The discrepancy (0.5-0.32) may probably be accounted for by the liberal coating of white paint that was put on the insulating material to protect it from the weather and to reflect as much direct radiation as possible. The paint doubtless permeated into the material so as to increase its conductivity. Moreover, in spite of the white paint, the outer surfaces of the slabs receiving direct solar radiation must have been kept at a temperature somewhat higher than the dry bulb. Since the prevailing direction of heat-leakage was <u>into</u> the water, this increase of temperature of the surface of the insulating material must have had the effect of an apparent increase in conductivity.

Figure 4 shows the relation between h as computed from the phototube-pyrheliometer readings and H as obtained from the pan.

It is not surprising that equation (2) yields fairly consistent results when we remember that in the morning and evening the light entering the window of the phototube comes chiefly from the sky and is therefore richer in short waves than it is during the middle part of the day. Toward the ends of the day there are, accordingly, pulses which are not accompanied by much heat. The kt-term subtracts these out.

The errors shown by Figure 4 are due to several causes, but perhaps one of the most serious is the warping of the walls of the pan due to temperature-changes. If some means can be found for measuring nocturnal back-radiation, the seriousness of this difficulty can be eliminated by making a continuous run of several days' duration. The total algebraic result of warping would then be negligible. Arrangements are being undertaken for carrying out this plan.

The writer desires to express appreciation to Dr. White of the Physics Department of the University of California for his suggestion of covering the thyratron with a hood in order to help minimize erratic behavior due to temperature-changes. Thanks are also extended to Dr. Nicholas Ricciardi, President of the San Bernardino Valley Junior College and to Dean Frank B. Lindsay for assigning Federal Emergency Relief Administration students to this work, as well as to the students themselves. Most of the observations were made by Newell Call and William Hand, but Guy Harris and Robert Whaling also helped. Hayden Gordon, technician of the College, assisted in various ways. Mr. Moore and Mr. Watson of the Smithsonian Institution at Table Mountain rendered valuable aid in some preliminary investigations.

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THE RELATION BETWEEN THE LOWERING OF THE PIEZOMETRIC SURFACE AND THE RATE AND DURATION OF DISCHARGE OF A WELL USING GROUND-WATER STORAGE

Charles V. Theis

When a well is pumped or otherwise discharged, water-levels in its neighborhood are lowered. Unless this lowering occurs instantaneously it represents a loss of storage, either by the unwatering of a portion of the previously saturated sediments if the aquifer is nonartesian or by release of stored water by the compaction of the aquifer due to the lowered pressure if the aquifer is artesian. The mathematical theory of ground-water hydraulics has been based, apparently entirely, on a postulate that equilibrium has been attained and therefore that water-levels are no longer falling. In a great number of hydrologic problems, involving a well or pumping district near or in which water-levels are falling, the current theory is therefore not strictly applicable. This paper investigates in part the nature and consequences of a mathematical theory that considers the motion of ground-water before equilibrium is reached and, as a consequence, involves time as a variable.

To the extent that Darcy's law governs the motion of ground-water under natural conditions and under the artificial conditions set up by pumping, an analogy exists between the hydrologic conditions in an aquifer and the thermal conditions in a similar thermal system. Darcy's law is analogous to the law of the flow of heat by conduction, hydraulic pressure being analogous to temperature, pressure-gradient to thermal gradient, permeability to thermal conductivity, and specific yield to specific heat. Therefore, the mathematical theory of heat-conduction developed by Fourier and subsequent writers is largely applicable to hydraulic theory. This analogy has been recognized, at least since the work of Slichter, but apparently no attempt has been made to introduce the function of time into the mathematics of ground-water hydrology. Among the many problems in heat-conduction analogous to those in ground-water hydraulics are those concerning sources and sinks, sources being analogous to recharging wells and sinks to ordinary discharging wells.

C. I. Lubin, of the University of Cincinnati, has with great kindness prepared for me the following derivation of the equation expressing changes in temperature due to the type of source or sink that is analogous to a recharging or discharging well under certain ideal conditions, to be discussed below.

