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An analytical model for flow induced by a constant-head pumping in a leaky unconfined aquifer system with considering unsaturated flow

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A B S T R A C T

A new mathematical model is developed to describe the flow in response to a constant-head pumping (or constant-head test, CHT) in a leaky unconfined aquifer system of infinite lateral extent with considering unsaturated flow. The model consists of an unsaturated zone on the top, an unconfined aquifer in the middle, and a second aquifer (aquitard) at the bottom. The unsaturated flow is described by Richard's equation, and the flows in unconfined aquifer and second layer are governed by the groundwater flow equation. The well partially penetrates the unconfined aquifer with a constant head in the well due to CHT. The governing equations of the model are linearized by the perturbation method and Gardner's exponential model is adopted to describe the soil retention curves. The solution of the model for drawdown distribution is obtained by applying the methods of Laplace transform and Weber transform. Then the solution for the wellbore flowrate is derived from the drawdown solution with Darcy's law. The issue of the equivalence of normalized drawdown predicted by the present solution for constant-head pumping and Tartakovsky and Neuman's (2007) solution for constant-rate pumping is discussed. On the basis of the wellbore flowrate solution, the results of the sensitivity analysis indicate that the wellbore flowrate is very sensitive to the changes in the radial hydraulic conductivity and the thickness of the saturated zone. Moreover, the results predicted from the present wellbore flowrate solution indicate that this new solution can reduce to Chang's et al. (2010a) solution for homogenous aquifers when the dimensionless unsaturated exponent approaches 100. The unsaturated zone can be considered as infinite extent in the vertical direction if the thickness ratio of the unsaturated zone to the unconfined aquifer is equal to or greater than one. As for the leakage effect, it can be ignored when the vertical hydraulic conductivity ratio (i.e., the vertical hydraulic conductivity of the lower layer over that of the unconfined aquifer) is smaller than 0.1. The present solution is compared with the numerical solution from FEMWATER for validation and the results indicate good match between these two solutions. Finally, the present solution is applied to a set of field drawdown data obtained from a CHT for the estimation of hydrogeologic parameters.

1. Introduction

A constant-head test (CHT) is one of aquifer tests for the determination of the hydraulic parameters (e.g., hydraulic conductivity, specific storage, and specific yield) when the test formation is of low permeability (e.g., Jones et al., 1992; Jones, 1993). During the test, the well water level is kept constant and the temporal wellbore flowrate is then measured at the test well. Chen and Chang (2003) mentioned that some landfills and groundwater contamination sites were found in shallow unconfined aquifers of low permeability, where the conventional constant rate pumping test is inappropriate because of producing a large drawdown or a great amount of pumped water for aquifer remediation. On the other hand, the CHT is an attractive alternative because a constant head pumping can avoid the problems of over-dewatering shallow aquifers (Hiller and Levy, 1994) or overdrawing the pumped well (Jones, 1993). Moreover, constant head pumping can be adopted to remove light nonaqueous phase liquids floating atop the water table (Abdul, 1992) or to hydraulically control the contaminated groundwater in the aquifer (Hiller and Levy, 1994). In the past, many studies addressed the issue of conducting CHT in confined aquifers (e.g., Jacob and Lohman, 1952; Hantush, 1964; Mishra and Guyonnet, 1992; Markle et al., 1995; Peng et al., 2002; Wang and Yeh, 2008; Yeh and Chang, 2013; Lin et al., 2016). In analyzing CHT data of leaky aquifer, Hantush (1964) developed a mathematical model to describe the drawdown distribution in the aquifer with considering the flow in overlying aquitard being entirely vertical.

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Nomenclature

(r, z,t)	radial, vertical distance and time
(K_r, K_z)	radial and vertical hydraulic conductivity in unconfined aquifer
(K_r', K_z')	radial and vertical hydraulic conductivity in second aquifer
(S_s, S_s')	specific storage in aquifer and second aquifer
(s, σ, s')	drawdown in unconfined aquifer, unsaturated zones and second aquifer
$(s_{\mathrm{skin}}, s_{\mathrm{loss}})$	drawdown in skin region and head loss
(b, b_u, b')	Thickness of unconfined aquifer, unsaturated zones and second aquifer
(ψ, ψ_a)	pressure head and air-entry pressure in unsaturated zone
k	relative hydraulic conductivity
C	specific moisture capacity
$(X_{i,k}, O_i)$	normalized sensitivity and ith output value
α	unsaturated exponent
z_1	pumping well bottom of screen
η	elevation of water table
r_w	radius of pumping well
ε	parameter of perturbation expansion
(K_s, r_s)	skin hydraulic conductivity and thickness of well skin
(Q, Q_c)	wellbore flowrate, and constant-pumping rate
S_k	skin factor
S_y	specific yield
(p, λ)	Laplace and Weber transform parameters
$(P_k, \Delta P_k)$	kth input parameter and 10^{-3} P_k

On the basis of Hantush's (1960) leaky aquifer model, Wen et al. (2011) developed a model for CHT in an aquifer bounded by overlying and underlying aquitards with a finite-thickness skin around the wellbore. Notice that the skin is usually developed near the wellbore as a result of well drilling and/or well development. The thickness of the skin is generally very small and may be considered as infinitesimal. Therefore, a factor, referred to as skin factor, is commonly introduced to represent the effect of skin zone on the pumping drawdown.

As regard to unconfined aquifers, their temporal drawdown behavior is totally different from the confined ones, since the water table elevation in unconfined aquifers under pumping declines at different rates (Neuman, 1972). In the early pumping period, the pumped water is mainly from the elastics storage of the aquifer. Then, the effect of the gravity drainage works on the aquifer flow and the drawdown curve appears to be stabilized. As the effect of delayed release of water gradually vanishes, the drawdown curve tends to exhibit confined behavior. Boulton (1954a) was the first scholar to propose a mathematical model with considering the delayed drainage. His governing equation includes a term of convolution integral to represent the delayed release of water. There are mainly two approaches to model unconfined flow: one is to linearize water table condition (e.g. Boulton, 1954b, 1973; Neuman, 1972, 1974, 1975; Moench, 1995, 1997; Zlotnik and Zhan, 2005; Malama et al., 2007; Malama et al., 2008), in which the water table is assumed always to stay at its initial position. On the basis of this approach, Malama et al. (2007) developed a flow model in a leaky unconfined aquifer system for constant-rate withdrawal at a fully penetrating well. Different from the typical leaky aquifer theory (e.g., Hantush and Jacob, 1955; Hantush, 1960), the flows in aquifer and aquitard can be both vertical and horizontal. Later, Malama et al. (2008) extended the work of Malama et al. (2007) to the case of well partial penetration in an unconfined aquifer–aquitard-confined aquifer system. The other approach is to include the unsaturated zone above the water table in the groundwater flow model (e.g., Kroszynski and Dagan, 1975; Mathias and Bulter, 2006; Tartakovsky and Neuman, 2007; Mishra and Neuman, 2010; Mishra and Neuman, 2011; Mishra et al., 2012; Sun, 2016, Liang et al., 2017). In the model, the flow in unsaturated zone is represented by Richards' (1931) equation and the unsaturated parameters of relative hydraulic conductivity and specific moisture capacity are described by Gardner's (1958) approach in terms of unsaturated exponent. As a leaky unconfined aquifer with considering the effect of unsaturated flow, Mishra et al. (2012) proposed a mathematical model to describe the drawdown distribution caused by constant-rate pumping at a finite-radius well. Their model adopted the four-parameter Gardner's model (Mishra and Neuman, 2010) for unsaturated flow and incorporated the effect of the wellbore storage in the saturated flow equation.

