HHU Düsseldorf
Math.-Nat. Fakultät
Prof. Dr. Nils Detering
Dr. Nicole Hufnagel

Heinrich Heine Universität Düsseldorf
Lehrstuhl Finanz- und Versicherungsmathematik

Summer semester 2024

Markov Chains

Problem sheet 5

Markov chains: stopping times, limiting distributions, stationary distributions

Problems to be handed in by:
Thursday, June 27th, 11:59 p.m., online via Ilias.

Problem 1 *(8 points)*
Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}[Y_n = 1] = p = 1 - \mathbb{P}[Y_n = -1]$ for some $0 < p < 1$. Define $X_n := \prod_{i=1}^n Y_i$ for all $n \geq 1$ and $X_0 = 1$.

(a) Show that $(X_n)_{n\geq 0}$ is a Markov chain and provide the corresponding transition matrix P.

(b) Argue that $P_{i,j}^{(n)}$ converges for $n \to \infty$ for all i and j in $\mathcal{S}$, and determine the corresponding limits $\lim_{n\to\infty} P_{i,j}^{(n)}$.

(c) Let $T_1 := \min\{n \geq 1 : X_n = 1\}$. Compute $\mathbb{E}_1[T_1] = \mathbb{E}[T_1 \,|\, X_0 = 1]$.

Solution:

(a) With $Y_n \in \{1, -1\}$, the same holds for X_n. We can write $X_n = X_{n-1}Y_n$ and see that Y_n is independent of $X_0, \ldots, X_{n-1}$. This gives us

$$\begin{aligned}\mathbb{P}(X_n = i_n \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) &= \mathbb{P}(i_{n-1}Y_n = i_n \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) \\ &\overset{\text{ind.}}{=} \mathbb{P}(i_{n-1}Y_n = i_n) = \mathbb{P}(X_{n-1}Y_n = i_n \,|\, X_{n-1} = i_{n-1}) = \mathbb{P}(X_n = i_n \,|\, X_{n-1} = i_{n-1})\end{aligned}$$

for all $i_0, \ldots, i_n \in \{1, -1\}$.
$(X_n)_{n\geq 0}$ is a Markov chain on the state space $\mathcal{S} = \{1, -1\}$ with transition probabilities

$$P_{ji} = \mathbb{P}(X_n = i \,|\, X_{n-1} = j) = \mathbb{P}(X_{n-1}Y_n = i \,|\, X_{n-1} = j) = \mathbb{P}(Y_n = \tfrac{i}{j}),$$

so the transition matrix is given by

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

(b) Because we know $p \in (0,1)$, the entries are all positive. This means that both states are aperiodic (can reach itself in 1 step) and communicating with each other. By Theorem 4.38 $P_{i,j}^{(n)}$ converges against the j-th entry of the unique stationary distribution π. In this case, we obtain

$$\pi = \pi P \qquad \Longleftrightarrow \qquad \begin{cases} \pi_{-1} = p\pi_{-1} + (1-p)\pi_1 \\ \pi_1 = (1-p)\pi_{-1} + p\pi_1 \end{cases} \qquad \Longrightarrow \qquad \pi_{-1} = \pi_1 = 0.5,$$

hence $\lim_{n\to\infty} P_{i,j}^{(n)} = 0.5$ for all $i, j \in \{1, -1\}$.

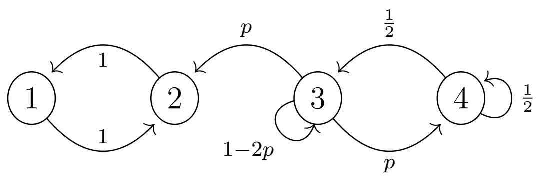
(c) There are two options to get back to state 1 after starting there. Either we stay in the same state and get $T_1 = 1$ or we swap to state -1 and swap back to 1 later:

$$\begin{aligned} \mathbb{P}(T_1 = k \mid X_0 = 1) &= \begin{cases} \mathbb{P}(X_1 = 1 \mid X_0 = 1), & k = 1 \\ \mathbb{P}(X_k = 1, X_{k-1}, \dots, X_1 = -1 \mid X_0 = 1), & k > 1 \end{cases} \\ &= \begin{cases} p, & k = 1 \\ (1-p)^2 p^{k-2}, & k > 1 \end{cases} \end{aligned}$$

