HHU Düsseldorf
Math.-Nat. Fakultät
Prof. Dr. Nils Detering
Dr. Nicole Hufnagel

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Markov Chains

Problem sheet 3

random walks, markov chains

Problems to be handed in by:
Thursday, May 23rd, 11:59 p.m., online via Ilias.

Problem 1 *(6 points)*
Consider the Gamblers Ruin problem covered in class. The basic setup is unchanged and you want to determine the ruin probabilty $\mathbb{P}(S_{T_{10,0,80}} = 0)$. However, instead of betting \$1 on red in each round, you bet \$2 on red in each round. What is the ruin probability now? Compare it with the situation when betting \$1 on red in each round.

Solution:
We consider the Gamblers Ruin problem from chapter 3.3 with $p \neq \frac{1}{2}$. Theorem 3.15 gives us the ruin probability

$$\mathbb{P}(S_{T_{10,0,80}} = 0) = \frac{\left(\frac{1-p}{p}\right)^{80} - \left(\frac{1-p}{p}\right)^{10}}{\left(\frac{1-p}{p}\right)^{80} - 1}$$

If we bet \$2 on red each round, we can simply rescale our game by the factor $\frac{1}{2}$ to normalize the gains and losses back to 1. This also grants us new values $\tilde{x} = \frac{10}{2}, \tilde{a} = \frac{0}{2}, \tilde{b} = \frac{80}{2}$. By Theorem 3.15:

$$\mathbb{P}(S_{T_{5,0,40}} = 0) = \frac{\left(\frac{1-p}{p}\right)^{40} - \left(\frac{1-p}{p}\right)^{5}}{\left(\frac{1-p}{p}\right)^{40} - 1}$$

Comparing these 2 expressions shows that

$$\mathbb{P}(S_{T_{5,0,40}} = 0) > \mathbb{P}(S_{T_{10,0,80}} = 0) \Leftrightarrow p > \frac{1}{2}$$

(In an advantageous game ($p > \frac{1}{2}$) smaller stakes result in a lower ruin probability. When the game is disadvantageous ($p < \frac{1}{2}$), it is better to bet higher values ... or not to play at all.)

Problem 2 *(8 points)*
Let $X_0, X_1, X_2, \dots$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$, transition probabilities

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/5 & 1/10 & 7/10 \end{pmatrix}$$

and initial distribution $\alpha^T = (1/3, 1/3, 1/3)$. Find the following probabilities:

(a) $\mathbb{P}[X_5 = 2 \,|\, X_4 = 1]$,

(b) $\mathbb{P}[X_1 = 1,\, X_2 = 2]$,

(c) $\mathbb{P}[X_1 = 2 \,|\, X_2 = 1]$,

(d) $\mathbb{P}[X_5 = 1 \,|\, X_1 = 2, X_2 = 3, X_3 = 2]$.

Solution:
We use the definition of the n-step transition matrix to get

(a) $\mathbb{P}(X_5 = 2 \,|\, X_4 = 1) = P_{12} = 1$

(b) $$\begin{aligned}\mathbb{P}(X_1 = 1,\, X_2 = 2) &= \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 2 \,|\, X_1 = 1) \\ &= (\alpha^T P)_1 P_{12} = \frac{8}{45} \cdot 1 = \frac{8}{45}\end{aligned}$$

(c) $$\begin{aligned}\mathbb{P}(X_1 = 2 \,|\, X_2 = 1) &\overset{\text{Bayes}}{=} \frac{\mathbb{P}(X_2 = 1 \,|\, X_1 = 2)\mathbb{P}(X_1 = 2)}{\mathbb{P}(X_2 = 1)} \\ &= \frac{P_{21}(\alpha^T P)_2}{(\alpha^T P^2)_1} = \frac{\frac{1}{3} \cdot \frac{11}{30}}{\frac{16}{75}} = \frac{55}{96}\end{aligned}$$

(d) $\mathbb{P}(X_5 = 1 \,|\, X_1 = 2, X_2 = 3, X_3 = 2) \overset{4.7}{=} \mathbb{P}(X_5 = 1 \,|\, X_3 = 2) \overset{4.7}{=} P^2_{21} = \frac{2}{15}$.

Problem 3 *(8 points)*
Consider a Markov chain $(X_n)_{n=0,1,2,\dots}$ with state space $\mathcal{S} = \{1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 0 & 1/2 & 1/2 \\ 3/10 & 7/10 & 0 \end{pmatrix}.$$

The initial distribution is given by $\alpha^T = (1/2, 1/6, 1/3)$. Compute

(a) $\mathbb{P}[X_2 = k]$ for all $k = 1, 2, 3$;

(b) $\mathbb{E}[X_2]$.

Does the distribution of X_2 computed in (a) depend on the initial distribution α? Does the expected value of X_2 computed in (b) depend on the initial distribution α? Give a reason for both of your answers.

Solution:

(a) With the given transition matrix and initial distribution, we obtain the distribution of X_2 by calculating

$$\alpha^T P^2 = (1/2, 1/6, 1/3) \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 0 & 1/2 & 1/2 \\ 3/10 & 7/10 & 0 \end{pmatrix}^2 = (19/200, 167/300, 209/600)$$

This means that we have

$$\mathbb{P}(X_2 = 1) = \frac{19}{200}, \quad \mathbb{P}(X_2 = 2) = \frac{167}{300}, \quad \mathbb{P}(X_2 = 3) = \frac{209}{600}$$

(b) $\mathbb{E}[X_2] = 1 \cdot \frac{19}{200} + 2 \cdot \frac{167}{300} + 3 \cdot \frac{209}{600} = \frac{169}{75} = 2.25\overline{3}$

Let $\tilde{\alpha}^T = (a_1, a_2, 1 - a_1 - a_2)$ be an arbitrary initial distribution. Just as before we obtain the distribution by calculating

$$\tilde{\alpha}^T P^2 = \frac{1}{100} \left(4a_1 + 9a_2 + 6, 3a_1 + 7a_2 + 53, -7a_1 - 16a_2 + 41\right),$$

which depends on the choice of a_1 and a_2. Similarly we get

$$\mathbb{E}[X_2] = \frac{-11a_1 - 25a_2 + 235}{100},$$

which also depends on the initial distribution.

Problem 4 *(8 points)*

A stochastic matrix is called *doubly stochastic* if its columns sum to 1. Let $X_0, X_1, ...$ be a Markov chain on the state space $\mathcal{S} = \{1, ..., k\}$ with doubly stochastic transition matrix P and initial distribution that is uniform on $\mathcal{S}$.

Show that the distribution of X_n is uniform on $\mathcal{S}$ for all $n \geq 0$.

Solution:

We use induction for this proof. We already know that the distribution α of X_0 is uniform on $\mathcal{S}$.

If for any $n \in \mathbb{N}_0$ the distribution of X_n is uniform on $\mathcal{S}$, X_{n+1} has the distribution

$$\mathbb{P}(X_{n+1} = i) = \left((\frac{1}{k}, \ldots, \frac{1}{k}) P \right)_i = \sum_{j=1}^{k} \frac{1}{k} P_{ji} = \frac{1}{k} \underbrace{\sum_{j=1}^{k} P_{ji}}_{\substack{=1, \text{doubly} \\ \text{stochastic}}} = \frac{1}{k} \quad \text{for all } i \in \mathcal{S}.$$

This means that the distribution of X_{n+1} is also uniform on $\mathcal{S}$. The result now follows by induction.