MATH350 – Statistical Inference

STATISTICS + MACHINE LEARNING + DATA SCIENCE

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Course Webpage: https://www.ctanujit.org/SI.html
Course for BSc Mathematics and Data Science Students.

- Oftentimes, a very good approximate answer emerges when n is large (in other words, you have many samples). We call results that rely on this type of approximation as asymptotic.

- Computerized simulations can also be carried out to approximate sampling distributions.

- With a model we can draw many random samples, compute the statistic, and characterize it's sampling distribution.
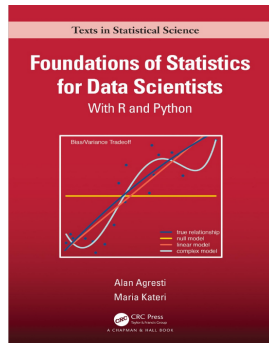

Figure: (Easy) Textbook

- **Bootstrap** is useful to compute the sampling distribution without a model! (to be discussed later)

- If we can just simulate, why do asymptotic analysis?

  1. Better understanding of the behavior. (Understanding the assumptions: What if X_i are not uniform? What if I don't really know the distribution of X_i? Understanding the scaling: What if $n = 1000$ instead of 100? What if $n = 1,000,000$?)
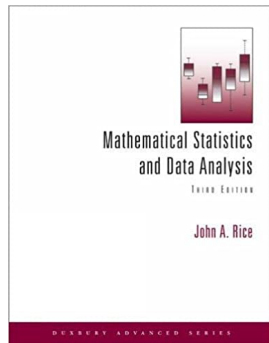
  2. Faster to get an answer.

Figure: See Chapter 5

In theory. we've discussed a few examples of how to determine the distribution of a statistic computed from data, assuming a certain probability model for the data.

For example, we have shown the following results: If $X_1, \ldots, X_n \overset{IID}{\sim} \mathcal{N}(0,1)$, then

$$\bar{X} \sim \mathcal{N}\left(0, \frac{1}{n}\right),$$

$$X_1^2 + \ldots + X_n^2 \sim \chi_n^2.$$

For many (seemingly simple) statistics, it's difficult to describe its PMF or PDF exactly. For example:

1. Suppose $X_1, \ldots, X_{100} \overset{IID}{\sim} Uniform\ (-1, 1)$. What is the distribution of $\bar{X}$?

2. Suppose $(X_1, \ldots, X_6) \sim \mathrm{Multinomial}\left(500, \left(\frac{1}{6}, \ldots, \frac{1}{6}\right)\right)$. What is the distribution of

$$T = \left(\frac{X_1}{500} - \frac{1}{6}\right)^2 + \ldots + \left(\frac{X_6}{500} - \frac{1}{6}\right)^2 ?$$

For questions that we don't know how to answer exactly, we'll try to answer them approximately.

If we fully specify the distribution of data, then we can always simulate the distribution of any statistic:

```
nreps = 10000
sample.mean = numeric(nreps)
n = 100
for (i in 1:nreps) {
X = runif(n, min = -1, max = 1)
sample.mean [i] = mean(X)
}
hist (sample.mean)
```
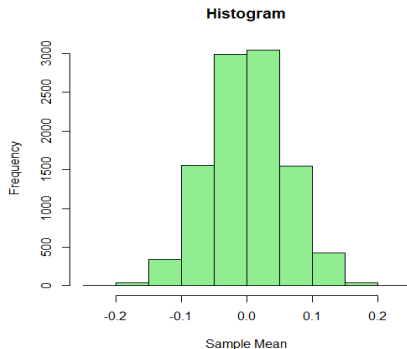

Figure: Histogram of sample.mean

Motivating Example : Dice Game

You are playing a dice game with your friend and he seems to be cheating (either that or you are really bad at this game). You deduce that the dice is not fair. This is, you expect each of the outcomes to be equally likely, but they do not seem to be coming up that way.

- Suppose $(Y_1, \ldots, Y_6) \sim \text{Multinomial}\left(n, \left(\frac{1}{6}, \ldots, \frac{1}{6}\right)\right)$. Y represents the number of times we obtain 1 through 6 when rolling a 6-sided die n times.

