HHU Düsseldorf
Math.-Nat. Fakultät
Prof. Dr. Nils Detering
Dr. Nicole Hufnagel

Heinrich Heine Universität Düsseldorf
Lehrstuhl Finanz- und Versicherungsmathematik

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Markov Chains

Problem sheet 4

markov chains

Problems to be handed in by:
Thursday, June 13th, 11:59 p.m., online via Ilias.

Problem 1 *(6 points)*
Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix} 0.4 & 0 & 0.6 \\ 0.3 & 0.1 & 0.6 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Compute the stationary distribution π.

Solution:
We need to find a distribution $\pi = (x_1, x_2, x_3)^T$ with $\pi^T P = \pi^T$ and $x_1+x_2+x_3 = 1$.

$$\begin{aligned} x_1 &= 0.4x_1 + 0.3x_2 + 0.2x_3 \\ x_2 &= 0.1x_2 + 0.2x_3 \\ x_3 &= 0.6x_1 + 0.6x_2 + 0.6x_3 \end{aligned}$$

The last equation gives us $x_3 = 0.6(x_1 + x_2 + x_3) = 0.6$. This implies

$$\begin{aligned} 0.9x_2 = 0.2x_3 = 0.2 \cdot 0.6 = 0.12 \\ \Leftrightarrow \quad x_2 = \frac{0.12}{0.9} = \frac{2}{15} \end{aligned}$$

and lastly

$$\begin{aligned} 0.6x_1 = 0.3x_2 + 0.2x_3 = 0.3 \cdot \frac{2}{15} + 0.2 \cdot 0.6 = 0.16 \\ \Leftrightarrow \quad x_1 = \frac{0.16}{0.6} = \frac{4}{15}. \end{aligned}$$

We obtain $\pi = (4/15, 2/15, 9/15)$ as the unique solution to $\pi^T P = \pi^T$ with the restriction of entries summing up to 1, which means that π is the stationary distribution.

Problem 2 *(8 points)*

Let $(X_n)_{n\geq 0}$ denote a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4\}$ and transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & p & 1-p & 0 \end{pmatrix}$$

where $p \in [0, 1]$ is some arbitrary probability.

(a) Provide the transition graph.

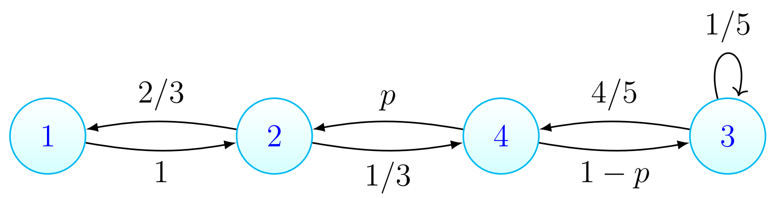
(b) Determine all communication classes. Which communication classes are closed?

Hint: You need to do a case-by-case analysis depending on the value of p.

(c) Fix $p = 0$. The initial distribution is given by $\alpha^T = (0, 0, \frac{5}{9}, \frac{4}{9})$. Compute the distribution of X_1. What is the distribution of X_{103}?

Solution:

(a) The transition graph has the form



(b) We can see that the states 1 and 2 are communicating for all $p \in [0, 1]$. Additionally state 4 is accessible from state 2, however the opposite is only true for $p > 0$. In this case these states are communicating. Likewise state 4 is accessible from state 3, but they are only communicating if state 3 is accessible from state 4, which is the case iff $p < 1$.

With the fact that the “communicating” property is an equivalence relation, we get the following communication classes:

(i) $p \in (0, 1)$: $\mathcal{S} = \{1, 2, 3, 4\}$. Since the Markov chain cannot leave the state space, this communication class is closed.

(ii) $p = 1$: $\{1, 2, 4\}$, $\{3\}$. $\{3\}$ is not closed since $P_{34} = 4/5 > 0$. On the other hand we have $P_{i3} = 0$ for $i \in \{1, 2, 4\}$, which means that this communication class is closed.

