Ex 29 (a) $E(X \cdot Y) = \iint x \cdot y \, f(x,y) \, dy \, dx$

$$= \iint x \cdot y \, f_{Y|X}(y|x) f(x) \, dy \, dx$$

$$= \int x f(x) \int y \, f_{Y|X}(y|x) \, dy \, dx$$

Since $X \perp Y$, $f_{Y|X}(y|x) = f_Y(y)$

then, $\int x f(x) \int y \, f_{Y|X}(y|x) \, dy \, dx$

$$= \int x f(x) \int y \, f_Y(y) \, dy \, dx$$

$$= \int x f(x) \, E(Y) \, dx$$

$$= E(Y) \int x f(x) \, dx$$

$$= E(Y) E(X)$$

(b) If X and Y are $\mathbb{R}^n$-valued random variable,

then $X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ and $Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$ $\quad Y^T = [Y_1 \cdots Y_n]$

$E(X) = \begin{bmatrix} EX_1 \\ \vdots \\ EX_n \end{bmatrix}$ $\quad E(Y^T) = [EY_1 \cdots EY_n]$

$$X \cdot Y^T = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} [Y_1 \cdots Y_n] = \begin{bmatrix} X_1 Y_1 \\ \vdots \\ X_n Y_n \end{bmatrix}$$

$E(X \cdot Y^T) = \begin{bmatrix} E X_1 Y_1 \\ \vdots \\ E X_n Y_n \end{bmatrix}$ since X and Y are independent

$$E(X \cdot Y^T) = \begin{bmatrix} EX_1 \, EY_1 \\ \vdots \\ EX_n \, EY_n \end{bmatrix} = \begin{bmatrix} EX_1 \\ \vdots \\ EX_n \end{bmatrix} [EY_1 \cdots EY_n] = E(X) \cdot E(Y^T)$$

c) $Cov(X) = Cov(X_i, X_j) = E[(X_i - EX_i)(X_j - EX_j)]$

$Cov(Y) = Cov(Y_i, Y_j) = E[(Y_i - EY_i)(Y_j - EY_j)]$

$Cov(X+Y) = Cov(X_i + Y_i, X_j + Y_j)$

$= E(((X_i + Y_i) - E(X_i + Y_i)) \cdot ((X_j + Y_j) - E(X_j + Y_j)))$

$= E[[(X_i - EX_i) + (Y_i - EY_i)] \cdot [(X_j - EX_j) + (Y_j - EY_j)]]$

$= E[(X_i - EX_i)(X_j - EX_j) + (X_i - EX_i)(Y_j - EY_j) + (Y_i - EY_i)(X_j - EX_j) + (Y_i - EY_i)(Y_j - EY_j)]$

$= E[(X_i - EX_i)(X_j - EX_j)] + E[(X_i - EX_i)(Y_j - EY_j)] + E[(Y_i - EY_i)(X_j - EX_j)] + E[(Y_i - EY_i)(Y_j - EY_j)]$

$= Cov(X) + Cov(X_i, Y_j) + Cov(X_j, Y_i) + Cov(Y)$

Since X and Y are independent, $Cov(X_i, Y_j) = 0$, $Cov(X_j, Y_i) = 0$.

Therefore $Cov(X+Y) = Cov(X) + Cov(Y)$

Ex 30 (a) $\int_{\mathbb{R}} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$

$= \lim_{b\to\infty} [-e^{-\lambda b}]_0^b$

$= \lim_{b\to\infty} (-e^{-\lambda b}+1)$

$= 1$

(b) $E(X) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \int_{-\infty}^{0} x\cdot 0\,dx + \int_0^{\infty} x\cdot\lambda e^{-\lambda x}\,dx$

$= 0 + \lambda\int_0^{\infty} x e^{-\lambda x}\,dx$

Let $u = x, dv = e^{-\lambda x}dx \Rightarrow du = dx$, $v = -\frac{1}{\lambda}e^{-\lambda x}$

$\lambda\int_0^{\infty} x e^{-\lambda x}\,dx = \lambda[x(-\frac{1}{\lambda}e^{-\lambda x}) - \int_0^{\infty} -\frac{1}{\lambda}e^{-\lambda x}\,dx]$

