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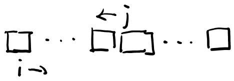
Exercise 02

Sunday, October 29, 2023 8:22 PM

Problem 1 to hand in: *Running Time*

The following pseudocode describes the `bubble_sort` algorithm. This is another way to solve the SORTING computation problem: Given a finite sequence A of pairwise distinct integers (and its length), return A sorted ascendingly.




```
bubble_sort(A[0..n − 1], n):
for j ← n − 1 down to 1 do
    for i ← 0 to j − 1 do
        if A[i] > A[i + 1] then
            key = A[i]
            A[i] = A[i + 1]
            A[i + 1] = key
return A
```

a) How does `bubble_sort` work in comparison to `insertion_sort`? Describe it intuitively in one or two sentences.

b) Analyse the asymptotic worst-case running time of `bubble_sort`:

- For each line of code, write down the number of running steps (dependent on the input size) in the worst case.
- Sum up the total number $T(n)$ of steps the algorithm needs in the worst case for an input of size n.
- Provide a function $f(n)$ as a representative upper bound in $\mathcal{O}$-notation.
- Proof formally that $T(n) \in \mathcal{O}(f(n))$ holds.

c) What is the (asymptotic) average running time? Briefly argue why.

a)

Bubble sort repeated compares the values of two adjacent numbers in an array and keeps pushing the larger number to the end, while insertion sort works by placing the numbers from the unsorted portion in the right place. Usually, bubble sort is less efficient than insertion sort, since it makes more comparisons.

Intuitively, the bubble-sort algorithm has a lot in common comparing to the insertion_sort. They both have two for-loops and the inner loop shows dependency on the first loop. On the other hand, bubble_sort algorithm starts the loop with a whole length of A, pushing current largest number to the end. ok

b) Worst case: Totally inversed sequence

```
bubble_sort(A[0..n − 1], n):
for j ← n − 1 down to 1 do
    for i ← 0 to j − 1 do
        if A[i] > A[i + 1] then
            key = A[i]
            A[i] = A[i + 1]
            A[i + 1] = key
return A
```

Line 1: $\overbrace{1 \sim n-1}^{n-1}$ | $n-1+1 = n$

Line 2: $0 \sim j-1$ | $\sum_{j=1}^{n-1} j + n - 1 = \frac{(n-1+1)(n-1)}{2} + n - 1 = \frac{1}{2}(n^2-n)+n-1 = \frac{1}{2}n^2+\frac{1}{2}n-1$

Lines 3–6: $0 \sim j-1$, $0 \sim j-1$ | $4 \times \sum_{j=1}^{n-1} j = 2n^2 - 2n$

Line 7: 1

- $T(n) = \cancel{n} + \cancel{\frac{1}{2}}n^2 + \cancel{\frac{1}{2}n} \cancel{-1} + \cancel{2}n^2 \cancel{-2n} \cancel{+1}$

$= \frac{5}{2}n^2 - \frac{1}{2}n \quad \Rightarrow T(n)$ can be bounded by $f(n) = n^2$

- It is enough to prove that

$\exists c>0, n_0>0$, s.t. $\forall n \geq n_0$: $\frac{5}{2}n^2 - \frac{1}{2}n \leq cn^2$

Let $c = \frac{7}{2}$, we construct a function

$$J(n) = f(n) - T(n)$$
$$= \frac{7}{2}n^2 - \frac{5}{2}n^2 + \frac{1}{2}n$$
$$= n^2 + \frac{1}{2}n = n(n+\frac{1}{2})$$

Let $n_0 = 10$

$\forall n \geq 10$, we have $n \geq 0$ and $n + \frac{1}{2} \geq 0$

$\Rightarrow J(n) \geq 0 \Rightarrow f(n) \geq T(n)$

Therefore $\exists c>0, n_0>0$, s.t. $\forall n \geq n_0$ $T(n) \leq cn^2$ $\square$

c) The average running time will still be in $O(n^2)$

The if-condition (IF $A[i] > A[i+1]$) in the inner loop has to be executed no matter how many inversions are in the sequence (independent on the input)

Its number of steps is $\sum_{j=1}^{n-1} j = \frac{(n-1+1)\cdot(n-1)}{2} = \frac{1}{2}n(n-1) \in O(n^2)$

nice