A two-day workshop on

"High Performance Computing Methods for complex and moving geometries"

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Jointly Organized by **IIT Kanpur and Nodal Centre for Training in HPC and AI (IIT Kharagpur)**

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Finite Difference Method (FDM)

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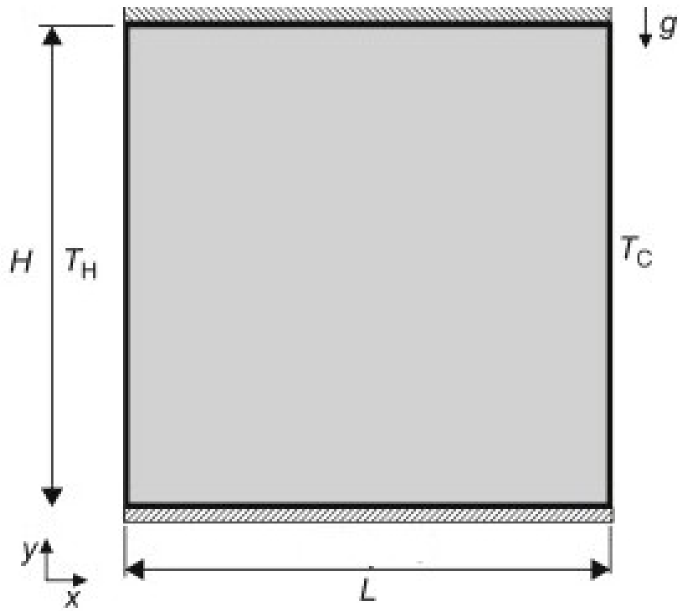
Basic Approach in Solving a Problem by CFD Techniques

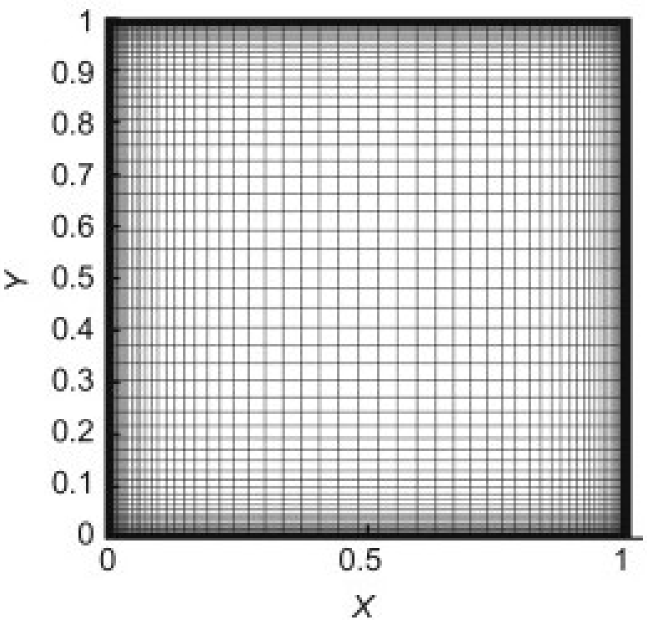
Name of the Approach	Process
Mathematical model	A set of partial differential equations and boundary conditions which include simplifications of exact conservation laws. A solution methodology is usually proposed for a particular set of equations.
Discretization scheme	A suitable discretization scheme is selected such as finite difference, finite element, or finite volume using which the differential equations are approximated by a system of algebraic equations for the variables at some set of discrete points in space and time
Coordinate system	A relevant coordinate system such as Cartesian, cylindrical, spherical, curvilinear orthogonal, or non-orthogonal, which may be fixed or moving, is chosen depending on the problem
Numerical grid	The computational domain is imagined to be filled with a grid that is a network of lines. The intersection of grid lines is called a grid point in the finite-difference scheme. A grid divides the solution domain into a number of subdomains. The conservation equations are assumed to be valid at each grid point in the computational domain in FDM. The position of a grid point is identified by a set of two such as (i, j) in 2D or three such as (i, j, k)in 3D indices. Each point has four surrounding neighbors in 2D and six in 3D. For an unsteady or a time-dependent problem, the additional indices are p and p + 1, indicating the present and future times, respectively.

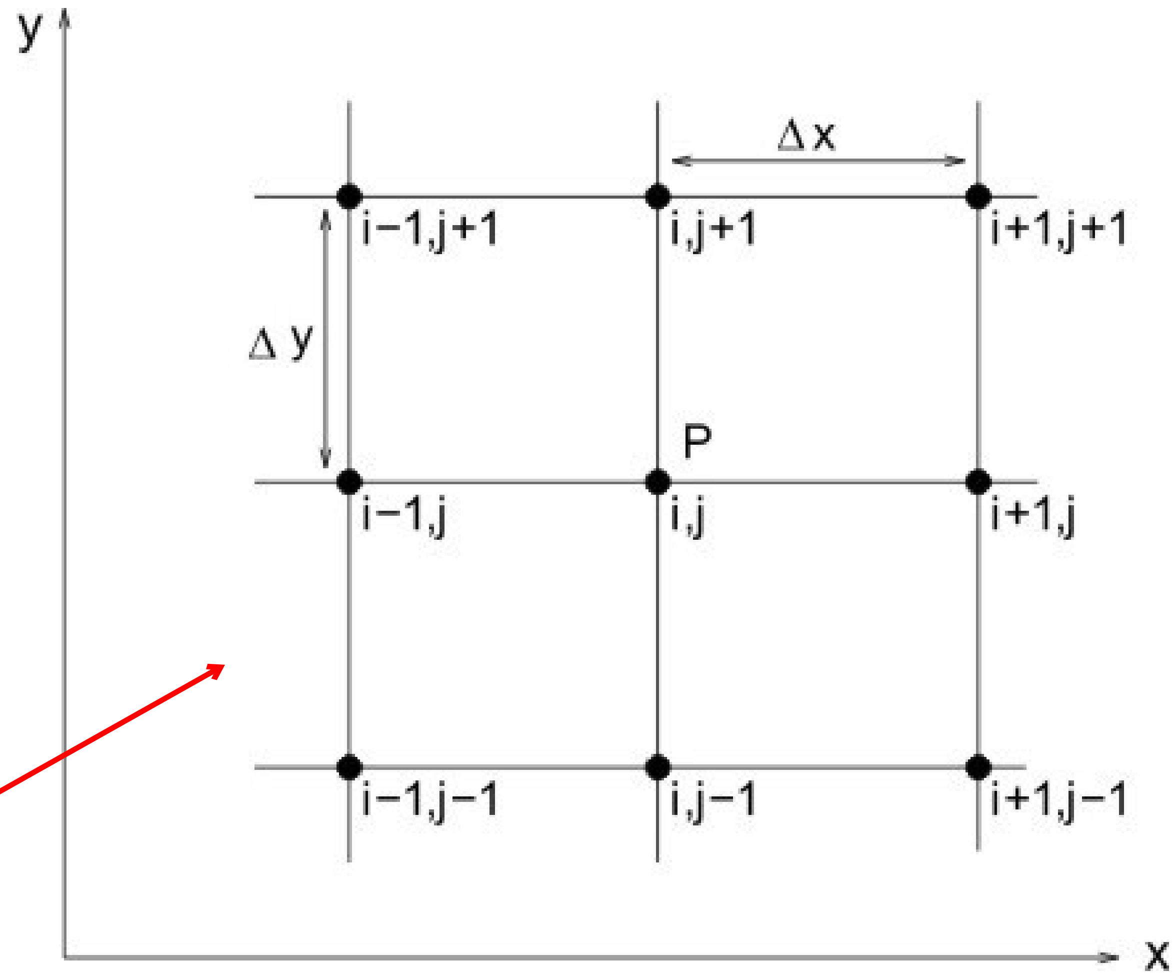
Methods of Discretization

Name of the Method	Process
Finite-Difference Method	The method includes the assumption that the variation of the unknown to be computed is somewhat like a polynomial in x, y, or z so that higher derivatives are unimportant.
Due to limitations of FDM in dealing with problems with increasing physical complexity, new advanced methods were developed. They can be divided in following 2 categories:	
a. **Finite element Method**	It finds solutions at discrete spatial regions (called elements) by assuming that the governing differential equations apply to the continuum within each element.
b. **Spectral Method**	The approximation is based on expansions of independent variables into finite series of smooth functions.
2. **Finite Volume Method**	The calculation domain is divided into a number of non-overlapping control volumes such that there is one control volume surrounding each grid point. The differential equation is integrated over each control volume. Piecewise profiles expressing the variation of the unknown between the grid points are used to evaluate the required integrals.

Basic concept of numerical grid points







Grid points

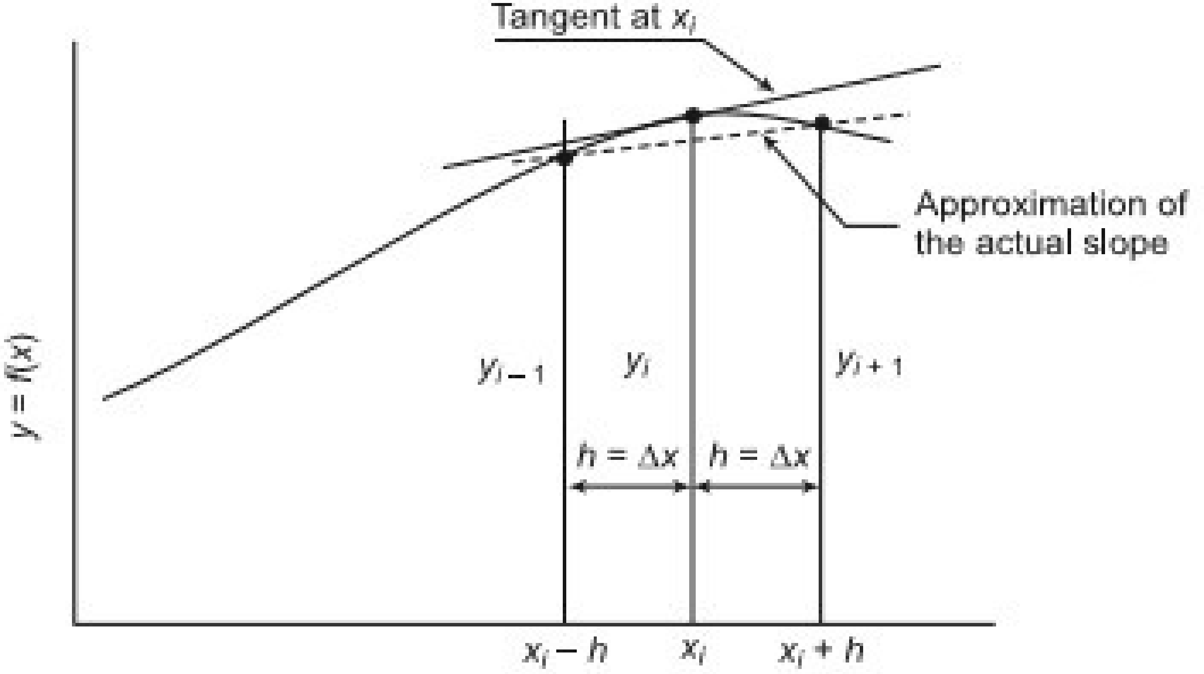
Introduction to the Finite Difference Method (FDM)

The finite difference method enables one to integrate a differential equation numerically by evaluating the values of the function at a discrete (finite) number of points. The origin of this method is Taylor series expansion, which assumes that the function is smooth, that is, continuous and differentiable.

Taylor series expansion

The Taylor series of a real or complex-valued function *f* (*x*) that is infinitely differentiable at a real or complex number *a* is the power series

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \frac{(x-a)^3}{3!}\,f'''(a) + \ldots.$$



Central difference

The Taylor series expansion of a function *f* (*x*) at x_i+h expanded about x_i is

$$f(x_i+h) = f(x_i) + (x_i+h-x_i)\, f'(x_i) + \frac{(x_i+h-x_i)^2}{2!} f''(x_i) + \frac{(x_i+h-x_i)^3}{3!} f'''(x_i) + \ldots.$$

Let, $y_i = f(x_i) \quad y_{i+1} = f(x_i+h) \quad y_{i-1} = f(x_i-h)$

$$y_{i+1} = y_i + h\, y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \ldots. \qquad (1)$$

Then, the Taylor series expansion of a function *f* (*x*) at x_i-h expanded about x_i is

$$y_{i-1} = y_i - h\, y_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \ldots. \qquad (2)$$

Subtracting (1)-(2), we get,

$$y_{i+1} - y_{i-1} = 2h\, y_i' + \frac{2h^3}{3!} y_i''' + \ldots.$$

$$\therefore y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{1}{6}\left(y_i''' h^2\right) + higher\ order\ terms$$

Therefore,

$$\therefore y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O\left(h^2\right)$$

Neglecting the O(h^2) terms, which is called the **truncation error (error due to truncated Taylor series)**, we obtain the central difference FDM expression

$$\therefore y_i' = \left(\frac{dy}{dx}\right)_{x=x_i} = \frac{y_{i+1} - y_{i-1}}{2h}$$

Now, adding Eqs. (1) and (2)

$$y_{i+1} + y_{i-1} = 2y_i + h^2 y_i'' + \frac{h^4}{12} y_i'''' + ...$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{1}{12}\left(y_i'''' h^2\right) + \text{higher order terms}$$

$$\therefore y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O\left(h^2\right)$$

Neglecting the O(h^2) terms

$$y_i'' = \left(\frac{d^2 y}{dx^2}\right)_{x=x_i} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Forward difference

From Taylor series expansions, it is also easy to obtain expressions for the derivatives that are entirely in terms of values of function at x_i and points to the right of x_i. These are called forward difference expressions.

$$y_{i+1} = y_i + h\,y_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \ldots.$$

$$\Rightarrow\; h\,y_i' = y_{i+1} - y_i - \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' - \ldots.$$

$$\Rightarrow\; y_i' = \frac{y_{i+1} - y_i}{h} - \frac{h}{2!}y_i'' - \frac{h^2}{3!}y_i''' - \ldots.$$

$$\therefore\, y_i' = \frac{y_{i+1} - y_i}{h} + O(h)$$

Neglecting the O(h) terms

$$\therefore\, y_i' = \left(\frac{dy}{dx}\right)_{x=x_i} = \frac{y_{i+1} - y_i}{h}$$

Similarly,

$$y_{i+2} = y_i + (2h)\, y_i' + \frac{(2h)^2}{2!} y_i'' + \frac{(2h)^3}{3!} y_i''' + \ldots. \qquad (1)$$

