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ANALYTICAL SOLUTIONS FOR PERIODIC WELL RECHARGE IN RECTANGULAR AQUIFERS WITH THIRD-KIND BOUNDARY CONDITIONS

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ABSTRACT

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Analytical solutions for groundwater flow in rectangular aquifers are presented in the case of single-well recharge. The problem concerns the impact of a seasonal (periodic) recharge scheme of variable duration on aquifers with various boundary conditions of which at least one is of the third kind. Results can be obtained for any combination of boundary conditions and can be used in a preliminary assessment of the groundwater response to artificial recharge schemes.

1. INTRODUCTION

Analytical solutions of groundwater flow in confined and unconfined homogeneous and isotropic aquifers are especially useful in understanding the physical processes on a local scale, as well as in checking numerical solutions for large-scale problems. The most commonly treated case in this particular field concerns the single-well pumping at a constant rate from an aquifer of either infinite, semi-infinite or finite extent. In the last case Chan et al. (1976) obtained analytical solutions for the drawdown in rectangular homogeneous aquifers by employing the technique of Mullineux et al. (1970).

The development of water-resource management techniques, with an overall task to balance nonuniformities of water distribution in both space and time and to augment existing water supplies, introduced the reverse functioning of wells, i.e. recharging upon the aquifer, or artificial recharge. Among others some important parameters in the application of this technique are the total duration of recharge, the spacing of recharge wells and the groundwater response to different recharge regimes. Analytical solutions for single recharge sites were obtained for one-dimensional problems with seasonal (periodic) recharge regimes of either fixed duration

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(Nutbrown, 1976) or varying duration (Latinopoulos, 1981), and for two-dimensional problems (recharge wells in rectangular aquifers) and varying recharge regimes (Latinopoulos, 1982).

In all the above problems the boundary conditions of the idealized flow field were either of the first type or Dirichlet boundary conditions, or of the second type or Neumann boundary conditions. It is true that in most of the cases the geological conditions and the aquifer properties and boundaries are such that a prescribed head boundary condition (Dirichlet) or a prescribed flux one (Neumann) can sufficiently describe the physical phenomenon. However, in some specific situations a third type or Cauchy boundary condition is needed. This is the case when the aquifer is separated from another formation by a relatively thin semipervious layer with no storage, e.g. in leaky aquifers. In addition horizontal Cauchy boundary conditions appear in perched aquifers (Bear, 1979, p.26). Finally, Cauchy boundary conditions are encountered in confined or unconfined aquifers in contact with a river with a clogged bed (Bear, 1979, p.118). So far analytical solutions for unsteady flow in rectangular aquifers with third-kind boundary conditions were obtained only by Corapcioglu et al. (1983) who studied the drawdown variation due to single-well pumping at a constant rate.

The purpose of this study is to obtain analytical solutions for the hydraulic head in rectangular aquifers with third-kind boundary conditions in the case of a single-well recharging seasonally under any form of continuously varying recharge regime. The solutions can be used for a preliminary assessment of the groundwater response to artificial recharge schemes upon either perched aquifers or aquifers in contact with clogged river beds.

2. STATEMENT OF THE PROBLEM

The main advantages from the application of artificial recharge upon a local aquifer are storage augmentation and low-flow supplement. In order that such an application could be a successful one, the design of the recharge scheme must be such that deficiencies during dry seasons could be beneficially supplemented from surplus water which would be artificially added during the wet period. As the natural groundwater recharge varies periodically (seasonally), with dry and wet seasons being repeated almost consecutively year after year, the most rational procedure to follow is to operate any artificial recharge scheme under an annual periodicity. In this sense the following approach is adopted.

