Ex 27

(a) (i) $f(x_1, x_2, x_3) := (x_1^2 + x_2^2 + x_3^2)\chi_I \qquad \chi_I : I : [0,1]^3$

$$P(X_1 \le \tfrac{1}{2}, X_3 \ge \tfrac{1}{2}) = \int_{\frac{1}{2}}^{1}\int_0^1\int_0^{\frac{1}{2}} (x_1^2 + x_2^2 + x_3^2)\, dx_1\, dx_2\, dx_3$$

$$= \int_{\frac{1}{2}}^{1}\int_0^1 \tfrac{1}{3}[x_1^3]_0^{\frac{1}{2}} + (x_2^2 + x_3^2)[x_1]_0^{\frac{1}{2}}\, dx_2\, dx_3$$

$$= \int_{\frac{1}{2}}^{1}\int_0^1 \tfrac{1}{24} + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_3^2\, dx_2\, dx_3$$

$$= \int_{\frac{1}{2}}^{1} \tfrac{1}{24}[x_2]_0^1 + \tfrac{1}{2}\cdot\tfrac{1}{3}[x_2^3]_0^1 + \tfrac{1}{2}x_3^2[x_2]_0^1\, dx_3$$

$$= \int_{\frac{1}{2}}^{1} (\tfrac{1}{24} + \tfrac{1}{6} + \tfrac{1}{2}x_3^2)\, dx_3$$

$$= \tfrac{5}{24}[x_3]_{\frac{1}{2}}^1 + \tfrac{1}{2}[\tfrac{1}{3}x_3^3]_{\frac{1}{2}}^1$$

$$= \frac{5}{48} + \frac{1}{2}(\frac{1}{3} - \frac{1}{24})$$

$$= \frac{1}{4}$$

(ii) $$f(X_1, X_2) = \int_0^1 f(x_1, x_2, x_3)\, dx_3$$

$$= \int_0^1 x_1^2 + x_2^2 + x_3^2\, dx_3$$

$$= x_1^2 + x_2^2 + \frac{1}{3}$$

(b) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^\infty e^{-x-y}\, dy \qquad = \int_0^\infty e^{-x-y}\, dx$$

$$= -e^{-x-y}\Big|_0^\infty = e^{-x} \qquad = -e^{-x-y}\Big|_0^\infty = e^{-y}$$

$$f_X(x)\cdot f_Y(y) = e^{-x-y} = f_{X,Y}(x,y)$$

Therefore, X and Y are independent.

(c) (i) According to the question

$$f_X(x) = \begin{cases} x = 1, & \frac{1}{2} \\ x = -1, & \frac{1}{2} \end{cases} \quad f_Y(y) = \begin{cases} y = 1, & \frac{1}{2} \\ y = -1, & \frac{1}{2} \end{cases}$$

$$Z = X \cdot Y \qquad P_{X,Y}(X = x, Y = y) = P_X(x) \cdot P_Y(y)$$

$$\left.\begin{aligned} P(Z=1) &= P(X=1, Y=1) + P(X=-1)P(Y=-1) = \tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{2} \\ P(Z=-1) &= P(X=-1, Y=1) + P(X=1)P(Y=-1) = \tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{2} \end{aligned}\right\} \Rightarrow$$

$$f_Z(z) = \begin{cases} z = 1, & \frac{1}{2} \\ z = -1, & \frac{1}{2} \end{cases}$$

When $X = 1$, $Z = 1$

$$P(X=1, Z=1) \underset{*}{=} P(X=1, Y=1) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} = P(X=1)P(Z=1)$$

$*$ (since $Z = X \cdot Z$)

When $X = 1$, $Z = -1$

$$P(X=1, Z=-1) = P(X=1, Y=-1) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} = P(X=1)P(Z=1)$$

When $X = -1$, $Z = 1$

$$P(X=-1, Z=1) = P(X=-1, Y=-1) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} = P(X=-1)P(Z=1)$$

When $X = -1$, $Z = -1$

$$P(X=-1, Z=-1) = P(X=-1, Y=1) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} = P(X=1)P(Z=1)$$

We can see that $P(X = x, Z = z) = P(X = x)P(Z = z)$ for all x, z in $\{1, -1\}$, which means X and Z are independent.

Similarly, we can show that Y and Z are independent by the same way.

(ii) Consider $a = b = c = 1$

$$P(X = 1, Y = 1, Z = 1) \underset{\uparrow}{=} P(X=1, Y=1) = P(X=1) \cdot P(Y=1) = \tfrac{1}{4}$$

$\uparrow$ since $Z = X \cdot Y$

$$P(X=1) \cdot P(Y=1) \cdot P(Z=1) = \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{8} \neq P(X=1, Y=1, Z=1)$$

Ex 28 (a) If Z is almost surely constant, then $\exists v \in \mathbb{R}^n$, $P(Z=v)=1$.

We know that $E(Z) = \sum_{i=1}^{n} z_i P(z_i)$

Since $\exists v \in \mathbb{R}^n$, $P(Z=v)=1$

$E(Z) = v \cdot 1 = v$

$E(Z^2) = (v^2) \cdot 1 = v^2$

Thus $Var(Z) = E(Z^2) - E(Z)^2 = v^2 - v^2 = 0$

Suppose $Var(Z) = 0$. Then $E(Z^2) = E(Z)^2$

Denote $\mu = E(Z)$, then we have $E((Z-\mu)^2) = 0$

Since $(Z-\mu)^2 \geq 0$, the only way for expected value to be 0 is if $(Z-\mu)^2 = 0$ almost surely.

This implies that $Z = \mu$ almost surely. Therefore z is almost surely constant.

(b) Closure under addition:

Let Z_1 and Z_2 be almost sure constant variable, i.e. $P(Z_1 = v_1) = 1 = P(Z_2 = v_2) = 1$ for some constant v_1 and v_2.

Consider $Z_1 + Z_2$, where $(Z_1 + Z_2)w := Z_1(w) + Z_2(w)$.

For any w in Ω:

$(Z_1 + Z_2)(w) = Z_1(w) + Z_2(w) = v_1 + v_2$

Since $v_1 + v_2$ is constant, $Z_1 + Z_2$ is almost surely constant.

Closure under scalar multiplication:

Let Z be almost sure constant variable, $P(Z=v)=1$

Consider the λZ for any scalar λ, where $(\lambda Z)(w) := \lambda Z(w)$

For any $w \in \Omega$: $(\lambda Z)(w) = \lambda \cdot Z(w) = \lambda \cdot v$

Since $\lambda \cdot v$ is constant, λZ is almost surely constant.

Contain the zero vector:

Let Z_0 be a random variable defined as $Z_0(w) = 0$ for all $w \in \Omega$.

Clearly Z_0 is almost surely constant.

(c) If Z is not almost sure constant and Z and Y are linearly dependent,
then $Z = cY$

c: constant

$$\rho = \frac{Cov(Z, Y)}{\sigma_Z \sigma_Y} = \frac{Cov(cY, Y)}{\sigma_{cY} \sigma_Y} = \frac{c\,Var(Y)}{|c|\sigma_Y^2}$$

If $c > 0$, then the correlation coefficient between Z and Y is 1.
If $c < 0$, then the correlation coefficient between Z and Y is -1.

(d) No. If X and Y are linearly independence, it means there is no linear relationship between X and Y.
But they could be dependt in some non-linear way.

(e) No. X and Y are independent, doesn't means they are linearly independt. E.g. $X = 2Y$, they are independt random variable but linearly dependent.

(f) No.