The equation given by H. S. Carslaw (Introduction to the mathematical theory of the conduction of heat in solids, 2nd ed., p. 152, 1921) for the temperature at any point in an infinite plane with initial temperature zero at any time due to an "instantaneous line-source coinciding with the axis of z of strength Q" (involving two-dimensional flow of heat) is

$$v = (Q/4\pi\kappa t) \; e^{-(x^2 + y^2)/4\kappa t}$$

(1)

where v = change in temperature at the point x,y at the time t; Q = the strength of the source or sink--in other words, the amount of heat added or taken out instantaneously divided by the specific heat per unit-volume; κ = Kelvin's coefficient of diffusivity, which is equal to the coefficient of conductivity divided by the specific heat per unit-volume; and t = time.

The effect of a continuous source or sink of constant strength is derived from equation (1) as follows: Let $Q = \varphi(t')dt'$; then $v_{(x,y,t)} = \int_0^t [\varphi(t')/4\pi\kappa(t-t')] \; e^{-(x^2+y^2)/4\kappa(t-t')} \, dt'$. Let $\varphi(t') = \lambda$, a constant; then $v_{(t)} = (\lambda/4\pi\kappa)\int_0^t [e^{-(x^2+y^2)/4\kappa(t-t')}/(t-t')] \, dt'$. Let $u = (x^2+y^2)/4\kappa(t-t')$; then

$$v_{(t)} = (\lambda/4\pi\kappa) \int_{(x^2+y^2)/4\kappa t}^\infty [e^{-u}/(t-t')] \, [(x^2+y^2)/4\kappa] \, [du/u^2]$$

$$= (\lambda/4\pi\kappa) \int_{(x^2+y^2)/4\kappa t}^\infty (e^{-u}/u)du$$

(2)

The definite integral, $\int_{(x^2+y^2)/4\kappa t}^\infty (e^{-u}/u)du$, is a form of the exponential integral, tables of which are available (Smithsonian Physical Tables, 8th rev. ed., table 32, 1933; the values to be used are those given for Ei (-x), with the sign changed). The value of the integral is given by the series

$$\int_x^\infty (e^{-u}/u)du = -0.577216 - \log_e x + x - x^2/2 \cdot 2! \; + x^3/3 \cdot 3! - x^4/4 \cdot 4! + \ldots\ldots$$

(3)

Equation (2) can be immediately adapted to ground-water hydraulics to express the draw-down at any point at any time due to pumping a well. The coefficient of diffusivity, , is analogous to the coefficient of transmissibility of the aquifer divided by the specific yield. (The term "coefficient of transmissibility" is here used to denote the product of Meinzer's coefficient of permeability and the thickness of the saturated portion of the aquifer; it quantitatively describes the ability of the aquifer to transmit water. Meinzer's coefficient of permeability denotes a characteristic of the material; the coefficient of transmissibility denotes the analogous characteristic of the aquifer as a whole.) The continuous strength of the sink is analogous to the pumping rate divided by the specific yield. Making these substitutions, we have
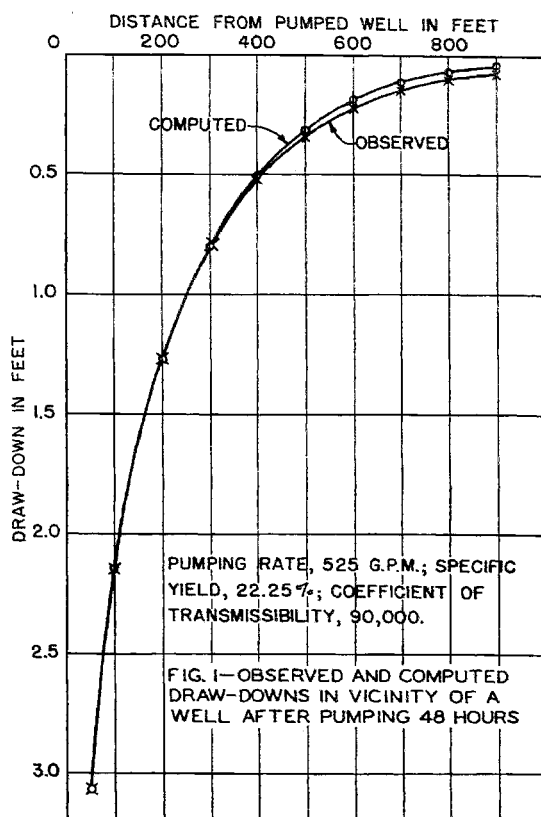
$$v = (F/4\pi\tau) \int_{r^2 s/4\tau t}^\infty (e^{-u}/u)du$$

