Step-drawdown tests and the Forchheimer equation

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[1] Step-drawdown tests (SDTs) typically involve pumping a well at a constant rate until a quasi-steady state (QSS) is observed in the drawdown response. The well is then pumped at a higher constant rate until a new QSS is achieved. The process is repeated for additional flow rates. Analysis involves plotting the QSS drawdowns against their corresponding abstraction rates and fitting a nonlinear empirical expression. A commonly used expression, the so-called Jacob method, contains a linear "formation loss" coefficient, *A*, and a nonlinear "well loss" coefficient, *B*. In this paper, an analytical formula is derived relating *B* to the Forchheimer parameter. The efficiency of the Forchheimer equation for simulating SDTs is demonstrated through four case studies. Quantitative guidance is given as to when the Jacob method can be used as an accurate alternative to numerical simulation of the Forchheimer equation. Finally, the four corresponding estimates of field-scale Forchheimer parameter are implicitly compared to those obtained from smaller scale laboratory experiments. Unfortunately, the comparison remains implicit due to uncertainty associated with quantifying effective well radius and aquifer formation thickness for the SDTs.

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1. Introduction

[2] An important groundwater abstraction assessment technique is the step-drawdown test (SDT). SDTs typically involve pumping at a constant rate, Q [L^3T^{-1}], while continuously monitoring water level drawdown within the abstraction well, s_w [L]. Eventually, the water level is expected to reach a quasi-steady state (QSS). Once this occurs, the abstraction rate is increased and held constant until a new QSS is achieved. Typically, standards suggest that SDTs should constitute a minimum of four steps. Subsequently, the four QSS drawdowns are plotted against abstraction rate and an empirical equation is fitted. The most popular of these is that proposed by *Jacob* [1946],

$$s_w = AQ + BQ^2, \tag{1}$$

where A [$L^{-2}T$] and B [$L^{-5}T^2$] are referred to as the formation-loss and well-loss coefficients, respectively.

[3] Another commonly used equation is that of *Rorabaugh* [1953],

$$s_w = AQ + BQ^c, \tag{2}$$

where c [−] is an empirical exponent and B now has the dimensions [$L^{1-3c}T^c$].

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[4] Often A is treated as a function of time, t [T], and is calculated from the *Theis* [1935] solution

$$A(t) = \frac{1}{4\pi T}\mathrm{E}_1\left(\frac{Sr_w^2}{4Tt}\right), \quad A(t \le 0) = 0, \tag{3}$$

where T [L^2T^{-1}] is transmissivity, S [−] is storativity, r_w [L] is well radius, and E_1 denotes the exponential integral [see *Gautschi and Cahill*, 1964, p. 229]. For large times this can be approximated by [*Cooper and Jacob*, 1946]

$$A(t) \approx \frac{1}{4\pi T}\left[\ln\left(\frac{4Tt}{Sr_w^2}\right) - 0.5772\right]. \tag{4}$$

Now consider a sequence of abstraction rates $Q_{n=1\ldots N}$ starting at $t = \tau_{n=0\ldots N-1}$ and terminating at $t = \tau_{n=1\ldots N}$, respectively. Note that $\tau_0 = 0$ and $Q_0 = 0$. Equation (2) then takes the form

$$s_w = \sum_{n=1}^{N}\left\{A(t-\tau_{n-1})(Q_n - Q_{n-1}) + B[H(t-\tau_{n-1}) - H(t-\tau_n)]Q_n^c\right\}, \tag{5}$$

where $H(t)$ denotes the Heaviside step-function. This is essentially the approach adopted by *Clark* [1977] and *Kawecki* [1995] (albeit with $c = 2$). Application of equation (5) implies that B quantifies a non-Darcy component of the well response. Hereafter, equation (5) with $c = 2$ and $A(t)$ calculated from equation (3) is referred to as the Jacob method. The Heaviside step-functions make the nonlinear Q_n^c terms zero for $t < \tau_{n-1}$ and $t > \tau_n$. It is only the linear terms that are being added in equation (5). Note that application of the principal of superposition is inappropriate for nonlinear systems.

[5] Considering the Darcy-Weisbach equation for steady turbulent flow in pipes [e.g., *Massey*, 1989], one would generally expect $c = 2$ (i.e., Equation (1)). Nevertheless, the modification of *Rorabaugh* [1953] to account for unknown c is frequently invoked. Additionally, it is often found necessary to treat B as a function of time [*Helweg*, 1994; *van Tonder et al.*, 2001] leading to complicated over-parameterized empirical functions.

[6] The term, BQ^c is generally attributed to losses within the well infrastructure (for a comprehensive discussion on sources, see *Barker and Herbert* [1992]). However, the term 'well-loss' coefficient is misleading as nonlinear losses also occur in the surrounding formation where flow velocities are enhanced due to the convergence of flow lines around the abstraction well [*Mathias et al.*, 2008; *Butler et al.*, 2009]. In recognition of this fact, an alternative approach to analyzing non-Darcy flow in SDTs is to use the *Forchheimer* [1901] equation [*Kohl et al.*, 1997; *Kolditz*, 2001]

$$\frac{\mu q}{k} + \rho b q^2 = -\frac{dP}{dr}, \tag{6}$$

where μ [ML^{-1}T^{-1}] is dynamic viscosity, q [LT^{-1}] is water flux, k [L^2] is intrinsic permeability, ρ [ML^{-3}] is density, P [ML^{-1}T^{-2}] is pressure, r [L] is radial distance from the pumping well, and b [L^{-1}] is known as the Forchheimer parameter. When $b = 0$, equation (6) reduces to Darcy's law. Note that transmissivity is calculated from $T = H\rho gk/\mu$ and drawdown from $s = (P_0 - P)/\rho g$, where H [L] is formation thickness, g [LT^{-2}] is acceleration due to gravity, and P_0 [ML^{-1}T^{-2}] is initial pressure.