$= [-xe^{-\lambda x}]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$

$= [-\frac{1}{\lambda}e^{-\lambda}]_0^{\infty}$

$= \frac{1}{\lambda}$

(c) $Y = g(X) = \frac{1}{2}X^{\frac{1}{3}}$

$g^{-1}(y) = 8y^3$

$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$

$= \lambda e^{-\lambda 8 y^3}\left|24y^2\right|$

$= 24\lambda y^2 e^{-8\lambda y^3} \qquad y \in [0,+\infty)$

(d) $Z = h(X) = e^X$

$h^{-1}(z) = \ln z$

$f_Z(z) = f_X(h^{-1}(z))\left|\frac{d}{dz}h^{-1}(z)\right|$

$= \lambda e^{-\lambda \ln z}\cdot\frac{1}{z}$

$= \lambda z^{-\lambda-1} \qquad z \in [1,+\infty)$

Ex3.1 (a) X and Y are independent iff $f(x,y) = g(x)\,h(y)$

(b) $f(x) = \int_x^{\infty} 2e^{-x-y}\,dy$

$= \left[-2e^{-x-y}\right]_x^{\infty}$

$= 2e^{-2x}$

$f(y) = \int_0^y 2e^{-x-y}\,dx$

$= \left[-2e^{-x-y}\right]_0^y$

$= -2e^{-2y} + 2e^{-y}$

$= 2e^{-2y}(e^{y}-1)$

$f(x)\cdot f(y) = 2e^{-2x}\cdot 2e^{-2y}(e^{y}-1) \neq 2e^{-x-y}$

Therefore, X and Y are not independent.

(c) $EX = \int_0^{\infty}\int_0^y X f(x)\,dx\,dy$

$= \int_0^{\infty}\int_0^y x\, 2e^{-x-y}\,dx\,dy$

$= \frac{1}{2}$

$EY = \int_0^{\infty}\int_0^y y\cdot 2e^{-x-y}\,dx\,dy$

$= \frac{3}{2}$

$Cov(X,Y) = E((X-EX)(Y-EY))$

$= E(XY - XEY - YEX + EXEY)$

$= \int_0^{\infty}\int_0^y (x\cdot y - x\cdot\frac{3}{2} - y\cdot\frac{1}{2} + \frac{1}{2}\times\frac{3}{2})\, 2e^{-x-y}\,dx\,dy$

$= \frac{1}{4}$

Ex32 Let $A, B \in \mathcal{B}$

Since f, g are $\mathbb{R} \Rightarrow \mathbb{R}$

there exist $A', B' \in \mathcal{B}$

such that $f^{-1}(A) = A'$ and $g^{-1}(B) = B'$.

Since X and Y are independent random variables,

we have that $X^{-1}(A')$ and $Y^{-1}(B')$ are independent

Thus, for any $A, B \in \mathcal{B}$

that $X^{-1}(f^{-1}(A))$ and $Y^{-1}(g^{-1}(B))$ are independent.

Therefore $f(X)$ and $g(Y)$ are independent

Ex331 a) $f(y|x) = \dfrac{f(x,y)}{\int f(x,y)\,dy} = \begin{cases} \dfrac{2}{\int_x^1 2\,dy} = \dfrac{1}{1-x} & \text{if } 0 \le x \le 1,\ 0 \le y \le 1,\ \text{and } x \le y \\ 0 & \text{otherwise} \end{cases}$

b) $E(Y|X) = \int_0^1 y f(y|x)\,dy = \int_0^1 y \cdot \frac{1}{1-x}\,dy$

$= \frac{1}{1-x} \int_0^1 y\,dy$

$= \frac{1}{1-x}\left[\frac{1}{2}y^2\right]_0^1$

$= \frac{1}{2(1-X)}$

c)

d) $Var(Y|X) = E(Y^2|X) - (E(Y|X))^2$

$E(Y^2|X) = \int_0^1 y^2 f(y|x)\,dy$

$= \int_0^1 y^2 \frac{1}{1-X}\,dy = \frac{1}{1-X}\int_0^1 y^2\,dy = \frac{1}{1-X}\left[\frac{1}{3}y^3\right]_0^1 = \frac{1}{3(1-X)}$

Therefore $Var(Y|X) = \frac{1}{3(1-X)} - \frac{1}{4(1-X)^2}$