Also,

$$y_{i+1} = y_i + (h)\, y_i' + \frac{(h)^2}{2!} y_i'' + \frac{(h)^3}{3!} y_i''' + \ldots. \qquad (2)$$

Now, (1) – 2X(2) gives

$$\therefore y_i'' = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} + O(h)$$

Neglecting the O(h) terms

$$y_i'' = \left(\frac{d^2 y}{dx^2} \right)_{x=x_i} = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2}$$

Backward difference

With this approach, one can easily obtain derivative expressions that are entirely in terms of values of the function at x_i and points to the left of x_i. These are known as backward difference expressions, which are given below for y'_i and y''_i.

$$y_{i-1} = y_i - h y_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \ldots.$$

$$\Rightarrow h y_i' = y_i - y_{i-1} + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \ldots.$$

$$\Rightarrow y_i' = \frac{y_i - y_{i-1}}{h} + \frac{h}{2!} y_i'' - \frac{h^2}{3!} y_i''' + \ldots.$$

$$\therefore y_i' = \frac{y_i - y_{i-1}}{h} + O(h)$$

Neglecting the O(h) terms

$$\therefore y_i' = \left(\frac{dy}{dx}\right)_{x=x_i} = \frac{y_{i+1} - y_i}{h}$$

Similarly, one can obtain

$$y_i'' = \left(\frac{d^2 y}{dx^2}\right)_{x=x_i} = \frac{y_i - 2y_{i-1} + y_{i-2}}{h^2}$$

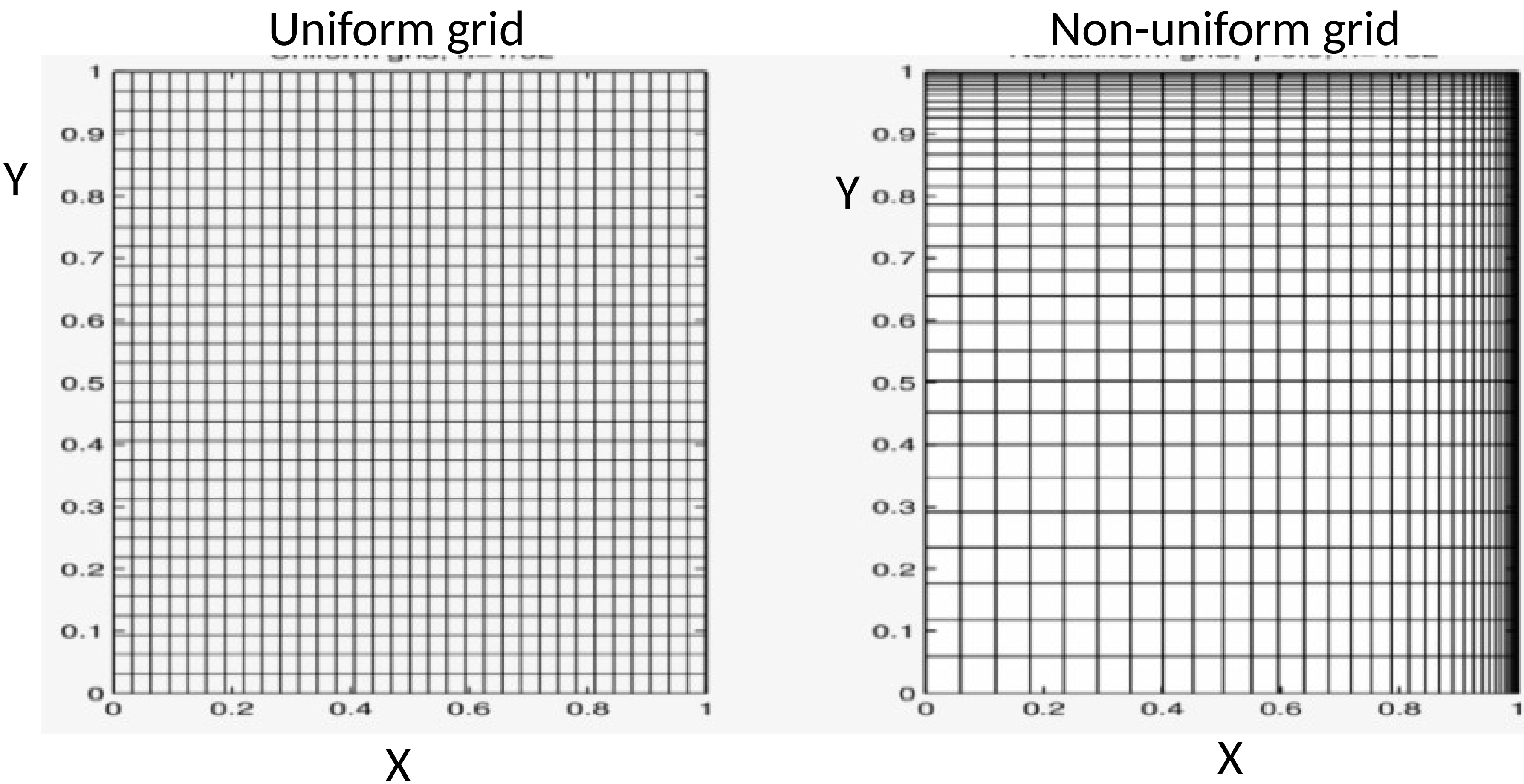
Employing forward, backward, and central difference expressions

I. Forward difference expressions are used when data to the left of a point at which a derivative is desired are not available.

II. Backward difference expressions are used when data to the right of the desired point are not available.

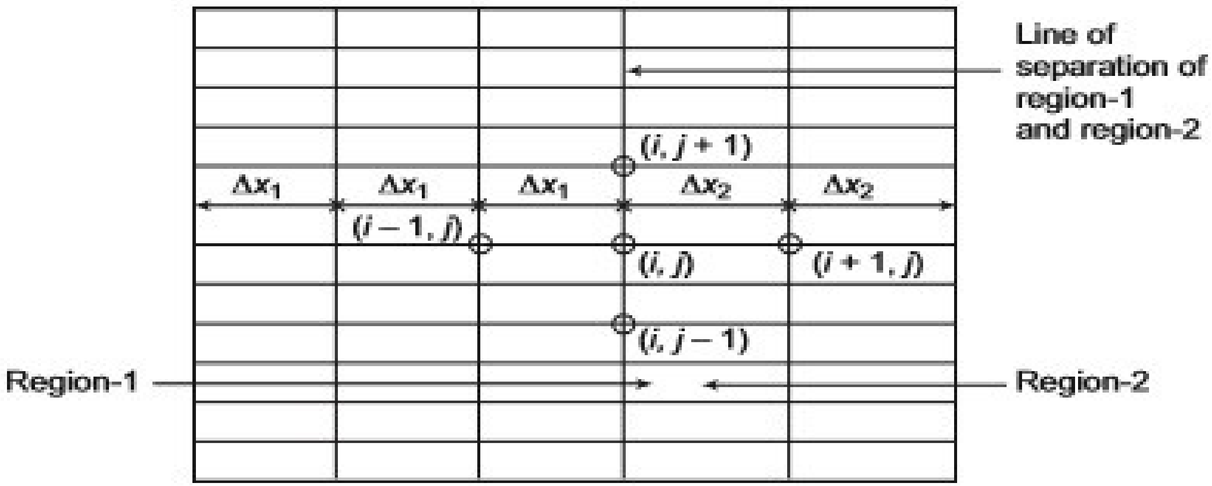
III. Central difference expressions are used when data on both sides of the desired point are available and are more accurate than either forward or backward difference expressions.

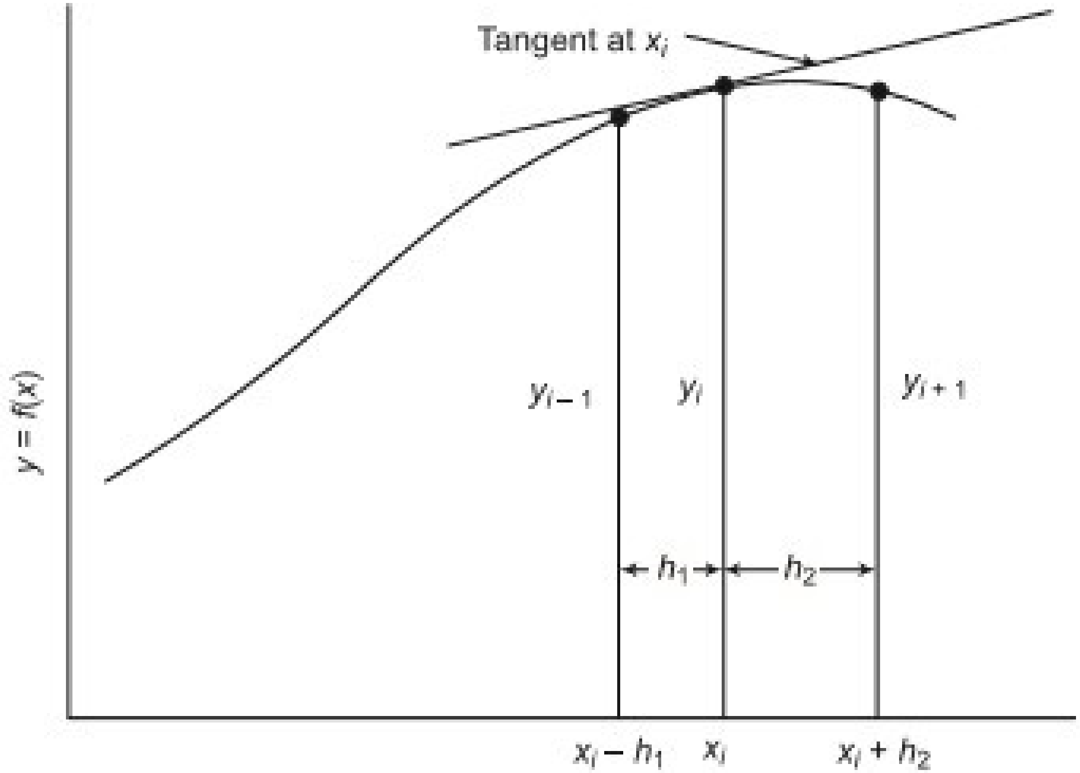
Concept of uniform and non-uniform grids



Non-uniform grid is used in the region where gradients of the are expected to be high. One can save on computational memory and execution time with the help of fine grid in the region.

Central difference expressions for a non-uniform grid





The Taylor series for the function y = f(x) at $x_i + h_2$ and $x_i - h_1$ expanded about x_i are:

$$y_{i+1} = y_i + h_2 y_i' + \frac{h_2^{\ 2}}{2!} y_i'' + \frac{h_2^{\ 3}}{3!} y_i''' + \ldots.$$

$$y_{i-1} = y_i - h_1 y_i' + \frac{h_1^{\ 2}}{2!} y_i'' - \frac{h_1^{\ 3}}{3!} y_i''' + \ldots.$$

In terms of $h_2/h_1 = R$, the equation can be written as,

$$y_{i+1} = y_i + (Rh_1) y_i' + \frac{(Rh_1)^2}{2!} y_i'' + \frac{(Rh_1)^3}{3!} y_i''' + \ldots. \qquad (1)$$

$$y_{i-1} = y_i - h_1 y_i' + \frac{h_1^{\ 2}}{2!} y_i'' - \frac{h_1^{\ 3}}{3!} y_i''' + \ldots. \qquad (2)$$

Now, (1) – R^2X(2) gives

$$y_{i+1} + R^2 y_{i-1} = y_i - R^2 y_i + (Rh_1) y_i' + \left(R^2 h_1\right) y_i' + O(h_1^{\ 3})$$

$$\therefore y_i' = \frac{y_{i+1} - y_i\left(1 - R^2\right) - R^2 y_{i-1}}{R(1+R)\, h_1} + O(h_1^{\ 2})$$

Checking for accuracy: For uniform grid, R = 1; hence, $h_1 = h_2 = h$. Substituting R = 1, we get

$$\therefore y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O\left(h^2\right)$$

Central difference for y''_i: Eliminating y'_i between the first two equations, we have

$$\therefore y_i'' = \frac{y_{i+1} - \left(1+R\right) y_i + R y_{i-1}}{\left(R/2\right)\left(1+R\right) h_1^2} + O(h_1)$$

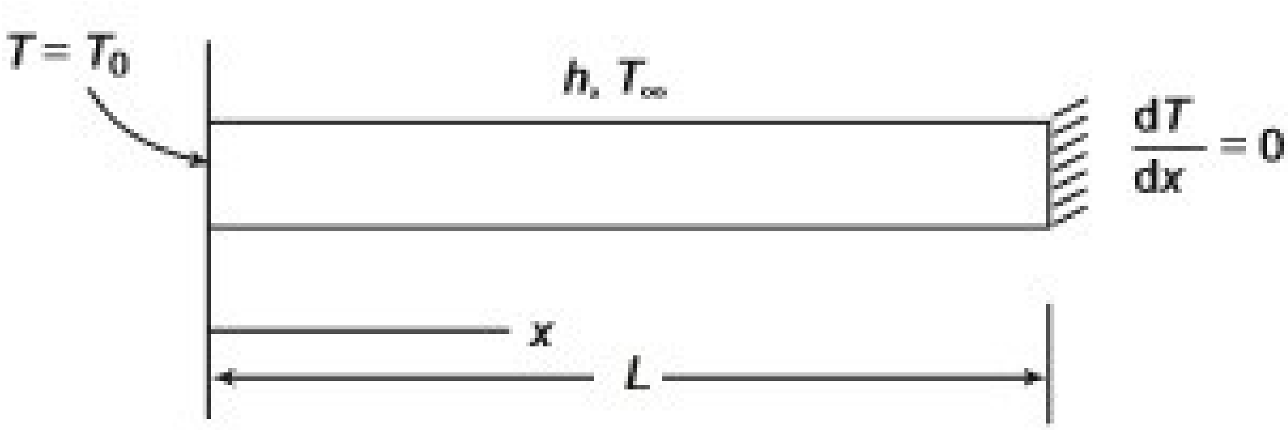
The order of accuracy for the central difference expression of is reduced by 1 in the case of non-uniform grid.

Checking for accuracy: For uniform grid, R = 1; hence, $h_1 = h_2 = h$. Substituting R = 1, we get

$$\therefore y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O\left(h^2\right)$$

Numerical solution methods

Step by step numerical solution of one dimensional steady conduction problem



Problem definition:

✓ Consider the one-dimensional steady-state heat conduction in an isolated rectangular horizontal fin. The base temperature is maintained at $T = T0$ and the tip of the fin is insulated.