- For each $I = 1, \ldots, n$, let $\mathbf{X}^{(I)} = (1, 0, 0, 0, 0, 0)$ if we got 1 on the Ith roll, $(0, 1, 0, 0, 0, 0)$ if we got 2 on the Ith roll, etc. Then $(Y_1, \ldots, Y_6) = \mathbf{X}^{(1)} + \ldots + \mathbf{X}^{(n)}$.

- Let's apply the (multivariate) WLLN and CLT!

Let's write $\mathbf{X}^{(1)} = (X_1, \ldots, X_6)$, so $X_1, \ldots, X_6$ are random variables where exactly one them equals 1 (and the rest equal 0). Then:

$$\mathbb{E}\left[X_i\right] = \mathbb{P}\left[X_i = 1\right] = \frac{1}{6}$$

$$\text{Var}\left[X_i\right] = \mathbb{E}\left[X_i^2\right] - \left(\mathbb{E}\left[X_i\right]\right)^2 = \frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{5}{36},$$

$$\text{Cov}\left[X_i, X_j\right] = \mathbb{E}\left[X_i X_j\right] - \mathbb{E}\left[X_i\right]\mathbb{E}\left[X_j\right] = 0 - \left(\frac{1}{6}\right)^2 = -\frac{1}{36},$$

for $i \neq j$.

By the LLN, as $n \to \infty$,

$$\left(\frac{Y_1}{n}, \dots, \frac{Y_6}{n} \right) \to \left(\frac{1}{6}, \dots, \frac{1}{6} \right)$$

in probability. By the CLT, as $n \to \infty$,

$$\sqrt{n} \left(\frac{Y_1}{n} - \frac{1}{6}, \dots, \frac{Y_6}{n} - \frac{1}{6} \right) \to \mathcal{N}(0, \Sigma)$$

in distribution, where

$$\Sigma = \begin{pmatrix} \frac{5}{36} & -\frac{1}{36} & \cdots & -\frac{1}{36} \\ -\frac{1}{36} & \frac{5}{36} & \cdots & -\frac{1}{36} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{36} & -\frac{1}{36} & \cdots & \frac{5}{36} \end{pmatrix} \in \mathbb{R}^{6 \times 6}$$

(The negative values of Σ_{ij} for $i \neq j$ mean Y_i and Y_j are, as expected, slightly anti-correlated.)

Recall

$$nT_n = n\left(\frac{Y_1}{n} - \frac{1}{6}\right)^2 + \ldots + n\left(\frac{Y_6}{n} - \frac{1}{6}\right)^2.$$

The function $g(x_1, \ldots, x_6) = x_1^2 + \ldots + x_6^2$ is continuous, so

$$nT_n \to Z_1^2 + \ldots + Z_6^2.$$

in distribution, where $(Z_1, \ldots, Z_6) \sim \mathcal{N}(0, \Sigma)$.

Hence, when n is large, the distribution of T_n is approximately that of $\frac{1}{n}\left(Z_1^2 + \ldots + Z_6^2\right)$.

Finally, what is the distribution of $Z_1^2 + \ldots + Z_6^2$?

Using bilinearity of covariance, it is easy to show that if

$$W_1, \ldots, W_6 \overset{IID}{\sim} \mathcal{N}(0,1),$$

then

$$\frac{1}{\sqrt{6}} \left(W_1 - \bar{W}, \ldots, W_6 - \bar{W}\right) \sim \mathcal{N}(0, \Sigma)$$

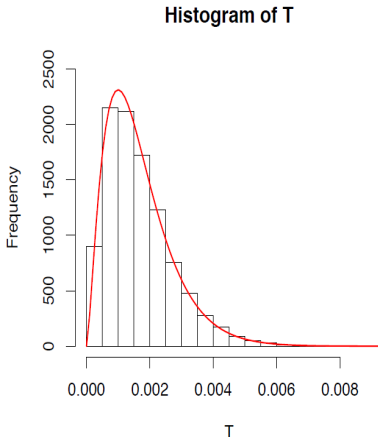(Here $\bar{W} = \frac{1}{6}(W_1 + \ldots + W_6)$.) So $Z_1^2 + \ldots + Z_6^2$ has the same distribution as

$$\frac{1}{6} \left(\left(W_1 - \bar{W}\right)^2 + \ldots + \left(W_6 - \bar{W}\right)^2 \right)$$

This is the sample variance of 6 IID standard normals, which we will show next week has distribution $\frac{1}{6}\chi_5^2$.

