1.Force-Free Response: This means observing the system's natural behavior without applying external forces like continuous pumping.

VOL. 12, NO. 1 WATER RESOURCES RESEARCH FEBRUARY 1976

Determining Aquifer Transmissivity by Means of Well Response Tests: The Underdamped Case

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2. Any fluctuations in pressure due to external factors like pumping or changes in water level would affect the entire filter simultaneously, rather than creating different pressure zones at different heights.The water wouldn't be flowing vertically within the filter itself, as there's no pressure difference to drive it (since the potential is uniform). The flow would primarily be into and out of the filter from the surrounding water column.This simplifies the analysis because we can treat the filter as a single point where the potential changes with time (hr(t)) but remains constant throughout its small height. This can be helpful for modeling the overall water column's dynamic behavior around the filter.

Well response tests, often referred to as slug tests, provide a relatively simple and inexpensive method for estimating aquifer transmissivity. An approximate theory is developed for the problem of under- 1 damped well response where the inertia of the water in the well column results in force-free oscillation of the water level in the well. This type of response is often encountered in conjunction with highly permeable aquifers. The theory is applicable for nearly all such cases except those where the oscillation is very quickly damped out. Theoretical predictions compare well with empirical results and indicate that the theory may be used to obtain estimates of aquifer transmissivity. This theory for the underdamped case together with existing theory for overdamped response makes it possible to obtain an estimate of transmissivity from well response tests for almost any aquifer.

INTRODUCTION

Well response tests, often referred to as 'slug tests,' measure the force-free response of a well-aquifer system to an abruptly induced change of the water level in the well. Since the publication of a mathematically and physically consistent theory by *Cooper et al.* [1967] (see also *Papadopulos et al.* [1973]) a fairly reliable determination of aquifer transmissivity by means of well response tests has become possible for many cases. For some types of groundwater problems, well response tests can be used in conjunction with pump tests and may even reduce the need for pump testing. The obvious advantage of well response tests is that they are simple and inexpensive, often requiring only a few minutes to complete.

The theory for well response tests developed by *Cooper et al.* [1967] neglects the effect of the inertia of the water in the well column. It is therefore not applicable to certain cases involving highly permeable aquifers or deep wells, where inertial effects are important. The purpose of this paper is to present an approximate theory for such cases and show how it may be used to determine aquifer transmissivities.

The problem of inertial effects has been studied by *Cooper et al.* [1965] in the context of an analysis of well response to seismic waves and by *Bredehoeft et al.* [1966] through an electric analog investigation. In general, the well-aquifer system responds in a manner analogous to the classical mechanical system of a mass on a spring in a viscous medium, the water in the well corresponding to the mass and the aquifer corresponding to the spring. Depending on the mass of water in the well and the hydraulic characteristics of the aquifer, especially its
*1 conductivity, the response of the water level in the well may be as an overdamped oscillator (the case studied by *Cooper et al.* [1967]) or as a critically damped or underdamped oscillator. When the system is overdamped, the water level returns to the equilibrium level in an approximately exponential manner. In the underdamped case the water level oscillates about the equilibrium level. The critical damping case is the transition between these two types of response. Some actual cases of field observations from three different aquifers are illustrated in Figure 1. For each of these aquifers both overdamped and underdamped responses were observed. The response of well 2A in the Cap Pelé aquifer can be characterized as critically damped. These observations will be discussed further below.

The problem is that although free oscillations of water level in a well-aquifer system are often encountered in the field, to date no theory has been published which allows a determination of aquifer transmissivity from such underdamped response. The usefulness of the theory for the overdamped case in this respect would lead one to expect that development of the theory for the underdamped case may also turn out to be useful. A full and general analytical solution of the problem of well response would probably be exceedingly complicated, but it seems that the approximate theory which is derived below for the underdamped case may well give an adequate description. This theory together with the theory given by *Cooper et al.* [1967] should make it possible to obtain an estimate of aquifer transmissivity from well response tests for almost any aquifer.