(4)

in which the symbols have the meanings given with equation (5). In equation (4) the same units must of course be used throughout. Equation (4) may be adapted to units commonly used

$$v = (114.6F/\tau) \int_{1.87r^2 s/\tau t}^\infty (e^{-u}/u)du$$

(5)

where v = the draw-down, in feet, at any point in the vicinity of a well pumped at a uniform rate; F = the discharge of the well, in gallons a minute; τ = the coefficient of transmissibility of aquifer, in gallons a day, through each 1-foot strip extending the height of the aquifer, under a unit-gradient--this is the average coefficient of permeability (Meinzer) multiplied by the thickness of the aquifer; r = the distance from pumped well to point of observation, in feet; s = the specific yield, as a decimal fraction; and t = the time the well has been pumped, in days.

Equation (5) gives the draw-down at any point around a well being pumped uniformly (and continuously) from a homogeneous aquifer of constant thickness and infinite areal extent at any time. The introduction of the function, time, is the unique and valuable feature of the equation. Equation (5) reduces to Thiem's or Slichter's equation for artesian conditions when the time of pumping is large.

FIG. I—OBSERVED AND COMPUTED DRAW-DOWNS IN VICINITY OF A WELL AFTER PUMPING 48 HOURS

Empirical tests of the equation are best made with the data obtained by L. K. Wenzel (Recent investigations of Thiem's method for determining permeability of water-bearing sediments, Trans. Amer. Geophys. Union, 13th annual meeting, pp. 313-317, 1932; also Specific yield determined from a Thiem's pumping test, Trans. Amer. Geophys. Union, 14th annual meeting, pp. 475-477, 1933) from pumping tests in the Platte Valley in Nebraska. Figure 1 presents the comparison of the computed and observed draw-downs after two days of pumping. The observed values are those of the generalized depression of the water-table as previously determined by Mr. Wenzel. The computed values are obtained by equation (5), using values of permeability and specific yield that are within one per cent of those determined by Mr. Wenzel by other methods. The agreement represented may be regarded as showing either that the draw-downs have been computed from known values of transmissibility and specific yield or that these factors have been computed from the known draw-downs.

Theoretically, the equation applies rigidly only to water-bodies (1) which are contained in entirely homogeneous sediments, (2) which have infinite areal extent, (3) in which the well penetrates the entire thickness of the water-body, (4) in which the coefficient of transmissibility is constant at all times and in all places, (5) in which the pumped well has an infinitesimal diameter, and (6) - applicable only to unconfined water-bodies - in which the water in the volume of sediments through which the water-table has fallen is discharged instantaneously with the fall of the water-table.

These theoretical restrictions have varying degrees of importance in practice. The effect of heterogeneity in the aquifer can hardly be foretold. The effect of boundaries can be considered by more elaborate analyses, once they are located. The effect of the well failing to penetrate the entire aquifer is apparently negligible in many cases. The pumped well used in the set-up that yielded the data for Figure 1 penetrated only 30 feet into a 90-foot aquifer. The coefficient of transmissibility must decrease during the process of pumping under water-table conditions, because of the diminution in the cross-section of the area of flow due to the fall of the water-table; however, it appears from Figure 1 that if the water-table falls through a distance equal only to a small percentage of the total thickness of the aquifer the errors are not large enough to be observed. In artesian aquifers the coefficient of transmissibility probably decreases because of the compaction of the aquifer, but data on this point are lacking. The error due to the finite diameter of the well is apparently always insignificant.

In heat-conduction a specific amount of heat is lost concomitantly and instantaneously with fall in temperature. It appears probable, analogously, that in elastic artesian aquifers a specific amount of water is discharged instantaneously from storage as the pressure falls. In nonartesian aquifers, however, the water from the sediments through which the water-table has fallen drains comparatively slowly. This time-lag in the discharge of the water made available from storage is neglected in the mathematical treatment here given. Hence an error is always present in the equation when it is applied to water-table conditions. However, inasmuch as the rate of fall of the water-table decreases progressively after a short initial period, it seems probable that as pumping continues the rate of drainage of the sediments tends to catch up with the rate of fall of the water-table, and hence that the error in the equation becomes progressively smaller.

For instance, although the draw-downs computed for a 24-hour period of pumping in Mr. Wenzel's test showed a definite lack of agreement with the observations, similar computations for a

48-hour period gave the excellent agreement shown in Figure 1. Unfortunately data for peri
of pumping longer than 48 hours have not been available.

