Exercise 02

Tuesday, May 7, 2024 10:56 PM

Problem 1 *(12 points)*

Let $(S_n)_{n\geq 0}$ be a simple random walk starting at 0 with $p = 0.4$ and $q = 1-p = 0.6$. Compute the following probabilities:

Non - symmetric

- $\mathbb{P}(S_2 = 0, S_4 = 0, S_5 = -1)$,
- $\mathbb{P}(\{S_4 = 4\} \cup \{S_4 = -2\})$,
- $\mathbb{P}(M_{17} \leq -5, S_7 = -5)$, where $M_{17} = \min_{0\leq i\leq 17} S_i$.

- $P(S_2 = 0, S_4 = 0, S_5 = -1)$ "$X_5 = -1$"

$= P(S_2 - 0 = 0)\, P(S_4 - 0 = 0)\, P(S_5 - S_4 = -1)$

$= (P(S_2 = 0))^2 q$

$= \left(\binom{2}{1} p \cdot q\right)^2 q = 4p^2q^3 = 0.13824$

- $P(\{S_4 = 4\} \cup \{S_4 = -2\})$ $\binom{4}{3}$

$\underline{\underline{\text{mutually exclusive}}}$ $P(S_4 = 4) + P(S_4 = -2)$

$= p^4 + \binom{4}{3} p q^3$

$= 0.3712$

- $P(M_{17} \leq -5, S_7 = -5)$ $S_7 = -5$: reach -5 at $n = 7$ at least once

$= P(S_7 = -5)$

$= \binom{7}{6} p q^6 = 7 \times 0.4 \times 0.6^6$

$= 0.1306$

Problem 2 *(6 points)*

For a simple symmetric random walk $(S_n)_{n=0,1,2,\ldots}$ starting in 0 ($S_0 = 0$), show that

$$\mathbb{P}(S_4 = 0) = \mathbb{P}(S_3 = 1).$$

$P(S_{2m} = 0) = \binom{2m}{m} p^m q^m$ $p = q = \frac{1}{2}$ $\begin{matrix} 2m-1 \\ m-1 \end{matrix}$

$= \frac{(2m)!}{m!\, m!} p^m q^{m-1} \cdot \frac{1}{2} \cdot$

$= \frac{\cancel{m}(2m-1)!}{m!\, \cancel{m!}} p^m q^{m-1}$

$\Downarrow (m-1)!$

$= \binom{2m-1}{m} p^m q^{m-1} = P(S_{2m-1} = 1)$

Let $m = 2$ $\Rightarrow P(S_4 = 0) = P(S_3 = 1)$

Sol. 2 :

$P(S_3 = 1) = P(S_3 = -1)$ symmetric: $\left(p^2 q \binom{3}{1} = p q^2 \binom{3}{1}\right)$

$P(S_4 = 0) = P(S_4 = 0 \mid X_4 = +1)\, P(\underbrace{X_4 = +1}_{0.5})$

$+ P(S_4 = 0 \mid X_4 = -1)\, P(\underbrace{X_4 = -1}_{0.5})$

$= P(S_3 = -1) P(X_4 = 1) + P(S_3 = 1) P(S_4 = -1)$

$= 2 P(S_3 = 1) \cdot 0.5 = P(S_3 = 1)$

Problem 3 *(6 points)*

Use the reflection principle to find the probability $\mathbb{P}(M_8 = 6)$, where $M_8 = \max_{0\leq i\leq 8} S_i$ and $(S_n)_{n\geq 0}$ is a simple symmetric random walk starting in 0 ($S_0 = 0$).

$P(M_8 = 6) = P(M_8 \geq 6) - P(M_8 \geq 7)$

$= P(S_8 \geq 6) + \cancel{P(S_8 > 6) - P(S_8 \geq 7)} - P(S_8 > 7)$

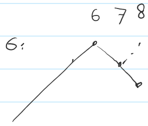
$= P(S_8 \geq 6) - P(S_8 \geq 8)$

$= P(S_8 = 6) + \cancel{P(S_8 = 7)} + \cancel{P(S_8 = 8)} - \cancel{P(S_8 = 8)}$

$\overset{\shortparallel}{0}$

6 7 8

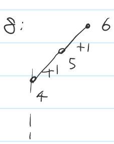
6:



2 possibilites

7: not possible

8:



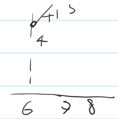
$\binom{6}{5} q^5 \cdot p$ 6 possibilites

$\cdot q^2$

$$= P(S_8=6) + \cancel{P(S_8=7)} + \cancel{P(S_8=8)} - \cancel{P(S_8=8)}$$

$$\quad\quad\quad\quad\quad\quad \overset{}{\underset{0}{\shortparallel}}$$

$$= \binom{8}{7} q^7 p = 0.03125$$



$\cdot\, q^2$

Problem 4 *(6 points)*

In an election candidate A receives 200 votes while candidate B only receives 100. Assume that the probability of getting a vote is identical (50% each) for A and B. What is the probability that A is always ahead throughout the count?

Define the sequence $X_1, X_2, \dots X_{300}$,

$$X_i = \begin{cases} +1, & B \text{ gets vote at this round} \quad p = 0.5 \\ -1, & A \text{ gets vote } \quad q = p = 0.5 \end{cases}$$

$$S_n = S_0 + \sum_{i=1}^{n} X_i ,$$

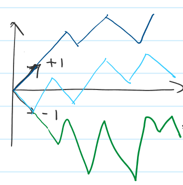
$$P\left(\max_{1 \le i \le 300} S_i \le -1 \mid S_{300} = -100\right)$$

$$= p^{300}\left[\underbrace{\binom{300}{100}}_{\text{total paths from 0 to }-100} - \overbrace{N^0_{300}(0,-100)}^{\text{paths that passes through 0 at least once}}\right]$$

$N^k_n(0, k-\ell)$

$= N_n(0, k+\ell)$

Take one step further ($X_1 = +1$ or $X_1 = -1$)



$$= p^{300}\left[\binom{300}{100} - \left(\underbrace{N^0_{299}(-1,-100)}_{\substack{\text{paths } X_1=-1 \\ \text{(touchs 0 at least once)}}} + \underbrace{\binom{299}{99}}_{\substack{\text{paths } X_1 = 1 \\ \text{(All the paths since they already crosses 0)}}}\right)\right]$$

$$N^0_{299}(-1,-100) = N'_{299}(0,-99) \qquad \left(\begin{matrix} k-\ell = -99, \ k=1, \ \ell = 100 \\ n = 299 \end{matrix}\right)$$

$$= N_{299}(0, 101) = \binom{299}{99}$$

$299 + 102$

Therefore

$$P\left(\max_{1\le i\le 300} S_i \le -1 \mid S_{300}=-100\right) = P\left(\max_{1 \le i \le 300} S \le -1, S_{300} = -100\right) / P(S_{300} = -100)$$

$$= \frac{p^{300}}{p^{300}}\left(\binom{300}{100} - 2\binom{299}{99}\right) \Big/ \binom{300}{100}$$

$$= \left(1 - 2 \cdot \frac{\cancel{299!}}{\cancel{200!}\,\cancel{99!}} \cdot \frac{\overset{100}{\cancel{100!}}\,\cancel{200!}}{\underset{300}{\cancel{300!}}}\right) = \frac{1}{3}$$