Exercise 08

Tuesday, December 5, 2023 10:55 PM

Exercise 29 (7 points)

(a) (3 points) Show that for two independent $\mathbb{R}$-valued random variables X, Y we have $E(X \cdot Y) = (EX) \cdot (EY)$

$X : \Omega \to \mathcal{X}$, $Y : \Omega \to \mathcal{Y}$

Let $Z = X \cdot Y$, so $Z(\omega) = X(\omega) Y(\omega)$

Continuous: probability density function: $f_{(X,Y)}(x, y) = f_X(x) f_Y(y)$

$$E(X \cdot Y) = \int_{\mathcal{X}} \int_{\mathcal{Y}} xy\, f_{(X,Y)}\, dx\, dy$$

$$= \int_{\mathcal{Y}} \int_{\mathcal{X}} x f_X(x) \underbrace{y f_Y(y)}_{\text{independent of } x} dx\, dy$$

$$= \int_{\mathcal{Y}} \underbrace{\int_{\mathcal{X}} x f(x) dx}_{EX}\, y f_Y(y) dy$$

$$= EX \cdot \int_{\mathcal{Y}} y f_Y(y) dy = EX \cdot EY$$

Discrete:

$\mathcal{X} = \{x_i \mid i \in \mathbb{N}\}$, $\mathcal{Y} = \{y_j \mid j \in \mathbb{N}\}$

Too hard for me.

$$E(X \cdot Y) = \sum_{\omega_k \in \Omega} z_k \; P(XY = z_k)$$

$$= \sum_{\omega_k \in \Omega} z_k P((X, Y) \in \overbrace{\{(x, y) \mid xy = z_k,\ x \in \mathcal{X},\ y \in \mathcal{Y}\}}^{:= A_k})$$

$$= \sum_{\omega_k \in \Omega} z_k \left(\sum_{(x_i, y_j) \in A_k} P((X, Y) = (x_i, y_j)) \right)$$

$$= \sum_{\omega_k \in \Omega} \left(\sum_{(x_i, y_j) \in A_k} x_i y_j \; P(X = x_i, Y = y_j) \right)$$

$$= \sum_{x_i \in \mathcal{X}} \sum_{y_j \in \mathcal{Y}} x_i y_j P(X = x_i, Y = y_j)$$

$$= \sum_{x_i \in \mathcal{X}} x_i \left(\sum_{y_j \in \mathcal{Y}} y_j P(X = x_i) P(Y = y_j) \right)$$

$$= \sum_{x_i \in \mathcal{X}} (x_i P(X = x_i)) \cdot \left(\sum_{y_j \in \mathcal{Y}} y_j P(Y = y_j) \right)$$

$$= EX \cdot EY$$

$\ast$ For $\forall$ independent random variables X, Y

$$Cov(X, Y) = E((X - EX)(Y - EY)) = E(XY - XEY - YEX - EXEY)$$
$$= E(XY) - EX \cdot EY = 0$$

(b) (2 points) Show that for two independent $\mathbb{R}^n$-valued random variables X, Y we have $E(X \cdot Y^T) = (EX) \cdot (EY)^T$

$X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)$ $\Rightarrow$ X_i and Y_j are independent for $\forall\, i, j \in 1, \dots n$.

$\Rightarrow Cov(X_i, Y_j) = E(X_i Y_j) - EX_i EY_j = 0$

$$Cov(X, Y) = E\,(X - EX)(Y - EY)^T)$$

$$= \begin{pmatrix} E((X_1 - EX_1)(Y - EY_1)) & \cdots & E((X_1 - EX_1)(Y_n - EY_n)) \\ & \ddots & \vdots \\ & & E((X_n - EX_n)(Y_n - EY_n)) \end{pmatrix}$$

$$= \begin{pmatrix} Cov(X_1, Y_1) & \cdots & Cov(X_1, Y_n) \\ & \ddots & \vdots \\ & & Cov(X_n, Y_n) \end{pmatrix} = \underline{\underline{0}}$$