Conclusion: T_n has approximate distribution $\frac{1}{6n}\chi_5^2$.

- Here's our simulated histogram of T_n, overlaid with the (appropriately rescaled) PDF of the $\frac{1}{6n}\chi_5^2$ distribution:

Histogram of T

Histogram of T

```
nreps = 10000
T = numeric (nreps)
n = 500
p=c(1/6,1/6,1/6,1/6,1/6,1/6)
for (i in 1 : nreps){
X = rmultinom(1, n, p)
T[i]=sum((X/n − p)^2)
}
hist(T)
```
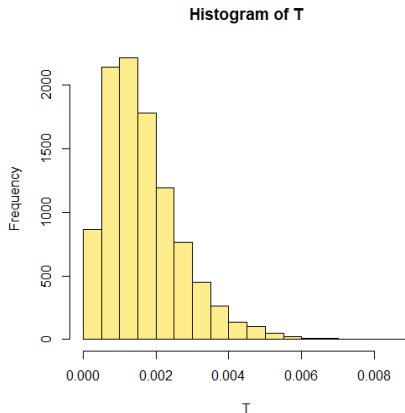
Figure: Histogram of sample.mean

EXAMPLES: SIMULATIONS IN R

Assume price $\sim$ Exponential (λ)

Consider samples of size $n = 201$

$E[\text{price}] = \lambda^{-1}$ and $\text{Var}[\text{price}] = \lambda^{-2}$ and therefore

$$\text{Var}(\overline{\text{price}}) = \sqrt{(1/\lambda^2)/201} \approx 0.0705/\lambda$$

Recall that The Normal approximation for the sampling distribution of the

average price suggests

$$1/\lambda \pm 1.96 \cdot 1/(\lambda\sqrt{n})$$

should contain 95% of the distribution.

We may use simulations in order to validate this approximation.

Assume $\lambda = 1/12{,}000$

```
X.bar = replicate(10^5,mean(rexp(201,1/12000)))
mean(abs(X.bar-12000) <= 1.96*0.0705*12000)


## [1] 0.95173
```

Which shows that the Normal approximation is adequate in this example.
How about other values of n or λ?

Simulations may also be used in order to compute probabilities in cases where the Normal approximation does not hold.

Consider the following statistic

$$(\min(x_i) + \max(x_i))/2$$

where $X_i \sim \text{Uniform}(3, 7)$ and $n = 100$

What interval contains 95% of the observations?

Let us carry out the simulation that produces an approximation of the central region that contains 95% of the sampling distribution of the mid-range statistic for the Uniform distribution:

```
mid.range <- rep(0,10^5)
for(i in 1:10^5) {
  X <- runif(100,3,7)
  mid.range[i] <- (max(X)+min(X))/2
}
quantile(mid.range,c(0.025,0.975))
```

```
##      2.5%     97.5%
## 4.9409107 5.0591218
```

Observe that (approximately) 95% of the sampling distribution of the statistic are in the range [4.941680, 5.059004].

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Simulations can be used in order to compute any numerical summary of the sampling distribution of a statistic.

To obtain the expectation and the standard deviation of the mid-range statistic of a sample of 100 observations from the $Uniform(3, 7)$ distribution:

```
mean(mid.range)
```

```
## [1] 4.9998949
```

```
sd(mid.range)
```

```
## [1] 0.027876151
```

```
n = 1000
data = rbinom(n, 1, .54) # true distr, usually unknown
estimates = rep(0,999)
for(i in 1:999) {
    id = sample(1:n, n, replace=T)
    estimates[i] = mean(data[id])
}
sd(estimates)
```

```
## [1] 0.015946413
```

```
sqrt(.54*(1-.54)/1000)   # true value, usually unknown
```

```
## [1] 0.015760711
```
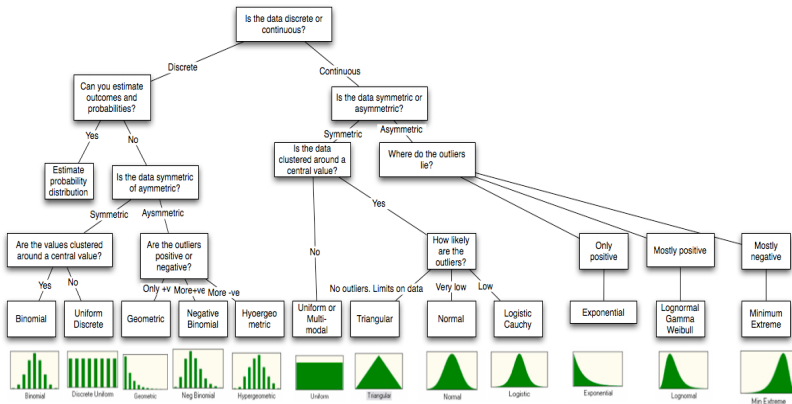
SIMULATIONS IN PYTHON

