HHU DÜSSELDORF
MATH.-NAT. FAKULTÄT
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Summer semester 2024

Markov Chains

Problem sheet 4

markov chains

Problems to be discussed (in parts) during the exercise sections:

Problem 1

You are given the following transition matrix for a Markov chain with state space $\mathcal{S} = \{1, 2\}$.

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) Does the Markov chain have stationary distributions? If so, determine all of them. If not, explain why.

(b) Does the Markov chain have a limiting distribution? If so, determine it. If not, explain why.

Problem 2

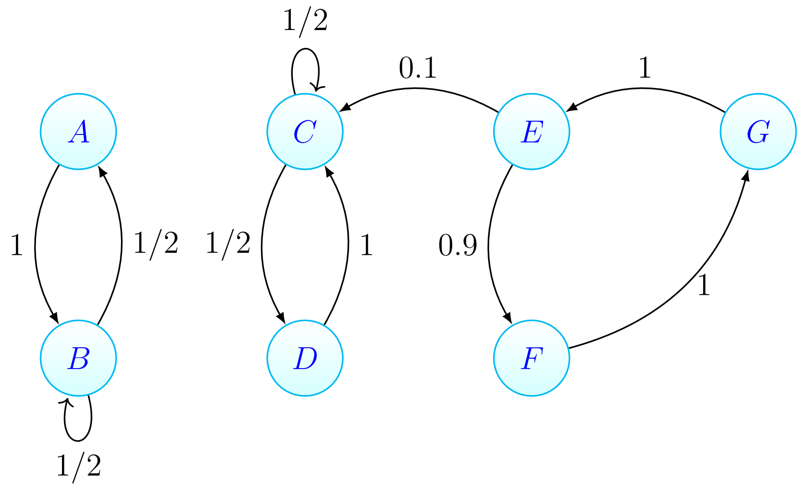
Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/4 & 0 & 3/4 \\ 0 & 1/2 & 1/2 \\ 2/3 & 1/3 & 0 \end{pmatrix}.$$

Compute the stationary distribution π.

Problem 3

A Markov chain $X_0, X_1, X_2, \ldots$ has the following transition graph:



(a) Provide the transition matrix for the Markov chain.

(b) Is there a limiting distribution? If so, determine it.

(c) Determine the set of stationary distributions if any exist. If not, explain why.

Problems to be handed in by:
Thursday, June 13th, 11:59 p.m., online via Ilias.

Problem 1 *(6 points)*
Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix} 0.4 & 0 & 0.6 \\ 0.3 & 0.1 & 0.6 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Compute the stationary distribution π.

Problem 2 *(8 points)*
Let $(X_n)_{n\geq 0}$ denote a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4\}$ and transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & p & 1-p & 0 \end{pmatrix}$$

where $p \in [0, 1]$ is some arbitrary probability.

(a) Provide the transition graph.

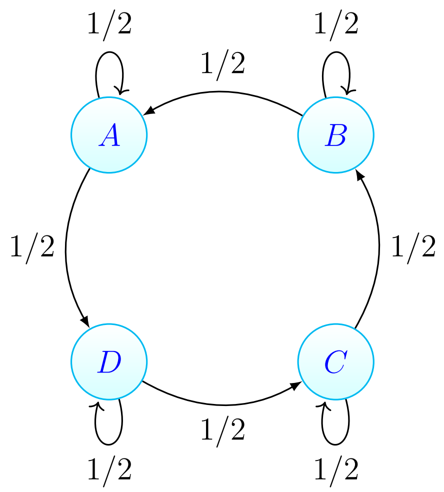
(b) Determine all communication classes. Which communication classes are closed?

Hint: You need to do a case-by-case analysis depending on the value of p.

(c) Fix $p = 0$. The initial distribution is given by $\alpha^T = (0, 0, \frac{5}{9}, \frac{4}{9})$. Compute the distribution of X_1. What is the distribution of X_{103}?

Problem 3 *(8 points)*

A Markov chain $X_0, X_1, X_2, \ldots$ has the following transition graph:



(a) Provide the transition matrix for the Markov chain.

(b) Classify all states (recurrent/transient).

(c) Find the communication classes. Is the chain irreducible?

(d) Find the stationary distribution.

(e) What do you know about the limiting distribution?

Problem 4 *(8 points)*

Consider a Markov chain on the state space $\mathcal{S} = \{1, 2, 3, 4, \ldots\}$ with the following transition matrix:

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 2/3 & 0 & 1/3 & 0 & 0 & \cdots \\ 3/4 & 0 & 0 & 1/4 & 0 & \cdots \\ 4/5 & 0 & 0 & 0 & 1/5 & \cdots \\ 5/6 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

That is, $P_{ij} = i/(i+1)$ if $j = 1$, $P_{ij} = 1/(i+1)$ if $j = i+1$, and $P_{ij} = 0$ otherwise.

(a) Classify all states of the Markov chain (transient, recurrent)

(b) Determine if the Markov chain is irreducible.