$$Cov(X, Y) = E(XY^T - EX \cdot Y^T - X \cdot (EY)^T + EX \cdot (EY)^T)$$

since EX, EY are constant $n \times 1$ matrices, and since expectation is linear

$$Cov(X, Y) = E(XY^T) - EX \cdot (EY)^T - EX \cdot (EY)^T + EX (EY)^T$$

since EX, EY are constant $n\times 1$ matrices, and since expectation is linear

$$Cov(X,Y) = E(XY^T) - EX\cdot(EY)^T - EX\cdot(EY)^T + EX(EY)^T$$

$$= E(XY^T) - EX(EY)^T = 0 \quad \Rightarrow \quad E(XY^T) = EX(EY)^T$$

(c) (2 points) For an $\mathbb{R}^n$-valued random variable $X = (X_1, \dots, X_n)$, the covariance matrix is defined as the matrix whose entry at the place i, j is $\mathrm{Cov}(X_i, X_j)$. It is denoted by $\mathrm{Cov}(X)$.
Show that for two $\boxed{\text{independent}}$ $\mathbb{R}^n$-valued random variables X, Y we have $\mathrm{Cov}(X+Y) = \mathrm{Cov}(X) + \mathrm{Cov}(Y)$ (a sum of covariance matrices)

$$Cov(X+Y) = E((X+Y)(X+Y)^T) - E(X+Y)(E(X+Y))^T$$

$$Cov(X) = E((X-EX)(X-EX)^T) = EXX^T - EXEX^T$$

$$= E(XX^T + XY^T + YX^T + YY^T) - (EX+EY)(EX+EY)^T$$

$$= E(XX^T) + E(XY^T) + E(YX^T) + E(YY^T) - (EX(EX)^T + EY(EX)^T + EX(EY)^T + EYEY^T)$$

$$= \underbrace{[E(XX^T) - EX(EX)^T]}_{Cov(X)} + \underbrace{[E(YY^T) - EY(EY)^T]}_{Cov(Y)}$$

$$+ \underbrace{[E(XY^T) - EX(EY)^T]}_{0} + \underbrace{[E(YX^T) - EY(EX)^T]}_{0} = Cov(X) + Cov(Y)$$

Exercise 30 (9 points) Let X be a random variable with exponential distribution with parameter λ, i.e. with density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(a) (2 points) Show that f_X is indeed a density function, i.e. that $\int_{\mathbb{R}} f_X(x)dx = 1$.

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx$$

$$= \cancel{\lambda}\cdot\left(-\frac{1}{\cancel{\lambda}}\right)e^{-\lambda x}\Big|_0^{+\infty}$$

$$= [(-0) - (-e^{-\lambda 0})] = 1 \qquad \text{iff } \lambda \geq 0$$

(b) (3 points) Show that the expectation of X is $\frac{1}{\lambda}$

$$EX = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

$$= \lambda \int_0^{+\infty} x e^{-\lambda x}\,dx$$

$$(xe^{-\lambda x})' = e^{-\lambda x} - \lambda x e^{-\lambda x}$$

$$\Rightarrow xe^{-\lambda x} = \frac{1}{\lambda}(e^{-\lambda x} - (xe^{-\lambda x})').$$

$$= \lambda \int_0^{+\infty} \frac{1}{\lambda}(e^{-\lambda x} - (xe^{-\lambda x})')\,dx$$

$$= \int_0^{+\infty} e^{-\lambda x}\,dx - (xe^{-\lambda x})\Big|_0^{+\infty}$$

$$\lim_{x\to\infty} \frac{x}{e^{\lambda x}} \overset{\text{L'Hospital}}{=\!=} \lim_{x\to\infty} \frac{1}{\lambda e^{\lambda x}} = 0$$

$$= \frac{1}{\lambda}\cdot 1 - (0 - 0) = \frac{1}{\lambda}$$

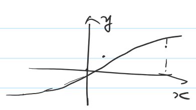
(c) (2 points) Let $Y := \frac{1}{2}X^{\frac{1}{3}}$. Compute the density function of Y.