✓ The fin is exposed to a convective environment (neglecting radiation heat transfer from the fin), which is at $T\infty$ ($T\infty < T0$). The average heat transfer coefficient of the fin to the ambient is h. The length of the fin is L, and the coordinate axis begins at the base of the fin.

✓ The one dimensionality arises from the fact that thickness of the fin is much small as compared to its length, and width can be considered either too long or the sides of the fin to be insulated.

Governing differential equation

The energy equation for the fin at the steady state (assuming constant k)

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0$$

where P= perimeter and A = cross-sectional area of the fin

Boundary conditions

Since the above equation is a linear, second-order ordinary differential equation, two boundary conditions are needed to completely describe this problem (which is a boundary value problem or elliptic problem). Boundary conditions are

$$BC1: \; at \; x = 0, \;\; T = T_0$$

$$BC2: \; at \; x = L, \;\; \frac{dT}{dx} = 0$$

Dimensionless form

Non-dimensionalizing using the dimensionless variables: $\theta = \dfrac{T - T_\infty}{T_0 - T_\infty}, \;\; X = \dfrac{x}{L}$

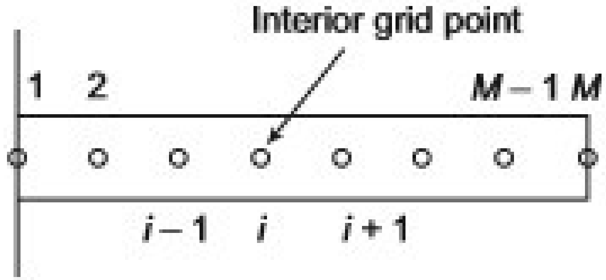
Hence, $$\frac{d^2\theta}{dX^2} - (mL)^2\,\theta = 0$$ Where, $m^2 = hP/kA$

And $$BC1: \; at \; X = 0, \;\; \theta = 1$$

$$BC2: \; at \; x = 1, \;\; \frac{d\theta}{dX} = 0$$

Discretization

The equation given below is discretized at any interior grid point *i* using central difference for $d^2\Theta/dX^2$ as follows:



$$\frac{d^2\theta}{dX^2} - (mL)^2\theta = 0$$

$$\Rightarrow \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta x)^2} - (mL)^2\theta_i = 0$$

$$\therefore \theta_{i-1} - D\theta_i + \theta_{i+1} = 0 \qquad \text{for } i = 1, 2, \ldots.., M$$

$$\text{where, } D = 2 + (mL)^2(\Delta x)^2$$

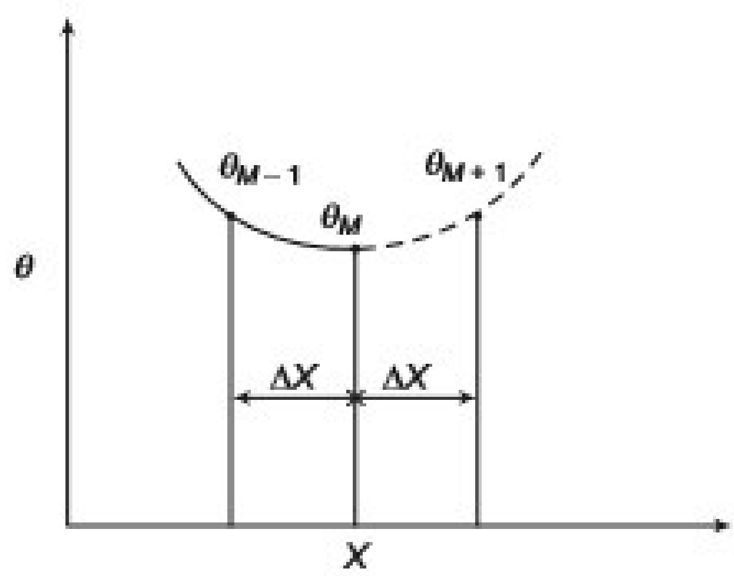
Handling of the boundary condition: At *x* = L, *i.e., at i* = M, reduces to:

$$\theta_{M-1} - D\theta_M + \theta_{M+1} = 0$$

By observing above equation it is revealed that Θ_{M+1} represents a fictitious temperature Ө at point M + 1, which lies outside the computational domain.

Method 1: image point technique

It is assumed that *Θ vs. X* curve extends beyond *X = 1* so that at *X = 1*, the condition *dΘ/dX = 0* is satisfied.



The dotted line represents the mirror-image extension of the solid line, indicating that a minima exists at X = 1. Mirror-image extension of Θ vs. X curve near the fin tip.

At point M
$$\left(\frac{d\theta}{dX}\right)_M = 0$$

$$\Rightarrow \frac{\theta_{M+1} - \theta_{M-1}}{2\Delta X} = 0$$

$$\therefore \theta_{M+1} = \theta_{M-1}$$

Hence,

$$\theta_{M-1} - D\theta_M + \theta_{M+1} = 0$$

$$\Rightarrow \theta_{M-1} - D\theta_M + \theta_{M-1} = 0$$

$$\therefore 2\theta_{M-1} - D\theta_M = 0$$

Method 2: use of higher order backward difference expression

An alternative to image point technique is to use a second-order backward difference for

$$\left(\frac{d\theta}{dX}\right)_M = 0$$

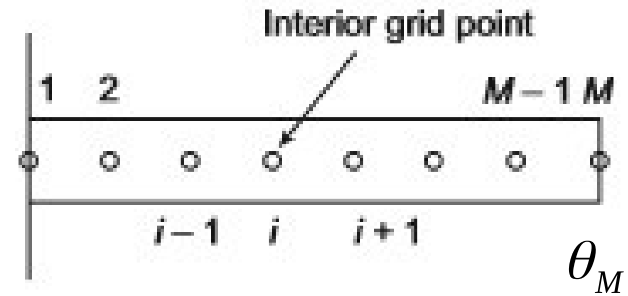
$$\Rightarrow \frac{3\theta_M - 4\theta_{M-1} + \theta_{M-2}}{2(\Delta X)} + O(\Delta X^2) = 0$$

$$\Rightarrow \frac{3\theta_M - 4\theta_{M-1} + \theta_{M-2}}{2(\Delta X)} = 0$$

$$\therefore 3\theta_M - 4\theta_{M-1} + \theta_{M-2} = 0$$

The above equation is valid at the grid point on the right boundary. The second-order scheme is used to match the order of accuracy of the central difference scheme used for interior points.

Matrix equations



$$\theta_1 = 1 \qquad \text{for } i = 1$$

$$\theta_{i-1} - D\theta_i + \theta_{i+1} = 0 \qquad \text{for } i = 2, \ldots\ldots, M-1$$

$$2\theta_{M-1} - D\theta_M = 0 \qquad \text{for } i = M$$

The above sets of algebraic equations can be written as,

$$\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 1 & -D & 1 & 0 & . & . & 0 \\ 0 & 1 & -D & 1 & . & . & 0 \\ 0 & 0 & 1 & -D & 1 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 2 & -D \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ . \\ . \\ \theta_{M-1} \\ \theta_M \end{Bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A\theta = b$$

Matrix equations

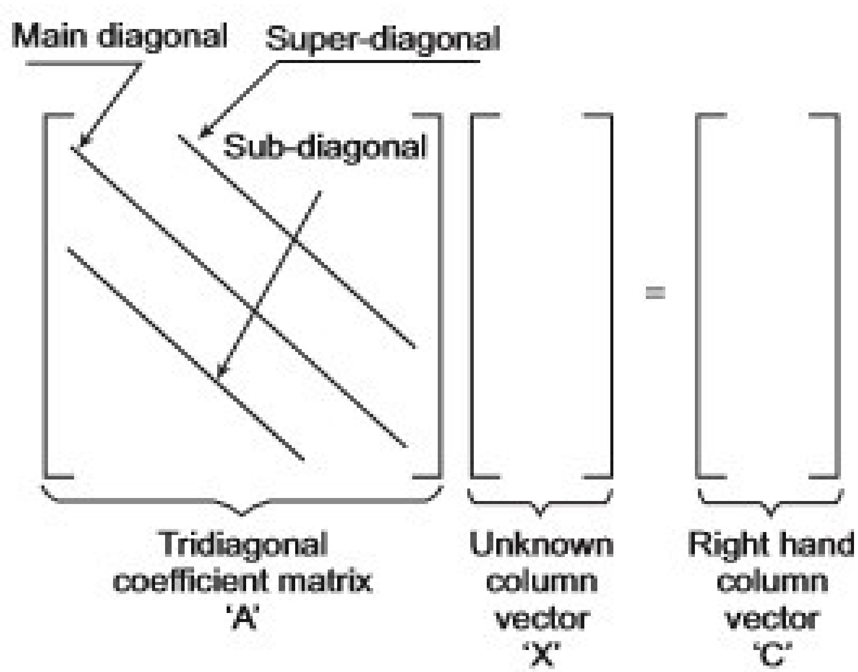
$$A=\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 1 & -D & 1 & 0 & . & . & 0 \\ 0 & 1 & -D & 1 & . & . & 0 \\ 0 & 0 & 1 & -D & 1 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 2 & -D \end{bmatrix}, \quad \theta=\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ . \\ . \\ \theta_{M-1} \\ \theta_M \end{Bmatrix}, \quad b=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Tridiagonal matrix

Tridiagonal system of equations

Method of Solution

The coefficient matrix has three diagonals: main diagonal, sub-diagonal, and super-diagonal; hence, the name tridiagonal matrix.



The set of equations can be solved by any of the following three methods:

1. Gaussian elimination.
2. Thomas algorithm (or tridiagonal matrix algorithm or simply TDMA).
3. Gauss−Seidel iterative method.

Relaxation (over-relaxation and under-relaxation)

One of the problems with G–S method is that it is relatively slow to converge to the solution. The rate of convergence can often be improved by relaxation method

$$x_i^{(p+1)} = \frac{\alpha\left[c_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(p+1)} - \sum_{j=i+1}^{N} a_{ij}x_j^{(p)}\right]}{a_{ii}} + (1-\alpha)\,x_i^{(p)} \qquad for\ i = 1, 2, 3, \ldots\ldots N$$

$0 < \alpha < 1$ Under-relaxation

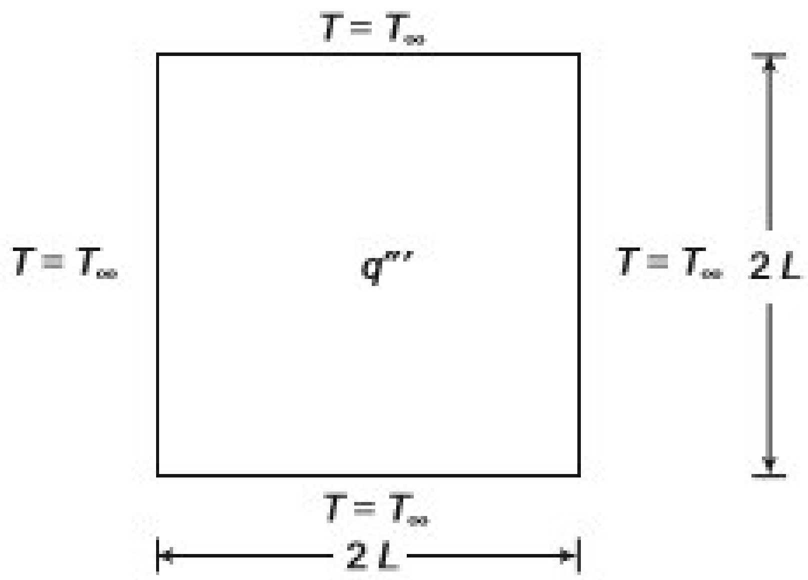
$1 < \alpha < 2$ Over-relaxation

- ✓Successive under-relaxation (SUR) or under-relaxation is generally used for non-linear equations and for systems that result in a divergent G–S iteration.
- ✓Successive over-relaxation (SOR) or over-relaxation is widely used for accelerating convergence in linear systems.

Summary of three methods

G-E	TDMA	G-S
Direct solver	Direct solver	Iterative
Based on foreword elimination and back substitution	Based on recursion formula	Based on initial guess of unknowns and subsequent improvement in each iteration
High round-off error	Low round-off error	Low round-off error
Number of arithmetic operation $O(N^3)$	Number of arithmetic operation $O(N)$	Number of arithmetic operation $O(N)$
Can be used low numbers of equations	Can be used high numbers of equations	Can be used for several thousands numbers of equations

Two-dimensional steady-state problem

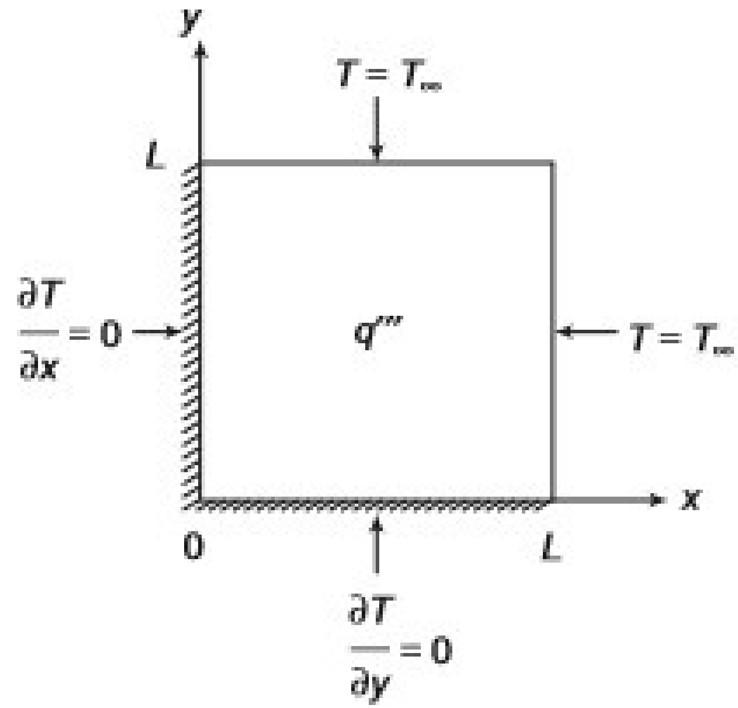


✓The case of steady heat conduction in a long square slab *(2L × 2L)* in which heat is generated at a uniform rate of q''' W/m3.

