

1. (a) To show B are the bases of \mathbb{R}^2 , we need to prove that $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent.

To prove these vectors are linearly independent,

we need to show that there are no scalars (other than zero) that can be multiplied to each vector to obtain the zero vector $(0, 0)$.
we can assume that some linear combination of these two vectors is zero:

$$\lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \mu \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we can write it as an Augmented matrix:

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right)$$

We can find that there exist no free variable and only the trivial solution ($\lambda = \mu = 0$).

So, these two vectors are linearly independent, and B is base of \mathbb{R}^2 .

(b) like (a), we can assume that some linear combination of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

We can find that there exist no free variable and only the trivial solution ($a = b = c = 0$).

So, these three vectors are linearly independent, and B is base of \mathbb{R}^3 .

$$(b) \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a-b \\ b+2a \\ 3b+2a \end{pmatrix}$$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad S' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow s'M(f)_S = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 2 & 3 \end{pmatrix}$$

$$B = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad S' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$f\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -1-2 \\ -2+2 \\ -2+6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = -3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 1+1 \\ 2-1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow s'M(f)_B = \begin{pmatrix} -3 & 2 \\ 0 & 1 \\ 4 & -1 \end{pmatrix}$$

$$B = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad B' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right\}$$

$$f\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = a_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_{21} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + a_{31} \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\Rightarrow 1 \cdot a_{11} - 1 \cdot a_{21} + 0 \cdot a_{31} = -3$$

$$1 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} = 0$$

$$0 \cdot a_{11} + 1 \cdot a_{21} - 2 \cdot a_{31} = 2$$

we can get $\left(\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$

Therefore we have $a_{31} = \frac{1}{3}$, $a_{21} = 3 - \frac{1}{3} = \frac{8}{3}$, $a_{11} = -3 + \frac{8}{3} = -\frac{1}{3}$

$$f\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 4 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{8}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Likewise, we need to find a_{12}, a_{22}, a_{32} such that

$$f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = a_{12}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_{22}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + a_{32}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

We can get $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & -1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right)$

Therefore we have $a_{32} = 0$, $a_{22} = -1$, $a_{12} = 1$

$$f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\Rightarrow BM(f)B' = \begin{pmatrix} -\frac{1}{3} & 1 \\ \frac{8}{3} & -1 \\ \frac{1}{3} & 0 \end{pmatrix}$$

(c) Assume there exists polynomial $p(x) = ax^3 + bx^2 + cx + d$, ($a, b, c, d \in \mathbb{R}$) then we can get $\begin{cases} p(0) = d = 1 \\ p(1) = a + b + c + d = 1 \\ p(2) = 8a + 4b + 2c + d = 0 \\ p(3) = -a + b - c + d = 1 \end{cases}$

$$\begin{cases} p(0) = d = 1 \\ p(1) = a + b + c + d = 1 \\ p(2) = 8a + 4b + 2c + d = 0 \\ p(3) = -a + b - c + d = 1 \end{cases}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 0 \\ -1 & 1 & -1 & 1 & 1 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

then we can know that $a = -\frac{1}{6}$, $b = 0$, $c = \frac{1}{6}$, $d = 1$.

The polynomial is $p(x) = \frac{1}{6}x^3 - \frac{1}{6}x + 1$ and there is only one polynomial.