Ex 20

minimize $(x-3)^2+(y-1)^2$

s.t. $x^2+1-y\le 0$

$y+3x-11\le 0$

$\mathcal{L}((x,y),\lambda) = (x-3)^2+(y-1)^2+\lambda_1(x^2+1-y)+\lambda_2(y+3x-11)$

KKT condition:

(I) $\dfrac{\partial \mathcal{L}((x,y),\lambda)}{\partial x} = 2(x-3)+2\lambda_1 x+3\lambda_2 \overset{!}{=} 0$

(II) $\dfrac{\partial \mathcal{L}((x,y),\lambda)}{\partial y} = 2(y-1)-\lambda_1+\lambda_2 \overset{!}{=} 0$

The complementary slackness conditions are:

(III) $\lambda_1(x^2+1-y)=0$

(IV) $\lambda_2(y+3x-11)=0$

(V) $\lambda_1\ge 0$

(VI) $\lambda_2\ge 0$

Case 1: $\lambda_1=0$

Then (II) implies $\lambda_2=2(1-y)$

Case 1.1: $\lambda_2=0$

Then we know $y=1$, $x=3$

Since $(3)^2+1-1>0$, not in feasible region

Case 1.2: $\lambda_2\ne 0$

Then we know $\begin{cases} y+3x-11=0 \\ x=3y \end{cases} \Rightarrow \begin{cases} x=\frac{33}{10} \\ y=\frac{11}{10} \end{cases} \Rightarrow \lambda_2=-\frac{1}{5}$

do not satisfy the KKT condition

Case 2: $\lambda_1\ne 0$

Then (III) implies $x^2+1-y=0$

Case 2.1 $\lambda_2=0$, implies $\lambda_1=2(y-1)$

Then we know $\begin{cases} y=x^2+1 \\ 2x-6+4(y-1)x=0 \end{cases} \Rightarrow 4x^3+2x-6=0$

$(x-1)(4x^2+4x-6)=0$

$\Rightarrow x=1,\ y=2$

$\Rightarrow \lambda_1=2$

case 2.2 $\lambda_2 \neq 0$

Then (III) implies $x^2+1-y=0$
(IV) implies $y+3x-11=0$ $\Bigg\} \Rightarrow \begin{cases} x=2, \ y=5 \\ \text{or} \\ x=-5, \ y=26 \end{cases}$

if $x=2$, $y=5$, then

$$\begin{cases} 4\lambda_1+3\lambda_2-2=0 \\ 8-\lambda_1+\lambda_2=0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{86}{7} \\ \lambda_2 = \frac{30}{7} \end{cases}$$

if $x=-5$, $y=26$

$$\begin{cases} -16-10\lambda_1+3\lambda_2=0 \\ 50-\lambda_1+\lambda_2=0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -\frac{166}{7} < 0 \\ \lambda_2 = -\frac{516}{7} < 0 \end{cases}$$

so, it do not satisfy KKT condition

Therefore, pairs $((1,2),2,0)$ and $((2,5),\frac{86}{7},\frac{30}{7})$ satisfy the KKT condition.

min $(x-3)^2+(y-1)^2$

if $x=1$, $y=2$, then objective function $=5$

if $x=2$, $y=5$, objective function $=17$

So pair $((1,2),2,0)$ attain the minimum.

Ex2.1 (a) Proof convexity of $(x-1)^2 \le 4$:

consider two points x_1 and x_2 in the set,

$(x_1-1)^2 \le 4$ and $(x_2-1)^2 \le 4$

Now, for any $\lambda \in [0,1]$,

$$(\lambda x_1 + (1-\lambda)x_2 - 1)^2 = \lambda^2 (x_1-1)^2 + 2\lambda(1-\lambda)(x_1-1)(x_2-1) + (1-\lambda)^2(x_2-1)^2$$

Since $(x_1-1)^2 \le 4$ and $(x_2-1)^2 \le 4$

$$\lambda^2(x_1-1)^2 + 2\lambda(1-\lambda)(x_1-1)(x_2-1) + (1-\lambda)^2(x_2-1)^2 \le 4\lambda^2 + 8\lambda(1-\lambda) + 4(1-\lambda)^2$$
$$= 4$$

So $(\lambda x_1 + (1-\lambda)x_2 - 1)^2 \le 4$, which means $\lambda x_1 + (1-\lambda)x_2$ is also in the set.

Thus $(x-1)^2 \le 4$ is convex

Proof convexity of $y \ge 0$

$\forall y_1 \ge 0, y_2 \ge 0, \forall \lambda \in [0,1], \lambda y_1 + (1-\lambda)y_2 \ge 0$

So $y \ge 0$ is convex

Since both constraints individually convex, the intersection of these sets is also convex.

(b) Lagrangian:

$$L(x, y, \lambda) = -4xy + 3x^2 + 2x + 4y + \lambda((x-1)^2 - 4) - \mu y$$

KKT:

$$\begin{cases} \dfrac{\partial L(x,y,\lambda)}{\partial x} = 6x + 2 + 2\lambda(x-1) = 0 \\ \dfrac{\partial L(x,y,\lambda)}{\partial y} = -4x + 4 - \mu = 0 \\ (x-1)^2 - 4 \le 0 \\ -y \le 0 \\ \lambda \ge 0, \mu \ge 0 \\ \lambda((x-1)^2 - 4) = 0 \\ \mu y = 0 \end{cases}$$