Exercise 03

Saturday, November 4, 2023 6:18 PM

Problem 1 to hand in: *Loop Invariant*

The following algorithm computes the symmetric difference $A \Delta B = (A \setminus B) \cup (B \setminus A)$, given two input sets A and B.

get_symmetric_difference(A, B):

```
1 C ← B
2 for a ∈ A do
3 |   if a ∈ C then
4 |   |   C ← C \ {a}
5 |   if a ∉ B then
6 |   |   C ← C ∪ {a}
7 return C
```

side note: the current proof is too lengthy.
It's not suitable for the exam at all.

a) State a loop invariant that holds at the beginning of each iteration of the for loop (lines 2 to 4).

b) Proof this loop invariant.

c) Use this loop invariant to show that indeed the algorithm returns the symmetric difference.

Suppose input set A has n elements $a_1, \dots, a_n \in A$.

The for-loop at line 2 iterates sequentially from $a_i (i=1)$ to $a_i (i=n)$

a) Claim $S[i]$: At the end of loop i, $C = (A_t \setminus B) \cup (B \setminus A_t)$

where $A_t = \{a_1, \cdots, a_i\}$

b) and c)

Initialization: $i = 1$:

line 1: $C = B$

line 2: pick a_1, $A_t = \{a_1\}$

line 3~4: if $a_1 \in B$. "$\Rightarrow$" $C = C \setminus \{a_1\} = B \setminus A_t$

"$\Leftarrow$" $(A_t \setminus B) \cup (B \setminus A_t) = \emptyset \cup (B \setminus A_t)$

$= B \setminus A_t = C$ ✓

line 5~6: if $a_1 \notin B$

"$\Rightarrow$" $C = C \cup \{a_1\} = B \cup A_t$

"$\Leftarrow$" $\underbrace{(A_t \setminus B)}_{A_t} \cup \underbrace{(B \setminus A_t)}_{B} = A_t \cup B = C$ ✓

so at the end of $i = 1$, $C = (A_t \setminus B) \cup (B \setminus A_t)$

Maintainence: $i \rightsquigarrow i+1$: $A_t(i) = \{a_1, \dots, a_i\}$, $C = (A_t(i) \setminus B) \cup (B \setminus A_t(i))$

line 2: pick a_{i+1}, $A_t(i+1) = \{a_1, \dots, a_{i+1}\} = A_t(i) \cup \{a_{i+1}\}$

Line 3~4: if $a_{i+1} \in B$.

"$\Rightarrow$" $C = C \setminus \{a_{i+1}\}$

$= \big((A_t(i) \setminus B) \cup (B \setminus A_t(i))\big) \setminus \{a_{i+1}\}$

$= (A_t(i) \setminus B) \cup B \setminus A_t(i+1)$

since $a_{i+1} \in B$, $A_t(i) \setminus B = (A_t(i) \cup \{a_{i+1}\}) \setminus B$

$= A_t(i+1)$

since $a_{i+1} \in B$, $A_t(i) \setminus B = (A_t(i) \cup \{a_{i+1}\}) \setminus B$
$= A_t(i+1)$

$C = (A_t(i+1) \setminus B) \cup (B \setminus A_t(i+1))$ ✓

line 5 ~ 6: if $a_{i+1} \notin B$, $B \setminus A_t(i) = B \setminus (A_t(i) \cup \{a_{i+1}\}) = B \setminus A_t(i+1)$

"$\Rightarrow$" $C = C \cup \{a_{i+1}\}$

$= (A_t(i) \setminus B) \cup (B \setminus A_t(i)) \cup \{a_{i+1}\}$

$= (A_t(i+1) \setminus B) \cup (B \setminus A_t(i))$

$= (A_t(i+1) \setminus B) \cup (B \setminus A_t(i+1))$ ✓

Termination: $i = n$: $S[n]$: At the end of Loop n, $C = (A_t(n) \setminus B) \cup (B \setminus A_t(n))$

where $A_t(n) = \{a_1, \cdots, a_n\} = A$

So $C = (A \setminus B) \cup (B \setminus A)$ at the end of the loop.

At line 7: return $C = (A \setminus B) \cup (B \setminus A) = A \Delta B$ $\square$

so indeed the algorithm returns the symmetric difference between A and B.