right)-\frac{\partial}{\partial y}(q_y c)$$
$$+\frac{\partial}{\partial z}\left(\eta D_{zx}\frac{\partial c}{\partial x}+\eta D_{zy}\frac{\partial c}{\partial y}+\eta D_{zz}\frac{\partial c}{\partial z}\right)-\frac{\partial}{\partial z}(q_z c)$$
$$+q_s c_s+\sum_{n=1}^{R}R_n.\quad\ldots\ldots(74.17)$$

⊙ Sorption:-

Adsorption and absorption.

Freundlich Isotherm can be described as,

$$\bar{c}=K_f c^a \approx k_d.c\quad\ldots\ldots(74.18)$$

$\bar{c}$ looks like $\frac{x}{m}$ in equation (74.20)

(?) But, why "$k_d.c$"? This was not the form of eqn.

Slope of isotherm, $\frac{\partial\bar{c}}{\partial c}=K_f a c^{a-1}=K_d\cdots\ (74.19)$.

Source: Wikipedia.

Freundlich adsorption isotherm:-
It is an empirical relationship between 'quantity of gas absorbed into a solid surface' and 'gas pressure'.

The same relationship is also applicable for 'the concentration of a solute adsorbed on the surface of a solid' and 'concentration of the solute in liquid phase'.

For solute case, Freundlich adsorption isotherm can be written as,

$$\frac{x}{m} = kc^{1/n} \ldots\ldots (74.20)$$

x = mass of absorbate
m = mass of absorbant
$\left(\frac{x}{m}\right)$ = Adsorbate concentration $= \bar{c}$ (!)

c = Equilibrium conc. of absorbate in case of experiments made with an aquous solution in contact with a dispersed solid phase.

k, n = constants for a given absorbate and absorbent at a given temperature.

Rate of change of sorbed concentration,

$$\frac{\partial \bar{c}}{\partial t} = \frac{\partial \bar{c}}{\partial c} \cdot \frac{\partial c}{\partial t} = k_d \cdot \frac{\partial c}{\partial t} \quad (\text{\# From } 74.19) \ldots (74.21)$$

Chemical source/sink term,

$$\sum_{n=1}^{N} R_n = -\rho_b \frac{\partial \bar{c}}{\partial t} = -\rho_b k_d \frac{\partial c}{\partial t} \ldots (74.22)$$

(what is ρ_b here)??

⊙ Decay:-
Chemical reaction, $A \to B$
For this reaction, using 1st order reaction kinetics, we can write,

$$-\frac{\partial c_A}{\partial t} = \frac{\partial c_B}{\partial t} = \lambda c_A \ldots\ldots (74.23)$$

[$\frac{\partial c_A}{\partial t}$ is itself negative, **because**, reaction occurs and conc. of A decreases with time].

Now, if we consider concentration of pollutant solute $= c$, under 1st order irreversible condition, (biodegradation, radioactive decay),

$$\frac{\partial c}{\partial t} = -\lambda c \ldots\ldots (74.24).$$

$\therefore$ Chemical Source/sink terms,

$$\sum_{n=1}^{N} R_n = -\lambda \eta c \ldots\ldots (74.25).$$

↳ what does it mean?

Putting values of $\sum_{n=1}^{N} R_n$ from (74.22) and (74.25) to equation (74.17),

$$\frac{\partial}{\partial t}(\eta c) = \frac{\partial}{\partial x}\left(\eta D_{xx}\frac{\partial c}{\partial x} + \eta D_{xy}\frac{\partial c}{\partial y} + \eta D_{xz}\frac{\partial c}{\partial z}\right) - \frac{\partial}{\partial x}(q_x c)$$
$$+ \frac{\partial}{\partial y}\left(\eta D_{yx}\frac{\partial c}{\partial x} + \eta D_{yy}\frac{\partial c}{\partial y} + \eta D_{yz}\frac{\partial c}{\partial z}\right) - \frac{\partial}{\partial xy}(q_y c)$$
$$+ \frac{\partial}{\partial z}\left(\eta D_{zx}\frac{\partial c}{\partial x} + \eta D_{zy}\frac{\partial c}{\partial y} + \eta D_{zz}\frac{\partial c}{\partial z}\right) - \frac{\partial}{\partial z}(q_z c)$$
$$+ q_s c_s - \rho_b K_d \frac{\partial c}{\partial t} - \lambda \eta c \ldots\ldots (74.26).$$

$$\frac{\partial}{\partial t}(\eta c) + \rho_b K_d \frac{\partial c}{\partial t}$$
$$= (\eta + \rho_b K_d)\frac{\partial c}{\partial t}$$
$$= \eta\left(1 + \frac{\rho_b}{\eta}K_d\right)\frac{\partial c}{\partial t} = \eta R \frac{\partial c}{\partial t}.$$

Putting, $\left(1 + \frac{\rho_b}{\eta}K_d\right) = R$ i.e Retardation factor. in (74.26), we get (74.27).

$$\therefore \eta R \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\eta D_{xx}\frac{\partial c}{\partial x} + \eta D_{xy}\frac{\partial c}{\partial y} + \eta D_{xz}\frac{\partial c}{\partial z}\right) - \frac{\partial}{\partial x}(q_x c)$$
$$+ \frac{\partial}{\partial y}\left(\eta D_{yx}\frac{\partial c}{\partial x} + \eta D_{yy}\frac{\partial c}{\partial y} + \eta D_{yz}\frac{\partial c}{\partial z}\right) - \frac{\partial}{\partial y}(q_y c)$$
$$+ \frac{\partial}{\partial z}\left(\eta D_{zx}\frac{\partial c}{\partial x} + \eta D_{zy}\frac{\partial c}{\partial y} + D_{zz}\eta\frac{\partial c}{\partial z}\right) - \frac{\partial}{\partial z}(q_z c)$$
$$+ q_s c_s - \lambda \eta c \ldots\ldots\ldots\ldots (74.27)$$

what is this? From 1st to last, it remains unchanged!

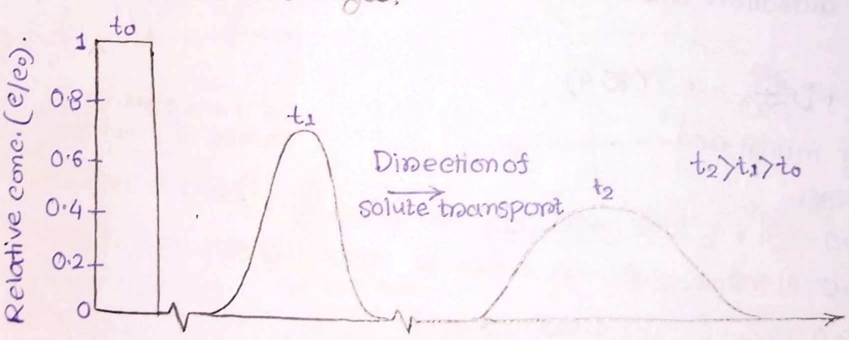


Fig: 20.3:- An instanteneous (Pulse) Source in a flow field creates a plume that spreads after moving towards down-gradient. (Shown in both 1D and 2D).

General continuity:

The pore space is filled by the sum of the fluids.

$$\eta = \theta_w + \theta_a + \theta_o \quad \ldots\ldots (75.1)$$

From AI ⇒ [θ_w = Volumetric water content,
θ_a = volumetric air content.
θ_o = Volumetric solid content
(I think, solutes here!)]

