Stochastic Analysis of Phreatic Aquifers

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Three analytical models are developed to describe the spectral response characteristics of phreatic aquifers subject to time variable accretion and fluctuations in adjacent stream stage. It is found that a linear reservoir model will approximate the behavior of a more complete distributed linear model based on the Dupuit approximation if a single parameter is adjusted appropriately. A linearized two-dimensional analysis including the effects of vertical flow shows, under conditions of large anisotropy with relatively narrow partially penetrating streams, differences from the Dupuit analysis that may be important under some field conditions. Possible applications of the results in aquifer evaluation are suggested.

Hydrologic phenomena are generally recognized as being affected by complex natural events, the details of which cannot be anticipated precisely. Hence the analysis of hydrologic systems is often viewed in terms of stochastic processes. However, the analysis of groundwater flow has traditionally been based on a deterministic approach to the solution of the governing partial differential equations. Natural variability, such as temporal fluctuations in groundwater recharge or water level in adjacent bodies of water and spatial variations in recharge and hydraulic conductivity, is usually dealt with only in terms of average conditions. Yet natural variability may be an important feature of groundwater flow in that it may be possible to infer aquifer properties from water table fluctuations. Also of interest are problems of aquifer management from a probabilistic point of view.

In the following analysis, temporal variability in a phreatic aquifer is treated by using a stochastic description rather than a deterministic approach. Considering a stationary random process, we analyze several different models of a homogeneous phreatic aquifer that receives variable recharge and is connected to a stream (Figure 1). The linear reservoir model is a lumped parameter representation that is governed by a time-dependent linear ordinary differential equation. The Dupuit aquifer model is a linearized version of the Dupuit-Forchheimer approximation [see, e.g., *Bear*, 1972] in which dependence on a horizontal space coordinate is included as an independent variable in the partial differential equation. Finally, a two-dimensional representation, the Laplace aquifer, with linearized phreatic surface boundary conditions is developed to evaluate the nonhydrostatic effects of vertical flow.

For each model the analysis yields frequency domain relationships involving the spectral density function of the input and output. The results of the three models are compared to illustrate the degree of modeling sophistication that is necessary to represent certain features. The application of the results in the evaluation of aquifer parameters is discussed.

Although there appears to have been no previous work dealing directly with analytical modeling of groundwater systems in a stochastic sense, there have been several related studies. The interpretation of groundwater level fluctuations has been explored [see, e.g., *Jacob*, 1943; *Tison*, 1965; *Pinder et al.*, 1969], and linear reservoir models of subsurface flow have been used by *Kraijenhoff van de Leur* [1958], *Dooge* [1960], J. Van Schilfgaarde (unpublished manuscript, 1965), and *Eriksson* [1970]. Linearized Dupuit aquifer models have been applied extensively in deterministic problems [*Cooper and Rorabaugh*, 1963; *Glover*, 1967; *Venetis*, 1971; *Hall and Moench*, 1972]. *Dagan* [1964] has developed a linearized Laplace aquifer model for the case of deterministic transient recharge.

LINEAR RESERVOIR

One of the simplest models that can be used to represent a phreatic aquifer is the lumped parameter linear reservoir system indicated schematically in Figure 1*a*. In this model we neglect all spatial variation of water level and consider the average thickness of the saturated zone $h(t)$ to be represented solely as a function of time. In addition, it is assumed that the outflow per unit area can be represented by $q = a(h - H)$, where a is an outflow constant and H represents the elevation of the water surface in some adjacent body of water. A water balance can then be stated as

$$S\, dh/dt + a(h - H) = \epsilon \tag{1}$$

where ϵ is the accretion or recharge rate and S is the average storage coefficient. For this analysis we will consider S and a to be constants and h, H, and ϵ to be stationary random functions of time. Under conditions of stationarity the average of (1) indicates that $a(\langle h\rangle - \langle H\rangle) = \langle\epsilon\rangle$, where the angle brackets denote the expected values. Thus it follows that the fluctuation about the mean also satisfies (1) and that we may consider (1) to govern h, H, and ϵ with zero mean.

We will use generalized harmonic analysis [*Papoulis*, 1965] with stochastic Fourier-Stieltjes integrals to represent the random functions in the form

$$h(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_h(\omega) \tag{2}$$

where ω is frequency and

$$Z_h(\omega_2) - Z_h(\omega_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega_2 t} - e^{-i\omega_1 t}}{-it}\, h(t)\, dt$$

is a generalized Fourier transform of $h(t)$. The differential $dZ_h(\omega)$ can be thought of as the generalized Fourier

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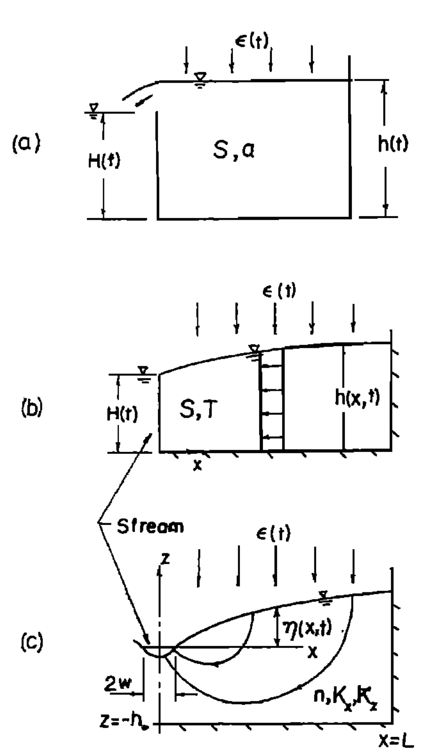