The groundwater flow domain is assumed to be a rectangular confined aquifer with dimensions a and b in a system of Cartesian coordinates as shown in Fig. 1a. The aquifer is also assumed to be non-leaky, homogeneous with respect to storativity and homogeneous and isotropic with respect to transmissivity. A recharge well penetrating fully the aquifer is located at the point (ξ, η), where $0 < \xi < a$ and $0 < \eta < b$. It is assumed that water is

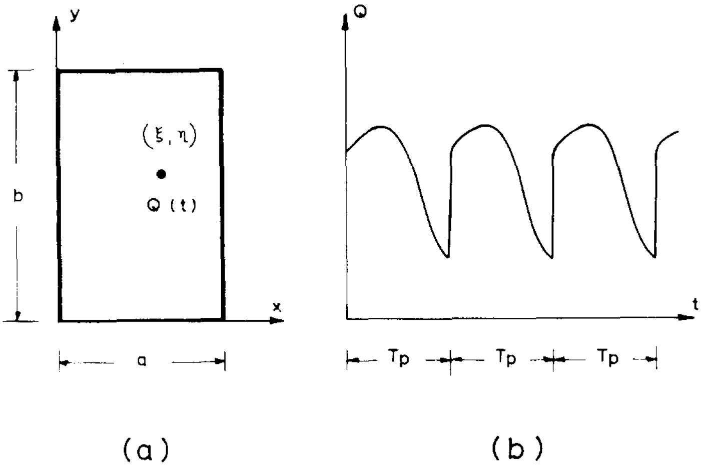

Fig. 1. Statement of the problem: (A) idealized aquifer; and (B) periodic recharge regime.

recharged through this well at a time-dependent rate $Q(t)$ which is periodic, with a period $T_p = 1$ yr. (Fig. 1b), so that:

$$Q(t) = Q(t + mT_p), \qquad m = 1, 2, \ldots \tag{1}$$

Assuming horizontal flow, the governing partial differential equation for the two-dimensional groundwater flow problem under the above assumptions is:

$$S\frac{\partial h(x,y,t)}{\partial t} = T\left[\frac{\partial^2 h(x,y,t)}{\partial x^2} + \frac{\partial^2 h(x,y,t)}{\partial y^2}\right] + q(x,y,t) \tag{2}$$

where h is the hydraulic head above datum level; S and T denote the storativity and transmissivity of the aquifer, respectively; and q is a source function. For the case of a recharge well at the point (ξ, η), this function reads:

$$q(x,y,t) = Q(t)\,\delta(x-\xi)\delta(y-\eta) \tag{3}$$

where δ is the Dirac delta function.

The three types of boundary conditions associated with the problem and noted earlier can be expressed by the relations:

$$\text{Type } \mathit{1} \text{ (Dirichlet):} \quad h = 0 \tag{4}$$

$$\text{Type } \mathit{2} \text{ (Neumann):} \quad \partial h/\partial n = 0 \tag{5}$$

$$\text{Type } \mathit{3} \text{ (Cauchy):} \quad \pm\,\partial h/\partial n + C_i h = 0, \qquad i = 1, 4 \tag{6}$$

where n is the direction normal to the ith boundary; and C_i is a parameter that equals the reciprocal of the product of the hydraulic conductivity times the resistance of the semipervious layer. The sign of ($\partial h/\partial n$) depends on both the boundary position and the direction of the flow. Note that eqs. 4—6

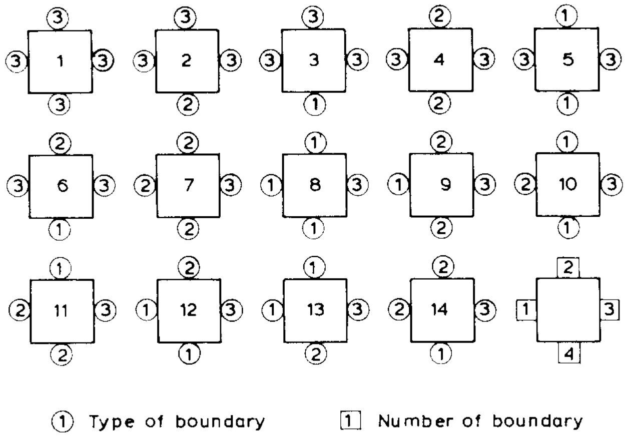

Fig. 2. The types of aquifers considered.

can be implemented in the mathematical model with a non-zero right-hand side part but this is omitted here for the sake of simplicity in the final solutions.

The solutions obtained in this study concern with any of the 14 types of aquifers shown in Fig. 2. These cases comprise every possible combination of the three boundary conditions provided that at least one boundary condition is of the third kind. Solutions for aquifers with boundary conditions of the first and second kind only and a constant recharge rate have been presented elsewhere (Latinopoulos, 1982) and they are omitted here.