THEORY

Statement of the Problem

The geometric parameters involved in the problem of well response are illustrated in Figure 2. According to *Cooper et al.* [1965], the effective length L of the water column is given by

$$L = L_c + \tfrac{3}{8}L_f \qquad (1)$$

where L_c is the length of the water column inside the casing and L_f is the length of the well filter. In practice aquifers are always inhomogeneous, and the magnitude of L will depend on the distribution of permeability along the filter. For most wells the filter radius r_f is approximately equal to the casing radius r_c, but caving and flushing or clogging of fractures may change the effective radius of the filter.

For the present purposes it will be assumed that the filter may be represented by a very short open section of well at 2
depth L below the water level so that the transient part $h_f(t)$ of the hydraulic potential of the water in the filter does not vary with height along the filter. The variation of the water level in the well about the steady state position is denoted by w. Its magnitude is usually very much smaller than L.

The equation for the balance of forces within the well filter, including the effects of inertia, i.e., the rate of change of mo-

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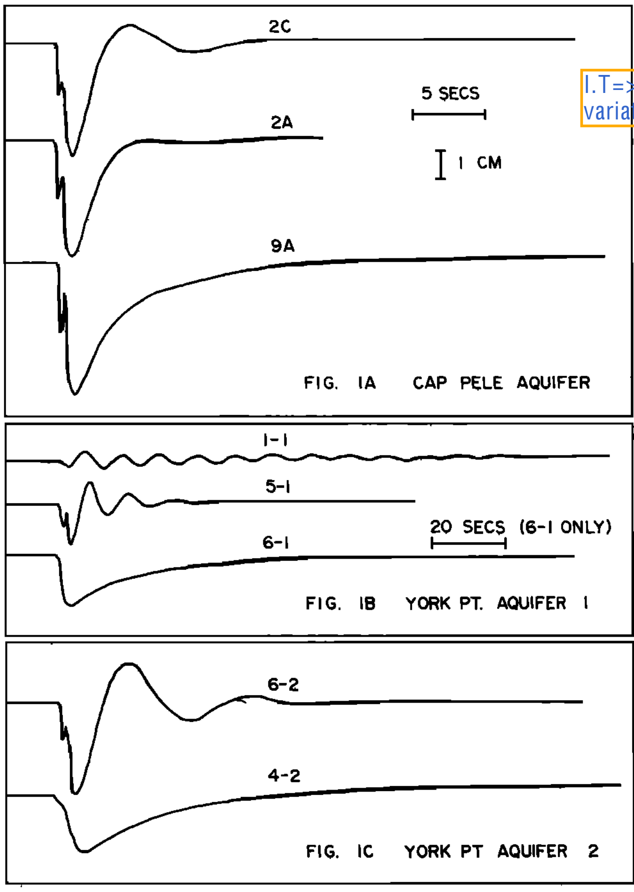

Fig. 1. Response of the water level in various wells after the sudden removal of a small volume of water. Height and time scales are the same for all the measurements except for well 6-1.

L+W~L

mentum of the water column, can be written as

$$\frac{d}{dt}\left[\rho(L + w)\frac{dw}{dt}\right] + \rho g(L + w) = \rho g(L + h_f) \qquad (2)$$

I.T => Considering unit c/s area

It may be safely assumed that

$$|w| \ll L \qquad (3)$$

and then (2) reduces to

$$\frac{d^2w}{dt^2} + \frac{g}{L}w = \frac{g}{L}h_f \qquad (4)$$

A similar equation was derived by *Cooper et al.* [1965]. It is shown in the appendix to this paper that the effects of friction in the casing, which have not been considered here, are negligible for all except very small diameter wells or very slowly damped oscillations.

The transient part of the hydraulic potential of the water in the aquifer is denoted by $h(r, t)$ and is assumed to be governed by the standard equation for time-dependent groundwater flow,

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t} \qquad (5)$$

where it is assumed that the flow is perpendicular to the axis of the well and radially symmetric.

In addition to (4) and (5), w, h_f, and h must satisfy a number of other conditions, written as follows:

$$\frac{dw}{dt} = \frac{2r_fT}{r_c^2}\left[\frac{\partial}{\partial r}h\right]_{r=r_f} \qquad (6a)$$

I.T => there is no spatial variation of head in filter.

$$h(r_f, t) = h_f(t) \qquad (6b)$$

$$h(\infty, t) = 0 \qquad (6c)$$

$$h(r, 0) = 0 \qquad r > r_f \qquad (6d)$$

$$w(t = 0) = w_0 \qquad (6e)$$

$$w(t = \infty) = 0 \qquad (6f)$$

where w_0 is the initial displacement of the water level.

