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1. (a) To show B are the bases of $\mathbb{R}^2$, we need to prove that $\begin{pmatrix}-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ are linearly independent.

To prove these vectors are linearly independent,

we need to show that there are no scalars (other than zero) that can be multiplied to each vector to obtain the zero vector (0,0)

we can assume that some linear combination of these two vectors is zero:

$$\lambda\cdot\begin{pmatrix}-1\\2\end{pmatrix} + \mu\cdot\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

we can write it as an Augmented matrix:

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

We can find that there exist no free variable and only the trival solution ($\lambda = \mu = 0$).

So, these two vectors are linearly independent, and B is base of $\mathbb{R}^2$.

(b) like (a), we can assume that some linear combination of $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}-1\\0\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\-2\end{pmatrix}$ is zero.

$$a\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} + b\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} + c\cdot\begin{pmatrix}0\\1\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

We can find that there exist no free variable and only the trival solution ($a = b = c = 0$).

So, these three vectors are linearly independent, and $B'$ is base of $\mathbb{R}^3$.

✓

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(b) $f: \mathbb{R}^2 \to \mathbb{R}^3, \quad \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a-b \\ b+2a \\ 3b+2a \end{pmatrix}$

$S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = -1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \quad {}_{S'}M(f)_S = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 2 & 3 \end{pmatrix}$ ✓

$B = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$f\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -1-2 \\ -2+2 \\ -2+6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = -3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 1+1 \\ 2-1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \quad {}_{S'}M(f)_B = \begin{pmatrix} -3 & 2 \\ 0 & 1 \\ 4 & -1 \end{pmatrix}$ ✓

$B = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad B' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$

$f\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = a_{11} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_{21} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + a_{31} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$\Rightarrow \quad 1 \cdot a_{11} - 1 \cdot a_{21} + 0 \cdot a_{31} = -3$

$1 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} = 0$

$0 \cdot a_{11} + 1 \cdot a_{21} - 2 \cdot a_{31} = 2$

we can get $\left(\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$

Therefore we have $a_{31}=\frac{1}{3}$, $a_{21}=3-\frac{1}{3}=\frac{8}{3}$, $a_{11}=-3+\frac{8}{3}=-\frac{1}{3}$

$$f\left(\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right)=\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}=-\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+\frac{8}{3}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}+\frac{1}{3}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$\frac{8}{3}-\frac{2}{3}=\frac{6}{3}=2\neq 4$

Likewise, we need to find $a_{12}, a_{22}, a_{32}$ such that

$-1$

$$f\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right)=\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}=a_{12}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+a_{22}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}+a_{32}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

We can get $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$

Therefore we have $a_{32}=0$, $a_{22}=-1$, $a_{12}=1$

$$f\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right)=\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}=1\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}-1\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}+0\cdot\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$\Rightarrow {}_B M(f)_{B'}=\begin{pmatrix} -\frac{1}{3} & 1 \\ \frac{8}{3} & -1 \\ \frac{1}{3} & 0 \end{pmatrix}$

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$\boxed{\text{EX2}}$

(i) Assume there exists polynomial $p(x)=ax^3+bx^2+cx+d$, $(a,b,c,d\in\mathbb{R})$

then, we can get $\begin{cases} P(0)=d=1 \\ P(1)=a+b+c+d=1 \\ P(2)=8a+4b+2c+d=0 \\ P(3)=-a+b-c+d=1 \end{cases}$

$$\Rightarrow \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 0 \\ -1 & 1 & -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

then we can know that $a=-\frac{1}{6}$, $b=0$, $c=\frac{1}{6}$, $d=1$.

The polynomial is $p(x)=\frac{1}{6}x^3-\frac{1}{6}x+1$.

$-1$

$=-\frac{1}{6}x^3+\frac{1}{6}x+1$

There is no more polynomials which satisfy the four conditions. To prove it, we assume the contradition.

Assume there exists such polynomial $Q(x) = a_1x^3 + b_1x^2 + c_1x + d_1$

From $Q(0) = 1 \Rightarrow a_1x^3 + b_1x^2 + c_1x = 0$, $d_1 = 1$

By the conditions, we got: $P(1) - Q(1) = a+b+c-a_1-b_1-c_1 = 0$ (1)

$P(2) - Q(2) = 8a+4b+2c-8a_1-4b_1-2c_1 = 0$ (2)

$P(-1) - Q(-1) = a+b-c+a_1-b_1+c_1 = 0$ (3)

we add the left and right side of (1) and (2), get the equation

$2b - 2b_1 = 0 \Rightarrow b = b_1$ (4)

by (4) and (1) $\Rightarrow a - a_1 = c - c_1$ (5)

substitute (5), (4) into (2) $\Rightarrow 6c_1 - 6c = 0 \Rightarrow c_1 = c$ (6)

substitute (6) into (5) we have $a_1 = a$

therefore $Q(x) = P(x) = ax^3 + bx^2 + cx + d$, contradict to our assumption.

Hence such polynomial is unique. ✓

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GOOD JOB!