[7] Recently, *Mathias et al.* [2008] showed that for large times, the drawdown response of a well-being pumped at a constant rate, according to the Forchheimer equation, can be approximated by

$$s_w = \frac{Q}{4\pi T}\left[\ln\left(\frac{4Tt}{Sr_w^2}\right) - 0.5772\right] + \frac{bQ^2}{(2\pi H)^2 r_w g}. \tag{7}$$

Mathias et al. [2008] were only interested in Forchheimer flow to wells for constant abstraction rates and therefore did not consider SDTs or Jacob's method. However, comparison of equations (4) and (7) would suggest that equation (5) (when $c = 2$) is a large time approximation for Forchheimer flow to a well with a piecewise constant abstraction rate, whereby Jacob's well-loss coefficient, B, is related to the Forchheimer parameter by

$$B = \frac{b}{(2\pi H)^2 r_w g}. \tag{8}$$

In this paper, following the work of *Helweg* [1994] and *van Tonder et al.* [2001], it is proposed that the main shortcoming of the Jacob method is due to the assumption of a constant B coefficient. The assumption of constant B is equivalent to approximating only the large time behavior of the Forchheimer equation. Proper numerical solution of the Forchheimer equation leads to a transient B coefficient.

[8] In the work presented subsequently, the necessary mathematical equations for simulating SDTs using the Forchheimer equation are presented. Guidance is given concerning an efficient procedure for calibrating the numerical solution to observed SDT drawdown data. The efficiency of the Forchheimer equation for simulating SDTs is demonstrated through four case studies [*Clark*, 1977; *Shapiro et al.*, 1998; *van Tonder et al.*, 2001; *Karami and Younger*, 2002]. Quantitative guidance is given as to when the Jacob method can be used as an accurate alternative to numerical simulation of Forchheimer's equation. Finally, the four corresponding estimates of Forchheimer parameter are compared to those obtained from smaller scale laboratory experiments [*Cornell and Katz*, 1953; *Geertsma*, 1974; *Thiruvengadam and Kumar*, 1997; *Sidiropoulou et al.*, 2007]. A limitation of the work presented here is that nonlinear losses associated with the well infrastructure are not accounted for. This is discussed further in the summary and conclusions.

2. Mathematical Formulation of the Flow Problem

[9] The governing equation of flow to a fully penetrating well in a homogenous, isotropic, and confined aquifer is [*Papadopulos and Cooper*, 1967]

$$\frac{S}{\rho g H}\frac{\partial P}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rq) = 0, \tag{9}$$

subjected to the initial and boundary conditions

$$\begin{aligned} P &= P_0, \quad r \geq r_w, \quad t = 0, \\ P &= P_w, \quad r = r_w, \quad t > 0, \\ P &= P_0, \quad r = \infty, \quad t > 0, \end{aligned} \tag{10}$$

where q [LT^{-1}] is the water flux found from equation (6) and P_w [ML^{-1}T^{-2}] is the pressure in the wellbore found from [*Papadopulos and Cooper*, 1967]

$$\frac{\pi r_c^2}{\rho g}\frac{dP_w}{dt} + Q(t) + 2\pi H r_w q(r = r_w) = 0, \tag{11}$$

subjected to

$$P_w = P_0, \quad t = 0, \tag{12}$$

where r_c [L] is the radius of the well casing [see *Papadopulos and Cooper*, 1967, Figure 1]. Applying the following dimensionless transformations:

$$P_D = \frac{2\pi T(P_0 - P)}{\rho g Q_1}, \quad P_{wD} = \frac{2\pi T(P_0 - P_w)}{\rho g Q_1}, \quad Q_D = \frac{Q}{Q_1}, \tag{13}$$

$$q_D = -\frac{2\pi H r_w q}{Q_1}, \quad b_D = -\frac{Q_1 T b}{2\pi H^2 r_w g} \equiv -2\pi Q_1 TB, \tag{14}$$

$$t_D = \frac{Tt}{Sr_w^2}, \quad r_D = \frac{r}{r_w}, \quad r_{cD} = \frac{r_c}{S^{1/2} r_w}, \tag{15}$$

then leads to

$$\frac{\partial P_D}{\partial t_D} + \frac{1}{r_D}\frac{\partial}{\partial r_D}(r_D q_D) = 0, \tag{16}$$

$$q_D + b_D q_D^2 = -\frac{\partial P_D}{\partial r_D}, \tag{17}$$

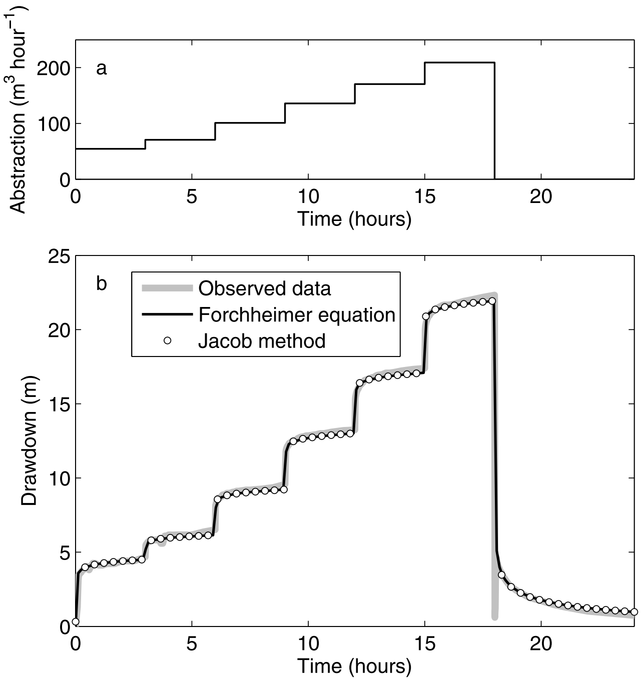

Figure 1. The SDT of *Clark* [1977]. (a) The abstraction history. (b) Observed and simulated (using Stage 3 parameters from Table 1) drawdown.

$$\begin{aligned} &P_D = 0, && r_D \geq 1, && t_D = 0, \\ &P_D = P_{wD}, && r_D = 1, && t_D > 0, \\ &P_D = 0, && r_D = \infty, && t_D > 0, \end{aligned} \tag{18}$$

$$\frac{r_{cD}^2}{2}\frac{dP_{wD}}{dt_D} - Q_D + q_D(r_D = 1) = 0, \tag{19}$$

$$P_{wD} = 0, \quad t_D = 0. \tag{20}$$

Detailed guidance on how to solve the above dimensionless boundary value problem using finite differences and MATLAB's ODE15s is given by *Mathias et al.* [2008].

