Ex35 | a) $MSE(\hat{\theta}) = E((\hat{\theta}-\theta)^2)$

$$= E(\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2)$$

$$= E(\hat{\theta}^2) - 2E(\hat{\theta}\theta) + E(\theta^2) - E(\hat{\theta})^2 + E(\hat{\theta})^2$$

$$= E(\hat{\theta}^2) - E(\hat{\theta})^2 + E(\hat{\theta}^2) - 2E\hat{\theta}\theta + \theta^2$$

$$= Var(\hat{\theta}) + B(\hat{\theta})^2$$

b) $MSE(\bar{X}) = E[(\bar{X}-\mu)^2] = (\frac{\sigma}{\sqrt{n}})^2 = \frac{\sigma^2}{n}$

Ex36 | $X \sim N(\mu, \sigma^2)$, $L(\mu, \sigma^2) = \prod_{i=1}^{k} N(x_i \mid \mu, \sigma^2)$

$$\hat{\sigma}^2_{ML} = \underset{\sigma^2}{\operatorname{argmax}}\, L(\sigma^2, x_1, \cdots x_n) = \underset{\sigma^2}{\operatorname{argmax}}\, \log L(\sigma^2, x_1 \cdots x_n)$$

$$l(\sigma^2) = \ln\left((2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x-\mu)^2\}\right)$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x-\mu)^2$$

$$\frac{\partial l}{\partial \sigma^2} = \frac{\partial\left(-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x-\mu)^2\right)}{\partial \sigma^2}$$

$$= -\frac{n}{2}\frac{1}{\sigma^2}\cdot 2\sigma + \frac{1}{2}\sum_{i=1}^{n}(x-\mu)^2 \cdot 2\frac{1}{\sigma^3} = 0$$

$$\Rightarrow \frac{n}{\sigma} = \sum_{i=1}^{n}(x-\mu)^2 \cdot \frac{1}{\sigma^3}$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x-\mu)^2$$

Ex3.11

$y \sim Geo(p_j)$

$p_j \sim Beta(a, b)$

$$P(p \mid y) \propto \underbrace{P(p \mid a,b)}_{\text{prior}} \cdot \underbrace{P(y \mid p, a, b)}_{\text{posterior}}$$

$$\propto \prod_{j=1}^{J} \frac{1}{B(a,b)} p_j^{a-1} (1-p_j)^{b-1} \cdot p_j (1-p_j)^{k-1}$$

$$\propto p_j^{a} (1-p_j)^{k+b-2}$$

Given a and b, the components of P have independent posterior densities that are of the form $p_j^{A} (1-p_j)^{B}$ that is beta densities. And the joint density is:

$$P(p \mid y) = \prod_{i=1}^{J} \frac{1}{B(a+1, k+b-1)} p_j^{a} (1-p_j)^{b+k-2}$$

$$= Beta(a+1, k+b-1)$$

Thus $a' = a+1$, $b' = b+k-1$

Ex 38 (a) $L = L(x, y, z, P_X, P_Y, P_Z) = \frac{n!}{X!\,Y!\,Z!} P_X^X \; P_Y^Y \; P_Z^Z$

$$\log L = \log n! \left(\frac{P_X^X}{X!} \cdot \frac{P_Y^Y}{Y!} \cdot \frac{P_Z^Z}{Z!} \right)$$

$$= \log n! + X \log P_X - \log X! + Y \log P_Y - \log Y! + Z \log P_Z - \log Z!$$

$$P_X + P_Y + P_Z = 1$$

$$\mathcal{L}(X, Y, Z, P_X, P_Y, P_Z, \lambda) = \log L + \lambda (1 - (P_X + P_Y + P_Z))$$

$$\frac{\partial}{\partial P_X} \mathcal{L}(X, Y, Z, P_X, P_Y, P_Z, \lambda) = \frac{\partial}{\partial P_X} \log L + \frac{\partial}{\partial P_X} \lambda (1 - (P_X + P_Y + P_Z))$$

$$= \frac{X}{P_X} - \lambda = 0 \Rightarrow \frac{X}{P_X} = \lambda \Rightarrow P_X = \frac{X}{\lambda}$$

$$\frac{\partial}{\partial P_Y} \mathcal{L}(X, Y, Z, P_X, P_Y, P_Z, \lambda) = \frac{\partial}{\partial P_X} \log L + \frac{\partial}{\partial P_X} \lambda (1 - (P_X + P_Y + P_Z))$$

$$= \frac{Y}{P_Y} - \lambda = 0 \Rightarrow \frac{Y}{P_Y} = \lambda \Rightarrow P_Y = \frac{Y}{\lambda}$$

$$\frac{\partial}{\partial P_Z} \mathcal{L}(X, Y, Z, P_X, P_Y, P_Z, \lambda) = \frac{\partial}{\partial P_X} \log L + \frac{\partial}{\partial P_X} \lambda (1 - (P_X + P_Y + P_Z))$$

$$= \frac{Z}{P_Z} - \lambda = 0 \Rightarrow \frac{Z}{P_Z} = \lambda \Rightarrow P_Z = \frac{Z}{\lambda}$$

$$P_X + P_Y + P_Z = \frac{X}{\lambda} + \frac{Y}{\lambda} + \frac{Z}{\lambda}$$

$$1 = \frac{1}{\lambda} \underbrace{(X + Y + Z)}_{n}$$

$$\Rightarrow \lambda = n$$

$$\Rightarrow P_X = \frac{X}{n}, \quad P_Y = \frac{Y}{n}, \quad P_Z = \frac{Z}{n}$$