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NON-STEADY RADIAL FLOW IN AN INFINITE LEAKY AQUIFER

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Abstract--The non-steady drawdown distribution near a well discharging from an infinite leaky aquifer is presented. Variation of drawdown with time and distance caused by a well of constant discharge in confined sand of uniform thickness and uniform permeability is obtained. The discharge is supplied by the reduction of storage through expansion of the water and the concomitant compression of the sand, and also by leakage through the confining bed. The leakage is assumed to be at a rate proportional to the drawdown at any point. Storage of water in the confining bed is neglected. Two forms of the solution are developed. One is suitable for computation for large values of time and the other suitable for small values of time. This solution is compared with earlier solutions for slightly different boundary conditions.

Introduction--The differential equation for radial flow in an elastic artesian aquifer with linear leakage has been given by JACOB [1946]. He also obtained the non-steady drawdown distribution produced by a well of constant discharge situated in the center of a circular region whose outer boundary is maintained at constant head. The head distribution in his problem is initially uniform.

In this paper the solution is obtained for the problem in which the outer boundary is removed to infinity.

Statement of the problem--The problem is to determine the variation with time of the drawdown induced by a well steadily discharging from an infinite leaky aquifer in which the initial head is uniform. Leakage into the aquifer is assumed vertical and proportional to the drawdown. Stated mathematically the boundary-value problem is

$$\partial^2 s/\partial r^2 + (1/r)\, \partial s/\partial r - s/B^2 = (S/T)\, \partial s/\partial t \qquad (1)$$

$$s(r, 0) = 0 \qquad r \geq 0 \qquad (2a)$$

$$s(\infty, t) = 0 \qquad t \geq 0 \qquad (2b)$$

$$\lim_{r \to 0} r\, \partial s(r, t)/\partial r = -Q/2\pi T \qquad t > 0 \qquad (2c)$$

where

$s(r, t)$ is the drawdown at any time and any distance from the well.
r is the distance to any point measured from the axis of the well.
S is the storage coefficient of the artesian aquifer (a non-dimensional constant) defined as 'the product of the thickness of the artesian bed and the relative volume of water released from storage by a unit decline of head' [JACOB, 1946].
K and K' are the hydraulic conductivities (or 'permeabilities') of the artesian sand and confining bed, respectively. They have the dimension L/t.
b and b' are the thicknesses of the artesian sand and confining bed, respectively.
$T = Kb$ is the transmissibility (of dimension L^2/t) of the artesian sand. The ratio K'/b' may be termed 'specific leakage' or leakance [HANTUSH, 1949, p. 8]. It has the dimension 1/t. The transmissibility divided by the leakance (of dimension L^2) is symbolized by B^2.
Q is the discharge of the well.

Solution of the problem--After separating the variables, it can be shown that

$$J_0(\alpha r/B)\, \exp[-(\alpha^2 + 1)\, Tt/SB^2] \quad \text{and} \quad K_0(r/B)$$

are particular solutions of (1), where J_0 and K_0 are respectively the Bessel function of the first kind of zero order and the modified Bessel function of the second kind of zero order, and where α is any real constant.

Due to the linearity and homogeneity of (1), the above particular solutions can be combined linearly to obtain the following solution

$$s = c\left[\int_0^\infty A(\alpha)\, J_0(\alpha r/B) \exp[-(\alpha^2 + 1)\, Tt/SB^2]\, d\alpha + K_0(r/B)\right] \quad \ldots\ldots\ldots (3)$$

in which c is an arbitrary constant, and in which $A(\alpha)$ is a function depending upon α alone and hence constant or independent of the variables of (1).

That (3) satisfies condition (2b) is seen immediately. Recalling that

$$\lim_{r \to 0} K_0(r/B) = -0.5772 - \ln(r/2B)$$

and applying condition (2c), the value of c is found to be $Q/2\pi T$. The function $A(\alpha)$ may be found by using condition (2a) as follows. When $t=0$, $s=0$ and (3) becomes

$$0 = K_0(r/B) + \int_0^\infty A(\alpha)\, J_0(\alpha r/B)\, d\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Using the integral relation [WATSON, 1944, p. 425]

$$K_0(r/B) = \int_0^\infty [\alpha/(\alpha^2 + 1)]\, J_0(\alpha r/B)\, d\alpha$$

Eq. (4) reduces to

$$\int_0^\infty J_0(\alpha r/B)\, [A(\alpha) + \alpha/(\alpha^2 + 1)]\, d\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots (5)$$