✓The problem can be assumed to be a two dimensional as the dimension of the slab is much longer in the direction normal to the cross-sectional plane; therefore, end effects can be neglected.

✓ All four sides are maintained at $T = T_\infty$, temperature of the surrounding fluid, assuming a large heat transfer coefficient.

Consideration of symmetry



- A close look at the physics of the problem reveals that the problem is geometrically and thermally symmetric.
- Therefore, from the temperature distribution in any quarter of the physical domain by mirror-imaging, one can get the solution for the entire region.
- The use of symmetry enables the numerical analyst to obtain the solution much faster as the number of grid points is greatly reduced.

Governing differential equation

The energy equation at the steady state (assuming constant k)

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+q''' = 0$$

Boundary conditions

Boundary conditions are

$$BC1:\ at\ x=0,\ \ \frac{\partial T}{\partial x}=0$$

$$BC2:\ at\ x=L,\ T=T_\infty$$

$$BC3:\ at\ y=0,\ \ \frac{\partial T}{\partial y}=0$$

$$BC4:\ at\ y=L,\ T=T_\infty$$

Dimensionless form

$$\theta=\frac{T-T_\infty}{\left(q'''L^2/k\right)},\ \ X=\frac{x}{L}, Y=\frac{y}{L}$$

Non-dimensionalizing using the dimensionless variables:

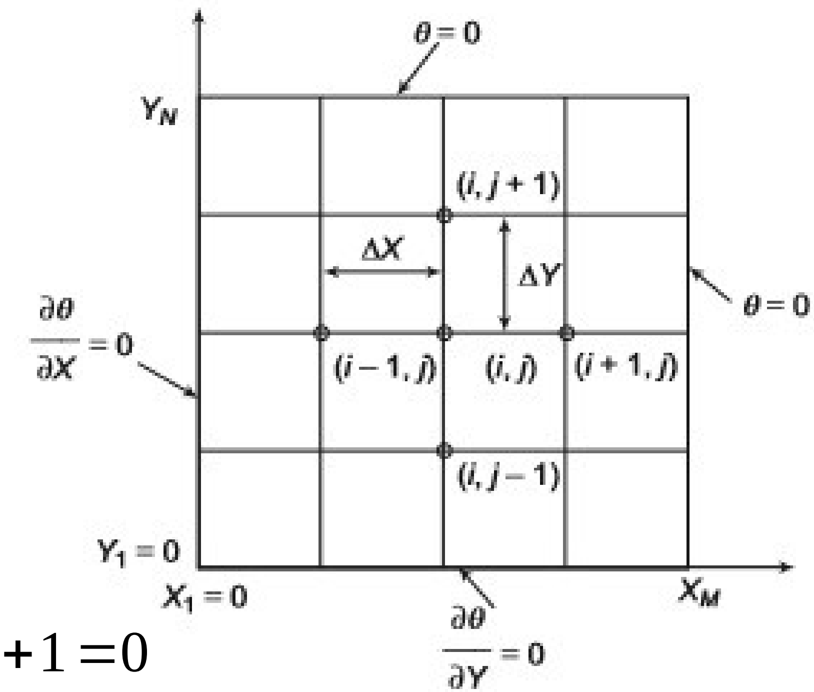
$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+1=0$$

$$BC1:\ at\ X=0,\ \frac{\partial\theta}{\partial X}=0$$

$$BC2:\ at\ X=1,\ \theta=0$$

$$BC3:\ at\ Y=0,\ \frac{\partial\theta}{\partial Y}=0$$

$$BC4:\ at\ Y=1,\ \theta=0$$



Discretization

The equation is discretized at any interior grid point (*i,j)* using central difference as follows:

$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+1=0$$

$$\Rightarrow\ \frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{(\Delta X)^2}+\frac{\theta_{i,j+1}-2\theta_{i,j}+\theta_{i,j-1}}{(\Delta Y)^2}+1=0$$

For a uniform grid, ΔX=ΔY

$$\Rightarrow\ -\theta_{i-1,j}-\theta_{i,j-1}+4\theta_{i,j}-\theta_{i,j+1}-\theta_{i+1,j}=(\Delta X)^2 \qquad \text{(a)}$$

- **Boundary condition along X = 0:**

Using image point technique $\theta_{i-1,j} = \theta_{i+1,j}$

Setting i=1 in Eq. (a), we have

$$-2\theta_{2,j} - \theta_{1,j-1} + 4\theta_{1,j} - \theta_{1,j+1} = (\Delta X)^2$$

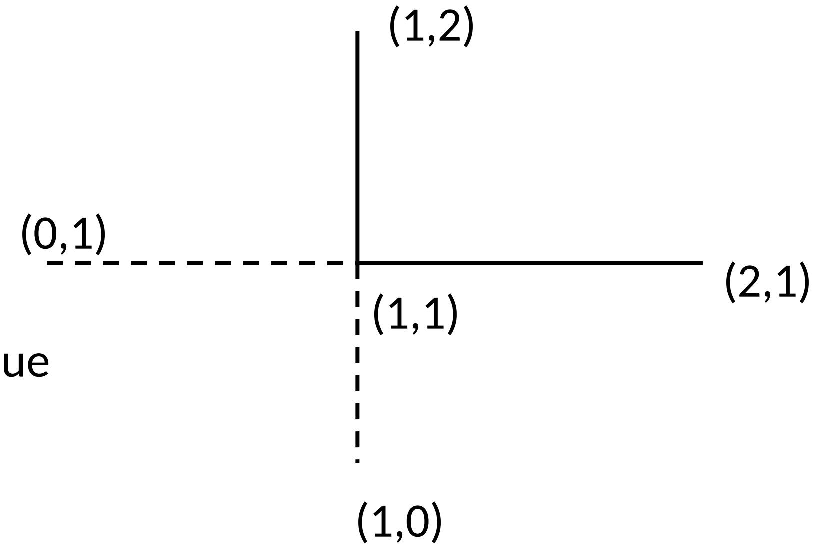
- **Boundary condition along Y= 0:**

Using image point technique $\theta_{i,j-1} = \theta_{i,j+1}$

Setting j=1 in Eq. (a), we have

$$-\theta_{i+1,1} - 2\theta_{i,2} + 4\theta_{i,1} - \theta_{i-1,1} = (\Delta X)^2$$

Handling of corner points



Using image point technique

$$\theta_{0,1} = \theta_{2,1}$$

$$\theta_{1,0} = \theta_{1,2}$$

$$\left(\frac{\partial^2\theta}{\partial X^2}\right)_{1,1} + \left(\frac{\partial^2\theta}{\partial Y^2}\right)_{1,1} + 1 = 0$$

$$\Rightarrow \frac{\theta_{2,1} - 2\theta_{1,1} + \theta_{0,1}}{(\Delta X)^2} + \frac{\theta_{1,2} - 2\theta_{1,1} + \theta_{1,0}}{(\Delta Y)^2} + 1 = 0$$

$$\Rightarrow \frac{\theta_{2,1} - 2\theta_{1,1} + \theta_{2,1}}{(\Delta X)^2} + \frac{\theta_{1,2} - 2\theta_{1,1} + \theta_{1,2}}{(\Delta Y)^2} + 1 = 0$$

For a uniform grid, ΔX=ΔY

$$2\theta_{2,1} - 4\theta_{1,1} + 2\theta_{1,2} + (\Delta X)^2 = 0$$

Solution method Gauss-Seidel method

Equations:

$$-\theta_{i-1,j}-\theta_{i,j-1}+4\theta_{i,j}-\theta_{i,j+1}-\theta_{i+1,j}=(\Delta X)^2$$

$$-2\theta_{2,j}-\theta_{1,j-1}+4\theta_{1,j}-\theta_{1,j+1}=(\Delta X)^2$$

$$-\theta_{i+1,1}-2\theta_{i,2}+4\theta_{i,1}-\theta_{i-1,1}=(\Delta X)^2$$

$$2\theta_{2,1}-4\theta_{1,1}+2\theta_{1,2}+(\Delta X)^2=0$$

Pseudo code:

$$for\ i=1,\ j=1$$

$$\theta_{1,1}=\frac{1}{4}\left(2\theta_{2,1}+2\theta_{1,2}+(\Delta X)^2\right)$$

$$for\ i=M,\ j=2,N-1$$

$$\theta_{1,j}=\frac{1}{4}\left[(\Delta X)^2+2\theta_{2,j}+\theta_{1,j-1}+\theta_{1,j+1}\right]$$

$$for\ i=2,M-1,\ j=1$$

$$\theta_{i,1}=\frac{1}{4}\left[(\Delta X)^2+\theta_{i+1,1}+2\theta_{i,2}+\theta_{i-1,1}\right]$$

$$for\ i=2,M-1,\ j=2,N-1$$

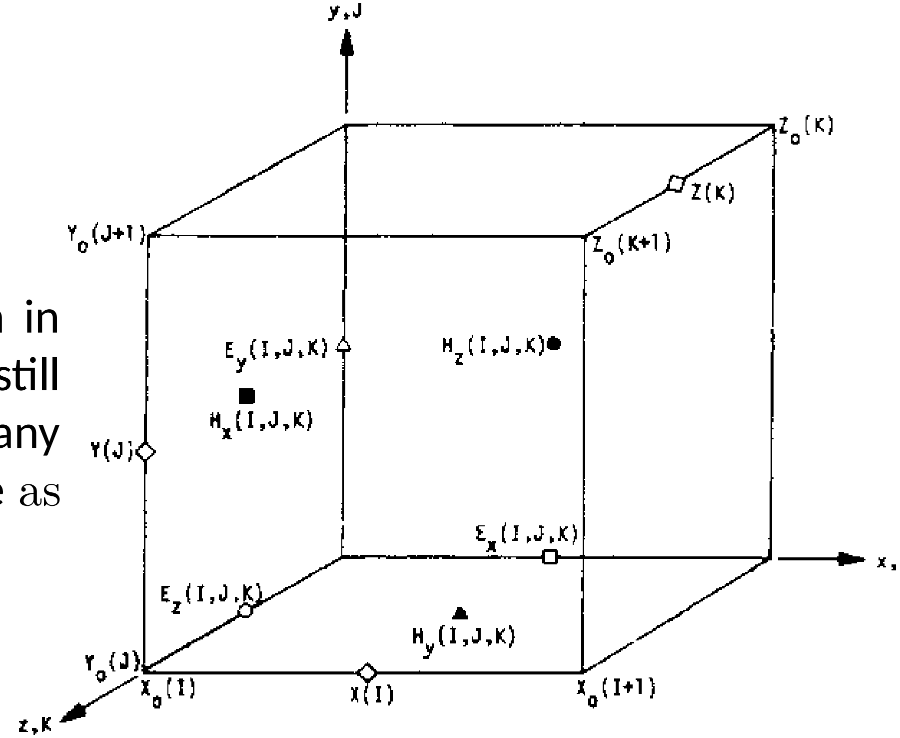
$$\theta_{i,j}=\frac{1}{4}\left[(\Delta X)^2+\theta_{i-1,j}+\theta_{i,j-1}+\theta_{i,j+1}+\theta_{i+1,j}\right]$$

Initial guess,

$$for\ i=1,M,\ J=1,N$$

$$\theta_{i,j}=0$$

Three-dimensional problems



For three-dimensional steady heat conduction in Cartesian coordinates, the basic approach is still the same. The equation is discretized at any interior grid point (*i,j,k*) using central difference as follows:

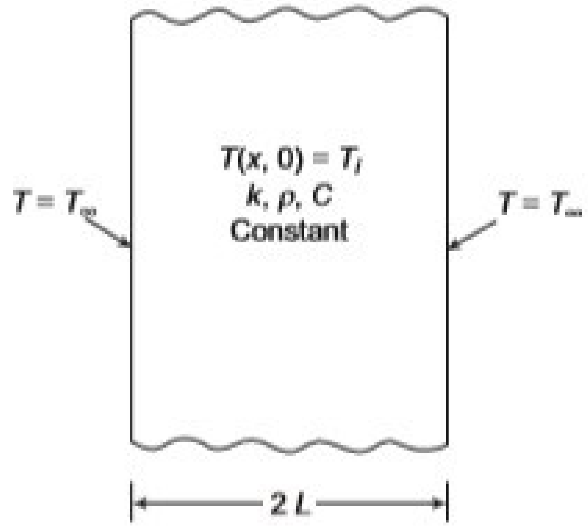
$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+\frac{\partial^2\theta}{\partial Z^2}1=0$$

$$\Rightarrow \frac{\theta_{i+1,j,k}-2\theta_{i,j,k}+\theta_{i-1,j,k}}{(\Delta X)^2}+\frac{\theta_{i,j+1,k}-2\theta_{i,j,k}+\theta_{i,j-1,k}}{(\Delta Y)^2}+\frac{\theta_{i,j,k+1}-2\theta_{i,j,k}+\theta_{i,j,k-1}}{(\Delta Z)^2}+1=0$$

For a uniform grid, ΔX=ΔY= ΔZ

$$\Rightarrow \theta_{i+1,j,k}+\theta_{i-1,j,k}+\theta_{i,j+1,k}+\theta_{i,j-1,k}+\theta_{i,j,k+1}+\theta_{i,j,k-1}-6\theta_{i,j,k}+(\Delta X)^2=0$$

Transient one-dimensional problem



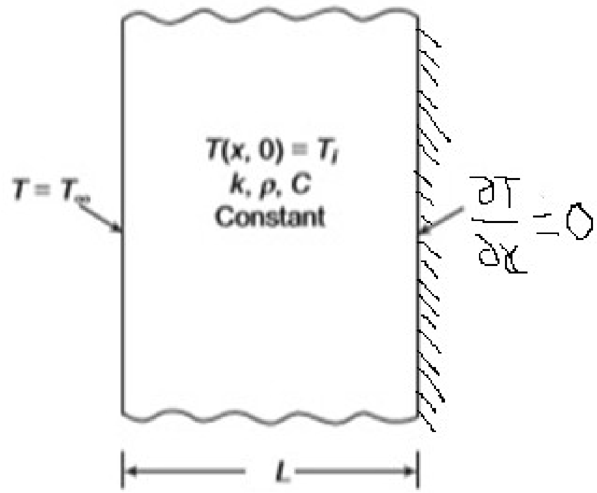
✓The case of a hot infinite plate of finite thickness 2L, is suddenly exposed to an atmosphere of temperature $T = T_\infty$.

✓The initial temperature is $T = T_i$.

