Exercise 02

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Exercise 3 (8 points, eigenvalues will be discussed on Tuesday)

Compute the eigenvalues of the following matrix (4 points):

$$A = \begin{pmatrix} 2 & 12 & 17 \\ 0 & 0 & 3 \\ 0 & 2 & -1 \end{pmatrix}$$

For each eigenvalue find a basis for its space of eigenvectors (4 points).

Suppose $\lambda \in \mathbb{R}$ is the eigenvalue of matrix A:

then $\det(\lambda I - A) = 0$

$$\det \begin{pmatrix} \lambda-2 & -12 & -17 \\ 0 & \lambda & -3 \\ 0 & -2 & \lambda+1 \end{pmatrix} = (\lambda-2) \det \begin{pmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{pmatrix}$$

$$= \lambda(\lambda-2)(\lambda+1) - 6(\lambda-2)$$

$$= (\lambda-2)[\lambda(\lambda+1) - 6]$$

$$= (\lambda-2)(\lambda^2+\lambda-6)$$

$$= (\lambda-2)(\lambda+3)(\lambda-2) = 0$$

$\Rightarrow \lambda = 2$ or $\lambda = -3$

suppose eigenvector $v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$

For $\lambda = 2$

$$(\lambda I - A)v = \begin{pmatrix} 0 & -12 & -17 \\ 0 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} 12x_2 + 17x_3 = 0 \\ 2x_2 + 3x_3 = 0 \end{matrix}\right\} \Rightarrow x_2 = x_3 = 0$$

let $x_1 = 1$

eigenvector of $\lambda = 2$ can be $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which is a basis for eigenvector space $\left\{ v = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}$

For $\lambda = -3$

$$(\lambda I - A)v = \begin{pmatrix} -5 & -12 & -17 \\ 0 & -3 & -3 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(\lambda I - A)v = \begin{pmatrix} -5 & -12 & -17 \\ 0 & -3 & -3 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_2 - 3x_3 = 0$$

$$x_2 = -x_3$$

$$5x_1 + 12x_2 + 17x_3 = 5x_1 - 5x_2 = 0$$

$$x_1 = x_2$$

let $x_1 = 1$

then the eigenvector of $\lambda = -3$ can be $(1, 1, -1)^T$, which is a basis for eigenvector space $\{v = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix} \mid \lambda \in \mathbb{R}\}$

Exercise 4 (10 points)

Let A be an $n \times m$-matrix (i.e. n rows, m columns) of rank n.

(a) (3 points) Show that $x \mapsto A^T x$ is an injective map.

(b) (5 points) Show that $A \cdot A^T$ is an invertible $n \times n$-matrix.

(c) (2 points) Conclude from (b) that if B is an $n \times m$-matrix with m linearly independent columns, then $B^T \cdot B$ is an invertible $m \times m$-matrix.

[*Hint for (b):* Prove and use that for a vector v we have $v^T \cdot v = 0$ if and only if $v = 0$.]

A is an $n \times m$-matrix of rank n

$\Rightarrow m \geq n$

(a) $x \mapsto A^T x$ is a linear map $f: \mathbb{R}^n \to \mathbb{R}^m$

$\text{rank}(A) = n$

$\Leftrightarrow \text{rank}(A^T) = n$

$\Leftrightarrow \dim(\text{Im} f) = n$

According to Dimension Formula:

$\dim \mathbb{R}^n = \dim(\text{Ker} f) + \dim(\text{Im} f) = n$

$\Rightarrow \dim(\text{Ker} f) = n - n = 0$

so $\text{Ker} f = \{0\}$

$\Leftrightarrow f(v) = A^T \cdot v = 0$ iff $v = 0$

$\Rightarrow A^T \cdot v$ is injective

(b) $x \mapsto A \cdot x$ is a linear map $g: \mathbb{R}^m \to \mathbb{R}^n$

so $x \mapsto A \cdot A^T x \;:\; (g \circ f)\, \mathbb{R}^n \to \mathbb{R}^n$

prep. Given $v \in \mathbb{R}^n$, prove $\langle v, v \rangle = \|v\|^2$ iff $v = 0$

let $v = (v_1, \dots, v_n)^T$, suppose $\exists v \neq 0$,

and $\langle v, v \rangle = v^T \cdot v = v_1^2 + \cdots + v_n^2 = 0$

Since $v \neq 0$, $\exists\, v_k \in \{v_1, \cdots, v_n\}$ s.t. $v_k \neq 0 \Rightarrow v_k^2 > 0$

$$\langle v, v\rangle = v_k^2 + \sum_{i=1, i\neq k}^{n} v_i^2$$

$$\left.\begin{array}{l} \sum_{i=1, i\neq k}^{n} v_i^2 \geq 0 \\ v_k^2 > 0 \end{array}\right\} \Rightarrow \langle v, v\rangle > 0 \perp \text{ contradiction!}$$