- The constituent mass density or bulk concentration, m can be represented as,

(# Bulk conc. = Mass of constituent per bulk volume)

$$m = \theta_w c_w + \theta_a c_a + \theta_o c_o + \theta_b c_s \quad \ldots (75.2)$$

[I.T ⇒ c_w, c_a, c_o, c_s are concentrations (mass/volume) of water, air, solutes and solid media (soil or rocks) respectively]

∴ continuity eqn can be written as,

$$\frac{\partial m}{\partial t} + \nabla \cdot \vec{F} = S^+ \quad \ldots\ldots (75.3)$$

where, $\vec{F}$ is mass flux vector
$S^+ \rightarrow$ Source/Sink term.
(see page-38).

Ogata and Banks (1961) Analytical solution:

It solves the ADE (Advective-dispersion equation) for a continuous source of infinite duration and a 1D domain:-

$$R\frac{\partial c}{\partial t} = -v\frac{\partial c}{\partial x} + D\frac{\partial^2 c}{\partial x^2} \quad \ldots\ldots (75.4)$$

Or it is a continuous source of pollution?

with the following initial and boundary conditions,

$C(0,t) = C_o$, $t>0$ → At any time, how conc. can be $= C_o$ (?) → doesn't it vary with time

$C(\infty,t) = 0$, $t>0$ (At infinite distance, solute is completely spreaded. So, $C=0$.)

$C(x,0) = 0$; $x \geq 0$ (Because, at t=0, initially solute is present at x=0).

(?)

! (I Think, only $x>0$, not $x \geq 0$.)

> If we consider, (74.27) for only x direction (i.e 1D), and consider advection and dispersion related terms only, (though, adsorption was also considered because, 'R' contains adsorption related term 'K_d'. See 74.22, 74.26, 74.27 for more).

$$\eta R\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\eta D_{xx}\frac{\partial c}{\partial x}\right) - \frac{\partial}{\partial x}(q_x c)$$

as it is only for 1D,

$$\eta R\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\eta D\frac{\partial c}{\partial x}\right) \mp \frac{\partial}{\partial x}(qc)$$

$$\Rightarrow \eta R\frac{\partial c}{\partial t} = \eta D\frac{\partial^2 c}{\partial x^2} \mp q\frac{\partial c}{\partial x}$$

(For constant hydraulic gradient and homogenous media, q does not vary spatially).

$$\Rightarrow R\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - \left(\frac{q}{\eta}\right)\frac{\partial c}{\partial x} \quad \left(\frac{q}{\eta} = v = \text{Seepage velocity}\right)$$

$$\Rightarrow R\frac{\partial c}{\partial t} = -v\frac{\partial c}{\partial x} + D\frac{\partial^2 c}{\partial x^2}$$

This is equation (75.4)

Where, c = Concentration $[ML^{-3}]$
x = distance $[L]$
R = Retardation factor
D = Effective dispersion/diffusion coefficient $[L^2T^{-1}]$.
v = flow velocity $[LT^{-1}]$
C_0 = Concentration at the upstream boundary. $[ML^{-3}]$.

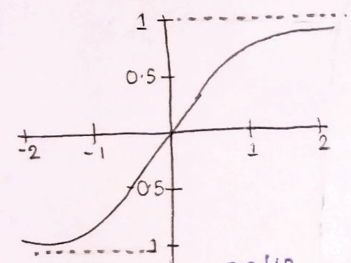
The analytical solution of (75.4) is

$$c(x,t) = \frac{C_0}{2}\left[\operatorname{erfc}\left(\frac{x - vt/R}{2\sqrt{Dt/R}}\right) + \exp\left(\frac{vx}{D}\right)\operatorname{erfc}\left(\frac{x+vt/R}{2\sqrt{Dt/R}}\right)\right] \dots (75.5)$$ → Try later!

$$\operatorname{erf}(\varphi) = \frac{2}{\sqrt{\pi}}\int_0^{\varphi} e^{-w^2}dw \dots\dots (75.6)$$

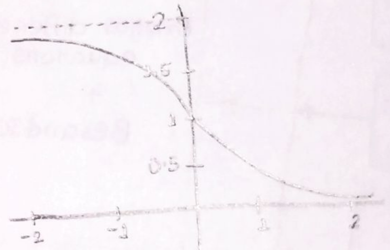
$$\operatorname{erfc}(\varphi) = 1 - \operatorname{erf}(\varphi) \dots\dots (75.7).$$

⊙ Graphs of erf(φ) and erfc(φ):- (See P-59 for more).



erf φ

Graph: 20.4



erfc(φ) = 1 − erf(φ)

Graph 20.5

⊙ If the source is active for time $t = T_0$ (I think, for 75.5 it was a continuous source),

$$c(x,t) = \begin{cases} \frac{C_0}{2}\left[\operatorname{erfc}\left(\frac{x-vt/R}{2\sqrt{Dt/R}}\right) + \exp\left(\frac{vx}{D}\right)\operatorname{erfc}\left(\frac{x+vt/R}{2\sqrt{Dt/R}}\right)\right], & \text{for } t<T_0 \text{ (Same as 75.5)} \dots (75.8) \\ \frac{C_0}{2}\left[\operatorname{erfc}\left(\frac{x-vt/R}{2\sqrt{Dt/R}}\right) + \exp\left(\frac{vx}{D}\right)\operatorname{erfc}\left(\frac{x+vt/R}{2\sqrt{Dt/R}}\right)\right] \\ -\frac{C_0}{2}\left[\operatorname{erfc}\frac{x-v(t-T_0)/R}{2\sqrt{D(t-T_0)/R}} + \exp\left(\frac{vx}{D}\right)\operatorname{erfc}\left(\frac{x+v(t-T_0)/R}{2\sqrt{D(t-T_0)/R}}\right)\right], & \text{when } t>T_0 \dots (75.9) \end{cases}$$

[# Take a look, at 75.8 and 75.9
If (75.8) is = f(t) then (75.9) is = f(t) − f(t−T₀)]! 😀

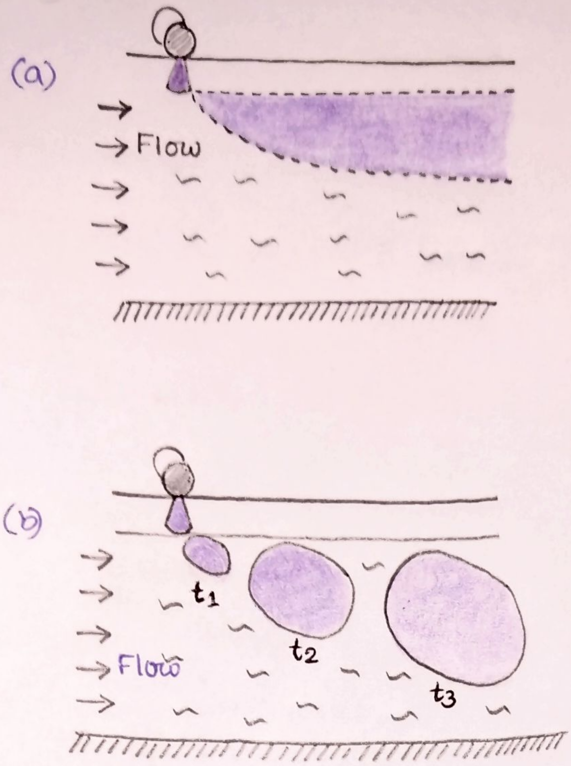


Figure: 20.6
(a) Continuous release (leaching),
(b) Instanteneous release or injection (spill),
of contaminant from a point source into an aquifer with isotropic sand in a 2D uniform field.