Equations (4), (5), and (6) together give a complete description of the problem of well response, the major simplification being that h_f has been assumed to be independent of position within the filter. *Cooper et al.* [1967] derived the solution for the case when $w = h_f$; i.e., they assumed in effect that the second time derivative of w in (4) is negligibly small. A complete solution will not be attempted here. Instead an approximate solution will be given for the underdamped case which should give an adequate description for all such cases except those approaching critical damping conditions.

The Approximate Solution

It is known from observation that the water level in the well for the underdamped case, except near critical damping conditions, can be approximately described as an exponentially damped cyclic fluctuation; in other words, the water level fluctuation may be assumed to be given by

$$w(t) = w_0e^{-\gamma t}\cos\omega t \qquad (7)$$

where γ and ω are the damping constant and angular frequency of the oscillation, respectively. Since (7) describes the observed form of $w(t)$ to a good approximation, a solution to (4) and (5) and the appropriate boundary conditions corresponding to (7) will in fact be a good approximation to the general solution, except for the initial moments of the displacement. Such a solution will now be derived.

Using complex representation, we assume that w is given by

$$w = \text{Re}\,[W] \qquad (8)$$

a.t

where

$$W(t) = w_0e^{pt} \qquad (9)$$

The quantity p is complex and is given by

$$p = -\gamma + i\omega \qquad (10)$$

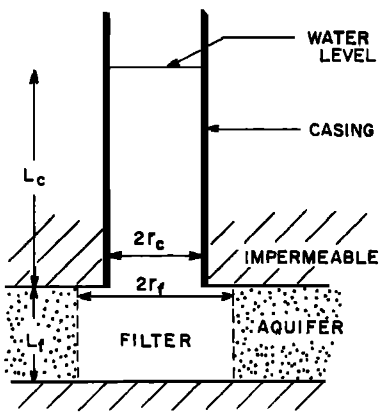

Fig. 2. Diagram of a well-aquifer system.

It follows from (4) that h_f may be similarly represented:

$$h_f(t) = \mathrm{Re}\,[H_f(t)] \tag{11}$$

where

$$H_f(t) = H_f(0)e^{pt} \tag{12}$$

Here $H_f(0)$ may be complex.

The corresponding solution for the hydraulic potential in the aquifer h can be attempted on the assumption that it is given by

$$h = \mathrm{Re}\,[H] \tag{13}$$

where

$$H(r, t) = R(r)e^{pt} \tag{14}$$

With this form of h, boundary condition (6d) cannot be satisfied. In effect the approximation involved in the present solution consists in replacing (6d) by (14).

The solution will be carried out in terms of the complex variables H, H_f, and W. If these satisfy (4), (5), and (6), then so will their real parts h, h_f, and w, since the equations are linear.

With the form of H given by (14), (5) becomes

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - q^2 R = 0 \quad \text{a.t.} \tag{15}$$

where

$$q^2 = p(S/T) \tag{16}$$

Conditions (6b) and (6c) are written

Corresponding to
h(r_f,t)=h_f(t)
h(\infty,0)=0

$$R(r_f) = H_f(0) \tag{17a}$$

$$R(\infty) = 0 \quad \text{a.t.} \tag{17b}$$

The solution of (15) subject to conditions (17) is

$$R(r) = \frac{K_0(qr)}{K_0(qr_f)} H_f(0) \tag{18}$$

where K_0 is a modified Bessel function of the second kind of zeroth order [*McLachlan*, 1955, p. 172]. Equation (6a) together with (12) then leads to

?????

$$\frac{dW}{dt} = \frac{2Tqr_f K_0'(qr_f)}{r_c^2 K_0(qr_f)} H_f \tag{19}$$

Equation (4) can then be written

$$\frac{d^2W}{dt^2} - \frac{g}{L} A \frac{dW}{dt} + \frac{g}{L} W = 0 \tag{20}$$

where

$$A = \frac{r_c^2 K_0(qr_f)}{2Tqr_f K_0'(qr_f)} \tag{21}$$

With the form of W given in (9), (20) leads to

????

$$p = \frac{gA}{2L} \pm \left(\frac{g^2A^2}{4L^2} - \frac{g}{L}\right)^{1/2} \tag{22}$$

Thus a solution for h, h_f, and p has now been found corresponding to an exponentially damped cyclic fluctuation of the water level as described by (7). Further analysis will concentrate on obtaining expressions for γ and ω through (22).