3. Calibration Procedure

[10] In the next section, the efficiency of the Forchheimer equation at simulating SDTs is demonstrated through four case studies [*Clark*, 1977; *Shapiro et al.*, 1998; *van Tonder et al.*, 2001; *Karami and Younger*, 2002]. For each case study, the numerical Forchheimer equation model (NFEM) (discussed in the previous section) was calibrated to the observed drawdown data. The calibration procedure used is described as follows:

[11] Stage 1: First, preliminary estimates of T and the product Sr_w^2 are obtained by fitting the *Cooper and Jacob* [1946] approximation of the Theis solution (recall equation (4)) to the large time data from the first step of the SDT using linear regression.

[12] Stage 2: These preliminary estimates are refined and a preliminary estimate of B is obtained by calibrating the Jacob method (Equation (5) with c = 2) to the full set of SDT data. Calibration is achieved by minimizing the mean absolute error (MAE):

$$\mathrm{MAE} = \frac{1}{N}\sum_{n=1}^{N} |s_{wm,n} - s_{wo,n}|, \tag{21}$$

where N [−] is the number of observed data points and $s_{wm,n}$ [L] and $s_{wo,n}$ [L] are the nth modeled and observed drawdowns, respectively. Minimization is achieved using MATLAB's nonlinear minimization routine, FMINSEARCH. The preliminary estimates of T and Sr_w^2 are used as seed values for the search. The seed value for the B coefficient is fixed (somewhat arbitrarily) to 1000 m^{-4} s^2.

[13] Stage 3: Final estimates of T, Sr_w^2, and B are obtained by calibrating the NFEM in exactly the same way as for the Jacob method (as described previously in Stage 2) except that the T, Sr_w^2, and B values from the Jacob method calibration are used as the seed values for FMINSEARCH.

[14] In principal Stages 1 and 2 are avoidable and calibration should involve only Stage 3. However, to ensure adequate accuracy of the numerical solution to the Forchheimer equation, it is necessary to approximate the infinite far-field boundary at $r_D = 10^8$ and discretize the space $10^0 \leq r_D \leq 10^8$ with 2000 logarithmically spaced nodes [*Mathias et al.*, 2008]. Consequently, the numerical simulations are computationally expensive to resolve. The implementation of Stages 1 and 2 avoids unnecessary computations by providing FMINSEARCH with seed values that already yield low MAE.

4. Step-Drawdown Test Case Studies

[15] The first case study to be considered is that presented by *Clark* [1977] whereby an SDT, consisting of six 3 hour steps, was conducted in a confined sandstone aquifer. Figure 1 shows plots of abstraction flow rate and drawdown during the SDT. There is excellent correspondence between the observed (thick gray line) and modeled (solid black line) data for all steps, including the recovery period. Also of interest is that the Jacob method (using parameters from Stage 3) accurately predicts the numerical simulation of the Forchheimer equation (compare the circular markers and solid black line, respectively). Note that application of the numerical model also requires knowledge of the well-casing radius, which was 0.25 m in this case.

[16] The second case study is the SDT originally presented by *Shapiro et al.* [1998], which involved step pumping the Madison limestone aquifer near Rapid City, South Dakota. The well-casing radius was r_c = 0.17 m. Figure 2 shows plots of flow rate and drawdown during the SDT. There is good correspondence between the observed (thick gray line) and modeled (solid black line) data for all steps. Also shown, as a dashed black line, is modeled output from the Jacob method (using parameters from Stage 3). In this scenario, the Jacob method consistently overestimates the response of the Forchheimer equation. See Table 1 for parameter values.

[17] The third case study is an SDT from a fractured sandstone aquifer in South Africa, originally presented by *van Tonder et al.* [2001]. The well-casing radius was r_c = 0.08 m (van Tonder, personal communications, 2009). Figure 3 shows plots of flow rate and drawdown during the SDT. There is excellent correspondence between the

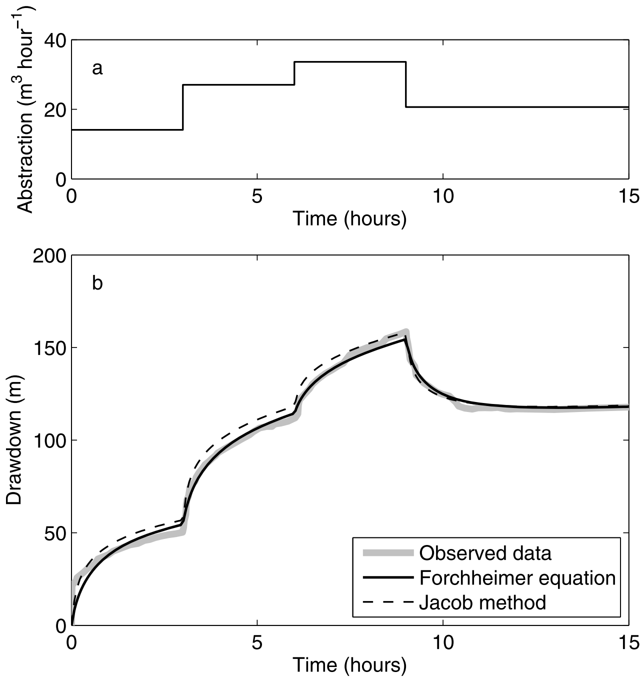

Figure 2. The SDT of *Shapiro et al.* [1998]. (a) The abstraction history. (b) Observed and simulated (using Stage 3 parameters from Table 1) drawdown.

