Math 20D: Ordinary Differential Equations

Suhas Arehalli

December 13, 2015

1 1st Order ODEs

1.1 Integrating Factor

1.1.1 Given

A 1st order linear differential equation of the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

1.1.2 Technique

Begin by calculating the integrating factor (feel free to ignore the constant of integration - it's irrelevant),

$$\mu(t) = e^{\int p(t)dt}$$

Then multiply both sides by it

$$\frac{dy}{dt}e^{\int p(t)dt} + p(t)e^{\int p(t)}y = g(t)e^{\int p(t)}$$

$$\frac{dy}{dt}\mu(t) + \frac{d\mu}{dt}y = g(t)\mu(t)$$

Note that the right-hand side is just $\frac{d}{dt}[y\mu(t)]$

$$\frac{d}{dt}[y\mu(t)] = g(t)\mu(t)$$

$$y\mu(t) = \int g(t)\mu(t)dt$$

$$y = \frac{\int g(t)\mu(t)dt}{\mu(t)}$$

1.2 Separable Equations

1.2.1 Given

A 1st order differential equation of the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

These are called **separable**.

1.2.2 Technique

Separate your variables

$$N(y)\frac{dy}{dx} = M(x)$$

And integrate

$$\int N(y)dy = \int M(x)dx$$

And then solve for y.

1.3 Exact Equations

1.3.1 Given

A 1st order differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Such that

$$\frac{d}{dy}[M(x,y)] = \frac{d}{dx}[N(x,y)]$$

1.3.2 Technique

Calculate $\psi(x,y)$ from

$$\psi(x,y) = \int M(x,y)dx + C_1(y)$$
$$\psi(x,y) = \int N(x,y)dy + C_2(x)$$

There should be one function that satisfies both of these equations with varying functions $C_1(x)$ and $C_2(y)$: That is $\psi(x,y)$. Then we can rewrite the differential equation as

$$\frac{d\psi}{dx} + \frac{d\psi}{dy}\frac{dy}{dx} = 0$$
$$\frac{d}{dx}[\psi(x,y)] = 0$$
$$\psi(x,y) = C$$

Which we can proceed to algebraically solve for y.

2 Second Order ODEs

2.1 The Wronskian

In first order ODEs, we know there is only one solution to each Differential Equation (plus all of it's scalar multiples). With 2nd Order ODEs, we expect 2 linearly independent solutions. How do we determine if these solutions are independent? By calculating the Wronskian:

$$W = det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

If $W \neq 0$, then the 2 solutions are linearly independent enough.

2.2 Characteristic Equation

2.2.1 Given

A homogenous 2nd order differential equation of the form

$$y'' + ay' + b = 0$$

2.2.2 Technique

Assume our solution is of the form $y = e^{rt}$. Substituting it in gives us

$$\begin{aligned} r^2e^{rt} + are^{rt} + be^{rt} &= 0 \\ (r^2 + ar + b)e^{rt} &= 0 \end{aligned}$$

However, since $e^{rt} \neq 0, \forall r \in \mathbb{R}$, we can divide by e^{rt} and obtain the characteristic equation

$$r^2 + ar + b = 0$$

and solve for the r_1, r_2 that satisfy the equation (the roots of the polynomial on the LHS). Depending on the kind of roots obtained, follow one of the techniques below:

2.2.3 Technique: Real and Unique Roots

In this case, your solution is simply

$$y = C_1e^{r_1t} + C_2e^{r_2t}$$

Why? If you followed along above, you see that we assumed that solutions would be of the form $y = e^{rt}$. However, since there are 2 solutions for that equations, by the Law of Superposition, every linear combination of those solutions is also a solution.

2.2.4 Technique: Complex Roots

Here we can say that out solution is simply the same as above. However, we don't care about imaginary solutions: The DE was real-valued, so so should our solution. Thus, we will inspect one solution (which is complete, as we could show with the Wronskian) and remove the imaginary terms. Consider

$$\begin{aligned} y = Ce^{rt} &= Ce^{(a+bi)t} \\ &= Ce^{at}e^{(bt)i} \\ &= Ce^{at}(cos(bt) + isin(bt)) \\ &= (C_1 + C_2i)e^{at}(cos(bt) + isin(bt)) \\ &= e^{at}(C_1cos(bt) - C_2sin(bt)) + ie^{at}(C_1sin(bt) + C_2cos(bt)) \end{aligned}$$

If we swap the sign on C_2 and drop the imaginary part, we obtain our real solution

$$y = e^{at}(C_1cos(bt) + C_2sin(bt))$$

2.2.5 Technique: Repeated Roots

If we only have a single root, we can simply say the answer is

$$y = C_1 e^{rt} + C_2 t e^{rt}$$

Why? This is a special case of the next technique, Reduction of Order.