A considerable simplification of (21) and (22) is possible. It follows from the series expansion for K_0 [*McLachlan*, 1955, p. 203] that if

$$|qr_f| \ll 0.1 \tag{23}$$

then

$$K_0(qr_f) \approx \log_e (0.89qr_f) \tag{24}$$

From the definition of q (equations (16) and (10)) a.t.

$$q = \left(\frac{S}{T}\right)^{1/2} (\omega^2 + \gamma^2)^{1/4} \exp \frac{i}{2}\left(\pi - \arctan \frac{\omega}{\gamma}\right) \tag{25}$$

Thus

Taking log in both sides of (25),

$$\log_e (0.89qr_f) = \log_e \alpha + \frac{i}{2}\left(\pi - \arctan \frac{\omega}{\gamma}\right) \tag{26}$$

where

$$\alpha = 0.89\,(S/T)^{1/2}\,(\omega^2 + \gamma^2)^{1/4} r_f \tag{27}$$

Condition (23) can be written approximately as

$$\alpha \ll 0.1 \tag{28}$$

On the right-hand side of (26) the magnitudes of the real and imaginary parts are respectively greater than 4.6 and less than $\pi/2$. Thus for a first approximation the imaginary part can be neglected and

$$K_0(qr_f) = \log_e (\alpha) \tag{29}$$

The approximation involved in (29) is not essential to the analysis, but with respect to practical applications the resulting simplification of the equations more than makes up for the loss of accuracy. Condition (28) holds for nearly all cases of practical interest.

With (24) and (29) the expression for A (equation (21)) reduces to

??? Not matching....

$$A = (r_c^2/2T) \log_e (\alpha) \tag{30}$$

so that in view of (28), A is a negative real number.

It follows from (22) that p is complex, i.e., that there is an oscillation, if

$$d^2 < 1 \tag{31}$$

where

$$d = -(g/L)^{1/2} A/2 \tag{32}$$

Equation (22) then yields

$$\gamma = (g/L)^{1/2} d \tag{33a}$$

$$\omega = (g/L)^{1/2}(1 - d^2)^{1/2} \tag{33b}$$

and when these two equations are combined,

$$\omega^2 + \gamma^2 = g/L \tag{34}$$

With these results, (32) can be rewritten as

$$d = \frac{-r_c^2 (g/L)^{1/2} \log_e [0.79 r_f^2 (S/T)(g/L)^{1/2}]}{8T} \tag{35}$$

Equations (33) and (35) together relate the frequency and damping of the free water level oscillation in the well to the geometric and hydraulic characteristics of the well-aquifer system, subject to condition (28). Thus an approximate solution for the underdamped case has been found. The equations may be used to predict the magnitude of γ and ω, or they can be used to determine L and T if γ and ω are known from measurements.

Applicability of the Solution

The above analysis of the underdamped case is based on the assumption of an exponentially damped cyclic fluctuation. It is therefore not applicable for the initial moments of the fluctuation and consequently for cases when the fluctuation is very rapidly damped out, i.e., when γ is only slightly less than ω.

Equation (22) seems to imply that critical damping occurs when $d = 1$. However by the argument of the preceding paragraph this equation cannot be expected to give an adequate description of the critical damping case, and therefore the criterion $d = 1$ may not in fact define critical damping. But one might expect that a valid criterion for critical damping would in fact involve the parameter d. Now it happens that *Bredehoeft et al.* [1966] by means of an electrical analog determined a critical damping condition for a 6-in. (15.24 cm) diameter well with $S = 10^{-4}$. Their results can be almost exactly reproduced if d is put equal to 0.69. It is interesting to note that if one defines critical damping as occurring when $\gamma = \omega$, then (27) leads to $d = 1/(2)^{1/2} = 0.707$, almost exactly the same value as that obtained from the results of Bredehoeft et al. Thus it seems reasonable to hypothesize that critical damping occurs when $d = 0.7$. In practice it may be more meaningful to define critical damping as occurring over the range of values of d centered around $d = 0.7$, where neither the theory for the underdamped case as derived above nor the theory for the overdamped case as given by *Cooper et al.* [1967] is applicable.