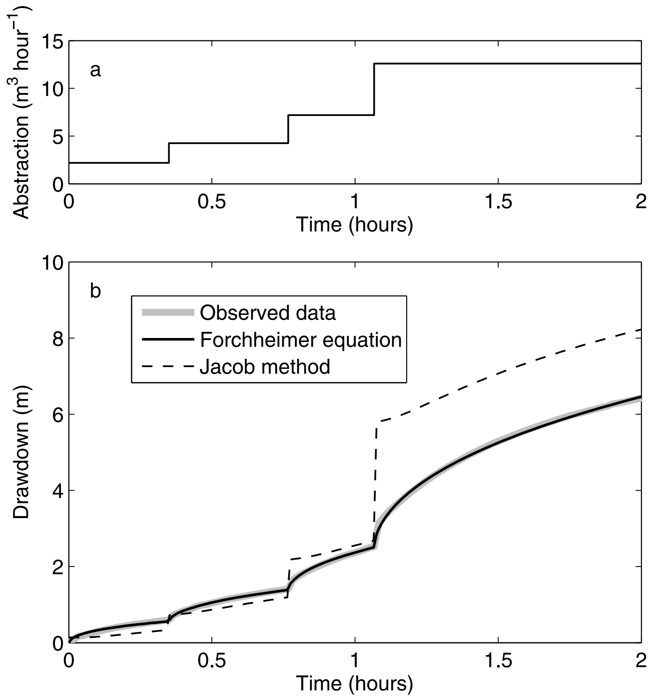

Figure 3. The SDT of *van Tonder et al.* [2001]. (a) The abstraction history. (b) Observed and simulated (using Stage 3 parameters from Table 1) drawdown.

observed (thick gray line) and modeled (solid black line) data for all steps. However, the Jacob method (using parameters from Stage 3) is seen to underestimate the response of the Forchheimer equation during the smaller flow rate steps and overestimate during the larger flow rate steps (compare solid and dashed black lines). The discrepancy during smaller flow rate steps is due to the Jacob method not accounting for wellbore storage. In their original analysis, *van Tonder et al.* [2001] were particularly concerned with flow geometry and fracture deformation. As a result, they developed a complex empirical equation including six unknown parameters. However, Figure 3 demonstrates that their data set is adequately simulated by the four-parameter (T, Sr_w^2, B, and r_c) NFEM.

[18] The fourth case study is that originally presented by *Karami and Younger* [2002] whereby a SDT was applied to a carboniferous limestone aquifer in southern Ireland. The well-casing radius was r_c = 0.1 m [*Karami*, 2002]. Figure 4 shows plots of flow rate and drawdown during the SDT. There is good correspondence between the observed (thick gray line) and modeled (solid black line) data for all steps. Except during the very beginning of the test, there is also good correspondence between the Jacob method and the Forchheimer equation. Again, the discrepancy during early times is due to the Jacob method not accounting for wellbore storage. Note that both the Forchheimer equation and the Jacob method are unable to replicate the change in linear-log slope noted in the large-time portion of the final step in the observed drawdown data. *Karami and Younger* [2002] attributed this change in slope to heterogeneity in transmissivity.

[19] The parameter values and corresponding MAE for the four case studies are presented in Table 1. It can be seen that the MAE is only marginally improved by Stage 3 (calibration of the NFEM). However, the calibration of the NFEM led to significantly reduced MAE for Case Studies 3 and 4. The difference between modeled output from the Jacob method and the Forchheimer equation is very small for Case Studies 1 and 2 (compare Figures 1 and 2). For Case Study 4 the difference is significant only for the early time part of the test. As stated earlier, this is largely due to wellbore effects. The difference is most substantial for Case

Table 1. Calibrated Parameter Values and Mean Absolute Error (MAE) for the Four Case Studies for Each Stage of the Calibration Procedure

	Step 1	Step 2	Step 3
Case Study 1: *Clark* [1977]			
T (ms^{-1})	2.72E-03	4.55E-03	4.55E-03
Sr_w^2 (-)	1.78E-03	1.46E-05	1.46E-05
B (m^{-5}s^2)	N/A	1.40E+03	1.40E+03
MAE (m)	N/A	2.29E-01	2.54E-01
Case Study 2: *Shapiro et al.* [1998]			
T (ms^{-1})	3.60E-05	2.19E-05	2.43E-05
Sr_w^2 (-)	2.98E-03	1.21E-02	7.00E-03
B (m^{-5}s^2)	N/A	8.82E+03	4.09E+03
MAE (m)	N/A	2.19E+00	2.46E+00
Case Study 3: *van Tonder et al.* [2001]			
T (ms^{-1})	1.98E-04	1.26E-04	1.01E-04
Sr_w^2 (-)	5.09E-02	1.05E-01	3.28E-01
B (m^{-5}s^2)	N/A	6.63E+04	3.75E+05
MAE (m)	N/A	3.28E-02	1.94E-02
Case Study 4: *Karami Younger* [2002]			
T (ms^{-1})	5.92E-04	4.09E-04	7.92E-04
Sr_w^2 (-)	3.76E-07	8.05E-04	2.67E-04
B (m^{-5}s^2)	N/A	6.31E+04	1.24E+05
MAE (m)	N/A	1.29E+00	8.17E-01