☑ Basic framework:-

- Equation of motion: Darcy's law (Momentum equation!?)
- Equation of continuity: Mass balance
- ? Advection-Dispersion equation (I think, it also comes from mass balance. See below eq^n (75.4))

→ Partial differential equations + BC's and IC's

→ Analytical Solution / Numerical Solution.

⊙ erf(x) and erfc(x): (See P-57),

erf(x) is called error function.

It is defined as gaussian or normal distribution from 0 to x.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

In other words, erf(x) gives the probability that a random variable ~~(here, x)~~ with a standard normal distribution is less than or equal to x.

It's an odd function i.e $\operatorname{erf}(-x) = -\operatorname{erf}(x)$.

Its value ranges from +1 to -1. (See graph 20.4).

On the other hand, erfc(x) is called complementary error function, defined as,

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

It provides probability of an event occurring beyond a certain value of in a Gaussian distribution.

In other words, it quantifies the tail probability of a normal distribution.

Its value ranges from 0 to 2.

Surface Investigations of Groundwater

Remote sensing:

Geology, Soils, soil moisture, vegetation, land use.

Fracture pattern in rocks ⇒ Related to porosity, permeability, ultimate yeild of well.

- Vegetation types:

<u>Phreatophytes</u>:- Transpire water from shallow water table.

<u>Halophytes</u>:- Plant with a high tolerance of soluble salts. White efflorescences of salt at ground surface indicates the presence of shallow brackish or saline groundwater.

<u>Xerophytes</u>: Desert plant, suggests a considerable depth of GW.

- Non-visible portion of electromagnetic spectrum:-

<u>Infrared imagery</u>:- Records difference in apparent surface temperature ⇒ Provides information on soil moisture, GW circulation, faults functioning as aquiclude.

<u>Near-infrared imagery</u>:- Seepage pattern from canals, mapping coastal Submarine spring.

<u>Radar imagery</u>:- Presence of moisture on or at shallow depths below ground surface.

<u>Low frequency electromagnetic Aerial survey</u>:- Buried canals and Zone of seawater intrusion.

Geophysical Exploration:

Investigates mineral deposits or geological structures.

Geophysical methods detects differences or anamolies of physical properties (density, magnetism, elasticity, electrical resistivity etc) within the earth's crust.

⊙ Electric Resistivity Method

Electric resistivity:- Resistance (in Ω) between opponent faces of a unit cube of a material.

Resistivity, $\rho = \frac{RA}{L}$ ...... (76.1)

Unit of ρ is ohm-m.

Resistivity of rocks vary over wide range. Depends on material, density, porosity, pore size and shape, water content and quality, temperature.

Igneous and metamorphic rocks ⇒ $10^2 - 10^8$ ohm-m.

Sedimentary and unconsolidated rocks = $10^0 - 10^4$ ohm-m.

I.I⇒ # Dense material has more ρ.

⊙ For aquifers composed on unconsolidated materials ⇒ resistivity↓, Degree of saturation↑.

resistivity↓, Salinity↑.

⊙ clay:- clay minerals carry current through it. Also, clay has higher WHC, so typically wet. Also, clay is highly porous. For all these reasons, clay has very low resitivity, (almost 5-30 ohm-m) wet sand and gravel has resistivities 5-10 times higher than that.

Therefore, relatively higher resistivity zones are of interest as shallow aquifers.

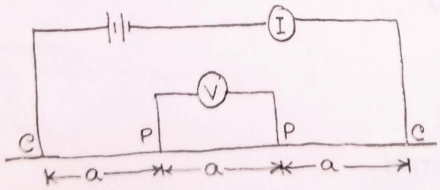


Fig: 2.1.1:- Wenner's arrangement

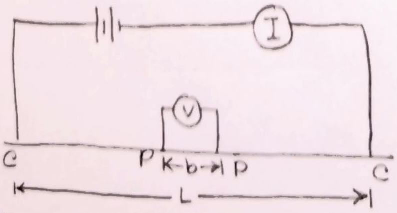


Fig: 2.1.2:- Schlumberger's arrangement.

Apparent resistivity by Wenner's arrangement,

$\rho_a = 2\pi a \cdot \frac{V}{I}$ ...... (76.2)

By Schlumberger's arrangement,

$\rho_a = \pi \frac{(L/2)^2 - (b/2)^2}{b} \cdot \frac{V}{I}$ .... (76.3).

For good results $L \geq 5b$ .... (76.4).

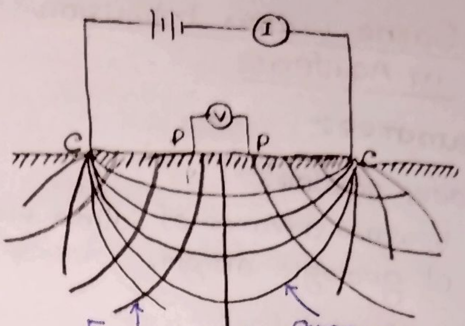


Fig: 2.1.3.

In (76.2), a = distance between adjacent electrodes

[For Wenner's arrangement, all adjacent distances are same]

⊙ Apparent resistivity vs electrode space curve and it's interpretation:

Apparant resistivity is plotted against electrode spacing ('a' for Wenner, $\frac{L}{2}$ for Schlumberger)

(i) Interpretation in terms of various layers of actual (as distinguished from apparent) resistivities in terms of Depth:-

Accomplished with theoritically computed resistivity-spacing curves for 2,3 or 4 layer cases of various ratio of resistivities. Curve-matching techniques are there.

(ii) Interpretation of actual resistivities in terms of subsurface geological and groundwater condition:-

It depends on supplemental data. Comparing actual resistivity variations with depth to data from a nearby logged test holes enables a corelation to be established with subsurface geology and groundwater conditions. The information can then be applied for interpretation of resistivity measurement of surrounding area.

Saline Water Introusion in Aquifers:

☑ **Occurrence:-**

In deep aquifers:-

Upward advance of saline waters of geologic origin.

In shallow aquifers:-

Discharge of surface wastes.

Coastal aquifers:-

From an invasion of seawater.

- Typically, shallow freshwater overlies saline water because, "the flushing action of recent times" removes salt from ancient marine deposit.

But, at greater depth, G/W movement is much less. So, displacement of saline water is much slower.

☑ **Source of saline water in aquifers:-**

(i) Encroachment of sea-water in coastal areas.

(ii) Seawater entered aquifers during past geological times.

(iii) Salt in salt-doms, thin beds, disseminated in geological formation.

(iv) Water concentrated by evaporation of tidal lagoons, playas and other enclosed areas.

(v) Return flow from irrigated lands to stream.

(vi) Human saline wastes.

☑ **Mechanisms of Saline water introusions:**

(i) Pumping of wells in coastal aquifer (which has hydraulic continuity with sea).

↓

Disturbs natural hydrodynamic balance

↓

Reduction or reversal of hydraulic gradient

↓

Denser saline water displaces fresh water.

(ii) Second method, destruction of natural barriers that seperates fresh and saline water.

Eg. Construction of coastal drainage canal that enables tide water to advance inland and to percolate into a freshwater aquifer.

(iii) 3rd mechanism,
Subsurface disposal of waste saline water.
Eg. Disposal wells, landfill etc.

☑ Ghyben-Herzberg relation between fresh and saline water:- (This is for equilibrium condition)

Salt water occured underground
→ Below sea level 40 times depth of difference between sea level and fresh water level.