3. ANALYTICAL SOLUTIONS

The problem of a single-well recharging periodically over a homogeneous rectangular confined aquifer is defined through eqs. 1–6. Analytical solutions of this problem, and for any combination of boundary condition types, are possible by using in succession the methods of Laplace and double-integral (Fourier) transforms.

In the following we indicate the Laplace transform of $h(x,y,t)$ by $\bar{h}(x,y,r)$ and the double-integral transform of $\bar{h}(x,y,r)$ by $\bar{\bar{h}}(\alpha_m, \beta_n, r)$, where α_m and β_n are the corresponding eigenvalues which depend upon the types of boundary conditions.

The Laplace transform of eq. 2 with respect to time t, taking into account eq. 3, leads to:

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2} - r\frac{S}{T}\bar{h} + \frac{S}{T}h(x,y,0) = -\frac{\bar{Q}}{T}\,\delta(x-\xi)\delta(y-\eta) \tag{7}$$

where $\bar{Q}(r)$ is the Laplace transform of the periodic function $Q(t)$. We next assume that the initial heads are zero, i.e. $h(x,y,0) = 0$. If we let $u^2 = rS/T$, eq. 7 reads now:

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2} - u^2 \bar{h} = -\frac{\bar{Q}}{T}\delta(x-\xi)\delta(y-\eta) \tag{8}$$

Due to the specific boundary conditions of the problem (eqs. 4—6), eq. 8 can be solved by applying the double-integral transform (Ozisik, 1968, ch. 2). The transforms of the left-hand side terms of eq. 8 are:

$$\frac{\partial^2 \bar{\bar{h}}}{\partial x^2} + \frac{\partial^2 \bar{\bar{h}}}{\partial y^2} = \int_{x=0}^{a} \int_{y=0}^{b} K(\alpha_m, x)K(\beta_n, y)\left(\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2}\right) \mathrm{d}y\,\mathrm{d}x = -(\alpha_m^2 + \beta_n^2)\bar{\bar{h}} \tag{9}$$

$$u^2 \bar{\bar{h}} = u^2 \int_{x=0}^{a} \int_{y=0}^{b} K(\alpha_m, x)K(\beta_n, y)\bar{h}\,\mathrm{d}y\,\mathrm{d}x \tag{10}$$

while using the property of the Dirac delta function the transform of the right-hand side term of eq. 8 is:

$$\frac{\bar{Q}}{T}\int_{x=0}^{a} \int_{y=0}^{b} \delta(x-\xi)\delta(y-\eta)K(\alpha_m, x)K(\beta_n, y)\,\mathrm{d}y\,\mathrm{d}x = \frac{\bar{Q}}{T}K(\alpha_m, \xi)K(\beta_n, \eta) \tag{11}$$

In eqs. 9—11, α_m and β_n are the eigenvalues in x- and y-directions, respectively; and $K(\alpha_m, x)$ and $K(\beta_n, y)$ are the kernels of the transform. The eigenvalues α_m and β_n can be computed as consecutive positive roots of a transcendental equation, the type of which depends on the combination of the various boundary conditions, and so does the type of the kernels. Ozisik (1968, pp. 50—51) presented a table for the above parameters and functions for one-dimensional problems. For our case of double-integral transforms (two space variables), a similar table has been prepared for all the types of aquifers considered in this paper and is presented as Table I.

Finally, applying the double-integral transformation to eq. 8 and substituting for the values given in eqs. 9—11 we obtain:

$$(\alpha_m^2 + \beta_n^2)\bar{\bar{h}} + u^2 \bar{\bar{h}} = \frac{\bar{Q}}{T}K(\alpha_m, \xi)K(\beta_n, \eta) \tag{12}$$

Solving eq. 12 for $\bar{\bar{h}}$ leads to:

$$\bar{\bar{h}} = \frac{(\bar{Q}/T)K(\alpha_m, \xi)K(\beta_n, \eta)}{\alpha_m^2 + \beta_n^2 + u^2} \tag{13}$$

which inverts to (Ozisik, 1968):

$$\bar{h} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K(\alpha_m, x)K(\beta_n, y)\bar{\bar{h}}$$

or

$$\bar{h} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\bar{Q}/T)K(\alpha_m, x)K(\beta_n, y)K(\alpha_m, \xi)K(\beta_n, \eta)}{\alpha_m^2 + \beta_n^2 + u^2} \tag{14}$$