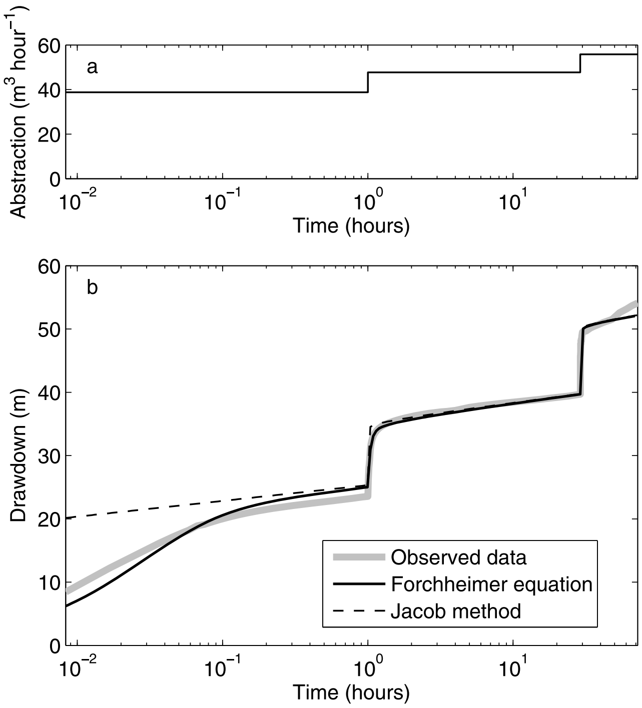

Figure 4. The SDT of *Karami and Younger* [2002]. (a) The abstraction history. (b) Observed and simulated (using Stage 3 parameters from Table 1) drawdown.

Study 3 where the Jacob method is seen to significantly overestimate drawdown for large flow rates.

[20] The reason for the latter is as follows. In the NFEM, non-Darcy losses grow in time before reaching a steady state [*Wu*, 2002]. With the Jacob method, the non-Darcy loss is assumed constant and equal to the steady-state value (recall equation (5)). If the steps in van Tonder's test (Case Study 3) were sufficiently long, the NFEM and the Jacob method would have ultimately converged. Given that the Jacob method is much more convenient to implement than the NFEM, it is useful to know when its use is appropriate.

5. Extent of Validity of the Jacob Method

[21] Figure 5 shows plots of dimensionless drawdown response of the NFEM for a constant abstraction rate for a set of different r_{cD} and b_D values alongside the large time approximation given in equation (7) (the thick black line). Also shown in Figure 5 are a set of circular markers. These mark the dimensionless times, t_{cD} [−] at which the dimensionless error in drawdown between equation (7) and the NFEM has become less than 0.02 (a value <0.02 was considered unnecessarily accurate and led to too large a t_{cD}). It is speculated that the Jacob method becomes appropriate for a given change in abstraction rate when the error between equation (7) and the NFEM is sufficiently low. This requires that the dimensionless time for which the change in abstraction rate is implemented is at least greater than t_{cD}.

[22] The value of t_{cD} is dependant on two parameters, the dimensionless Forchheimer parameter, b_D, and the dimensionless well-casing radius, r_{cD}. Variations of t_{cD} with b_D for various values of r_{cD} are plotted in Figure 6. An estimate

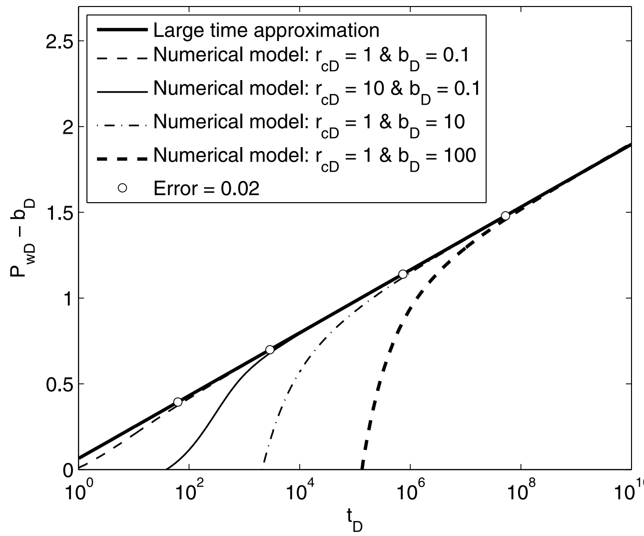

Figure 5. Plots of dimensionless drawdown for a constant abstraction rate using the large time approximation and the numerical Forchheimer equation model (NFEM). The circular markers mark the dimensionless times, t_{cD}, at which the dimensionless error in drawdown between the large time approximation and the NFEM has become less than 0.02.

of t_{cD} can be obtained graphically from Figure 6. Alternatively, a reasonable approximation can be obtained from

$$t_{cD} = \left\{ \left[7.97 r_{cD}^{2.32}\right]^{1/2} + \left[\frac{1}{(7\times10^3)b_D^2} + \frac{1}{(3\times10^7)b_D^{1/2}}\right]^{-1/2} \right\}^2, \tag{22}$$

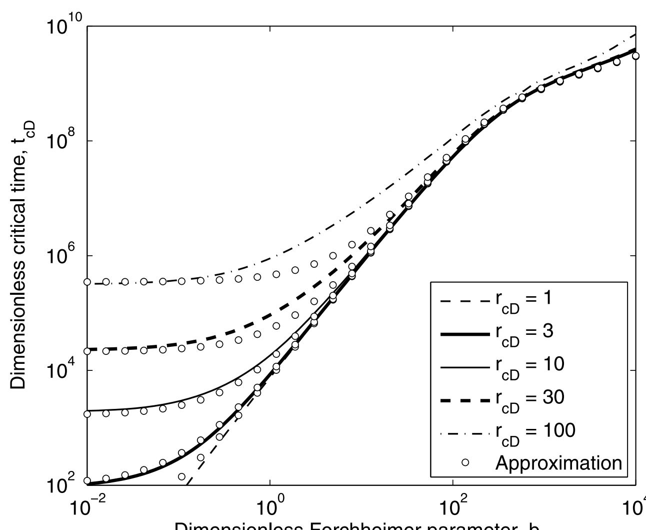

Figure 6. Plots of dimensionless time at which the error between the large time approximation and the numerical Forchheimer equation model has become less than 0.02, t_{cD}, against dimensionless Forchheimer parameter, b_D, for various values of dimensionless well-casing radius, r_{cD}. Also shown as circular markers is the approximate expression given in equation (22).