COMPARISON WITH EMPIRICAL RESULTS

In this section the theory presented above will be compared with some empirical observations, both to check whether the theory does in fact give a satisfactory description of actual cases and to illustrate the use of the theory for determining transmissivity.

Well response tests were carried out for a number of wells in Prince Edward Island and New Brunswick, Canada, at locations where pump test data were also available from the same wells. These wells are all completed in sandstone formations and are directly open to the formation without a filter lining. The water level fluctuations were induced by means of an object, such as a sealed piece of pipe, lowered into the water, held in position for sufficient time to allow the water level to stabilize, and then abruptly hoisted up. The fluctuations were observed by means of a pressure transducer positioned a meter or so below the water level. Some typical results of these response tests are illustrated in Figure 1. Here it should be pointed out that even for highly permeable aquifers, successful response tests may be carried out by methods such as the one described by *Walton* [1963] utilizing ordinary water level recorders.

The relevant geometric characteristics of the wells and the hydraulic characteristics as determined by pump tests are summarized in Table 1. Also listed are the data for a well near Perry, Florida, as given by *Cooper et al.* [1965] and for an idealized well simulated by an electrical analog as given by *Bredehoeft et al.* [1966]. For these last two cases the authors have also given diagrams of the well response similar to those in Figure 1, and the same analysis can therefore be applied. For each of the wells the radius of the casing and that of the filter are roughly equal. For some of the wells the storage coefficient S is not known and has been estimated. Such estimates are indicated by parentheses in Table 1 and the following tables.

With these characteristics of the well-aquifer system known the parameter d, as defined by (35), can be calculated. It was hypothesized above that if d is smaller than 0.7, the system is underdamped. The calculated values of d are listed in Table 1, and in the column headed 'type of response' the letters U, C, or O indicate whether the system was observed to be underdamped, critically damped, or overdamped. These results indicate that the transition from the underdamped to the overdamped case occurs at values of d between about 0.4 and 1.2. The uncertainties in the calculated value of d, due especially to the uncertainty in the value of T, preclude a more precise test.

Equations (33) allow a prediction of the values of the damping constant γ and the angular frequency ω once d has been determined by independent means. The results of such a calculation for the wells which were observed to be underdamped are tabulated in Table 2 together with the measured values of γ and ω. (The values of d used in the calculation are those listed in Table 1.) For well 2C (Cap Pelé), d is greater than unity, and γ and ω cannot be calculated. Also listed in Table 2 are values of $(g/L)^{1/2}$, which should be approximately equal to ω unless the oscillation is almost critically damped. The measured values of γ and ω were estimated directly from the traces of the water level fluctuations as given in Figure 1 and in similar figures given by *Cooper et al.* [1965] (well near Perry, Florida) and *Bredehoeft et al.* [1966] (electrical analog). The determination of γ and ω is based on the assumption that after the first maximum or minimum the water level fluctuation is described by (7), i.e., that it is an exponentially damped sinusoidal mo-

TABLE 1. Summary of Well-Aquifer System Characteristics

Aquifer	Well No.	r_f, r_c, m	L_c, m	L_f, m	L, m	T, m²/s	S, ×10⁴	d	Type of Response
Cap Pelé,									
New Brunswick	2C	0.051	5	15	11	0.0023	1.4	1.2	U
	2A	0.051	8	12	13	0.0019	1.1	1.4	C
	9A	0.051	7	11	11	0.0017	1.1	1.6	O
York Point, Prince									
Edward Island (1)	1–1	0.051	1.4	2.8	2.4	>0.1	(0.3)	<0.09	U
	5–1	0.051	0.4	2.8	1.3	?	(0.3)	?	U
	6–1	0.051	0	2.5	0.9	?	(0.3)	?	O
York Point, Prince									
Edward Island (2)	6–2	0.051	6.5	15	12	0.007	0.8	0.40	U
	4–2	0.076	7.2	14	13	0.007	0.8	0.89	O
Perry, Florida		0.153	57.6	13	61	0.13	(1)	(0.11)	U
Electrical analog		0.076			30.5	0.014	1.0	0.32	U

Parenthetical values are estimated.
U is underdamped, C is critically damped, and O is overdamped.