This distribution was attributed to hydrostatic equilibrium between two fluids of different densities.

This phenomenon was illustrated by a U-tube. (See figure 22.1)

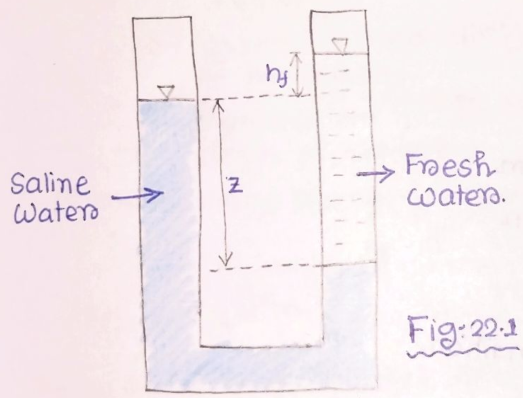
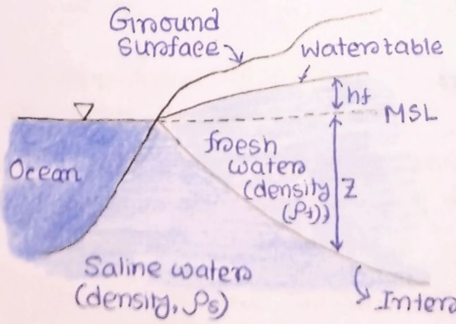


Fig: 22.1

Pressure on each side of the tube should be equal.

So, $\rho_s g z = \rho_f g (z+h_f)$ .... (77.1)



h_f = Elevation of water table above sea level.

Z = Depth to fresh-saline water interface below the sea-level.

Figure 22.2:- Translating U tube to a coastal situation of figure 22.1.

From (77.1),

$$\rho_s z = \rho_f z + \rho_f h_f$$

$$\Rightarrow z(\rho_s - \rho_f) = \rho_f h_f$$

$$\Rightarrow z = \left(\frac{\rho_f}{\rho_s - \rho_f}\right) h_f \quad \cdots\cdots (77.2)$$

For typical seawater conditions, let, $\rho_s = 1.025\ g/cm^3$, $\rho_f = 1.00\ g/cm^3$

$$\therefore z = \frac{1}{1.025 - 1} h_f$$

$$= 40 h_f \quad \cdots\cdots (77.3)$$

Where the flow is nearly horizontal, the Ghyben-Herzberg relation gives satisfactory results. (Results in terms of position of interface) Only near the shoreline, where vertical flow components are pronounced, do significant error in terms of position of interface.

Applicability of Ghyben-Herzberg relation:-

Equation (77.1) holds when,
water table or piezometric surface,
(i) Lie above sea level
(ii) Slope downward to the ocean.
Without these two conditions, Seawater will advance directly inland. (Here, comes from downward, see fig 22.2).

- Luszczynski generalized G-H relation where underlying saline water is in motion with heads below or above sea-level.

The result for non-equilibrium condition,

$$z = \frac{\rho_f}{\rho_s - \rho_f} h_f - \frac{\rho_f}{\rho_s - \rho_f} h_s \quad \cdots\cdots (78.1)$$

h_f = Altitude of water level (from MSL) in a well filled with fresh water. (Well is terminated at depth z, i.e interface between sea-water and freshwater).

h_s = Altitude of water level in a well filled with saline water (also terminated at depth = z).

[When $h_s = 0$, Saline water is in equilibrium with sea, then (78.1) becomes (77.2) (for that case, there will be no flow between sea water and aquifer affected by saline zone). i.e equilibrium condition.]

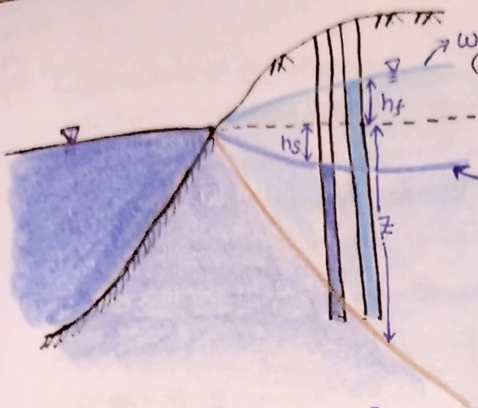


Fig: 22.3:- Different heads for non-equilibrium conditions in fresh and saline water in an unconfined coastal aquifer.

(Compare with 22.2. There ocean water level and pressure surface at saline water was same i.e MSL. Thats why it was an equilibrium condition).

Shape of the fresh-salt water interface:

Developed by potential flow theory (by Glover),

$$Z^2 = \frac{2\rho q x}{\Delta\rho K} + \left(\frac{\rho q}{\Delta\rho K}\right)^2 \quad \cdots\cdots (78.2)$$

where, z and x is shown in fig (22.4). I.T ⇒ [x = Distance from the shore of that point. z = Depth of interface from MSL for that point]

$$\Delta\rho = \rho_s - \rho_f \quad \cdots\cdots (78.3)$$

ρ = density of freshwater (i.e $= \rho_f$!)

K = Hydraulic conductivity of the aquifer.

q = Freshwater flow per unit length of shoreline.

Corresponding shape for water table:-

$$h_f = \left[\frac{2\Delta\rho q x}{(\rho+\Delta\rho)K}\right]^{1/2} \quad \cdots\cdots (78.4)$$

x_0 = width of the submarine zone (where z=0)

From (78.2),

$$0 = \frac{2\rho q x_0}{\Delta\rho K} + \frac{\rho^2 q^2}{(\Delta\rho)^2 K^2}$$

$$\Rightarrow x_0 = -\frac{\rho q}{2\Delta\rho K} \quad \cdots\cdots (78.5)$$

(x_0 is negative because, it is in opposite direction to positive x axis).

Now, Z_0 = Depth of the interface beneath the shoreline (i.e for x=0)

From (78.2), $Z = \frac{\rho q}{\Delta\rho K} \quad \cdots\cdots (78.6)$

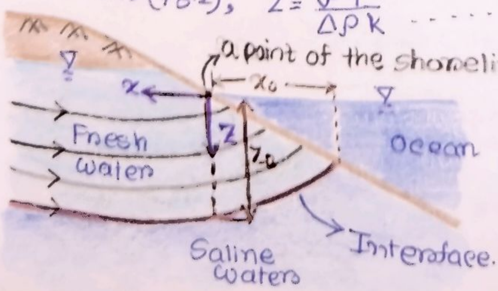


Figure 22.4: Flow pattern of fresh water in an unconfined coastal aquifer.

Note:-

Shape of the fresh-salt water interface by Ghyben- Herzberg (i.e 78.2's preliminary form) (see below 77.3) theory.

$$z^2 = \frac{2\rho q x}{\Delta\rho K} \cdots\cdots (78.7)$$

If we compare GH theory and Glover's eq^n (78.2 & 78.7), in 78.2, extra constant term i.e $\left(\frac{\rho q}{\Delta\rho K}\right)^2$ is added. This term accounts for missing seepage face that allows for the vertical components of flow and discharge of fresh water onto the sea floor.

Near shoreline (where, x value is less, so $\left(\frac{\rho q}{\Delta\rho K}\right)^2$ term predominants), GH eq^n (78.7) gives incorrect depth of z values or interface profile.