Calculate CDF of Y:

$$x \to \frac{1}{2}x^{\frac{1}{3}}$$
$$\mathbb{R} \to \mathbb{R} \quad \text{bijection}$$

$X \in \mathbb{R} \Rightarrow Y \in \mathbb{R}$

For $y \in \mathbb{R}$:



$$P(Y \leq y) = P(\tfrac{1}{2}X^{\frac{1}{3}} \leq y)$$

$$= P(X \leq 8y^3) = \int_{-\infty}^{8y^3} f_X(x)\,dx$$

for $8y^3 < 0 \Rightarrow y < 0$

$$P(Y \leq y) = \int_{-\infty}^{8y^3} 0\,dx = 0$$

$$(-e^{-\lambda x})' = \lambda e^{-\lambda x}$$

for $8y^3 \geq 0$,

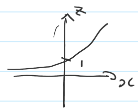
$$P(Y \leq y) = \int_0^{8y^3} \lambda e^{-\lambda x}\,dx = 1 - e^{-8\lambda y^3}$$

$$P(Y \leq y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-8\lambda y^3}, & y \geq 0 \end{cases}$$

$$f_Y(y) = \frac{\partial}{\partial y} P(X \le y) = \begin{cases} 0, & y<0 \\ +24\lambda y^2 e^{-8\lambda y^3}, & y \ge 0 \end{cases}$$

(d) (2 points) Let $Z := e^X$. Compute the density function of Z.

$X \in \mathbb{R} \Rightarrow Z \in (0, +\infty)$



For $\forall z \in (0, +\infty)$

$$P(Z \le z) = P(e^X \le z)$$
$$= P(X \le \lg(z))$$
$$= \int_{-\infty}^{\lg(z)} f_X(x)\,dx$$

for $z \in (0,1)$, $x \in (-\infty, 0)$

$$P(Z \le z) = \int_{-\infty}^{\lg(z)} 0\,dx = 0$$

for $z \ge 1$

$$P(Z \le z) = \int_0^{\lg(z)} \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda \lg z} = 1 - z^{-\lambda}$$

$$\Rightarrow \quad P(Z \le z) = \begin{cases} 0, & z \in (0,1) \\ 1 - z^{-\lambda}, & z \in [1, +\infty) \end{cases}$$

$$f_Z(z) = \frac{\partial}{\partial z} P(Z \le z) = \begin{cases} 0, & z \in (0,1) \\ \lambda z^{-\lambda-1}, & z \in [1, +\infty) \end{cases}$$

Exercise 31 (10 points)

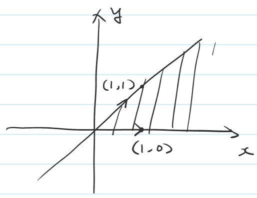
Let X, Y be $\mathbb{R}$-valued random variables with joint density function

$$f(x,y) = \begin{cases} 2e^{-x-y} & \text{if } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

(a) (4 points) Are X and Y independent? Justify your answer.

(b) (4 points) Compute the marginal density functions of X and Y.

(c) (2 points) Compute the covariance of X and Y.



$$(x,y) \overset{\phi}{\longmapsto} (x, y-x)$$

(a) and (b)

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$$
$$= \underbrace{\int_0^x f(x,y)\,dy}_{0} + \int_x^{+\infty} f(x,y)\,dy \qquad \text{for } x \in [0, +\infty)$$
$$= \int_x^{+\infty} 2e^{-x-y}\,dy = -2e^{-x-y}\Big|_x^{+\infty}$$
$$= 2e^{-2x}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$$
$$= \int_0^y f(x,y)\,dx + \underbrace{\int_y^{+\infty} f(x,y)\,dx}_{0} \qquad \text{for } y \in [0, +\infty)$$
$$= -2e^{-x-y}\Big|_0^y$$
$$= -2e^{-2y} - (-2e^{-y})$$
$$= -2e^{-2y} + 2e^{-y}$$

