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RADIAL FLOW IN A LEAKY ARTESIAN AQUIFER

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Abstract--A partial differential equation is set up for radial flow in an elastic artesian aquifer into which there is vertical leakage in proportion to the drawdown. This differential equation is integrated to obtain two steady state solutions, one for the case of a well in an infinite aquifer, and the other for the case where the head is maintained constant along an outer boundary concentric with the well. In the second case, the solution of the non-steady state is also obtained for flow towards a well discharging at a steady rate, the initial state being one of uniform head distribution. A table and some curves are given for one set of assumed values of three of the parameters of the system.

Introduction--The theory of an elastic artesian aquifer [see "References" at end of paper, JACOB, 1940] has in general proved successful in dealing with the non-steady flow of water in actual artesian beds. It is framed on the hypothesis that "a specific amount of water is discharged instantaneously from storage (within the aquifer) as the pressure falls" [THEIS, 1945]. In other words, the water-producing bed is idealized as a perfectly elastic aquifer confined between parallel impermeable planes. As the head declines, the water expands and the aquifer is compressed. As the head is restored, the water is compressed and the aquifer expands, assumedly with no permanent loss of volume.

The volume of water released from "storage" within an elastic aquifer is proportional to the decline of head and to the "storage coefficient" (S). This coefficient is the product of the thickness of the bed, m, and the relative volume of water released from storage by a unit decline of head. That is,

$$S = m(dV_w/ds)/V \qquad (1)$$

where dV_w is the small volume of water released from storage within the volume V. The small decline of head is ds. Taking into account the porosity, θ, of the bed, $V = V_w/\theta$. Making this substitution and replacing ds by its equivalent $(-dp/\gamma)$, p being pressure and γ the specific weight of the water,

$$S = \gamma \theta m(-dV_w/V_w dp) \qquad (2)$$

or

$$S = \gamma \theta m \beta' \qquad (3)$$

The quantity in parentheses in equation (2), symbolized by β' in equation (3), is the apparent compressibility of the water, which includes the actual compressibility of the water, β, and the compressibility of the solid "skeleton" of the aquifer itself. Experience has shown [GUYTON, 1941; JACOB, 1941] that β' is generally several times as large as β.

The storage coefficient is to be determined empirically from observations of the rate of decline of head produced by pumping. In practice minor departures from the elastic theory are absorbed into this empirical coefficient. More serious departures generally are reflected in a systematic variation of S with time.

Commonly an artesian aquifer is confined by beds, one or both of which may be semi-permeable and which may therefore permit leakage to occur. This principle was recognized by CHAMBERLIN [1885] in his paper on the "requisite and qualifying conditions of artesian wells," and was later elaborated upon somewhat by FULLER [1905, 1908] who cited unusual examples showing that a confining bed is not always a requisite for artesian flow. CHAMBERLIN'S ideas regarding leakage through confining beds were confirmed by the work of DARTON and others [NORTON, 1912] in Iowa, and of MEINZER [1911] in the drift area of southern Minnesota.

An artesian bed resting on an impermeable base at relatively shallow depth may be overlain by a semi-pervious "clay" or silt in which the ground-water is continuously being replenished by rainfall and in which a more or less uniform head is maintained supplying the leakage that is induced by the lowering of artesian head. This condition is found in the polder areas of Holland where the problem was first submitted to analysis by DE GLEE [1930], STEGGEWENTZ and VAN NES [1939], and others. Very commonly the semi-pervious confining bed is in turn overlain by another sand, with a capacity for lateral flow sufficient to maintain essentially constant head in spite of the downward leakage into the underlying artesian sand. This situation is illustrated diagrammatically by Figure 1.

Upward leakage through a semi-pervious bed from a second artesian sand underlying the first gives rise to the same physical problem, as does a combination of leakage from above and below. In either case the semi-confined condition may be effected on a large scale by lenticular zones in which clays predominate, as in many coastal plain deposits, rather than by a single well-defined layer.

To describe the flow more precisely under these various conditions, it is necessary to modify the elastic theory to take leakage into account. Fortunately, however, the same mathematical model may be applied to the several examples cited. For convenience in developing the theory, the simple case shown in Figure 1 is chosen.

