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Markov Chains

Problem sheet 1

Review concepts probability, conditional expectation, random walk, markov chains

Problems to be handed in by:
Thursday, April 25th, 11:59 p.m., online via Ilias.

Problem 1 *(10 points)*

Let $(X_n)_{n\geq 0}$ be a simple random walk starting in $X_0 = 0$ with probability $p \in (0,1)$ to move up. Determine the following probabilities:
$\mathbb{P}(X_6 = 4 | X_5 = 3, X_4 = 2)$,
$\mathbb{P}(X_6 = 4 | X_5 = 3)$,
$\mathbb{P}(X_6 = 4 | X_5 = 3, X_4 \text{ is even})$,
$\mathbb{P}(X_6 = 4 | X_5 \text{ is odd}, X_4 = 2)$,
$\mathbb{P}(X_6 = 4 | X_5 \text{ is odd})$.

Solution:
We write X_n as the sum $\sum_{i=1}^n Y_i$ of i.i.d. random variables $(Y_i)_{i\in\mathbb{N}}$ with $\mathbb{P}(Y_i = 1) = 1 - \mathbb{P}(Y_i = -1) = p$. In the lecture we proved that $(X_n)_{n\geq 0}$ is a Markov Chain. Because we move either up or down by 1 in each step, we always swap between even and odd values. This means that the conditions "X_4 is even" and "X_5 is odd" are always true and can be omitted. With this argument and the Markov property, we get:

$$
\begin{aligned}
&\mathbb{P}(X_6 = 4 | X_5 = 3) = \mathbb{P}(Y_6 = 1) = p \\
&\mathbb{P}(X_6 = 4 | X_5 = 3, X_4 = 2) = \mathbb{P}(X_6 = 4 | X_5 = 3) = p \\
&\mathbb{P}(X_6 = 4 | X_5 = 3, X_4 \text{ is even}) = \mathbb{P}(X_6 = 4 | X_5 = 3) = p \\
&\mathbb{P}(X_6 = 4 | X_5 \text{ is odd}, X_4 = 2) = \mathbb{P}(X_6 = 4 | X_4 = 2) = \mathbb{P}(Y_5 + Y_6 = 2) \\
&= \mathbb{P}(Y_5 = 1)\mathbb{P}(Y_6 = 1) = p^2
\end{aligned}
$$

For the last case, we observe that X_6 equals 4 if and only if our random walk moves up 5 out of 6 times. The number of upward steps at time n is $B(n,p)$-distributed, which gives us

$$
\mathbb{P}(X_6 = 4 | X_5 \text{ is odd}) = \mathbb{P}(X_6 = 4) = \binom{6}{1} p^5 (1-p) = 6p^5(1-p).
$$

Problem 2 *(10 points)*

Let $(X_n)_{n\in\mathbb{N}_0}$ be a simple random walk with probability $p \in (0,1)$ to move up. Determine

(a) $\mathbb{P}(X_n - X_0 = k)$ for $k \in \mathbb{Z}$,

(b) $\mathbb{E}[X_n|X_{n-1}]$,

(c) $\mathbb{E}[|X_n||X_{n-1}]$.

Solution:
We again write $X_n = \sum_{i=1}^n Y_i$ with i.i.d. random variables $(Y_i)_{i\in\mathbb{N}}$, where $\mathbb{P}(Y_i = 1) = 1 - \mathbb{P}(Y_i = -1) = p$.

(a) We note that $Y_i = \mathbb{1}_{\{Y_i=1\}} - \mathbb{1}_{\{Y_i=-1\}}$. Using this, we get

$$\begin{aligned}
\mathbb{P}(X_n - X_0 = k) &= \mathbb{P}\left(\sum_{i=1}^n Y_i = k\right) = \mathbb{P}\left(\sum_{i=1}^n (\mathbb{1}_{\{Y_i=1\}} - \mathbb{1}_{\{Y_i=-1\}}) = k\right) \\
&= \mathbb{P}\left(\sum_{i=1}^n (2\cdot \mathbb{1}_{\{Y_i=1\}} - 1) = k\right) = \mathbb{P}\left(\underbrace{\sum_{i=1}^n \mathbb{1}_{\{Y_i=1\}}}_{\sim B(n,p)} = \frac{n+k}{2}\right) \\
&= B(n,p)\left(\left\{\frac{n+k}{2}\right\}\right) = \begin{cases} \binom{n}{\frac{n+k}{2}} p^{\frac{n+k}{2}}(1-p)^{\frac{n-k}{2}} & -n \le k \le n,\ \frac{n+k}{2} \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