Table 2. Actual and Ideal Step Durations (i.e., Δt and t_c, Respectively) for the Four Case Studies Along With Associated Auxiliary Parameters[a]

Step	1	2	3	4	5	6
Case Study 1: *Clark* [1977]						
Δt (hours)	3	3	3	3	3	3
ΔQ (m^3hour^{-1})	54	16	30	35	35	39
b_D (-)	0.61	0.18	0.34	0.39	0.39	0.43
r_{cD} (-)	65	65	65	65	65	65
t_{cD} (-)	1.7E+05	1.4E+05	1.5E+05	1.5E+05	1.5E+05	1.6E+05
t_c (hours)	0.15	0.13	0.13	0.14	0.14	0.14
Case Study 2: *Shapiro et al.* [1998]						
Δt (hours)	3	3	3	15	N/A	N/A
ΔQ (m^3hour^{-1})	14	13	7	13	N/A	N/A
b_D (-)	0.0024	0.0022	0.0011	0.0022	N/A	N/A
r_{cD} (-)	2.0	2.0	2.0	2.0	N/A	N/A
t_{cD} (-)	44	44	42	44	N/A	N/A
t_c (hours)	3.5	3.5	3.4	3.5	N/A	N/A
Case Study 3: *van Tonder et al.* [2001]						
Δt (hours)	0.35	0.42	0.30	0.93	N/A	N/A
ΔQ (m^3hour^{-1})	2	2	3	5	N/A	N/A
b_D (-)	0.14	0.14	0.19	0.36	N/A	N/A
r_{cD} (-)	0.14	0.14	0.14	0.14	N/A	N/A
t_{cD} (-)	154	135	275	904	N/A	N/A
t_c (hours)	139	122	248	818	N/A	N/A
Case Study 4: *Karami and Younger* [2002]						
Δt (hours)	1	28	43	N/A	N/A	N/A
ΔQ (m^3hour^{-1})	39	9	8	N/A	N/A	N/A
b_D (-)	6.7	1.5	1.4	N/A	N/A	N/A
r_{cD} (-)	6.1	6.1	6.1	N/A	N/A	N/A
t_{cD} (-)	3.36E+05	2.31E+04	1.96E+04	N/A	N/A	N/A
t_c (hours)	31	2	2	N/A	N/A	N/A

[a]Calculations are based on the Stage 3 parameter sets detailed in Table 1.

which is plotted in Figure 6 as circular markers. The first term in equation (22) was obtained by fitting a power law to values of t_{cD} when $b_D = 0$. The second term and third terms were obtained by fitting power laws to the intermediate and large b_D parts, respectively, of the curve when $r_{cD} = 1$. The structural form of the equation was designed to ensure that the first term is emphasized for small b_D, the second term for intermediate b_D, and the third term for large b_D. The approximation is reasonably accurate except for intermediate values of b_D when $r_{cD} > 30$.

[23] Table 2 compares actual step durations, Δt [T], and values of $t_c = Sr_w^2 t_{cD}/T$ [T] (as calculated from equation (22)) for each of the steps in each of the case studies. Note that b_D is calculated by assuming $Q_1 = -\Delta Q$. Providing, $\Delta t > t_c$, the step should be sufficiently large that the NFEM is adequately approximated by the Jacob method. Following parameter estimation by applying the Jacob method to observed data, the suitability of the Jacob method can be verified by checking whether $\Delta t > t_c$ for each step.

[24] For Case Study 1 [*Clark*, 1977], Δt is consistently greater than t_c by at least a factor of 20. Correspondingly, it is seen in Figure 1 that the drawdown response due to the Forchheimer equation and the Jacob method are indistinguishable. For Case Study 2 [*Shapiro et al.*, 1998] all the step durations are consistently 14% less than t_c. Correspondingly, Figure 2 shows a minor discrepancy between the Forchheimer equation and the Jacob method. For Case Study 3 [*van Tonder et al.*, 2001], it is found that the step durations are two to three orders of magnitude lower than necessary with the duration of the final step being a factor of 880 lower than t_c. Not surprisingly, Figure 3 shows the Jacob method to be a poor approximation of the Forchheimer equation in this scenario. For Case Study 4 [*Karami and Younger*, 2002], the second and third steps are sufficiently long but the duration of the first step is a factor of 30 lower than t_c. A corresponding discrepancy between the Jacob method and the Forchheimer equation during the first step is apparent in Figure 4. Clearly, equation (22) is a useful tool for assessing when the Jacob method may be inappropriate for analyzing step-drawdown test.

[25] Figure 7 shows the expected drawdown response for van Tonder's test (Case Study 3) if values of t_c had been used for the step durations. There is now very good correspondence between the Jacob method and the numerical solution of Forchheimer's equation. However, the expense manifests itself in that the test would need to be 1330 hours long. An obvious alternative is to keep the step durations as they were and use the NFEM to analyze the results (instead of the Jacob method).