The equation implies that any two observations of draw-down, whether at different plac
at the same place at different times, are sufficient to allow the computation of specific y .d
and transmissibility. However, more observations are always necessary in order to guard against
the possibility that the computations will be vitiated by the heterogeneity of the aquifer.
Moreover, it appears that the time-lag in the drainage of the unwatered sediments makes it impos-
sible at present to compute transmissibility and specific yield from observations on water-
levels in only one observation-well during short periods of pumping. Good data from artesian
wells have not been available, but such data as we have hold out the hope that transmissibility
and specific yield may be determined from data from only one observation-well.

A useful corollary to equation (5) may be derived from an analysis of the recovery of a
pumped well. If a well is pumped for a known period and then left to recover, the residual
draw-down at any instant will be the same as if pumping of the well had been continued but a re-
charge well with the same flow had been introduced at the same point at the instant pumping
stopped. The residual draw-down at any instant will then be

$$v' = (114.6F/\tau) \; [\int_{1.87r^2s/\tau t}^{\infty} (e^{-u}/u)du \; - \int_{1.87r^2s/\tau t'}^{\infty} (e^{-u}/u)du \qquad (6)$$

where t is the time since pumping started and t' is the time since pumping stopped.

In and very close to the well the quantity $(1.87r^2s/\tau t')$ will be very small as soon as t'
ceases to be small, because r is very small. In many problems ordinarily met in ground-water
hydraulics, all but the first two terms of the series of equation (3) may be neglected, so that,
if $Z = (1.87r^2s/\tau t)$ and $Z' = (1.87r^2s/\tau t')$ equation (6) may be approximately rewritten

$$v' = (114.6F/\tau) \; [-0.577 - \log_e Z + 0.577 + \log_e Z(t/t')] = (114.6F/\tau) \log_e(t/t')$$

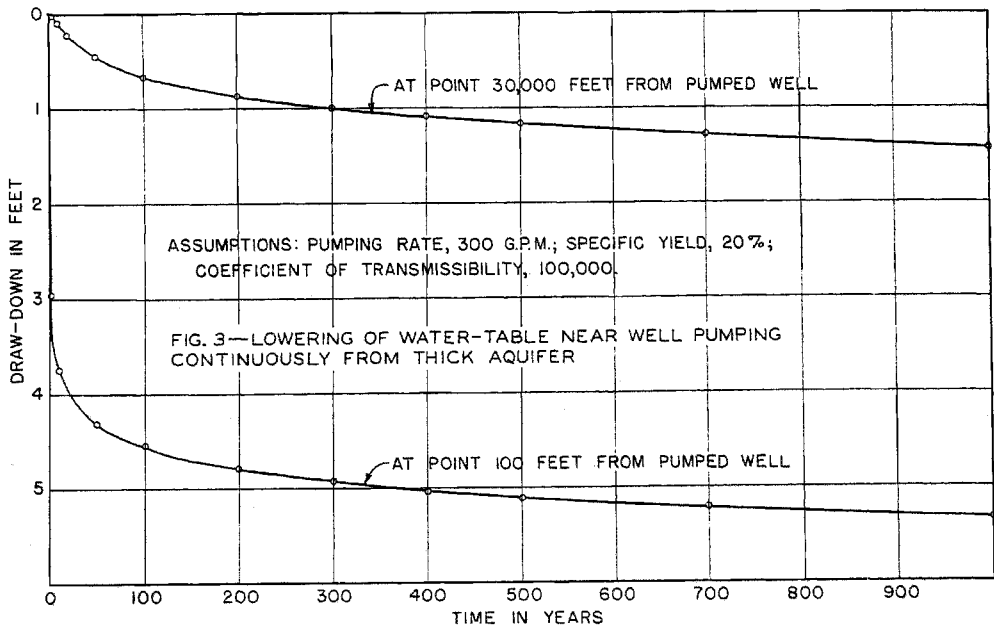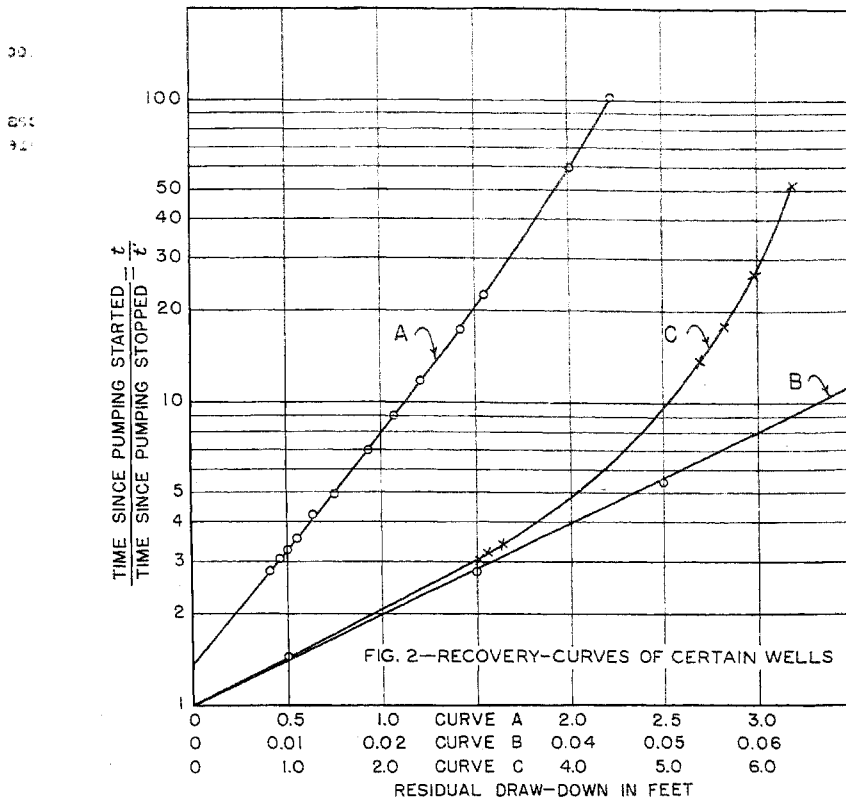Transposing and converting to common logarithms, we have