Fig. 1. Phreatic aquifer models: (*a*) linear reservoir, (*b*) Dupuit aquifer, and (*c*) Laplace aquifer.

amplitudes of the process. Parameter $Z_h(\omega)$ represents a random process with nonoverlapping increments, i.e.,

$$\begin{aligned} \langle dZ_h(\omega)\, dZ_h^*(\omega')\rangle &= 0 \qquad \omega \neq \omega' \\ \langle dZ_h(\omega)\, dZ_h^*(\omega')\rangle &= S_{hh}(\omega)\delta(\omega - \omega')\, d\omega\, d\omega' \end{aligned} \tag{3}$$

where the asterisk denotes the complex conjugate and $\delta(\omega - \omega')$ is the Dirac delta function. The parameter $S_{hh}(\omega)$ is the spectral density function of $h(t)$, which is related to the autocorrelation function $R_{hh}(\tau)$ by the Fourier transform pair

$$S_{hh}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} R_{hh}(\tau)\, d\tau \tag{4}$$

$$R_{hh}(\tau) = \langle h(t+\tau)h^*(t)\rangle = \int_{-\infty}^{\infty} S_{hh}(\omega) e^{i\omega\tau}\, d\omega \tag{5}$$

Similarly, H and ϵ are represented by

$$H(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_H(\omega) \qquad \epsilon(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_\epsilon(\omega) \tag{6}$$

When the representations of (2) and (6) are used in (1), the generalized Fourier amplitudes are given by

$$dZ_h(\omega) = [a\, dZ_H(\omega) + dZ_\epsilon(\omega)]/(i\omega S + a) \tag{7}$$

Through the use of (3) the spectral density function becomes

$$\begin{aligned} S_{hh} = [a^2 S_{HH}(\omega) &+ aS_{H\epsilon}(\omega) \\ &+ aS_{\epsilon H}(\omega) + S_{\epsilon\epsilon}(\omega)]/(\omega^2 S^2 + a^2) \end{aligned} \tag{8}$$

$$\langle dZ_H\, dZ_\epsilon^*\rangle = S_{H\epsilon}\delta(\omega - \omega')\, d\omega\, d\omega'$$

where $S_{H\epsilon}$ is the cross spectrum, which is related to the cross-correlation function $R_{H\epsilon}(\tau)$ by

$$S_{H\epsilon} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} R_{H\epsilon}(\tau)\, d\tau = C_{H\epsilon}(\omega) - iQ_{H\epsilon}(\omega)$$

The real and imaginary parts of $S_{H\epsilon}(\omega)$ are, respectively, the cospectrum and the quadrature spectrum. The cross-correlation function is given by

$$R_{H\epsilon}(\omega) = \langle H(t+\tau)\epsilon^*(t)\rangle = \int_{-\infty}^{\infty} e^{i\omega\tau} S_{H\epsilon}(\omega)\, d\omega$$

from which it follows that $S_{H\epsilon}(\omega) = S_{\epsilon H}^*(\omega)$. Therefore the spectral density of h can be written, by using (8), as

$$S_{hh}(\omega) = [S_{\epsilon\epsilon}(\omega) + a^2 S_{HH}(\omega) + 2aC_{H\epsilon}(\omega)]/(S^2\omega^2 + a^2) \tag{9}$$

The mean square fluctuation $\langle h^2\rangle$ is obtained from (5) with $\tau = 0$ as

$$\langle h^2\rangle = \int_{-\infty}^{\infty} S_{hh}(\omega)\, d\omega$$

where S_{hh} is given by (9). Equation 9 illustrates how the spectral density functions of the inputs (ϵ and H), along with their cospectrum, determine the spectrum of the output h for this simple lumped parameter system.

In a similar fashion the input-output cross spectra are found, from (7), to be

$$S_{\epsilon h}(\omega) = \{[aS_{\epsilon H}(\omega) + S_{\epsilon\epsilon}(\omega)](a + i\omega S)\}/(a^2 + \omega^2 S^2) \tag{10}$$

$$S_{Hh}(\omega) = \{[aS_{HH}(\omega) + S_{H\epsilon}(\omega)](a + i\omega S)\}/(a^2 + \omega^2 S^2) \tag{11}$$

When only a single input is considered (e.g., $H = 0$ or $\epsilon = 0$), the results given by (9), (10), and (11) can be obtained directly by using the convolution theorem from classical linear systems analysis. For example, when $H = 0$, (9) reduces to

$$S_{hh}(\omega) = S_{\epsilon\epsilon}(\omega)/(a^2 + S^2\omega^2) \tag{12}$$

In the context of linear systems analysis the factor $(a^2 + S^2\omega^2)^{-1}$ is the square of the modulus of the frequency response function [see, e.g., *Bendat and Piersol*, 1971]. The result in (12) illustrates a potential application of this analysis in aquifer evaluation. If the spectral density functions S_{hh} and $S_{\epsilon\epsilon}$ are observed, the frequency dependence of the ratio $S_{hh}/S_{\epsilon\epsilon}$ might be used to infer values of the parameters a and S for a groundwater system. The ratio S/a is the response time of the first-order linear system. It is of interest to compare the results of this simple lumped parameter analysis with those of more realistic distributed systems in the following.