(iii) $p = 0$: $\{1,2\}$, $\{3,4\}$. $\{1,2\}$ is not closed since $P_{2,4} = 1/3 > 0$. $\{3,4\}$ is closed since $P_{ij} = 0$ for $i \in \{3,4\}$ and $j \in \{1,2\}$.

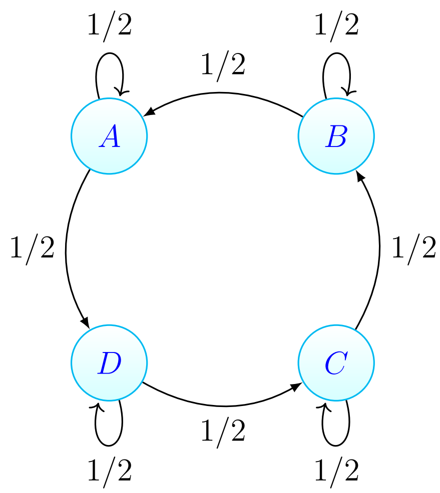
(c) With $p = 0$ and the given initial distribution we get

$$\alpha^T P = \left(0,\, 0,\, \frac{5}{9} \cdot \frac{1}{5} + \frac{4}{9},\, \frac{5}{9} \cdot \frac{4}{5}\right) = \left(0,\, 0,\, \frac{5}{9},\, \frac{4}{9}\right) = \alpha^T$$

This means that α is a stationary distribution and with our observation from the lecture the distribution of X_n is equal to α for all $n \in \mathbb{N}_0$, in particular for X_{103}.

Problem 3 *(8 points)*

A Markov chain $X_0, X_1, X_2, \ldots$ has the following transition graph:



(a) Provide the transition matrix for the Markov chain.

(b) Classify all states (recurrent/transient).

(c) Find the communication classes. Is the chain irreducible?

(d) Find the stationary distribution.

(e) What do you know about the limiting distribution?

Solution:

(a) The transition matrix for this transition graph is

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b) Due to the circular nature of the Markov chain, we never revisit a state $s \in \mathcal{S}$ iff we swap states at most 3 times, which implies that we stay in one state $t \neq s$ forever. However

$$\mathbb{P}(X_m = X_{m-1} = \cdots = X_1 = t \,|\, X_0 = t) = \left(\frac{1}{2}\right)^m \xrightarrow{m\to\infty} 0$$

This means that for all $s \in \mathcal{S}$ we have

$$\mathbb{P}(T_s < \infty) = \mathbb{P}(X_m = s \text{ for some } m \in \mathbb{N} \,|\, X_0 = s) = 1$$

All states are recurrent.

(c) There is only one communication class ($\mathcal{S}$ itself), so the Markov chain is irreducible. This can be deduced from looking at the transition graph or by considering

$$P^3 = \frac{1}{8}\begin{pmatrix} 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \\ 3 & 3 & 1 & 1 \\ 1 & 3 & 3 & 1 \end{pmatrix},$$

which has only positive entries, which means that every state is accessible from any other state within 3 steps.

(d) For the stationary distribution we need to solve the equations

$$\begin{aligned} x_1 &= 0.5x_1 + 0.5x_4 \\ x_2 &= 0.5x_1 + 0.5x_2 \\ x_3 &= 0.5x_2 + 0.5x_3 \\ x_4 &= 0.5x_3 + 0.5x_4 \end{aligned}$$

This implies

$$x_1 = x_2 = x_3 = x_4$$

and with the restriction of being a distribution, we obtain the uniform distribution $\mu^T = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

(e) If $(X_n)_{n\in\mathbb{N}_0}$ has a limiting distribution, it is stationary by Lemma 4.16. Following from (d) this would have to be the uniform distribution since it is the only stationary distribution for this Markov chain.
We will learn later in the lecture (Markov's theorem) that the limiting distribution does exist under certain conditions, which this Markov chain fulfills.

