1. (a) The pixel space has 800,000 dimensions ($[0, 255]^{800000}$)

(b) Each pixel has 256 possibilities of values (8 bits)

We can randomly choose a value for 8×10^5 pixels.

so together we have $M = 256^{800000}$ images

(c) Since the number of plausible images (assume N) is much less than the number of random images,

we only need $(N+1)$ linear independent vectors in the pixel space to uniquely idetify the plausible image points. Therefore the images lie in a much lower-dimensional manifold (spanned by the vectors) in the pixel space.

(d) If noise is reduced, for each plausible image, the pixels of it become more deterministic, leading to a reduction of possible points in the pixel space.

Therefore, the dimension of the manifold will be even lower.

(e) Like the example in the lecture, the image will be a 50% transparent overlap between the two original images I_1 and I_2.

(f) It is not a good generator.

The interpolated image is a mirage of the images interpolated

the camera never capture such image (unless there's motion relation between images like in a film)

In other words, the generated image is often out of distribution.

2. CDF of exponential function $\quad -e^{-\lambda x}\Big|_0^a = -e^{-\lambda a} - (-1)$

$$F(x) = P(X \le x) = \int_0^x p(z;\lambda)\,dz = \begin{cases} 1 - e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases} \quad \in [0,1]$$

$F(x)$ is not inventible in $(-\infty, +\infty)$, we choose $x \in [0, \infty)$.

since $P(X<0) = 0$

$u = F(x) = 1 - e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda}\ln(1-u) = F^{-1}(u)$

For uniform samples y_i, $x_i = F^{-1}(y_i) = -\frac{1}{\lambda}\ln(1-y_i)$

3. (a) for $\forall x \in \mathbb{R}$

$M q(x;b) \geq p(x)$

$\frac{M}{2b}\exp(-\frac{|x|}{b}) \geq \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2)$

Note: No need to derive this. It's already given in the sheet!

since $b>0$, $\exp(-\frac{|x|}{b}) > 0$

$M \geq \sqrt{\frac{2}{\pi}}\, b \exp(\frac{|x|}{b} - \frac{x^2}{2})$, for $\forall x \in \mathbb{R}$

therefore $M \geq \max_x (\sqrt{\frac{2}{\pi}}\, b \exp(\frac{|x|}{b} - \frac{x^2}{2}))$

Let $x^* = \arg\max_x (\frac{\sqrt{2}}{\sqrt{\pi}} b \exp(\frac{|x|}{b} - \frac{x^2}{2}))$

$= \arg\max_x (\ln(\frac{\sqrt{2}}{\sqrt{\pi}} b) + \frac{|x|}{b} - \frac{x^2}{2})$

$= \arg\max_x (\frac{|x|}{b} - \frac{x^2}{2})$

for $x \geq 0$ $\quad x_1^* = \arg\max_x (\frac{-bx^2+2x}{2}) = \frac{1}{b}$

for $x < 0$ $\quad x_2^* = \arg\max_x (\frac{-bx^2-2x}{2}) = -\frac{1}{b}$

$M^* = \max(\frac{p(x_1^*)}{q(x_1^*;b)}, \frac{p(x_2^*)}{q(x_2^*;b)}) = \frac{p(x_1^*)}{q(x_1^*;b)} = \frac{p(x_2^*)}{q(x_2^*;b)}$

$= \sqrt{\frac{2}{\pi}}\, b \exp(\frac{1}{b^2} - \frac{1}{2b^2}) = \sqrt{\frac{2}{\pi}}\, b \exp(\frac{1}{2b^2})$

prove global ...

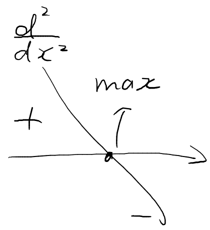
Rewrite in the next page

3. $M(b;x) = \frac{p(x)}{q(x;b)} = \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{|x|}{b} - \frac{x^2}{2}\right)$

$$\frac{\partial M}{\partial x} = \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{|x|}{b} - \frac{x^2}{2}\right) \cdot \left(\frac{\operatorname{sign}(x)}{b} - x\right)$$

$$\frac{\partial^2 M}{\partial x^2} = \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{|x|}{b} - \frac{x^2}{2}\right) \cdot (-1) + \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{|x|}{b} - \frac{x^2}{2}\right) \cdot \left(\frac{\operatorname{sign}(x)}{b} - x\right)^2$$

$$= \underbrace{\sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{|x|}{b} - \frac{x^2}{2}\right)}_{:= C > 0} \left[\left(\frac{\operatorname{sign}(x)}{b} - x\right)^2 - 1\right]$$



Find possible x^*. Let $\frac{\partial M}{\partial x} = 0$

for $x \geq 0$, $x = \frac{\operatorname{sign}(x)}{b} = \frac{1}{b}$.