Figure: Source: https://pages.stern.nyu.edu/ adamodar/

Application:

- Tossing uneven coin
- Rolling a dice
- Counting errors/successes
- Trying until success
- Countable, rare events whose occurrence is independent
- Random "noise", sums of many variables

Distribution:

- Bernoulli
- Uniform
- Binomial
- Geometric
- Poisson
- Normal

Function:

- scipy.stats.bernoulli
- scipy.stats.randint
- scipy.stats.binom
- scipy.stats.geom
- scipy.stats.poisson
- scipy.stats.norm

Stochastic Method

In [1]:   import scipy.stats

In [2]:   u = scipy.stats.uniform(90, 10)
          u.rvs(5)
Out [2]:   array([93.42, 93.97, ..., 94.46])

In [3]:   u.rvs(1000).mean()
Out [3]:   94.91863434778571

How many values are less than 92?

In [4]:   (u.rvs(1000) < 92).sum() / 1000.0
Out [4]:   0.202

Analytical Method

In [5]:   import sympy.stats

In [6]:   u = sympy.Symbol("u")
          u = sympy.stats.Uniform(u, 90, 100)
          sympy.stats.sample(u)

In [7]:   sympy.stats.E(u)
Out [7]:   95

How many values are less than 92?

In [8]:   sympy.stats.P(u < 92)
Out [8]: $\frac{1}{5}$

- Consider a Gaussian distribution centered in 5, standard deviation of 1
- Plot the distribution
- Check that half the distribution is located to the left of 5
- Find the first percentile (value of x which has 1% of realizations to the left)
- Check that it is equal to the 99% quantile

```
In [1]:   import scipy.stats
In [2]:   norm = scipy.stats.norm(5, 1)
          x = numpy.linspace(1, 9, 100)
          plt.plot(x, norm.pdf(x));
In [3]:   norm.cdf(5)
Out [3]:  0.5
In [4]:   norm.ppf(0.5)
Out [4]:  5.0
In [5]:   norm.isf(0.99)
Out [5]:  2.6736521259591592
In [6]:   norm.cdf(norm.isf(0.99))
Out [6]:  0.01
```

The Central Limit Theorem states that the mean (also true of the sum) of a set of random measurements will tend to a normal distribution, no matter the shape of the original measurement distribution.

```
In [1]:   N = 10000
          sim = numpy.zeros(N)
          for i in range(N):
              sim[i] = numpy.random.uniform(30, 40, 100).mean()
          plt.hist(sim, bins=20, alpha=0.5, density=True);
```

Out [1]:
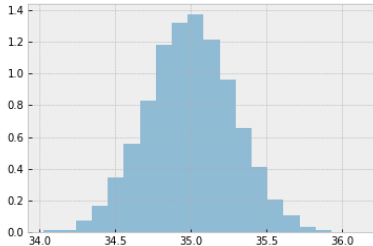
Simulate the number of heads when tossing a coin twice. Do this 1000 times, calculate the expected value of the number of heads, and plot the distribution of results (the Probability Mass Function).

```
In [1]:   N = 1000
          heads = numpy.zeros(N, dtype=int)
          for i in range(N):
              heads[i] = numpy.random.randint(0, 2, 2).sum()
          heads.mean()
Out [1]:  0.987
```

```
In [2]:   values, counts = numpy.unique(heads, return_counts = True)
          plt.stem(values, counts, use_line_collection = True)
          plt.show()
```
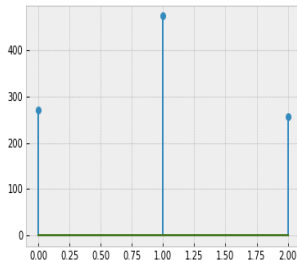
```
In [1]:   N = 1000
          roll = numpy.zeros(N, dtype=int)
          expectation = numpy.zeros(N)
          for i in range(N):
              roll[i] = numpy.random.randint(1, 7)
          for i in range(1, N):
              expectation[i] = numpy.mean(roll[0:i])
          plt.plot(expectation)
          plt.title("Expectation of a dice roll")
          plt.show()
```


Expectation of a dice roll
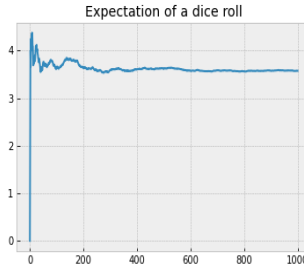
- Simulate thousand dice throws.

