Exercise 11

Thursday, January 18, 2024 12:08 PM

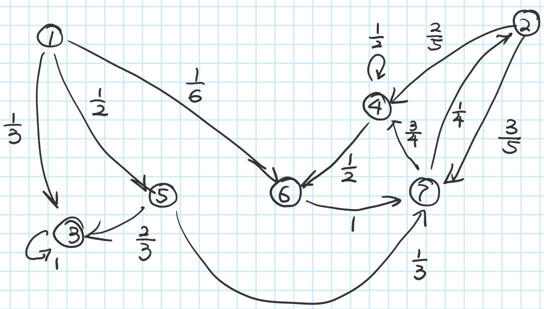
Exercise 43 (11 points)

Consider a Markov chain with state space $\{1, \dots, 7\}$ given by the following transition matrix:

$$\begin{array}{c|c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix} \end{array}$$

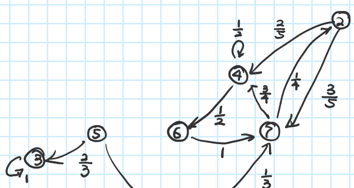
$$P = \begin{array}{c|c} & \begin{array}{ccccccc} 1 & 5 & 3 & 2 & 4 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 5 \\ 3 \\ 2 \\ 4 \\ 6 \\ 7 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \end{pmatrix} \end{array}$$

(a) (1 point) Draw the transition diagram.

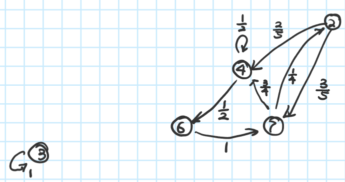


(b) (4 point) Say for each state whether it is transient or recurrent and justify your answer.

① is transient, since it can't be visited again after leaving.



⑤ is transient for the same reason, after excluding ①



③ is recurrent since it's the only state in the rest sub Markov chain.

②, ④, ⑥, ⑦ are reccurent since there must be a recurrent state in the sub chain and the three states can reach each other freely.

(c) (4 points) Determine $\lim_{n\to\infty} p_{ii}(n)$ for all i.

Rewite the transition matrix

$$P = \begin{array}{c|c} & \begin{array}{ccc|cccc} 1 & 5 & 3 & 2 & 4 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 5 \\ 3 \\ 2 \\ 4 \\ 6 \\ 7 \end{array} & \left(\begin{array}{ccc|cccc} 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \end{array}\right) \end{array} = \begin{array}{c|c} & \begin{array}{cc} 153 & 2467 \end{array} \\ \begin{array}{c} 153 \\ 2467 \end{array} & \begin{pmatrix} P_{I} & P_{II} \\ 0 & P_{IV} \end{pmatrix} \end{array}$$

$$P = \begin{array}{c|ccccccc} & 1 & 5 & 3 & 2 & 4 & 6 & 7 \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 \\ 5 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 4 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 7 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \end{array} = \begin{array}{c|cc} & 153 & 2467 \\ \hline 153 & P_{I} & P_{II} \\ 2467 & 0 & P_{IV} \end{array}$$

$P^n = \begin{pmatrix} P_I^n & \tilde{P}_{II} \\ 0 & P_{IV}^n \end{pmatrix}$, where $\tilde{P}_{II}$ is a complex matrix multiplication of P_I, P_{II} and P_{IV}

$$P_I^n = \begin{pmatrix} p_{11}^n & ? & ? \\ 0 & p_{55}^n & ? \\ 0 & 0 & p_{33}^n \end{pmatrix} = \begin{pmatrix} 0 & ? & ? \\ 0 & 0 & ? \\ 0 & 0 & 1 \end{pmatrix}$$

, so $\lim_{n\to\infty} p_{11}^n = \lim_{n\to\infty} p_{55}^n = 0$

and $\lim_{n\to\infty} p_{33}^n = 1$

$$P_{IV} = \begin{array}{cccc} 2 & 4 & 6 & 7 \end{array} \begin{pmatrix} 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \end{pmatrix}$$

The sub-Markov chain is irreducible

and has non-zero entry $\frac{1}{2}$ on the diagonal

$\Rightarrow$ The Markov chain is regular.