Chen and Chang (2003) developed a mathematical model for CHT at a fully penetrating well based on linearized water table approach with considering the skin effect but neglecting the effect of unsaturated flow. They presented the Laplace domain solutions for drawdown distribution and wellbore flowrate and also provided early-time and late-time approximate solutions to describe the drawdown distribution. For a partially penetrating well (PPW), it has screened and unscreened sections; thus, pumping at a PPW causes vertical flow and results in different drawdown behavior near the wellbore. For a CHT, a constant-head condition is generally specified at the screened section while a no-flow condition is adopted for the unscreened section. Mathematically, the well face is treated as an inner boundary in the groundwater flow model and a boundary condition is imposed on there. The flow induced by a CHT in a PPW is therefore a mixed boundary value problem (MBVP) (Duffy, 1998). Chang et al. (2010a) presented a finite-thickness skin model with a PPW in the aquifer and they assumed that the flux across the wellbore is uniform to approximate the constant-head condition without solving it as a MBVP. Later, Chang et al. (2010b) applied the method of domain decomposition to solve the MBVP for flow due to CHT in a PPW.

To our knowledge, there is no research on the CHT conducted in an unconfined aquifer with considering the effect of flows from the unsaturated zone on the top and the second aquifer/layer at the bottom. This study aims at developing an analytical model to describe the flow due to CHT in an unconfined aquifer subject to the effects of unsaturated flow, wellbore skin, and leakage from an underlying layer. The solution of the model for drawdowns in unconfined aquifer, unsaturated zone and second layer is derived by the Laplace transform technique and Weber transform; then, the wellbore-flowrate solution is obtained based on Darcy's law. The issue of equivalence between the constant-rate pumping and the constant-head pumping is also addressed. Additionally, sensitivity analysis is also performed to explore the behavior of wellbore flowrate in response to the change in each of parameters in a leaky unconfined aquifer system. The effects of unsaturated exponent, thickness of unsaturated zone, and hydraulic conductivity of the second layer on the wellbore flowrate are investigated. Also, the present solution is validated by comparing with FEMWATER (Yeh and Ward, 1980) which is capable of simulating the flow in saturated–unsaturated system. Finally, the present solution is used to analyze the measured drawdown data from the CHT conducted by Jones et al. (1992) in Wisconsin age weathered till in Iowa.

2. Method

Fig. 1 shows the schematic diagram for the conceptual model describing the drawdown distribution due to CHT at a PPW in a leaky unconfined aquifer system of infinite lateral extent. The model consists of unsaturated zone on the top, unconfined aquifer

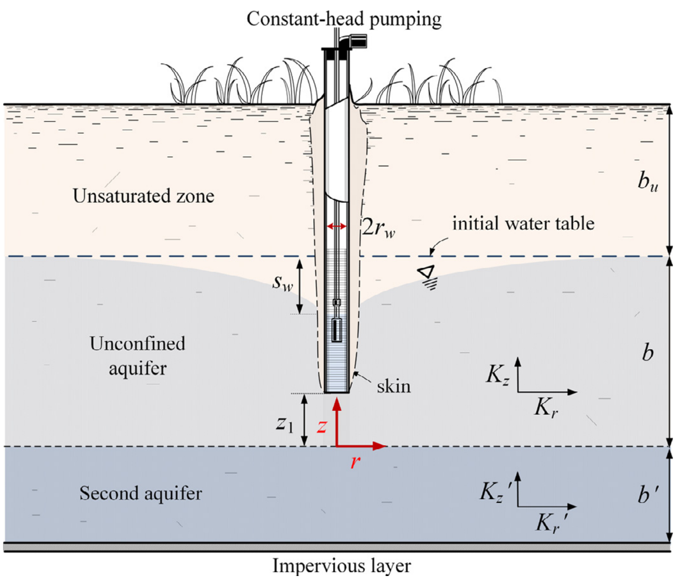

Fig. 1. Schematic representation for a CHT conducted at a partially penetrating well in a leaky unconfined aquifer system with considering unsaturated flow.

in the middle, and second layer at the bottom with uniform thicknesses b_u, b, and b', respectively. The well has a radius r_w and partially penetrates the aquifer with a screen installed from z_1 to b. A skin zone is considered to present adjacent to the wellbore in the unconfined aquifer. The origin of the coordinate is set at the center of the well and the bottom of the aquifer. Both the aquifer and second layer are anisotropic with saturated vertical and radial hydraulic conductivities, i.e., K_z and K_r for the aquifer as well as K_z' and K_r' for the second layer. The water extracted by the pumping is assumed instantaneously from the specific storage of the aquifer, S_s and the second layer specific storage, S_s'. The hydraulic properties of the unsaturated zone associated with soil moisture characteristic curves are introduced in the following section. The boundary conditions at the top of unsaturated zone and the bottom of the second layer are both under no-flow condition.

2.1. Mathematical model

The groundwater flow in unconfined aquifer obeys the following governing equation for aquifer drawdown:

$$K_r\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)+K_z\frac{\partial^2 s}{\partial z^2}=S_s\frac{\partial s}{\partial t},\quad r_w<r<\infty,\ 0<z<\eta \tag{1}$$

where s is drawdown in the aquifer [L]; r is radial distance from the center of the well [L]; z is vertical distance from the bottom of the aquifer [L]; η is the elevation of the water table [L]; and t is well operating time [T].

Before the test, the drawdown in the aquifer is considered zero. The initial condition is written as:

$$s=0,\ t=0 \tag{2}$$

At the edge of the wellbore, the flux across the screen is assumed uniform (Yang and Yeh, 2005). The inner boundary condition is therefore denoted as

$$K_r\frac{\partial s}{\partial r}=\begin{cases}q(t), & z_1<z<b\\ 0, & 0<z<z_1\end{cases},\quad r=r_w \tag{3}$$

Near the test well, the head loss may occur due to the presence of well skin. The head loss at the interface between the skin and aquifer may be expressed as:

$$\frac{1}{b-z_1}\int_{z_1}^{b}s_{loss}dz=s_w-\frac{1}{b-z_1}\int_{z_1}^{b}sdz \tag{4}$$

where s_{loss} is head loss in terms of drawdown [L] and s_w is constant drawdown induced by a CHT at the well face [L].

Assume that the flux between the skin and the aquifer is continuous and written as

$$K_s\frac{\partial s_{skin}}{\partial r}=K_r\frac{\partial s}{\partial r} \tag{5}$$

where s_{skin} is drawdown in skin region [L] and K_s is skin hydraulic conductivity [LT^{-1}]. If the skin thickness is quite small, the term on the left-hand side of Eq. (5) may be approximated as:

$$K_s\frac{\partial s_{skin}}{\partial r}=-K_s\frac{\partial s_{loss}}{\partial r_s} \tag{6}$$

where r_s is the thickness of well skin [L]. Substituting Eq. (6) into Eq. (5) results in

$$-K_s\frac{\partial s_{loss}}{\partial r_s}=K_r\frac{\partial s}{\partial r} \tag{7}$$

Furthermore, substituting Eq. (7) into Eq. (4) yields

$$\frac{1}{b-z_1}\int_{z_1}^{b}-\frac{K_r r_s}{K_s}\frac{\partial s}{\partial r}dz=s_w-\int_{z_1}^{b}sdz \tag{8}$$

The dimensionless skin factor [–] is therefore introduced as

$$S_k=\frac{K_r r_s}{K_s r_w} \tag{9}$$

The drawdown considering the skin effect at the well face can then be obtained by substituting Eq. (9) into Eq. (8) and arranged as:

$$\frac{1}{b-z_1}\int_{z_1}^{b}s-r_wS_k\frac{\partial s}{\partial r}dz=s_w,\ \ r=r_w \tag{10}$$

At remote boundary, i.e., $r\to\infty$, the drawdown is assumed zero; thus, the outer boundary condition is written as

$$\lim_{r\to\infty}s=0 \tag{11}$$

To explore the behavior of water flow in unsaturated zone, volumetric water content θ [–], relative hydraulic conductivity k [–], and unsaturated pressure head ψ [L] are key properties of the unsaturated soils. Both the parameters θ and k are dependent on ψ. The relationship between the θ and ψ for a soil is called soil retention curve. The curve for $k$in terms of ψ (i.e., $k(\psi)$) and the retention curve $\theta(\psi)$ are commonly referred to as the soil characteristic curves.

