Ex 3.11

$y \sim Geo(p_j)$

$p_j \sim Beta(a, b)$

$$P(p|y) \propto \underbrace{P(p|a,b)}_{\text{prior}} \cdot \underbrace{P(y|p,a,b)}_{\text{posterior}}$$

$$\propto \prod_{j=1}^{J} \frac{1}{B(a,b)} p_j^{a-1}(1-p_j)^{b-1} \cdot p_j(1-p_j)^{k-1}$$

$$\propto p_j^{a}(1-p_j)^{k+b-2}$$

Given a and b, the components of P have independent posterior densities that are of the form $p_j^A(1-p_j)^B$ that is beta densities. And the joint density is:

$$P(p|y) = \prod_{i=1}^{J} \frac{1}{B(a+1, k+b-1)} p_j^{a}(1-p_j)^{b+k-2}$$

$$= Beta(a+1, k+b-1)$$

Thus $a' = a+1$, $b' = b+k-1$

Ex 40)

$$P(p \mid x) = \frac{P(x \mid p) \cdot P(p)}{P(x)}$$

$$\propto P(x \mid p) \cdot P(p)$$

$$= p(1-p)^{n-1} \cdot 3(p-1)^2$$

$$= 3p(1-p)^{n+1}$$

$$P(p \mid 2,5,1) = 3p(1-p)^3 \cdot 3p(1-p)^6 \cdot 3p \cdot (1-p)^2$$

$$= 27p^3(1-p)^{11}$$

$$0 = \frac{\partial}{\partial p} P(p \mid 2,5,1) = -27p^2(14p-3)(1-p)^{10}$$

$$\Rightarrow p = 0 \text{ or } \frac{3}{14} \text{ or } 1$$

$$P(1 \mid 2,5,1) = 0$$

$$P(0 \mid 2,5,1) = 0$$

$$P\left(\frac{3}{14} \mid 2,5,1\right) = 27 \times \left(\frac{3}{14}\right)^3 \times \left(1 - \frac{3}{14}\right)^{11} > 0$$

So the maximum posterior estimate of p is $\frac{3}{14}$.

Ex 4.1 (a) $\theta \sim Beta(\alpha, \beta)$

$x \sim Bin(n, \theta)$

$$P(\theta | x) \propto P(x|\theta) \cdot P(\theta)$$

$$= \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{k+\alpha-1} (1-p)^{n-k+\beta-1}$$

$$\propto p^{k+\alpha-1} (1-p)^{n-k+\beta-1}$$

$$= Beta(\alpha+k, \beta+n-k)$$

in this case, the distribution is

$$Beta(\alpha+12, \beta+3)$$

(b) The MAP for p is the mode of the posterior Beta distribution:

$$\hat{p}_{MAP} = \frac{k+\alpha-1}{(k+\alpha-1)+(n-k+\beta-1)} = \frac{k+\alpha-1}{n+(\alpha-1)+(\beta-1)}$$

in this case $\hat{p}_{MAP} = \frac{11+\alpha}{13+\alpha+\beta}$