$\Rightarrow$ for $\pi P_{IV} = \pi$, π is unique and

$\pi = \lim_{n\to\infty} (p_{22}^{(n)}\ p_{44}^{(n)}\ p_{66}^{(n)}\ p_{77}^{(n)})$

(d) (1 point) Find two different invariant distributions.

$\pi = (\pi_1, \pi_2 \cdots \pi_7)$

for any invariant distributions

the transient state has prob. 0

$\Rightarrow \quad \pi_1 = \pi_5 = 0$

1. Since ③ forms a sub Markov chain

let $\pi_3 = 1$, $\pi_i = 0$ for $i \neq 3$

$\Rightarrow \pi P = (0\ 0\ 1\ 0\ 0\ 0\ 0) P = \pi$

2. Let $\pi_3 = 0$

$(\pi_2\ \pi_4\ \pi_6\ \pi_7) P_{IV} = (\pi_2\ \pi_4\ \pi_6\ \pi_7)$

$$\pi' \begin{pmatrix} -1 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 & -1 \end{pmatrix} = 0 \Rightarrow \begin{cases} 4\pi_2 = \pi_7 \\ \pi_4 = 2\pi_6 \\ \pi_6 = \frac{17}{5}\pi_2 \\ \Sigma\pi = 1 \end{cases}$$

$\pi_2 + \frac{34}{5}\pi_2 + \frac{17}{5}\pi_2 + 4\pi_2 = 1$

$25 + 34 + 17$

$51 + 25 = \frac{76}{5}$

$\frac{34}{76} = \frac{17}{38}$

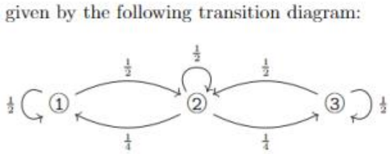
$\Rightarrow \pi' = (\frac{5}{76}, \frac{17}{38}, \frac{17}{76}, \frac{5}{19})$

$\lim_{n\to\infty} p_{22}^{(n)}$ $\lim_{n\to\infty} p_{44}^{(n)}$ $\lim_{n\to\infty} p_{66}^{(n)}$ $\lim_{n\to\infty} p_{77}^{(n)}$

$\pi = (0\ 0\ 0\ \frac{5}{76}\ \frac{17}{38}\ \frac{17}{76}\ \frac{5}{19})$

Exercise 44 (11 points)

Consider the Markov chain given by the following transition diagram:



(a) (1 point) Write down the transition matrix of the Markov chain

$$P = \begin{array}{ccc} 1 & 2 & 3 \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \ldots \end{pmatrix}$$

$$P = \begin{array}{ccc} 1 & 2 & 3 \\ \end{array}$$
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b) (2 points) Say, with proof, which states are transient and which states are recurrent.

All the states are accessable to any other states,

since the finte states Markov chain must have at least one recurrent states,

states ① ② ③ are all recurrent.

(c) (4 points) Show that there is a unique stationary distribution and compute it.

$$P = \frac{1}{4}\begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$P^2 = \left(\frac{1}{4}\right)^2 \begin{pmatrix} 4+2+0 & 4+4+0 & 0+2+0 \\ 2+2+0 & 2+4+2 & 0+2+2 \\ 0+2+0 & 0+4+4 & 0+2+4 \end{pmatrix}$$

$$= \left(\frac{1}{4}\right)^2 \begin{pmatrix} 6 & 8 & 2 \\ 4 & 8 & 4 \\ 2 & 8 & 6 \end{pmatrix}$$