$f(x,y) \ne f_X(x) f_Y(y) \Rightarrow X$ and Y are dependent

(c) $EX = \int_{-\infty}^{+\infty} x f_X(x)\,dx \qquad EY = \int_{-\infty}^{+\infty} y(-2e^{-2y} + 2e^{-y})\,dy$

$$= \int^{+\infty} x \cdot 2e^{-2x}\,dx$$

(c) $EX = \int_{-\infty}^{+\infty} x f_X(x)\,dx$

$= \int_0^{+\infty} x \cdot 2e^{-2x}\,dx$

$= \frac{1}{\lambda}$, $\lambda = 2$

$= \frac{1}{2}$

$EY = \int_{-\infty}^{+\infty} y\,(-2e^{-2y} + 2e^{-y})\,dy$

$= -\frac{1}{2} + 2 = \frac{3}{2}$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\tfrac{1}{2})(y-\tfrac{3}{2}) f(x,y)\,dx\,dy$$
$$= \int_0^{+\infty}\int_0^{y} (x-\tfrac{1}{2})(y-\tfrac{3}{2})\cdot 2e^{-x-y}\,dx\,dy$$
$$= 2\int_0^{+\infty} (y-\tfrac{3}{2})e^{-y}\left(\int_0^{y}(x-\tfrac{1}{2})e^{-x}\,dx\right)dy$$
$$= 2\cdot\int_0^{+\infty} (y-\tfrac{3}{2})e^{-y}(-\tfrac{1}{2}e^{-y} - ye^{-y} + \tfrac{1}{2})\,dy$$
$$= \int_0^{+\infty} (y-\tfrac{3}{2})(-e^{-2y} - 2ye^{-2y} + \tfrac{1}{2}e^{-y})\,dy$$
$$= \int_0^{+\infty} (-ye^{-2y} - 2y^2e^{-2y} + \tfrac{1}{2}ye^{-y} - \tfrac{3}{2}e^{-2y} + 3ye^{-2y} - \tfrac{3}{4}e^{-y})\,dy$$
$$= \int_0^{+\infty} (-2y^2e^{-2y} + 2ye^{-2y} + \tfrac{1}{2}ye^{-y} - \tfrac{3}{4}e^{-y} - \tfrac{3}{2}e^{-2y})\,dy$$
$$= \underbrace{(y^2e^{-2y})'\Big|_0^{+\infty}}_{0} + \tfrac{1}{2}e^{-y}\Big|_0^{+\infty} + \tfrac{3}{4}e^{-y}\Big|_0^{+\infty} + \tfrac{3}{4}e^{-2y}\Big|_0^{+\infty}$$
$$= 0 + \tfrac{1}{2} + \tfrac{3}{4} + \tfrac{3}{4} = 2$$

$Cov(X,Y) = E((X-EX)(Y-EY))$

$= E(XY - EX\cdot Y - X\cdot EY + EX\cdot EY)$

$= E(XY) - EX\cdot EY$

$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f(x,y)\,dx\,dy - \frac{1}{2}\cdot\frac{3}{2}$

$= \int_0^{+\infty}\int_0^{y} xy\cdot 2e^{-x-y}\,dx\,dy - \frac{3}{4}$

$(2xye^{-x-y})_x = 2ye^{-x-y} - 2xye^{-x-y}$

$2xye^{-x-y} = 2ye^{-x-y} + (2xye^{-x-y})_x$

$= (-2ye^{-x-y})_x + (2xye^{-x-y})_x$

$= \int_0^{+\infty} -2ye^{-2y} + \underbrace{2y^2e^{-2y}}_{0} - (-2ye^{-y})\,dy - \frac{3}{4}$

$= -\frac{1}{2} + 0 + 2 - \frac{3}{4}$

$= \frac{1}{4}$

Exercise 32 (5 points)

Show that if X, Y are independent $\mathbb{R}$-valued random variables, and $g, h: \mathbb{R} \to \mathbb{R}$ are functions, then $g(X), h(Y)$ are also independent from each other.