LINEAR DUPUIT AQUIFER

We consider the linearized form of the classical Dupuit approximation given by

$$S\frac{\partial h}{\partial t} = T\frac{\partial^2 h}{\partial x^2} + \epsilon \tag{13}$$

where $h(x, t)$ is the thickness of the saturated zone, x is the horizontal position, S is the storage coefficient, T is the transmissivity, and ϵ represents accretion. An aquifer of finite length L that is connected to a fully penetrating stream (Figure 1*b*) will be analyzed for the case of variable accretion $\epsilon(t)$ and stream stage $H(t)$. For a stationary random process the mean conditions are given from (13) by

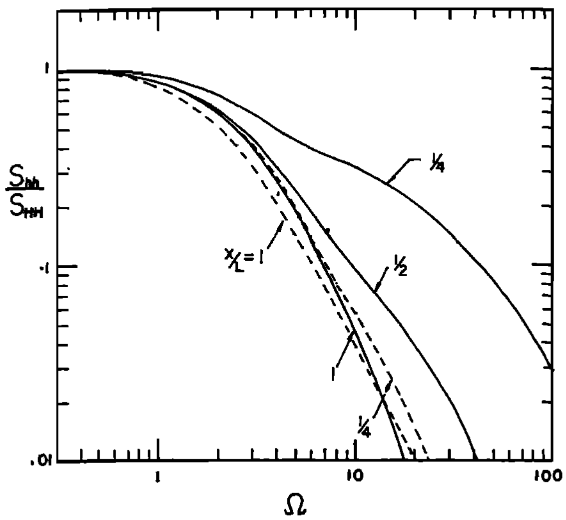

Fig. 2. Spectral response of the Dupuit aquifer affected by stream stage, where $\Omega = \omega L^2 S/T$, the solid lines represent the Dupuit aquifer (19), and the dashed lines represent the linear reservoir (20).

$$T\, d^2\langle h\rangle/dx^2 = -\langle \epsilon\rangle \tag{14}$$

for which the solution is

$$\langle h\rangle - \langle H\rangle = (\langle \epsilon\rangle/2T)x(2L - x) \tag{15}$$

with the boundary conditions $x = 0$, $\langle h\rangle = \langle H\rangle$ and $x = L$, $d\langle h\rangle/dx = 0$. By subtracting (14) from (13) it is seen that the fluctuations of h and ϵ about the mean are also governed by (13). In the following analysis we deal only with those fluctuations denoted again by h, ϵ, and H. With the representation

$$h(x, t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_h(\omega, x) \tag{16}$$

and the representations given by (6), (13) reduces to an ordinary differential equation for the generalized Fourier amplitudes $dZ_h(\omega, x)$ in the form

$$i\omega S\, dZ_h = T\, d^2/dx^2[dZ_h(\omega, x)] + dZ_\epsilon \tag{17}$$

With the application of the boundary conditions

$$x = 0 \qquad h = H(t) \qquad dZ_h = dZ_H$$

$$x = L \qquad \partial h/\partial x = 0 \qquad d/dx(dZ_h) = 0$$

the solution of (17) is

$$dZ_h(\omega, x) = F(x)[dZ_H + i\, dZ_\epsilon/(\omega S)] - i\, dZ_\epsilon/(\omega S)$$

$$F = \frac{\cosh b(x - L)}{\cosh bL} \qquad b = (1 + i)(|\omega|/2\alpha)^{1/2} \qquad \alpha = T/S$$

Through the use of (3) the spectral density function S_{hh} becomes

$$\begin{aligned} S_{hh}(\omega, x) = {} & FF^* S_{HH}(\omega) \\ & + [FF^* - (F + F^*) + 1]S_{\epsilon\epsilon}(\omega)/(S^2\omega^2) \\ & - [2FF^* - (F + F^*)]Q_{H\epsilon}(\omega)/(S\omega) \\ & - (F - F^*)C_{H\epsilon}(\omega)/(iS\omega) \end{aligned} \tag{18}$$

It is seen that this result involves a complicated interaction between the two inputs H and ϵ.

Some features of the result are illustrated by considering the special case involving only stream-aquifer interaction ($\epsilon = 0$, $dZ_\epsilon = 0$). In this case the spectral relationship reduces to

$$S_{hh}(\omega, x)/S_{HH}(\omega) = FF^* = f(\Omega, x/L) \qquad \Omega = \omega L^2/\alpha \tag{19}$$

where Ω is the dimensionless frequency. This relationship is shown in Figure 2; also shown is the corresponding relationship for the linear reservoir (equation 9 with $S_{\epsilon\epsilon} = C_{H\epsilon} = 0$)

$$S_{hh}(\omega)/S_{HH}(\omega) = [1 + (\Omega/\beta)^2]^{-1} \tag{20}$$

where a has been taken in the form $a = \beta T/L^2$, β being a numerical constant. This form for a is obtained by evaluating the outflow and spatial average water table height from the steady result in (15). In that case, the spatial average being denoted by $[\![\ \ ]\!]$,

$$a = \langle \epsilon\rangle/[\![\langle h\rangle - \langle H\rangle]\!] = 3T/L^2 \qquad \beta = 3$$