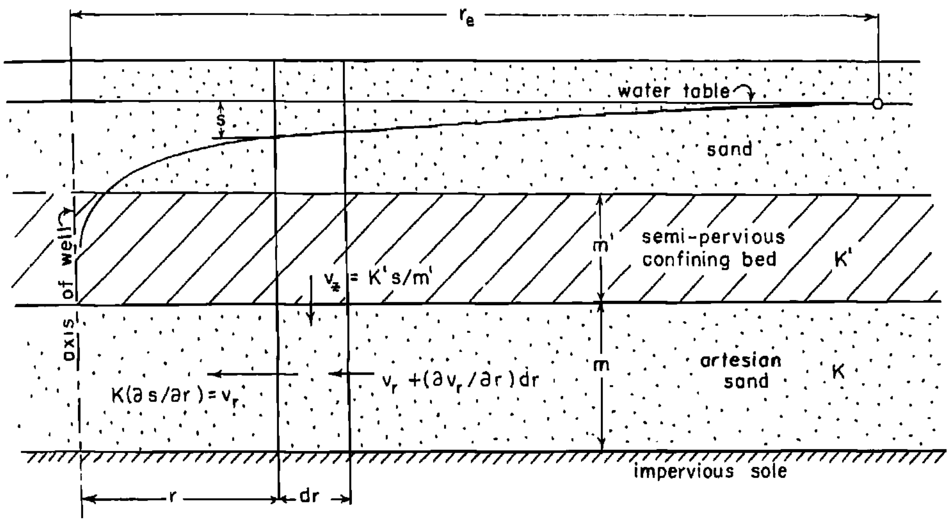

Fig. 1--Radial flow in a leaky artesian aquifer

Differential equation for radial flow in a leaky elastic artesian aquifer--Figure 1 shows an artesian sand of thickness m resting on an impervious sole and overlain by a semi-pervious confining bed of thickness m′. Overlying the confining bed is another sand, in which there is a "phreatic surface" or water table. If a well, screened in the artesian sand, is pumped, the induced flow is three-dimensional. However, provided the artesian sand is much more permeable than the confining bed (that is, provided the ratio (K/K′) is very large) the actual flow may be replaced by an idealized radial flow that increases in magnitude proceeding toward the axis of the well as the flow is augmented by the assumedly vertical leakage through the confining bed. The flow-lines are considered to be refracted a full right angle as they cross the contact between the artesian sand and its confining bed. Actually the tangents of the angles of incidence and refraction are in the same ratio as the respective permeabilities.

Consider the elemental cylindrical shell of inner radius r and thickness dr which is co-axial with the well. The flow per unit area through the inner cylindrical surface of that shell is

$$v_r = K(\partial s/\partial r) \qquad (4)$$

where s is the drawdown and K is the "permeability" or transmission constant (defined as "the quantity [volume] of water that would be transmitted in unit time through a cylinder of the soil of unit length and unit cross section under unit difference in head at the ends"[SLICHTER, 1899]. Similarly the flow per unit area through the outer cylindrical surface is

$$v_r + (\partial v_r/\partial r)\,dr = K[(\partial s/\partial r) + (\partial^2 s/\partial r^2)\,dr] \quad \text{(5)}$$

The flow per unit area through the upper bounding plane of the aquifer is

$$v_z = K's/m' \quad \text{(6)}$$

where K' is the transmission constant of the semi-pervious confining bed. The z is taken positive downwards.

Multiplying each of these by the respective surface areas and combining, one gets for the net inward flux into the cylindrical shell

$$\begin{aligned} 2\pi r m v_r - 2\pi m(r+dr)[v_r + (\partial v_r/\partial r)\,dr] + 2\pi r v_z dr &= \\ = 2\pi r dr[-(mv_r/r) - m(\partial v_r/\partial r) + v_z] &= \\ = 2\pi r dr[-(Km/r)(\partial s/\partial r) - Km(\partial^2 s/\partial r^2) + (K'/m')s] & \quad \text{(7)} \end{aligned}$$

The second member of equation (7) is obtained by simplifying the first and eliminating the differential of higher order. The substitutions indicated by equations (4) to (6) lead to the third member of equation (7).

By the principle of conservation of matter the net inward flux into the cylindrical shell must equal the time rate of increase of the volume of water within the shell. According to equation (1) the latter is given by

$$-(\partial V_w/\partial t) = -(VS/m)(\partial s/\partial t) = -A_z S(\partial s/\partial t) \quad \text{(8)}$$

where A_z ($=2\pi r dr$) is the area of the upper annular surface of the shell.

Equating the right-hand members of equations (7) and (8), dividing through by ($2\pi r dr$), and then rearranging,

$$Km[(\partial^2 s/\partial r^2) + (1/r)(\partial s/\partial r)] = S(\partial s/\partial t) + (K'/m')s \quad \text{(9)}$$

The product (Km) is termed the "transmissibility" of the bed and is symbolized by T. The ratio (T/S), symbolized here by a^2 is called "diffusivity" by reason of its analogy with KELVIN'S thermal diffusivity. It may be noted that

$$a^2 = (T/S) = (Km)/(\gamma\theta m\beta') = K/\gamma\theta\beta' \quad \text{(10)}$$

K is analogous to thermal conductivity and the triple product in the denominator of the last member is analogous to thermal capacitance per unit volume.