My Observation:

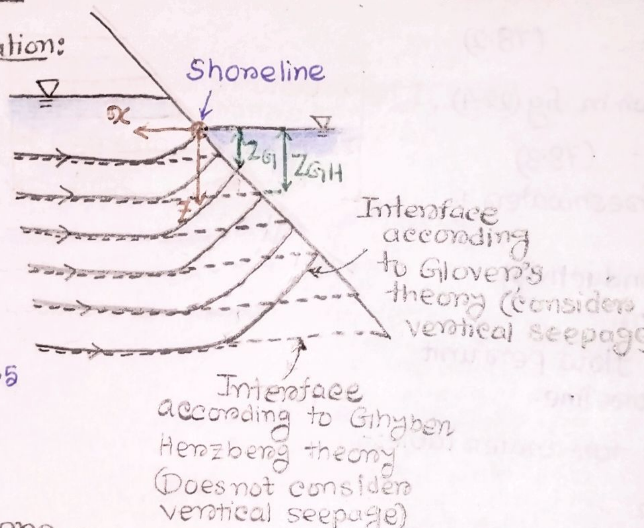


Fig: 22.5

Here, $Z_G \rightarrow$ According to Glover's equation (78.2)

$Z_{GH} \rightarrow$ According to Ghyben- Herzberg eq^n (78.7)

Here, $Z_G < Z_{GH}$, because, x is negative at the right-side of the shoreline. (Put $-ve$ x at 78.2, you can see).

$|Z_G - Z_{GH}|$ gives a higher difference upto 20 m distance from the shoreline. Using Glover's eq^n is better.

Structure of Fresh-Salt water interface:-

Previously, we considered the interface as sharp boundary or as a line.
Instead, in field condition, a brackish transition zone of ~~of~~ finite thickness seperates two fluids.
Observed thickness may vary from 1 to more than 100 m.

Generally, highly permeable coastal aquifers subjected to heavy pumping have greatest thickness of transition zone.

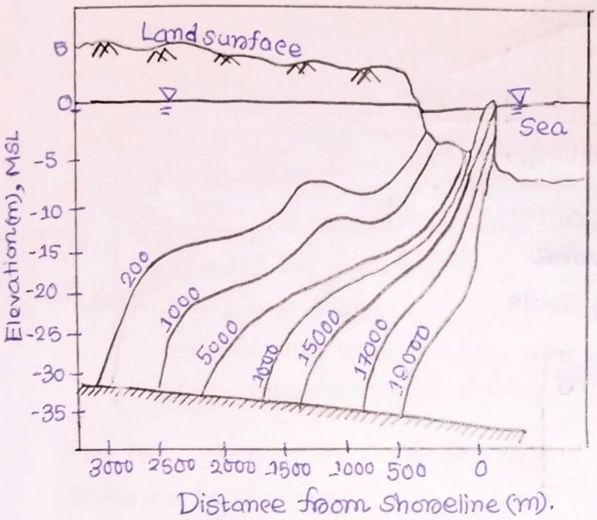


Fig 22.6: cross section of aquifer through transition zone. Numbered lines are isochlors in mg/L.

Note: Isochlors approach base of the aquifer vertically because, the flow is parallel to the base of the aquifer, restricts vertical mixing.

Saline water originates from underlying portion and seaward flow (i.e transport of saline water towards sea) occurs.

Tidal action is predominant mixing mechanism, (horizontal or vertical mixing)? causes fluctuations of groundwater, hence thickness of transition zone is greatest near shoreline.

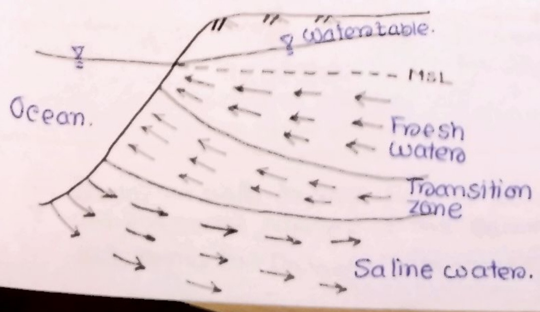


Fig:22.7:- Flow patterns of fresh and saline water in unconfined coastal aquifer.

→ why water is flowing from aquifer to ocean here?

J.T ⇒ In figure 22.7, water flowing from fresh water zone of aquifer to sea, because, water table has higher elevation than MSL. So, it has more hydrostatic pressure.

But, as ocean water has more density. So, after a certain depth, pressure is more for ocean water. So, water flows from Ocean to Saline water zone of aquifer.

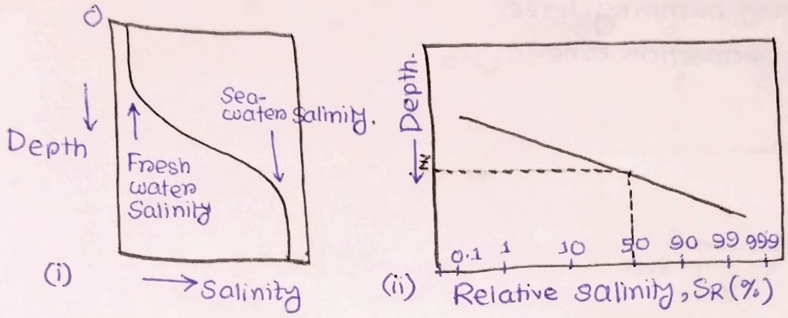


Fig: 22.8:- Increase in salinity with depth through the transition zone (i) Linear scale (ii) Probability Scale.

Typically, distribution of salinity with depth varies as an error function (see P-59)

$$S_R(\%) = 100\left(\frac{C - C_f}{C_s - C_f}\right) \quad \cdots\cdots\cdots (79).$$

where, C_s, C_f represents Salinity of Saline and fresh water.

C = Salinity at a particular depth within the transition zone.

The 50% value of S_R, representing the midline of the transition zone, defines the position of the interface without mixing → why no mixing ?! (See fig 22.8-(ii)).

Effect of Wells in Sea-water Intrusion:

- Single well is located at a distance x_w from the shoreline (fig 22.9).

Assumptions:-

- Isotropic, homogenous Aquifer.
- Horizontal impervious base.
- Horizontal fresh water flow.
- No flow in the salt-water zone.

Relationship used for determining the position of the toe of an interface under steady conditions. (By Strack),

$$\frac{1}{2}(1+\delta)\frac{B^2}{\delta^2} = \frac{Q'_{ox}}{K}x + \frac{Q_W}{4\pi K}\ln\left[\frac{(x-x_w)^2+y^2}{(x+x_w)^2+y^2}\right] \quad \ldots (80.1)$$

where,

$$\delta = \frac{\rho_f}{\rho_s-\rho_f} \quad \ldots\ldots (80.1.2)$$

B = depth to bedrock below MSL.

Q'_{ox} = Freshwater flow per unit length of shoreline. (Assumed, uniform value from infinity to the coast.

Q_W = Constant pumping rate of well superimposed on Q'_{ox}

K = Aquifer's hydraulic conductivity.

x_w = Distance between the well and the shoreline.

(x, y) : x-y co-ordinates of the toe of the interface.

J.T ⇒ x-y plane is aquifer bed or plane parallel to the aquifer bed. 'Toe of the interface' means where interface line of salt-water and fresh-water meets the aquifer bed

Origin of the (x, y) plane is the point of intersection of 'shoreline' and 'line perpendicular to shoreline passing through the well'. (see fig 22.9 and 22.10)

For the stagnation point (x_s, y_s), (Obtained by Strack)

$$x_s = x_w\left[1-\frac{Q_W}{\pi Q'_{ox}x_W}\right]^{1/2} \quad \ldots (80.2)$$

$$y_s = 0$$

J.T ⇒ Fresh water moves from aquifer to the sea. On the other hand, well pumps out water from the aquifer. At stagnation point: effect of these opposite discharges cancelled out.