whence

$$A(\alpha) = -\alpha/(\alpha^2 + 1)$$

Substituting the values of c and $A(\alpha)$ thus obtained, the formal solution of the problem is

$$s/(Q/2\pi T) = K_0(r/B) - \int_0^\infty [\alpha/(\alpha^2 + 1)]\, J_0(\alpha r/B) \exp[-p(\alpha^2 + 1)]\, d\alpha \quad \ldots\ldots (6)$$

where

$$p = Tt/SB^2$$

As pumping continues indefinitely, that is, as $t \to \infty$, the integral in (6) becomes zero and the drawdown is represented by $s/(Q/2\pi T) = K_0(r/B)$, which is the steady-state solution of the problem [JACOB, 1946].

To evaluate the infinite integral of (6), hereafter represented by I, one proceeds as follows. The integral representation of $1/(\alpha^2 + 1)$ is

$$\int_0^\infty \exp[-(\alpha^2 + 1)\, x]\, dx$$

Thus, replacing $1/(\alpha^2 + 1)$ by its integral form and changing the order of integration, one obtains

$$I = \int_0^\infty \exp[-(p + x)]\, dx \int_0^\infty J_0(\alpha r/B) \exp[-(p + x)\,\alpha^2]\alpha\, d\alpha \quad \ldots\ldots\ldots (7)$$

Reversing the order of integration is justified, since the integral I is convergent [BROMWICH, 1947, p. 504]. Integration with respect to α gives [WATSON, 1944, p. 394]

$$\int_0^\infty J_0(\alpha r/B) \exp[-(p+x)\alpha^2]\alpha \, d\alpha = [1/2(p+x)] \exp[-(r^2/B^2)/4(p+x)]$$

Hence, (7) becomes

$$I = (1/2)\int_0^\infty [1/(p+x)] \exp[-p-x-(r^2/B^2)/4(p+x)]\, dx$$

and upon the substitution of y for $(p + x)$, it becomes

$$I = (1/2)\int_p^\infty (1/y) \exp[-y-(r^2/B^2)/4y]\, dy \quad \dots\dots\dots\dots\dots (8)$$

If, in the integral of (8), $\exp[-r^2/4B^2y]$ is replaced by the absolutely and uniformly convergent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (r^2/4B^2)^n}{n!\, y^n}$$

and if the order of integration and summation are changed, integration by parts will result in

$$I = (1/2)\exp(-p) \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \frac{(-1)^{n+m}(n-m-1)!\,(r^2/4B^2)^n}{(n!)^2\, p^{n-m}}$$

$$+ (1/2)\int_p^\infty \left\{[\exp(-y)]/y\right\} dy \left[\sum_{n=0}^{\infty} \frac{(r/2B)^{2n}}{(n!)^2}\right]$$

$$I = -(1/2)\exp[-r^2/4B^2u] \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^{n+m}(n-m)!\,(r^2/4B^2)^m u^{n-m+1}}{(n+1)!^2}$$

$$+ (1/2)[-\mathrm{Ei}(-r^2/4B^2u)]\, I_0(r/B) \quad \dots\dots\dots\dots\dots (9)$$

where $I_0(r/B)$ is the modified Bessel function of the first kind and zero order, and where

$$u = r^2/4B^2p = r^2S/4Tt$$

That the series in (9) is uniformly and absolutely convergent can be seen by applying the ratio test for any given value of m. Because of the absolute and uniform convergence of this double series, then, summation in any manner is justified. Separating the first column $m = 0$ and the diagonal $m = n$ from the double series, the series of (9) becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n u^{n+1}}{(n+1)(n+1)!} + \sum_{n=1}^{\infty} \frac{(r^2/4B^2)^n u}{(n+1)!^2} + \sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{(-1)^{n+m}(n-m)!}{(n+1)!^2}(r^2/4B^2)^m u^{n-m+1}$$

$$\dots\dots\dots\dots\dots (10)$$

Recognizing the first series in (10) as $0.5772 + \ln u + [-\mathrm{Ei}(-u)]$, and the second series as $-u + u[I_0(r/B) - 1]/(r^2/4B^2)$, and by changing the index of summation from $n = 2$ to $n = 1$ in the third series; and then substituting in (9), and then in (6), one obtains

$$s/(Q/4\pi T) = 2K_0(r/B) - I_0(r/B)[-Ei(-r^2/4B^2u)] + [\exp(-r^2/4B^2u)]\Big\{0.5772 + \ln u + [-Ei(-u)] - u + u[I_0(r/B) - 1]/(r^2/4B^2) - u^2 \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+m}(n-m+1)!}{(n+2)!^2}(r^2/4B^2)^m u^{n-m}\Big\} \qquad (11)$$

