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Summer semester 2024

Markov Chains

Problem sheet 3

random walks, markov chains

Problems to be discussed (in parts) during the exercise sections:

Problem 1

Let $X_0, X_1, X_2, \dots$ be a Markov chain with state space $\mathcal{S} = \{1,2,3\}$, transition matrix

$$P = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix}$$

and initial distribution $\alpha^T = (0.1, 0.6, 0.3)$. Compute

(a) $\mathbb{P}[X_7 = 1 \,|\, X_6 = 3]$,

(b) $\mathbb{P}[X_9 = 3 \,|\, X_1 = 1, X_5 = 2, X_7 = 1]$,

(c) $\mathbb{P}[X_0 = 2 \,|\, X_1 = 3]$,

(d) $\mathbb{E}[X_2]$.

Let $X_0, X_1, \dots$ be MC with transition matrix P. We can interpret it as a random walk on a weighted, directed graph whose vertex set is the state space $\mathcal{S}$. The graph is called transition graph and constructed as follows:

1. Vertex set is $\mathcal{S}$
2. For $P_{i,j} > 0$ draw directed edge from i to j with weight $P_{i,j}$.

Problem 2

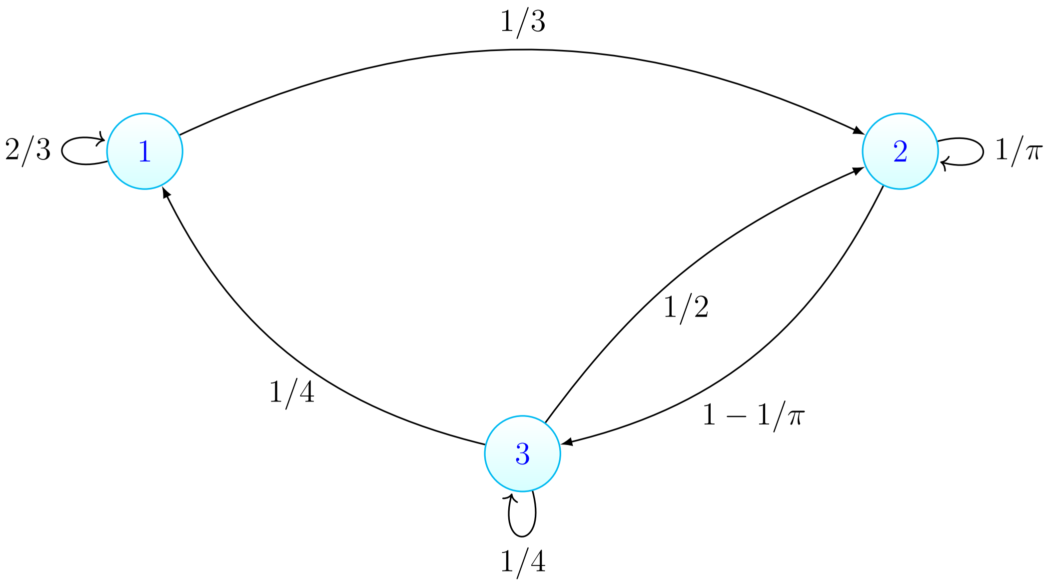
Consider again the Markov chain from Problem 1.

(a) Draw the transition graph for the Markov chain.

(b) Define the Markov chain $Y_0, Y_1, Y_2, \dots$ by $Y_k = X_{2k}$ for all $k \geq 0$. Draw the transition graph for this new Markov chain.

Problem 3

A Markov chain $X_0, X_1, X_2, \dots$ has the following transition graph:



(a) Provide the transition matrix for the Markov chain.

(b) Given the initial distribution $\alpha^T = (0, 0, 1)$ calculate $\mathbb{E}[X_2]$.

Problems to be handed in by:

Thursday, May 23rd, 11:59 p.m., online via Ilias.

Problem 1 *(6 points)*

Consider the Gamblers Ruin problem covered in class. The basic setup is unchanged and you want to determine the ruin probabilty $\mathbb{P}(S_{T_{10,0,80}} = 0)$. However, instead of betting \$1 on red in each round, you bet \$2 on red in each round. What is the ruin probability now? Compare it with the situation when betting \$1 on red in each round.

Problem 2 *(8 points)*

Let $X_0, X_1, X_2, \dots$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$, transition probabilities

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/5 & 1/10 & 7/10 \end{pmatrix}$$

and initial distribution $\alpha^T = (1/3, 1/3, 1/3)$. Find the following probabilities:

(a) $\mathbb{P}[X_5 = 2 \,|\, X_4 = 1]$,

(b) $\mathbb{P}[X_1 = 1,\ X_2 = 2]$,

(c) $\mathbb{P}[X_1 = 2 \,|\, X_2 = 1]$,

(d) $\mathbb{P}[X_5 = 1 \,|\, X_1 = 2, X_2 = 3, X_3 = 2]$.

Problem 3 *(8 points)*

Consider a Markov chain $(X_n)_{n=0,1,2,...}$ with state space $\mathcal{S} = \{1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 0 & 1/2 & 1/2 \\ 3/10 & 7/10 & 0 \end{pmatrix}.$$

The initial distribution is given by $\alpha^T = (1/2, 1/6, 1/3)$. Compute

(a) $\mathbb{P}[X_2 = k]$ for all $k = 1, 2, 3$;

(b) $\mathbb{E}[X_2]$.

Does the distribution of X_2 computed in (a) depend on the initial distribution α? Does the expected value of X_2 computed in (b) depend on the initial distribution α? Give a reason for both of your answers.

Problem 4 *(8 points)*

A stochastic matrix is called *doubly stochastic* if its columns sum to 1. Let $X_0, X_1, ...$ be a Markov chain on the state space $\mathcal{S} = \{1, ..., k\}$ with doubly stochastic transition matrix P and initial distribution that is uniform on $\mathcal{S}$.

Show that the distribution of X_n is uniform on $\mathcal{S}$ for all $n \geq 0$.