Suppose X, Y have joint density function $f_{X,Y}(x,y)$, marginal density function $f_X(x)$, $f_Y(y)$, then $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Note: $g(X), h(Y)$ of course must be invertible

Suppose $A, B \subseteq \mathbb{R}$ are arbitrary

$P(g(X)\in A,\ h(Y)\in B) = P(X \in g^{-1}(A),\ Y \in h^{-1}(B))$

$= \int_{g^{-1}(A)}\int_{h^{-1}(B)} f_{X,Y}(x,y)\,dx\,dy$

$= \int_{g^{-1}(A)}\int_{h^{-1}(B)} f_X(x) f_Y(y)\,dx\,dy$

$= \int_{g^{-1}(A)} f_X(x)\,dx \int_{h^{-1}(B)} f_Y(y)\,dy$

$= P(X\in g^{-1}(A)) \cdot P(Y\in h^{-1}(B))$

$= P(g(X)\in A) \cdot P(h(Y) \in B)$

$\Rightarrow g(X)$ and $h(Y)$ are independent

Exercise 33 (9 points)

For discrete random variables X, Y and a value b in the range of X with $P(X = b) \neq 0$, one can define $P(Y \in A \mid X = b) := \frac{P(Y\in A, X=b)}{P(X=b)}$. This defines a new distribution on the range of possible values of Y, and thus a new random variable denoted $(Y \mid X = b)$.

For continuous $\mathbb{R}$-valued random variables X, Y with joint density function $f(x, y)$, one can similarly define a continuous random variable $(Y \mid X = b)$ with the density function $f(y \mid b) := \frac{f(b,y)}{\int_{-\infty}^{\infty} f(b,y)dy}$

This new random variable has an expectation, concretely $E(Y \mid X = b) = \int_{-\infty}^{\infty} y \cdot f(y \mid b)dy$.

Now we can calculate this value for every b and from this get a random variable which is a function of X! This random variable is denoted $E(Y \mid X)$ and called *conditional expectation of Y given X*.

Let X, Y have joint density function given by $f(x, y) := \begin{cases} 2 & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \text{ and } x \le y \\ 0 & \text{otherwise} \end{cases}$

(a) (3 points) Compute the conditional density function $f(y|x)$.

Exercise 33 (9 points)

For discrete random variables X, Y and a value b in the range of X with $P(X=b) \neq 0$, one can define $P(Y \in A \mid X = b) := \frac{P(Y \in A, X=b)}{P(X=b)}$. This defines a new distribution on the range of possible values of Y, and thus a new random variable denoted $(Y \mid X = b)$.

For continuous $\mathbb{R}$-valued random variables X, Y with joint density function $f(x, y)$, one can similarly define a continuous random variable $(Y \mid X = b)$ with the density function $f(y \mid b) := \frac{f(b,y)}{\int_{-\infty}^{\infty} f(b,y)dy}$.

This new random variable has an expectation, concretely $E(Y \mid X = b) = \int_{-\infty}^{\infty} y \cdot f(y \mid b)dy$.

Now we can calculate this value for every b and from this get a random variable which is a function of X! This random variable is denoted $E(Y \mid X)$ and called *conditional expectation of Y given X*.

Let X, Y have joint density function given by $f(x,y) := \begin{cases} 2 & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \text{ and } x \le y \\ 0 & \text{otherwise} \end{cases}$

(a) (3 points) Compute the conditional density function $f(y|x)$.

for $0 \le x \le 1$

$\int_{-\infty}^{+\infty} f(x,y)\,dy = \int_0^x \cancel{f(x,y)\,dy} + \int_x^1 f(x,y)\,dy$

$= \int_x^1 2\,dy = 2y\Big|_x^1 = 2-2x$

$\int_0^y 2\,dx$

$\int_0^1 2y^2\,dy$

for $x \notin [0,1]$ $\int_{-\infty}^{+\infty} f(x,y)\,dy = 0$, there is no $f(y|x)$ in this region

Therefore

$f(y|x) = \frac{2}{2-2x} = \frac{1}{1-x}$, $x \in [0,1)$, $y \in [0,1]$ and $x \le y$

(b) (3 points) Compute the function $E(Y|X)\colon [0,1] \to \mathbb{R}$.