TABLE I

Kernels and eigenvalues for use in the solutions of various aquifer types

Type of aquifer	Kernel $K(\alpha_m, x)$	Transcendental equation for α_m's	Kernel $K(\beta_n, y)$	Transcendental equation for β_n's
1	F_1^{*1}	$\tan \alpha a = \dfrac{\alpha(C_1 + C_3)}{\alpha^2 - C_1 C_3}$	F_3^{*1}	$\tan \beta b = \dfrac{\beta(C_2 + C_4)}{\beta^2 - C_2 C_4}$
2	F_1	$\tan \alpha a = \dfrac{\alpha(C_1 + C_3)}{\alpha^2 - C_1 C_3}$	$F_4 \cos \beta_n y^{(*1)}$	$\beta \tan \beta b = C_2$
3	F_1	$\tan \alpha a = \dfrac{\alpha(C_1 + C_3)}{\alpha^2 - C_1 C_3}$	$F_4 \sin \beta_n y$	$\beta \cot \beta b = - C_2$
4	F_1	$\tan \alpha a = \dfrac{\alpha(C_1 + C_3)}{\alpha^2 - C_1 C_3}$	$\sqrt{\dfrac{2}{b}} \cos \beta_n y^{(*2)}$	$\sin \beta b = 0$
5	F_1	$\tan \alpha a = \dfrac{\alpha(C_1 + C_3)}{\alpha^2 - C_1 C_3}$	$\sqrt{\dfrac{2}{b}} \sin \beta_n y$	$\sin \beta b = 0$
6	F_1	$\tan \alpha a = \dfrac{\alpha(C_1 + C_3)}{\alpha^2 - C_1 C_3}$	$\sqrt{\dfrac{2}{b}} \sin \beta_n y$	$\cos \beta b = 0$
7	$F_2 \cos \alpha_m x^{(*1)}$	$\alpha \tan \alpha a = C_3$	$\sqrt{\dfrac{2}{b}} \cos \beta_n y^{(*2)}$	$\sin \beta b = 0$
8	$F_2 \sin \alpha_m x$	$\alpha \cot \alpha a = - C_3$	$\sqrt{\dfrac{2}{b}} \sin \beta_n y$	$\sin \beta b = 0$
9	$F_2 \sin \alpha_m x$	$\alpha \cot \alpha a = - C_3$	$\sqrt{\dfrac{2}{b}} \cos \beta_n y^{(*2)}$	$\sin \beta b = 0$
10	$F_2 \cos \alpha_m x$	$\alpha \tan \alpha a = C_3$	$\sqrt{\dfrac{2}{b}} \sin \beta_n y$	$\sin \beta b = 0$
11	$F_2 \cos \alpha_m x$	$\alpha \tan \alpha a = C_3$	$\sqrt{\dfrac{2}{b}} \cos \beta_n y$	$\cos \beta b = 0$
12	$F_2 \sin \alpha_m x$	$\alpha \cot \alpha a = - C_3$	$\sqrt{\dfrac{2}{b}} \sin \beta_n y$	$\cos \beta b = 0$
13	$F_2 \sin \alpha_m x$	$\alpha \cot \alpha a = - C_3$	$\sqrt{\dfrac{2}{b}} \cos \beta_n y$	$\cos \beta b = 0$
14	$F_2 \cos \alpha_m x$	$\alpha \tan \alpha a = C_3$	$\sqrt{\dfrac{2}{b}} \sin \beta_n y$	$\cos \beta b = 0$

*1 These functions are:

$$F_1 = \sqrt{2}\, \frac{\alpha_m \cos \alpha_m x + C_1 \sin \alpha_m x}{[(\alpha_m^2 + C_1^2)\{a + C_3/(\alpha_m^2 + C_3^2)\} + C_1]^{1/2}}$$

TABLE I (*continued*)

$$F_2 = \sqrt{2}\left[\frac{\alpha_m^2 + C_3^2}{a(\alpha_m^2 + C_3^2) + C_3}\right]^{1/2}$$

$$F_3 = \sqrt{2}\ \frac{\beta_n \cos\beta_n y + C_4 \sin\beta_n y}{[(\beta_n^2 + C_4^2)\{b + C_2/(\beta_n^2 + C_2^2)\} + C_4]^{1/2}}$$

$$F_4 = \sqrt{2}\left[\frac{\beta_n^2 + C_2^2}{b(\beta_n^2 + C_2^2) + C_2}\right]^{1/2}$$

*2 For this particular case, replace $(2/b)^{1/2}$ by $(1/b)^{1/2}$ if β is zero.