The volumetric water content can be expressed as

$$\theta(\psi)=\theta_r+(\theta_s-\theta_r)S_e \tag{12}$$

where S_e is effective saturation [–]; θ_r and θ_s represent the residual content [–] and saturated moisture content [–], respectively. Thus, the term $\theta_s-\theta_r$ represents drainable porosity in soils and is considered to equal specific yield S_y[–]. Eq. (12) can then be written as:

$$\theta(\psi)=\theta_r+(\theta_s-\theta_r)S_y \tag{13}$$

In the past, several studies proposed the formulas for the soil characteristic curves (e.g., Gardner, 1958; Brooks and Corey, 1964; van Genuchten, 1980). For the development of the unsaturated flow equation, Warrick (2003) assumed:

$$k(\psi)=S_e(\psi) \tag{14}$$

with k expressed as (Gardner, 1958):

$$k(\psi)=\begin{cases}e^{\alpha(\psi-\psi_a)}, & \psi\le\psi_a\\ 1, & \psi>\psi_a\end{cases} \tag{15}$$

where α is unsaturated exponent [L^{-1}] representing the ability of the water holding in the unsaturated zone; ψ is pressure head [L] and ψ_a is air entry pressure head [L]. Note that Philip (1969, p. 256) mentioned the value of α reflecting the relative importance of gravity and capillarity for water movement in soils, and its value for most applications may range from 0.2 to 5 m^{-1}.

Substituting Eqs. (14) and (15) into Eq. (13), the $\theta(\psi)$ based on Gardner's (1958) model can be obtained as

$$\theta(\psi) = \theta_r + S_y e^{\alpha(\psi-\psi_a)} \tag{16}$$

The specific moisture capacity C [L^{-1}] reflecting the storage properties of an unsaturated soil is expressed as:

$$C(\psi) = \frac{d\theta(\psi)}{d\psi} = S_y \alpha e^{\alpha(\psi-\psi_a)} \tag{17}$$

More discussion of soil characteristic curves can be referred to Appendix A in supplemental material (SM). On the basis of Eqs. (12)–(17), the drawdown distribution described by Richards' equation in terms of pressure head can then be expressed as (Tartakovsky and Neuman, 2007):

$$K_r \frac{1}{r}\frac{\partial}{\partial r}\left(k(\psi) r \frac{\partial \sigma}{\partial r}\right) + K_z \frac{\partial}{\partial z}\left(k(\psi)\frac{\partial \sigma}{\partial z}\right) = C(\psi)\frac{\partial \sigma}{\partial t},$$
$$r_w < r < \infty,\ \ \eta < z < b + b_u \tag{18}$$

where σ is drawdown (i.e., loss in pressure head) in unsaturated zone [L].

The initial condition in the unsaturated zone is written as

$$\sigma = 0,\ \ t = 0 \tag{19}$$

The conditions for inner and outer boundaries are expressed, respectively, as

$$\frac{\partial \sigma}{\partial r} = 0,\ \ r = r_w \tag{20}$$

and

$$\lim_{r\to\infty} \sigma = 0 \tag{21}$$

On the top of the unsaturated zone, there is no water source. Thus, the upper boundary condition is subject to no-flow condition and expressed as:

$$\frac{\partial \sigma}{\partial z} = 0,\ \ z = b + b_u \tag{22}$$

Similar to Tartakovsky and Neuman (2007), the continuity conditions at the interface between the unconfined aquifer and unsaturated zones are, respectively, denoted as:

$$s = \sigma,\ \ z = \eta \tag{23}$$

and

$$\nabla s \cdot n = \nabla \sigma \cdot n,\ \ z = \eta \tag{24}$$

where ∇ is the gradient operator and n is the unit normal vector to the water table.

For the second layer, the flow equation with hydraulic properties differs from those of unconfined aquifer is written as

$$K_r'\left(\frac{\partial^2 s'}{\partial r^2} + \frac{1}{r}\frac{\partial s'}{\partial r}\right) + K_z'\frac{\partial^2 s'}{\partial z^2} = S_s'\frac{\partial s'}{\partial t},\ \ r_w < r < \infty,\ \ -b' < z < 0 \tag{25}$$

where s' is drawdown in the second layer [L]. Its initial condition is:

$$s' = 0,\ \ t = 0 \tag{26}$$

The conditions of the inner and outer boundaries for the second layer can be written, respectively, as

$$\frac{\partial s'}{\partial r} = 0,\ \ r = r_w \tag{27}$$

and

$$\lim_{r\to\infty} s' = 0 \tag{28}$$

The bottom of the second layer is assumed bounded by an impermeable layer; therefore, the boundary condition is expressed as:

$$\frac{\partial s'}{\partial z} = 0,\ \ z = -b' \tag{29}$$

At the interface between the second layer and unconfined aquifer, the continuity requirements for the drawdown and flow rate are, respectively, written as

$$s' = s,\ \ z = 0 \tag{30}$$

and

$$K_z'\frac{\partial s'}{\partial z} = K_z \frac{\partial s}{\partial z},\ \ z = 0 \tag{31}$$

2.2. Linearization

Following Kroszynski and Dagan's (1975) approach, the unconfined aquifer flow will depart slightly from equilibrium, if the drawdown caused by pumping is small (Dagan, 1964). In such circumstance, the water table can be assumed fixed at the initial position. Additionally, Kroszynski (1975) mentioned that the recovery of water movement after the pumping is not considered in order to avoid hysteretic effects. A small parameter ε, defined as s_w which is small compared with b, is then chosen to expand the non-linear model based on the perturbation method:

$$\begin{aligned}
&s = s^{(0)} + \varepsilon s^{(1)} + \varepsilon^2 s^{(2)} + \cdots,\ \ s^{(0)} = 0\\
&s' = s'^{(0)} + \varepsilon s'^{(1)} + \varepsilon^2 s'^{(2)} + \cdots,\ \ s'^{(0)} = 0\\
&\sigma = \sigma^{(0)} + \varepsilon \sigma^{(1)} + \varepsilon^2 s\sigma^{(2)} + \cdots,\ \ \sigma^{(0)} = 0\\
&\psi = \psi^{(0)} + \varepsilon \psi^{(1)} + \varepsilon^2 \psi^{(2)} + \cdots,\ \ \psi^{(0)} = b + \psi_a - z\\
&\theta = \theta^{(0)} + \varepsilon \theta^{(1)} + \varepsilon^2 \theta^{(2)} + \cdots,\ \ \theta^{(0)} = \theta_r + S_y e^{\alpha(b-z)}\\
&\eta = \eta^{(0)} + \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \cdots,\ \ \eta^{(0)} = b
\end{aligned} \tag{32}$$

where the superscript (m) with $m=0$, 1, 2 … represent the mth term of the perturbation series and $m=0$ denotes for variables at their initial states. Note that as ε is small enough, the second order and higher order terms on the right-hand side of Eq. (32) are negligible, implying that the water table is to stay at its initial position. The pressure head ψ can be expressed as $\psi = \eta + \psi_a - z$ because the elevation of the water table is at $\eta + \psi_a$. Since both ψ and η are fixed at the initial water table, thus one can write that $\psi = b + \psi_a - z$. Substituting this relationship into Eqs. (15) and (17) results in Eqs. (33) and (34), respectively.