(b) Using the measurability of X_{n-1} and the independence of Y_n and X_{n-1}, we get

$$\begin{aligned}
\mathbb{E}[X_n|X_{n-1}] &= \mathbb{E}[X_{n-1} + Y_n|X_{n-1}] \overset{\text{lin.}}{=} \mathbb{E}[X_{n-1}|X_{n-1}] + \mathbb{E}[Y_n|X_{n-1}] \\
&= X_{n-1} + \mathbb{E}[Y_n] = X_{n-1} + p - (1-p) = X_{n-1} + 2p - 1
\end{aligned}$$

(c) We first want to understand how $|X_n|$ depends on X_{n-1} by looking at the conditional probabilities. For $k \in \mathbb{N}$ we get

$$\begin{aligned}
&\mathbb{P}(|X_n| = 1|X_{n-1} = 0) = 1, \\
&\mathbb{P}(|X_n| = k+1|X_{n-1} = k) = p, \\
&\mathbb{P}(|X_n| = k-1|X_{n-1} = k) = 1-p, \\
&\mathbb{P}(|X_n| = k+1|X_{n-1} = -k) = 1-p, \\
&\mathbb{P}(|X_n| = k-1|X_{n-1} = -k) = p.
\end{aligned}$$

This gives us

$$\begin{aligned}
\mathbb{E}[|X_n||X_{n-1}] &= \mathbb{1}_{\{X_{n-1}>0\}}\mathbb{E}[|X_n||X_{n-1} > 0] \\
&+\mathbb{1}_{\{X_{n-1}=0\}}\mathbb{E}[|X_n||X_{n-1} = 0] + \mathbb{1}_{\{X_{n-1}<0\}}\mathbb{E}[|X_n||X_{n-1} < 0] \\
&=\mathbb{1}_{\{X_{n-1}>0\}}\big(p(X_{n-1}+1) + (1-p)(X_{n-1}-1)\big) \\
&+\mathbb{1}_{\{X_{n-1}=0\}}\cdot 1 + \mathbb{1}_{\{X_{n-1}>0\}}\big(p(|X_{n-1}|-1) + (1-p)(|X_{n-1}|+1)\big) \\
&=\mathbb{1}_{\{X_{n-1}>0\}}(|X_{n-1}| + 2p - 1) + \mathbb{1}_{\{X_{n-1}=0\}} + \mathbb{1}_{\{X_{n-1}=0\}}(|X_{n-1}| + 1 - 2p)
\end{aligned}$$

Problem 3 *(10 points)*
Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. $B(1,p)$-distributed random variables. Let

$$Y_i := 1_{\{X_i=1\}} + 1_{\{X_i=X_{i-1}=X_{i-2}=1\}},\ i \geq 3$$

$$Z_i := 1_{\{X_i=1\}} + 1_{\{X_i=X_{i-1}=1\}} + 1_{\{X_i=X_{i-1}=X_{i-2}=1\}},\ i \geq 3$$

(a) Determine $\mathbb{P}(Z_{n+1} = j_1|Z_n = i_1)$ and $\mathbb{P}(Y_{n+1} = j_2|Y_n = i_2)$ for $i_1, j_1 \in \{0,1,2,3\}$ and for $i_2, j_2 \in \{0,1,2\}$ and $n \geq 3$.

(b) Is $(Y_n)_{n\geq 3}$ respectively $(Z_n)_{n\geq 3}$ a Markov Chain? Justify your answer.

Solution:

(a) We will first take a closer look at the given random variables and see that the indicators depend on their respective predecessors:

$$\{X_i = 1\} \supseteq \{X_i = X_{i-1} = 1\} \supseteq \{X_i = X_{i-1} = X_{i-2} = 1\} \quad , i \geq 3$$

Because of this, we can come up with a more practical version of Z_i and Y_i:

$$Z_i = \begin{cases} 3, & X_i, X_{i-1}, X_{i-2} = 1, \\ 2, & X_i, X_{i-1} = 1, X_{i-2} = 0, \\ 1, & X_i = 1, X_{i-1} = 0, \\ 0, & X_i = 0 \end{cases} \qquad Y_i = \begin{cases} 2, & X_i, X_{i-1}, X_{i-2} = 1, \\ 1, & X_i = 1 \wedge \big(X_{i-1} = 0 \vee X_{i-2} = 0\big), \\ 0, & X_i = 0 \end{cases}$$

Now it is much easier to calculate the conditional probabilities using the independence:

$$\mathbb{P}(Z_{n+1} = 0|Z_n = i) = \mathbb{P}(X_{n+1} = 0|Z_n = i) \overset{\text{ind.}}{=} \mathbb{P}(X_{n+1} = 0) = 1 - p$$