$$\tau = (264F/v') \log_{10} (t/t') \qquad (7)$$

This equation permits the computation of the coefficient of transmissibility of an aquifer from
an observation of the rate of recovery of a pumped well.

Figure 2 shows a plot of observed recovery-curves. The ordinates are log (t/t'); the ab-
scissas are the distances the water-table lies below its equilibrium-position. According to
equation (7) the points should fall on a straight line passing through the origin. Curve A is a
plot of the recovery of a well within 3 feet of the well pumped for Mr. Wenzel's test, previously
mentioned. Most of the points lie on a straight line, but the line passes below the origin.
This discrepancy is probably due to the fact that the water-table rises faster than the sur-
rounding pores are filled. The coefficient of permeability computed from the equation is about
1200, against a probably correct figure of 1000. Curve B is plotted from data obtained from an
artesian well near Salt Lake City. The points all fall according to theory.

Curve C shows the recovery of a well penetrating only the upper part of a nonartesian aqui-
fer of comparatively low transmissibility. It departs markedly from a straight line. This
curve probably follows equation (6), but it does not follow equation (7), for in this case
$(1.87r^2s/\tau t')$ is not small. Equation (6), involving r and s, neither of which may be known in
practice, is not of practical value for the present purpose. Further empirical tests may show
that it is feasible to project the curve to the origin, in the neighborhood of which
$(1.87r^2s/\tau t')$ becomes small, owing to the increase in t and t', and apply equation (7) to the
extrapolated values so obtained in order to determine at least an approximate value of the
transmissibility.

The paramount value of equation (5) apparently lies in the fact that it gives part of the
theoretical background for predicting the future effects of a given pumping regimen upon the
water-levels in a district that is primarily dependent on ground-water storage. Such districts
may include many of those tapping extensive nonartesian bodies of ground-water. Figure 3 shows
the vertical rate of fall of the water-level in an infinite aquifer, the water being all taken
from storage. The curves are plotted for certain definite values of pumping rate, transmissi-
bility, and specific yield, but by changing the scales either curve could be made applicable to
any values set up.

FIG. 2—RECOVERY-CURVES OF CERTAIN WELLS

		CURVE A			
0	0.5	1.0	2.0	2.5	3.0
0	0.01	0.02 CURVE B	0.04	0.05	0.06
0	1.0	2.0 CURVE C	4.0	5.0	6.0

RESIDUAL DRAW-DOWN IN FEET

(Vertical axis: TIME SINCE PUMPING STARTED $=\frac{t}{t'}$ / TIME SINCE PUMPING STOPPED)

AT POINT 30,000 FEET FROM PUMPED WELL

ASSUMPTIONS: PUMPING RATE, 300 G.P.M.; SPECIFIC YIELD, 20%;
COEFFICIENT OF TRANSMISSIBILITY, 100,000.

