Problem 1 *(8 points)*

Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}[Y_n = 1] = p = 1 - \mathbb{P}[Y_n = -1]$ for some $0 < p < 1$. Define $X_n := \prod_{i=1}^{n} Y_i$ for all $n \geq 1$ and $X_0 = 1$.

(a) Show that $(X_n)_{n\geq 0}$ is a Markov chain and provide the corresponding transition matrix P.

(b) Argue that $P_{i,j}^{(n)}$ converges for $n \to \infty$ for all i and j in $\mathcal{S}$, and determine the corresponding limits $\lim_{n\to\infty} P_{i,j}^{(n)}$.

(c) Let $T_1 := \min\{n \geq 1 : X_n = 1\}$. Compute $\mathbb{E}_1[T_1] = \mathbb{E}[T_1 \mid X_0 = 1]$.

(a) The sequence (Y_n) consists of i.i.d. random variables where $P[Y_n = 1] = p$ and $P[Y_n = -1] = 1-p$

X_n is $\prod_{i=1}^{n} Y_i$ with $X_0 = 1$

Since Y_n can only take the values of 1 or -1, X_n can only take the values of 1 and -1 as well.

If $X_n = 1$, then $X_{n+1} = X_n \cdot Y_{n+1} = 1 \cdot Y_{n+1} = Y_{n+1}$

If $X_n = -1$, then $X_{n+1} = -Y_{n+1}$

Therefore, X_{n+1} only depends on X_n and Y_{n+1}, and not on any X_k $(k < n)$

Thus, (X_n) is a Markov Chain because the feature state only depends on the current state.

Transition probabilities:

$$P[X_{n+1} = 1 \mid X_n = 1] = P[Y_{n+1} = 1] = p$$

$$P[X_{n+1} = 1 \mid X_n = -1] = P[Y_{n+1} = -1] = 1-p$$

$$P[X_{n+1} = -1 \mid X_n = 1] = P[Y_{n+1} = -1] = 1-p$$

$$P[X_{n+1} = -1 \mid X_n = -1] = P[Y_{n+1} = 1] = p$$

Thus the transition matrix P is:

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

(b) Given the transition matrix P and since the states are symmetric, when n goes infinity, the Markov chain will reach a stationary distribution if it exists.

Let stationary distribution $\pi = [\pi_1, \pi_2]$

Then:

$$\begin{cases} p\pi_1 + (1-p)\pi_2 = \pi_1 \\ (1-p)\pi_1 + p\pi_2 = \pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{1}{2} \\ \pi_2 = \frac{1}{2} \end{cases}$$

Thus, the chain converges to:

$$\lim_{n \to \infty} P^{(n)} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

(c) Given $X_0 = 1$, since it already starts at 1, the expected number of steps to return to 1, if it departs from 1, follows a geometric distribution with prob. p.

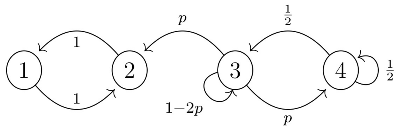
Thus $E_1[T_1] = 1 + (1-p)E_1[T_1]$

$$\Rightarrow E_1[T_1] = \frac{1}{p}$$

This is the expected time to return to 1.

Problem 2 *(8 points)*

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition graph



where $p \in [0, 0.5]$.

(a) Provide the transition probability matrix P.

(b) Determine all communication classes.

Hint: You need to do a case-by-case analysis depending on the value of p.

Now, let $p = 0$:

(c) Which states are recurrent and which are transient?

(d) Compute all stationary distributions.

(a)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & p & 1-2p & p \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b) case 1: $p=0$

state 1 and 2 are communication and closed

state 3 is communication and closed.

case 2: $0 < p \leq 0.5$

state 1 and 2 are communication and closed

state 3 and state 4 are communication

(c) if $P=0$

state 1 and 2 are recurrent.

state 3 is recurrent

state 4 is transient

(d) Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ be the stationary distribution

$\pi P = \pi$

$\Rightarrow$

$$\begin{cases} \pi_2 = \pi_1 & ① \\ \pi_1 = \pi_2 & ② \\ \pi_3 + \frac{1}{2}\pi_4 = \pi_3 & ③ \\ \frac{1}{2}\pi_4 = \pi_4 & ④ \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 & ⑤ \end{cases}$$

Simplifies ③ :

$$\pi_4 = 0$$

with $\pi_4 = 0$, we solve $\pi_1 = \pi_2$, $\pi_3 = \pi_3$

So, the general form of the stationary distribution π is:

$$\pi = (\pi_1, \pi_1, 1-2\pi_1, 0)$$

Bring them into ⑤ :

$$2\pi_1 + \pi_3 = 1$$

where $2\pi_1 \in [0, 1]$.

Thus $\pi_1 \in [0, 0.5]$

For $\pi_1 = 0.5$, $\pi = (0.5, 0.5, 0, 0)$

For $\pi_1 = 0$, $\pi = (0, 0, 1, 0)$

Conclusion :

There is a sequence of stationary distributions ranging from $(0, 0, 1, 0)$ to $(0.5, 0.5, 0, 0)$.

Problem 3 *(8 points)*

The yearly premium for your car insurance depends on your status. There are three possible states 0,1,2 for your status and your status will be decreased by one (if possible) in the next year if you had an accident during the year. If you had no accident during a year, your status will be increased by one (if possible). Let $p \in (0,1)$ be the probability of having an accident during a year. Your yearly premium in EUR is given by

state	0	1	2
premium	364	182	91

(a) Give the transition matrix P of the HMC corresponding to your car insurance status over the years.

