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Markov Chains

Problem sheet 2

random walks

Problems to be handed in by:
Thursday, May 9th, 11:59 p.m., online via Ilias.

Problem 1 *(12 points)*
Let $(S_n)_{n\geq 0}$ be a simple random walk starting at 0 with $p = 0.4$ and $q = 1-p = 0.6$. Compute the following probabilities:

- $\mathbb{P}(S_2 = 0, S_4 = 0, S_5 = -1)$,
- $\mathbb{P}(\{S_4 = 4\} \cup \{S_4 = -2\})$,
- $\mathbb{P}(M_{17} \leq -5, S_7 = -5)$, where $M_{17} = \min_{0\leq i\leq 17} S_i$.

Solution:
With the multiplication rule and the Markov property, we get

$$\begin{aligned}
\mathbb{P}(S_2 = 0, S_4 = 0, S_5 = -1) &= \mathbb{P}(S_5 = -1 | S_4 = 0, S_2 = 0)\mathbb{P}(S_4 = 0, S_2 = 0)\\
&= \mathbb{P}(S_5 = -1 | S_4 = 0, S_2 = 0)\mathbb{P}(S_4 = 0 | S_2 = 0)\mathbb{P}(S_2 = 0)\\
&\overset{\text{MC}}{=} \mathbb{P}(S_5 = -1 | S_4 = 0)\mathbb{P}(S_4 = 0 | S_2 = 0)\mathbb{P}(S_2 = 0)\\
&\overset{3.6}{=} q \cdot 2pq \cdot 2pq = 4p^2q^3 = 0.13824
\end{aligned}$$

For the second probability, we notice that $\{S_4 = 4\}$ and $\{S_4 = -2\}$ are disjoint.

$$\begin{aligned}
\mathbb{P}(\{S_4 = 4\} \cup \{S_4 = -2\}) &= \mathbb{P}(S_4 = 4) + \mathbb{P}(S_4 = -2)\\
&\overset{3.6}{=} p^4 + \binom{4}{1} pq^3 = p^4 + 4pq^3 = 0.3712
\end{aligned}$$

For the last probability, $M_{17} \leq -5$ implies the existence of an $i \leq 17$ with $S_i \leq -5$. This means that $\{S_7 = -5\} \subseteq \{M_{17} \leq -5\}$.

$$\begin{aligned}
\mathbb{P}(M_{17} \leq -5, S_7 = -5) &= \mathbb{P}(\{M_{17} \leq -5\} \cap \{S_7 = -5\})\\
&= \mathbb{P}(S_7 = -5) \overset{3.6}{=} \binom{7}{1} pq^6 \approx 0.1306
\end{aligned}$$

Problem 2 *(6 points)*
For a simple symmetric random walk $(S_n)_{n=0,1,2,\ldots}$ starting in 0 ($S_0 = 0$), show that

$$\mathbb{P}(S_4 = 0) = \mathbb{P}(S_3 = 1).$$

Solution: Using Proposition 3.6 from the lecture, we note that $4+0$ and $3+1$ are both even. This means that we can use the given formula with $p = 0.5$ to get

$$\mathbb{P}(S_4 = 0) = \mathbb{P}(S_4 - S_0 = 0) \stackrel{3.6}{=} \binom{4}{\frac{1}{2}(4+0)} \left(\frac{1}{2}\right)^4 = \binom{4}{2} \cdot \frac{1}{16} = 6 \cdot \frac{1}{16} = \frac{3}{8}$$

$$\mathbb{P}(S_3 = 1) = \mathbb{P}(S_3 - S_0 = 1) \stackrel{3.6}{=} \binom{3}{\frac{1}{2}(3+1)} \left(\frac{1}{2}\right)^3 = \binom{3}{2} \cdot \frac{1}{8} = 3 \cdot \frac{1}{8} = \frac{3}{8}$$

Problem 3 *(6 points)*
Use the reflection principle to find the probability $\mathbb{P}(M_8 = 6)$, where $M_8 = \max_{0 \le i \le 8} S_i$ and $(S_n)_{n \ge 0}$ is a simple symmetric random walk starting in 0 ($S_0 = 0$).