TABLE 2. Comparison of Measured and Predicted Values of Damping Constant γ and Angular Frequency ω

Aquifer	Well No.	$(g/L)^{1/2}$, s^{-1}	ω, s^{-1} Predicted	ω, s^{-1} Measured	γ, s^{-1} Predicted	γ, s^{-1} Measured
Cap Pelé, New Brunswick	2C	0.94		0.73		0.30
York Point, Prince Edward Island (1)	1–1	2.0	2.0	2.6	<0.18	0.082
	5–1	2.7	?	2.2	?	0.20
York Point, Prince Edward Island (2)	6–2	0.81	0.74	0.76	0.32	0.26
Perry, Florida		0.40	0.40	0.37	0.044	0.017
Electrical analog		0.57	0.54	0.60	0.18	0.18

tion. The agreement between the predicted and measured values of γ and ω is seen to be quite satisfactory. To this extent therefore the theory that has been developed seems to give a fairly good description of actual cases.

More interesting from a practical point of view is the possibility of determining aquifer transmissivity from well response tests. For the underdamped case both the effective length L and the transmissivity T can be determined from the measured values of γ and ω. To this end (34) can be rewritten as

$$L = g/(\omega^2 + \gamma^2) \tag{36}$$

Thus the calculation of the effective length is straightforward. With this value of L the parameter d can be calculated with (equation (33a))

$$d = \gamma/(g/L)^{1/2} \tag{37}$$

To allow a calculation of T, (35) can be rewritten in the form

$$T = b + a \log_e T \tag{38}$$

where

$$a = r_c^2(g/L)^{1/2}/8d \tag{39a}$$

$$b = -a \log_e [0.79 r_f^2 S(g/L)^{1/2}] \tag{39b}$$

Calculation of the quantity b through (39b) requires prior knowledge of the filter radius r_f and the storage coefficient S. However, large variations in the values of r_f and S result in only small variations in the value of b, and therefore a reasonably good estimate of r_f and S is sufficient for most practical purposes.

The value of T can be determined from (38) through an iterative procedure starting with a first estimate

$$T_0 = b \tag{40a}$$

and proceeding to

$$T_n = b + a \log_e T_{n-1} \qquad n \geq 1 \tag{40b}$$

Convergence to the final value of T generally occurs within three or four iterations.

The results of such a calculation of L and T for the wells with underdamped response are summarized in the first part of Table 3. The values of r_f, r_c, and S used for this calculation are those given in Table 1. For purposes of comparison the values of L and T as estimated from well geometry and pump tests are also given in Table 3. For the wells with overdamped response a value of T can be determined from the observed response by the method described by *Cooper et al.* [1967]. The results of such a calculation are also given in Table 3. Well 2A (Cap Pelé) responds in a critically damped manner. Therefore neither the theory for overdamped response nor the theory for underdamped response is applicable, and a value of T cannot be obtained from the observed response of this well by the methods described above.

Also given in Table 3 are the values of the parameters d and α, incorporating the results of the response tests as far as possible. For the underdamped cases, d was calculated directly through (37). For the overdamped case, (35) was used to arrive at a value of d on the basis of the value of T from response tests. The parameter α was calculated for the underdamped cases through (27), the response test value of T again being used.

It can be seen that the theory is self-consistent insofar as the

TABLE 3. Comparison of the Values of L and T Calculated From Response Tests With the Values Obtained by Other Means

Aquifer	Well No.	Type of Response	L, m Predicted From Well Dimensions	L, m Calculated From Response Tests	T, m^2/s Determined by Pump Tests	T, m^2/s Calculated From Response Tests	d	α
Cap Pelé	2C	U	11	16	0.0023	0.0068	0.38	0.006
York Point (1)	1–1	U	2.4	1.5	>0.1	0.38	0.032	(0.0006)
	5–1	U	1.3	2.0	?	0.043	0.19	(0.002)
York Point (2)	6–2	U	12	15.2	0.007	0.0090	0.32	0.004
Perry, Florida		U	61	72	0.13	0.30	0.046	(0.002)
Electrical analog		U	30.5	25	0.014	0.017	0.29	0.004
York Point (1)	6–1	O	0.9		?	0.00041	20	
York Point (2)	4–2	O	13		0.007	0.0036	1.6	
Cap Pelé	9A	O	11		0.0017	0.0022	1.3	