Dividing equation (9) through by Km or T, and putting $(K'/m'S) = b^2$,

$$(\partial^2 s/\partial r^2) + (1/r)(\partial s/\partial r) = (1/a^2)(\partial s/\partial t) + (b^2/a^2)s \quad \text{(11)}$$

This is the differential equation for the radial flow of water in a leaky elastic artesian aquifer. To find a particular solution that is useful it is first necessary to formulate the problem by specifying the boundary conditions that are to be satisfied.

Steady-state solutions--If the flow remains unchanged with time, $(\partial s/\partial t) = 0$, and a permanent or steady state is said to exist. Then the differential equation is

$$(\partial^2 s/\partial r^2) + (1/r)(\partial s/\partial r) - (b^2/a^2)s = 0 \quad \text{(12)}$$

A simple problem, yet one of practical interest is that of determining the drawdown produced by a well discharging at a steady rate from an infinite aquifer. The well can conveniently be idealized as a mathematical sink. The drawdown at infinity is zero. As the sink is approached the gradient increases in such fashion that $2\pi r T(\partial s/\partial r) \rightarrow -Q$, Q being the discharge of the well. In this case, then, the boundary conditions are

$$\left.\begin{aligned} &s \to 0 \ \text{as}\ r \to \infty \\ &\lim_{r\to 0} r(\partial s/\partial r) = -Q/2\pi T \end{aligned}\right\} \quad (13)$$

The solution [STEGGEWENTZ and VAN NES, 1939; ANONYMOUS, 1939] is found to be

$$s = C\cdot K_0(br/a) \quad (14)$$

where C is a constant whose value is to be determined from the second of equations (13), and K_0 is the modified BESSEL function of the second kind of zero order, which satisfies equation (12). As r becomes infinite the K_0 approaches zero, satisfying the first boundary condition. As r approaches zero the slope of K_0 approaches the negative reciprocal of its argument. Therefore, $C = Q/2\pi T$, and

$$s = (Q/2\pi T)K_0(br/a) \quad (15)$$

Another problem of interest is that of a steadily discharging well (idealized as a sink) in the center of a circular aquifer along whose outer boundary the head remains constant. The boundary conditions of this problem are

$$\left.\begin{aligned} &s = 0,\ r = r_e \\ &\lim_{r\to 0} r(\partial s/\partial r) = -Q/2\pi T \end{aligned}\right\} \quad (16)$$

Here r_e is the radius of the outer boundary of the aquifer.

To obtain the solution, try

$$s = C[I_0(br_e/a)K_0(br/a) - K_0(br_e/a)I_0(br/a)] \quad (17)$$

which satisfies the first boundary condition (equations 16).

The function I_0 is the modified BESSEL function of the first kind of zero-order. I_0 and K_0 are fundamental solutions of equation (12). Any linear combination of these functions, such as in equation (17), is also a solution because of the linearity of that differential equation. The constant C is to be determined so as to satisfy the second boundary condition. This requires that

$$-CI_0(br_e/a)\lim_{r\to 0}[(br/a)K_1(br/a)] - CK_0(br_e/a)\lim_{r\to 0}[(br/a)I_1(br/a)] = -Q/2\pi T \quad (18)$$

I_1 and K_1 are first-order modified BESSEL functions of the first and second kinds, respectively

The first limit in equation (18) is unity and the second zero. Therefore

$$C = (Q/2\pi T)/I_0(br_e/a) \quad (19)$$

and

$$s = (Q/2\pi T)[K_0(br/a) - K_0(br_e/a)I_0(br/a)/I_0(br_e/a)] \quad (20)$$

A graph of this equation is given in Figure 2 for r_e = 100,000 ft and (a/b) = 20,000 ft. In making these and other computations, the tables of the British Association for the Advancement of Science [ANONYMOUS, 1937] have proved very useful.

Non-steady state solution--A practical problem is that of determining the variable drawdown produced by a well starting from rest and discharging at a steady rate thereafter. At a distance r_e, equal to the radius of the outer circular boundary, the head is assumed to remain constant. Initially the drawdown is uniformly zero throughout the aquifer. The problem is to determine the variation of the drawdown with time at a given distance from the well. For convenience the solution will be broken up into two parts, a steady-state solution, s, and a transient solution, s′, the combined solution satisfying the boundary conditions

$$\left.\begin{aligned} &(s + s') = 0,\ t = 0 \\ &(s + s') = 0,\ r = r_e \\ &\lim_{r\to 0} r[\partial(s + s')/\partial r] = -Q/2\pi T \end{aligned}\right\} \quad (21)$$