$$k(\psi) = e^{\alpha(b-z)} \tag{33}$$

and

$$C(\psi) = S_y \alpha e^{\alpha(b-z)} \tag{34}$$

Note that Mishra and Neuman (2010) made a comparison of temporal distribution curve of drawdown predicted by the analytical solution with Gardner' model and the numerical solution based on the computer code STOMP (White and Oostrom, 2000) with van Genuchten–Mualem soil constitutive model for the problem of constant rate pumping in a coupled saturated and unsaturated flow system. They found good agreement at early and late time values in the predicted time drawdown curve.

The first-order representation for unsaturated flow described by Eq. (18) can be expressed as

$$K_r\left(\frac{\partial^2 \sigma}{\partial r^2} + \frac{1}{r}\frac{\partial \sigma}{\partial r}\right) + K_z\left(\frac{\partial^2 \sigma}{\partial z^2} - \alpha\frac{\partial \sigma}{\partial z}\right) = S_y \alpha \frac{\partial \sigma}{\partial t},$$
$$r_w < r < \infty,\ \ b < z < b + b_u \tag{35}$$

Table 1
Definition of dimensionless variables or parameters.

Name	Definition
Hydraulic conductivity ratio	$\kappa = K_z/K_r$, $\kappa' = K_z'/K_r'$, $\kappa_r' = K_r'/K_r$, $\kappa_z' = K_z'/K_z$
Dimensionless vertical distance	$\zeta = z/b$, $\zeta_1 = z_1/b$, $\zeta_u = b_u/b$, $\zeta' = b'/b$
Dimensionless drawdown	$s_D = s/s_w$, $\sigma_D = \sigma/s_w$, $s'_D = s'/s_w$
Dimensionless flowrate/flux	$Q_D = Q/2\pi K_r b s_w$, $q_D = bq/K_r s_w$
Storativity ratio	$\gamma = S_s b/S_y$, $\gamma' = S_s' b/S_s$
Dimensionless radial distance	$\rho = r/b$, $\rho_w = r_w/b$
Dimensionless time	$\tau = K_r t/S_s b^2$
Dimensionless unsaturated exponent	$\alpha_D = \alpha b$

Note that both Eqs. (33) and (34) are evaluated at zero order and Eq. (35) is at first order.

Similarly, the continuity conditions, Eqs. (23) and (24), can be obtained, respectively, as:

$$s = \sigma, \; z = b \tag{36}$$

and

$$\frac{\partial s}{\partial z} = \frac{\partial \sigma}{\partial z}, \; z = b \tag{37}$$

Notice that the range in the z-domain for Eq. (1) is linearized as $0 < z < b$.

2.3. Solutions in Laplace and Weber domains

The dimensionless parameters are defined in Table 1 and the properties of Weber transform are introduced in the SM as Appendix B. The Weber transform can be considered as a Hankel transform with a more general kernel function. It can be applied to diffusion-type problems with a radial-symmetric region from a finite distance to infinity. The solutions for the drawdown and wellbore flowrate are developed based on the methods of Laplace transform and Weber transform. Details of derivation for both solutions are listed in the SM as Appendix C. The resulting solutions for drawdown in the unconfined aquifer, unsaturated zone, and second layer are expressed, respectively, as

$$\bar{s}_D(\rho, \zeta, p) = \bar{q}_D(p) \int_0^\infty \bar{\bar{s}}_D(\lambda, \zeta, p) \times \frac{J_1(\lambda\rho_w)Y_0(\lambda\rho) - Y_1(\lambda\rho_w)J_0(\lambda\rho)}{\sqrt{{J_1}^2(\lambda\rho_w) + {Y_1}^2(\lambda\rho_w)}} \lambda d\lambda \tag{38}$$

$$\bar{\sigma}_D(\rho, \zeta, p) = \bar{q}_D(p) \int_0^\infty \bar{\bar{\sigma}}_D(\lambda, \zeta, p) \times \frac{J_1(\lambda\rho_w)Y_0(\lambda\rho) - Y_1(\lambda\rho_w)J_0(\lambda\rho)}{\sqrt{{J_1}^2(\lambda\rho_w) + {Y_1}^2(\lambda\rho_w)}} \lambda d\lambda \tag{39}$$

and

$$\bar{s}'_D(\rho, \zeta, p) = \bar{q}_D(p) \int_0^\infty \bar{\bar{s'}}_D(\lambda, \zeta, p) \times \frac{J_1(\lambda\rho_w)Y_0(\lambda\rho) - Y_1(\lambda\rho_w)J_0(\lambda\rho)}{\sqrt{{J_1}^2(\lambda\rho_w) + {Y_1}^2(\lambda\rho_w)}} \lambda d\lambda \tag{40}$$

The solution for wellbore flowrate can then be obtained as

$$Q_D(\tau) = -\rho_w(1 - \zeta_1) q_D(\tau) \tag{41}$$

The solution in time domain for drawdowns in the unsaturated zone, unconfined aquifer, and second layer can be obtained by using numerical Laplace inverse transform method developed by Crump (1976). The Weber inverse transform can be achieved by numerical integration.

Table 2
Default parameter values in sensitivity analysis.

Parameter	Value	Parameter	Value
K_r	0.0006 m/min	α	0.15 m^{-1}
K_z	0.0003 m/min	b	10 m
S_s	0.00001 m^{-1}	z_1	2 m
K_r'	0.000006 m/min	b_u	5 m
K_z'	0.000003 m/min	b'	5 m
S_s'	0.00001 m^{-1}	S_k	0.1
S_y	0.1	s_w	1.5 m

3. Results and discussion

3.1. Equivalence of normalized drawdown predicted by present solution and Tartakovsky and Neuman's (2007) solution

A figure given in Jacob and Lohman (1952) showed a comparison of the ratio of wellbore flowrate to drawdown at large time for CHT and constant-rate test (CRT). They mentioned that the difference in the ratio is only one percent when $K_r t/S_s r_w^2 \approx 80,000$ or $t \approx 80,000 S_s r_w^2/K_r$. Mishra and Guyonnet (1992) showed that the drawdown at the observation well, normalized by the wellbore flowrate, can be the same for both CHT and CRT for pumping in confined aquifers. They mentioned that such an equivalence between CHT and CRT was originally pointed out by Jacob and Lohman (1952) for the case of a transient response at the wellbore and the Jacob and Lohman semi-log approximation (also called the Jacob equation in Charbeneau, 2000, Eq. (3.3.9)) is applicable for the problems of CHT under the time criterion $t \geq 5S_s r^2/K_r$.