A similar analysis for the deterministic solution of the Dupuit equation with falling sinusoidal water table gives $\beta = \pi^2/4$. These two estimates indicate the order of magnitude to be expected for β if the results of the Dupuit aquifer analysis (19) and the linear reservoir analysis (20) are to be equivalent. An explicit relationship for β can be obtained by comparing the low-frequency limit ($\Omega \ll 1$) of (19)

$$S_{hh}/S_{HH} = 1 + [(x/L - 1)^4 - 1]\Omega^2/4 + O(\Omega^4)$$

with (20). The relationship is $\beta^2 = 4/[1 - (x/L - 1)^4]$, which indicates the dependence of β on its relative position in the aquifer. The linear reservoir result (20) is shown in Figure 2 for two locations with corresponding values of β from the above relationship ($x/L = 1$, $\beta = 2$ and $x/L = 1/4$, $\beta = 2.42$). Near the water table divide ($x/L \sim 1$) the linear reservoir result is quite similar to the Dupuit aquifer solution; however, near the stream the linear reservoir does not adequately represent the aquifer response characteristics.

Applications of the result of (19) can be noted with reference to Figure 2. If the stream stage and aquifer water levels at some point are observed, the aquifer parameter α can be determined from a logarithmic plot of S_{hh}/S_{HH} versus ω by using the fact that $\log \Omega = \log \omega + \log (L^2/\alpha)$.

For aquifers affected primarily by fluctuations in accretion the case $H = 0$ ($dZ_H = 0$) is of interest; then (18) reduces to

$$(\omega S)^2 S_{hh}/S_{\epsilon\epsilon} = [FF^* - (F + F^*) + 1] = g(\Omega, x/L) \tag{21}$$

The results are presented graphically in Figure 3. This form illustrates the method of aquifer evaluation that can be developed from the result. A logarithmic plot of $\omega^2 S_{hh}/S_{\epsilon\epsilon}$ versus ω based on observations with known L and x/L is overlaid on the analytical result of Figure 3, and the ordinates of the two plots would be related by

$$\log (\omega S)^2 S_{hh}/S_{\epsilon\epsilon} = \log \omega^2 S_{hh}/S_{\epsilon\epsilon} + 2 \log S$$

whereas the abscissas are related as was noted above. The storage coefficient S and the parameter α could be determined, and the transmissivity $T = \alpha S$ is then known.

This result for accretion is compared with the corresponding linear reservoir result from (12) in the form

$$(S\omega)^2 S_{hh}/S_{\epsilon\epsilon} = (\Omega/\beta)^2/[1 + (\Omega/\beta)^2]$$

The expression for the linear reservoir becomes equivalent to the low-frequency limit of (21) if

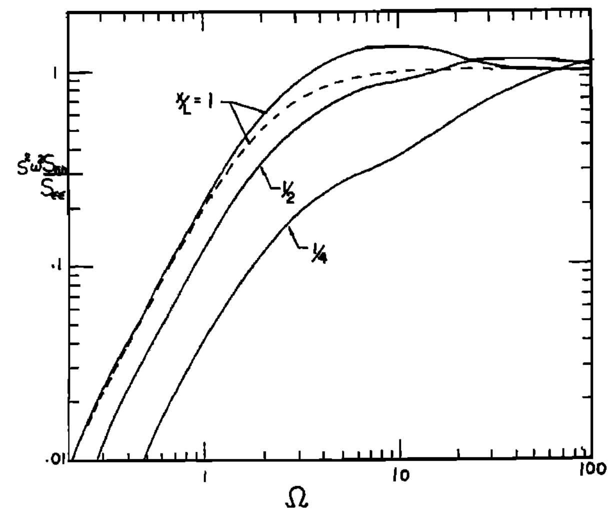

Fig. 3. Spectral response of the Dupuit aquifer affected by accretion, where $\Omega = \omega L^2 S/T$, the solid lines represent the Dupuit aquifer (21), and the dashed line represents the linear reservoir ($x/L = 1, \beta = 2$).

$$\beta^2 = 4/[1 - 2(x/L - 1)^2 + (x/L - 1)^4]$$

The linear reservoir expression for $x/L = 1$, $\beta = 2$ is also shown in Figure 3. The results for the Dupuit aquifer with accretion are also shown in Figure 4 in terms of the dependent variable $\psi \equiv \Omega^{-1}g(\Omega, x/L) = (TS\omega/L^2)(S_{hh}/S_{\epsilon\epsilon})$. This form of presentation provides a direct graphical indication of the importance of different frequencies in the transfer relationship $S_{hh}/S_{\epsilon\epsilon}$. Because ψ is proportional to $\omega S_{hh}/S_{\epsilon\epsilon}$ and

$$\langle h^2 \rangle = \int_{-\infty}^{\infty} S_{hh}\, d\omega = 2 \int_{-\infty}^{\infty} \omega S_{hh}\, d \ln \omega$$

the incremental area under the curves in Figure 4 directly represents the contribution of a given frequency to the total mean square fluctuation when the input accretion is represented by white noise ($S_{\epsilon\epsilon}$ = const). Also shown in Figure 4 are the linear reservoir results with β based on the low-frequency equivalence noted above.