Note that the above successive transformations do not affect the form of the homogeneous boundary conditions (4)—(6).

In order to obtain the final solution we apply the inversion theorem and the Cauchy theorem of residues. Assuming for the moment that the function $\bar{Q}(r)$ in eq. 14 is known one can use the inversion theorem to compute h. Thus the solution can be written as:

$$h(x,y,t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{rt}\bar{h}(x,y,r)\mathrm{d}r \tag{15}$$

where γ is a large quantity. Next we use the theorem of residues which states that the value of the above integral (eq. 15) is equal to the product of $2\pi\mathrm{i}$ and the sum of the integral residues at its poles (singularities).

The poles of the function $\bar{h}(x,y,r)$ given in eq. 14 and appearing in the integral of eq. 15 are first a single pole at $u^2 = -(\alpha_m^2 + \beta_n^2)$ and a number of other poles which are due to the function $\bar{Q}(r)$. The final solution consists of the sum of all these poles and it can be split in two parts: one of them concerns the sum of the residues at the singularities of the function $\bar{Q}(r)$ (which depend solely upon the type of this function), and this sum provides the steady-state periodic solution. The second part of the sum concerns the pole $u^2 = -(\alpha_m^2 + \beta_n^2)$ and this part provides the transient term. Having assumed that $h = 0$ at $t = 0$, these parts become equal and therefore the final solution for the steady periodic flow, when the transient effects have vanished, can be described by any of the two. For the sake of convenience this final solution can be written in terms of the above-mentioned single pole and in its general form reads:

$$h(x,y,t) = \frac{1}{S}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R(\alpha_m,\beta_n,t)K(\alpha_m,x)K(\beta_n,y)K(\alpha_m,\xi)K(\beta_n,\eta) \tag{16}$$

Eq. 16 has been derived after manipulations related to the calculation of the relevant residue. Also the new function R depends both on the form of the periodic variation of $Q(t)$ and on the eigenvalues α_m and β_n.

The major advantage of the general form of solution given in eq. 16 is that it applies to any type of periodically varying recharge regime. The applicability of this solution is shown below where examples for two special cases of recharge are examined.

4. APPLICATION OF THE GENERAL SOLUTION

In the following paragraphs two examples are given for the calculation of the function R appearing in eq. 16. The periodic recharge function Q for the two cases is given in Fig. 3.

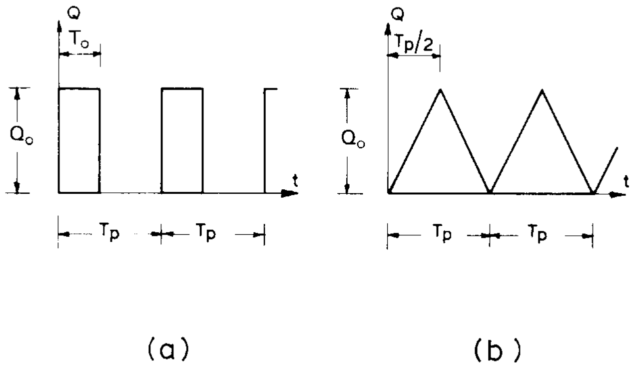

Fig. 3. Recharge regimes for the two study cases: (A) case *1*: and (B) case *2*.

4.1. Case 1

As shown in Fig. 3a this case concerns a constant recharge at a rate equal to Q_0 during a time interval T_0 of an annual period T_p. This regime can be described for a typical year as:

$$\begin{aligned} Q(t) &= Q_0, \quad \text{for} \quad 0 \leqslant t < T_0 \\ Q(t) &= 0, \quad \text{for} \quad T_0 \leqslant t < T_p \end{aligned} \tag{17}$$

The Laplace transform of the periodic function $Q(t)$ can be evaluated from the relation:

$$\bar{Q}(r) = \frac{1}{1 - e^{-rT_p}} \int_0^{T_p} e^{-rt}\, Q(t)\, dt \tag{18}$$

Due to the specific form of this function (eq. 17) the evaluation of the above transform is performed as proposed by Carslaw and Jaeger (1959, pp. 130–131). Thus the integral of eq. 18 is evaluated separately for the two time intervals defined in eq. 17.

