nice

Exercise 03

Saturday, November 4, 2023 6:18 PM

Problem 1 to hand in: *Loop Invariant*

The following algorithm computes the symmetric difference $A \Delta B = (A \setminus B) \cup (B \setminus A)$, given two input sets A and B.

```
get_symmetric_difference(A, B):
1 C ← B
2 for a ∈ A do
3 |  if a ∈ C then
4 |  |  C ← C \ {a}
5 |  if a ∉ B then
6 |  |  C ← C ∪ {a}
7 return C
```

side note: the current proof is too lengthy. It's not suitable for the exam at all.

How long did it take to write the proof down? How would you abbreviate it?

a) State a loop invariant that holds at the beginning of each iteration of the for loop (lines 2 to 4).

b) Proof this loop invariant.

c) Use this loop invariant to show that indeed the algorithm returns the symmetric difference.

Suppose input set A has n elements $a_1, \dots, a_n \in A$.

The for-loop at line 2 iterates sequentially from $a_i (i=1)$ to $a_i (i=n)$ ok

a) Claim $S[i]$: At the end of loop i, $C = (A_t \setminus B) \cup (B \setminus A_t)$

where $A_t = \{a_1, \cdots, a_i\}$

b) and c)

Initialization: $i = 1$:

line 1: $C = B$

line 2: pick a_1, $A_t = \{a_1\}$

line 3~4: if $a_1 \in B$, "=>" $C = C \setminus \{a_1\} = B \setminus A_t$

"<=" $(A_t \setminus B) \cup (B \setminus A_t) = \emptyset \cup (B \setminus A_t)$

notation: which equivalence do you show here? $= B \setminus A_t = C$

line 5~6: if $a_1 \notin B$

"⇒" $C = C \cup \{a_1\} = B \cup A_t$

"<=" $\underbrace{(A_t \setminus B)}_{A_t} \cup \underbrace{(B \setminus A_t)}_{B} = A_t \cup B = C$

so at the end of $i=1$, $C = (A_t \setminus B) \cup (B \setminus A_t)$

Maintainence: $i \rightsquigarrow i+1$: $A_t(i) = \{a_1, \dots, a_i\}$, $C = (A_t(i) \setminus B) \cup (B \setminus A_t(i))$

line 2 : pick a_{i+1}, $A_t(i+1) = \{a_1, \dots, a_{i+1}\} = A_t(i) \cup \{a_{i+1}\}$

Line 3~4: if $a_{i+1} \in B$. $a_{i+1} \in C$?

"⇒" $C = C \setminus \{a_{i+1}\}$

$= \big((A_t(i) \setminus B) \cup (B \setminus A_t(i))\big) \setminus \{a_{i+1}\}$

$= (A_t(i) \setminus B) \cup B \setminus A_t(i+1)$

since $a_{i+1} \in B$, $A_t(i) \setminus B = (A_t(i) \cup \{a_{i+1}\}) \setminus B$

$= A_t(i+1)$? $A_t(i) \setminus B$

since $a_{i+1} \in B$, $A_t(i) \setminus B = (A_t(i) \cup \{a_{i+1}\}) \setminus B$
$= A_t(i+1)$

$C = (A_t(i+1) \setminus B) \cup (B \setminus A_t(i+1))$ ✓ ✓

line 5~6: if $a_{i+1} \notin B$, $B \setminus A_t(i) = B \setminus (A_t(i) \cup \{a_{i+1}\}) = B \setminus A_t(i+1)$ ok

"$\Rightarrow$" $C = C \cup \{a_{i+1}\}$

$= (A_t(i) \setminus B) \cup (B \setminus A_t(i)) \cup \{a_{i+1}\}$

$= (A_t(i+1) \setminus B) \cup (B \setminus A_t(i))$

$= (A_t(i+1) \setminus B) \cup (B \setminus A_t(i+1))$ ✓

Termination: $i = n$: $S[n]$: At the end of loop n, $C = (A_t(n) \setminus B) \cup (B \setminus A_t(n))$

where $A_t(n) = \{a_1, \cdots, a_n\} = A$

So $C = (A \setminus B) \cup (B \setminus A)$ at the end of the loop.

At line 7: return $C = (A \setminus B) \cup (B \setminus A) = A \Delta B$ $\square$

so indeed the algorithm returns the symmetric difference between A and B.