Exercise 10

Friday, December 22, 2023 5:00 AM

Exercise 37 (10 points)

A coin shows heads with probability p and tails with probability $(1-p)$. The random variable X counts how many times the coin is thrown until heads is shown for the first time. Thus X takes values in the natural numbers with distribution $P(X=k) = p(1-p)^{k-1}$ (a so-called geometric distribution).

In a series of n experiments, the numbers of throws until the first appearance of tails are: $k_1+1, \dots, k_n+1$ (i.e. $k_1, \dots, k_n$ are the numbers of tails before the first head).

Let $a, b > 0$. Compute the a posteriori distribution for p on $(0,1)$, where the prior distribution on $(0,1)$ has the Beta(a,b)-distribution, i.e. density function

$$h(p) := \begin{cases} \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1} & \text{if } 0 < p < 1 \\ 0 & \text{otherwise} \end{cases}$$

Here $B(a,b) := \int_0^1 u^{a-1}(1-u)^{b-1} du$ is the normalizing constant ensuring that the above is really a density function on $(0,1)$. Show that the a posteriori distribution is again a Beta(a',b')-distribution. What are the new parameters a', b'?

Likelihood function $\quad L = \prod_{i=1}^{n} P(X = k_i + 1)$

$$= \prod_{i=1}^{n} p\,(1-p)^{k_i}$$

$$= p^n (1-p)^{k_1+\cdots+k_n}$$

Posterior: $\quad f(p \mid x_1, \dots, x_n) = \dfrac{p^n(1-p)^{k_1+\cdots k_n} \cdot h(p)}{\int_0^1 v^n (1-v)^{k_1+\cdots+k_n} dv} \quad$ $:= B(n+1, \sum_{i=1}^{n} k_i + 1)$

$$= \underbrace{\frac{1}{B(a,b)\, B(n+1, \sum_{i=1}^{n} k_i + 1)}}_{\text{normalizing constant}} \; p^{a+n-1} (1-p)^{b + \sum_{i=1}^{n} k_i - 1}$$

$$= \text{Beta}\left(a+n,\ b + \sum_{i=1}^{n} k_i\right)$$

$$\Rightarrow a' = a+n, \quad b' = b + \sum_{i=1}^{n} k_i$$

Exercise 40 (6 points)

In a game a coin is thrown repeatedly until it shows heads for the first time. Let the random variable X count the total number of coin throws in the game. If the probability of showing heads in a single throw is p, then X has the geometric distribution:

$$P(X = n) = p(1-p)^{n-1}$$

You think that the heads side shows less than the tails side. More precisely, your beliefs about the bias of the coin are expressed by the density function $p \mapsto 3(p-1)^2$.

Now you play the game three times and get 2, 5 and 1 coin throws. Compute the maximum a posteriori estimate for p using this data.

Likelihood function

$$f(X_1=2, X_2=5, X_3=1 \mid p)$$

$$= p(1-p)^{2-1} p(1-p)^{5-1} p(1-p)^{1-1}$$

$$= p^3(1-p)^5$$

posterior $f(p \mid 2,5,1) = \dfrac{f(2,5,1,p)\, h(p)}{\int_0^1 f(2,5,1,u)\,du := C_1}$

Note: p is continuous

$\Rightarrow f(p \mid 2,5,1)$ is a PDF

$$= \frac{3}{C_1} p^3(1-p)^7$$

$$\overset{C := \frac{3}{C_1}}{=} C p^3(1-p)^7$$

to find the MAP for p:

$$\frac{\partial f(p \mid 2,5,1)}{\partial p} = C\left(3p^2(1-p)^7 - 7p^3(1-p)^6\right) = 0$$

$$\Rightarrow 3p^2(1-p)^7 = 7p^3(1-p)^6$$

$$\overset{p\neq 0, p\neq 1}{\Rightarrow} 3(1-p) = 7p$$

$$3 - 3p = 7p$$

$$p = 0.3$$

$f(0.3 \mid 2,5,1) > 0$

$f(0 \mid 2,5,1) = f(1 \mid 2,5,1) = 0$

$\Rightarrow p$ is the MAP estimate

Exercise 41 (14 points)

The Beta distribution with parameters α, β is the distribution on the unit interval $(0,1)$ with density function $B(\alpha, \beta)$, given by

$$B(\alpha, \beta)(x) = C \cdot x^{\alpha-1}(1-x)^{\beta-1}.$$

The binomial distribution for n trials with parameter p is a discrete distribution on $\{0, \ldots, n\}$, giving probabilities for the number of successes in n experiments with success probability p. Its probability mass function is:

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

The family of Beta distributions (parametrized by α, β) forms a conjugate family for the family of binomial distibutions (parametrized by p).

$\prod_{i=1}^{3}$ n

Suppose you plant 3 batches of 5 seeds each, and count how many of them sprout. From the first batch all 5 seeds sprout. From the second batch 3 seeds sprout. From the third batch 4 seeds sprout.

k_1 k_2 k_3

(a) (7 points) If your prior distribution was $B(\alpha, \beta)$ for some fixed numbers α, β, which member of the Beta family is your posterior distribution?

(b) (7 points) Compute the maximum a posteriori estimate for p as a function of α, β.

(a)
$$f(X_1=5,\ X_2=3,\ X_3=4 \mid p)$$
$$= \binom{5}{5} p^5 \cdot \binom{5}{3} p^3 (1-p)^2 \cdot \binom{5}{4} p^4 (1-p)$$
$$= C_1 p^{12} (1-p)^3$$

$$f(p \mid 5,3,4) = \frac{C_1 p^{12}(1-p)^3 \cdot C_2 p^{\alpha-1}(1-p)^{\beta-1}}{C_3}$$
$$= \underbrace{\left(\frac{C_1 C_2}{C_3}\right)}_{:=C} p^{\alpha+12-1} (1-p)^{\beta+3-1} = B(\alpha+12, \beta+3)$$

(b)
$$\frac{\partial f(p \mid 5,3,4)}{\partial p} = C\left[(\alpha+11) p^{\alpha+10} (1-p)^{\beta+2} - (\beta+2) p^{\alpha+11} (1-p)^{\beta+1}\right] = 0$$

$$\overset{p\neq 0,\, p \neq 1}{\Longrightarrow} \quad (\alpha+11)(1-p) = (\beta+2)p$$
$$\alpha+11 - (\alpha+11)p = (\beta+2)p$$
$$p = \frac{\alpha+11}{\alpha+\beta+13} := p^*$$

for $\alpha, \beta > 0$, $\quad 0 < \frac{\alpha+11}{\alpha+\beta+13} < 1 \quad \Rightarrow f(p^* \mid 5,3,4) > 0$

$$f(0 \mid 5,3,4) = f(1 \mid 5,3,4) = 0 < f(p^* \mid 5,3,4)$$

$\Rightarrow p^*$ is the MAP estimate

$\Rightarrow p^*$ is the MAP estimate