This gives us

$$\begin{aligned} \mathbb{E}_1[T_1] &= \sum_{k=1}^{\infty} k\mathbb{P}(T_1 = k \mid X_0 = 1) = p + \sum_{k=2}^{\infty} k(1-p)^2 p^{k-2} \\ &= p + (1-p)\sum_{k=1}^{\infty}(k+1)\underbrace{(1-p)p^{k-1}}_{\text{PMF of Geo(1-p)}} \\ &= p + (1-p)\left(\sum_{k=1}^{\infty} k(1-p)p^{k-1} + \sum_{k=1}^{\infty}(1-p)p^{k-1}\right) \\ &= p + (1-p)\left(\tfrac{1}{1-p} + 1\right) = p + 1 + (1-p) = 2 \end{aligned}$$

Problem 2 *(8 points)*
Let $(X_n)_{n\geq 0}$ be a Markov chain with transition graph



where $p \in [0, 0.5]$.

(a) Provide the transition probability matrix P.

(b) Determine all communication classes.

Hint: You need to do a case-by-case analysis depending on the value of p.

Now, let $p = 0$:

(c) Which states are recurrent and which are transient?

(d) Compute all stationary distributions.

Solution:

(a) The transition matrix is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & p & 1-2p & p \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

(b) The states 1 and 2 are always communicating. 3 and 4 are communicating iff $p > 0$. There are no further pairs of communicating states, so the communication classes are given by

(i) $p = 0$: $\{1, 2\}, \{3\}, \{4\}$

(ii) $p \in (0, 0.5]$: $\{1, 2\}, \{3, 4\}$

(c) We use the fact that every finite, closed communication class is recurrent. Since $\{1, 2\}$ and $\{3\}$ fulfill these conditions, these states are recurrent.
State 4 on the other hand cannot be reached from any other state. This means that

$$\mathbb{P}(T_4^+ < \infty \,|\, X_0 = 4) = \mathbb{P}(T_4^+ = 1 \,|\, X_0 = 4) = P_{44} = 0.5 < 1,$$

hence 4 is transient.

(d) For $p = 0$ we have the transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

We now look for solutions to $\pi P = \pi$.

$$\pi = \pi P \qquad \Longleftrightarrow \qquad \begin{cases} \pi_1 = \pi_2 \\ \pi_2 = \pi_1 \\ \pi_3 = \pi_3 + 0.5\pi_4 \\ \pi_4 = 0.5\pi_4 \qquad \Longrightarrow \qquad \pi_4 = 0 \end{cases}$$

This grants us the set of stationary distributions

$$\{(\mu, \mu, 1 - 2\mu, 0) : \mu \in [0, 0.5]\}.$$

Problem 3 *(8 points)*

The yearly premium for your car insurance depends on your status. There are three possible states 0,1,2 for your status and your status will be decreased by one (if possible) in the next year if you had an accident during the year. If you had no accident during a year, your status will be increased by one (if possible). Let $p \in (0,1)$ be the probability of having an accident during a year. Your yearly premium in EUR is given by

state	0	1	2
premium	364	182	91

(a) Give the transition matrix P of the HMC corresponding to your car insurance status over the years.

(b) Calculate the (unique) stationary distribution.

(c) What is the average yearly premium you will have to pay in the long run if $p = 0.1$?

Solution:

(a) The transition matrix is given by

$$P = \begin{pmatrix} p & 1-p & 0 \\ p & 0 & 1-p \\ 0 & p & 1-p \end{pmatrix}.$$

There is only one recurrent class, which is also finite and aperiodic and hence there exists a unique stationary distribution π.

(b) For the stationary distribution we have

$$\pi = \pi P \quad \Longleftrightarrow \quad \begin{cases} \pi_0 = p\pi_0 + p\pi_1 & \Longrightarrow \quad \pi_0 = \frac{p}{1-p}\pi_1 \\ \pi_1 = (1-p)\pi_0 + p\pi_2 & \\ \pi_2 = (1-p)\pi_1 + (1-p)\pi_2 & \Longrightarrow \quad \pi_2 = \frac{1-p}{p}\pi_1 \end{cases}$$

