Exercise 6

Problem 20

Lagrange function:

$$L(x, y, \alpha, \beta) = (x-3)^2 + (y-1)^2 + \alpha(x^2+1-y) + \beta(y+3x-11)$$

KKT condition:

$$\begin{cases} \nabla_{x,y} L(x,y,\alpha,\beta) = 0 \Rightarrow \begin{pmatrix} 2x + 2\alpha x + 3\beta \\ 2y - \alpha + \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \alpha, \beta \geq 0, \quad x^2+1-y \leq 0, \quad y+3x-11 \leq 0 \quad \alpha(x^2+1-y) = 0, \quad \beta(y+3x-11) = 0 \end{cases}$$

From the gradient condition and complementary slackness conditions we have:

$$2(1+\alpha)x + 3\beta = 0$$
$$y = \frac{\alpha-\beta}{2}$$
$$\alpha(x^2+1-y) = 0$$
$$\beta(y+3x-11) = 0$$

Case 1. $\alpha = 0$, then $y = \frac{-\beta}{2}$, $x = \frac{-3\beta}{2}$

Case 1.1: $\beta = 0$. then $y = x = 0$. since $x^2+1-y = 1 \nleq 0$, not feasible.

Case 1.2: $\beta \neq 0$, $x = 3y$, $y + 9y - 11 = 0 \Rightarrow y = \frac{11}{10}$, $x = \frac{33}{10}$, not feasible.

Case 2. $\alpha \neq 0$,

Case 2.1: $\beta = 0$. then $2(1+\alpha)x = 0$, $y = \frac{\alpha}{2}$, $x = 0, y = 1$, feasible.

Case 2.2: $\beta \neq 0$, then $x=2, y=5$, not feasible $(2 \cdot 2(1+\alpha)+3\beta = 0 \Rightarrow 4+4\alpha+3\beta=0$, α, β

$x=-5, y=26$, not feasible, contradict to $\alpha, \beta \geq 0$

hence $((x=0, y=1), \alpha=2, \beta=0)$ satisfy the KKT condition, and the minimum value is 9.

Problem 21

(a) Hessian matrix of $f(x,y)$ is $\begin{pmatrix} 6 & -4 \\ -4 & 0 \end{pmatrix}$, it is positive semi-definite.

$g = (x-1)^2 - 4$ are convex, $-y \leq 0$ is affine. so their intersections are convex

(b) $L(x, y, \alpha, \beta) = -4xy + 3x^2 + 2x + 4y + \alpha((x-1)^2 - 4) + \beta(-y)$

KKT conditions:

$$\nabla_{x,y} L(x,y,\alpha,\beta) = \begin{pmatrix} -4y + 6x + 2 + 2\alpha(x-1) \\ -4x + 4 - \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(x-1)^2 - 4 \leq 0, \quad -y \leq 0$

(c) Case 1 $\alpha=0$

Case 1.1. $\beta=0$: $x=1, y=2$ feasible.

Case 1.2. $\beta\neq 0$: $x=-\frac{1}{3}, y=0$ feasible.

~~Case 2 $\alpha\neq 0$, $x=3$ or $x=-1$, not feasible~~

~~Case 2.1. $\beta=0$: $x=3$ or $x=-1$ not feasible $(-4x+4-0\neq 0)$~~

~~Case 2.2. $\beta\neq 0$.~~

Case 2 $\alpha\neq 0$

Case 2.1. $\beta=0$: $x=3$ or $x=-1$ ~~not~~ feasible $(-4x+4-0\neq 0)$

not feasible, $-4x+4-\beta=0 \Rightarrow \beta=-8$ contradict to $\beta\geq 0$

Case 2.2 $\beta\neq 0$: $y=0$, $x=3$ or $x=-1$ ~~feasible~~ $\Rightarrow \alpha=-1$ contradict to $\alpha\geq 0$.

$((1,2),0,0)$, $((-\frac{1}{3},0),0,5\frac{1}{3})$ ~~, ((−1,0)~~ are the points satisfy KKT.

d) $(x,y)=(-\frac{1}{3},0)$ is an optimal point. By Proposition 3.7.2

e) As it shows in (a), the Hessian matrix of the objective function is positive semi-definite, hence it is convex.