2.3 Reduction of Order

2.3.1 Given

A 2nd order linear DE of the form

$$y'' + p(t)y' + q(t)y = 0$$

and a single solution $y_1(t)$.

2.3.2 Technique

Assume that

$$y_2(t) = v(t)y_1(t)$$

for some function $v(t)$. Our goal is to find this function $v(t)$, and thus find $y_2(t)$. Now simply substitute into the original equation, first finding that

$$\begin{aligned} y_2(t) &= v(t)y_1(t) \\ y_2'(t) &= v'(t)y_1(t) + v(t)y_1'(t) \\ y_2''(t) &= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) \end{aligned}$$

And then substiting fully, getting

$$\begin{aligned} \left[v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)\right] + p(t)\left[v'(t)y_1(t) + v(t)y_1'(t)\right] + q(t)\left[v(t)y_1(t)\right] &= 0 \\ \left[v''(t)y_1(t) + 2v'(t)y_1'(t)\right] + p(t)\left[v'(t)y_1(t)\right] + v(t)\left[y_2''(t) + p(t)y_1'(t) + q(t)y_1(t)\right] &= 0 \\ \left[v''(t)y_1(t) + 2v'(t)y_1'(t)\right] + p(t)\left[v'(t)y_1(t)\right] &= 0 \end{aligned}$$

At this point, since all terms with v(t) were eliminated, we can define a new variable $w(t) = v'(t)$. Now We're left with

$$\left[w'(t)y_1(t) + 2w(t)y_1'(t)\right] + p(t)\left[w(t)y_1(t)\right] = 0$$

And since $y_1(t)$ is known, we have a 1st order DE that we can solve using the previous methods, finding $w = v'(t)$, from which we can find $v(t) = \int w(t)dt$, which in turn lets us find $y_2(t) = v(t)y_1(t)$.

2.4 Method of Undetermined Coefficients

2.4.1 Given

A second order nonhomogenous ODE of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

with a g(t) that is a sum/product of sin/cos, polynomials, or exponentials, plus homogenous solutions $y_1(t)$ and $y_2(t)$ that satisfy

$$y'' + p(t)y' + q(t)y = 0$$

We could obtain these with the other homogenous 2nd order techniques discussed above.

2.4.2 Technique

We need to add an additional term, $Y(t)$ to our homogenous solution to account for the function $g(t)$ on the RHS. To find $Y(t)$, first, guess a generalized version of $g(t)$: Replace an nth order polynomial with $\sum_{i=0}^{n} A_i t^i$, sine or cosine with $Acos(Bt) + Csin(Dt)$, and any exponential with Ae^{Bt}.

Then find $Y'(t), Y''(t)$ and substitute into the differential equation. The equate all of the coefficients of equal terms, obtaining a system of linear equations which you can proceed to solve for the undetermined coefficients you got from "generalizing" when guessing $Y(t)$. Then our final solution will be

$$y = y_1(t) + y_2(t) + Y(t)$$

Sometimes the system we find will have no solutions. In this case, we can multiply any Ae^{Bt} terms by t and try again, and if that doesn't work repeat. Don't ask me why this works. **TIP**: If something of the form Ae^{Bt} appears in the homogenous solution, multiply by t. If something of the form Ate^{Bt} appears in the homogenous solution, multiply by t again. To recap:

1. Construct a guess Y(t)
2. Differentiate Y(t) and substitute into the DE
3. Equate coefficients of similar terms, getting a system of linear equations for the coefficients
4. Solve the system to get the coefficients
5. If the system has no solutions, multiply Ae^{Bt} terms by t and go back to 2.
6. Write the final solution

2.5 Variation of Parameters

2.5.1 Given

A second order nonhomogenous ODE of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

with homogenous solutions $y_1(t)$ and $y_2(t)$ that satisfy

$$y'' + p(t)y' + q(t)y = 0$$

We could obtain these with the other homogenous 2nd order techniques discussed above.

