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Math for AI
Winter term 2023/24

Exercise sheet 8

Exercise 29 (7 points)

(a) (3 points) Show that for two independent $\mathbb{R}$-valued random variables X, Y we have $E(X \cdot Y) = (EX) \cdot (EY)$

(b) (2 points) Show that for two independent $\mathbb{R}^n$-valued random variables X, Y we have $E(X \cdot Y^T) = (EX) \cdot (EY)^T$

(c) (2 points) For an $\mathbb{R}^n$-valued random variable $X = (X_1, \ldots, X_n)$, the covariance matrix is defined as the matrix whose entry at the place i, j is $\mathrm{Cov}(X_i, X_j)$. It is denoted by $\mathrm{Cov}(X)$.

Show that for two independent $\mathbb{R}^n$-valued random variables X, Y we have $\mathrm{Cov}(X+Y) = \mathrm{Cov}(X) + \mathrm{Cov}(Y)$ (a sum of covariance matrices)

Exercise 30 (9 points) Let X be a random variable with exponential distribution with parameter λ, i.e. with density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(a) (2 points) Show that f_X is indeed a density function, i.e. that $\int_{\mathbb{R}} f_X(x)dx = 1$.

(b) (3 points) Show that the expectation of X is $\frac{1}{\lambda}$

(c) (2 points) Let $Y := \frac{1}{2} X^{\frac{1}{3}}$. Compute the density function of Y.

(d) (2 points) Let $Z := e^X$. Compute the density function of Z.

Exercise 31 (10 points)

Let X, Y be $\mathbb{R}$-valued random variables with joint density function

$$f(x, y) = \begin{cases} 2e^{-x-y} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(a) (4 points) Are X and Y independent? Justify your answer.

(b) (4 points) Compute the marginal density functions of X and Y.

(c) (2 points) Compute the covariance of X and Y.

Exercise 32 (5 points)

Show that if X, Y are independent $\mathbb{R}$-valued random variables, and $g, h \colon \mathbb{R} \to \mathbb{R}$ are functions, then $g(X), h(Y)$ are also independent from each other.

Exercise 33 (9 points)

For discrete random variables X, Y and a value b in the range of X with $P(X = b) \neq 0$, one can define $P(Y \in A \mid X = b) := \frac{P(Y \in A, X=b)}{P(X=b)}$. This defines a new distribution on the range of possible values of Y, and thus a new random variable denoted $(Y \mid X = b)$.

For continuous $\mathbb{R}$-valued random variables X, Y with joint density function $f(x, y)$, one can similarly define a continuous random variable $(Y \mid X = b)$ with the density function $f(y \mid b) := \frac{f(b,y)}{\int_{-\infty}^{\infty} f(b,y)dy}$

This new random variable has an expectation, concretely $E(Y \mid X = b) = \int_{-\infty}^{\infty} x \cdot f(y \mid b)dy$.

Now we can calculate this value for every b and from this get a random variable which is a function of X! This random variable is denoted $E(Y \mid X)$ and called *conditional expectation of* Y *given* X.

Let X, Y have joint density function given by $f(x, y) := \begin{cases} 2 & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \text{ and } x \leq y \\ 0 & \text{otherwise} \end{cases}$

(a) (3 points) Compute the conditional density function $f(y|x)$.

(b) (3 points) Compute the function $E(Y|X)\colon [0, 1] \to \mathbb{R}$.

(c) (0 points) Draw the possible values of the joint variable (X, Y) and the graph of the function $E(Y|X)$ and compare with the statement of Theorem 4.7.9.

(d) (3 points) Compute the function $\mathrm{Var}(Y \mid X) := E(Y^2 \mid X) - (E(Y \mid X))^2$

Exercise 34 (optional, if you want a challenge - **? points** - you can replace any other exercise by this one)

The conditional variance of two $\mathbb{R}$-valued random variables X, Y is defined by $\mathrm{Var}(Y \mid X) := E(Y^2 \mid X) - (E(Y \mid X))^2$. It is a map from the possible outcomes of X to $\mathbb{R}$, and thus again a random variable (since we have a probability distribution on the range of X).

(a) (3 points) Show that $\mathrm{Var}(Y \mid X) = E(\,(Y - E(Y \mid X))^2 \mid X)$.

[If this helps: show it pointwise, i.e. for each x in the range of X show that $\mathrm{Var}(Y \mid X = x) = E(\,(Y - E(Y \mid X = x))^2 \mid X = x)$.]

(b) (0 points) Stare at the definition and the result from (a) and try to come up with a description in words of what conditional variance measures.

(c) (5 points) Show that the usual variance of Y decomposes into the variance of the conditional expectation and the expectation of the conditional variance:

$$\mathrm{Var}(E(Y \mid X)) + E(\mathrm{Var}(Y \mid X)) = \mathrm{Var}(Y)$$

(d) (1 points) What are the summands in the equation of (c) if X and Y are independent?

(e) (4 points) For discrete random variables X, Y show that $Var(Y|X) = 0$ if and only if Y is a function of X.

[Remark: Propositions 4.7.6 and 4.7.8 might help.]

Deadline: Friday 8th of December, 10:00.
Upload your solution to this link.