Another method of determining the value of β is to require mean square equivalence of the two analyses. For a white noise input $S_{\epsilon\epsilon}$ = const the linear reservoir spectrum is integrated to give

$$\langle h^2 \rangle = \pi L^2 S_{\epsilon\epsilon}/\alpha\beta S^2$$

and by numerical integration of ψ in the logarithmic domain of Figure 4, numerical values of $\langle h^2 \rangle \alpha S^2/L^2 S_{\epsilon\epsilon}$ are found, and β is determined. This procedure is equivalent to requiring equal area under the curves in Figure 4 for a given x/L. The values of β for this mean square equivalence are shown in Table 1 with those based on low-frequency equivalence. The curve for mean square equivalence at $x/L = 1$ ($\beta = 1.70$) is also shown in Figure 4.

The above analysis of the Dupuit aquifer can be expected to provide a reasonable representation of relatively shallow aquifers at locations away from the stream. However, for relatively deep anisotropic aquifers with partially penetrating streams the assumption of predominantly horizontal flow becomes inadequate. An analysis of the dynamic effects of vertical flow will be developed in the next section.

Laplace Aquifer

The flow in a vertical section of a two-dimensional homogeneous anisotropic phreatic aquifer, as depicted in Figure 1c, is described by

$$K_x \frac{\partial^2 \phi}{\partial x^2} + K_z \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (22)$$

where $\phi(x, z, t)$ is the piezometric head and K_x and K_z are the principal components of the hydraulic conductivity tensor whose principal axes coincide with x and z. The linearized phreatic surface boundary condition is

$$n \frac{\partial \phi}{\partial t} + K_z \frac{\partial \phi}{\partial z} = \epsilon \qquad z = \eta(x, t) \cong 0 \qquad (23)$$

where n is the effective porosity. It is consistent with this linearization to apply the condition at $z = 0$, as was done by *Tóth* [1962], *Dagan* [1964], and *Hunt* [1971] in related deterministic problems. The shape of the phreatic surface $\eta(x, t)$ is related to the piezometric head through the constant pressure condition $\phi(x, \eta, t) = \eta(x, t)$, and under this linearization it is consistent to apply this condition at $z = 0$, i.e.,

$$\phi(x, 0, t) = \eta(x, t) \qquad (24)$$

The conditions of no flux through impervious boundaries are

$$z = -h_0 \qquad \partial\phi/\partial z = 0 \qquad (25)$$

$$x = L \qquad \partial\phi/\partial x = 0 \qquad (26)$$

In addition, at the stream of width $2w$ we require that

$$\phi = \eta = 0 \qquad \text{at} \quad z = 0 \qquad x = w \qquad (27)$$

which implies that any seepage face at the stream is negligible.

The mean flow condition is found by taking an ensemble average of (22)–(27). Explicit solutions [*Dagan*, 1964, equation 14] have been given for the steady mean flow in the case $h_0 \to \infty$.

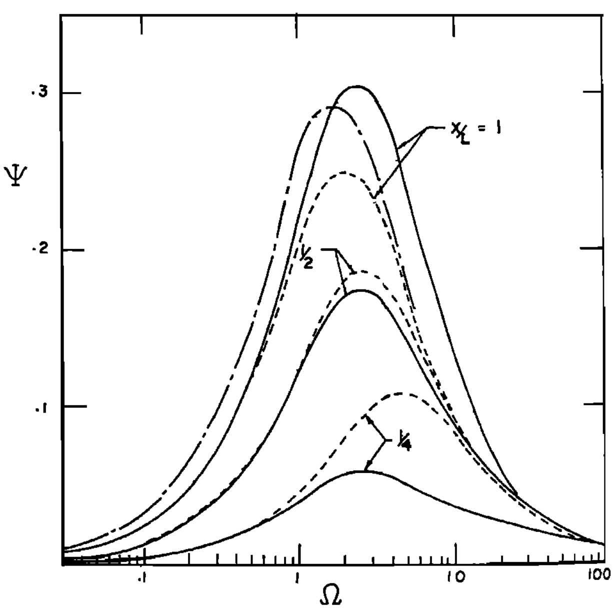

Fig. 4. Dupuit aquifer spectral response with accretion, where $\psi = (TS\omega/L^2)(S_{hh}/S_{\epsilon\epsilon})$, $\Omega = \omega L^2 S/T$, the solid lines represent the Dupuit aquifer, the dashed lines represent the linear reservoir with low-frequency equivalence, and the dash dot line represents the linear reservoir with mean square equivalence ($x/L = 1$, $\beta = 1.70$).

TABLE 1. Comparison of Values of the Linear Reservoir Parameter β

Relative Location x/L	β Based on Low-Frequency Equivalence	β Based on Mean Square Equivalence
1/4	4.57	7.42
1/2	2.67	2.83
3/4	2.13	1.89
1	2.00	1.70

However, for a stationary random process the fluctuations of ϕ, η, and ϵ are also governed by the linear system (22)–(27), and the mean flow is not explicitly involved.

The functions ϕ, η, and ϵ now denoting fluctuations with zero mean, the representations

$$\phi = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_\phi(\omega, x, z) \tag{28}$$

$$\eta = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_\eta(\omega, x) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_\phi(\omega, x, 0) \tag{29}$$

are introduced along with

$$\epsilon(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ_\epsilon(\omega) - \delta(x)\int_{-\infty}^{\infty} e^{i\omega t}\, dZ_q(\omega) \tag{30}$$

where $\delta(x)$ is the Dirac delta function. The last term in (30) represents outflow to the stream that is approximated in the form of a point sink. Note that this representation of the stream does not introduce the shape of the stream cross section explicitly. The shape of the stream is given implicitly by the condition $\phi = 0$. Because equipotentials near a sink are circular, the stream boundary will be practically circular in the transformed (y, z) system or elliptical in (x, z) when $w \ll L$.