2.5.2 Technique

Again we begin by making an assumption about $Y(t)$:

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

From this, we again differentiate

$$\begin{aligned} Y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ Y'(t) &= u_1(t)y_1'(t) + u_1'(t)y_1(t) + u_2(t)y_2'(t) + u_2'(t)y_2(t) \end{aligned}$$

And now, since that looks disgusting, and because we have the freedom to, let's impose an additional constraint:

$$u_1'(t)y_1(t) + u_2'(t)y_1(t) = 0 \tag{1}$$

which gets us

$$\begin{aligned} Y'(t) &= u_1(t)y_1'(t) + u_2(t)y_2'(t) \\ Y''(t) &= u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t) \end{aligned}$$

Now substitute into the DE, getting

$$\begin{aligned} &[u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)] \\ &\qquad +p(t)[u_1(t)y_1'(t) + u_2(t)y_2'(t)] \\ &\qquad\qquad +q(t)\,[u_1(t)y_1(t) + u_2(t)y_2(t)] = g(t) \end{aligned}$$

$$\begin{aligned} &u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] \\ &\qquad +u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] \\ &\qquad\qquad + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{aligned}$$

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \tag{2}$$

This paired with (1) gives us a system of equations we can use to solve for $u_1'(t)$ and $u_2'(t)$, giving us

$$\begin{aligned} u_1'(t) &= -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \\ u_2'(t) &= \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \end{aligned}$$

Which, of course leads to

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \tag{3}$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \tag{4}$$

As part of

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Which gives us our final answer

$$y = y_1(t) + y_2(t) + Y(t)$$

For normal use of this technique:

1. Find $u_1(t)$ and $u_2(t)$ using equations (3) and (4)
2. Write the final solution

Not actually that bad.

3 Systems of ODEs

3.1 The Wronskian for Systems

Again we must ask ourselves the question: How do we know if we've found all of the solutions. Again we answer by checking is $W \neq 0, \forall t$. How do we compute the Wronskian for a system of equations? Since our solutions $y_1, ..., y_n$ are vector valued functions $y_i : \mathbb{R} \to \mathbb{R}^n$, we say that

$$W(y_1, ..., y_n) = det\,[y_1, ..., y_n]$$

In this case, we literally mean that our solutions are linearly independent over all t.

3.2 Characteristic Equation

3.2.1 Given

Given a system of n linear ODEs of order n with coefficients that can be written in the form

$$x' = Ax$$

With A being an n x n matrix and x being a vector-valued functions $x : \mathbb{R} \to \mathbb{R}^n$

3.2.2 Technique

Assume that solutions are of the form

$$y = \xi e^{\lambda t}$$

Then we get

$$\lambda \xi e^{\lambda t} = A \xi e^{\lambda t}$$
$$\lambda \xi = A \xi$$

Thus we discover that ξ must be an eigenvector of A with eigenvalue λ. To find the eigenvalue, we must first find the roots of the characteristic polynomial of A, since

$$A\xi - \lambda \xi = 0$$
$$(A - \lambda I)\xi = 0$$

And in order for this system to have a solution for ξ other than the 0 vector, we must have that

$$det(A - \lambda I) = 0$$

Depending on the λ's we find, we can use different techniques to find the individual solutions for each eigenvalue, and use the Principle of Superposition (the linear combination of all solutions is the general solution) to give the general solution.

Note that these directly parallel the 2nd order techniques discussed. If we formulate a 2nd order equation

$$y'' + ay' + by = 0$$

as

$$x_1' = x_2$$
$$x_2' = -ax_2 - bx_1$$

With $y = x_1$.

3.2.3 Technique: Real and Unique Eigenvalues

The solution is simply

$$x(t) = \sum_{i=1}^{n} C_i \xi_i e^{\lambda_i t}$$

for each real eigenvalue λ_i and associated eigenvector ξ_i.

3.3 Technique: Imaginary Eigenvalues

For reasons similar to the 2nd order ODEs, we'll drop the imaginary portion. Let $\lambda = a + bi$ be one of the conjugate pair of imaginary eigenvalues and ξ be the corresponding eigenvector. Their

$$\begin{aligned} x(t) &= C\xi e^{(a+bi)t} \\ &= C\xi e^{at}(cos(bt) + isin(bt)) \\ \dots \\ &= y_1 + iy_2 \end{aligned}$$

Since ξ can have imaginary parts, you must multiply through and separate imaginary and real parts. Then, since the coefficient $C \in \mathbb{C}$, you may take both the real and imaginary parts of this solution as your x_1 and x_2.

