Exercise 03

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Problem 1 *(6 points)*
Consider the Gamblers Ruin problem covered in class. The basic setup is unchanged and you want to determine the ruin probabilty $\mathbb{P}(S_{T_{10,0,80}} = 0)$. However, instead of betting \$1 on red in each round, you bet \$2 on red in each round. What is the ruin probability now? Compare it with the situation when betting \$1 on red in each round.

The new set up equals betting \$1 on red in each round.

starting at \$5 and end at \$40

Let $Y_1, Y_2, \ldots$ be the sequence of gain in each round

$P(Y_k=+2)=p$, $P(Y_k=-2)=1-p$

$P(Y_k=m)=0$, $m \neq \pm 2$, $i=1,2\ldots$

Random walk $S'_k = S'_0 + \sum_{i=1}^{k} Y_k$ $k=1,2,\ldots$

Let $X_k = Y_k/2 \Rightarrow S'_k = S'_0 + 2\sum_{i=1}^{k} X_k$

Since $\sum_{i=1}^{k} X_k \in \mathbb{Z}$, $S'_k = S'_0 + 2n$, $n \in \mathbb{Z}$

$S'_k - S'_0 = 2n$

Let $S_k = S_0 + \sum_{i=1}^{k} X_k$, $S_0 = S'_0/2$

$S'_k - S'_0 = 2\sum_{i=1}^{k} X_k = 2(S_k - S_0)$

$$\boxed{S'_k = 2S_k - 2S_0 + S'_0 = 2S_k}$$

$T_{x,a,b} = \min\{n \geq 0 : S' \quad \text{or } S'_n \geq b\}$ given $S'_0 = x$

Problem 2 *(8 points)*
Let $X_0, X_1, X_2, \ldots$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$, transition probabilities

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/5 & 1/10 & 7/10 \end{pmatrix}$$

and initial distribution $\alpha^T = (1/3, 1/3, 1/3)$. Find the following probabilities:

(a) $\mathbb{P}[X_5 = 2 \mid X_4 = 1]$,

(b) $\mathbb{P}[X_1 = 1,\ X_2 = 2]$,

(c) $\mathbb{P}[X_1 = 2 \mid X_2 = 1]$,

(d) $\mathbb{P}[X_5 = 1 \mid X_1 = 2, X_2 = 3, X_3 = 2]$.

(a) $P(X_5=2 \mid X_4=1) = P_{12} = 1$

(b) $P(X_1=1, X_2=2) = P(X_2=2 \mid X_1=1)\, P(X_1=1)$

$$= P_{12}(\alpha P)_1 = \frac{8}{45}$$

(c) Bayes rule:

$$P(X_1=2 \mid X_2=1) = \frac{P(X_2=1 \mid X_1=2)\, P(X_1=2)}{P(X_2=1)}$$

$$= \frac{P_{21}(\alpha P)_2}{(\alpha P^2)_1} = 1/3.$$

(d) $P(X_5=1 \mid X_1=2, X_2=3, X_3=2)$

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$$= P(X_5=1 \mid X_3=2)$$

$$= (\alpha P^2)_{21}$$

Problem 3 *(8 points)*
Consider a Markov chain $(X_n)_{n=0,1,2,...}$ with state space $\mathcal{S} = \{1,2,3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 0 & 1/2 & 1/2 \\ 3/10 & 7/10 & 0 \end{pmatrix}.$$

The initial distribution is given by $\alpha^T = (1/2, 1/6, 1/3)$. Compute

(a) $\mathbb{P}[X_2 = k]$ for all $k = 1,2,3$;

(b) $\mathbb{E}[X_2]$.

(a) $P(X_2 = k) = (\alpha P^2)_k$

(b) $E(X_2) = \sum_{k=1}^{3} k P(X_2 = k)$ $\qquad V^T = (1, 2, 3)$

$= \sum_{k=1}^{3} V_k (\alpha P^2)_k$

$= V^T(\alpha P^2)$

Problem 4 *(8 points)*
A stochastic matrix is called *doubly stochastic* if its columns sum to 1. Let $X_0, X_1, ...$ be a Markov chain on the state space $\mathcal{S} = \{1, ..., k\}$ with doubly stochastic transition matrix P and initial distribution that is uniform on $\mathcal{S}$.

Show that the distribution of X_n is uniform on $\mathcal{S}$ for all $n \geq 0$.

$\sum_{i=1}^{k} P_{ij} = 1$, for $j = 1, \dots, k$

$\sum_{j=1}^{k} P_{ij} = 1$, for $j = 1, \dots, k$

$P(X_n = l) = (\alpha^T P^n)_l$

$\alpha^T = (\frac{1}{k}, \dots, \frac{1}{k})$

$\alpha^T P = (\sum_{i=1}^{k} \frac{1}{k} P_{i1} \quad \sum_{i=1}^{k} \frac{1}{k} P_{i2} \quad \cdots \quad \sum_{i=1}^{k} \frac{1}{k} P_{ik})$

$= (\frac{1}{k} \underbrace{\sum_{i=1}^{k} P_{i1}}_{1}, \frac{1}{k} \underbrace{\sum_{i=1}^{k} P_{i2}}_{1}, \dots, \frac{1}{k} \underbrace{\sum_{i=1}^{k} P_{ik}}_{1})$

$= (\frac{1}{k}, \cdots \frac{1}{k}) = \alpha^T \quad \Rightarrow \quad \alpha^T P^n = \alpha^T$ for $n = 0, 1, \cdots$

$P(X_n = l) = (\alpha^T P^n)_l = (\alpha^T)_l = \frac{1}{k} \quad \Rightarrow$ The distribution of X_n is uniform