all the elements of P^2 is positive

$\Rightarrow$ the Markov chain is regular

$\Rightarrow$ there is a unique stationary distribution

$\pi = (\pi_1 \ \pi_2 \ \pi_3)$

$\pi P = \pi$

$$\Rightarrow \quad \pi(P-I) = \pi \cdot \frac{1}{4}\begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} = 0$$

$$\begin{cases} \pi_2 = 2\pi_1 \\ \pi_2 = 2\pi_1 \\ \sum \pi = 1 \end{cases} \quad \Rightarrow \quad \pi = \left(\frac{1}{4} \ \ \frac{1}{2} \ \ \frac{1}{4}\right)$$

(d) (1 point) If one starts in state ②, is the probability distribution of the states after n steps equal to the stationary distribution for some n? Justify your answer.

for $n=1$

$$(0 \ 1 \ 0)\, P^1 = \left(\frac{1}{4} \ \ \frac{1}{2} \ \ \frac{1}{4}\right) = \pi \quad \checkmark$$

(e) (2 points) (tricky!) If one starts in state ①, is the probability distribution of the states after n steps equal to the stationary distribution for some n? Justify your answer.

diagonalize P:

$$\det(P - \lambda I) = \frac{1}{4}\det\begin{pmatrix} 2-4\lambda & 2 & 0 \\ 1 & 2-4\lambda & 1 \\ 0 & 2 & 2-4\lambda \end{pmatrix} = 0$$

$$8(1-2\lambda)^3 - 8(1-2\lambda) = 0 \qquad 4\lambda^2 - 4\lambda$$

$8(1-2\lambda)^3 - 8(1-2\lambda) = 0$ $\qquad 4\lambda^2 - 4\lambda$

$(1-2\lambda)(1-4\lambda+4\lambda^2-1) = 0$

$(1-2\lambda)\lambda(\lambda-1) = 0$

$\lambda = 0, 1, \frac{1}{2}$

2

$\lambda_1 = 0, \quad v_1 = (1 \quad -1 \quad 1)$

$\lambda_2 = \frac{1}{2}, \quad v_2 = (-1 \quad 0 \quad 1)$

$\lambda_3 = 1, \quad v_2 = (1 \quad 1 \quad 1)$

$$U = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & & \\ & \frac{1}{2} & \\ & & 1 \end{pmatrix}$$

$$P^n = U\Sigma^n U^{-1}$$

$$= U \begin{pmatrix} 0 & & \\ & (\frac{1}{2})^n & \\ & & 1 \end{pmatrix} U^{-1}$$

$$(1 \ 0 \ 0) P^n = (1 \quad -1 \quad 1) \begin{pmatrix} 0 & & \\ & (\frac{1}{2})^n & \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$= (0 \quad -(\tfrac{1}{2})^n \quad 1) \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$= \left((\tfrac{1}{2})^{n+1} + \tfrac{1}{4} \quad \tfrac{1}{2} \quad -(\tfrac{1}{2})^{n+1} + \tfrac{1}{4} \right)$$

$$\overset{0}{=} \left(\tfrac{1}{4} \quad \tfrac{1}{2} \quad \tfrac{1}{4} \right)$$

so $(1\,0\,0)P^n$ iff. $n \to \infty$

there is no n s.t. $(1\,0\,0)P^n = (\frac{1}{4}\ \frac{1}{2}\ \frac{1}{4})$

Exercise 45 (9 points)

The AI researcher J. Doe thinks that Markov chains are awesome. While thinking about those, Doe switches between two states, E (excited enthusiasm) and C (calm satisfaction). In state E, Doe sits in an armchair smiling (S) with probability $\frac{1}{2}$ and dances through the office (D) with probability $\frac{1}{2}$. In state C, Doe sits in an armchair smiling (S) with probability 1 and dances (D) with probability 0.

From one minute to the next, Doe switches between the states E and C with probability α, and stays in the same state with probability $1-\alpha$.