6. Comparison of Laboratory and Field-Scale Forchheimer Parameters

[26] Previously, Forchheimer parameters, b, have mostly been obtained from laboratory-scale one-dimensional permeameter experiments [e.g., *Cornell and Katz*, 1953; *Ward*, 1964; *Geertsma*, 1974]. Exceptions to this are the studies of *Kohl et al.* [1997] and *Kolditz* [2001], who applied the Forchheimer equation to pumping test data from an artificially hydraulically fractured geothermal site in Soultz, France. Most of the laboratory-scale experiments have involved gravelly type unconsolidated materials with the exception of *Cornell and Katz* [1953], who looked at a range of consolidated rocks, including sandstones, limestones, and dolomites. More recently, some two-dimensional experiments have also been conducted using radially con-

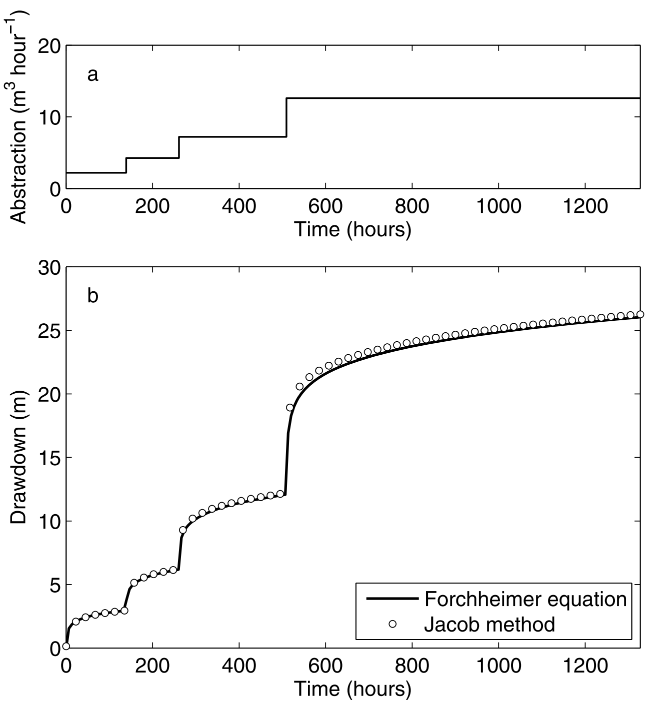

Figure 7. Simulation of the *van Tonder et al.* [2001] SDT using the Stage 3 parameter set in Table 1 and step durations calculated from t_{cD} (see Table 2). (a) The abstraction history. (b) Simulated drawdown.

vergent permeameters [*Thiruvengadam and Kumar*, 1997; *Venkataraman and Rao*, 2000; *Reddy and Rao*, 2006].

[27] Many empirical relationships have been proposed linking b [L^{-1}] to other parameters include permeability, k [L^2], porosity, ϕ [−], and particle size, D [L]. By straightforward dimensional analysis, *Ward* [1964] shows that b should be a function of $k^{-1/2}$. A popular relationship linking this with porosity is that of *Geertsma* [1974] who proposed

$$b = 0.005\phi^{-5.5}k^{-0.5}. \tag{23}$$

The *Geertsma* [1974] data set mostly included data from their own experiments and those of *Cornell and Katz* [1953]. More recently, *Sidiropoulou et al.* [2007] compiled an extensive alternative data set from literature dating back to 1901. We have incorporated the data of *Sidiropoulou et al.* [2007] with that of *Geertsma* [1974] and *Cornell and Katz* [1953] and the data from radially converging permeameter tests of *Thiruvengadam and Kumar* [1997] (see Figure 8). Note that only data with accompanying porosity and permeability values are used. From Figure 8, it is clear the additional data supports the robustness of equation (23). Furthermore, it can be seen that the results from the radially converging experiments of *Thiruvengadam and Kumar* [1997] (the square markers in Figure 8) show no noticeable visual difference from the other one-dimensional permeameter data.

[28] It would be interesting to also plot the values obtained from the four SDT case studies. However, because of uncertainty associated with the value of well radius, r_w, and formation thickness, H, it is not possible to directly calculate values of b and k. We also have no appropriate knowledge of the formation porosities. Note that well radius is essentially an unknown parameter due to problems associated with well skin effects [see discussion of *Shapiro et al.* 1998].

[29] However, the laboratory results from the radially convergent permeameter tests can be converted to the well-loss coefficient, B, and transmissivity, T, providing information is known regarding r_w and H. For apparatus used by *Thiruvengadam and Kumar* [1997], $r_w = 0.15$ m and $H = 0.5$ m. Figure 9 shows plots of T against B for the laboratory experiments of *Thiruvengadam and Kumar* [1997] along with the results from the four SDT case studies. Figure 9 also shows a power law relationship fitted to the *Thiruvengadam and Kumar* [1997] data using linear regression along with the corresponding 95% confidence limits. Interestingly, the confidence limits fully encompass the field-scale SDT results. Given the correspondence between the *Thiruvengadam and Kumar* [1997] data and the *Geertsma* [1974] correlation (recall Figure 8), the results presented in Figure 9 lend good support to the use of the *Geertsma* [1974] correlation for field-scale problems. Note that simply applying a rule based on $B \propto T^{-0.5}$ (which one might lead to when considering equation (23)) does not work well due to the additional influence of formation thickness, H, and porosity, ϕ.

7. Summary and Conclusions

[30] In this paper, it has been proposed to use the Forchheimer equation for the analysis of SDTs. In particular, by equating the Jacob method ($s_w = AQ + BQ^2$) with a large time approximation for Forchheimer flow to a well [*Mathias et al.*, 2008] a direct relationship was ascertained between the so-called well-loss coefficient, B, and the Forchheimer parameter, b (recall equation (8)). The necessary governing equations for transient Forchheimer flow were presented. An NFEM was then calibrated to drawdown data from four SDT case studies. Note that equation (8) was not previously discussed or presented by *Mathias et al.* [2008].