- What is the probability of a die rolling 4?

- What is the probability of rolling 4 or below?

- What is the probability of rolling between 2 and 4 (inclusive)?

```
In [1]:   dice = scipy.stats.randint(1, 7)
In [2]:   dice.pmf(4)
Out [2]:   0.166
In [3]:   dice.cdf(4)
Out [3]:   0.666
In [4]:   dice.cdf(4) - dice.cdf(1)
Out [4]:   0.5
```
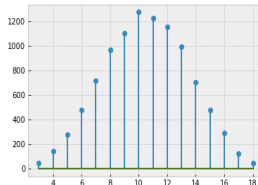
Estimate the expected value and the probability mass function for the sum of three dice.

```
In [1]:   N = 10000
          roll = numpy.zeros(N, dtype=int)
          for i in range(N):
              roll[i] = scipy.stats.randint(1, 7).rvs(3).sum()
          roll.mean()
Out [1]:   10.5018
```

```
In [2]:   values, counts = numpy.unique(roll, return_counts = True)
          roll = numpy.zeros(N, dtype=int)
          for i in range(N):
              roll[i] = scipy.stats.randint(1, 7).rvs(3).sum()
          roll.mean()
```

Two people are stranded on an island with only one banana to eat. To decide who gets it, they agree to play a game. Each of them will roll a fair 6-sided dice. If the largest number rolled is a 1, 2, 3, or 4, then Player 1 gets the banana. If the largest number rolled is a 5 or 6, then Player 2 gets it. Which player has the better chance?

```
In [1]:  N = 1000
         player1 = 0
         for i in range(N):
             roll1 = scipy.stats.randint(1, 7).rvs(1)
             roll2 = scipy.stats.randint(1, 7).rvs(1)
             if max(roll1, roll2) <= 4:
                 player1 += 1
         player1 / float(N)

Out [1]:  0.439
```

Player 1 gets the banana when both the dice rolls are smaller than 5. The probability of a roll smaller than 5 is 4/6, the two dice rolls are independent events, so their combined probability is the product of their individual probabilities: Player 1 gets the banana with probability $\frac{4}{6} \times \frac{4}{6} = 0.444$.

- Antoine Gombaud, chevalier de Méré, a french gambler and the mathematician Blaise Pascal discussed a problem.

- This discussion was a little daring for the time, because it went against established doctrine of the Catholic church that people should not attempt to predict the future (an activity that was reserved for the deity).

- The discussion and collaboration of Pascal with another mathematician, Pierre de Fermat, helped in the development of probability theory.

The first problem: is it a good idea to gamble on the appearance of at least one 6 when a dice is thrown 4 times?

The second problem: is it a good idea to gamble on the appearance of at least one double six when two dice are thrown 24 times?

The first problem: is it a good idea to gamble on the appearance of at least one 6 when a dice is thrown 4 times?

The probability of losing this gamble is easy to calculate analytically: each throw has 5 chances out of 6 of not seeing a 6, and the events are independent. So the probability of winning is

```
In [1]:   1 - (5/6.0)**4
Out [1]:  0.517746913580247
```

The probability of winning is greater than 0.5. We can also check this problem with SymPy:

```
In [2]:   import sympy.stats
          D1 = sympy.stats.Die("D1", 6)
          D2 = sympy.stats.Die("D2", 6)
          D3 = sympy.stats.Die("D3", 6)
          D4 = sympy.stats.Die("D4", 6)
          sympy.stats.P(sympy.Or(D1 > 5, sympy.Or(D2 > 5, sympy.Or(D3 > 5,
          D4 > 5)))) + 0.0
Out [2]:  0.517746913580247
```