The series of (11) is rapidly convergent for the values $r/2B \leq 1$ and $u \leq 1$. Only a few terms are required to obtain results accurate to four decimal places. When $r/2B \leq 0.10$ and $u \leq 1$, the solution can be reduced to

$$s/(Q/4\pi T) = 2K_0(r/B) - I_0(r/B)[-Ei(-r^2/4B^2u)] + [\exp(-r^2/4B^2u)][0.5772 + \ln u + [-Ei(-u)] + (u/4)(r^2/4B^2)(1 - u/9)] \qquad (12)$$

which gives values of the drawdown accurate to four decimal places.

For the range of values $r/2B \leq 0.10$ and $u \leq 0.10$, the solution may be approximated by

$$s/(Q/4\pi T) = 2K_0(r/B) - I_0(r/B)[-Ei(-r^2/4B^2u)] + [\exp(-r^2/4B^2u)][(u/4)(r^2/4B^2)(1 - u/9) + u - u^2/2.2! + u^3/3.3!] \qquad (13)$$

Solution for short times--A solution that is more convenient for computations when t is small can be developed as follows. Making use of the integral relation [WATSON, 1944, p. 183]

$$2K_0(r/B) = \int_0^{\infty} (1/y)\exp[-y - r^2/4B^2y]\,dy$$

combining (8) with (6) and making the substitutions, $r^2/4B^2y = Z$, (6) reduces to

$$s/(Q/4\pi T) = \int_u^{\infty} (1/Z)\exp[-Z - r^2/4B^2Z]\,dZ \qquad (14)$$

where

$$u = r^2S/4Tt$$

It is of interest to note that if $B \to \infty$, that is, either when leakage is very small or when the time is very small so that leakage does not have time to enter the main flow, (14) becomes

$$s/(Q/4\pi T) = \int_u^{\infty} (1/Z)\exp(-Z)\,dZ \qquad (15)$$

which is the solution to the problem if leakage is not present [THEIS, 1935].

The integral of (14), being similar to that of (8), can be evaluated in the same way. The final expression is

$$s/(Q/4\pi T) = I_0(r/B)[-Ei(-u)] - e^{-u}\Big\{0.5772 + \ln(r^2/4B^2u) + [-Ei(-r^2/4B^2u)] - (r^2/4B^2u) + [I_0(r/B) - 1]/u\Big\} + (e^{-u}/u^2)\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^{n+m}(n-m+1)!\,(r^2/4B^2)^n}{(n+2)!^2\,u^{n-m}} \qquad (16)$$

which is in a form convenient for computation with values of $u \geq 1$. Very few terms of the series need be used when $r/2B \leq 1$ and $u \geq 1$. For values of $u \geq 5$ and values of $r/2B \leq 1$, the series may be safely neglected and results obtained accurate to four decimal places. If $r/2B \leq 0.10$ and $u \geq 1$, the solutions can be approximated by

$$s/(Q/4\pi T) = I_0(r/B)\,[-\,\mathrm{Ei}(-u)] - e^{-u}\,[(r^2/4B^2)(1/u - 1/36u^2) + (r^2/4B^2)^2\,(1/4u - 1/4u^2)] \quad \ldots\ldots (17)$$

Drawdown at the well--The drawdown at the face of the well is found by substituting r_w for r in any of the previous solutions, r_w being the radius of the well. However, if the ratio $r_w/B \leq 0.01$ the drawdown at the face of the well will, for all practical purposes, be given by

$$s_w/(Q/4\pi T) = -\,\mathrm{Ei}(-u_w) \quad \ldots\ldots (18)$$

which is the drawdown at the well if there is no leakage. Eq. (18) is obtained by making r/B very small in either (12) or (17).

Calculated example--Tables 1 and 2 were prepared to facilitate evaluation of (11) and (16). It gives a few coefficients of the double series that appears in both equations. The numbers in brackets in the body of the table are negative powers of ten by which the other numbers are multiplied.