Then your solution will be

$$x(t) = C_1 y_1 + C_2 y_2$$

3.3.1 Technique: Repeated Eigenvalues

If there is a repeated eigenvalue λ with corresponding eigenvector ξ, it is tempting to follow our logic from 2nd order DEs and simply multiply our solution by t. However, that is **wrong**. Instead, we must assume our second solution is of the form

$$x_2(t) = \xi t e^{\lambda t} + \eta e^{\lambda t}$$

With vector η. Derive, getting

$$x_2'(t) = (\lambda t + 1)\xi e^{\lambda t} + \lambda \eta e^{\lambda t}$$

And substitute into our system

$$\begin{aligned} x' &= Ax \\ (\lambda t + 1)\xi e^{\lambda t} + \lambda \eta e^{\lambda t} &= A(t\xi e^{\lambda t} + \eta e^{\lambda t}) \\ \lambda t \xi e^{\lambda t} + \xi e^{\lambda t} + \lambda \eta e^{\lambda t} &= \lambda \xi t e^{\lambda t} + A\eta e^{\lambda t} \\ \xi e^{\lambda t} + \lambda \eta e^{\lambda t} &= A\eta e^{\lambda t} \\ \xi + \lambda \eta &= A\eta \\ (A - \lambda I)\eta &= \xi \end{aligned}$$

This will create a linear system, which we can solve for η. Then we have that

$$x(t) = C_1 \xi e^{\lambda t} + C_2(\xi t e^{\lambda t} + \eta e^{\lambda t}) \tag{5}$$

To summarize:

1. Find eigenvalues of the matrix A, and determine that one of the roots is repeated.
2. Solve $(A - \lambda I)\eta = \xi$ to find η.
3. The solution for that repeated eigenvalue is as given in (5).

3.4 Undetermined Coefficents

3.4.1 Given

A non-homogenous 2nd order system of ODEs of the form

$$x' = At + g(t)$$

And a homogenous solution, $x_h(t)$ presumably obtained using the characteristic equation method above.

3.4.2 Technique

Using the same generalization techniques as for 2nd order ODEs, craft a guess for an additional term we'll call $v(t)$, except instead of multiplying a term by t, add the higher order term. e.g.

$$ae^{Bt} \to ate^{Bt} + be^{Bt}$$

The process is then pretty much the same:

1. Derive and substitute into the original equation
2. Equate coeffients of similar terms to get systems of systems of equations, some corresponding to eigenvalue equations, others solvable by row reduction, and so on.
3. Solve them. If you decide to go down this painful road and the system of systems is unsolvable, then go back and do the "add a higher order" thing mentioned above. The tips on that above are relevant here too.
4. Now you know $v(t)$, so write the final solution: $x(t) = x_h(t) + v(t)$

3.5 Diagonalization

3.5.1 Given

A system of ODEs of the form

$$x' = Ax + g(t)$$

No homogenous solutions needed!

3.5.2 Technique

Remember that given a matrix A and an eigenbasis $\{\xi_1, ..., \xi_n\}$, we can construct a Change of Basis matrix $T = [\xi_1, ..., \xi_n]$, with

$$T^{-1}AT = D = \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \dots & \dots & \dots \end{bmatrix} \tag{6}$$

Given these properties, we can reduce a system of ODEs into a series of 1st order DEs. How? First let

$$x = Ty$$

For some y. We then get

$$Ty' = ATy + g(t)$$

Then we simply multiply on the left (because matrix multiplication order matters!) by T^{-1}

$$y = T^{-1}ATy + T^{-1}g(t)$$

And, using (6) we get

$$y = Dy + T^{-1}g(t)$$

Now, since D is diagonal, we have a series of 1st order ODEs! Split the one matrix equation into it's component system, and the ith equation/row will only have

$$y_i' = \lambda_i y_i + \left[T^{-1}g(t)\right]_i$$

Using a method like the Integrating factor to solve each of these, we can obtain y, and can now reverse the transformation to get back the solution to our first equation, x. Our final solution will be

$$x(t) = Ty$$

To summarize

1. Find the eigenvectors and the eigenbasis for A (solving the characteristic equation and doing row reduction is the simplest method)

2. Construct T by normalizing each eigenvector and making them the columns of T.

3. Let $x = Ty$, and multiply to the left by T^{-1}. Remember $T^{-1}AT = D$, the matrix with $D_{i,i} = \lambda_i$ and all else 0.

4. Separate this into n 1st order linear equations with the form $y_i' = \lambda_i y_i + \left[T^{-1}g(t)\right]_i$. Solve them using whatever technique (Integrating factor always works).