With representations (28), (29), and (30) and the notation $\Phi \equiv dZ_\phi(\omega, y, z)$, $N \equiv dZ_\eta(\omega)$, $c \equiv dZ_\epsilon(\omega)$, $q \equiv dZ_q(\omega)$, $y = (K_z/K_x)^{1/2}x$, and $l = (K_z/K_x)^{1/2}L$, the system (22)–(27) becomes

$$\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{31}$$

$$i\omega n\Phi + K_z\, \partial\Phi/\partial z = c - \delta(y)q \qquad z = 0 \tag{32}$$

$$\Phi = N \qquad z = 0 \tag{33}$$

$$\partial\Phi/\partial z = 0 \qquad z = -h_0 \tag{34}$$

$$\partial\Phi/\partial y = 0 \qquad y = l \tag{35}$$

$$\Phi = 0 \qquad z = 0 \qquad y = w' = w(K_z/K_x)^{1/2} \tag{36}$$

With symmetry about $y = x = 0$ the solution of the system (31)–(36) will be of the form

$$\Phi = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\lambda_m y \cosh\lambda_m(z + h_0) \tag{37}$$

which satisfies (31), (34), and (35) with $\lambda_m = m\pi/l$. With the use of the Fourier series representation of the delta function

$$\delta(y) \sim \frac{b_0}{2} + \sum_{m=1}^{\infty} b_m \cos\lambda_m y \qquad b_m = 1/l \tag{38}$$

(32) requires that

$$a_0 = (2c - q/l)/(i\omega n) \tag{39}$$

$$a_m = -(q/l)(i\omega n \cosh\lambda_m h_0 + K_z\lambda_m \sinh\lambda_m h_0) \qquad m > 0 \tag{40}$$

and from (36)

$$0 = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\lambda_m w' \cosh\lambda_m h_0 \tag{41}$$

which, with (39), yields

$$a_0 = \frac{4bl\sigma/K_z}{1 + i2\xi\sigma} \qquad \sigma = \sum_{m=1}^{\infty} \cos\lambda_m w'(m\pi \tanh\lambda_m h_0 + i\xi)^{-1} \qquad \xi = nl\omega/K_z \tag{42}$$

The complete solution for Φ then becomes, from (39), (40), and (42),

$$dZ_\phi \equiv \Phi = \frac{2lb}{(1 + i2\xi\sigma)K_z} \cdot\left(\sigma - \sum_{m=1}^{\infty} \frac{\cosh\lambda_m(z + h_0)\cos\lambda_m y}{m\pi \sinh\lambda_m h_0 + i\xi \cosh\lambda_m h_0}\right) \tag{43}$$

The generalized Fourier amplitudes of η are found from (43) with $z = 0$, $c = dZ_\epsilon$ and (33) as

$$dZ_\eta(\omega, y) = dZ_\epsilon(\omega)\frac{2l/K_z}{1 + i2\xi\sigma}\sum_{m=1}^{\infty}\frac{\cos\lambda_m w' - \cos\lambda_m y}{m\pi \tanh\lambda_m h_0 + i\xi}$$

Thus from (3) the spectral density functions are related by

$$\frac{S_{\eta\eta}K_zK_z}{S_{\epsilon\epsilon}4L^2} = G(\Omega, x/L, h_0/l, w/L)$$
$$= r^{-1}\sum_{m=1}^{\infty} A_m \sum_{m=1}^{\infty} A_m^* = r^{-1}\left[\left(\sum_{m=1}^{\infty} u_m\right)^2 + \left(\sum_{m=1}^{\infty} v_m\right)^2\right] \tag{44}$$
$$r = 1 - 4\xi\sigma_i + 4\xi^2(\sigma_r^2 + \sigma_i^2)$$
$$\sigma = \sigma_r + i\sigma_i \qquad \xi = \Omega h_0/l \qquad \Omega = nL^2\omega/K_x h_0$$
$$A_m = u_m + iv_m = \frac{\cos\lambda_m w' - \cos\lambda_m y}{m\pi \tanh\lambda_m h_0 + i\xi}$$

The variable Ω is the equivalent of the dimensionless frequency variable used for the Dupuit aquifer when $n = S$ and $K_x h_0 = T$. The sums in (44) were evaluated by digital computer; the series are convergent, but several hundred terms may be required to obtain accuracy of a few percent, especially when w/L is very small. Some typical results of the numerical evaluation are shown in Figures 5 and 6. In Figure 5 the dependent variable $\psi = 4\Omega(h_0/l)^2G$ is used because it is equivalent to the dependent variable for the Dupuit aquifer in Figure 4 when $T = K_x h_0$ and $S = n$. Figure 5 illustrates the effects of the additional parameters for the Laplace aquifer (w/L, h_0/l) in comparison with the Dupuit result. The response curves for the Laplace aquifer are seen to have shapes quite similar to those of the curves for the Dupuit aquifer, but the amplitudes can differ significantly depending on the parameters. The trends indicated in Figure 5 are intuitively reasonable; an increase in the stream width w/L reduces the amplitude of the fluctuations, and an increase in the relative thickness of the aquifer h_0/l increases the