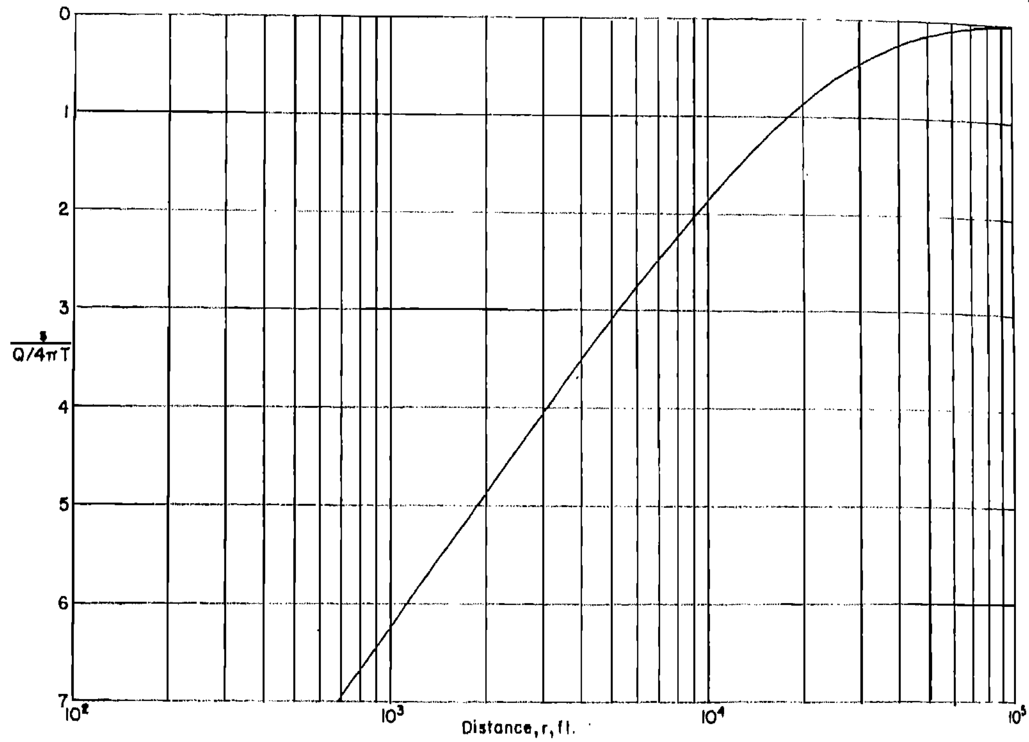

Fig..2--Ultimate distribution of drawdown in a leaky artesian aquifer, where $r_e = 100{,}000$ feet, $T = 20{,}000$ square feet per day, and $(a/b) = 20{,}000$ feet

The steady-state solution in question is the one given by equation (20), which, it will be recalled, satisfies the differential equation (12) and the boundary equations (16). Comparing those boundary conditions with the ones given by equations (21) it is seen that the boundary conditions of the transient problem are

$$\left.\begin{aligned} s' &= -(Q/2\pi T)[K_0(br/a) - K_0(br_e/a)\,I_0(br/a)/I_0(br_e/a)],\ t = 0 \\ s' &= 0,\quad r = r_e \\ \lim_{r\to 0}\, &r\,(\partial s'/\partial r) = 0 \end{aligned}\right\} \quad \text{(22)}$$

The transient solution may be obtained by assuming that it will be in the form

$$s' = R(r)\cdot T(t) \quad \text{(23)}$$

where R is a function that is independent of t and T is another function (not to be confused with transmissibility) that is independent of r. Proceeding on this postulate equation (11) becomes

$$T[(\partial^2R/\partial r^2) + (1/r)(\partial R/\partial r)] = R[(1/a^2)(\partial T/\partial t) + (b^2/a^2)\,T] \quad \text{(24)}$$

Dividing equation (24) through by RT one gets on the left-hand side a function of the single variable r, and on the right-hand side a function of the other variable t. These can be equated only provided they are equal to some constant, $-\alpha^2$. Thus

$$(1/R)[(\partial^2R/\partial r^2) + (1/r)(\partial R/\partial r)] = -\alpha^2 = (1/a^2T)(\partial T/\partial t) + (b/a)^2 \quad \text{(25)}$$

From the first two members of equation (25) one has

$$(\partial^2R/\partial r^2) + (1/r)(\partial R/\partial r) + \alpha^2R = 0 \quad \text{(26)}$$

A fundamental solution of this equation is

$$R = A \cdot J_0(\alpha r) \quad \ldots\ldots (27)$$

where A is a constant whose value is to be determined from the boundary conditions. J_0 is the BESSEL function of the first kind of zero order.