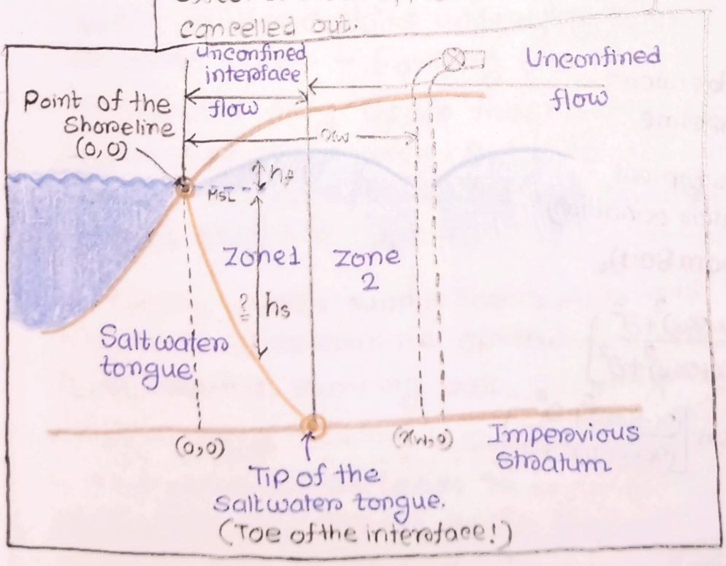


Fig: 22.9

? In figure 22.3, h_s was distance between MSL and pressure surface of the saline water.
But, here, h_s = Distance between MSL and salt-water, fresh-water interface. (why?)

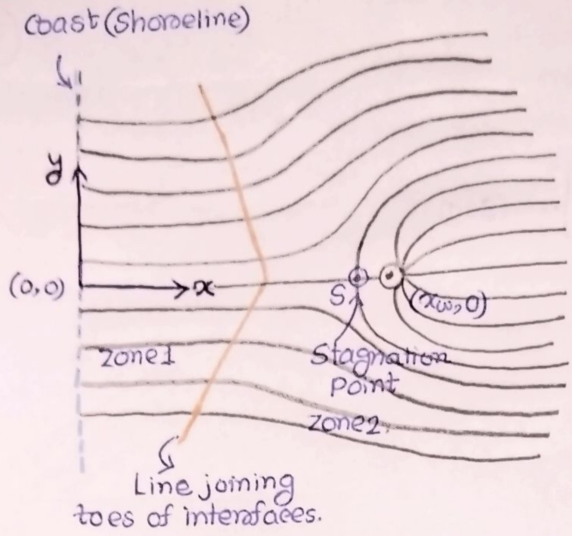


Fig 22.10:- Shallow coastal aquifer with a well (J.T ⇒ Top view).

Well takes water only from the right side of the stagnation point in this figure.

If, for extreme well discharge, toe of interface advance landward and stabilizes at the right side of the stagnation point, brackish water will come from pumping well. (See example @ P-72)

⊙ Assuming an unstable critical situation occurs (why unstable?) when toe of the interface passes through the stagnation point, following expression was obtained for computing the critical well discharge(Q_w) for unconfined aquifer, → What is critical well discharge?? (See P-73).

$$\lambda = 2\left(1-\frac{\mu}{\pi}\right)^{1/2} + \frac{\mu}{\pi}\ln\left[\frac{1-(1-\mu/\pi)^{1/2}}{1+(1-\mu/\pi)^{1/2}}\right] \quad \ldots\ldots (81.1)$$

Where,

$$\lambda = \left(\frac{KB^2}{Q'_{0x}x_w}\right)\left(\frac{1+\delta}{\delta^2}\right) \quad \ldots\ldots (81.2)$$

$$\text{and } \mu = \frac{Q_W}{Q'_{0x}x_w} \quad \ldots\ldots (81.3)$$

(See 80.1 for more)

Example 14.5.1 (Todd P-615)

Given data, $B = 57m$

$Q'_{0x} = 1.6\ m^3/day/m$ (Freshwater flow per unit length of the shoreline).

$Q_W = 432\ m^3/day$

$K = 8.6\ m/day$

$x_w = 800\ m$ = Distance between the well and shoreline.

Putting, $\delta = \frac{\rho_f}{\rho_s - \rho_f} = \frac{1}{1.025-1} = 40$ (for typical sea water condition)

and $y=0$, we have, (from 80.1),

$$\frac{1}{2}(1+\delta)\frac{B^2}{\delta^2} = \frac{Q'_{0x}}{K}x + \frac{Q_W}{4\pi K}\ln\left[\frac{(x-x_w)^2+y^2}{(x+x_w)^2+y^2}\right]$$

$$\Rightarrow \frac{1}{2}(1+40)\frac{57^2}{40} = \frac{1.6}{8.6}x + \frac{432}{4\pi \times 8.6}\ln\left[\frac{(x-800)^2+0^2}{(x+800)^2+0^2}\right]$$

⇒ Solving, we get, $x = 252m$.

Given that,

The distance of 'toe of the interface' from shoreline when there was no pumping = 215m

#Now due to pumping, toe of the interface moves inland by
$= (252-215) = 37m$.

To determine if brackish water will occur in the well or not, we must determine critical well discharge beyond which interface will advance rapidly untill a new equilibrium is reached, with the interface toe landward of the well.

From (81.2),

$$\lambda = \left(\frac{KB^2}{Q_{0x}' x_w}\right)\left(\frac{1+\delta}{\delta^2}\right)$$

$$= \frac{8.6\times57^2}{1.6\times800}\times\frac{1+40}{40^2}$$

$$= 0.559.$$

Substituting value of λ in equation (81.1),

$$0.559 = 2\left(1-\frac{\mu}{\pi}\right)^{1/2} + \frac{\mu}{\pi}\ln\left[\frac{1-(1-\mu/\pi)^{1/2}}{1+(1-\mu/\pi)^{1/2}}\right]$$

Solving, we get,

$$\mu = 1.524.$$

Using (81.3), $1.524 = \dfrac{Q_w}{1.6\times800}$

$$\Rightarrow Q_w = 1951\ m^3/day.$$

Since the proposed pumping rate ($432 m^3/s$) is less than critical pumping rate ($1951\ m^3/s$), the toe of the interface will stabilize some distance away from stagnation point. So, there won't be any brackish water in the pumping well.

☐ Upconing of Saline Water:

When aquifer contains underlying layer of saline water. – Pumping well is penetrated only upper freshwater portion of the aquifer. But, a local rise of interface below the well occurs, that is known as upconing.

From a water-supply standpoint, it is important to determine optimum location, depth, spacing, pumping rate, pumping sequence (eg. if pumping is stopped, the denser saline water tends to settle downward and return to its former position) that will ensure the production of largest quantity of groundwater.