✓ Heat transfer coefficient is large.

We may choose the consideration of symmetry.

Consideration of symmetry



Dimensionless form of governing equation

The energy equation in dimensionless form for constant thermophysical properties,

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2}$$

Where, $$\theta = \frac{T - T_\infty}{T_i - T_\infty}, \; X = \frac{x}{L}, \tau = \frac{\alpha t}{L^2}$$

Initial and boundary conditions

$$IC:\ at\ \tau = 0,\ \ \theta = 1,\ for\ all\ X$$

$$for\ \tau > 0,$$

$$BC1:\ at\ X = 0,\ \theta = 0$$

$$BC2:\ at\ X = 1,\ \frac{\partial \theta}{\partial X} = 0$$

Discretization

For any interior grid point, the FDM formulation gives

$$\frac{\partial \theta}{\partial \tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2} \qquad for\ \ i = 1, 2, \ldots\ldots, M$$

At the boundary point, *i*=*M*, we employ image-point technique to obtain,

$$\theta_{M+1} = \theta_{M-1}$$

$$\frac{\partial \theta_M}{\partial \tau} = \frac{2\theta_{M-1} - 2\theta_M}{(\Delta X)^2}$$

Methods of solution

There are three ways by which the initial-value problem can be solved. These are

(i) Euler method (or explicit method)

(ii) Crank–Nicolson method

(iii) Pure implicit method

Euler method (or explicit method)

Solution of temperature at the present time τ^{p} *is* θ^{p}

We seek the solution for the temperature at the future time, τ^{p+1} *is* θ^{p+1} with an increment in time as

$$\tau^{p+1} = \tau^{p} + \Delta\tau$$

The solution at the future time is obtained as

$$\theta^{p+1} = \theta^{p} + \left(\frac{d\theta}{d\tau}\right)^{p} \Delta\tau$$

Therefore, the discretized form,

$$\frac{\partial \theta}{\partial \tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

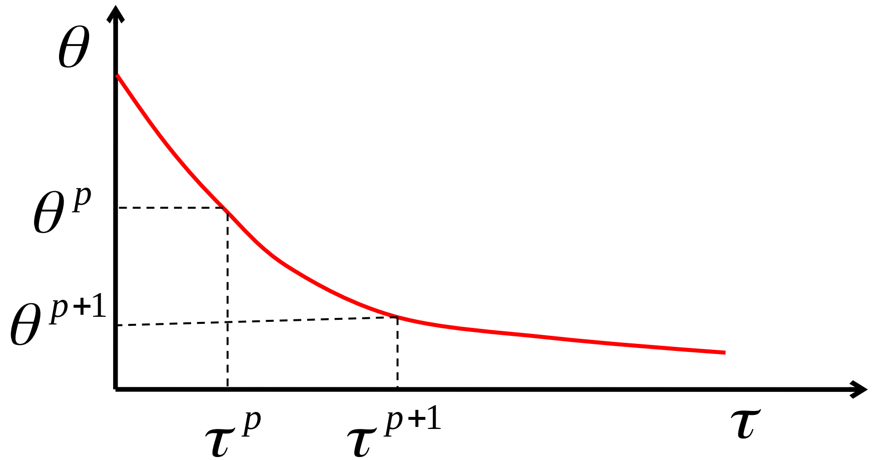
$$\Rightarrow \frac{\theta_i^{p+1} - \theta_i^p}{\Delta \tau} = \frac{\theta_{i+1}^p - 2\theta_i^p + \theta_{i-1}^p}{(\Delta X)^2}$$

$$\therefore \theta_i^{p+1} = \left\{\frac{\Delta \tau}{(\Delta X)^2}\right\}\theta_{i+1}^p + \left(1 - \frac{2\Delta \tau}{(\Delta X)^2}\right)\theta_i^p + \left\{\frac{\Delta \tau}{(\Delta X)^2}\right\}\theta_{i-1}^p$$

This method has some disadvantage as the solution becomes unstable when,

$$\left(1 - \frac{2\Delta \tau}{(\Delta X)^2}\right) \leq 0$$

Stability criteria, restriction on time-step



Crank–Nicolson method

Solution of temperature at the present time τ^{p} is θ^{p}

We seek the solution for the temperature at the future time, τ^{p+1} is θ^{p+1} with an increment in time is assumed as an arithmetic mean value of the derivates between the beginning and end of the time interval as

$$\theta^{p+1} = \theta^{p} + \frac{1}{2}\left[\left(\frac{d\theta}{d\tau}\right)^{p} + \left(\frac{d\theta}{d\tau}\right)^{p+1}\right]\Delta\tau$$

Therefore, the discretized form,

$$\frac{\partial\theta}{\partial\tau} = \frac{\theta_{i+1} - 2\theta_{i} + \theta_{i-1}}{(\Delta X)^{2}}$$

$$\Rightarrow \frac{\theta_{i}^{p+1} - \theta_{i}^{p}}{\Delta\tau} = \frac{1}{2}\left[\frac{\theta_{i+1}^{p} - 2\theta_{i}^{p} + \theta_{i-1}^{p}}{(\Delta X)^{2}} + \frac{\theta_{i+1}^{p+1} - 2\theta_{i}^{p+1} + \theta_{i-1}^{p+1}}{(\Delta X)^{2}}\right]$$

$$\therefore \left\{1 + \frac{\Delta\tau}{(\Delta X)^{2}}\right\}\theta_{i}^{p+1} = \left\{\frac{\Delta\tau}{2(\Delta X)^{2}}\right\}\theta_{i+1}^{p} + \left(1 + \frac{\Delta\tau}{(\Delta X)^{2}}\right)\theta_{i}^{p}$$

$$+ \left\{\frac{\Delta\tau}{2(\Delta X)^{2}}\right\}\theta_{i-1}^{p} + \left\{\frac{\Delta\tau}{2(\Delta X)^{2}}\right\}\left(\frac{\theta_{i+1}^{p+1} + \theta_{i-1}^{p+1}}{(\Delta X)^{2}}\right)$$

The Crank–Nicolson method is **unconditionally stable** and therefore no restriction in the time step.

Pure Implicit method:

Solution of temperature at the present time τ^{p} *is* θ^{p}

We seek the solution for the temperature at the future time, τ^{p+1} *is* θ^{p+1} is assumed that the time derivates at the new time is used to move ahead with the time. It can be expressed as

$$\theta^{p+1} = \theta^{p} + \left(\frac{d\theta}{d\tau}\right)^{p+1} \Delta\tau$$

Therefore, the discretized form,

$$\frac{\partial\theta}{\partial\tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

$$\Rightarrow \frac{\theta_i^{p+1} - \theta_i^{p}}{\Delta\tau} = \frac{\theta_{i+1}^{p+1} - 2\theta_i^{p+1} + \theta_{i-1}^{p+1}}{(\Delta X)^2}$$

$$\therefore \left[1 + \frac{2\Delta\tau}{(\Delta X)^2}\right]\theta_i^{p+1} = \theta_i^{p} + \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i+1}^{p+1} + \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i-1}^{p+1}$$

The implicit method is **unconditionally stable** and therefore no restriction in the time step.

The Crank–Nicolson method is **unconditionally stable** and therefore no restriction in the time step.

Pure Implicit method:

Solution of temperature at the present time τ^p is θ^p

We seek the solution for the temperature at the future time, τ^{p+1} is θ^{p+1} is assumed that the time derivates at the new time is used to move ahead with the time. It can be expressed as

$$\theta^{p+1} = \theta^p + \left(\frac{d\theta}{d\tau}\right)^{p+1} \Delta\tau$$

Therefore, the discretized form,

$$\frac{\partial\theta}{\partial\tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

$$\Rightarrow \frac{\theta_i^{p+1} - \theta_i^p}{\Delta\tau} = \frac{\theta_{i+1}^{p+1} - 2\theta_i^{p+1} + \theta_{i-1}^{p+1}}{(\Delta X)^2}$$

$$\therefore \left[1 + \frac{2\Delta\tau}{(\Delta X)^2}\right]\theta_i^{p+1} = \theta_i^p + \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i+1}^{p+1} + \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i-1}^{p+1}$$

The implicit method is **unconditionally stable** and therefore no restriction in the time step.

Accuracy of the methods

Euler method (or explicit method): FTCS

Space accuracy: $(\Delta X)^2$

Time accuracy: $(\Delta \tau)$

Crank–Nicolson method: CTCS

Space accuracy: $(\Delta X)^2$

Time accuracy: $(\Delta \tau)^2$

Pure Implicit method: BTCS

Space accuracy: $(\Delta X)^2$

Time accuracy: $(\Delta \tau)$

Consistency of Numerical Scheme

$$Original\ PDE = \text{Discretized PDE} + O(\Delta X, \Delta Y, \Delta\tau)^n$$

Now, a numerical scheme is consistent, if for $(\Delta X, \Delta Y, \Delta\tau) \to 0$

$$\text{Discretized PDE} = Original\ PDE$$

Now, a numerical scheme is inconsistent, if for $(\Delta X, \Delta Y, \Delta\tau) \to 0$

$$\text{Discretized PDE} \neq Original\ PDE$$

Two-dimensional transient problem

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

Euler or explicit method of solution leads to

$$\frac{T_{i,j}^{p+1} - T_{i,j}^{p}}{\Delta t} = \alpha \left[\frac{T_{i+1,j}^{p} - 2T_{i,j}^{p} + T_{i-1,j}^{p}}{(\Delta x)^2} + \frac{T_{i,j+1}^{p} - 2T_{i,j}^{p} + T_{i,j-1}^{p}}{(\Delta y)^2} \right]$$

The solution to the above equation face no difficulties if the stability condition is satisfied

$$\Delta t \le \frac{1}{2\alpha \left[(\Delta x)^{-2} + (\Delta y)^{-2} \right]}$$

Similarly pure implicit method of solution leads to

$$\frac{T_{i,j}^{p+1} - T_{i,j}^{p}}{\Delta t} = \alpha \left[\frac{T_{i+1,j}^{p+1} - 2T_{i,j}^{p+1} + T_{i-1,j}^{p+1}}{(\Delta x)^2} + \frac{T_{i,j+1}^{p+1} - 2T_{i,j}^{p+1} + T_{i,j-1}^{p+1}}{(\Delta y)^2} \right]$$

For $\Delta x = \Delta y$

$$-rT_{i-1,j}^{p+1} - rT_{i,j-1}^{p+1} + (1+4r)T_{i,j}^{p+1} - rT_{i,j+1}^{p+1} - rT_{i+1,j}^{p+1} = T_{i,j}^{p}, \quad \text{where, } r = \alpha \Delta t / (\Delta x)^2$$

Alternating Direction Implicit (ADI) method

This method employs two difference equations, which are used in turn over successive time-steps of duration Δt/2. The first equation is implicit in x-direction, whereas, the second one is implicit y-direction.

Let, $T^*_{i,j}$ is an intermediate value at the end of first Δt/2 time step, then,

$$\frac{T^*_{i,j} - T^p_{i,j}}{(\Delta t/2)} = \alpha\left[\frac{T^*_{i+1,j} - 2T^*_{i,j} + T^*_{i-1,j}}{(\Delta x)^2} + \frac{T^p_{i,j+1} - 2T^p_{i,j} + T^p_{i,j-1}}{(\Delta y)^2}\right]$$

Followed by,

$$\frac{T^{p+1}_{i,j} - T^*_{i,j}}{(\Delta t/2)} = \alpha\left[\frac{T^*_{i+1,j} - 2T^*_{i,j} + T^*_{i-1,j}}{(\Delta x)^2} + \frac{T^{p+1}_{i,j+1} - 2T^{p+1}_{i,j} + T^{p+1}_{i,j-1}}{(\Delta y)^2}\right]$$

For $\Delta x = \Delta y$

$$-T^*_{i-1,j} + 2\left(\frac{1}{r}+1\right)T^*_{i,j} - T^*_{i+1,j} = T^p_{i,j-1} + 2\left(\frac{1}{r}-1\right)T^p_{i,j} + T^p_{i,j+1}, \quad \text{where, } r = \alpha\Delta t/(\Delta x)^2$$

$$-T^{p+1}_{i,j-1} + 2\left(\frac{1}{r}+1\right)T^{p+1}_{i,j} - T^{p+1}_{i,j+1} = T^*_{i-,j} + 2\left(\frac{1}{r}-1\right)T^*_{i,j} + T^*_{i+1,j}$$

Accuracy $O\left[(\Delta x)^2, (\Delta y)^2, (\Delta t)^2\right]$

False transient approach

In this method, the solution of an elliptic problem can be obtained by using the methods for the solution of parabolic problem.

Example: For a steady elliptic problem,

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=0 \quad \text{in a region } R$$

Can be solved by assuming,

$$\frac{\partial T}{\partial t}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

Some initial condition is first applied throughout *R*.

Here, all the schemes of transient problem can be applied and can be time-stepped till the steady state is reached.

False transient: Although the problem is not time-dependent, it is solved in a manner as if it is time-dependent.

Numerical methods for incompressible fluid flow

Types of flow problems

- Three classes of flow problems are:

(i) Creeping flow (the limiting case of very large viscosity, that is, very small Reynolds number)

(ii) Boundary layer flow (the limiting case of very small viscous forces, that is, very large Reynolds number)

(iii) Inviscid flow or frictionless flow [ideal fluid, ($\mu = 0$)]. In all three cases, the flow geometry is taken as rectangular.

- Flow is assumed as laminar and isothermal, and viscosity is not a function of temperature. It may also be noted that gases may be treated as incompressible fluids when Mach Number < 0.3.

Governing Equations

Governing equations are the Navier-Stokes equations and can be written for a two-dimensional case as

Continuity:
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

x-Momentum:
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

y-Momentum:
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Difficulties in solving Navier–Stokes equations

Nonlinearity: The convection part of the momentum equations involves nonlinear terms.

- ❖Starting with a guessed velocity field, one could iteratively solve the momentum equation to arrive at the converged solution for the velocity components.
- ❖ Therefore, nonlinearity poses no problems as such. It only makes the computations more involved.

Pressure gradient: The main hurdle to overcome in the calculation of velocity field is the unknown pressure field.

- ❖The pressure gradient behaves like a source term for a momentum equation. But, there is no equation for obtaining pressure. The challenging task is to determine the correct pressure distribution.
- ❖The pressure field is indirectly linked with the continuity equation. When the correct pressure field is plugged into the momentum equations, the resulting velocity field satisfies the continuity equation.

