MW82: Time Series Analysis, Tutorial III

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Recap: Tutorial II

We did:

- Covariance-Stationarity
- ACF
- Properties of AR(1)
- Estimated AR(1) with OLS

Today we will:

- Properties of the MA(1) model
- PACF
- Use ACF and PACF empirically

Moving Average (MA) models

MA-processes depend on current and past values of a stochastic process (white noise $\varepsilon_t \sim N(0, \sigma^2)$).
MA(1) process:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

MA(2) process:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

MA(q) process:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

Theoretical properties of MA(1) process

Mean:

$$E(y_t) = \mu$$

Variance:

$$Var(y_t) = \sigma^2(1 + \theta_1^2)$$

Autocorrelation function, ACF(s) is:

$$\rho_1 = \frac{\theta_1}{1 + \theta_1^2}$$

$$\rho_{s>1} = 0$$

- Note: The only non-zero value in the theoretical ACF is for lag 1
- All other autocorrelations are zero
- Implies: A sample ACF with a significant correlation only at lag 1 is an indicator for an MA(1) model.

Properties for MA(2)

Mean is still μ.
Variance is:

$$Var(y_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2)$$

ACF(s) is given by:

$$\rho_1 = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$
$$\rho_2 = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2}$$
$$\rho_{s>2} = 0$$

In general: autocorrelations are non-zero for the first q lags, then 0 for lags $>$ q.
In general: MA(q) processes are covariance-stationary for any values of μ, θ.

Invertibility I

- There is a non-uniqueness of connection between values of θ_1 and ρ_1 in MA(1) models.
- In an MA(1) model, the reciprocal $\frac{1}{\theta_1}$ gives the same value as θ_1 for the ACF ρ_1
- Example (plug in $1/2$ or 2 for θ_1):

$$\rho_1 = \frac{\theta_1}{1+\theta_1^2} = \frac{\frac{1}{2}}{1+(\frac{1}{2})^2} = \frac{2}{5} = \frac{2}{1+2^2} = \frac{2}{5}$$

- We will have to choose only one of the models $\longrightarrow$ We will choose the model with an infinite AR representation.
- Such a process is called an invertible process.
- In the above example, $\theta_1 = 1/2$ is allowed while $\theta_1 = 2$ is not.

Invertibility II

- It is possible to write any stationary AR(p) model as an MA(∞) model.
- The reverse result holds if we impose some constraints on the MA parameters:
- An MA(q) model is said to be invertible if it is algebraically equivalent to a *converging, infinite order AR model*
- Converging here means that the AR coefficients go to 0 as we increase the lag

Invertibility III

- Example, write MA(1), $y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$ as an AR(∞):

$$\begin{aligned}
\varepsilon_t &= y_t - \theta_1 \varepsilon_{t-1} \\
&= y_t - \theta_1 [y_{t-1} - \theta_1 \varepsilon_{t-2}] \\
&= y_t - \theta_1 y_{t-1} + \theta_1^2 \varepsilon_{t-2} \\
&= \dots \\
&= \sum_{j=0}^{\infty} (-\theta_1)^j y_{t-j}
\end{aligned}$$

- which is just an AR(∞) model with parameters $\phi_j = (-\theta_1)^j$ converging to zero.
- $|\theta_1| < 1$ is required for that sum to be finite!

Invertibility III

- MA(2): $|\theta_2| < 1$, $\theta_1| + \theta_2 > -1$, and $\theta_1| - \theta_2 < 1$
- Invertibility is a restriction programmed into most time series software
- For manual estimation of MA models, we need to take this into account

Partial Correlation / Partial Autocorrelation Function (PACF)

- Conditional correlation. It is the correlation between two variables, taking into account some other set of variables.
- The partial correlation between y and x_3 is the correlation between the variables determined taking into account how both y and x_3 are related to x_1 and x_2.
- Correlation the “parts” of y and x_3 that are not predicted by x_1 and x_2.
- In time series: This is the correlation between values two time periods apart conditional on knowledge of the value in between
- Partial Autocorrelation Function (PACF)