From the last two members of equation (25) one has

$$(\partial T/\partial t) = -(\alpha^2 a^2 + b^2)T \quad \ldots\ldots (28)$$

A solution of this equation is

$$T = e^{-(\alpha^2 a^2 + b^2)t} \quad \ldots\ldots (29)$$

Substituting in equation (23) from equations (27) and (29), it is found that the transient solution of equation (11) is of the form

$$s' = A \cdot J_0(\alpha r)\, e^{-(\alpha^2 a^2 + b^2)t} \quad \ldots\ldots (30)$$

This satisfies the third boundary condition (equations 22), for $\partial[J_0(\alpha r)]/\partial r = -\alpha J_1(\alpha r)$ and $J_1(0) = 0$, J_1 being the BESSEL function of the first kind of the first order.

The second boundary condition is also satisfied by equation (30) provided α is so chosen that (αr_e) is a root of $J_0(\alpha r_e) = 0$. Actually there are an unlimited number of such roots, a typical one being designated $(\alpha_n r_e)$. Thus a solution of equation (11) which satisfies the second and third boundary conditions (equations 22) is

$$s' = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)\, e^{-(\alpha_n^2 a^2 + b^2)t} \quad \ldots\ldots (31)$$

When t = 0 the exponential becomes unity and equation (31) is thereby simplified. If the initial condition, given by the first of equations (22), is to be satisfied, then

$$\sum_{n=1}^{\infty} A_n J_0(\alpha_n r) = -(Q/2\pi T)[K_0(br/a) - K_0(br_e/a)I_0(br/a)/I_0(br_e/a)] \quad \ldots\ldots (32)$$

The A_n may be determined by multiplying both sides of equation (32) by $[rJ_0(\alpha_m r)dr]$ and integrating from 0 to r_e. When $m \neq n$, the integral of the left side is zero. When $m = n$, the integration of equation (32) yields

$$(A_n r_e^2/2) J_1^2(\alpha_n r_e) = -(Q/2\pi T)/[\alpha_n^2 + (b/a)^2] \quad \ldots\ldots (33)$$

Solving for A_n

$$A_n = -(Q/\pi T)/\left\{(\alpha_n r_e)^2 J_1^2(\alpha_n r_e)[1 + (b/\alpha_n a)^2]\right\} \quad \ldots\ldots (34)$$

Substituting in equation (31) the value of A_n given by equation (34),

$$s' = -(Q/\pi T)\sum_{n=1}^{\infty}[J_0(\alpha_n r)\, e^{-(\alpha_n^2 a^2 + b^2)t}]/\left\{(\alpha_n r_e)^2 J_1^2(\alpha_n r_e)[1 + (b/\alpha_n a)^2]\right\} \quad \ldots\ldots (35)$$

This is the transient solution which is to be combined with the steady-state solution obtained in the preceding section, giving

$$s = (Q/2\pi T)\left\{[K_0(br/a) - K_0(br_e/a)I_0(br/a)/I_0(br_e/a)] - 2\Sigma\right\} \quad \ldots\ldots (36)$$

in which Σ is an abbreviation for the summation indicated in equation (35).

To recapitulate, equation (36) gives the drawdown at time t and at a distance r from a well of constant discharge Q at the center of a circular aquifer of radius r_e. The drawdown is initially zero over the entire aquifer and remains zero along the outer boundary $(r = r_e)$. The discharge of the well begins at t = 0.

Figure 3 gives curves computed from equation (36) with r_e = 100,000 ft, T = 20,000 sq ft per day, and (a/b) = 20,000 ft. The drawdown is plotted in units of $(Q/4\pi T)$. For comparison the drawdown curve for an elastic aquifer without leakage is given. The function W(u) is the negative exponential integral of a negative argument, which for small values of u may be approximated by a logarithmic term minus EULER'S constant, as indicated in Figure 3. The reciprocal of the abscissa

