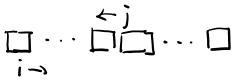
Exercise 02

Sunday, October 29, 2023 8:22 PM

Problem 1 to hand in: *Running Time*

The following pseudocode describes the `bubble_sort` algorithm. This is another way to solve the SORTING computation problem: Given a finite sequence A of pairwise distinct integers (and its length), return A sorted ascendingly.




```
bubble_sort(A[0..n − 1], n):
for j ← n − 1 down to 1 do
    for i ← 0 to j − 1 do
        if A[i] > A[i + 1] then
            key = A[i]
            A[i] = A[i + 1]
            A[i + 1] = key
return A
```

a) How does `bubble_sort` work in comparison to `insertion_sort`? Describe it intuitively in one or two sentences.

b) Analyse the asymptotic worst-case running time of `bubble_sort`:

- For each line of code, write down the number of running steps (dependent on the input size) in the worst case.
- Sum up the total number $T(n)$ of steps the algorithm needs in the worst case for an input of size n.
- Provide a function $f(n)$ as a representative upper bound in $\mathcal{O}$-notation.
- Proof formally that $T(n) \in \mathcal{O}(f(n))$ holds.

c) What is the (asymptotic) average running time? Briefly argue why.

a) Intuitively, the bubble-sort algorithm has a lot in common comparing to the insertion_sort. They both have two for-loops and the inner loop shows dependency on the first loop. On the other hand, bubble-sort algorithm starts the loop with a whole length of A, pushing current largest number to the end.

b) Worst case: Totally inversed sequence

bubble_sort(A[0..n − 1], n):		
for j ← n − 1 down to 1 do	$\overbrace{1 \sim n-1}^{n-1}$	$n-1+1 = n$
for i ← 0 to j − 1 do	$0 \sim j-1$	$\sum_{j=1}^{n-1} j + n - 1 = \frac{(n-1+1)(n-1)}{2} + n - 1 = \frac{1}{2}(n^2-n)+n-1 = \frac{1}{2}n^2 + \frac{1}{2}n - 1$
if A[i] > A[i + 1] then	$0 \sim j-1$	$4 \times \sum_{j=1}^{n-1} j = 2n^2 - 2n$ (lines 3–6)
key = A[i]	$0 \sim j-1$	
A[i] = A[i + 1]		
A[i + 1] = key		
return A		1

- $T(n) = \cancel{n} + \cancel{\frac{1}{2}}n^2 + \cancel{\frac{1}{2}n} \cancel{-1} + \cancel{2}n^2 - \cancel{2}n \cancel{+1}$

  $= \frac{5}{2}n^2 - \frac{1}{2}n \quad \Rightarrow T(n)$ can be bounded by $f(n) = n^2$

- It is enough to prove that

$\exists c > 0, n_0 > 0$, s.t. $\forall n \geq n_0$: $\frac{5}{2}n^2 - \frac{1}{2}n \leq cn^2$

Let $c = \frac{7}{2}$, we construct a function

$$J(n) = f(n) - T(n)$$
$$= \frac{7}{2}n^2 - \frac{5}{2}n^2 + \frac{1}{2}n$$
$$= n^2 + \frac{1}{2}n = n(n + \frac{1}{2})$$

Let $n_0 = 10$

$\forall n \geq 10$, $n \geq 0$ and $n + \frac{1}{2} \geq 0$

$\Rightarrow J(n) \geq 0 \Rightarrow f(n) \geq T(n)$

Therefore $\exists c > 0, n_0 > 0$, s.t. $\forall n \geq n_0$ $T(n) \leq cn^2$ $\square$

c) The average running time will still be in $O(n^2)$

The if-condition in the inner loop has to be executed

no matter how many inversions are in the sequence (independent on the input)

It's number of steps is $\sum_{j=1}^{n-1} j = \frac{(n-1+1)\cdot(n-1)}{2} = \frac{1}{2}n(n-1) \in O(n^2)$