Stream function–Vorticity method

In this method, the difficulty associated with the computation of pressure is circumvented by eliminating the pressure gradient terms from the momentum equations by cross-differentiation, which leads to a vorticity–transport equation. This, when coupled with the definition of stream function for steady two-dimensional situations, is the basis of the well-known stream function–vorticity method.

Advantages

1. The pressure makes no appearance.
2. Instead of having to deal with the continuity and two momentum equations, we need to solve only two equations to obtain stream function and vorticity.

Disadvantages

1. Calculation of pressure.
2. Difficulty in specification of vorticity at a wall.
3. The method is **valid for two-dimensional** problems as the definition of stream function applies to two-dimensional flow field.

Stream function–Vorticity method

In this method, the vorticity is eliminated by differentiating the N-S equation consecutively with respect to x and y and thereby subtracting it.

x-Momentum: $$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Differentiating x-momentum equation with respect to y, we obtain,

$$\rho\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial^2 p}{\partial x\partial y}+\mu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (1)$$

y-Momentum: $$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Differentiating y-momentum equation with respect to x, we obtain,

$$\rho\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial^2 p}{\partial x\partial y}+\mu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad (2)$$

Now, subtracting (1)-(2), we have

$$\rho\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-\rho\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)$$

$$=-\frac{\partial^2 p}{\partial x\partial y}+\mu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial^2 p}{\partial x\partial y}-\mu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right]$$

$$=\mu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right] \qquad (3)$$

Let us define vorticity field,

$$-\xi=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} \qquad (4)$$

Substituting Eq.(4) in Eq. (3), we obtain,

$$\rho\left(\frac{\partial \xi}{\partial t}+u\frac{\partial \xi}{\partial x}+v\frac{\partial \xi}{\partial y}\right)=\mu\left(\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}\right) \qquad (5)$$

$$\rho\frac{D\xi}{Dt}=\mu\nabla^2\xi$$

Vorticity-transport equation

Let us stream function,

$$v = -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \psi}{\partial y}$$

The stream function automatically satisfies continuity equation

Now, the vorticity field, $$-\xi = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

On substituting the definition of stream function in the vorticity field we obtain,

$$-\xi = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\xi$$

$$\nabla^2 \psi = -\xi$$

Therefore, the stream function and vorticity equations:

$$\nabla^2\psi = -\xi$$

$$\rho\frac{D\xi}{Dt} = \mu\nabla^2\xi$$

For inviscid flow, $\mu = 0, \Rightarrow \dfrac{D\xi}{Dt} = 0$

Therefore, for a steady inviscid flow, ξ is constant along a streamline.

For irrotational flow, $\nabla^2\psi = 0$

For rotational flow, $\nabla^2\psi = -\xi$

The vorticity transport equation becomes,

$$\rho\left(\frac{\partial\xi}{\partial t} + u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y}\right) = \mu\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right) = \mu\nabla^2\xi$$

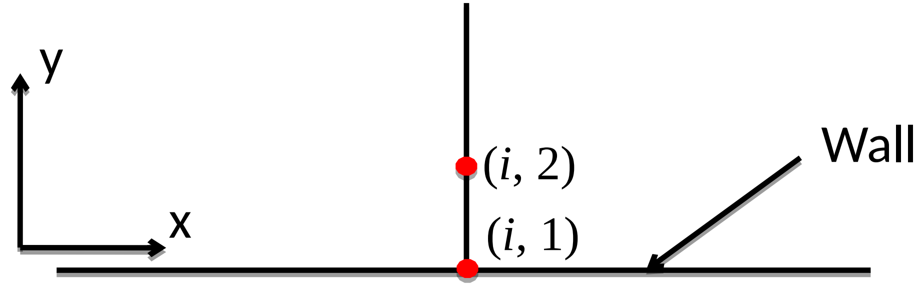
$$\Rightarrow \rho\left(\frac{\partial\xi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\xi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\xi}{\partial y}\right) = \mu\nabla^2\left(-\nabla^2\psi\right)$$

$$\therefore \frac{\partial}{\partial t}\left(\nabla^2\psi\right) - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} = \nu\nabla^4\psi$$

Boundary conditions for stream function and vorticity

- ✓ Stream function boundary conditions are obtained from the velocity distribution.
- ✓ Vorticity boundary conditions are also obtained from velocity distributions except at the walls where a special treatment is required.

a) Vorticity boundary condition at a stationary non-sloping wall



To determine ξ at the wall, a Taylor series expansion of ψ about the wall point (i, 1) reads,

$$\psi_{i,2} = \psi_{i,1} + \left.\frac{\partial \psi}{\partial y}\right|_{i,1} \Delta y + \frac{(\Delta y)^2}{2} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,1} + O(\Delta y)^3$$

Since, no-slip wall satisfies,

$$u_{i,1} = \left.\frac{\partial \psi}{\partial y}\right|_{i,1} = 0$$

Therefore, $\left.\dfrac{\partial^2\psi}{\partial y^2}\right|_{i,1} = \left.\dfrac{\partial u}{\partial y}\right|_{i,1}$ and, $\xi_{i,1} = \left.\dfrac{\partial v}{\partial x}\right|_{i,1} - \left.\dfrac{\partial u}{\partial y}\right|_{i,1}$

Since, for non-porous wall, $v = 0 \Rightarrow \left.\dfrac{\partial v}{\partial x}\right|_{i,1} = 0$

Hence, $\xi_{i,1} = -\left.\dfrac{\partial u}{\partial y}\right|_{i,1} = -\left.\dfrac{\partial^2\psi}{\partial y^2}\right|_{i,1}$

Now, $\psi_{i,2} = \psi_{i,1} + \left.\dfrac{\partial \psi}{\partial y}\right|_{i,1}\Delta y + \dfrac{(\Delta y)^2}{2}\left.\dfrac{\partial^2\psi}{\partial y^2}\right|_{i,1} + O(\Delta y)^3$

$$\Rightarrow \psi_{i,2} = \psi_{i,1} + 0 \times \Delta y + \frac{(\Delta y)^2}{2}(-\xi_{i,1}) + O(\Delta y)^3$$

$$\therefore \xi_{i,1} = \frac{2(\psi_{i,1} - \psi_{i,2})}{(\Delta y)^2} + O(\Delta y)$$

First-order accurate expression.

Therefore, in general, $$\therefore \xi_s = \frac{2(\psi_s - \psi_{s+1})}{(\Delta n)^2} + O(\Delta n)$$

Here, Δn is the normal distance between the grid points s at the wall and one grid point (s+1) away from the wall.

b) Vorticity boundary condition for moving wall

If the wall is moving with a velocity u=U and non-porous, v=0, one can adopt similar procedure of Taylor series expansion to obtain vorticity boundary condition for a moving wall as,

$$\therefore -\xi_{i,j} = \frac{2(\psi_{i,j+1} - \psi_{i,j} - U\Delta y)}{(\Delta y)^2} + O(\Delta y)$$

Here, y is positive away from the wall, and (*i*, *j*) represents the point at the wall.

c) Vorticity boundary condition at corners of a block

At corners, the velocity derivatives are not continuous at corners and vorticity becomes singular. The remedy is to exclude corners during computations and local refinement of the mesh at corners.

General algorithm for solution by Stream function-Vorticity method

Time marching sequential procedure to obtain velocity field:

Step 1: Specify the initial values for ξ and Ψ at t=0.

Step 2: Solve vorticity transport equation for ξ at each grid point at time t+Δt.

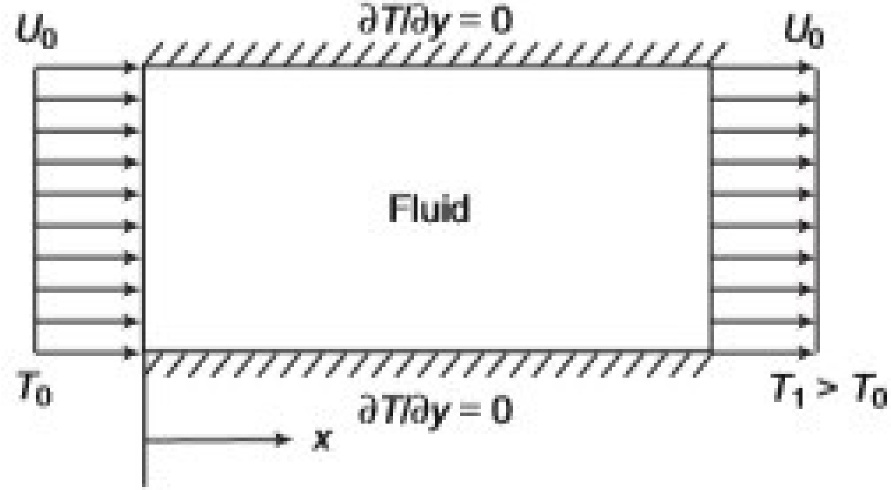
Step 3: Solve for new ψ values at all points by solving the Poisson equation using new ξ at each grid point.

Step 4: Find the velocity components using $u = \partial\psi/\partial x,\ v = -\ \partial\psi/\partial y$

Step 5: Update the wall vorticity using vorticity equation at the wall.

Step 6: Return to **Step 2** if the steady state is not reached.

Convection–diffusion (Steady, One Dimensional)



Convection diffusion problem for slug flow where the combined influences of steady axial convection and diffusion is examined. The fluid is flowing with a velocity U_0 and is subjected to specified temperatures as follows:

$$T_0 \quad \text{for} \;\; x \le 0$$

$$T_1 \quad \text{for} \;\; x \ge L$$

To determine, $\qquad T(x)$

Governing equation,

$$U_0 \frac{dT}{dx} = \alpha \frac{d^2T}{dx^2}$$

mensionless form

Non-dimensionalizing using the dimensionless variables:

$$\theta = \frac{T - T_0}{T_1 - T_0}, \quad X = \frac{x}{L}, \quad Pe = \frac{U_0 L}{\alpha}$$

Hence,

$$Pe \frac{d\theta}{dX} = \frac{d^2\theta}{dX^2}$$

$$BC1: \ at \ X = 0, \quad \theta = 0$$

$$BC2: \ at \ X = 1, \ \theta = 1$$

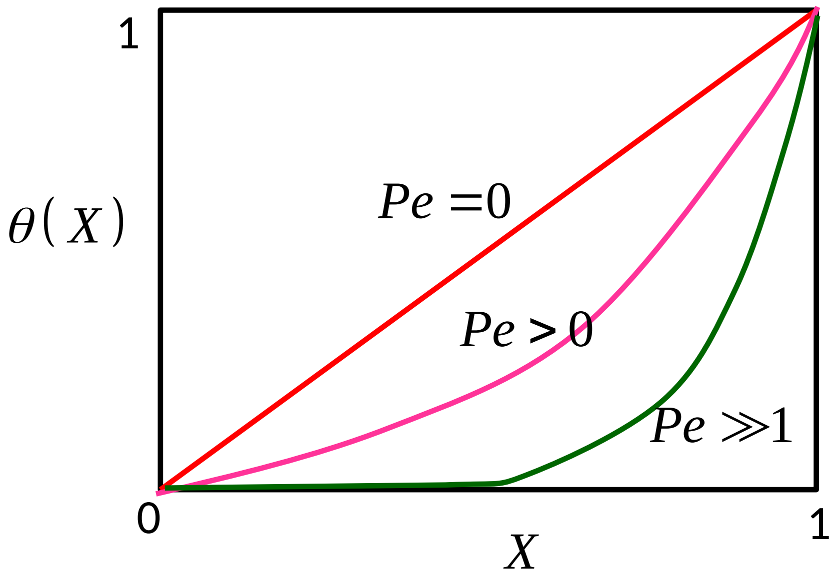
Analytical solution,

$$\theta(X) = \frac{\exp(Pe \cdot X) - 1}{\exp(Pe) - 1}$$

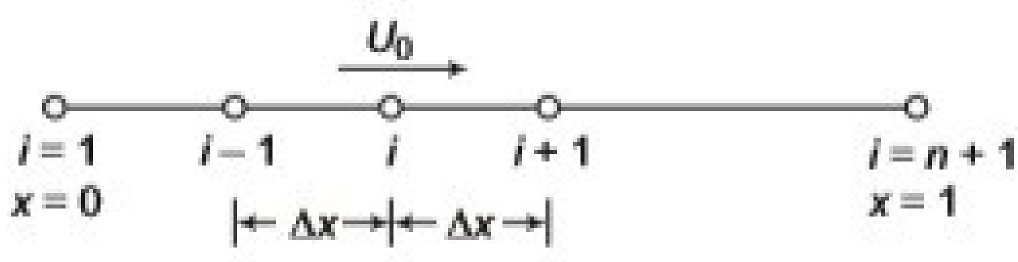
When, $Pe \to 0$ $\quad \theta(X) = X$ (Pure condition) And

When, $Pe \gg 1$ Convection dominates and the temperature changes from ϴ=0 to ϴ=1 is confined to a thin region X=1. This implies steep temperature gradient.

$$\left[\frac{d\theta(X)}{dX}\right]_{X=1} = \left[\frac{Pe.\exp(Pe\cdot X)}{\exp(Pe)-1}\right]_{X=1} = \frac{Pe.\exp(Pe)}{\exp(Pe)-1} \approx Pe \text{ (for } Pe \gg 1)$$



Numerical solution



Using central differencing FDM, we have

$$Pe\frac{d\theta}{dX}=\frac{d^2\theta}{dX^2}$$

$$Pe\left(\frac{\theta_{i+1}-\theta_{i-1}}{2\Delta X}\right)=\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{(\Delta X)^2}$$

Let us define cell or grid Peclet number as,

$$Pe_c=Pe\cdot\Delta X$$

Hence,

$$\frac{Pe_c}{2}(\theta_{i+1}-\theta_{i-1})=\theta_{i+1}-2\theta_i+\theta_{i-1}$$

$$\Rightarrow\ \theta_i=\frac{1}{4}\left\{(2+Pe_c)\theta_{i-1}+(2-Pe_c)\theta_{i+1}\right\} \qquad (1)$$

When, $Pe_c\rightarrow 0 \quad \Rightarrow\ \theta_i=\frac{1}{2}\left\{\theta_{i-1}+\theta_{i+1}\right\}$

When, $Pe_c\gg 1 \quad \Rightarrow\ \theta_i=\frac{1}{4}Pe_c\left\{\theta_{i-1}-\theta_{i+1}\right\}$

This shows that for θ increasing with X yields absurd and even negative values of θ_i. Hence, Eq.(1) does not satisfy computational stability unless $Pe_c<2$.