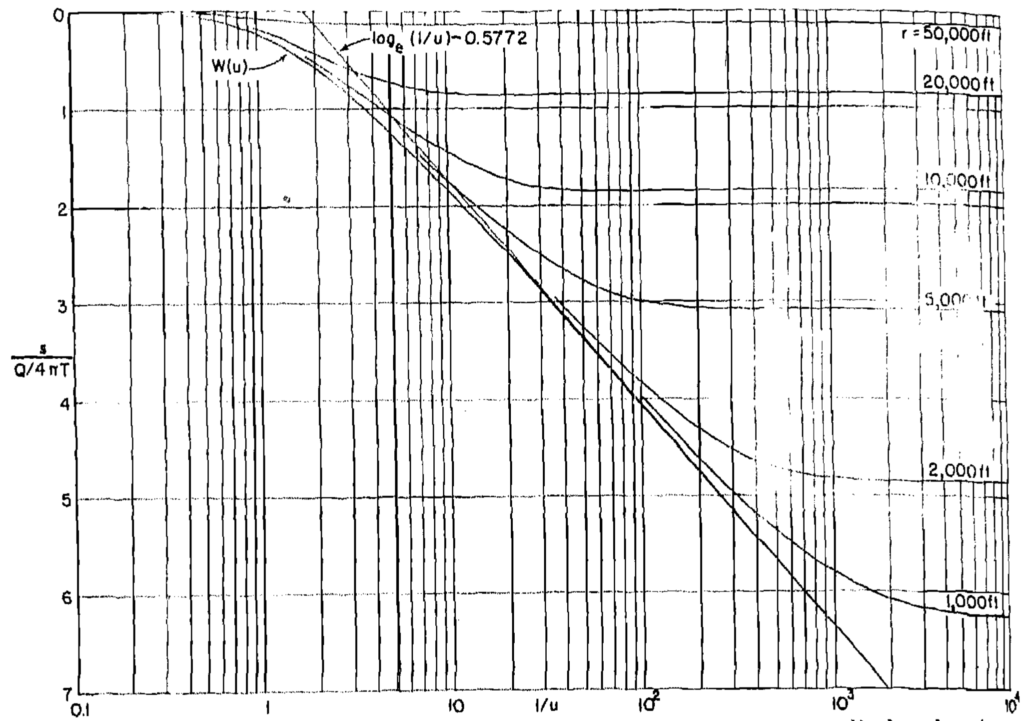

Fig. 3--Time drawdown curves for leaky artesian aquifer with well at center discharging at rate Q. Drawdown is 0 at r_e = 100,000 feet. T = 20,000 square feet per day, and (a/b) = 20,000 feet

is $u = r^2S/4Tt$, whence $(1/u) = 4Tt/r^2S$.

The data on which Figure 3 is based are given in the Table 1. As time becomes great, $u \rightarrow 0$ and the steady-state distribution is realized. Values of steady state drawdown (nondimensional, from equation 20) are given in the first line in the body of the table. For smaller values of time u increases correspondingly. Sufficient points on the non-steady time drawdown curves are given to aid in plotting those curves on semi-logarithmic coordinates.

From data on the steady state distribution of drawdown in the vicinity of a well or a center of pumping, if leakage is known to occur, values of the parameters T and (a/b) may be determined by a modification of the THEIS [JACOB, 1940, p. 582] graphical method, in which the "type curve" is a plot of $K_0(x)$ against x on logarithmic paper and the observational data are plotted on logarithmic paper of the same scale with r as abscissa and s as ordinate.

The value of the storage-coefficient, S, may be determined by application of the THEIS method in the usual manner to drawdowns observed during the early phase of the transient state. Simultaneously the value of T may be checked.

Just as with the theory of simple elastic aquifers, so not only does the theory presented here permit the determination of an additional parameter (the leakage) of an artesian system, but it also allows the prediction of drawdowns under conditions of operation different from those observed to determine that parameter. Thus is taken another step forward in the construction of useful mathematical models of confined ground water systems.

Table 1--Values of $s/(Q/4\pi T)$

u	Distance, r, in feet					
	1,000	2,000	5,000	10,000	20,000	50,000
0	6.228	4.854	3.083	1.849	0.842	0.124
0.0002	6.217	....	....	....	....	....
0.0005	6.082	....	....	....	....	....
0.001	5.796	4.829	....	....	....	....
0.002	5.354	4.708	....	....	....	....
0.005	4.608	4.296	3.072	....	....	....
0.01	3.980	3.815	2.993	....	....	....
0.02	....	3.245	2.766	1.838	....	....
0.05	....	....	2.230	1.708	....	....
0.1	....	....	1.715	1.443	0.819	....
0.2	....	....	....	1.060	0.715	....
0.5	....	....	....	0.521	0.421	0.117
1.	....	....	....	....	0.186	0.080
2.	....	....	....	....	....	0.027

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DISCUSSION

DON KIRKHAM (Naval Ordnance Laboratory, Washington, D. C., October 1, 1945)--This paper is a valuable contribution to the theory of hydrology. The mathematical development involves theory unfamiliar to many hydrologists, and for this reason in the discussion it is the purpose to point out aids in following the mathematics, and also to discuss the final equation in some detail, as it is believed that this equation involves much more than a reader will appreciate without considerable thought.