Solution: We can use the same argument from the proof of Lemma 3.11 to generalize the reflection principle to the form

$$N^b_{n-1}(0, b-m) = N_{n-1}(0, b+m) \quad \text{for } m \in \mathbb{N}.$$

Together with the symmetry of this random walk, this implies

$$\mathbb{P}(M_{n-1} \ge b, S_{n-1} = b-m) = \mathbb{P}(S_{n-1} = b+m) \quad \text{for } m \in \mathbb{N}.$$

Additionally we have the trivial equation

$$\mathbb{P}(M_{n-1} \ge b, S_{n-1} = b+m) = \mathbb{P}(S_{n-1} = b+m) \quad \text{for } m \in \mathbb{N}.$$

We now combine these two to calculate $\mathbb{P}(M_8 = 6)$:

$$\begin{aligned}
&\mathbb{P}(M_8 = 6) = \mathbb{P}(M_8 \ge 6) - \mathbb{P}(M_8 \ge 7) \\
&= \sum_{m=-2}^{2} \mathbb{P}(M_8 \ge 6, S_8 = 6-m) - \sum_{m=-1}^{1} \mathbb{P}(M_8 \ge 7, S_8 = 7-m) \\
&\stackrel{(1),(2)}{=} \sum_{m=-2}^{2} \mathbb{P}(S_8 = 6+|m|) - \sum_{m=-1}^{1} \mathbb{P}(S_8 = 7+|m|) \\
&= \mathbb{P}(S_8 \ge 6) + \underbrace{\mathbb{P}(S_8 > 6)}_{=\cancel{\mathbb{P}(S_8 \ge 7)}} - \cancel{\mathbb{P}(S_8 \ge 7)} - \mathbb{P}(S_8 > 7) \\
&= \mathbb{P}(S_8 = 6) + \mathbb{P}(S_8 = 8) - \mathbb{P}(S_8 = 8) \\
&= \mathbb{P}(S_8 = 6) \stackrel{3.6}{=} \binom{8}{7} \left(\frac{1}{2}\right)^8 = \frac{1}{32}
\end{aligned}$$

Problem 4 *(6 points)*

In an election candidate A receives 200 votes while candidate B only receives 100. Assume that the probability of getting a vote is identical (50% each) for A and B. What is the probability that A is always ahead throughout the count?

Solution:

This problem is also known as *Bertrand's Ballot Problem.*

We start by writing A_n, B_n for the number of votes for A and B respectively after $n+1$ votes have been counted. We then define $S_n := 1 + B_n - A_n$ for $n \in \mathbb{N}_0$ and formally add $S_{-1} = 1$. We do this so we can eventually condition on $S_0 = 0$.

We want to know the probability of A being ahead throughout the entire count. With our random variables defined above, this is equivalent to

$$M_{299} := \max_{0 \le i \le 299} S_i \le 0$$

This implies $S_0 = 0$, candidate A receives the first vote. Additionally we know $S_{299} = -99$, which gives us

$$\begin{aligned}\mathbb{P}(M_{299} \le 0 | S_{299} = -99, S_{-1} = 1) &= \mathbb{P}(S_0 = 0 | S_{299} = -99, S_{-1} = 1) \\ &\cdot \mathbb{P}(M_{299} \le 0 | S_{299} = -99, \underbrace{S_0 = 0, S_{-1} = 1}_{\overset{\text{MC}}{\to} S_0 = 0})\end{aligned}$$

For the first factor, we simply use the definition of conditional probability:

$$\begin{aligned}\mathbb{P}(S_0 = 0 | S_{299} = -99, S_{-1} = 1) &= \frac{\mathbb{P}(S_{-1} = 1, S_0 = 0, S_{299} = -99)}{\mathbb{P}(S_{-1} = 1, S_{299} = -99)} \\ = \frac{\mathbb{P}(S_{-1} = 1, S_0 - S_{-1} = -1, S_{299} - S_0 = -99)}{\mathbb{P}(S_{-1} = 1, S_{299} - S_{-1} = -100)} \\ \overset{\text{ind.}}{=} \frac{\mathbb{P}(S_0 - S_{-1} = -1)\mathbb{P}(S_{299} - S_0 = -99)}{\mathbb{P}(S_{299} - S_{-1} = -100)} \overset{3.6}{=} \frac{\frac{1}{2} \cdot \binom{299}{100} \left(\frac{1}{2}\right)^{299}}{\binom{300}{100} \left(\frac{1}{2}\right)^{300}} = \frac{200}{300}\end{aligned}$$

For the second factor, we can use the generalized reflection principle:

$$\begin{aligned}\mathbb{P}(M_{299} \le 0 | S_{299} = -99, S_0 = 0) &= 1 - \mathbb{P}(M_{299} \ge 1 | S_{299} = -99, S_0 = 0) \\ = 1 - \frac{\mathbb{P}(M_{299} \ge 1, S_{299} = -99 | S_0 = 0)}{\mathbb{P}(S_{299} = -99 | S_0 = 0)} &= 1 - \frac{\mathbb{P}(S_{299} = 101 | S_0 = 0)}{\mathbb{P}(S_{299} = -99 | S_0 = 0)} \\ \overset{3.6}{=} 1 - \frac{\binom{299}{200} 2^{-299}}{\binom{299}{100} 2^{299}} = 1 - \frac{100}{200} = \frac{1}{2}\end{aligned}$$

This gives us the final result

$$\mathbb{P}(M_{299} \le 0 | S_{299} = -99, S_{-1} = 1) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$