Table 1--Values of $(n - m + 1)!/(n + 2)!^2$ found in Equations (11) and (16)

n	m = 1	2	3	4	5	6	7	8
1	2.7778[-2]	......	......	......	......	......	......	......
2	3.4722[-3]	1.7361 [-3]	......	......	......	......	......	......
3	4.1667[-4]	1.3889 [-4]	6.9444 [-5]	......	......	......	......	......
4	4.6296[-5]	1.1574 [-5]	3.8580 [-6]	1.9290 [-6]	......	......	......	......
5	4.7240[-6]	9.4481 [-7]	2.3620 [-7]	7.8734 [-8]	3.9367 [-8]	......	......	......
6	4.4287[-7]	7.3814 [-8]	1.4763 [-8]	3.6907 [-9]	1.2302 [-9]	6.1512[-10]	......	......
7	3.8274[-8]	5.4677 [-9]	9.1129[-10]	1.8226[-10]	4.5564[-11]	1.5188[-11]	7.5940[-12]	......
8	3.0619[-9]	3.8274[-10]	5.4677[-11]	9.1129[-12]	1.8226[-12]	4.5564[-13]	1.5188[-13]	7.5940[-14]

Table 2--Values of $(r^2/4B^2)^m$ for B = 20,000 ft

r	r/2B	m = 1	2	3	4
ft					
1,000	0.025	6.25 [-4]	3.9062[-7]	....	....
2,000	0.05	2.5 [-3]	6.25 [-6]	1.5625[-8]	....
5,000	0.125	1.5625[-2]	2.4414[-4]	3.7903[-6]	5.9223[-8]
10,000	0.25	6.25 [-2]	3.9062[-3]	2.4414[-4]	1.5259[-5]
20,000	0.50	2.5 [-1]	6.25 [-2]	1.5625[-2]	3.9062[-3]
50,000	1.25	1.5625	2.4414	3.7903	5.9223

Drawdown values for one set of assumed values of the parameters are given in Table 3 and are represented graphically in Figure 1. The factor B is taken to be 20,000 ft. The abscissa is $4Tt/S = r^2/u$ and represents the time multiplied by a constant. The straight line marked 'zero leakage' shows the trend that the drawdown would take at a distance of 1000 ft if the leakage were zero ($B \rightarrow \infty$). The effect of leakage is shown by the departure of the drawdown curve from this straight line as time increases. The effect of leakage is shown also by the drawdown curves for the more remote distances.

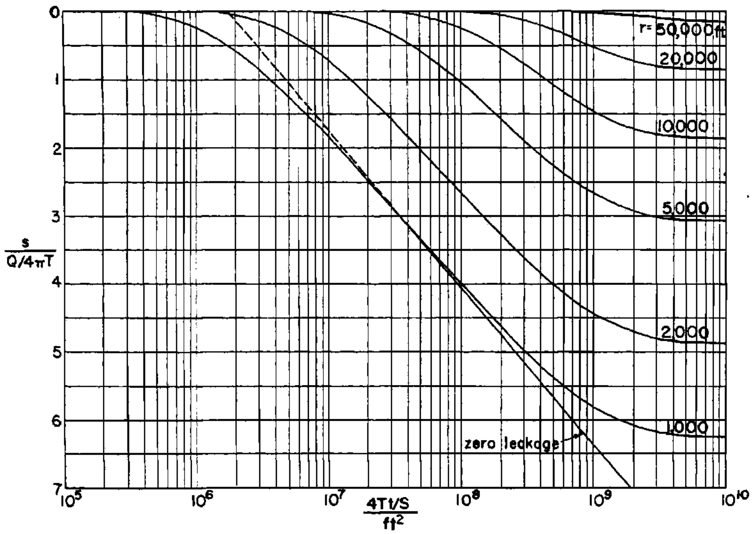

Fig. 1--Variation of drawdown with time at different distances in an infinite leaky aquifer with B = 20,000 ft

Table 3--Values of $s/(Q/4\pi T)$ for B = 20,000 ft

u	r (in feet)					
	1000	2000	5000	10,000	20,000	50,000
0.0002	6.217	....	....	....	....	....
0.0005	6.082	....	....	....	....	....
0.001	5.796	4.829	....	....	....	....
0.002	5.354	4.708	....	....	....	....
0.005	4.608	4.296	3.072	....	....	....
0.01	3.979	3.815	2.993	....	....	....
0.02	3.326	3.244	2.766	1.838	....	....
0.05	2.457	2.427	2.230	1.708	....	....
0.10	1.818	1.805	1.715	1.443	0.819	....
0.20	1.221	1.215	1.187	1.060	0.715	....
0.50	0.559	0.558	0.550	0.521	0.420	0.117
1.0	0.219	0.218	0.218	0.210	0.185	0.080

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