$\frac{\partial^2 M}{\partial x^2} = C[-1] = -C < 0 \Rightarrow$ local maximum at $x_1^* = \frac{1}{b}$

for $x < 0$, $x = -\frac{1}{b}$

$\frac{\partial M^2}{\partial x^2} = C[-1] = -C < 0 \Rightarrow$ local maximun at $x_2^* = -\frac{1}{b}$

$$M^* = \max\left(\frac{p(x_1^*)}{q(x_1^*;b)}, \frac{p(x_2^*)}{q(x_2^*;b)}\right) = \frac{p(x_1^*)}{q(x_1^*;b)} = \frac{p(x_2^*)}{q(x_2^*;b)}$$

$$= \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{1}{b^2} - \frac{1}{2b^2}\right) = \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{1}{2b^2}\right)$$

check bounday $x = 0$

$M(x=0) = \sqrt{\frac{2}{\pi}}\, b < M^*$ ✓

$\lim_{x \to -\infty} M(x) = -\infty < M^*$ ✓

$\lim_{x \to +\infty} M(x) = -\infty < M^*$ ✓

$M^* = \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{1}{2b^2}\right)$ is ineed the global maximum of $M(x)$

(b) $M(b) = \sqrt{\frac{2}{\pi}}\, b \exp\left(\frac{1}{2b^2}\right) \underline{> 0}$

$$\frac{\partial M}{\partial b} = \sqrt{\frac{2}{\pi}} \exp\left(\frac{1}{2b^2}\right) + \sqrt{\frac{2}{\pi}} \cdot b \exp\left(\frac{1}{2b^2}\right) \cdot \cancel{\frac{1}{2}} \cdot \cancel{(-2)} \cdot \frac{1}{b^3}$$

$$= \left(\frac{1}{b} - \frac{1}{b^3}\right) M$$

$$\frac{\partial^2 M}{\partial b^2} = \left(-\frac{1}{b^2} + \frac{3}{b^4}\right) M + \left(\frac{1}{b} - \frac{1}{b^3}\right)^2 M$$

Similarly, Let $\frac{\partial M}{\partial b} = 0$.

$$\frac{1}{b} - \frac{1}{b^3} = 0$$

$$\Rightarrow b^2 - 1 = 0 \overset{b>0}{\Rightarrow} b^* = 1$$

$$\frac{\partial^2 M}{\partial b^2}(b^*) = (-1+3) M + (1-1)^2 M$$

$= -2M < 0$ $\quad b^*$ is local minima in $(0, +\infty)$

Check boundary:

$\lim_{x \to +\infty} M(b) = +\infty$ ✓ $\quad \lim_{x \to 0^+} M(b) = +\infty$ ✓

So $b^* = \arg\min_b M(b) = 1$

$\underbrace{M^* = M(1) = \sqrt{\frac{2}{\pi}} \exp\left(\frac{1}{2}\right)}_{\text{(average iterations)}}$ is the global minimum in $(0, +\infty)$

The probability of acceptance in each iteration is $\frac{1}{M(1)} = \sqrt{\frac{\pi}{2}} \exp\left(-\frac{1}{2}\right)$

- Why is $\frac{1}{M}$ the acceptane rate?

$P\left(u \le \frac{p(X)}{Mq(X;b)}\right) = \frac{p(X)}{Mq(X;b)}$, here. X is r.v.

acceptance $= \frac{A_{shadow}}{A_{whole}}$

$E[P] = \int_x \frac{p(x)}{Mq(X;b)} \cdot \underbrace{q(X;b)\,dx}_{\text{prior distr. of } x} = \frac{1}{M} \int_x p(x)\,dx = \frac{1}{M}$

- Why is M the "average" iterations? Let p = acceptance rate $= \frac{1}{M}$

expected iterations =

$$1p + 2\cdot(1-p)p + \cdots = \underbrace{\sum_{k=1}^{\infty} k(1-p)^{k-1} p}_{\text{power series}} = \frac{1}{(1-(1-p))^2} \cdot p = \frac{1}{p} = \frac{1}{\frac{1}{M}} = M$$