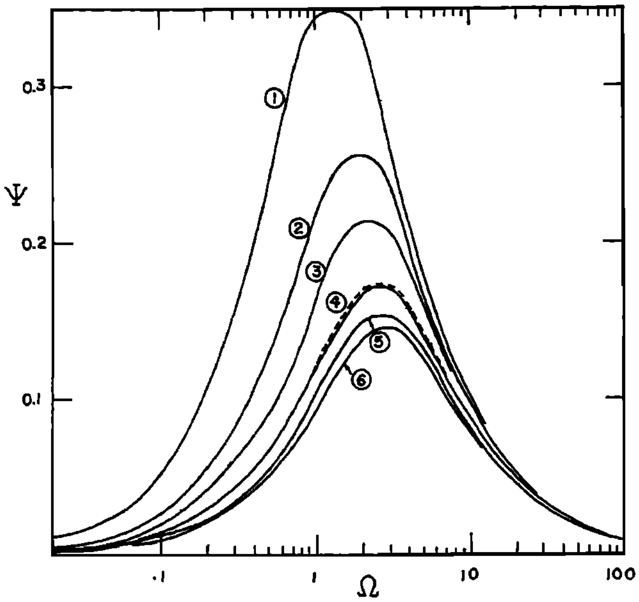

Fig. 5. Laplace aquifer spectral response at $x/L = 1/2$, where $\psi = 4\Omega(h_0/l)^2G$ and $\Omega = \omega L^2 n/(K_x h_0)$: curve 1, $h_0/l = 0.5$, $w/L = 0.05$; curve 2, $h_0/l = 0.1$, $w/L = 0.003$; curve 3, $h_0/l = 0.1$, $w/L = 0.01$; curve 4, $h_0/l = 0.02$ and 0.002, $w/L = 0.01$; curve 5, $h_0/l = 0.1$, $w/L = 0.05$; and curve 6, $h_0/l = 0.02$ and 0.002, $w/L = 0.05$. The dashed line represents the Dupuit aquifer ($x/L = 1/2$).

amplitude. Figure 6 shows, in a logarithmic form that is applicable for parameter determinations, the effects of relative horizontal position in the aquifer; the trends are similar to those for the Dupuit aquifer.

The rather significant differences between the Laplace and the Dupuit aquifers for some values of the parameters may have some bearing on the applicability of the Dupuit approximation. However, no precise conclusions can be drawn because the boundary conditions for the Laplace aquifer problem, with a partially penetrating stream of finite width, and the Dupuit aquifer, with a fully penetrating stream, are not exactly equivalent. On the basis of previous work on deterministic problems we would expect the Laplace and Dupuit aquifers to become equivalent when $h_0/l \ll 1$. From numerical solutions of nonlinear Dupuit and Laplace problems, *Verma and Brutsaert* [1971] found agreement when the length of the aquifer was 4 times its initial saturated thickness, even with complete drawdown at the stream.

A careful analysis of the solution for the Laplace aquifer (44) in the limit $h_0/l \to 0$ shows that there is a finite limit for the function ψ depending on the parameter w/L. The results in Figure 5 reflect this feature for the cases $w/L = 0.01$ and 0.05. The function ψ becomes independent of h_0/l when $h_0/l <$ 0.02. It is seen that the Dupuit aquifer is equivalent to the Laplace aquifer with $w/L = 0.01$ when h_0/l is very small.

In general, the results for the Laplace aquifer show that nonhydrostatic effects can be quite important in a natural system when the anisotropy K_x/K_z is large and the relative stream width w/L is very small.

CONCLUDING REMARKS

This series of analytical models, which has been developed to describe the spectral response characteristics of phreatic aquifers, demonstrates several common features that should characterize temporal variability in groundwater systems. The general shape of the response curves for the three models is similar. Through the use of appropriate values of a single parameter the linear reservoir model can approximate the Dupuit aquifer behavior either in the mean square sense or at low frequency. The Dupuit aquifer model shows behavior that is similar to that of the more complete Laplace aquifer. However, the behavior of the Laplace aquifer is much more complex, being quite sensitive to the relative stream width w/L and the aspect ratio h_0/l; simple parametric equivalence between the Dupuit and the Laplace aquifers is not found.

The three models provide an analytical basis for interpretation of spectral data from phreatic aquifers. One of the main applications of the results will be the determination of aquifer parameters via spectral analysis. Some parameter determination procedures have been suggested, but the utility of these methods remains to be established through detailed analyses of field data.

Some initial testing of these procedures using monthly groundwater levels and precipitation for a local site shows a reasonable correlation with the linear reservoir result of (12) when a linear relationship between precipitation and recharge is assumed. The simple linear reservoir model may be satisfactory for initial approximate analysis of many field situations.