Consider a case of constant pumping with a rate of 1.5 m^3/min with other aquifer parameters given in Table 2. In order to make the comparison with Tartakovsky and Neuman's (2007) solution, the unsaturated thickness is considered as infinite extent and the effects of leakage, well radius, and well skin are ignored. Fig. 2 shows the temporal distribution curves of normalized drawdown (ratio of drawdown to wellbore flowrate) predicted by the present solution and Tartakovsky and Neuman's (2007) solution observed at $r = 1$ m. The curves demonstrate that both solutions give a good match for S_y ranging from 0 to 0.3, indicating that Tartakovsky and Neuman's (2007) solution is also applicable to estimate the drawdown induced by a constant-head pumping.

3.2. Sensitivity analysis

Sensitivity analysis is a useful tool to assess the behavior of the wellbore flowrate or aquifer drawdown in response to the change in each of the hydrogeologic parameters. The normalized sensitivity can be defined as (Kabala, 2001)

$$X_{i,k} = P_k \frac{\partial O_i}{\partial P_k} \tag{42}$$

where $X_{i,k}$ is the normalized sensitivity of the ith output value O_i with respect to the kth input parameter P_k. The term on the right-hand side of Eq. (42) can be approximated by the finite-difference

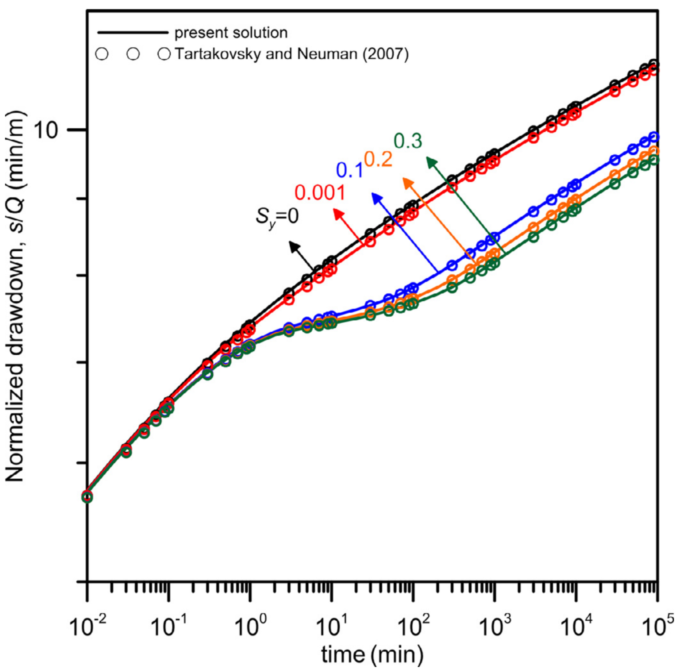

Fig. 2. Normalized drawdown versus time estimated by the present solution and Tartakovsky and Neuman's (2007) solution observed at $r=1$ for $S_y=0$, 0.01, 0.1, 0.2, and 0.3.

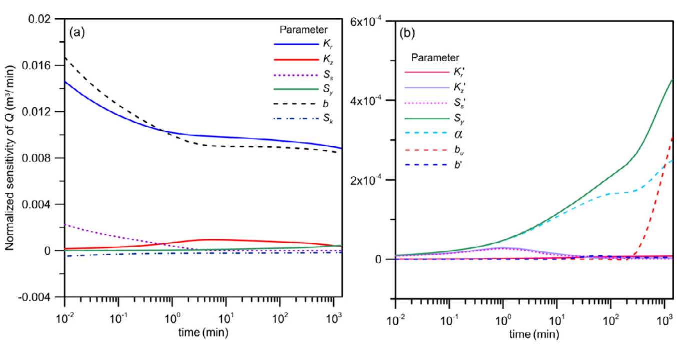

Fig. 3. Temporal distribution curves of normalized sensitivity of Q for parameters K_r, K_z, S_s, b, and S_k in case (a) and S_y, K_r', K_z', S_s', α, b_u, and b' in case (b) observed at $r=1$ in the unconfined aquifer.

formula as:

$$\frac{\partial O_i}{\partial P_k} = \frac{O_i(P_k + \Delta P_k) - O_i(P_k)}{\Delta P_k} \tag{43}$$

where ΔP_k may be approximated by $\Delta P_k = 10^{-3}\, P_k$ (Huang and Yeh, 2007). In the following case study, the default parameter values are given in Table 2.

Fig. 3 demonstrates the temporal normalized sensitivity curves for the wellbore flowrate Q in response to the change in each of the unconfined aquifer parameters K_r, K_z, S_s, b, and S_k in case (a) as well as the unsaturated and second layer parameters S_y, K_r', K_z', S_s', α, b_u, and b' in case (b) observed at the pumping well, $r=r_w$. The figure shows that the wellbore flowrate Q is much more sensitive to the change in each of the parameters in case (a) than that in case (b) in the leaky unconfined aquifer system. Fig. 3(a) indicates that Q is very sensitive to the change in parameters b and K_r. It may be attributed to the facts that the aquifer thickness reflects the amount of water stored in the unconfined aquifer and the magnitude of hydraulic conductivity indicates the ease of water flowing toward the pumping well. Accordingly, an aquifer with larger values of b and K_r can yield much more water from the well. The figure also shows that S_s and K_z produce positive influence on Q in distinct periods. The S_s has an effect in the period 0.01 min $< t <$ 10 min while K_z has an effect from 0.1 min to 1000 min, indicating that the water released from the aquifer storage in the early time and the effect of vertical hydraulic conductivity is significant only in the intermediate time. Fig. 3(b) shows that S_y is the most sensitive one in case (b) for the hydrogeologic parameters of the unsaturated zone and the second layer, but Q is less sensitive to its change as compared with those in case (a) for the unconfined aquifer. The S_y represents the ability of the water released from the unsaturated zone while α represents the ability of the water holding in the unsaturated zone. Fig. 3(b) shows that small positive perturbations in both parameters result in significant positive influences on Q. With larger values of S_y and α, the unsaturated zone provides more water to the pumping well. All the parameters of the second layer have minor positive effects on Q due to their relatively small values as compared to those of the unconfined aquifer. In short, Fig. 3 demonstrates that Q is much more sensitive in response to the change in each of the hydrogeologic parameters of the unconfined aquifer than those of the unsaturated zone and the second layer.

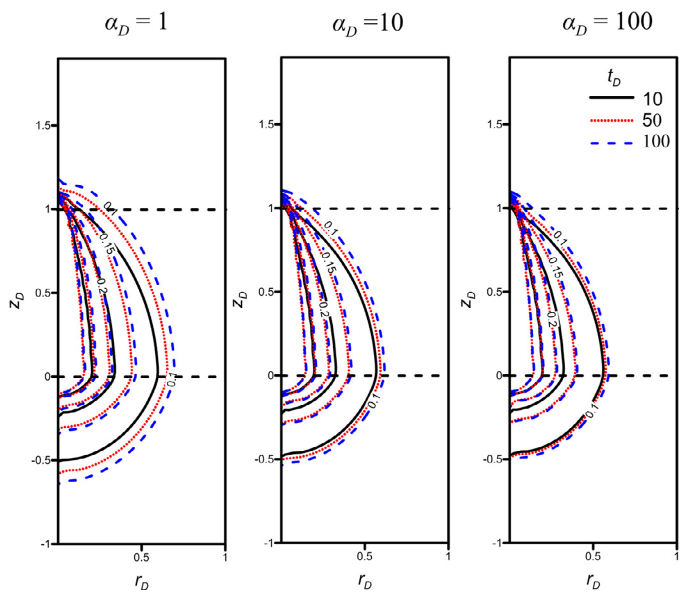

Fig. 4. Dimensionless drawdown contours for CHT in the leaky unconfined aquifer and unsaturated zone for $\alpha_D = 1$, 10, and 100 at various dimensionless time.