4.1.1. First time interval $(0 \leqslant t < T_0)$
This is the recharge period. Letting $t = 0$ at an arbitrary point as shown in Fig. 4a the integration in eq. 18 is split in three parts where:

$$\left.\begin{array}{llr} Q = Q_0, & \text{for} & 0 \leqslant t < T_0 - t' \\ Q = 0, & \text{for} & T_0 - t' \leqslant t < T_p - t' \\ Q = Q_0, & \text{for} & T_p - t' \leqslant t < T_p \end{array}\right| \quad (19)$$

Performing the integration leads to:

$$\bar{Q}(r) = \frac{Q_0}{r(1 - e^{-rT_p})} [1 - e^{-r(T_0 - t')} + e^{-r(T_p - t')} - e^{-rT_p}] \quad (20)$$

By eliminating primes (i.e. time measures from the start of the first time interval), and substituting:

$$w(\alpha_m, \beta_n) = -r = -u^2 \frac{T}{S} = (\alpha_m^2 + \beta_n^2) \frac{T}{S} \quad (21)$$

in eq. 20, where u^2 are the poles in eq. 15 the general function for this specific case reads:

$$R(\alpha_m, \beta_n, t) = \frac{Q_0}{w(e^{wT_p} - 1)} [1 - e^{w(T_0 - t)} + e^{w(T_p - t)} - e^{wT_p}] \quad (22)$$

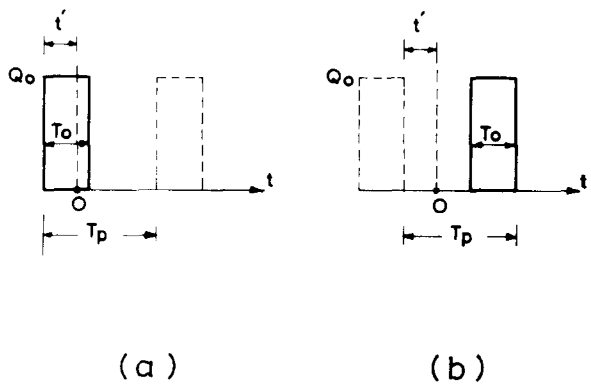

Fig. 4. Derivation of the Laplace transform of the periodic recharge function: case *1*.

4.1.2. Second time interval $(0 \leqslant t < T_p - T_0)$

For this recession period we measure time from its start. The relevant integration in eq. 18 is performed in a quite similar manner with that of the recharge period. Splitting the integration interval in three parts where (see also Fig. 4b):

$$\left.\begin{array}{llr} Q = 0, & \text{for} & 0 \leqslant t < T_p - T_0 - t' \\ Q = Q_0, & \text{for} & T_p - T_0 - t' \leqslant t < T_p - t' \\ Q = 0, & \text{for} & T_p - t' \leqslant t < T_p \end{array}\right| \quad (23)$$

and following the described procedure we have finally:

$$R(\alpha_m, \beta_n, t) = \frac{Q_0}{w(e^{wT_p} - 1)} [e^{w(T_p - T_0 - t)} - e^{w(T_p - t)}] \tag{24}$$

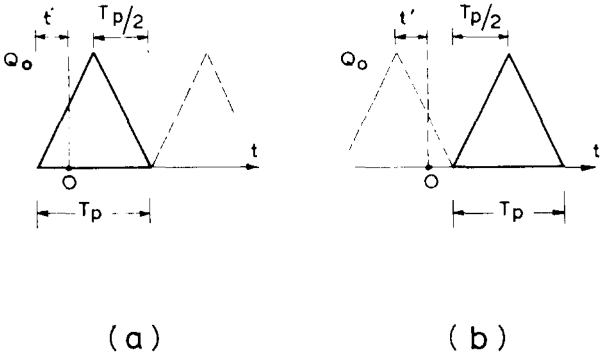

Fig. 5. Derivation of the Laplace transform of the periodic recharge function: case *2*.