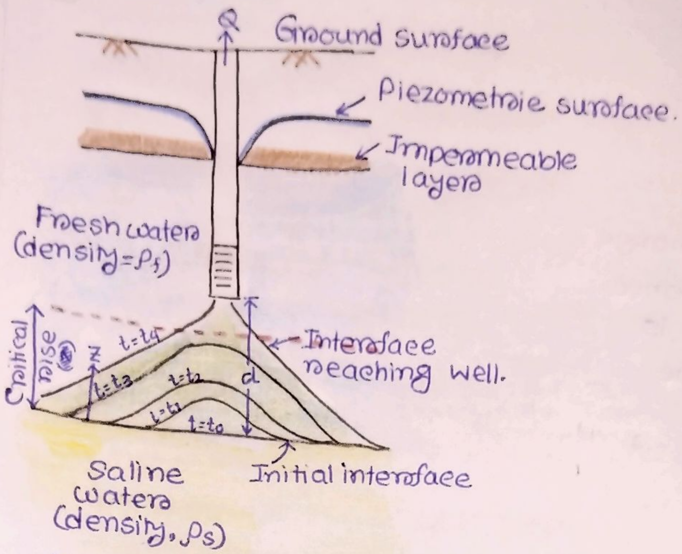


Fig: 22.11: Upconing of underlying saline water to a pumping well.

Rise of the interface is given by,

$$Z = \frac{Q}{2\pi d K (\Delta\rho/\rho_f)} \quad \cdots (82.1)$$

where, Q = Pumping rate of well
d = distance of intial interface below the pumping well (see fig. 22.11)
K = Hydraulic conductivity of aquifer.
$\Delta\rho = \rho_s - \rho_f$

Eq^n (82.1) holds if the rise of the interface < critical rise. After that water accelerates toward the well.

Critical rise is approximated as,
$\frac{Z}{d} = 0.3$ to 0.5 ⇒ Let's take, $\frac{Z}{d} \approx 0.5$ (maximum limit).

∴ Maximum possible pumping rate without salt entering, (From 82.1),

$$\Rightarrow 0.5d = \frac{Q_{max}}{2\pi d K(\Delta\rho/\rho_f)}$$

$$\Rightarrow Q_{max} = \pi d^2 K(\Delta\rho/\rho_f) \cdots (82.2)$$

For anisotropic aquifer, where, $K_{Vertical} < K_{Horizontal}$, Qmax is more than that of isotropic condition (i.e 82.2).

In reality, there is transition zone, not an interface. So, not abrupt, but a gradual variation of $\Delta\rho$ occurs. At the top of transition zone, $\Delta\rho \to 0$. From (82.1), we can say $Z \to \infty$. So, with any rate of continuous pumping, saline water must sooner or later will reach a well.

Arrival of salinity at a pumping well for an abrupt interface and for a transition zone:-

If $Q > Q_{max}$,

For abrupt interface, the salinity appears later, and but increases more rapidly than a transition zone.

J.T ⇒ [Abrupt interface assumed to be at the middle of the transition zone. So, salinity comes later]

If $Q < Q_{max}$,

For abrupt interface, there will be no salinity reaching in the well. But, a gradual invasion of saline water will occur from a rising transion zone.

[Remember:- Abrupt interface is an ideal case. Which does not exist in reality]

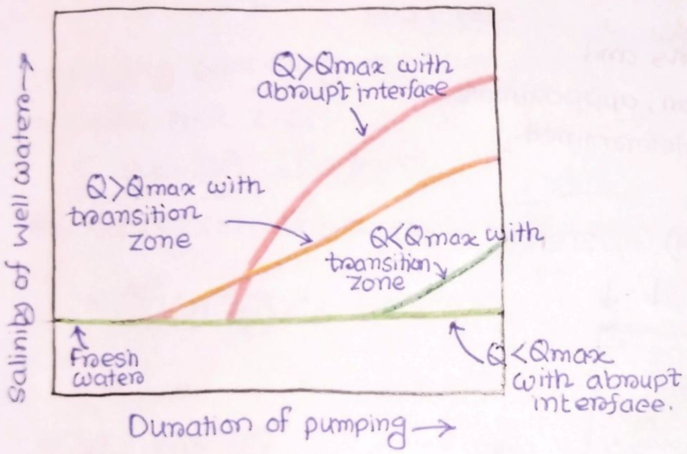


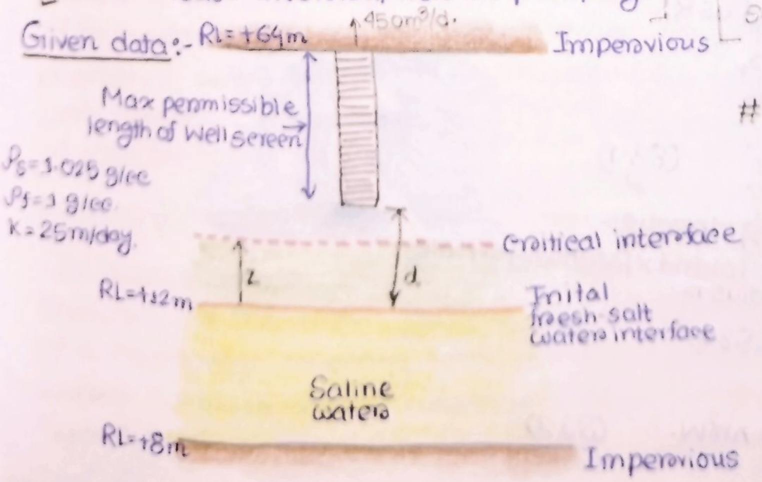
Figure: 22.12

Example (Todd P-618) - 14.6.2

[We need to determine the permissible length of well screen such that we can pump water at a rate of 450 m³/day without least interruption in pumping.]

[Interruption occurs, because if $Q > Q_{max}$, salt water invades to the well after pumping a period of time. After stopping, salt-water settles downward]

Given data:-



[To get that discharge, we need to maintain the length of 'd' in such a way that $\frac{z}{d} < 0.3$]

Figure 22.13:

Now, $z = 0.3d$.

Using (82.1), $z = \frac{Q}{2\pi d K (\Delta\rho/\rho_f)}$

$$\Rightarrow 0.3d = \frac{450}{2\pi \times d \times 25 \times \left(\frac{1.025-1}{1}\right)}$$

$$\Rightarrow d = 19.5m.$$

∴ RL of the well bottom $= 12 + 19.5 = 31.5m$

∴ Max permissible length of well screen $= (64 - 31.5) = 32.5m$.

Fresh-Salt water relation in Oceanic islands:

- Oceanic islands are permeable (consists of sand, coral, lava, limestones)
- Seawater is contact with GW from all sides. Source of fresh water is rainfall.
- Small fresh-water lens floats on the underlaying salt water.
- From Dupit's assumptions and Ghyben-Herzberg relation, approximate fresh-water boundary is determined.

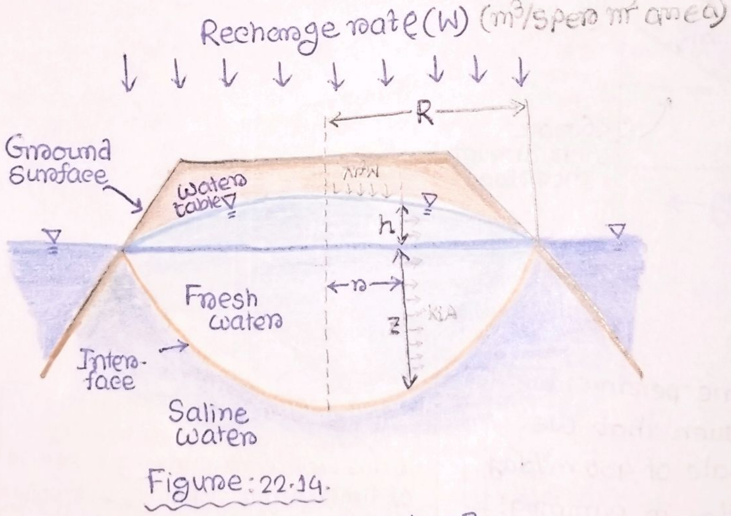


Figure: 22.14.

