Problem 1 *(12 points)*

Let $(S_n)_{n\geq 0}$ be a simple random walk starting at 0 with $p = 0.4$ and $q = 1-p = 0.6$
Compute the following probabilities:

- $\mathbb{P}(S_2 = 0, S_4 = 0, S_5 = -1)$,
- $\mathbb{P}(\{S_4 = 4\} \cup \{S_4 = -2\})$,
- $\mathbb{P}(M_{17} \leq -5, S_7 = -5)$, where $M_{17} = \min_{0\leq i\leq 17} S_i$.

$S_0 = 0$

1. $\mathbb{P}(S_2 = 0, S_4 = 0, S_5 = -1) = \mathbb{P}(S_2 = 0) \times \mathbb{P}(S_4 = 0 \mid S_2 = 0) \times \mathbb{P}(S_5 = -1 \mid S_4 = 0)$

$= \mathbb{P}(S_2 - S_0 = 0) \times \mathbb{P}(S_4 - S_2 = 0) \times \mathbb{P}(S_5 - S_4 = -1)$

$= \binom{2}{1} 0.4^1 0.6^1 \times \binom{2}{1} 0.4^1 0.6^1 \times 0.6$

$= 4 \cdot 0.4^2 \cdot 0.6^3$

$= 0.13824$

2. $\mathbb{P}(\{S_4 = 4\} \cup \{S_4 = -2\}) = \mathbb{P}(S_4 = 4) + \mathbb{P}(S_4 = -2)$

$= \mathbb{P}(S_4 - S_0 = 4) + \mathbb{P}(S_4 - S_0 = -2)$

$= \binom{4}{4} 0.4^4 0.6^0 + \binom{4}{1} 0.4^1 0.6^3$

$= 0.4^4 + 4 \times 0.4 \times 0.6^3$

$= 0.3712$

3. $\mathbb{P}(M_{17} \leq -5, S_7 = -5) = \mathbb{P}(S_7 = -5) \cdot \mathbb{P}(M_{17} \leq -5 \mid S_7 = -5)$

$\mathbb{P}(S_7 = -5) = \mathbb{P}(S_7 - S_0 = -5) = \binom{7}{1} 0.4^1 0.6^6$

$\mathbb{P}(M_{17} \leq -5 \mid S_7 = -5) = 1$

$\Rightarrow \mathbb{P}(M_{17} \leq -5, S_7 = -5) = \mathbb{P}(S_7 = -5) = 0.1306$

Problem 2 *(6 points)*

For a simple symmetric random walk $(S_n)_{n=0,1,2,...}$ starting in 0 ($S_0 = 0$), show that

$$\mathbb{P}(S_4 = 0) = \mathbb{P}(S_3 = 1).$$

Since it is simple symmetric random walk starting in 0, then $S_0 = 0$. $p = q = \frac{1}{2}$

$$\Rightarrow \quad \mathbb{P}(S_4) = \mathbb{P}(S_4 - S_0 = 0) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{6}{16} = \frac{3}{8}$$

$$\mathbb{P}(S_3) = \mathbb{P}(S_3 - S_0 = 1) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = \frac{3}{8}$$

$$\Rightarrow \quad \mathbb{P}(S_4 = 0) = \mathbb{P}(S_3 = 1)$$

Problem 3 *(6 points)*

Use the reflection principle to find the probability $\mathbb{P}(M_8 = 6)$, where $M_8 = \max_{0 \le i \le 8} S_i$ and $(S_n)_{n \ge 0}$ is a simple symmetric random walk starting in 0 ($S_0 = 0$).

By Theorem 3.12 (Hitting Theorem)

$$\mathbb{P}(M_8 = 6) = \mathbb{P}(T_{06} = 8) = \frac{6}{8}\binom{8}{\frac{1}{2}(8+6)}(0.5)^{\frac{1}{2}(8+6)}(0.5)^{\frac{1}{2}(8-6)}$$

$$= \frac{3}{4}\binom{8}{7}(0.5)^7(0.5)^1$$

$$= 0.0234375$$

Problem 4 *(6 points)*

In an election candidate A receives 200 votes while candidate B only receives 100. Assume that the probability of getting a vote is identical (50% each) for A and B. What is the probability that A is always ahead throughout the count?

The Ballot Theorem : $P(A \text{ is always head}) = \frac{n-m}{n-m}$

n: no. of votes of A

m: no. of votes of B

$n > m$

Proof:

Let $P(A$ is always head$)$ denote as $P_{n,m}$

By conditioning on which candidate receives the last vote counted we have:

$P_{n,m} = \mathbb{P}(A \text{ is always head} \mid \text{Last vote} = A) \cdot \mathbb{P}(\text{Last vote} = A)$

$+ \mathbb{P}(A \text{ is always head} \mid \text{Last vote} = B) \cdot \mathbb{P}(\text{Last vote} = B)$

$= P_{n-1,m} \cdot \frac{n}{n+m} + P_{n,m-1} \cdot \frac{m}{n+m}$

We can prove $P_{n,m} = \frac{n-m}{n+m}$ by induction on $n+m$

when $n+m=1$, we can know that $n=1$, $m=0$ $\quad P_{1,0} = \frac{1-0}{1+0} = 1$

by induction hypothesis, we have

$$P_{n,m} = \frac{(n-1)-m}{(n-1)+m} \cdot \frac{n}{n+m} + \frac{n-(m-1)}{n-(m-1)} \cdot \frac{m}{n+m}$$

$$= \frac{n^2-n-mn+mn-m^2+m}{(n-1+m)(n+m)} = \frac{n^2-m^2-n+m}{(n-1+m)(n+m)}$$

$$= \frac{(n-m)(n+m)-(n-m)}{(n-1+m)(n+m)} = \frac{(n-m)(n+m-1)}{(n-1+m)(n+m)} = \frac{n-m}{n+m}$$

the result is proven

So. $P_{200,100} = \frac{200-100}{200+100} = \frac{100}{300} = \frac{1}{3}$, the probability of A is always head is $\frac{1}{3}$.