4.2. Case 2

The case shown in Fig. 3b concerns a varying recharge rate. The methodology for obtaining the expressions for the function for the two time intervals, i.e.: $0 \leqslant t < T_p/2$ and $T_p/2 \leqslant t < T_p$, is exactly the same as for case *1*. The relevant integrals are performed according to Fig. 5 and the final expressions are:

4.2.1. First time interval ($0 \leqslant t < T_p/2$)

$$R(\alpha_m, \beta_n, t) = \frac{2Q_0}{w^2 T_p (1 - e^{wT_p})} [(1 - wt)(1 - e^{wT_p}) + 2\{e^{w(T_p - t)} - e^{w(T_p/2 - t)}\}] \tag{25}$$

4.2.2. Second time interval ($0 \leqslant t < T_p/2$)

$$R(\alpha_m, \beta_n, t) = \frac{2Q_0}{w^2 T_p (1 - e^{wT_p})} [\{1 + w(T_p/2 - t)\} \times (e^{wT_p} - 1) + 2\{e^{w(T_p/2 - t)} - e^{w(T_p - t)}\}] \tag{26}$$

Numerical results from the above cases are discussed in the following section.

5. NUMERICAL RESULTS AND DISCUSSION

The general analytical solution obtained in eq. 16 can be used to compute the hydraulic head for any type of rectangular aquifer under a periodic recharge regime. From the computational viewpoint the only subject worth of discussion is the convergence of the double series in eq. 16. Due to the time exponentials appearing in the function $R(\alpha_m, \beta_n, t)$ (e.g., eqs. 22 and 24—26) a large number of terms has to be taken into account in the numerical computations. This total number should equal $\sim 10^4$ ($m = 100$, $n = 100$).

The application of the solutions of this groundwater flow schematization can be useful in a preliminary evaluation of the response of groundwater to artificial recharge schemes.

In order to illustrate some basic advantages of these analytical solutions the case of a single recharge well in the centre of an aquifer is considered. The aquifer has three impermeable and one Cauchy-type boundary, i.e. aquifer type *7* in Fig. 2. Exactly the same schematization has been used in a previous study (Tolikas et al., 1983) to solve a real aquifer problem for steady-state conditions, where the analytical solutions were used in a linear programming management model.

The only outlet of the aquifer under study is at the boundary No. *3*, i.e. at $x = a$. Thus the response of groundwater of this particular aquifer can be

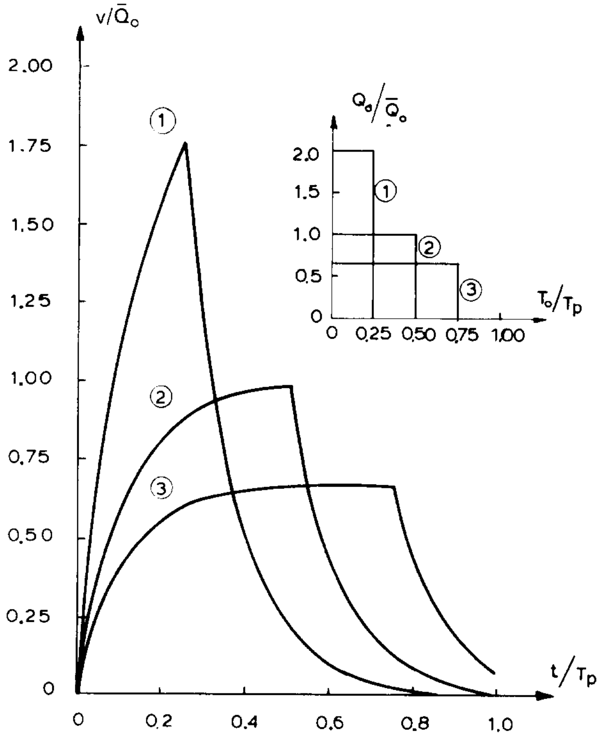

Fig. 6. Dimensionless total outflow at $x = a$ during an annual period for different recharge regimes: case *1* (aquifer type *7*, $T/Sa^2 = 0.1$, $C_3a = 0.25$, $a/b = 0.5$ and recharge at $\xi/a = 0.5$, $\eta/b = 0.5$).