These analyses are based on linearized forms of the governing aquifer equations, and hence some limitations are implied. In the case of the Dupuit aquifer some spectral distortion can be expected when the nonlinear terms become important. Numerical solution of related deterministic problems indicate that nonlinear effects are small when the relative change in the thickness of the saturated zone is small. For a base flow recession problem, *Singh* [1969] found agreement with the analytical solution of the linear equation for a relative change in thickness up to 2. In a flood wave interaction problem, *Hornberger et al.* [1970] indicate good comparisons with the linear solution for relative changes of thickness up to 1.5. An explicit evaluation of the nonlinear effect in the stochastic case is currently being developed by using numerical

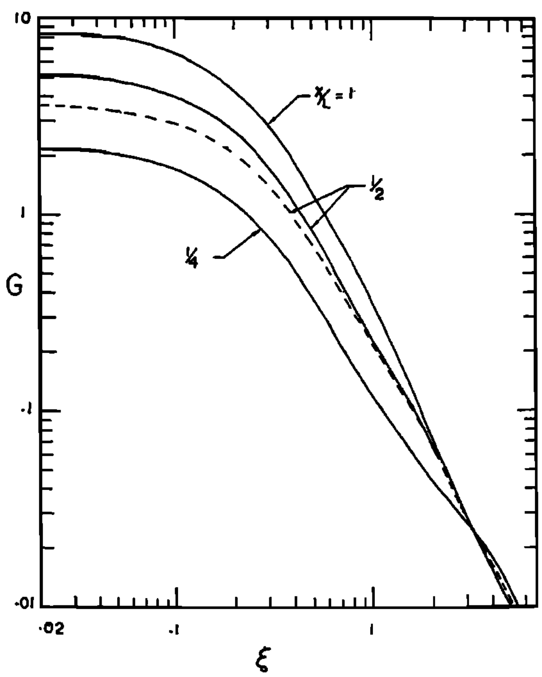

Fig. 6. Laplace aquifer spectral response at several locations, where the solid lines represent the Laplace aquifer, the dashed line represents the Dupuit aquifer, $G = S_{\eta\eta}K_xK_z/(4S_{\epsilon\epsilon}L^2)$, and $\xi = nl\omega/K_z$.

solutions of the nonlinear Dupuit model with random boundary conditions.

There are other spectral characteristics of aquifers that although they are not explicitly evaluated here, can be obtained routinely from the analytical results. Cross spectra between two observation wells in the same aquifer may be useful for parameter determinations that are not dependent on accretion. Also the spectral relationship between the outflow from the aquifer (stream base flow) and the water level in the aquifer is implied. An important nontrivial extension of this work is the inclusion of spatial variability of accretion and of medium properties. Such analyses can form the basis for the design of observation networks.

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REFERENCES

Bear, J., *Dynamics of Fluids in Porous Media,* pp. 361–437, Elsevier, New York, 1972.

Bendat, J. S., and A. G. Piersol, *Random Data: Analysis and Measurement Procedures,* pp. 136–141, Interscience, New York, 1971.

Cooper, H. H., Jr., and M. I. Rorabaugh, Changes in ground-water movement and bank storage caused by flood waves in surface streams, *U.S. Geol. Surv. Prof. Pap. 475-B,* B192–B195, 1963.

Dagan, G., Linearized solution of unsteady deep flow toward an array of horizontal drains, *J. Geophys. Res., 69*(16), 3361–3369, 1964.

Dooge, J. C. I., The routing of groundwater recharge through typical elements of linear storage, *Int. Ass. Sci. Hydrol. Helsinki Publ. 52,* 286-300, 1960.

Eriksson, E., Groundwater time-series; An exercise in stochastic hydrology, *J. Nord. Hydrol., 1*(3), 181–205, 1970.

Glover, R. E., Seasonal fluctuations of return flows, *Proc. Annu. Amer. Water Resour. Conf. 4th,* 272–278, 1967.

Hall, F. R., and A. F. Moench, Application of the convolution equation to stream-aquifer relationships, *Water Resour. Res., 8*(2), 487–493, 1972.

Hornberger, G. M., J. Ebert, and I. Remson, Numerical solution of the Boussinesq equation for aquifer–stream interaction, *Water Resour. Res., 6*(2), 601–608, 1970.

Hunt, B. W., Vertical recharge of unconfined aquifer, *J. Hydraul. Div. Amer. Soc. Civil Eng., 97*(HY7), 1017–1030, 1971.

Jacob, C. E., Correlation of ground-water levels and precipitation on Long Island, New York, *Eos Trans. AGU, 24,* 564–573, 1943.

Kraijenhoff van de Leur, D. A., A study of non-steady groundwater flow with special reference to a reservoir-coefficient, *Ingenieur,* no. 19, 87–94, 1958.

Papoulis, A., *Probability, Random Variables and Stochastic Processes,* pp. 453–473, McGraw-Hill, New York, 1965.

Pinder, G. F., J. D. Bredehoeft, and H. H. Cooper, Jr., Determination of aquifer diffusivity from aquifer response to fluctuations in river stage, *Water Resour. Res., 5*(4), 850–855, 1969.

Singh, K. P., Theoretical baseflow curves, *J. Hydraul. Div. Amer. Soc. Civil Eng., 95*(HY6), 2029–2048, 1969.

Tison, G., Jr., Fluctuations of ground-water levels, *Advan. Geophys., 2,* 303–325, 1965.

Tóth, J., A theory of groundwater motion in small drainage basins in central Alberta, Canada, *J. Geophys. Res., 67*(11), 4375–4387, 1962.

Venetis, C., Estimating infiltration and/or the parameters of unconfined aquifers from groundwater level observations, *J. Hydrol., 12,* 161–169, 1971.

Verma, R. D., and W. Brutsaert, Unsteady free surface ground water seepage, *J. Hydraul. Div. Amer. Soc. Civil Eng., 97*(HY8), 1213–1229, 1971.

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