FIG. 3—LOWERING OF WATER-TABLE NEAR WELL PUMPING
CONTINUOUSLY FROM THICK AQUIFER

AT POINT 100 FEET FROM PUMPED WELL

DRAW-DOWN IN FEET

TIME IN YEARS

These theoretical curves agree qualitatively with the facts generally observed when a well is pumped. The water-level close to the well at first falls very rapidly, but the rate of fall soon slackens. In the particular case considered in Figure 3 the water-level at a point 100 feet from the pumped well would fall during the first year of pumping more than half the distance it would fall in 1000 years. A delayed effect of the pumping is shown at distant points.

The water-level at a point about 6 miles from the pumped well of Figure 3 would fall only minute-
ly for about five years but would then begin to fall perceptibly, although at a much less rate
than the water-level close to the well. Incidentally the rate of fall after considerable pump-
ing is so small that it might easily lead to a false assumption of equilibrium. The danger in a
pumping district using ground-water storage lies in the delayed interference of the wells. For
instance, although in 50 years one well would cause a draw-down of only 6 inches in another well
6 miles away, yet the 100 wells that might lie within 6 miles of a given well would cause in it
a total draw-down of more than 50 feet.

In the preparation of this paper I have had the indispensable help not only of Dr. Lubin,
who furnished the mathematical keystone of the paper, but also of Dr. C. E. Van Orstrand, of the
United States Geological Survey, and of my colleagues of the Ground Water Division of the Survey,
who cordially furnished data and criticism.

U. S. Geological Survey,
    Washington, D. C.

THE PIEZOMETRIC SURFACE OF ARTESIAN WATER IN THE FLORIDA PENINSULA

V. T. Stringfield

Introduction

The ground-water of the Florida Peninsula constitutes one of its most valuable natural re-
sources and is of importance as a source of water-supplies throughout the area. The problems
relating to the development of ground-water supplies are both quantitative and qualitative.
They include such problems as the decline in yield of wells in areas of large withdrawals of
water and salt-water contamination of ground-water supplies. In order to facilitate a better
understanding and interpretation of hydrologic conditions relating to these local ground-water
problems, a general survey of the artesian water in the Florida Peninsula was made by the United
States Geological Survey in 1934.

As part of the results of the investigation a contour-map representing the piezometric sur-
face of the artesian water was prepared, and the general areal extent of highly mineralized
ground-water was outlined. The piezometric map shows the height to which water would rise above
sea-level in tightly cased wells in 1934. It indicates the hydraulic gradient and the direction
of movement of the ground-water, and therefore the areas of recharge and the areas of large dis-
charge.

General artesian conditions

The geologic formations exposed at the surface in different parts of the Peninsula probably
represent a thickness of about 1000 to 1500 feet of the geologic section and include the Ocala
limestone, of Eocene age, and the younger formations of the Miocene, Pliocene, Pleistocene, and
Recent. These formations constitute the rocks that yield the ground-water supplies. The forma-
tions of the Peninsula form an arch or broad anticline that trends southeast and plunges toward
the southern part of the State. The anticline constitutes the structural feature favorable for
artesian conditions in the Peninsula. A general outline of the structure is represented in Fig-
ure 1.

The principal formations that yield artesian water are the Ocala Limestone, the Tampa Lime-
stone, and the Hawthorn Formation.

The Ocala Limestone underlies all of Florida. In the northwestern part of the Peninsula it
lies at or near the surface on the eroded crest of the large anticline, and elsewhere it extends
under the younger formations that are exposed on the flanks of the anticline. It consists es-
sentially of limestone and has an estimated thickness of about 500 feet.

The Tampa Limestone, of Miocene age, overlies the Ocala Limestone. It is at or near the
surface in the west-central and northwestern parts and occurs as erosion remnants in a few areas
where the Ocala Limestone is near the surface. Well-records indicate that the formation is ab-
sent in the eastern and northeastern part of the Peninsula. The Formation consists essentially
of limestone and has an estimated thickness of about 200 feet.

The Hawthorn Formation, of Miocene age, overlies the Tampa Limestone or in some places rests