3.3. The effect of α_D on drawdown and wellbore flowrate

Fig. 4 shows the contours of dimensionless drawdown in the leaky unconfined aquifer and unsaturated zone induced by CHT for unsaturated exponent $\alpha_D = 1$, 10, and 100 at operating times $\tau = 10$, 50, and 100. The default parameter values of κ, κ', κ_r', κ_z', γ, γ', ρ_w, ζ_1, ζ_u and ζ' defined in Table 1 are set as 1, 1, 0.1, 0.1, 0.01, 1, 0.0001, 0, 1, and 1, respectively. The contours show that the drawdown increases dramatically with time and a smaller α_D (say $\alpha_D = 1$) has a larger drawdown near the well, indicating that the unsaturated zone with a smaller α_D provides less water to the unconfined aquifer. On the other hand, the drawdown near the well increases slowly because of much water drained from the unsaturated zone for a large value of α_D (say 100).

Fig. 5 demonstrates the temporal distribution curves of the dimensionless flowrate Q_D for α_D varying from 1 to 1000 for one-day CHT. The figure indicates that a larger value of α_D has a higher Q_D in the early time when $\tau < 40$ (i.e., $t = 1.1$ h). The Q_D, however, decreases rapidly for large α_D as $\tau > 40$, indicating that the flow from the unsaturated zone provides more water to the pumping

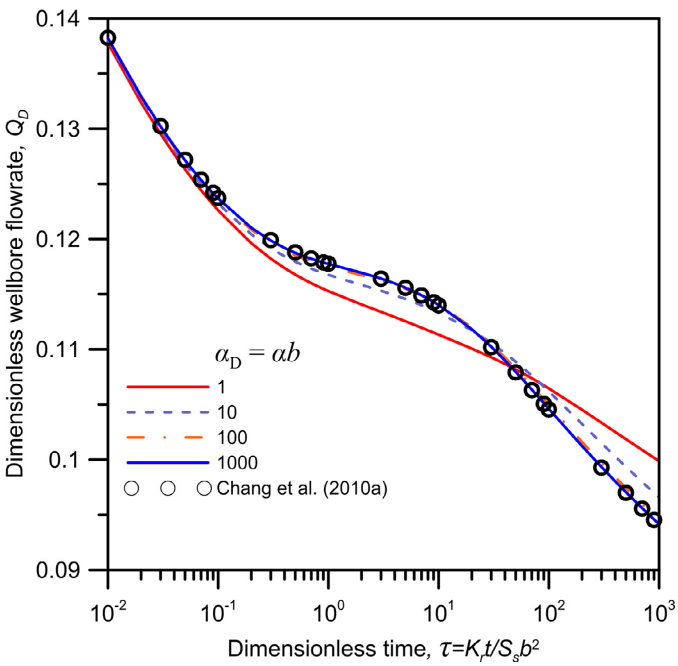

Fig. 5. Dimensionless wellbore flowrate versus dimensionless time for $\alpha_D = 1$, 10, 100, and 1000 predicted by the present solution and Chang et al. (2010a) solution.

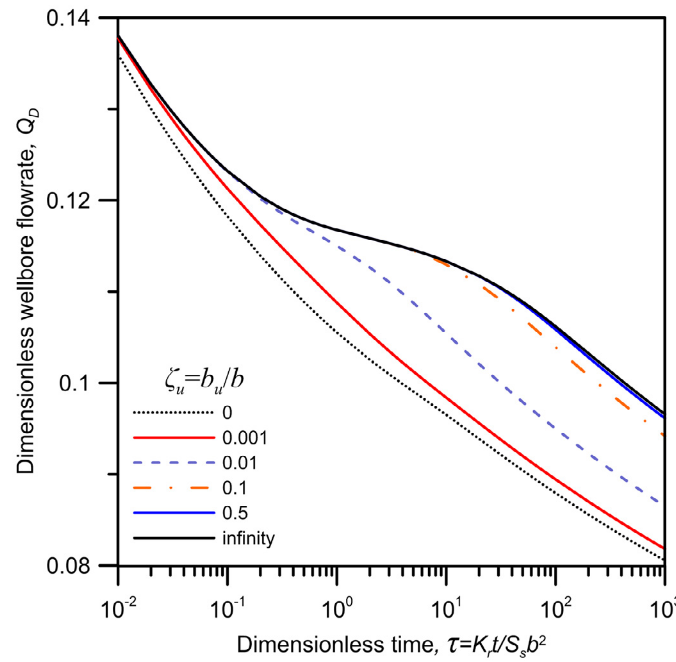

Fig. 6. Dimensionless wellbore flowrate versus dimensionless time for $\zeta_u = 0$, 0.001, 0.01, 0.1, 0.5, and infinity.

well only at early time for large α_D. This is because that a large α_D representing the case of coarse-textured soils is capable of supplying some amount of pore-water through the unconfined aquifer toward the pumping well at early time and resulting in large Q_D. At late time, on the other hand, the unsaturated zone is drying out and therefore the Q_D becomes very small. Note that the figure also indicates that the Q_D predicted by the present solution for $\alpha_D = 100$ matches with the solution of Chang et al. (2010a) for homogeneous aquifers derived based on the linearized water table condition (Neuman, 1974).

3.4. The effect of ζ_u on wellbore flowrate

The effect of the thickness of unsaturated zone ζ_u on Q_D is investigated herein. Fig. 6 exhibits the temporal distribution curves of Q_D for $\zeta_u = 0$, 0.001, 0.01, 0.1, 0.5, and infinity. It is noteworthy that the aquifer system can be considered under leaky confined condition for the case $\zeta_u = 0$ (i.e., the interface between the unconfined aquifer and unsaturated zone becomes under the no-flow condition). The default value of α_D is set to 10 hereinafter. The figure shows that the value of ζ_u can be considered as infinity as long as $\zeta_u \geq 0.5$ indicating that the effect of upper boundary condition on the aquifer flow can be ignored. On the other hand, the figure also indicates that the greater value of ζ_u is, the higher the Q_D will be and the different in estimated Q_D increases with time for the case of $\zeta_u < 0.5$ as compared to the case of $\zeta_u = 0.5$. In reality, a larger thickness contains more water than a thinner one; thus, the wellbore flowrate is higher due to much more water from the unsaturated zone. When the value of ζ_u decreases to 0.001, the temporal distribution curve of Q_D is close to that of the confined case. Both curves however do not agree with each other indicating that the effect of ζ_u is not negligible despite a very small value of ζ_u. In other words, the unsaturated zone cannot be ignored although the unsaturated zone is very thin.

3.5. The effect of κ_z' on wellbore flowrate

The dimensionless second layer permeability κ_z' is defined as the ratio of K_z' over K_z. Fig. 7 demonstrates the temporal distribution of Q_D for $\kappa_z' = 0$, 0.01, 0.1, 0.5, and 1. The figure shows that Q_D increases with κ_z', indicating that the second layer can provide more water flow to the well if the vertical hydraulic conductivity of the second layer is larger. If κ_z' is equal to zero, implying that the second layer is impermeable. The figure also shows that as the κ_z' is smaller than 0.01, the leakage effect of the second layer is ignorable. A small κ_z' represents the hydraulic conductivity of the second layer is relatively small as compared to that of the unconfined aquifer; hence, the water in the second layer is hard to move toward the pumping well.