Consider small island of radius R.

Outward flow at radius r,

$$Q = -KiA$$
$$= -K \cdot \frac{dh}{dr} \cdot 2\pi r (z+h) \quad \dots\dots (83.1)$$

Amount of flow entering through recharge = Area of the island upto radius r × Recharge rate

$$= \pi r^2 W \quad \dots\dots (83.2)$$

From continuity,

$$-K\frac{dh}{dr} \cdot 2\pi r(z+h) = \pi r^2 W \quad \dots\dots (83.3)$$

Now, using the concept of U-tube (figure 22.1 and 22.2),

$$\rho_f(h+z) = \rho_s z$$

$$\Rightarrow \frac{h+z}{z} = \frac{\rho_s}{\rho_f}$$

$$\Rightarrow \frac{h}{z} = \frac{\rho_s}{\rho_f} - 1$$

$$\Rightarrow \frac{h}{z} = \frac{\rho_s - \rho_f}{\rho_f}$$

$$\Rightarrow \frac{h}{z} = \frac{\Delta\rho}{\rho_f} \quad (\text{From } 83.3,\ \Delta\rho = \rho_s - \rho_f)$$

$$\Rightarrow h = \frac{\Delta\rho}{\rho_f} z \text{ .... } (83.4)$$

$$\Rightarrow \frac{dh}{dr} = \frac{\Delta\rho}{\rho_f}\frac{dz}{dr} \text{ ..... } (83.5)$$

From (83.3), and using (83.4) and (83.5),

$$-K \cdot \frac{\Delta\rho}{\rho_f}\frac{dz}{dr} \cdot 2\pi r\left(z + \frac{\Delta\rho}{\rho_f}z\right) = \pi r^2 W$$

$$\Rightarrow -2K\left(\frac{\Delta\rho}{\rho_f}\right)\cdot\left(1+\frac{\Delta\rho}{\rho_f}\right)\cdot\int z\,dz = W\cdot\int r\,dr$$

$$\Rightarrow -K\left(\frac{\Delta\rho}{\rho_f}\right)\left(1+\frac{\Delta\rho}{\rho_f}\right)z^2 = \frac{Wr^2}{2} + C \text{ .... } (83.6)$$

[where, C = constant of integration].

Applying boundary condition in (83.6),

When, $r = R$, $z = 0$,

$$C = -\frac{WR^2}{2} \text{ .... } (83.7),$$

So, eq^n (83.6) becomes,

$$-K\left(\frac{\Delta\rho}{\rho_f}\right)\left(1+\frac{\Delta\rho}{\rho_f}\right)z^2 = \frac{Wr^2}{2} - \frac{WR^2}{2}$$

$$= -\frac{W}{2}(R^2 - r^2)$$

$$\Rightarrow z^2 = \frac{W(R^2-r^2)}{2K\left[\frac{\Delta\rho}{\rho_f}\right]\left[1+\frac{\Delta\rho}{\rho_f}\right]} \text{ ..... } (83.8)$$

[In many places, density of fresh water is written as ρ instead of ρ_f. Don't be confused.]

[In ocenic islands storage of fresh water is very low. To avoid the danger of entrainment of saline water, small diameter wells should be employed with shallow, dispersed, and pumped at low uniform rates.]

Infiltration gallery:

In areas where water tables are shallow, an infiltration gallery, consisting of a horizontal collecting tunnel at the water table, is more advantageous. Sometimes, in this case, few centimeters drawdown can give plentiful water supply.

Control of Saline water intrusion:

[Only 2% of sea-water in fresh water can make water Un-potable.]

(i) Modification of pumping pattern:

Dispersing the locations of pumping wells in inland areas, helps to re-establish stronger sea-ward hydraulic gradient. So, flow from sea to aquifer not occurs.

(ii) Artificial recharge:

It raises the level of groundwater. But, problem is, it requires a Supplemental water source.

(iii) Extraction Barrier:-

Extraction barrier is created by maintaining continuous pumping trough (channel) with a line of wells adjacent to the sea. Water is pumped are generally from Salt-water zone and discharged into the sea

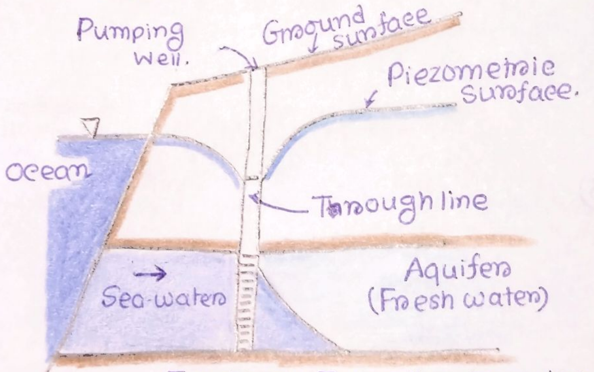


Fig: 22.15: Extraction barrier.

(iv) Injection Barrier:

This method maintains a pressure ridge along the coast by a line of recharge wells. Injected fresh water flows both sea ward and landward.

A combination of injection and extraction barriers are feasible, because this reduces 'required recharge and extraction rates'. But, problem is, it requires large number of wells.

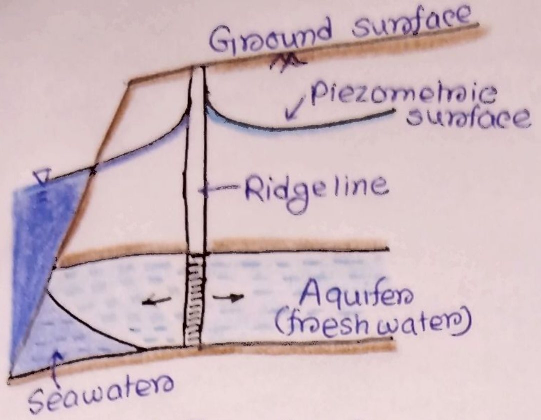


Fig: 22.16: Injection Barrier

(See fig 22.15 and 22.16):-
- Trough line of extraction barrier is situated at saline-water zone.
- Ridge line of injection Barrier is situated at fresh-water zone

(v) Sub-surface Barrier:-

Construction of impermeable subsurface barrier parallel to coast and through the vertical extent of the aquifer.
It can prevent inflow of sea-water into the basin.

Materials → Sheet piling, Puddled clay, emulsified asphalt, cement grout, bentonite, Silica gel, Calcium acrylated etc. which resists water percolation.

Problem ⇒ High construction cost, need to be resistant to earthquake and chemical erosion.

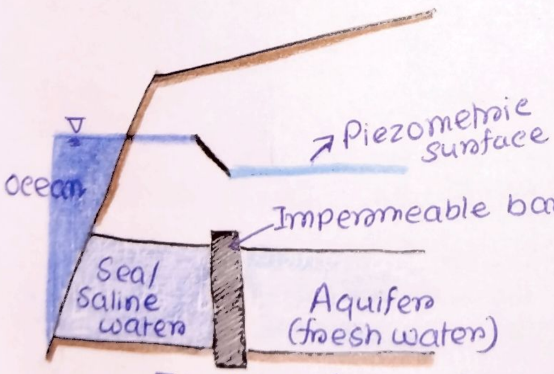


Fig: 22.17 (Subsurface barrier).

(Here saline water has higher piezometric head! But, barrier is there, you can not enter! Haha!!)