Sheet 07

Exercise 25

Since $C_k^* \geq 0$, f_k are convex, and also g_i are convex, the objective function is a positive linear combination of convex functions, hence it is also convex.

$$\min \quad f_c(x) = c_1^* f_1(x) + \cdots + c_l^* f_l(x)$$
$$\text{s.t.} \quad g_i(x) \leq 0 \quad (i = 1, \cdots, m)$$
$$h_j(x) = 0 \quad (j = 1, \cdots, p)$$

Assume the lagrangian of this optimization problem:

$$L(x, \lambda, u) = f_c(x) + \sum_i \lambda_i g_i(x) + \sum_j u_j h_j(x)$$

From the given condition, there exists a pair (x^*, λ^*, u^*) satisfying KKT conditions. By the convexity, this KKT pair is the unique optimal solution of the minimization problem, which implies $f_c(x^*)$ is the minimum, hence x^* is a Pareto optimal point, otherwise it contradicts to the minimum.

(Q: Is it supposed to prove x^* is also a pareto optimal point of $\min f_c(x)$?)
i.e. a pareto optimal point with non-negative c is also pareto optimal point with arbitrary c ?)

Exercise 27.

(a) (i) $P(X_1 \leq \frac{1}{2}, X_3 \geq \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^1 \int_{\frac{1}{2}}^1 (x_1^2 + x_2^2 + x_3^2)\, dx_3\, dx_2\, dx_1$

$= \int_0^{\frac{1}{2}} \int_0^1 (x_3 \cdot x_1^2 + x_3 \cdot x_2^2 + \frac{1}{3} x_3^3) \Big|_{\frac{1}{2}}^1 dx_2\, dx_1 = \boxed{\int_0^{\frac{1}{2}} \int_0^1 \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{7}{24}\, dx_2\, dx_1}$

$= \int_0^{\frac{1}{2}} (\frac{1}{2} x_2 \cdot x_1^2 + \frac{1}{2} \cdot \frac{1}{3} x_2^3 + \frac{7}{24} x_2) \Big|_0^1 dx_1 = \int_0^{\frac{1}{2}} \frac{1}{2} x_1^2 + \frac{11}{24}\, dx_1$

$= (\frac{1}{2} \cdot \frac{1}{3} x_1^3 + \frac{11}{24} x_1) \Big|_0^{\frac{1}{2}} = \frac{1}{4}$

(ii) $f(x_1, x_2) = \int_0^1 (x_1^2 + x_2^3 + x_3^2)\, dx_3 = (x_3 \cdot x_1^2 + x_3 \cdot x_2^2 + \frac{1}{3} x_3^3) \Big|_0^1$

$= x_1^2 + x_2^2 + \frac{1}{3}$

(b) $P(x \geq 0, y \geq 0) = \int_0^\infty \int_0^\infty e^{-x-y}\, dx\, dy = \int_0^\infty -e^{-x-y} \Big|_0^\infty dy = \int_0^\infty (0 - (-e^{-y}))\, dy$

$= \int_0^\infty e^{-y}\, dy = -e^{-y} \Big|_0^\infty = 0 - (-e^0) = 1$

$f(y) = \int_0^\infty f(x,y)\, dx = e^{-y}$, $f(x) = \int_0^\infty f(x,y)\, dx = e^{-x}$

$P(x \geq 0) = \int_0^\infty e^{-x}\, dx = 1$, $P(y \geq 0) = \int_0^\infty e^{-y}\, dy = 1$

$P(x \geq 0, y \geq 0) = 1 = P(x \geq 0) \cdot P(y \geq 0) \Rightarrow X, Y$ are independent (c)(i) $P(z=1) = P(x=1)\cdot P(y=1) + P(x=-1)\cdot P(y=-1) = \frac{1}{2}$

$P(z=-1) = P(x=-1)\cdot P(y=1) + P(x=1)\cdot P(y=-1) = \frac{1}{2}$

$P(z=1, x=1) = P(x=1)\cdot P(y=1) = \frac{1}{4}$, $P(z=-1, x=1) = P(x=1)\cdot P(y=-1) = \frac{1}{4}$

$P(z=1, x=-1) = P(x=-1)\cdot P(y=-1) = \frac{1}{4}$, $P(z=-1, x=-1) = P(x=-1)\cdot P(y=1) = \frac{1}{4}$

i.e. for all possible values of (z_i, x_j), $P(z_i, x_j) = \frac{1}{4} = P(z_i)\cdot P(x_j)$

similarly, $P(z_i, y_j) = P(z_i)P(y_j)$, hence Z and X, Z and Y are independent.

(ii) From (i), we have $P(z=-1) = \frac{1}{2}$.

$P(z=-1, x=1, y=1) = 0 \neq P(z=-1)\cdot P(x=1)\cdot P(y=1) = \frac{1}{8}$