U is underdamped, and O is overdamped.
Parenthetical values are estimated.

value of d is concerned, d being less than 0.7 for each of the underdamped cases and greater than 0.7 for the overdamped cases. Also the values of α satisfy the restriction expressed through condition (28) that α be much smaller than 0.1.

Comparison of the predicted values of L with the values calculated from the response tests shows a good agreement in view of the uncertainties in the predicted values. Usually of course L can be easily estimated from the known dimensions of the well. However in some cases a determination of L by means of a response test could be useful if the casing length were unknown or if the depth to the main water-conducting zone were to be determined.

The values of transmissivity T calculated from the response tests agree roughly with the values obtained by pump tests, but some of the discrepancies are rather large, amounting to a factor of 3 for the worst case. One problem here is that the value of T obtained from a response test reflects the conditions in the aquifer near the well and is therefore strongly affected by local inhomogeneities in the aquifer. Phenomena such as caving or washing out of the borehole, flushing of fractures, or additional fracturing caused by the drilling would tend to result in an increase of effective transmissivity near the well. These considerations may offer an explanation for most of the discrepancies. In addition it must be borne in mind that the theory for the underdamped case that has been developed in this paper may not be reliable when the fluctuation is strongly damped, i.e., when d is only slightly less than 0.7. Thus the discrepancy in the values of T for well 2C (Cap Pelé) may be due in part to the approximate nature of the theory for values of d approaching 0.7. On the other hand, it should be noted that good agreement is obtained for the case of the electrical analog (in effect the one well for which values of r_f, r_c, L, T, and S are known with good precision) even though the value of d for this case is 0.29.

For the critically damped well (2C, Cap Pelé) a calculation of T by the above methods is not possible because the theory is not applicable. However, if d is assumed to equal 0.7 for this well, then (39) and (40) yield a value for T of 0.0040 m²/s, which is probably a good estimate of the transmissivity near this well. In this manner it should be possible to obtain a rough estimate of transmissivity for wells with critically damped response.

In general these results indicate that the approximate theory for the underdamped response of a well-aquifer system which has been developed in this paper allows at least an approximate determination of aquifer transmissivity. It may in fact allow an accurate determination of the effective transmissivity near the well, but more stringent measurements and theoretical analysis would be required to establish such a conclusion.

SUMMARY

Approximate equations have been derived which relate the damping and frequency of force-free water level oscillations in a well to the dimensions and hydraulic characteristics of the well-aquifer system. The equations are not applicable near critical damping conditions, i.e., when the oscillations are very quickly damped out. With these equations it should be possible to estimate the transmissivity near a well through measurement of the well response.

Comparison of the theoretical predictions with observations for several different wells shows agreement to within the margin of possible error of the measurements. Estimates of transmissivity from underdamped well response obtained by the method described in this paper are probably accurate to within a factor of 2 or better.

The method for determining aquifer transmissivity from underdamped well response described in this paper together with the method for overdamped response described by *Cooper et al.* [1967] makes it possible to obtain a good estimate of transmissivity by means of well response tests for almost any aquifer, even if the transmissivity is very high.

APPENDIX: EFFECTS OF FRICTION WITHIN THE WELL

The head loss h_w due to friction within the well may be estimated with the Darcy-Weisbach formula,

$$h_w = fLv^2/4gr_c \tag{41}$$

where v is the velocity of the water in the well and the other symbols except for f are as defined previously. The factor f depends on the flow regime as governed by the Reynolds number N_R,

$$N_R = 2vr_c/\nu \tag{42}$$

where ν is the kinematic viscosity ($\nu = 1.31 \times 10^{-6}$ m²/s for water at 10°C). The transition from laminar to turbulent flow occurs when N_R is about 2000 [*Eskinazi*, 1962, p. 389]. For underdamped well response, N_R may be estimated by

$$N_R \approx 2\omega w_o r_c/\nu \tag{43}$$

Typical cases of response tests give values of N_R in a range from about 500 to 50,000.