If $Pe_c \geq 2$, the solution shows *wiggles*, a computational instability.

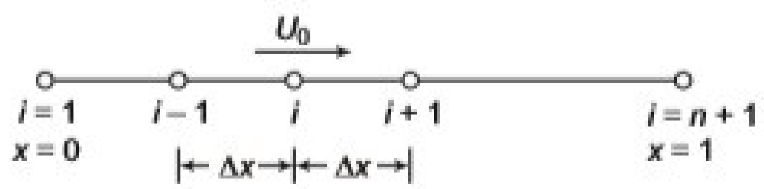
This problem in the numerical solution is circumvented by the **Upwind scheme** in the **Convection Term** .

Upwind scheme

Discretized form of equation,

$$Pe\frac{d\theta}{dX} = \frac{d^2\theta}{dX^2}$$

$$Pe\left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta X}\right) = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2} \qquad (2)$$



Upwind scheme tells that,

if U_0 >0, then $\theta_{i+1} = \theta_i$

(First order Upwinding scheme)

if U_0 <0, then $\theta_{i+1} = \theta_{i+2}$

Therefore, Eq. (2) can be written as, for $U_0>0$

$$Pe\left(\frac{\theta_i - \theta_{i-1}}{2\Delta X}\right) = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

$$\Rightarrow \theta_i = \frac{(1+Pe_c)\,\theta_{i-1} + \theta_{i+1}}{2+Pe_c} \qquad (3)$$

When $Pe_c \to 0 \;\Rightarrow\; \theta_i = \dfrac{\theta_{i-1} + \theta_{i+1}}{2}$, (exact solution)

When $Pe_c \gg 1 \qquad \Rightarrow\; \theta_i = \theta_{i-1}$, (exact solution)

Hence, Eq.(3) is computationally stable.

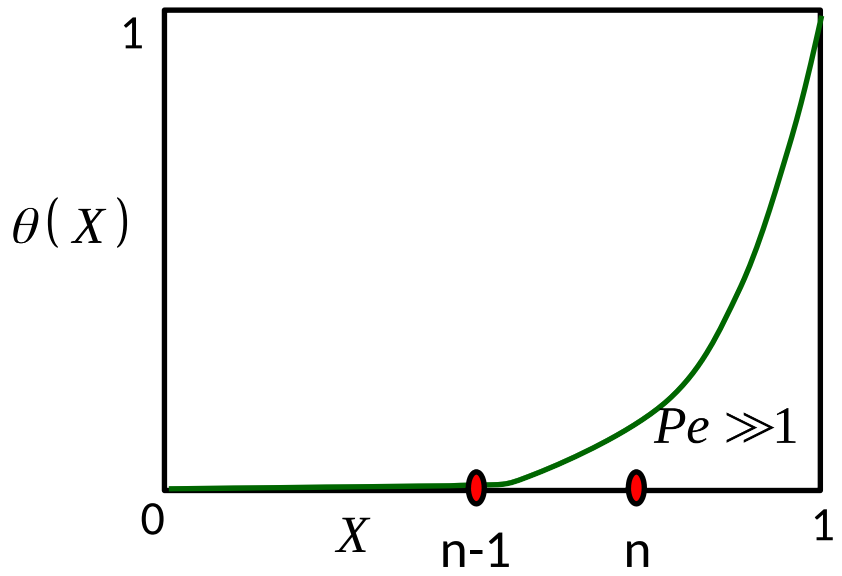
There are other higher order Upwind schemes,

a) Second Order Upwind scheme.

b) Third Order Upwind scheme or Quick.

Limitation of Upwind scheme: False Diffusion

In spite of the fact that the upwind scheme is computationally stable and gives physically realistic solution, it does not produce very accurate solution at high Peclet number (Pe >> 1) because of false diffusion.



When Pe is large, $d\theta/dX$ is nearly zero at X=0.5. Thus, the diffusion is almost absent. The upwind scheme always calculates the diffusion term and thus overestimates diffusion at high Peclet numbers. This is called False Diffusion.

To prevent such False Diffusion, local refinement in grid is necessary.

Primitive-variables approach

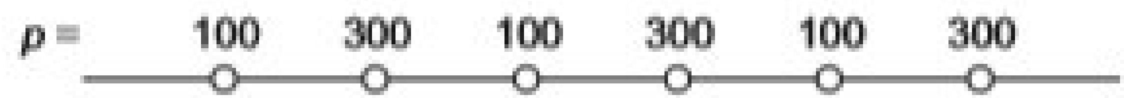
- The stream function−vorticity or biharmonic equation approach is suitable for solving two-dimensional flow problems. If the pressure distribution is also of interest, the problem becomes more involved.
- Stream function−vorticity formulation is difficult to implement in the problem of fluid flow with variable properties and in complex geometries.
- Also, ψ−ξ method is not applicable to three-dimensional problems.
- Therefore, if the pressure distribution is of interest and/or three-dimensional simulation is required, it is generally more efficient to solve the basic Navier−Stokes equations in terms of the primitive variables u, v, w, and p.

BASIC ISSUE WITH THE FDM GRID DURING PRIMITIVE VARIABLE APPROACH

Consider a 1D steady state momentum equation:

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2}$$

The FDM representation for the grid as shown in the figure below is:



Wavy pressure field for 1-D flow situation

$$\rho_i u_i \frac{u_{i+1} - u_{i-1}}{2\Delta x} = -\frac{p_{i+1} - p_{i-1}}{2\Delta x} + \mu_i \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

The momentum equation will contain the pressure difference between two alternate grid points and not between the adjacent ones.

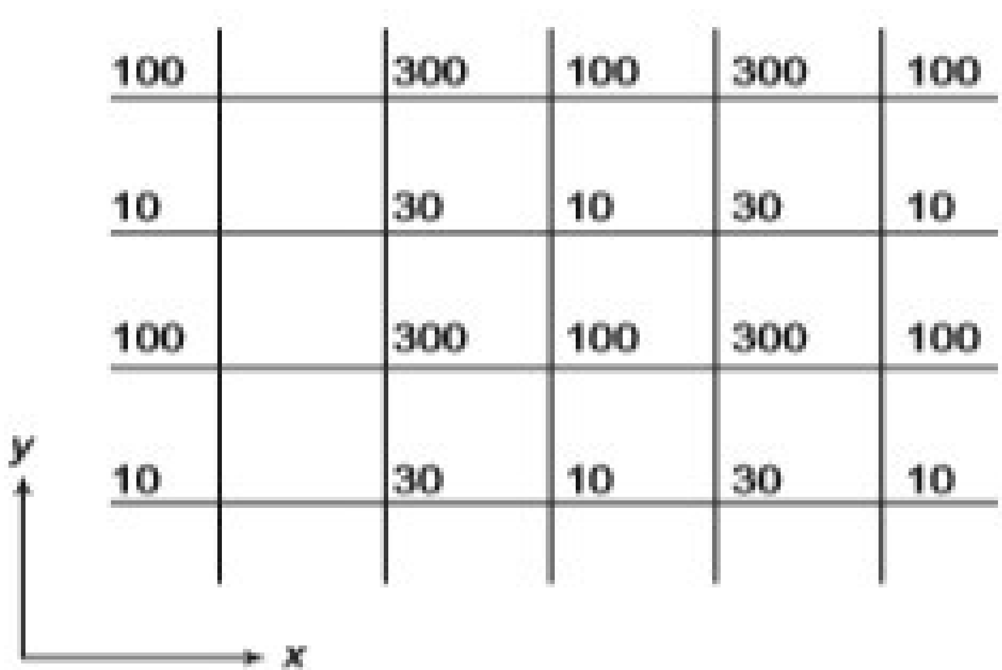
Implications

1. The pressure is taken from a coarser grid than the one actually employed. This would reduce the accuracy of the solution.
2. Even a wavy pressure (which can be realistic) field will be treated like a uniform pressure field by the momentum equation.

$$-\frac{p_{i+1}-p_{i-1}}{2\Delta x}=-\frac{100-100}{2\Delta x}=0 \text{ and } -\frac{p_{i+1}-p_{i-1}}{2\Delta x}=-\frac{300-300}{2\Delta x}=0$$

No pressure field at all!

Checker board pressure field for 2D flow situation



Implications

1. If a certain smooth pressure field is obtained as a solution, an infinite number of solutions can be constructed by adding a checker board pressure field to that source (Patankar, 1980).
2. The momentum equation would remain unaffected by this addition, since the checker board pressure field implies zero pressure force. A numerical method that allows such absurdities is certainly not desirable.

Representation of continuity equation

For 1D, steady, incompressible flow situation, the continuity equation is

$$\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0$$

$$\Rightarrow u_{i+1} = u_{i-1}$$

A consequence is that a wavy velocity field, which is not at all realistic, does satisfy the discretized continuity equation.

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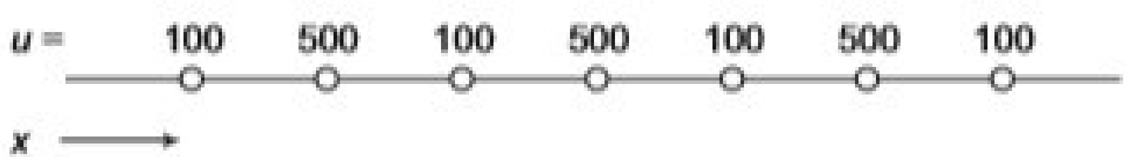
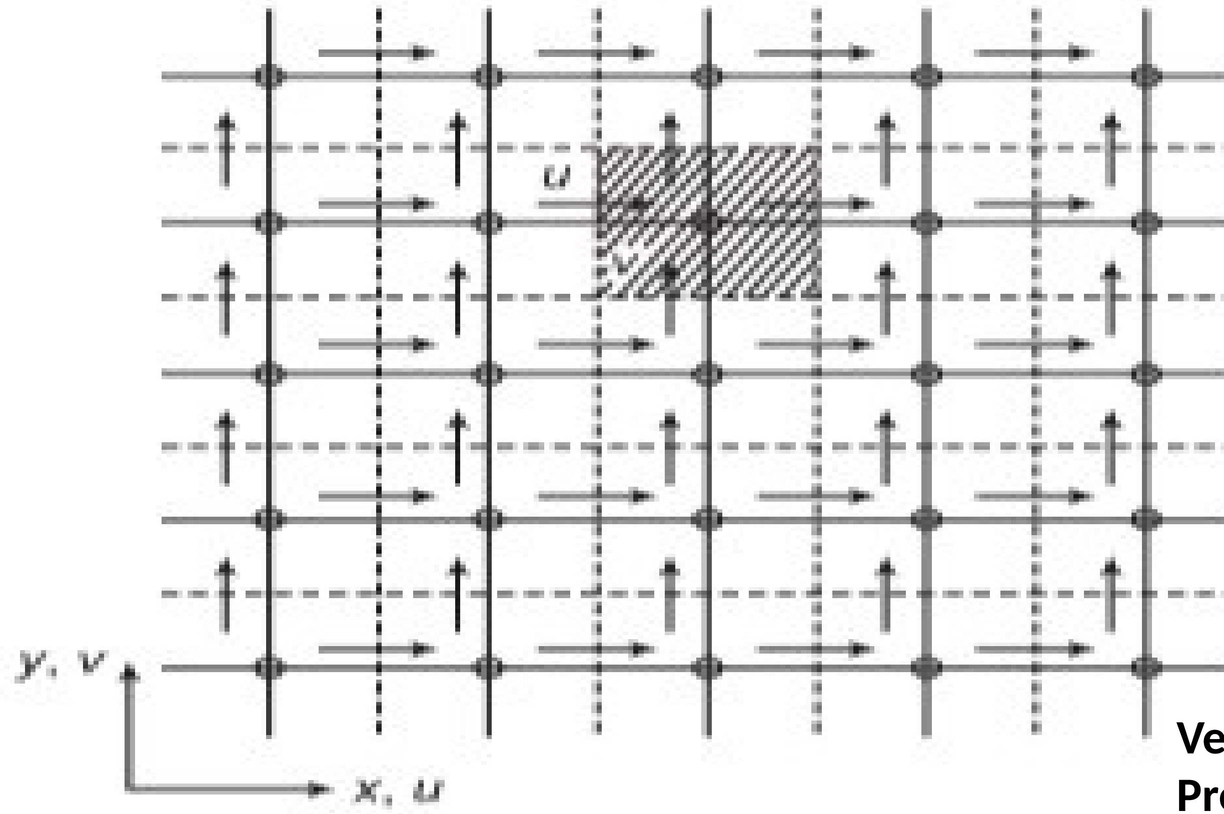
Pressure-velocity decoupling

A consequence is that a wavy velocity field, which is not at all realistic, does satisfy the discretized continuity equation.



A Remedy: The Staggered Grid



Velocities: Cell faces
Pressure: Cell centers

1. The aforementioned difficulties can be circumvented by recognizing that all the variables need not be computed for the same grid points.
2. It is possible to employ a different grid for each dependent variable. In the staggered grid, the velocity components are calculated for the points that are located on the faces of the control volume (shaded), whereas the pressure is calculated for the regular grid points.
3. For uniformly spaced grid points, the control volume faces are situated exactly at the midway between the grid points.

The SMAC (Simplified Marker And Cell) Algorithm

1. The procedure is based on a cyclic series of guess-and-correct operations to solve the governing equations.
2. The velocity components are first calculated from the momentum equations using a guessed pressure field.
3. The pressure and velocities are then corrected so as to satisfy continuity.
4. A staggered grid as shown is used.