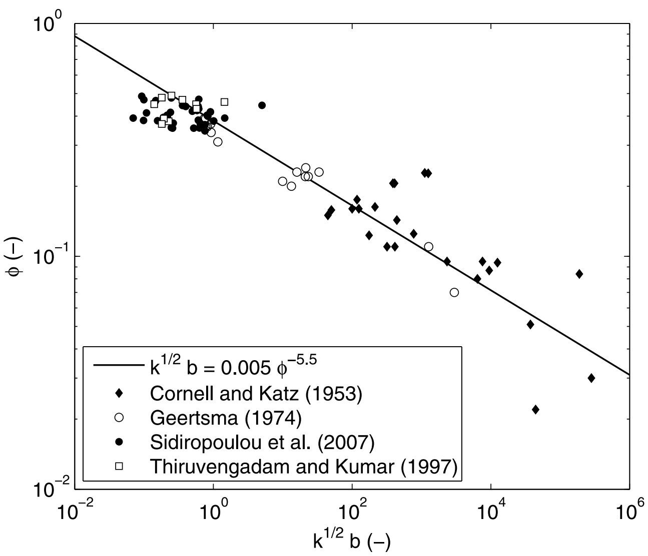

Figure 8. Plot of Forchheimer parameters obtained from laboratory-scale experiments found in the literature along with the Geertsma correlation (the solid black line).

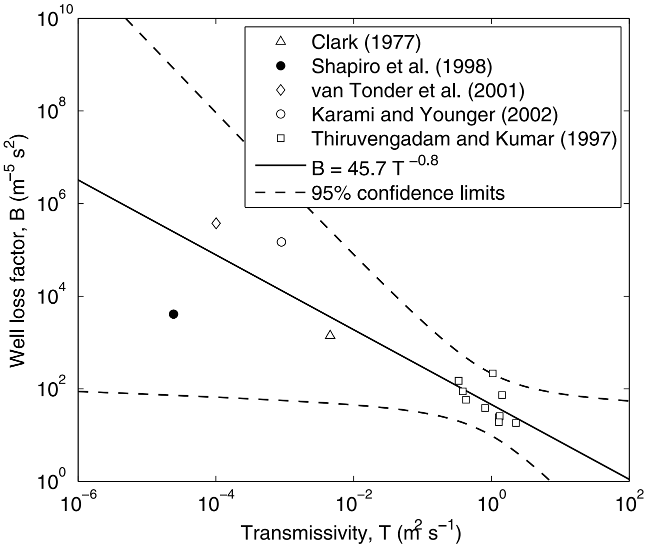

Figure 9. Plot of well-loss coefficient, B, against transmissivity, T, for the field-scale SDT [*Clark*, 1977; *Shapiro et al.*, 1998; *van Tonder et al.*, 2001; *Karami and Younger*, 2002] and the laboratory-scale results of *Thiruvengadam and Kumar* [1997]. The solid line (and corresponding confidence limits) is the result of applying linear regression to the *Thiruvengadam and Kumar* [1997] data only.

[31] There was reasonably good correspondence between the NFEM and the Jacob method for three of the Case Studies (1, 2, and 4). However, for Case Study 3 [*van Tonder et al.*, 2001], the Jacob method substantially overestimated the drawdown for the larger abstraction rate steps. The reason for this is that in the NFEM, non-Darcy losses grow in time before reaching a steady-state value. With the Jacob method, the non-Darcy loss is assumed constant and equal to the steady-state value. If the steps in Case Study 3 were sufficiently long, the NFEM and the Jacob method would have ultimately converged (see Figure 7).

[32] A method was provided for quantifying the necessary duration of steps in a SDT for the Jacob method and the Forchheimer equation to be equivalent. The NFEM requires considerable amounts of computation time (a typical SDT realization can take up to 10 min on a standard laptop computer in 2009). Therefore, it is preferable to use the Jacob method where appropriate. This was achieved by evaluating the dimensionless time, t_{cD}, at which the error associated with the large time approximation of *Mathias et al.* [2008] becomes less than 0.02 dimensionless head units. The value of t_{cD} is controlled by the dimensionless Forchheimer parameter, b_D, and the dimensionless well-casing radius, r_{cD}. A plot of t_{cD} against b_D for a range of r_{cD} values was presented (Figure 6) and a simple empirical expression was fitted (recall equation (22)). By application to the case studies it was demonstrated that providing the step duration is greater than $Sr_w^2 t_{cD}/T$, the Jacob method should be a good approximation of the NFEM.

[33] Having established that SDTs can be interpreted using the Forchheimer equation it follows that SDTs are also a potential source of field-scale Forchheimer parameters. Previously Forchheimer parameters have mostly been obtained from laboratory-scale experiments. It was not possible to directly compare these with the results from the field-scale SDTs because of uncertainty associated with formation thickness and effective well radius. Nevertheless, through some regression analysis it was demonstrated that the field-scale observations are visually consistent with measurements from laboratory-scale radially converging permeameter experiments.

[34] The main conclusions of this work are as follows: (1) The Forchheimer equation offers a parametrically simple method for interpreting SDTs where the step durations are insufficient to allow drawdowns to reach a quasi-steady state. (2) Providing the step durations are sufficiently large, the Forchheimer equation is equivalent to the Jacob method. (3) It is possible to obtain field-scale Forchheimer parameters from Jacob well-loss coefficients (the B parameter) and vice versa.

[35] A limitation of the above approach is that losses associated with the abstraction well infrastructure are not accounted for. Non-Darcian (or nonlinear) losses in the drawdown response are attributed exclusively to nonlinear losses within the geological formation via the nonlinear term in the Forchheimer equation. Additional nonlinear losses could be incorporated into the NFEM. However, it is apparent from the goodness of fit to the data presented in the four case studies (see Figures 1, 2, 3, and 4) that the information content of drawdown data in the abstraction well is insufficient to delineate between nonlinear losses in the formation and the abstraction well infrastructure. Further work is needed looking at calibrating the NFEM to SDTs with simultaneous observations of drawdown in the abstraction well and additional observation wells.

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