With respect to the theory for underdamped well response the effects of well losses can be accounted for through an expansion of (20) in the form

$$\frac{d^2W}{dt^2} + \frac{g}{L}\left(H_w - A\frac{dW}{dt}\right) + \frac{g}{L}W = 0 \tag{44}$$

where H_w is the complex representation of h_w. Thus well friction is negligible if

$$|H_w| << |A\ dW/dt| \tag{45}$$

For laminar flow, f is given by [*Eskinazi*, 1962, p. 389]

$$f = 64/N_R \tag{46}$$

and (41) leads to

How obtained??

$$H_w = (8L\nu/gr_c^2)(dW/dt) \tag{47}$$

Here the relationship

$$v = dw/dt \tag{48}$$

has been used. Substitution of the expression for H_w into condition (45) and use of (32) and (33*a*) lead to the conclusion that if the flow in the well is laminar, the effects of well friction may be neglected if

$$4\nu/r_c^2 << \gamma \tag{49}$$

Thus for laminar flow, well losses are likely to be negligible unless the well radius is very small or the oscillations are very slowly damped. A test of the wells studied in this paper shows that condition (49) is easily satisfied for each case and thus indicates that well losses for laminar flow are likely to be negligible for most normal cases.

For turbulent flow there is no simple relationship for f, but it generally has values between 0.02 and 0.06 [*Eskinazi*, 1962, p. 391]. Thus H_w may be estimated by

$$H_w = 0.04L(dW/dt)^2/4gr_c \tag{50}$$

Now

$$|dW/dt| < \omega w_0 \tag{51}$$

Condition (45) together with (32) and (33*a*) leads to the conclusion that for turbulent flow well losses are negligible if

$$0.005(w_0/r_c) << \gamma/\omega \tag{52}$$

For this case therefore well losses are likely to be unimportant unless the initial displacement of the water level is much larger than the well radius or the fluctuation is very slowly damped. A test of the wells described in this paper shows that only for the well near Perry, Florida, may well losses have had a significant influence on the water level oscillation.

NOTATION

d parameter defined by (35).
g acceleration of gravity.
h hydraulic head in aquifer.
h_f hydraulic head of water in well filter.
L effective length of water column, equal to $L_c + \frac{3}{8}L_f$.
L_c length of water column within casing.
L_f length of water column within filter.
r distance from center of well.
r_c radius of well casing.
r_f radius of well filter.
S coefficient of elastic storage.
t time.
T transmissivity of aquifer.
w water level in well.
w_0 initial water level displacement.
α parameter defined by (27).
γ damping constant of water level oscillation (see (7)).
ω angular frequency of water level oscillation.
ρ density of water.

Acknowledgments. The help of T. Maxim in preparing the pressure-transducer system for field measurements is gratefully acknowledged.

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(Received May 21, 1975;
revised August 6, 1975;
accepted August 12, 1975.)

*1. UNDERDAMPED, OVERDAMPED AND CRITICALLY-DAMPED SYSTEMS:

Imagine a well and its surrounding aquifer like a simple mechanical system:

Mass on a spring: The water in the well acts like the mass attached to a spring.

Spring: The aquifer itself behaves like the spring, providing resistance to changes in water level.

Viscous medium: The aquifer also acts like a viscous medium, resisting the flow of water (like air resistance to a moving object).Depending on the amount of wate the well (the mass) and the aquifer's properties (the spring's stiffness and the medium's viscosity), the water level in the well will respond in different ways when disturbed:

Overdamped: If the water is heavy (large mass) or the aquifer is very resistant (stiff spring and thick medium), the water level will slowly return to equilibrium witho any oscillations, like a ball rolling back to rest after a gentle push.

Underdamped: If the water is light (small mass) and the aquifer is less resistant (flexible spring and thin medium), the water level will bounce back and forth aroun equilibrium, like a yo-yo, before settling down.

Critically damped: This is the "just right" situation where the water returns to equilibrium quickly without any oscillations, like a perfectly balanced door closing smo