For $b \in [0,1)$

$E_Y(Y|X=b) = \int_{-\infty}^{+\infty} y\, f(y|b)\,dy$

$= \int_b^1 y \cdot \frac{1}{1-b}\,dy$

$= \frac{y^2}{2}\Big|_b^1 \cdot \frac{1}{1-b}$

$= \frac{1-b^2}{2(1-b)} = \frac{1}{2}(1+b)$

$\star$ $E(Y) = E_b(E_Y(Y|X=b)) = \int_0^1 [\frac{1}{2}(1+b)]\, f_X(b)\,db$

$\cancel{\int_0^1 \frac{1}{2} b(1+b)\,db}$

$\cancel{= \frac{1}{2}(\frac{b^2}{2} + \frac{b^3}{3})\Big|_0^1}$

$\cancel{= \frac{5}{12}}$

$= \int_0^1 \frac{2}{2}(1-b^2)\,db$

$= 1 - \frac{1}{3} = \frac{2}{3}$

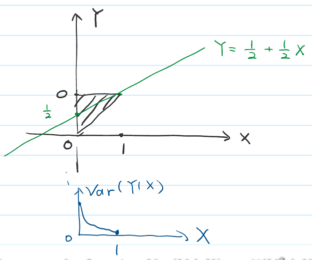
$E(Y) = \int_0^1 y \int_0^y 2\,dx\,dy$

$= \int_0^1 y \cdot 2y\,dy = \int_0^1 2 \cdot \frac{1}{3} y^3\,dy = \frac{2}{3} y^3\Big|_0^1$

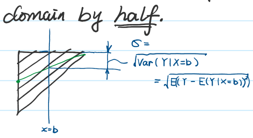
$= \frac{2}{3}$

✓

(c) (0 points) Draw the possible values of the joint variable (X, Y) and the graph of the function $E(Y|X)$ and compare with the statement of Theorem 4.7.9.



The line $Y = \frac{1}{2} + \frac{1}{2}X$ seperates the domain by half.



For $x=b$,

(d) (3 points) Compute the function $\mathrm{Var}(Y \mid X) := E(Y^2 \mid X) - (E(Y \mid X))^2$

conditional

$y \in [0,1]$, $y^2 \in [0,1]$.

$E(Y^2|X) = \int_{-\infty}^{+\infty} y^2 f(y|x)\,dy$

$= \frac{y^3}{3}\Big|_x^1 \cdot \frac{1}{1-x} = \frac{1}{3}(x^2+x+1)$

$(E(Y|X))^2 = \frac{1}{4}(1+2x+x^2)$

$Var(Y|X) = \frac{1}{12}(4x^2+4x+4-3-6x-3x^2)$

$= \frac{1}{12}(x^2-2x-1) = \frac{1}{12}(x-1)^2$

$\star$ $Var(Y|X=1) = 0$ ✓

Exercise 34 (optional, if you want a challenge - **? points** - you can replace any other exercise by this one)

The conditional variance of two $\mathbb{R}$-valued random variables X, Y is defined by $\mathrm{Var}(Y \mid X) := E(Y^2 \mid X) - (E(Y \mid X))^2$. It is a map from the possible outcomes of X to $\mathbb{R}$, and thus again a random variable (since we have a probability distribution on the range of X).

(a) (3 points) Show that $\mathrm{Var}(Y \mid X) = E((Y - E(Y \mid X))^2 \mid X)$.

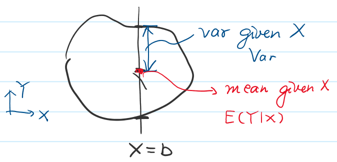
The expectations are all to Y

$$E((Y - E(Y|X))^2 | X)$$

$$= E(Y^2 - 2\underbrace{E(Y|X)}_{\text{constant}}Y + E(Y|X)^2 | X)$$

linearity 4.7.8 (ii) $$= E(Y^2|X) - (E(Y|X))^2 = Var(Y|X)$$

(b) (0 points) Stare at the definition and the result from (a) and try to come up with a description in words of what conditional variance measures.