evaluated by computing the variation of the outflow at this boundary. The total outflow from the aquifer can be computed by the relation:

$$V(t) = -T\int_0^b \frac{\partial h}{\partial n}\bigg|_{x=a} \mathrm{d}y \tag{27}$$

if the solution for h given by eq. 16 is introduced in eq. 27. Thus for the particular case of aquifer type 7 the outflow reduces to a single series as:

$$V(t) = \frac{-\sqrt{2b}\,T}{S}\sum_{m=1}^{\infty} R(\alpha_m,\beta_1,t)\alpha_m \sin\alpha_m a K(\alpha_m,\xi)K(\beta_1,\eta)$$
$$\times\left[\frac{\alpha_m^2 + C_3^2}{a(\alpha_m^2 + C_3^2) + C_3}\right]^{1/2} \tag{28}$$

Results from this application are given in dimensionless form in Figs. 6 and 7 for the two cases of recharge considered in the previous paragraphs (Fig. 3a and b). Thus numerical values can be obtained by using the expressions for the function R given in eqs. 22 and 24—26. The total amount of recharged water is assumed equal for the two regimes by assuming $T_0 = T_p/2$ in the constant-rate case (first case: Fig. 3a and eqs. 22 and 24). This total recharge volume equals $\bar{Q}_0 \cdot T_p/2$ where $\bar{Q}_0$ is an average recharge rate.

The effect of the duration of recharge upon the aquifer response is shown in Fig. 6 where results for three different recharge regimes with a constant recharge rate (first case) are presented. A common classification of aquifers, as far as their response is concerned, can be made by considering the

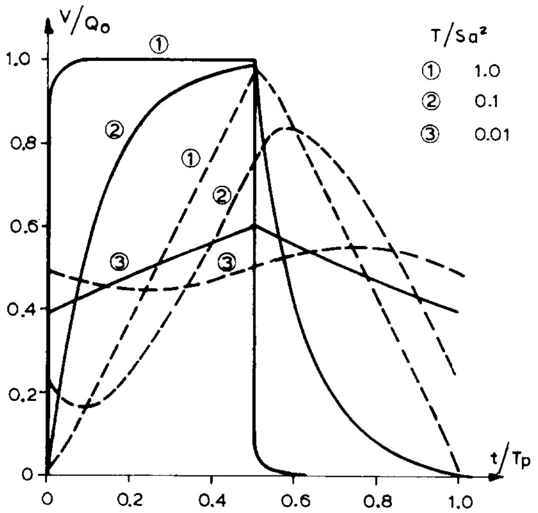

Fig. 7. Dimensionless total outflow at $x = a$ during an annual period: —— = case *1*; – – – – = case *2* (aquifer type *7*, $C_3 a = 0.25$, $a/b = 0.5$, $T_0/T_p = 0.5$ and recharge at $\xi/a = 0.5$, $\eta/b = 0.5$).

parameter T/Sa^2 which is often called the "aquifer response time" (Downing et al., 1974). The importance of this parameter as well as the effects of the relative size of recharge and its duration have been already outlined in previous works (Nutbrown, 1976; Latinopoulos, 1981, 1982).

Some interesting conclusions can be obtained from the results appearing in Fig. 7, where curves for different aquifer response time values have been drawn for the two examples of recharge regimes considered in this paper. It can be seen that for "fast" aquifers, e.g. for $T/Sa^2 > 10^{-1}$, the aquifer response is very quick with its time variation, almost the same as with the recharge rate variation. In contrast with the "slow" aquifer, that is for aquifers with $T/Sa^2 < 10^{-2}$, the response variation tends to be independent of that of the recharge rate. Thus depending upon the desired final management goals, the most suitable recharge regime can be at a first stage selected by trying various types of recharge rate variations.

6. CONCLUSIONS

Analytical solutions for periodic recharge from a single well on a rectangular aquifer with third-kind boundary conditions were obtained by applying Laplace and finite Fourier transforms.

Numerical results for two typical cases show that the analytical solutions are very useful in deriving preliminary estimates of the future consequences from the application of artificial recharge schemes. These solutions can also be used to check numerical solutions of large and more complex problems.

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