So $\langle v, v\rangle = 0 \Rightarrow v = 0$

and if $v = 0$, $\langle v, v\rangle = (0, \cdots, 0)\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0$

Therefore. $\langle v, v\rangle = 0$ iff $v = 0$

To prove AA^T is invertible, it's enough to prove

$g \circ f : x \mapsto AA^T$ is bijective/injective
$\mathbb{R}^n \to \mathbb{R}^n$

Since $(g \circ f)$ is a linear map: $\{0\} \subseteq Ker(g \circ f)$

Suppose $v \in \mathbb{R}^n$, s.t. $A \cdot A^T \cdot v = 0$

$\Rightarrow v^T \cdot A \cdot A^T \cdot v = \langle v, AA^T v\rangle = \langle v, 0\rangle = 0$

$\Rightarrow \langle A^T v, A^T v\rangle = 0$

$\Rightarrow A^T v = 0$

since $x \mapsto A^T x$ is injective $A^T v = 0 \Leftrightarrow v = 0$

so $Ker(g \circ f) \subseteq \{0\} \Rightarrow Ker(g \circ f) = \{0\}$

$\Rightarrow (g \circ f)\, x \to AA^T x$ is injective

Since the map is from $\mathbb{R}^n \to \mathbb{R}^n$, the map is also bijective. Therefore matrix $A \cdot A^T$ is invertible.

(c) Since B has m linear independent columns:

$rank(B) = m$

$\Rightarrow$ if $m \geq n$, $rank(B) = \min\{m, n\} = n = m$

$\Rightarrow$ B is an $n \times n$ square matrix

Let $B^T = A$, which is an $m \times n (m = n)$ matrix
with $rank(B^T) = rank(B) = n$

according to b):
$A \cdot A^T = (B^T) \cdot (B^T)^T = B^T \cdot B$ is invertible $m \times m$ matrix

$\Rightarrow$ More generally, since $rank(B) = m$, $m \leq n$

let $B^T = A$, which is an $m \times n$ matrix
$(m \leq n)$

according to b)
$A \cdot A^T = (B^T) \cdot (B^T)^T = B^T \cdot B$ is an
invertible $m \times m$ matrix

Exercise 5 (12 points)

A pseudoinverse of a matrix A is a matrix A^+ such that all of the following equations hold:

(i) $AA^+A = A$

(ii) $A^+AA^+ = A^+$

(iii) $(AA^+)^T = AA^+$

(iv) $(A^+A)^T = A^+A.$

Let A be an $n \times m$-matrix with linearly independent columns. Show that $(A^TA)^{-1}A^T$ is a pseudoinverse of A (2 points for each property).

Show that for invertible matrices B one has $(B^T)^{-1} = (B^{-1})^T$ and use it on the way (2 points).

[Warning: By exercise 4 above, A^TA is indeed invertible – but A need not be invertible (not even quadratic), so you can in general not form A^{-1}.]

(i) $AA^+A = A((A^TA)^{-1}A^T)A$

$= A((A^TA)^{-1}A^TA)$

$= A(\underbrace{(A^TA)^{-1}(A^TA)}_{I}) = A$

(ii) $A^+AA^+ = ((A^TA)^{-1}A^T)AA^+$

$= (\underbrace{(A^TA)^{-1}(A^TA)}_{I})A^+$

$= A^+$

—

If B is invertible, $\exists\, B^{-1}$ s.t. $B^{-1}\cdot B = B \cdot B^{-1} = I$

$(B^{-1}\cdot B)^T = I^T \qquad (B\cdot B^{-1})^T = I^T$

$B^T \cdot (B^{-1})^T = I \qquad (B^{-1})^T \cdot B^T = I$

so B^T is also invertible and

$(B^T)^{-1} = (B^{-1})^T$

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(iii) $(AA^+)^T =$

$(A(A^TA)^{-1}A^T)^T$

$= (A^T)^T((A^TA)^T)^{-1}A^T$

$= A\cdot(A^T\cdot A)^{-1}\cdot A^T = AA^+$

(iv) $(A^+A)^T = ((A^TA)^{-1}A^TA)^T$

$= A^TA((A^TA)^{-1})^T$

$= A^TA(A^TA)^{-1} = I$

$= (A^TA)^{-1}A^TA = A^+A$