The basic functions used in the solution of the author's problem are the BESSEL-functions J_0, J_1, I_0, I_1, K_0, and K_1, the notation being that used in his reference [ANONYMOUS, Mathematical tables, v. 6, British Assoc. for the Advancement of Science, Cambridge Univ. Press, 1937]. One of the best discussions covering essential points, and using the same notation as that employed by the author, is found in W. R. SMYTHE'S "Treatise on dynamic and static electricity," (McGraw-Hill, New York, 1939), pp. 168-196. McLACHLAN'S "Bessel functions for Engineers," (Oxford, 1934), is also valuable, for in it the reader will find that the use of equation (108) on page 164, and equation (132) on page 166, makes easy the verification of the author's equations (33) and (34).

As an example of the above functions - and since there will be occasion to refer to them later - the functions I_0 and K_0 will be given here. They are

$$I_0(x) = 1 + x^2/2^2\,(1!)^2 + x^4/2^4\,(2!)^2 + x^6/2^6\,(3!)^2 + \ldots \qquad (1)$$

$$K_0(x) = -[\gamma + \log_e(x/2)]\,I_0(x) + (1/1!)^2(x/2)^2 + (1/2!)^2(x/2)^4(1+1/2) + (1/3!)^2(x/2)^6(1+1/2+1/3) + \ldots \qquad (2)$$

In the last expression, γ is the usual EULER constant having the value 0.5772 to four places of decimals. This constant appears in the definition of a number of functions, for example, in the definition of the exponential integral functions, equation (4).

In connection with equation (36) of the author, giving the final result, it will first be pointed out that in the steady state ($t=\infty$) the equation shows the flow to be independent of the elasticity.

In the steady state, equation (36) reduces to equation (20). Now since, $a^2 = Km/S = T/S$ and $b^2 = K'/m'S$, although b and a each involves the elasticity constant S, their ratio--the only manner in which b and a occur in equation (20)--does not. In fact, $b/a = (K'/Kmm')^{1/2}$. Thus in the steady state the drawdown-curve, equation (20), is independent of the elasticity. The theory therefore indicates that in the steady state two aquifers having K, K', m, m', Q, T, and r_e all equal will have identical drawdown curves regardless of what the elasticity constants of these aquifers may be.

It is next shown that when $t=\infty$ (again the steady state) and, in addition, the overlying clay is strictly impermeable, equation (36) reduces to a well-known result. It is necessary in this case to determine the value of the right-hand side of equation (36) when $b=0$.

Substituting $b=0$ in equation (36) the result, $s=0$, an obviously erroneous answer, is obtained. The corrrect result is obtained by letting b approach zero, whereby it is seen from equations (1), (2), and the author's equation (36) that

$$\lim_{b\to 0} s = (Q/2\pi T)\left\{-[\gamma + \log_e(br/a)] + [\gamma + \log_e(br_e/a)]\right\}$$

that is,

$$s = (Q/2\pi T)\log_e(r_e/r), \text{leakage zero, steady state} \qquad (3)$$

the familiar result sought (see MUSKAT, "The flow of homogeneous fluids through porous media," (McGraw-Hill, pp. 150-154, 1937).

In order to determine next what happens in the steady state ($t=\infty$) when the leakage is finite ($b>0$), but when the extent of the aquifer is very large, ($r_e\to\infty$), the following relations (found in treatises on BESSEL-functions) are needed:

$$\underset{x\to\infty}{K_0(x)} = (\pi/2x)^{1/2}\,e^{-x}$$

$$\underset{x\to\infty}{I_0(x)} = (1/2\pi x)^{1/2}\,e^{x}$$

Using these in equation (36) it is found that

$$\lim_{r_e \to \infty} s = (Q/2\pi T)\,[K_0(br/a) - I_0(br/a) \cdot \pi e^{-2br_e/a}]$$

that is

$$s = (Q/2\pi T)\, K_0(br/a)\,, \quad r_e \to \infty$$

which is the steady state result for an infinite aquifer given by the author in equation (5). It is noted again that this result, since it contains b/a as a ratio, does not involve the elasticity of the aquifer.

So far, equation (36) has been considered for cases when $t=\infty$, that is, when the Σ terms are zero. The remainder of this discussion will be concerned with the solution of the non-steady state.

From physical intuition when the distances r from the well at which drawdown measurements are made are small compared to the extent of the aquifer, and when the elapsed time of pumping is small compared to the pumping time required to obtain the steady state, the drawdown should be the same for a leaky aquifer of finite extent as for a non-leaky aquifer ($b^2=0$) of infinite extent. That this is true may be seen in the author's Figure 2. It might also be expected that if two such aquifers (one of infinite extent and the other of finite extent r_e) each had zero leakage, the drawdown-curve would be the same for larger values of elapsed pumping time than when the aquifer of finite extent is leaky. I have prepared Figure 1 to bring out this point.