Pressure-velocity coupling is achieved

The SMAC (Simplified Marker And Cell) Algorithm

Predictor step: Predict velocities u* and v* by explicitly

X-momentum equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{u^{p+1}-u^{p}}{\Delta t}+\left(\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}\right)^{p}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)^{p+1}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)^{p} \qquad \text{(i)}$$

Y-momentum equation

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\Rightarrow \frac{v^{p+1}-v^{p}}{\Delta t}+\left(\frac{\partial vu}{\partial x}+\frac{\partial v^2}{\partial y}\right)^{p}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)^{p+1}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)^{p} \qquad \text{(ii)}$$

Velocity predictor equations, explicit form

$$\Rightarrow \frac{u^{*}-u^{p}}{\Delta t}+\left(\frac{\partial u^{2}}{\partial x}+\frac{\partial uv}{\partial y}\right)^{p}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)^{p}+\nu\left(\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}\right)^{p} \quad \text{(iii)}$$

$$\Rightarrow \frac{v^{*}-v^{p}}{\Delta t}+\left(\frac{\partial vu}{\partial x}+\frac{\partial v^{2}}{\partial y}\right)^{p}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)^{p}+\nu\left(\frac{\partial^{2} v}{\partial x^{2}}+\frac{\partial^{2} v}{\partial y^{2}}\right)^{p} \quad \text{(iv)}$$

Corrector step: Correct velocities and pressure by solving pressure correction equation

By (i)-(iii) and (ii)-(iv), we obtain

$$\Rightarrow \frac{u^{p+1}-u^{*}}{\Delta t}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)^{p+1}+\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)^{p}=-\frac{1}{\rho}\frac{\partial}{\partial x}\left(p^{p+1}-p^{p}\right)=-\frac{1}{\rho}\left(\frac{\partial p'}{\partial x}\right)$$

$$\therefore u^{p+1}=u^{*}-\frac{\Delta t}{\rho}\left(\frac{\partial p'}{\partial x}\right) \quad \text{(v)}$$

Hence,

$$\left(\frac{\partial u}{\partial x}\right)^{p+1}=\frac{\partial u^{*}}{\partial x}-\frac{\Delta t}{\rho}\left(\frac{\partial^{2} p'}{\partial x^{2}}\right) \quad \text{(vi)}$$

Also,

$$\Rightarrow \frac{v^{p+1} - v^{*}}{\Delta t} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)^{p+1} + \frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)^{p} = -\frac{1}{\rho}\frac{\partial}{\partial y}\left(p^{p+1} - p^{p}\right) = -\frac{1}{\rho}\left(\frac{\partial p^{'}}{\partial y}\right)$$

$$\therefore v^{p+1} = v^{*} - \frac{\Delta t}{\rho}\left(\frac{\partial p^{'}}{\partial y}\right) \qquad \text{(vii)}$$

Hence,

$$\left(\frac{\partial v}{\partial y}\right)^{p+1} = \frac{\partial v^{*}}{\partial y} - \frac{\Delta t}{\rho}\left(\frac{\partial^{2} p^{'}}{\partial y^{2}}\right) \qquad \text{(viii)}$$

$$p^{'} = p^{p+1} - p^{p} = \text{Pressure correction}$$

From continuity equation we have,

$$\left(\frac{\partial u}{\partial x}\right)^{p+1}+\left(\frac{\partial v}{\partial y}\right)^{p+1}=0 \qquad \text{(ix)}$$

Substituting Eq. (vi) and Eq. (viii) in Eq. (ix), we have

$$\left(\frac{\partial u}{\partial x}\right)^{p+1}+\left(\frac{\partial v}{\partial y}\right)^{p+1}=0$$

$$\Rightarrow \frac{\partial u^*}{\partial x}-\frac{\Delta t}{\rho}\left(\frac{\partial^2 p'}{\partial x^2}\right)+\frac{\partial v^*}{\partial y}-\frac{\Delta t}{\rho}\left(\frac{\partial^2 p'}{\partial y^2}\right)=0$$

$$\Rightarrow \frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}=\frac{\Delta t}{\rho}\left(\frac{\partial^2 p'}{\partial y^2}\right)^{p+1}+\frac{\Delta t}{\rho}\left(\frac{\partial^2 p'}{\partial x^2}\right)^{p+1}$$

$$\Rightarrow \left(\frac{\partial^2 p'}{\partial x^2}\right)+\left(\frac{\partial^2 p'}{\partial y^2}\right)=\frac{\rho}{\Delta t}\left(\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}\right)$$

$$\therefore \nabla^2(p')=\frac{\rho}{\Delta t}\nabla\cdot\vec{u}^{\,*} \qquad \text{(x)}$$

Pressure correction equation

Step by Step Algorithm

1. Initialization, set boundary conditions at the present time.
2. Solve velocity predictor equations explicitly to obtain predicted velocities u* and v* with the boundary conditions.
3. Solve pressure correction equation to obtain p'.
4. Correct velocity and pressure.
5. Check for convergence:

   $Sum[RMS(u^{p+1}-u^{p}, v^{p+1}-v^{p})]<error (10^-6)$
6. If achieved, STOP, If Not then
7. Return to Step No. 2 by setting $u^{p}=u^{p+1}$, $v^{p}=v^{p+1}$, and $p^{p}=p^{p+1}$

SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) Algorithm of Patankar and Spalding (1972)

1. The procedure is based on a cyclic series of **inner loops** containing guess-and-correct operations to solve the governing equations.
2. The velocity components are first calculated from the momentum equations using a guessed pressure field implicitly.
3. The pressure and velocities are then corrected so as to satisfy continuity.
4. A staggered grid as shown is used.

First Inner Loop: Solve for the predicted variable velocity u* implicitly by keeping the coefficient velocity v at the previous time step value till convergence

$$\Rightarrow \frac{u^{*} - u^{p}}{\Delta t} + \left(\frac{\partial u^{*2}}{\partial x} + \frac{\partial u^{*} v^{p}}{\partial y} \right) = - \frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)^{p} + \nu \left(\frac{\partial^{2} u^{*}}{\partial x^{2}} + \frac{\partial^{2} u^{*}}{\partial y^{2}} \right)$$

Second Inner Loop: Solve for the predicted variable velocity v* implicitly by keeping the coefficient velocity u at the previous time step value till convergence

$$\Rightarrow \frac{v^{*} - v^{p}}{\Delta t} + \left(\frac{\partial v^{*} u^{p}}{\partial x} + \frac{\partial v^{*2}}{\partial y} \right)^{p} = - \frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right)^{p} + \nu \left(\frac{\partial^{2} v^{*}}{\partial x^{2}} + \frac{\partial^{2} v^{*}}{\partial y^{2}} \right)$$

Corrector step: Correct velocities and pressure by solving pressure correction equation

Pressure correction equation

$$\nabla^2(p') = \frac{\rho}{\Delta t}\nabla\cdot\vec{u}^*$$

Velocity correction equation

$$u^{p+1} = u^* - \frac{\Delta t}{\rho}\left(\frac{\partial p'}{\partial x}\right)$$

$$v^{p+1} = v^* - \frac{\Delta t}{\rho}\left(\frac{\partial p'}{\partial y}\right)$$

$$p' = p^{p+1} - p^{p} = \text{Pressure correction}$$

Flowchart of the SIMPLE algorithm

Step 1: Initialize velocities at the staggered grid points. Guess p^* at the grid points.

Step 2:

First and Second inner loops: Solve velocity prediction equations with boundary conditions for u^* and v^* till convergence.

Corrector Step: Solve pressure correction equation with boundary conditions for p' with the predicted velocities till convergence.

Step 3: Correct velocities and pressure:

$$u^{p+1} = u^* - \frac{\Delta t}{\rho}\left(\frac{\partial p'}{\partial x}\right)^{p+1}$$

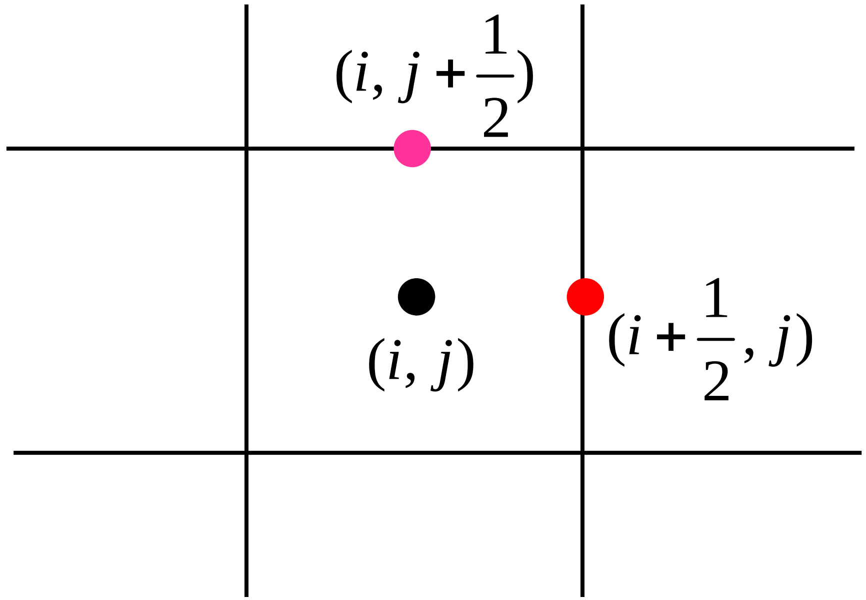
$$v^{p+1} = v^* - \frac{\Delta t}{\rho}\left(\frac{\partial p'}{\partial y}\right)^{p+1}$$

$$p^{p+1} = p' + p^*$$

Step 4: Go to step 2 and iterate till the convergence is achieved, that is all corrections becomes negligibly small and the mass balance, $\nabla \cdot u^* < \varepsilon$ is satisfied.

Step 5: Go to the next time step, t=t+Δt and repeat the cycle till steady state is achieved.

Discretizatied equations using the SIMPLE Algorithm



Predictor equation

X-momentum equation

$$\frac{u^{*}_{i+\frac{1}{2},j} - u^{p}_{i+\frac{1}{2},j}}{\Delta t} + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right)^{p+1}_{i+\frac{1}{2},j} = -\frac{1}{\rho}\left(\frac{\partial p^{*}}{\partial x} \right)^{p+1}_{i,j} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^{p+1}_{i+\frac{1}{2},j}$$

Now,

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)^{p+1}_{i+\frac{1}{2},j}=\left(\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}\right)^{p+1}_{i+\frac{1}{2},j}=\left(\frac{\partial u^2}{\partial x}\right)^{p+1}_{i+\frac{1}{2},j}+\left(\frac{\partial uv}{\partial y}\right)^{p+1}_{i+\frac{1}{2},j}$$

$$\left(\frac{\partial u^2}{\partial x}\right)^{p+1}_{i+\frac{1}{2},j}=\frac{\left(u^2_{i+\frac{1}{2}+1,j}\right)^{p+1}-\left(u^2_{i+\frac{1}{2}-1,j}\right)^{p+1}}{2\Delta x}=\frac{\left(u^2_{i+\frac{3}{2},j}\right)^{p+1}-\left(u^2_{i-\frac{1}{2},j}\right)^{p+1}}{2\Delta x}$$

$$\left(\frac{\partial uv}{\partial y}\right)^{p+1}_{i+\frac{1}{2},j}=\frac{(uv)^{p+1}_{i+\frac{1}{2},j+1}-(uv)^{p+1}_{i+\frac{1}{2},j-1}}{2\Delta y}=\frac{u^{p+1}_{i+\frac{1}{2},j+1}v^{p+1}_{i+\frac{1}{2},j+1}-u^{p+1}_{i+\frac{1}{2},j-1}v^{p+1}_{i+\frac{1}{2},j-1}}{2\Delta y}$$

Were, (by linear interpolation)

$$v^{p+1}_{i+\frac{1}{2},j+1}=\frac{v^{p+1}_{i+\frac{1}{2},j+\frac{1}{2}}+v^{p+1}_{i+\frac{1}{2},j+\frac{3}{2}}}{2}$$

$$v^{p+1}_{i+\frac{1}{2},j+\frac{1}{2}}=\frac{v^{p+1}_{i,j+\frac{1}{2}}+v^{p+1}_{i+1,j+\frac{1}{2}}}{2}$$

Again,

$$-\frac{1}{\rho}\left(\frac{\partial p^*}{\partial x}\right)_{i,j}^{p+1} = -\frac{1}{\rho}\left(\frac{p^*_{i+1,j} - p^*_{i,j}}{\Delta x}\right)^{p+1}$$

and,

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{i+\frac{1}{2},j}^{p+1} = \left(\frac{\partial^2 u}{\partial x^2}\right)_{i+\frac{1}{2},j}^{p+1} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i+\frac{1}{2},j}^{p+1}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i+\frac{1}{2},j}^{p+1} = \frac{u^{p+1}_{i+\frac{1}{2}+1,j} - 2u^{p+1}_{i+\frac{1}{2},j} + u^{p+1}_{i+\frac{1}{2}-1,j}}{(\Delta x)^2} = \frac{u^{p+1}_{i+\frac{3}{2},j} - 2u^{p+1}_{i+\frac{1}{2},j} + u^{p+1}_{i-\frac{1}{2},j}}{(\Delta x)^2}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i+\frac{1}{2},j}^{p+1} = \frac{u^{p+1}_{i+\frac{1}{2},j+1} - 2u^{p+1}_{i+\frac{1}{2},j} + u^{p+1}_{i+\frac{1}{2},j-1}}{(\Delta y)^2}$$

Similarly for v momentum equation

Pressure correction equation

$$\left(\frac{\partial^2 p'}{\partial x^2}\right)^{p+1} + \left(\frac{\partial^2 p'}{\partial y^2}\right)^{p+1} = \frac{\rho}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right)$$

$$\Rightarrow \left(\frac{p'_{i+1,j} - 2p'_{i,j} + p'_{i-1,j}}{(\Delta x)^2}\right)^{p+1} + \left(\frac{p'_{i,j+1} - 2p'_{i,j} + p'_{i,j-1}}{(\Delta y)^2}\right)^{p+1}$$

$$= \frac{\rho}{\Delta t}\left(\frac{u^*_{i+\frac{3}{2},j} - u^*_{i-\frac{1}{2},j}}{2\Delta x}\right) + \frac{\rho}{\Delta t}\left(\frac{v^*_{i,j+\frac{3}{2}} - v^*_{i,j-\frac{1}{2}}}{2\Delta y}\right)$$

Velocity correction equation

$$u^{p+1} = u^* - \frac{\Delta t}{\rho}\left(\frac{\partial p'}{\partial x}\right)^{p+1}$$

$$\Rightarrow u^{p+1}_{i+\frac{1}{2},j} = u^*_{i+\frac{1}{2},j} - \frac{\Delta t}{\rho}\left(\frac{p'_{i+1,j} - p'_{i,j}}{\Delta x}\right)^{p+1}$$

Boundary conditions:

At walls

$$u = 0, \; u^* = 0$$

$$v = 0, v^* = 0$$

No-slip

$$\frac{\partial p}{\partial n} = 0, \frac{\partial p^{'}}{\partial n} = 0$$

At outlet

$$\frac{\partial u}{\partial n} = 0$$

Fully-developed

$$\frac{\partial v}{\partial n} = 0$$

$$p = specified \Rightarrow \; p^{'} = 0$$

(*n* is the surface normal)

Thank You