5. Reconstruct the vector function y. The final solution will be $x(t) = Ty$, as we used before in 2.

4 More Powerful Techniques

4.1 Power Series

4.1.1 Given

Anything. Seriously. Any ODE. We can get *a* solution. Possibly a disgusting one. For the sake of making the example easy, we'll assume a 2nd order equation of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

4.1.2 Technique

Assume our solution is a **Power Series**. This is what makes this technique so **power**ful. Get it? No? Ok.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Note that

$$y' = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$
$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

Then note these 2 techniques to shift indices of summations and the power of the $(x - x_0)^n$ term.

$$\sum_{n=k}^{\infty} a_n (x - x_0)^{n-k} = \sum_{n=k}^{\infty} a_{n+k} (x - x_0)^n \tag{7}$$

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + ... + a_k(x - x_0)^k + \sum_{n=k+1}^{\infty} a_n (x - x_0)^n \tag{8}$$

With these techniques and derivatives, we have one goal: rearrange the differential equation into the form

$$a_0 + ... + a_k(x - x_0)^k + \sum_{n=k+1}^{\infty} F(a_n, a_{n+1}, ..., a_{n+l}) = 0$$

From which we learn that $a_0 = ... = a_k = 0$ and $F(a_n, ..., a_{n+l}) = 0, \forall n > k$. From this second equation, we can derive a **Recurrence Relation** that relates a_{n+l} to the previous l terms. We can always let a_0 and a_1 vary (since it's a 2nd order ODE), analagous to our C_1 and C_2 in previous techniques, and solve for all higher a_n through these. Sometimes we will be able to find a closed form solution for a_n, and in even rarer circumstances, we may be able to identify that as a taylor series of a function, in which case we can remove summations from our answer. But in general, our solution will look like what we originally guessed,

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

With some sort of recurrance relation for a_n. Yeah, this is pretty ugly. To sum up

1. Substitute in a general power series.
2. Manipulate sums and powers until we get one large sum equal to zero.
3. Set all coefficients equal to 0. Obtain a recurrence relation for a_n
4. Attempt to find a closed source form for a_n (this is tricky - have fun). Further, see if this is a taylor series for an elementary function. Even if not, we "solved" it, to an arbitrary precision.

4.2 Laplace Transform

4.2.1 Given

An ODE. That's really about it. One caveat is that the functions involved must have Laplace and Inverse Laplace transforms that we know of. We'll just say it looks like

$$\sum_{i=0}^{n} a_i(t)y^{(i)} = g(t)$$

Yeah. That general.

4.2.2 Technique

We define the Laplace Transform of a function as

$$\mathscr{L}\{f(x)\} = \int_0^\infty e^{-st}f(t)dt$$

Note that it's linear: that

$$\mathscr{L}\{af(x) + bg(x)\} = a\mathscr{L}\{f(x)\} + b\mathscr{L}\{g(x)\}$$

Most importantly, if we evaluate this for a derivative,

$$\begin{aligned}
\mathscr{L}\{f'(x)\} &= \int_0^\infty e^{st}f'(t)dt \\
\mathscr{L}\{f'(x)\} &= \left[e^{-st}f(t)\right]_0^\infty - \left(-s\int_0^\infty e^{-st}f(t)dt\right) \\
\mathscr{L}\{f'(x)\} &= -f(0) + s\int_0^\infty e^{-st}f(t)dt \\
\mathscr{L}\{f'(x)\} &= s\mathscr{L}\{f(x)\} - f(0)
\end{aligned}$$

With $f(0)$ and $f'(0)$ either being arbitrary constants of the general solution, or initial conditions.

Applying this further

$$\mathscr{L}\{f^{(n)}\} = s^n\mathscr{L}\{f(x)\} - \sum_{i=0}^{n} s^{n-i-1}f^{(i)}(0)$$

But we probably don't want to use this since it's so ugly.

Now the amazing thing is that this transforms differential equations into algebraic equations. That means that given any differential equation, we can get some algebraic equation we can solve for $\mathscr{L}\{y\}$. Transforming something merely requires an indefinite integral. however, the inverse transform $\mathscr{L}^{-1}$ is a lot more difficult to compute. For that reason, we typically have look-up tables to transform $\mathscr{L}\{y\}$ back into y.

To summarize the technique:

1. Apply the Laplace Transform to both sides of the DE.
2. Solve for $\mathscr{L}\{y\}$ algebraically.
3. Manipulate the RHS until we can use a lookup table to apply the inverse transform (typically partial fractions).

AND WE'RE DONE HERE FAM.