Problem 4 *(8 points)*

Consider a Markov chain on the state space $\mathcal{S} = \{1, 2, 3, 4, \dots\}$ with the following transition matrix:

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 2/3 & 0 & 1/3 & 0 & 0 & \cdots \\ 3/4 & 0 & 0 & 1/4 & 0 & \cdots \\ 4/5 & 0 & 0 & 0 & 1/5 & \cdots \\ 5/6 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

That is, $P_{ij} = i/(i+1)$ if $j = 1$, $P_{ij} = 1/(i+1)$ if $j = i+1$, and $P_{ij} = 0$ otherwise.

(a) Classify all states of the Markov chain (transient, recurrent)

(b) Determine if the Markov chain is irreducible.

Solution:

(a) We first show that 1 is recurrent:

$$\begin{aligned} f_1 &= \mathbb{P}(X_m = 1 \text{ for some } m \in \mathbb{N} \,|\, X_0 = 1) \\ &= 1 - \mathbb{P}(X_m = m+1 \text{ for all } m \in \mathbb{N} \,|\, X_0 = 1) \\ &= 1 - \prod_{m=1}^{\infty} \mathbb{P}(X_m = m+1 \,|\, X_{m-1} = m) \\ &= 1 - \prod_{m=1}^{\infty} \frac{1}{m+1} = 1 \end{aligned}$$

Note that the Markov chain either falls back down to 1 or increases by 1 in each step. Now for $n \geq 2$:
Starting from 1 we either reach n in $n-1$ steps with probability

$$\mathbb{P}(X_{n-1} = n \,|\, X_0 = 1) = \prod_{k=1}^{n-1} \mathbb{P}(X_k = k+1 \,|\, X_{k-1} = k) = \prod_{k=1}^{n-1} \frac{1}{k+1} = \frac{1}{n!}$$

or we go back to 1 along the way and try again, which happens with probability $1 - \frac{1}{n!}$. To revisit n we first need to drop to 1 and then come back up.

$$\begin{aligned} f_n &= \mathbb{P}(X_m = n \text{ for some } m \in \mathbb{N} \,|\, X_0 = n) \\ &= \mathbb{P}(X_m = 1 \text{ for some } m \in \mathbb{N} \,|\, X_0 = n) \cdot \mathbb{P}(X_m = n \text{ for some } m \in \mathbb{N} \,|\, X_0 = 1) \\ &= \underbrace{\left(1 - \prod_{m=1}^{\infty} \frac{1}{m+n}\right)}_{=1} \cdot \left(\sum_{k=0}^{\infty} \left(1 - \frac{1}{n!}\right)^k \cdot \frac{1}{n!}\right) \\ &= \frac{1}{n!} \underbrace{\sum_{k=0}^{\infty} \left(1 - \frac{1}{n!}\right)^k}_{\text{Geometric series}} = \frac{1}{n!} \cdot \frac{1}{1 - (1 - \frac{1}{n!})} = 1 \end{aligned}$$

This means that all states of the Markov chain are recurrent.

(b) This Markov chain is irreducible because all states are communicating. Fix $n, m \in \mathcal{S}$. We can reach n in n steps with positive probability:

$$
\begin{aligned}
&\mathbb{P}(X_k = n \text{ for some } k \in \mathbb{N} \mid X_0 = m) \\
&\geq \mathbb{P}(X_n = n, X_{n-1} = n-1, \ldots X_1 = 1 \mid X_0 = m) \\
&= \mathbb{P}(X_1 = 1 \mid X_0 = m) \cdot \prod_{k=2}^{n} \mathbb{P}(X_k = k \mid X_{k-1} = k-1) \\
&= \frac{m}{m+1} \cdot \prod_{k=2}^{n} \frac{1}{k} = \frac{m}{m+1} \cdot \frac{1}{n!} > 0
\end{aligned}
$$