Hence $\pi = \pi_1\big(\frac{p}{1-p}, 1, \frac{1-p}{p}\big)$ and we must have

$$\begin{aligned} \pi_1 &= \Big(\tfrac{p}{1-p} + 1 + \tfrac{1-p}{p}\Big)^{-1} = \Big(\tfrac{p^2+p(1-p)+(1-p)^2}{p(1-p)}\Big)^{-1} \\ &= \Big(\tfrac{p^2-p+1}{p(1-p)}\Big)^{-1} = \tfrac{p(1-p)}{p^2-p+1} \end{aligned}$$

which shows that

$$\pi = \Big(\tfrac{p^2}{p^2-p+1}, \tfrac{p(1-p)}{p^2-p+1}, \tfrac{(1-p)^2}{p^2-p+1}\Big).$$

(c) For $p = 1/10$ we get

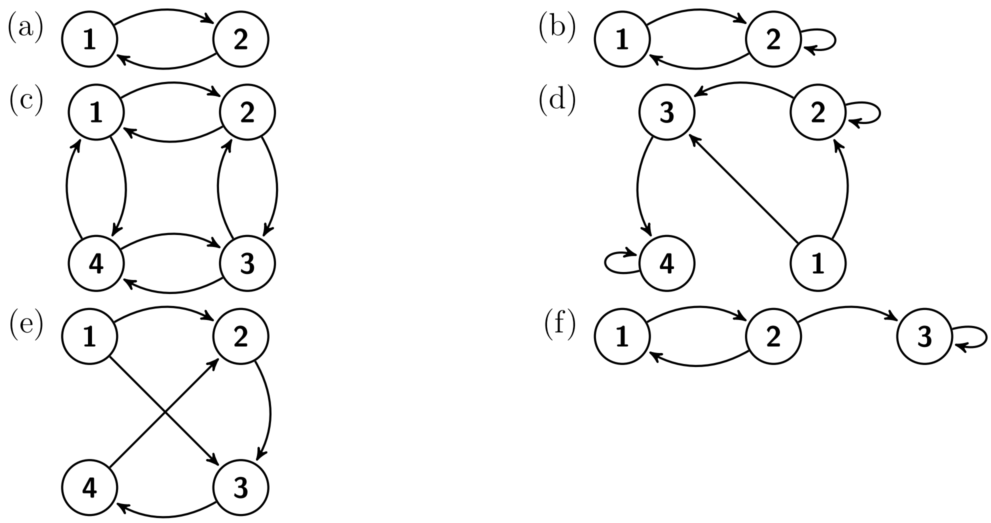
$$\pi = \left(\frac{\frac{1}{100}}{\frac{91}{100}}, \frac{\frac{9}{100}}{\frac{91}{100}}, \frac{\frac{81}{100}}{\frac{91}{100}}\right) = \left(\tfrac{1}{91}, \tfrac{9}{91}, \tfrac{81}{91}\right)$$

We now have the function r, which maps the state to the corresponding premium. With Theorem 4.42 we get

$$S = 364\,\tfrac{1}{91} + 182\,\tfrac{9}{91} + 91\,\tfrac{81}{91} = 4 + 18 + 81 = 103 \text{ EUR}$$

as the average yearly premium.

Problem 4 *(6 points)*
Which of the following Markov chains with corresponding transition graph are irreducible and aperiodic? Give a short explanation of your answer.



In which cases can we say something about convergence of the Markov chain against its stationary distribution?

Remark: The transition graphs of the Markov chains are without the corresponding transition probabilities. An arrow pointing from one vertex to another means that the transition probability is strictly positive. The actual probabilities are not relevant for the considered problem.

Solution:

(a) Irreducible, since both states are communicating. Not aperiodic, since it always takes 2 steps to come back to a state.

(b) Irreducible, since both states are communicating. Aperiodic, since state 2 can be revisited in 1 step, so the period of both states is 1.

(c) Irreducible due to the cyclic structure. Not aperiodic, since in each step we swap from an even to an odd state or vice versa, so we can only come back to a state in an even number of steps.

(d) Not irreducible, since there is no pair of communicating states. Aperiodicity was only defined for irreducible Markov chains.

(e) Not irreducible: State 1 doesn't communicate with any other state.

(f) Not irreducible: State 3 doesn't communicate with the other states.

Only the Markov chain in (b) fulfills the conditions of Theorem 4.38, so it converges against its stationary distribution. However there are other Markov chains which also converge against a limiting distribution: (d) and (f). In the long run we eventually leave the transient states and end up in a single closed, recurrent class with only aperiodic states. ($\{4\}$ in (d), $\{3\}$ in (f)).
In general we can ignore additional transient states, as long as there is only one recurrent class, which is also aperiodic.