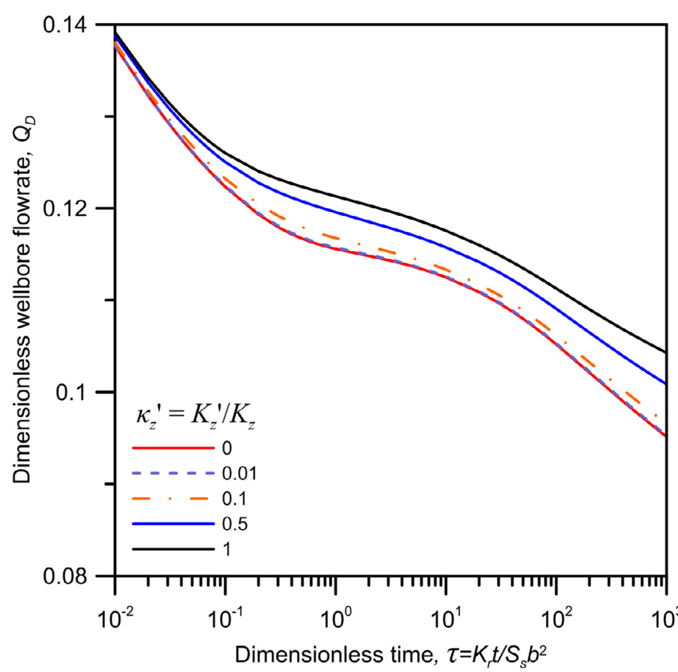

Fig. 7. Dimensionless wellbore flowrate versus dimensionless time for $\kappa_z' = 0$, 0.01, 0.1, 0.5 and 1.

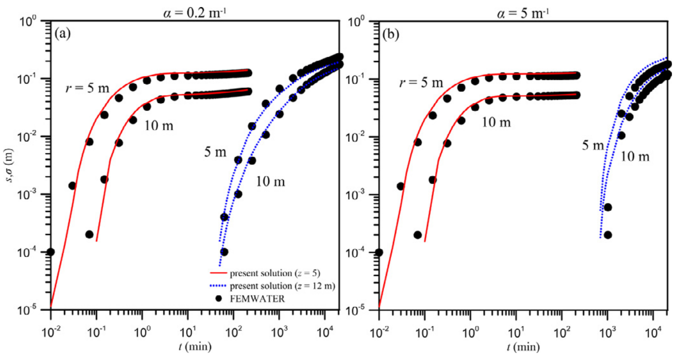

Fig. 8. Temporal drawdown curves predicted by present solution and FEMWATER with $\alpha =$(a) 0.2 m^{-1} and (b) 5 m^{-1} observed at $r = 5$ m and 10 m for $z = 5$ m in unconfined aquifer (solid line) and for $z = 12$ m in unsaturated zone (dotted line).

The effect of ζ' (defined as b'/b) on Q_D is also discussed and the details are in Appendix D given in the SM. Obviously, the effect of leakage on the wellbore flowrate may be negligible when the ζ' is less than 0.01 for $\kappa_z' = 0.5$ and 0.1 for $\kappa_z' = 0.1$.

3.6. Validation of present solution

The computer code FEMWATER developed by Yeh and Ward (1980) is chosen to simulate coupled saturated–unsaturated flow for the validation of the present analytical model. The FEMWATER is capable of modeling unconfined aquifer flow for the moving phreatic surfaces under transient conditions. Consider an aquifer of 15 m thick with $K_r = 6.0 \times 10^{-4}$ m/min, $K_z = 3.0 \times 10^{-4}$ m/min, $S_s = 1.0 \times 10^{-5}$ min^{-1}, and $S_y = 0.2$. The initial thickness of the unconfined aquifer is 10 m and therefore the initial unsaturated thickness is 5 m. The initial pressure head (ψ) in unsaturated zone is assumed to be $\psi = b - z$. Two cases of unsaturated exponent are considered, i.e., $\alpha = 0.2$ m^{-1} and 5 m^{-1}. The constitutive relationship of unsaturated parameters is described by Gardner's (1958) model in Eqs. (15)–(17) with a zero for both θ_r and ψ_a. The pumping well with $r_w = 0.01$ m and $s_w = 1$ m fully penetrates the aquifer. The effects of wellbore skin and leakage from the second layer are ignored herein. The region of the simulation is conducted based on a two-dimensional cylindrical plane with the outer boundary of 50 m. The finite element mesh has non-uniform grids with smaller grid sizes near the pumping well. There are 1980 quadrilateral elements and 2072 nodes used in the simulations. The curves of drawdown versus time predicted by the present solution and FEMWATER in the saturated–unsaturated flow system observed at $r = 5$ m and 10 m in the radial direction for $z = 5$ m in the unconfined aquifer and $z = 12$ m in the unsaturated zone are shown in Fig. 8. Fig. 8(a) shows that the drawdown curves for the case $\alpha = 0.2$ m^{-1} observed in both unconfined aquifer and unsaturated zone predicted by the present solution are close to those of numerical simulations. On the other hand, the predicted drawdowns for the case $\alpha = 5$ m^{-1} by the present solution have good match in the unconfined aquifer but small deviations (up to 0.05 m) in the unsaturated zone as compared with the results from FEMWATER exhibited in Fig. 8(b). Such deviations caused mainly by the assumption of constant water table may be acceptable from an engineering viewpoint.

3.7. Field data analysis

Jones et al. (1992) conducted a 24-h CHT with four observation wells in Wisconsin age weathered till in Iowa. The pumping well, maintained a 1.5 m constant drawdown, has a radius of 0.051 and fully penetrates the saturated thickness of 2.5 m. The observation wells, wells 5–9, were installed separately at 0.87 m, 1.8 m 2.71 m, and 3.62 m from the pumping well and drilled 4.6 m deep from the surface indicating that the thickness of unsaturated zone is 2.1 m. More details of the field geological condition for the test site can be referred to Jones et al. (1992) and the observed field drawdown data are available in Kao (2004).

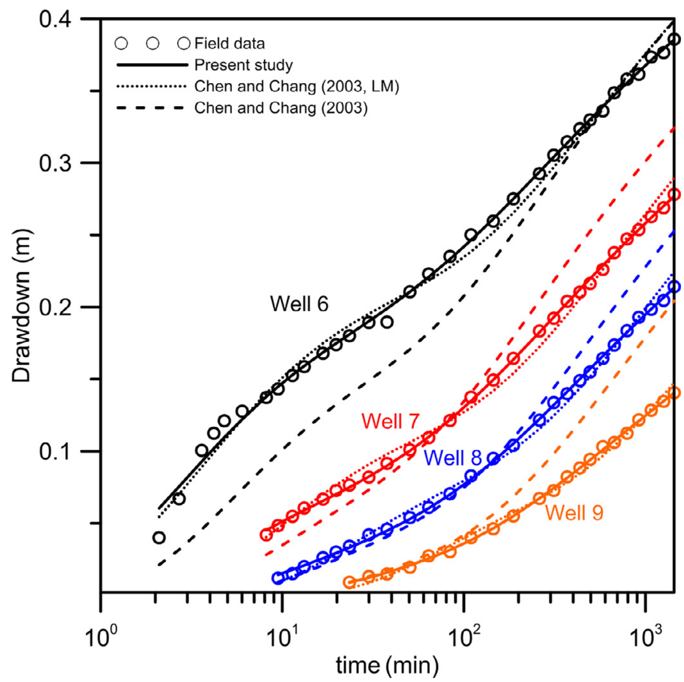

Fig. 9. Measured drawdowns at wells 6–9 and temporal drawdown curves predicted by present solution and Chen and Chang's (2003) solution.