(b) Calculate the (unique) stationary distribution.

(c) What is the average yearly premium you will have to pay in the long run if $p = 0.1$?

(a) From state 0, $P(0,0) = p$, $P(0,1) = 1-p$

From state 1, $P(1,0) = p$, $P(1,2) = 1-p$

From state 2, $P(2,1) = p$, $P(2,2) = 1-p$

Therefore, the transition matrix P is:

$$P = \begin{bmatrix} p & 1-p & 0 \\ p & 0 & 1-p \\ 0 & p & 1-p \end{bmatrix}$$

(b) To find the stationary distribution π, we have

$$\begin{cases} \pi_0 (1-p) + \pi_1 (1-p) = \pi_0 & ① \\ \pi_0 p + \pi_2 (1-p) = \pi_1 & ② \\ \pi_1 p + \pi_2 p = \pi_2 & ③ \\ \pi_0 + \pi_1 + \pi_2 = 1 & ④ \end{cases}$$

Simplifies ①:

$$\pi_1 = \frac{p}{1-p} \pi_0 \quad ⑤$$

Simplifies ③:

$$\pi_2 = \frac{p}{1-p} \pi_1 \quad ⑥$$

Bring ⑤ into ⑥:

$$\pi_2 = \frac{p^2}{(1-p)^2}\pi_0 \quad ⑦$$

Substituting ⑤ and ⑦ in ④:

$$\pi_0 + \frac{p}{1-p}\pi_0 + \frac{p^2}{(1-p)^2}\pi_0 = 1$$

$$\pi_0\left(1 + \frac{p}{1-p} + \frac{p^2}{(1-p)^2}\right) = 1$$

$$\pi_0 = \frac{(1-p)^2}{p^2-p+1}$$

$$\pi_1 = \frac{p}{\cancel{1-p}} \cdot \frac{(1-p)^{\cancel{2}}}{p^2-p+1} = \frac{p-p^2}{p^2-p+1}$$

$$\pi_2 = \frac{p^2}{\cancel{(1-p)^2}} \cdot \frac{\cancel{(1-p)^2}}{p^2-p+1} = \frac{p^2}{p^2-p+1}$$

So, the stationary distribution is $\pi = \left[\frac{(1-p)^2}{p^2-p+1}, \frac{p-p^2}{p^2-p+1}, \frac{p^2}{p^2-p+1}\right]$

(c) when $p = 0.1$;

$$\pi_0 = \frac{(1-0.1)^2}{1-0.1+0.1^2} = \frac{81}{91} \qquad \pi_1 = \frac{1}{9}\cdot\frac{81}{91} = \frac{9}{91} \qquad \pi_2 = \frac{1}{9}\cdot\frac{9}{91} = \frac{1}{91}$$

$$\text{Average Premium} = 364\pi_0 + 182\pi_1 + 91\pi_2$$

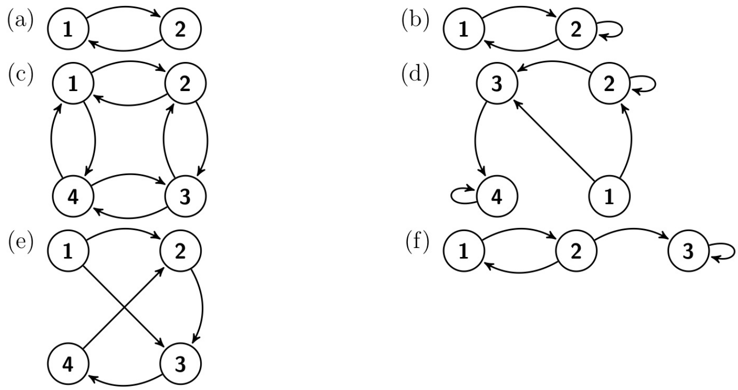
$$= \cancel{364}^{4} \times \frac{81}{\cancel{91}} + \cancel{182}^{2} \times \frac{9}{\cancel{91}} + \cancel{91} \times \frac{1}{\cancel{91}}$$

$$= 343$$

So, the average yearly premium you will have to pay in the long run is 343 €.

Problem 4 *(6 points)*

Which of the following Markov chains with corresponding transition graph are irreducible and aperiodic? Give a short explanation of your answer.



In which cases can we say something about convergence of the Markov chain against its stationary distribution?

Remark: The transition graphs of the Markov chains are without the corresponding transition probabilities. An arrow pointing from one vertex to another means that the transition probability is strictly positive. The actual probabilities are not relevant for the considered problem.

(a) irreducible: yes. State 1 and 2 communicate with each other.
aperiodic: no, it is periodic with a period of 2

(b) irreducible: yes. same reason as (a)
aperiodic: yes. the self-loop at state 2 allows for returning to state 2 in 1 step. This breaks the fixed cycle length.

(c) irreducible: yes. Each of these states can reach any other state in the chain.
aperiodic: no. The gcd of the chain is 2, larger than 1.

(d) irreducible: no. State 1 can not back to itself.
aperiodic: yes. state 2 and 4 can return to itself in 1 step.
State 1 can reach 2 and 3, but cannot return to itself, it doesn't form cycle.
state 3 can only reach 4.

(e) irreducible: no. state 4 cannot reach to state 1.
aperiodic: no. State 2, 3 and 4 form a cycle.
the length of the cycle is 3.

(f) irreducible: no. state 3 cannot reach to state 2
aperiodic: no. state 1 and 2 form a cycle
the length of the cycle is 2.

We can say the convergence of the Markov chain only if it is irreducible and aperiodic.