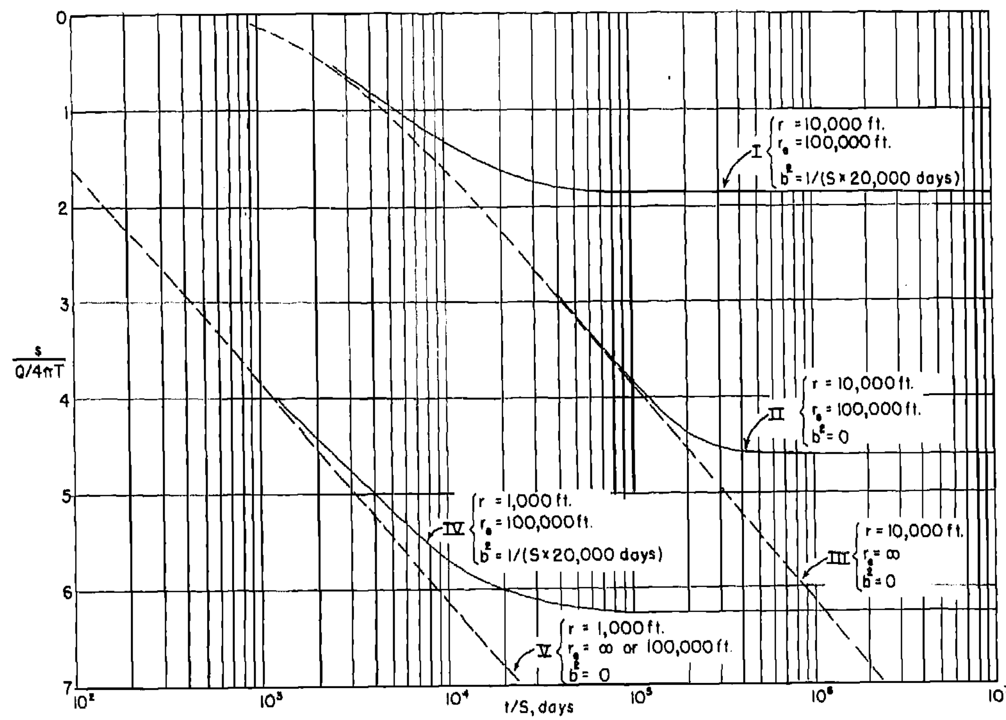

Fig. 1--Variation of drawdown with time

In Figure 1 (where the drawdown is plotted in units of $(Q/4\pi T)$, and the time as t in days divided by S), Curve I shows the variation of drawdown with time for $b^2 = 1/(20{,}000\ S\ \text{ft})$ and r_e = 100,000 ft at a distance 10,000 ft from the well, and Curve II shows the drawdown for the same case except that now $b^2 = 0$. Curve III is the drawdown curve for the infinite non-leaky aquifer. It is seen that Curves I and III correspond up to about $t/S = 3\times 10^3$ days, whereas Curves II and III correspond up to about $t/S = 10^5$ days. Curves IV and V show results for r = 1000 ft. Curve V is equally valid for the case $r_e = \infty$, $b^2 = 0$, and the case r_e = 100,000 ft, $b^2 = 0$, insofar as the curve is contained on the paper. A calculation from equation (3) shows that the curve for r_e=100,000, $b^2 = 0$, does not become flat until $S/(Q/4 \quad T) = 9.21$.

In Figure 1 the data for Curves I and IV are taken from the author's Figure 2, and Curves III and IV are calculated from the expression (where $u = r^2S/4Tt$)

$$s/(Q/4\pi T) = -\mathrm{Ei}(-u) = \int_u^\infty (e^{-y}/y)\,dy$$
$$= -0.5772 + \log_e(1/u) + u - u^2/2.2! + \ldots \quad \ldots\ldots\ldots(4)$$

the expression for drawdown in an infinte elastic aquifer of zero leakage given by the author in reference [1], and again mentioned near the end of his paper. In this paper, he uses the notation W(u) for the more conventional notation -Ei(-u), used in his paper of 1940. With regard to the function -Ei(-u), it may be noted that extensively tabulated values of -Ei(-u) are available in the Federal Works Project "Tables of Sine, Cosine, and Exponential Integrals," Vols. I and II, volumes not available when the author wrote his 1940 article.

A final point is noted. Although mathematically $s/(Q/4\pi T)$ appears to go to infinity (see the author's Figure 2 and curves III and V of Figure 1) when $t=\infty$; and also appears in equation (3) to go to infinity when $r_e \to \infty$; from physical considerations this cannot occur. Equation (36) of the author, of which equations (3) and (4) may be considered special cases, will, however, be valid for large values of r_e and/or t, if the parameter Q is small.