Sheet 12

Exercise 47

(a) $H(Y|X) = \Sigma_{x\in\mathcal{X}} P(x)\cdot H(Y|X=x)) = \Sigma_{x\in\mathcal{X}}\Sigma_{y\in\mathcal{Y}} P(x,y)\cdot \log P(y|x)$

$X, Y$ are independent r.v. $\Rightarrow$ $= \Sigma_{x\in\mathcal{X}}\Sigma_{y\in\mathcal{Y}} P(y)\, p(x)\, \log P(y) = \Sigma_{y\in\mathcal{Y}}\Sigma_{x\in\mathcal{X}} P(x) P(y) \log P(y)$

$= \Sigma_{y\in\mathcal{Y}} P(y)\cdot \log P(y) = H(Y)$

(b) By Proposition 7.1.6 : $H(X,Y) = H(X) + H(Y|X) = H(X) + H(Y)$

(c) $H(Y,X) = -\Sigma_{x\in\mathcal{X},\, y\in f(x)} P(x,y)\cdot \log P(x,y)$

$P(x,y) = P(y|x)\cdot P(x) \xRightarrow{y=f(x)} P(y|x) = 1 \Rightarrow P(x,y) = P(x)$

Hence $H(Y,X) = -\Sigma_{x\in\mathcal{X}} P(x)\cdot \log P(x,y) = H(X)$

(d) $H(Y|X) = H(X,Y) - H(X) \overset{\text{by (c)}}{=\!=\!=} 0$.

(e) $H(X) = -\Sigma_{x\in\mathcal{X}} P(x)\cdot \log(P(x))$, $P(x)\in[0,1] \Rightarrow \log P(x) \leq 0$

Since $P(x) \geq 0$, $P(x)\cdot \log P(x) \leq 0$. $H(X) = 0$ implies for all $x\in\mathcal{X}$, $P(x)\cdot \log P(x) = 0$.

which means either $P(x) = 0$ or $\log P(x) = 0$.

We know that $X$ ~~$=\{x_1, \dots, x_n\}$~~ $x_i \in$ are r.v., it's impossible for all $x_i$. $P(x_i) = 0$.

$\Rightarrow \log P(X) = 0$, i.e., $P(X) = 1$, which means $X$ takes one value with possibility 1.

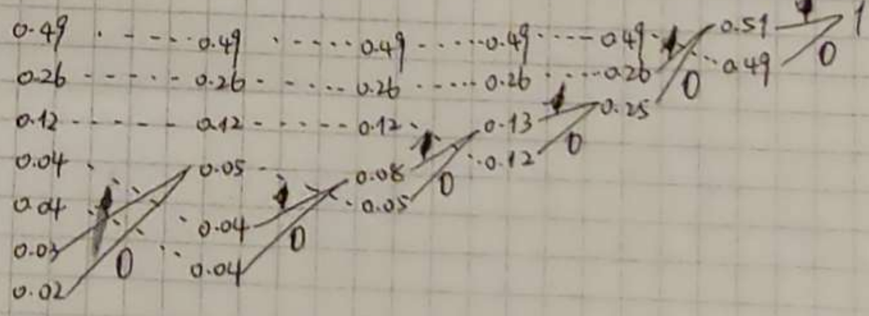
$X_i$ is the only value with possibility 1 $\Rightarrow \Sigma_{x\in\mathcal{X}} P(x)\cdot \log P(x) = P(x)\cdot \log P(x) = 0 = H(X)$

(f)

Exercise 48

(a.1)

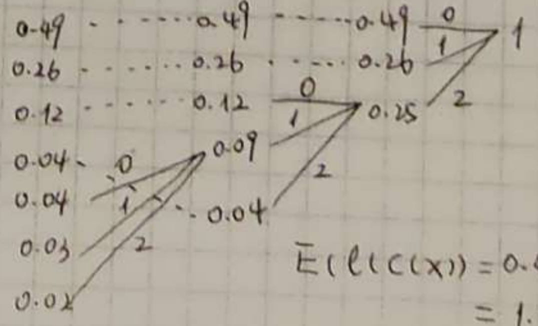
| | | |
|---|---|---|
| 0 | 1 | 1 |
| 11 | 00 | 2 |
| 100 | 011 | 3 |
| 10111 | 01000 | 4 |
| 10110 | 01001 | 5 |
| 10101 | 01010 | 6 |
| 10100 | 01011 | 7 |




$E(\ell(c(X))) = 0.49\times 1 + 0.26\times 2 + 0.12\times 3 + 0.13\times 5 = 2.02$

(a.2)

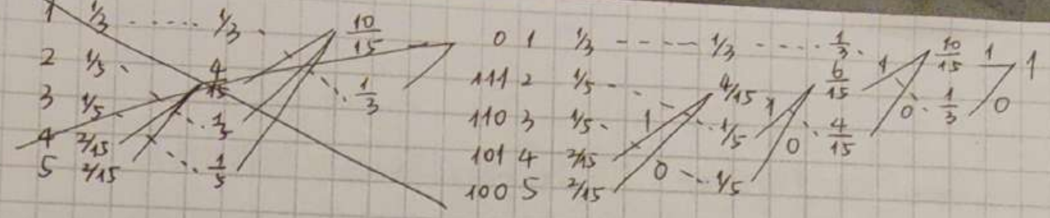
| | |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 20 | 3 |
| 22 | 4 |
| 210 | 5 |
| 211 | 6 |
| 212 | 7 |




$E(\ell(C(X))) = 0.49 + 0.26 + 0.24 + 0.08 + 0.27$

$= 1.34$

(b)



Exercise 50.

$$h(f) = -\int_{\mathbb{R}} f(x) \log f(x) dx = -\int_{\mathbb{R}} \frac{1}{2}\lambda e^{-\lambda|x|} \cdot \log \frac{1}{2}\lambda e^{-\lambda|x|} dx$$

$$= -\int_{\mathbb{R}} \frac{1}{2}\lambda e^{-\lambda|x|} \cdot \left(\log\frac{1}{2} + \log\lambda - \cancel{\log}\,\lambda|x|\right) dx$$

$$= -2\left(\int_0^\infty \frac{1}{2}\left(\log\frac{1}{2} + \log\lambda\right)\cdot\lambda\cdot e^{-\lambda x} dx - \frac{1}{2}\lambda^2 \int_0^\infty x\cdot e^{-\lambda x} dx\right)$$

$$= -\lambda\left(\log\frac{1}{2} + \log\lambda\right)\cdot\left(-\frac{1}{\lambda}\cdot e^{-\lambda x}\right)\Big|_0^\infty + \frac{1}{2}\lambda\cdot(-2)$$

$$= -\log\frac{1}{2} - \log\lambda - \lambda = -\left(\ln\frac{1}{2} + \ln\lambda + 1\right) = -\ln\frac{\lambda e}{2} \text{ nats}$$