Chen and Chang (2003) applied their solution to analyze Jones et al. (1992) drawdown over flowrate data obtained from wells 6–9 and estimate four aquifer parameters (i.e., K_r, K_z, S_s, and S_y) via the graphical approach (i.e., type curve method). We also analyze the same drawdown data set for the same aquifer parameters based on Chen and Chang's (2003) solution coupled with Levenberg–Marquardt (LM) algorithm (Press et al., 1992) in Mathematica routine FindFit (Wolfram, 1991). Two statistics, the standard error of estimate (*SEE*) and mean error (*ME*) defined in Yeh (1987), are used to assess the performance of parameter estimation. Both estimated results are given in Table 3. The table shows that the estimated parameters from different observation wells by both approaches are close (on the same order of magnitude). However, the values of *SEE* and *ME* indicate that estimated parameters by Chen and Chang (2003) are in poor accuracy. Fig. 9 show the measured drawdowns at wells 6–9 and temporal drawdown curves predicted by Chen and Chang's (2003) solution with parameters estimated using the graphical approach and LM algorithm. The figure shows that the predicted drawdowns by Chen and Chang (2003) with the parameters obtained from the graphical approach are obviously different from the field data. The inaccuracy in the estimated parameters by Chen and Chang (2003) may be attributed to the problems of graphical approach and the use of drawdown over flowrate data. There will be less weights at early times and more weights in late times in the drawdown over flowrate data since the wellbore flowrate appeared in the denominator decreases with time. The use of weighted drawdown data distorts the results in parameter estimation.

The present solution, which considers the unsaturated zone, is coupled with LM algorithm to estimate those four aquifer parameters along with an extra parameter α representing the unsaturated

Table 3
Estimated parameters by present solution and Chen and Chang's (2003) solution.

Observation well	Estimated hydrogeological parameters					Prediction errors	
	$K_r \times 10^{-4}$ (m/min)	$K_z \times 10^{-4}$ (m/min)	$S_s \times 10^{-3}(\mathrm{m}^{-1})$	S_y(-)	$\alpha(\mathrm{m}^{-1})$	$SEE(\times 10^{-3})$	$ME(\times 10^{-4})$
			Present solution[1]				
6	2.41	5.23	2.07	0.040	2.91	5.91	− 0.93
7	2.87	5.76	1.49	0.027	1.94	1.75	0.76
8	2.58	4.36	1.59	0.021	1.43	1.75	0.42
9	1.79	3.14	1.72	0.027	0.45	1.28	0.03
			Chen and Chang's (2003) solution[a]				
6	1.75	1.92	1.75	0.047	–	8.81	− 4.14
7	1.84	1.29	1.48	0.027	–	6.88	− 2.76
8	1.75	0.81	1.67	0.020	–	5.19	− 2.24
9	1.48	0.46	2.42	0.023	–	3.90	− 3.70
			Chen and Chang's (2003) solution[b]				
6	1.67	4.03	3.08	0.042	–	37.3	265
7	1.93	2.73	1.88	0.020	–	29	− 125
8	2.02	2.04	1.76	0.014	–	20.4	95.4
9	2.50	2.10	2.64	0.015	–	38.5	25.9

[a] Parameters are estimated based on Levenberg–Marquardt algorithm.
[b] Parameters are estimated using type-curve method.

flow. The estimated results by the present solution are also given in Table 3. The estimated unsaturated exponent α ranges from $0.45\,\mathrm{m}^{-1}$ to $2.91\,\mathrm{m}^{-1}$ implying that the unsaturated zone consists of fine- to coarse-textured soils. The estimated S_y by the present solution and Chen and Chang's (2003) solution coupled with LM algorithm are close; however, the estimated hydraulic conductivities of K_r and K_z by the present solution are all higher than those by Chen and Chang's (2003) solution. It may indicate that the hydraulic conductivities would be underestimated if the unsaturated flow is neglected. Fig. 9 shows the drawdown predicted by our solution (solid line) match well with the field data. It can also be observed that the drawdown predicted by Chen and Chang's (2003) solution with the parameters estimated by LM algorithm (dotted line) is higher at the early times and lower at the late times as compared with the field data, indicating that the neglect of unsaturated zone incurs errors in the predicted drawdowns.

4. Conclusions

This study presents a novel analytical model for describing groundwater flow induced by constant-head test/pumping in a leaky unconfined aquifer system which considers unsaturated flow. The drawdown solution of the model is developed using the approaches of the Laplace transform and Weber transform. The wellbore-flowrate solution is then derived based on the drawdown solution and Darcy's law. The issue of equivalence of the constant-rate pumping solution and the constant-head pumping solution is also studied. The sensitivity analysis is performed to investigate the influence of the change in each of the aquifer parameters on the wellbore flowrate. In addition, the effects of dimensionless parameters such as unsaturated exponent, thickness of unsaturated zone, and vertical hydraulic conductivity of the second layer on the wellbore flowrate are assessed. Moreover, the present solution is validated by numerical solution and applied to estimate the aquifer parameters via Levenberg–Marquardt algorithm for constant-head test in an unconfined aquifer. The main findings of this study can be summarized as follows:

(1) The temporal distribution curves of the aquifer drawdown, normalized by wellbore flowrate, predicted by the present solution and Tartakovsky and Neuman's (2007) solution match very well for various values of specific yield if the time criterion $t \geq 5S_s r^2/K_r$ is met. When satisfying this criterion, Tartakovsky and Neuman's (2007) solution can also be used to predict the aquifer drawdown induced by a constant-head pumping at a partially penetrating well in an unconfined aquifer system with considering unsaturated flow. On the other hand, our solution can be used to predict the drawdown induced by constant-rate pumping when the criterion is satisfied.
(2) The results of sensitivity analysis indicate that the wellbore flowrate is very sensitive to the change in parameters of radial hydraulic conductivity and saturated aquifer thickness. Those results conform to the fact that is well recognized.
(3) A larger unsaturated exponent provides more water from the unsaturated zone at the early times, and the effect of the unsaturated flow diminishes at the late times. When the dimensionless unsaturated exponent α_D approaches 100, the present solution gives predictions very close to those of Chang et al. (2010a) solution for non-leaky unconfined aquifers. This result indicates that the unsaturated zone can be replaced by the linearized water table condition (Neuman, 1974) as $\alpha_D \geq 100$.
(4) The thickness of the unsaturated zone may be treated as infinite if the unsaturated zone thickness is largely greater than a half of the thickness of unconfined aquifer.
(5) The effect of leakage on the wellbore flowrate may be neglected if the ratio of K_z'/K_z is less than 0.01.
(6) The present solution is successfully validated through the comparison with the simulation results from the finite element model FEMWATER, which can deal with the problems of moving phreatic surface in saturated–unsaturated media.
(7) The predicted drawdown curves based on the present solution and estimated parameters show very good match with measured data from four observation wells due to the consideration of unsaturated flow. On the other hand, the solution without considering the effect of unsaturated flow leads to underestimate the radial and vertical hydraulic conductivities of the unconfined aquifer in parameter estimations and overestimate the results at the early times and underestimate them at the late times in drawdown predictions.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.advwatres.2017.05.018.

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