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Math for AI
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Exercise sheet 9

Exercise 35 (4 points)

(a) (3 points) Let $\hat{\theta}$ be an estimator of θ. Show that $MSE(\hat{\theta}) = Var(\hat{\theta}) + B(\theta)^2$.

(b) (1 point) Let $X_1, \ldots, X_n$ be i.i.d. random variables with expectation μ and variance σ^2. Compute the mean squared error of $\frac{1}{n}\Sigma_{i=1}^n X_i$ as an estimator of μ in terms of σ^2.

Exercise 36 (10 points)

Let $X_1, \ldots, X_k$ be i.i.d. $\mathbb{R}$-valued random variables with a normal distribution $\sim \mathcal{N}(\mu, \sigma^2)$. We know that their joint distribution has density function $\Pi_{i=1}^k \mathcal{N}(x_i|\mu, \sigma^2)$.
The maximum likelihood estimator for σ^2 is the random variable $\widehat{\sigma^2}_{ML} : \omega \mapsto \operatorname{argmax}_{\sigma^2} \Pi_{i=1}^k \mathcal{N}(X_i(\omega)|\mu, \sigma^2)$.
That is, for a sample ω giving rise to values $x_1 = X_1(\omega), \ldots, x_k = X_k(\omega)$ the value $\widehat{\sigma^2}_{ML}(\omega)$ is the σ^2 for which $\Pi_{i=1}^k \mathcal{N}(x_i|\mu, \sigma^2)$ assumes its maximum.

Show that $\widehat{\sigma^2}_{ML} = \frac{1}{k}\Sigma_{i=1}^k (X_i - \mu)^2$.

Exercise 37 (9 points)

A coin shows heads with probability p and tails with probability $(1-p)$. The random variable X counts how many times the coin is thrown until heads is shown for the first time. Thus X takes values in the natural numbers with distribution $P(X = k) = p(1-p)^{k-1}$ (a so-called geometric distribution).

In a series of n experiments, the numbers of throws until the first appearance of tails are: $k_1 + 1, \ldots, k_n + 1$ (i.e. $k_1, \ldots, k_n$ are the numbers of tails before the first head).

Let $a, b > 0$. Compute the a posteriori distribution for p on $(0,1)$, where the prior distribution on $(0,1)$ has the Beta(a,b)-distribution, i.e. density function

$$h(p) := \begin{cases} \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1} & \text{if } 0 < p < 1 \\ 0 & \text{otherwise} \end{cases}$$

Here $B(a,b) := \int_0^1 u^{a-1}(1-u)^{b-1} du$ is the normalizing constant ensuring that the above is really a density function on $(0,1)$. Show that the a posteriori distribution is again a Beta(a',b')-distribution. What are the new parameters a', b'?

Exercise 38 (10 points)

In a pond there are yellow, silver and black fish. We catch n fish (gently, by hand, nobody is hurt) and throw them back each time. Let X, Y, Z be the total numbers of yellow/silver/black fish that we caught. If p_X, p_Y, p_Z are the probabilities of catching a yellow/silver/black fish, then the joint distribution of X, Y, Z is given by $P(X = x, Y = y, Z = z) = \frac{n!}{x!y!z!} p_X^x p_Y^y p_Z^z$ (a multinomial distribution).

In this exercise you should calculate the maximum likelihood estimates for p_X, p_Y, p_Z, if we have caught x yellow fish, y silver fish and z black fish.

This means that you have to solve the constrained optimization problem of maximizing the likelihood function $l = l(x, y, z, p_X, p_Y, p_Z)$ with respect to p_X, p_Y, p_Z, and fixed given parameters x, y, z. You can do this however you want, but here is a sequence of hints that might make it easier:

(a) Calculate the partial derivatives of the likelihood function $l = l(x, y, z, p_X, p_Y, p_Z)$ with respect to p_X, p_Y, p_Z.

(b) Express each of these partial derivatives as multiples of the likelihood function; $\frac{\partial}{\partial p_i} l = c_i \cdot l$ for some c_i, $i \in \{X, Y, Z\}$

(c) Write down the equations with Lagrange multiplier λ for the constraint $p_X + p_Y + p_Z = 1$

(d) Show that $\lambda = n \cdot l$.

(e) Substitute this into the Lagrange multiplier equations to compute your estimates for p_X, p_Y, p_Z.

[Remark: If you follow these instructions: 2 points for each step, otherwise 10 altogether]

Exercise 39 (7 points)

The provided notebook shows you an estimator for the covariance matrix of a multvariate distribution that is given as an algorithm, instead of a formula. Look at the notebook and follow the instructions.

Deadline: Friday 15th of December, 10:00.
Upload your solution to this link.