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Lehrstuhl Finanz- und Versicherungsmathematik

Summer semester 2024

Markov Chains

Problem sheet 5

Markov chains: stopping times, limiting distributions, stationary distributions

Problems to be discussed (in parts) during the exercise sections:

Problem 1

A Branching process / Galton-Watson process is a simple model for a population evolving in time.

- Each individual within a population gives birth to a number of children independently but with the same distribution.
- Let $(p_i)_{i\geq 0}$, the offspring distribution, i.e.

$$p_i = \mathbb{P}(\text{individual has } i \text{ children})$$

and

$$Z_n := \#\text{individuals in the n-th generation} \qquad Z_0 = 1$$

- Then $Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}$, where $X_{n,i} \sim X$ and X has distribution $(p_i)_{i\geq 0}$.

One easily observes that the process $Z_1, Z_2, ...$ is in fact a Markov Chain. Answer the following questions. Note that you will have to consider different situations for the offspring distribution.

(a) How does the transition matrix P of the Markov Chain $Z_1, Z_2, ...$ look like? It is sufficient to give the upper left 4×4 entries corresponding to the states $0, 1, 2, 3$ to see what happens.

(b) Determine the communication classes. Is this Markov Chain irreducible?

(c) Since 0 is an absorbing state, obviously $\pi = (1, 0, 0, 0, \dots)$ is a stationary distribution. Is this the only stationary distribution?

Problem 2
Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov Chain with Transitionmatrix P on the state space $\mathcal{S}$. For $B \subseteq \mathcal{S}$ define the Stopping time

$$T_B = \inf\{n \geq 0 : X_n \in B\}$$

which is the first entry to the set B. Now define for $n \in \mathbb{N}_0$

$$Y_n := \begin{cases} X_{T_B} & \text{, if } n \geq T_B \\ X_n & \text{, otherwise.} \end{cases}$$

(a) Show that $(Y_n)_{n\in\mathbb{N}_0}$ is again a Markov Chain.

(b) Write down the transition matrix P of $(Y_n)_{n\in\mathbb{N}_0}$. For this, write P in the following block form:

$$P = \begin{array}{cc} \begin{array}{cc} B & B^c \end{array} & \\ \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} & \begin{array}{c} B \\ B^c \end{array} \end{array}$$

Problems to be handed in by:
Thursday, June 27th, 11:59 p.m., online via Ilias.

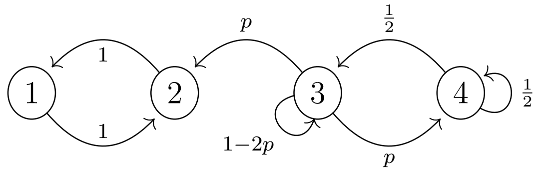
Problem 1 *(8 points)*
Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}[Y_n = 1] = p = 1 - \mathbb{P}[Y_n = -1]$ for some $0 < p < 1$. Define $X_n := \prod_{i=1}^n Y_i$ for all $n \geq 1$ and $X_0 = 1$.

(a) Show that $(X_n)_{n\geq 0}$ is a Markov chain and provide the corresponding transition matrix P.

(b) Argue that $P_{i,j}^{(n)}$ converges for $n \to \infty$ for all i and j in $\mathcal{S}$, and determine the corresponding limits $\lim_{n\to\infty} P_{i,j}^{(n)}$.

(c) Let $T_1 := \min\{n \geq 1 : X_n = 1\}$. Compute $\mathbb{E}_1[T_1] = \mathbb{E}[T_1 \mid X_0 = 1]$.

Problem 2 *(8 points)*
Let $(X_n)_{n\geq 0}$ be a Markov chain with transition graph



where $p \in [0, 0.5]$.

(a) Provide the transition probability matrix P.

(b) Determine all communication classes.

Hint: You need to do a case-by-case analysis depending on the value of p.

Now, let $p = 0$:

(c) Which states are recurrent and which are transient?

(d) Compute all stationary distributions.

Problem 3 *(8 points)*

The yearly premium for your car insurance depends on your status. There are three possible states 0,1,2 for your status and your status will be decreased by one (if possible) in the next year if you had an accident during the year. If you had no accident during a year, your status will be increased by one (if possible). Let $p \in (0, 1)$ be the probability of having an accident during a year. Your yearly premium in EUR is given by

state	0	1	2
premium	364	182	91

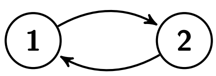
(a) Give the transition matrix P of the HMC corresponding to your car insurance status over the years.

(b) Calculate the (unique) stationary distribution.

(c) What is the average yearly premium you will have to pay in the long run if $p = 0.1$?

Problem 4 *(6 points)*

Which of the following Markov chains with corresponding transition graph are irreducible and aperiodic? Give a short explanation of your answer.

(a)


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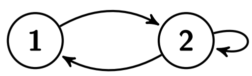
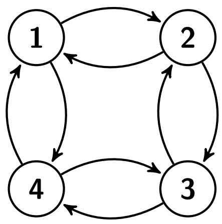


(c)


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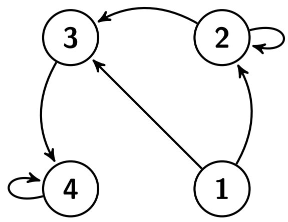

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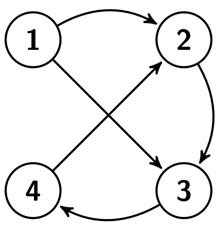
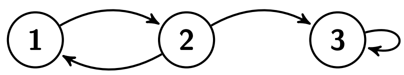


(f)


In which cases can we say something about convergence of the Markov chain against its stationary distribution?

Remark: The transition graphs of the Markov chains are without the corresponding transition probabilities. An arrow pointing from one vertex to another means that the transition probability is strictly positive. The actual probabilities are not relevant for the considered problem.