The second problem: is it a good idea to gamble on the appearance of at least one double six when two dice are thrown 24 times?
The probability of losing this gamble is also easy to calculate: there are 35 chances out of 36 (6 * 6) of not seeing a double 6 on each double throw, so the probability of winning is

```
In [1]:   1 - (35/36.0)**24
Out [1]:  0.491403876130903
```

So this is not a good gamble. We can also calculate this analytically with SymPy, by modelling a Binomial random variable:

```
In [2]:   import sympy.stats
          A = sympy.stats.Die("A", 6)
          B = sympy.stats.Die("B", 6)
          doublesix = sympy.stats.Binomial("DoubleSix", 24,
          sympy.stats.P(sympy.And(A > 5, B > 5)))
          sympy.stats.P(doublesix >= 1) + 0.0
Out [2]:  0.491403876130903
```

Samuel Pepys was a great diarist of the English language and a friend of Isaac Newton's. Pepys was a gambler and wrote to Newton to ask which of three events is the most likely:

- at least one six comes up when six fair dice are rolled;
- at least two sixes come up when 12 dice are rolled;
- at least three sixes come up when 18 dice are rolled.

Now Newton wasn't able to use Python, and spent a while working out the answers, but we can stand on the shoulders of giants and figure this out quite easily.

Possibility 1: the probability of rolling at least one six is 1 minus the probability of zero sixes, which is

> *In [1]:* $1 - (\frac{5}{6})^6$
>
> *Out [1]:* 0.665102023319616

Possibility 2: the probability of at least two sixes is 1 minus the probability of zero sixes, minus the probability of a single six. The probability of zero sixes is easy to calculate; here are two ways of calculating it.

Throw a non-6 12 times

In [1]: $\left(\frac{5}{6}\right)^{12}$
Out [1]: 0.112156654784615

Probability mass at 0 of a binomial distribution with n=12, p=$\frac{1}{6}$

In [2]: scipy.stats.binom(12, 1/6.0).pmf(0)
Out [2]: 0.112156654784615

So the final answer is
In [3]: roll12 = scipy.stats.binom(12, 1/6.0)
        1 - roll12.pmf(0) - roll12.pmf(1)
Out [3]: 0.618667373732309

Possibility 3: in the same way, the probability of at least three sixes when rolling 18 dice is one minus the probability of zero sixes, minus the probability of one six, minus the probability of two sixes.

```
In [1]:   roll18 = scipy.stats.binom(18, 1/6.0)
          1 - roll18.pmf(0) - roll18.pmf(1) - roll18.pmf(2)

Out [1]:   0.597345685947723
```

EXAMPLE: MODELING EARTHQUAKE

- Suppose we live in an area where there are typically 0.03 earthquakes of intensity 5 or more per year.

- Assume earthquake arrival is a Poisson process
  - interval between earthquakes follows an exponential distribution
  - events are independent

- Simulate the random intervals between the next earthquakes of intensity 5 or greater.

- What is the 25-th percentile of the interval between 5+ earthquakes?

```
> from scipy.stats import expon

> expon(scale=1/0.03).rvs(size=15)
array([23.23763551, 28.73209684, 29.7729332,
       46.66320369, 4.03328973, 84.03262547,
       42.22440297, 14.14994806, 29.90516283,
       87.07194806, 11.25694683, 15.08286603,
       35.72159516, 44.70480237, 44.67294338])

> expon(scale=1/0.03).ppf(0.25)
9.5894024150593644
# answer is "around 10 years"
```

Simulating earthquake occurrences

- **Worldwide:** 144 earthquakes of magnitude 6 or greater in 2013 (one every 60.8 hours on average)

- **Rate:** $= 1/60.8$ per hour

- What's the probability that an earthquake of magnitude 6 or greater will occur (worldwide) in the next day?
  - right: plot of the cdf of the corresponding exponential distribution
  - $scipy.stats.expon(scale = 60.8).cdf(24) = 0.326$

- **Data source:** earthquake.usgs.gov/earthquakes/search/

Earthquake locations

Course GitHub: https://github.com/tanujit123/MATH350
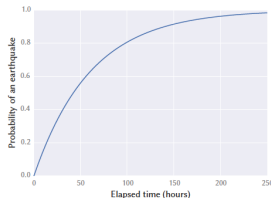
Other resources:

- SciPy lecture notes: scipy-lectures.org

- Book "Statistics done wrong" available online at statisticsdonewrong.com
- Risk Analysis course at risk-engineering.org