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Math for AI
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Exercise sheet 4

Exercise 12 (10 points)

(a) (6 points) Show that the spectral norm of a matrix A is equal to its largest singular value.

[Remark: Here you are allowed to use without proof that for real numbers $a_1, \dots, a_n$, a vector $x = (x_1, \dots, x_n)^T$ with $\|x\| = 1$ that maximizes $\sum_{i=1}^n a_i x_i^2$ is the standard basis vector e_k where k is the index with $a_k = \max\{a_i \mid i = 1 \dots n\}$. All the necessary arguments ocurred in the proof of Theorem 1.7.12.]

(b) (4 points, tricky) Prove that the matrix Σ in a singular value decomposition $A = U\Sigma V^T$ is unique (if one demands that the diagonal entries are all positive and ordered by size).

[Hint: Use the Eckart-Young-Mirsky theorem in its version for the spectral norm, together with (a).]

Exercise 13 (10 points)

Compute a singular value decomposition of the following matrix:

$$A = \begin{pmatrix} 2 & -2 & 1 \\ -4 & -8 & -8 \end{pmatrix}$$

Exercise 14 (10 points)

Find the polynomial of degree 2 in one variable that best approximates (in the sense of least squares) a function whose graph passes through the points $(-1,0), (2,1), (1,-1)$ and $(0,1)$ in $\mathbb{R}^2$. Use the idea of exercise 2 for this!

[Remark: It is up to you how you solve the exercise. One way is to use that the best approximate solution x to the equation $Ax = y$ is given by A^+y, where A^+ denotes the pseudoinverse of A – we will discuss this on Tuesday, it's Theorem 1.8.4. If you want to solve the exercise before Tuesday, you can simply use that. In the lecture I briefly showed the proof of Thm 1.8.2, i.e. how to compute a pseudoinverse in general. Alternatively, you have already seen how to compute a pseudoinverse for a full rank matrix in exercise 5. You can do it either way, but if you use the approach of exercise 5, don't forget to check that you actually have a full rank matrix. The method of exercise 5 then involves inverting a matrix – you don't need to show your calculations on how to find that inverse (e.g. you can let a computer find it). You can simply write down the inverse, preferredly in the form $\frac{1}{20}$ times an integer matrix.]

Exercise 15 (10 points)

See the notebooks for instructions! These are the subjects of the exercises:

(a) (4 points) Linear regression in Python
(b) (2 points) Linear regression by pseudoinverse in Python
(c) (4 points) Polynomial regression and overfitting

[Remark: You know everything for parts a) and c). The point of part b) is just to check, in a concrete example, the statement that the best approximate solution x to the equation $Ax = y$ is given by A^+y, where A^+ denotes the pseudoinverse of A. This statement will be proven on Tuesday, and then you will have more context, but you should be able to do the exercise beforehand.]

Deadline: Friday 10th of November, 10:00.
Upload your solution to this link.