$Var(Y|X=x)$ measures the variance of Y given the condition that $X=x$

$Var(Y|X)$ is then the variable which maps the outcome to the variance of Y given $X(\omega)$

(c) (5 points) Show that the usual variance of Y decomposes into the variance of the conditional expectation and the expectation of the conditional variance:

$$\mathrm{Var}(E(Y \mid X)) + E(\mathrm{Var}(Y \mid X)) = \mathrm{Var}(Y)$$

$$E_X E_Y(Y|X) = E_Y E_X(Y|X) = E_Y(E_Y Y) = E_Y Y$$

var of x

$$Var(E(Y|X)) = E_X[(E_Y(Y|X) - \overbrace{E_X[E_Y(Y|X)]}^{E_Y Y})^2]$$

$$= E_X[(E_Y(Y|X))^2 - 2E_Y(Y|X)\underbrace{E_Y Y}_{\text{const to } x} + (E_Y Y)^2]$$

$$= E_X[(E_Y(Y|X))^2] - 2(E_Y(Y))^2 + (E_Y Y)^2$$

$$= E_X[(E_Y(Y|X))^2] - (E_Y Y)^2$$

$$E(Var(Y|X)) = E_X(E_Y(Y^2|X) - (E_Y(Y|X))^2)$$

$$= E_Y(Y^2) - E_X[(E_Y(Y|X))^2]$$

So $Var(E(Y|X)) + E(Var(Y|X)) = E(Y^2) - (EY)^2 = Var(Y)$ ∎

(d) (1 points) What are the summands in the equation of (c) if X and Y are independent?

Independent: $E(Y|X) = E(Y)$ $\quad E(Y^2|X) = E(Y^2)$

$$Var(Y|X) = E((Y - E(Y|X))^2 | X)$$

$$= E(Y^2|X) - (E(Y|X))^2$$

$$= E(Y^2) - (E(Y))^2 = Var(Y)$$

$$Var(E(Y|X)) = Var(\overbrace{EY}^{\text{Constant in } X}) = 0$$

$$E(Var(Y|X)) = E(\underbrace{Var(Y)}_{\text{constant in } X}) = Var(Y)$$

(e) (4 points) For discrete random variables X, Y show that $Var(Y|X) = 0$ if and only if Y is a function of X.

[Remark: Propositions 4.7.6 and 4.7.8 might help.]

"$\Rightarrow$" $\qquad E_X(Y|X)$

$X \in \mathcal{X}$

$Var(Y|X) = 0$

Elementwise $Var(Y|X = x_i) = 0$, for $\forall x_i \in \mathcal{X}$, $\Rightarrow$ $(Y|X = x_i)$ is almost surely constant

$\Leftrightarrow$ for $\forall x_i \in \mathcal{X}$, $\exists r_i \in \mathbb{R}$ s.t. $P((Y|X = x_i) = r_i) = P(Y = r_i, X = x_i) = 1$

Note $g(X) = r$, s.t. $\forall x_i \in \mathcal{X}$, $P(Y = r_i = g(x_i), X = x_i) = 1$

$(Y = r_i$ iff $X = x_i)$

$\Rightarrow \forall X(\omega) \in \mathcal{X}$, $Y = g(X)$

$\Rightarrow g(X)$ is a function

$\Rightarrow Y$ is a function of X

"$\Leftarrow$" $Y = g(X)$, $E(g(X)|X) = E(1 \cdot g(X)|X)$

$= g(X)\, E(1|X)$

$= g(X)$

similarly $E(Y^2|X) = E(g^2(X)|X)$

$= E(h(X)|X) = h(X)$

let $h = g^2$

$= g^2(X)$

$Var(Y|X) = E(g(X)^2|X) - (E(g(X)|X))^2$

$= g^2(X) - g^2(X) = 0$ $\square$