Exercise 7

To prove these statements are equivalent for a symmetric matrix A, we should prove:

1. A is positive semidefinite $(v^T A v \geq 0)$ $\Rightarrow$ All eigenvalues of A are nonnegative.
2. All eigenvalues of A are nonnegative. $\Rightarrow$ There exists a matrix B s.t. $A = BB^T$
3. There exists a matrix B s.t. $A = BB^T$ $\Rightarrow$ A is positive semidefinite $(v^T A v \geq 0)$

Pf. 1: consider an eigenvector v of A with eigenvalue λ

$$Av = \lambda v$$

$$\Rightarrow v^T A v = v^T \lambda v$$

since $v^T A v \geq 0$, then $v^T \lambda v \geq 0$

since λ is a real number,

we can get: $v^T \lambda v = \lambda v^T v = \lambda \langle v, v \rangle = \lambda \|v\|^2$

since $\|v\|^2 \geq 0 \Rightarrow \lambda \geq 0$

So, all eigenvalue of A are nonnegative.

Pf 2: Since A is symmetric, it can be diagonalise the matrix A, while Q is an orthogonal matrix, D is a diagonal matrix:

$$A = QDQ^T$$

since D contains no negative value,

$$D = D^{\frac{1}{2}} (D^{\frac{1}{2}})^T$$

Then, $A = QD^{\frac{1}{2}}(D^{\frac{1}{2}})^T Q^T$

Let $B = QD^{\frac{1}{2}}$

Therefore $A = BB^T$

Pf 3: If $A = BB^T$, then for any vector x, we have:

$$x^TAx = x^T(B^TB)x = x^TB^TBx = (Bx)^T(Bx) = \langle Bx, Bx \rangle$$

$$= \|Bx\|^2 \geq 0$$

Therefore, matrix A is positive semidefinte.

Exercise 8

(a) $A = [a_1 | a_2, \cdots | a_m]$ ($a_1, a_2, \ldots a_m$ are columns of A)

$B = [b_1 | b_2, \cdots | b_m]$ ($b_1, b_2, - a_m$ are columns of B)

$$vec(A) = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \quad vec(B) = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$< vec(A), vec(B) > = vec(A)^T vec(B)$$

$$= \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = [a_1 | \cdots | a_m] \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$= \sum_{i=1}^{m} a_i b_i$$

$$A^T = [a_1^T | \cdots | a_m^T] \quad B = [b_1 | \cdots | b_m] = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} [b_1 | \cdots b_m]$$

$$A^T B = \begin{pmatrix} a_1 b_1 & \cdots & a_m b_1 \\ \vdots & & \\ a_1 b_m & \cdots & a_m b_m \end{pmatrix} \quad tr(A^T B) = \sum_{i=1}^{m} a_i b_i$$

So, $< A, B >_F = tr(A^T B) = < vec(A), vec(B) >$

(b) (i) $< A, \lambda B + \mu C >_F$

$= tr(A^T(\lambda B + \mu C))$

$= tr(\lambda A^T B) + tr(\mu A^T C)$

$= \lambda \, tr(A^T B) + \mu \, tr(A^T C)$

$= \lambda < A, B >_F + \mu < A, C >_F$

(ii) $< \lambda B + \mu C, D >_F$

$= tr((\lambda B + \mu C)^T D)$

$= tr((\lambda B^T + \mu C^T) D)$

$= tr(\lambda B^T D) + tr(\mu C^T D)$

$= \lambda \, tr(B^T D) + \mu \, tr(C^T D)$

$= \lambda < B, D >_F + \mu < C, D >_F$

(iii) $\langle A, B \rangle_F$

$= tr(A^T B)$

$= tr((A^T B)^T)$

$= tr(B^T A)$

$= \langle B, A \rangle_F$

(iv) $\langle A, A \rangle_F$

$= tr(A^T A)$

(c) B is orthogonal matrix $\Rightarrow B^T = B^{-1}$, $B^T B = B B^T = I$

Given $\|A\| := \sqrt{\langle A, A \rangle_F}$,

$\|A\|_F = \sqrt{\langle A, A \rangle_F} = \sqrt{tr(A^T A)} = \sqrt{tr(A^T I A)} = \sqrt{tr(A^T B^T B A)}$

$= \sqrt{tr((BA)^T (BA))} = \sqrt{\langle BA, BA \rangle_F} = \|BA\|_F$

$\|AB\|_F = \sqrt{\langle AB, AB \rangle_F} = \sqrt{tr((AB)^T (AB))} = \sqrt{tr((AB)(AB)^T)}$

$= \sqrt{tr(ABB^T A^T)} = \sqrt{tr(AA^T)}$

$= \sqrt{tr(A^T A)}$

$= \langle A, A \rangle_F$

Exercise 9

(a) ① if the two points lies on the same vertical line

the distance: $d := |a_2 - b_2|$

② else

the distance: $d := |a_1 - b_1| + |a_2| + |b_2|$

(b) in case ① $d(a, b) = |a_2 - b_2|$

$d(a+z, b+z) = |(a_2 + z_2) - (b_2 + z_2)| = |a_2 - b_2|$

in case ② $d(a, b) = |a_1 - b_1| + |a_2| + |b_2|$

$d(a+z, b+z) = |(a_1 + z_1) - (b_1 + z_1)| + |a_2 + z_2| + |b_2 + z_2|$

$= |a_1 - b_1| + |a_2 + z_2| + |b_2 + z_2| \neq d(a, b)$

Therefore the metric are translation variant, it is not the metric associated to any norm.

Exercise 10:

First we compute the mean of each coordinate:

$$\bar{x}_1 = \frac{1}{3}(0+\sqrt{3}-\sqrt{3}) = 0$$

$$\bar{x}_2 = \frac{1}{3}(1+4+1) = 2$$

Then we subtract the mean vector from the original vector, we get

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} \sqrt{3} \\ 2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix}$$

$$D := \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ -1 & 2 & -1 \end{pmatrix}$$

Then, calculate the covariance matrix

$$C := \frac{1}{3-1} \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \sqrt{3} & 2 \\ -\sqrt{3} & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 6 & 3\sqrt{3} \\ 3\sqrt{3} & 6 \end{pmatrix}$$

$$\chi_C(\lambda) = \det \begin{pmatrix} \lambda-6 & -3\sqrt{3} \\ -3\sqrt{3} & \lambda-6 \end{pmatrix} = (\lambda-6)^2 - (3\sqrt{3})^2 = 0$$

$$\Rightarrow \lambda^2 - 12\lambda + 36 - 27 = 0$$

$$\Rightarrow \lambda^2 - 12\lambda + 9 = 0$$

$$\Rightarrow \lambda = 6 \pm 3\sqrt{3}$$

The biggest of these eigenvalues is $6+3\sqrt{3}$

So, we look for an eigenvectors for the eigenvalue, we need to solve:

$$\begin{pmatrix} 6 & 3\sqrt{3} \\ 3\sqrt{3} & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6+3\sqrt{3} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 6x + 3\sqrt{3}y = (6+3\sqrt{3})x \\ 3\sqrt{3}x + 6y = (6+3\sqrt{3})y \end{cases} \Rightarrow x = y$$

general solution: $\left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \right\}$ and the solution set : $\left\{ \alpha \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

Let $\alpha = 1$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$