With clever choice of the next pairs (i,j), we can show that for $Z_n = i$, Z_{n+1} only has 2 possible values:

$$\begin{aligned}
\mathbb{P}(Z_{n+1} = 1|Z_n = 0) &= \mathbb{P}(X_{n+1} = 1, X_n = 0|X_n = 0) \\
&\overset{\text{ind.}}{=} \mathbb{P}(X_{n+1} = 1) = p \\
\mathbb{P}(Z_{n+1} = 2|Z_n = 1) &= \mathbb{P}(X_{n+1}, X_n = 1, X_{n-1} = 0|X_n = 1, X_{n-1} = 0) \\
&\overset{\text{ind.}}{=} \mathbb{P}(X_{n+1} = 1) = p \\
\mathbb{P}(Z_{n+1} = 3|Z_n = 2) &= \mathbb{P}(X_{n+1}, X_n, X_{n-1} = 1|X_n, X_{n-1} = 1, X_{n-2} = 0) \\
&\overset{\text{ind.}}{=} \mathbb{P}(X_{n+1} = 1) = p \\
\mathbb{P}(Z_{n+1} = 3|Z_n = 3) &= \mathbb{P}(X_{n+1}, X_n, X_{n-1} = 1|X_n, X_{n-1}, X_{n-2} = 1) \\
&\overset{\text{ind.}}{=} \mathbb{P}(X_{n+1} = 1) = p
\end{aligned}$$

For $i \in \{0, 1, 2, 3\}$ we know $\sum_{j=1}^{3} \mathbb{P}(Z_{n+1} = j | Z_n = i) = 1$, thus the remaining conditional probabilities are equal to 0.

For Y_i, we get $\mathbb{P}(Y_{n+1} = 0 | Y_n = i) = 1 - p$ with the same argument as for Z_i. Furthermore we get

$$\begin{aligned}
\mathbb{P}(Y_{n+1} = 2 | Y_n = 2) &= \mathbb{P}(X_{n+1}, X_n, X_{n-1} = 1 | X_n, X_{n-1}, X_{n-2} = 1) \\
&\overset{\text{ind.}}{=} \mathbb{P}(X_{n+1} = 1) = p \\
\mathbb{P}(Y_{n+1} = 2 | Y_n = 0) &= \mathbb{P}(X_{n+1}, X_n, X_{n-1} = 1 | X_n = 0) = 0.
\end{aligned}$$

This implies $\mathbb{P}(Y_{n+1} = 1 | Y_n = 2) = 0$ and $\mathbb{P}(Y_{n+1} = 1 | Y_n = 0) = p$ analogous to before. For the last case $i = 1$ we simply use the definition of conditional probability:

$$\begin{aligned}
\mathbb{P}(Y_{n+1} = 2 | Y_n = 1) &= \mathbb{P}(X_{n+1}, X_n, X_{n-1} = 1 | X_n = 1, X_{n-1} = 0 \vee X_{n-2} = 0) \\
&= \frac{\mathbb{P}(X_{n+1}, X_n, X_{n-1} = 1, X_{n-2} = 0)}{\mathbb{P}(X_n = 1, X_{n-1} = 0 \vee X_{n-2} = 0)} = \frac{p^3(1-p)}{p(1-p^2)} = \frac{p^2}{1+p}
\end{aligned}$$

This leaves us with

$$\mathbb{P}(Y_{n+1} = 1 | Y_n = 1) = 1 - (1 - p) - \frac{p^2}{1+p} = \frac{p}{1+p}.$$

(b) $(Y_n)_{n \geq 3}$ is not a Markov Chain. To show this, let $n \geq 4$ and

$$\begin{aligned}
\mathbb{P}(Y_{n+1} = 2 | Y_n = 1, Y_{n-1} = 0) &= \mathbb{P}(X_{n+1}, X_n, X_{n-1} = 1 | X_n = 1, X_{n-1} = 0) \\
&= 0 \neq \frac{p^2}{1+p} = \mathbb{P}(Y_{n+1} = 2 | Y_n = 1)
\end{aligned}$$

$(Z_n)_{n \geq 3}$ is a Markov Chain. We can easily see that Z_{n+1} only depends on the random variables X_{n-1}, X_n and X_{n+1}. Due to this, Z_{n+1} must already be independent from Z_{n-2}, Z_{n-3}, etc. While Z_{n+1} and Z_{n-1} technically intersect on X_{n-1}, Z_{n+1} only depends on X_{n-1}, if X_{n+1} and X_n equal 1. But if this is the case, Z_n already provides this information by $Z_n \geq 2$, if $X_{n-1} = 1$, and $Z_n = 1$ otherwise. With all relevant random variables being either independent from the past or already included in the predecessor Z_n, Z_{n+1} fulfills the Markov property.