Example AR(1)

$$y_t = \phi y_{t-1} + \varepsilon_t$$
$$y_{t-1} = \phi y_{t-2} + \varepsilon_{t-1}$$

- $ACF(2) = corr(y_t, y_{t-2}) = \phi^2$ since $y_{t-2} \xrightarrow{\phi} y_{t-1} \xrightarrow{\phi} y_t$
- Effect of y_{t-2} on y_t only through y_{t-1} (only one lag in y_t), so no "direct" effect. $PACF(2) = 0$

PACF I (general)

In time series: partial correlation (α) (ρ denotes the ACF):

- $\alpha(0) = \rho(0) = 1$
- $\alpha(1) = \rho(1)$
- $\alpha(2) = \frac{\rho(2)-\rho(1)^2}{1-\rho(1)^2}$

Example for an AR(1) process:

$$\alpha(2) = \frac{\phi_1^2 - \phi_1^2}{1 - \phi_1^2} = 0$$

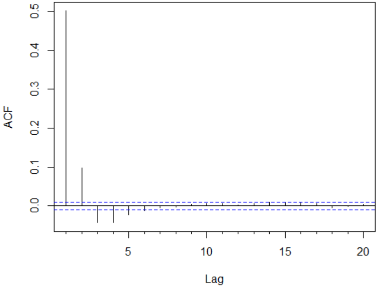
The ACF accounts for all correlation, including the indirect ones. The PACF on the other hand only accounts for the direct relationships.

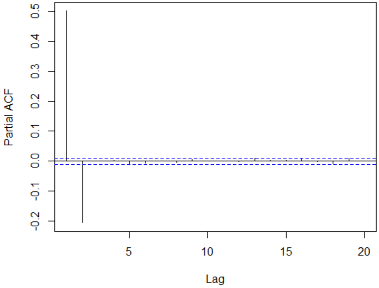
Useful facts for AR(p)

Choosing number of AR lags is done with the PACF:

- the theoretical PACF 'shuts off' after p lags.
- 'shuts off': in theory, all partial autocorrelations beyond p are equal to zero

Intuition: values beyond p only have an indirect effect through newer observations. Below: AR(2) with $\phi = (0.6, -0.2)$



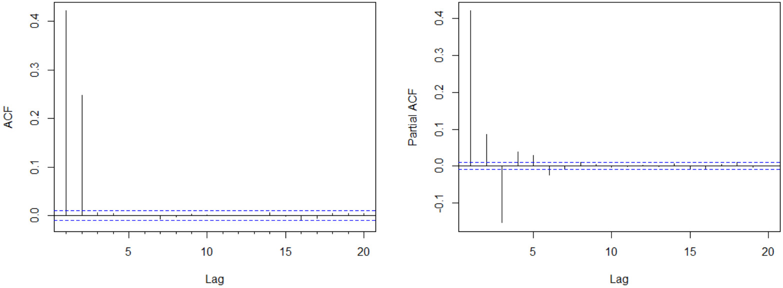


Useful facts for MA(q)

Identification of an MA model is best done with the ACF:

- the theoretical ACF 'shuts off' after q lags.

Below: MA(2) model with $\theta = (0.4, 0.3)$



ACF and PACF implementation in R

Table 3.2 Properties of the ACF and the PACF

Processes	ACF	PACF
AR(p)	Declines exponentially (monotonically or oscillating) to zero	$\alpha(h) = 0$ for $h > p$
MA(q)	$\rho(h) = 0$ for $h > q$	Declines exponentially (monotonically or oscillating) to zero

ACF and PACF can therefore be used to identify models and lag orders (in theory)!

- For ACF: use `acf()` (includes "zero lag" by default) or `forecast::Acf()` (does not)
- For PACF: use `pacf()` or `forecast::Pacf()`

Exercise I

See *exercise3.pdf*