At the beginning Doe is enthusiastic.

(a) (3 point) Describe the situation by a hidden Markov model, linking the observed sequence of Doe's behaviour $\{Y_t\}_{t\in\mathbb{N}}$ (where the Y_t take values S, D) and the latent sequence of Doe's states $\{X_t\}_{t\in\mathbb{N}}$ (where the X_t take values E, C). Concretely: Write down a distribution vector for the initial state π, a transition matrix P and an emission matrix B

(b) (7 points) Let $\alpha = \frac{1}{4}$. Suppose, starting at time $t = 1$ you see Doe sit, then dance, then sit again (i.e. $Y_1 = S, Y_2 = D, Y_3 = S$). What is the most likely sequence of states that Doe was in? [i.e. which sequence of states (x_1, x_2, x_3) maximizes $P(X_1 = x_1, X_2 = x_2, X_3 = x_3 \mid Y_1 = S, Y_2 = D, Y_3 = S)$?]

(a) $O = \{S, D\}, \quad S = \{E, C\}$

$\pi = (1 \quad 0) = (\pi_E \quad \pi_C)$

$$P = \begin{pmatrix} p_{EE} & p_{EC} \\ p_{CE} & p_{CC} \end{pmatrix} = \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix}$$

$B = (b_{ES} \quad b_{ED}$ ... $= E\ ($ S D ...

$p_{CE} \quad p_{CC} \qquad \alpha \quad 1-\alpha$

$$B = \begin{pmatrix} b_{ES} & b_{ED} \\ b_{CS} & b_{CD} \end{pmatrix} = \begin{array}{c|cc} & S & D \\ \hline E & \frac{1}{2} & \frac{1}{2} \\ C & 1 & 0 \end{array}$$

(b) $\alpha = \frac{1}{4}$, $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

Bayes' formula:

$$P(x_1, x_2, x_3 \mid S, D, S) = \frac{P(S, D, S \mid \overbrace{x_1, x_2, x_3}^{2^3 \text{ possibilities}}) \cdot P(x_1, x_2, x_3)}{P(S, D, S)}$$

$P(S,D,S) \rightarrow$ no need to calculate

Since $b_{CD} = 0$, $P(Y_2 = D \mid X_2 = C) = 0$. 4 possibilities left

$P(Y_1 = S, Y_2 = D, Y_3 = S \mid X_1 = E, X_2 = E, X_3 = E)$

$= b_{ES} \cdot b_{ED} \cdot b_{ES} = \frac{1}{8}$

$P(X_1 = E, X_2 = E, X_3 = E)$

$= \pi_E \cdot p_{EE}^2 = \frac{9}{16}$

$P(S, D, S \mid C, E, E)$

$= b_{CS} \cdot b_{ED} \cdot b_{ES} = \frac{1}{4}$

$P(C, E, E)$

$= \pi_C \, p_{CE} \, p_{EE} = 0$, not likely

$P(S, D, S \mid E, E, C)$

$= b_{ES} \cdot b_{ED} \cdot b_{CS} = \frac{1}{4}$

$P(E, E, C)$

$= \pi_E \cdot p_{EE} \cdot p_{EC} = \frac{3}{16}$

$P(S, D, S \mid C, E, C)$

$= b_{CS} \cdot b_{ED} \cdot b_{CS} = \frac{1}{2}$

$P(C, E, C)$

$= \pi_C \, p_{CE} \, p_{EC} = 0$, not likely

In summary

$$P(S, D, S \mid E, E, E)\, P(E, E, E) = \frac{1}{8} \cdot \frac{9}{16}$$

$$P(S, D, S \mid E, E, C)\, P(E, E, C) = \frac{1}{4} \cdot \frac{3}{16} = \frac{1}{8} \cdot \frac{6}{16} < \frac{1}{8} \cdot \frac{9}{16}$$

So Doe is most likely in excited enthusiasm all the time.