Problem 1 *(6 points)*

Consider the Gamblers Ruin problem covered in class. The basic setup is unchanged and you want to determine the ruin probabilty $\mathbb{P}(S_{T_{10,0,80}} = 0)$. However, instead of betting \$1 on red in each round, you bet \$2 on red in each round. What is the ruin probability now? Compare it with the situation when betting \$1 on red in each round.

For betting \$1:

$$\mathbb{P}(S_{T=10,0,80}=0) = \frac{(1-p)^{80} - (1-p)^{10}}{(1-p)^{80} - \left(\frac{1-p}{p}\right)^{80}}$$

For betting \$2:

$$\mathbb{P}(S_{T=10,0,80}=0) = \frac{(1-p)^{\frac{80}{2}} - (1-p)^{\frac{10}{2}}}{(1-p)^{\frac{80}{2}} - \left(\frac{1-p}{p}\right)^{\frac{80}{2}}}$$

$p = \frac{18}{37}$

$\Rightarrow$ Betting \$1 : $\mathbb{P} = 0.00169\%$

Betting \$2 : $\mathbb{P} = 0.41\%$

Problem 2 *(8 points)*

Let $X_0, X_1, X_2, \ldots$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$, transition probabilities

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/5 & 1/10 & 7/10 \end{pmatrix} \end{array}$$

and initial distribution $\alpha^T = (1/3, 1/3, 1/3)$. Find the following probabilities:

(a) $\mathbb{P}[X_5 = 2 \mid X_4 = 1]$,

(b) $\mathbb{P}[X_1 = 1,\ X_2 = 2]$,

(c) $\mathbb{P}[X_1 = 2 \mid X_2 = 1]$,

(d) $\mathbb{P}[X_5 = 1 \mid X_1 = 2, X_2 = 3, X_3 = 2]$.

(a) $\mathbb{P}(X_5 = 2 \mid X_4 = 1) = P_{12} = 1$

(b) $\mathbb{P}(X_1 = 1, X_2 = 2) = \mathbb{P}(X_2 = 2 \mid X_1 = 1) \cdot \mathbb{P}(X_1 = 1)$

$= P_{12} (\alpha^T P)_1 = 1 \cdot \left(\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{5}\right) = \frac{8}{45}$

(c) $\mathbb{P}(X_1 = 2 \mid X_2 = 1) = \dfrac{\mathbb{P}(X_2 = 1 \mid X_1 = 2)\, \mathbb{P}(X_1 = 2)}{\mathbb{P}(X_2 = 1)} = \dfrac{P_{21} (\alpha^T P)_2}{(\alpha^T P^2)_1} = \dfrac{\frac{1}{3} \cdot \frac{11}{30}}{\frac{16}{75}}$

$= \frac{55}{96}$

(d) $\mathbb{P}(X_5 = 1 \mid X_1 = 2, X_2 = 3, X_3 = 2)$

$= \mathbb{P}(X_5 = 1 \mid X_3 = 2)$

$= P^2_{21} = \frac{2}{15}$

$$P^2 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{15} & \frac{2}{5} & \frac{7}{15} \\ \frac{13}{75} & \frac{27}{100} & \frac{167}{300} \end{bmatrix}$$

Problem 3 *(8 points)*

Consider a Markov chain $(X_n)_{n=0,1,2,\ldots}$ with state space $\mathcal{S} = \{1,2,3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 0 & 1/2 & 1/2 \\ 3/10 & 7/10 & 0 \end{pmatrix}.$$

The initial distribution is given by $\alpha^T = (1/2, 1/6, 1/3)$. Compute

(a) $\mathbb{P}[X_2 = k]$ for all $k = 1,2,3$;

(b) $\mathbb{E}[X_2]$.

Does the distribution of X_2 computed in (a) depend on the initial distribution α? Does the expected value of X_2 computed in (b) depend on the initial distribution α? Give a reason for both of your answers.

(a) $\mathbb{P}(X_2 = k) = \alpha^T \cdot P^2$

$$P^2 = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{10} & \frac{7}{10} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{10} & \frac{7}{10} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{14}{25} & \frac{17}{50} \\ \frac{3}{20} & \frac{3}{5} & \frac{1}{4} \\ \frac{3}{30} & \frac{53}{100} & \frac{41}{100} \end{pmatrix}$$

$$\mathbb{P}(X_2 = k) = \left(\tfrac{1}{2} \quad \tfrac{1}{6} \quad \tfrac{1}{3}\right) \begin{pmatrix} \frac{1}{10} & \frac{14}{25} & \frac{17}{50} \\ \frac{3}{20} & \frac{3}{5} & \frac{1}{4} \\ \frac{3}{30} & \frac{53}{100} & \frac{41}{100} \end{pmatrix} = \left(\frac{19}{200} \quad \frac{167}{300} \quad \frac{209}{600}\right)$$

$\mathbb{P}(X_2 = 1) = \frac{19}{200}$ $\quad\mathbb{P}(X_2 = 2) = \frac{167}{300}$ $\quad\mathbb{P}(X_2 = 3) = \frac{209}{600}$

(b) $E[X_2] = 1 \cdot \mathbb{P}(X_2 = 1) + 2 \cdot \mathbb{P}(X_2 = 2) + 3 \cdot \mathbb{P}(X_2 = 3)$

$= 1 \cdot \frac{19}{200} + 2 \times \frac{167}{300} + 3 \times \frac{209}{600}$

$= \frac{169}{75}$

The Probability $\mathbb{P}(X_2 = k)$ directly depend on α as we can see the formula above.

The Expected value depends on $\mathbb{P}(X_2 = k)$, since $\mathbb{P}(X_2 = k)$ is dependent on α, the $E[X_2]$ also depends on the initial distribution.

Problem 4 *(8 points)*

A stochastic matrix is called *doubly stochastic* if its columns sum to 1. Let $X_0, X_1, \ldots$ be a Markov chain on the state space $\mathcal{S} = \{1, \ldots, k\}$ with doubly stochastic transition matrix P and initial distribution that is uniform on $\mathcal{S}$.

Show that the distribution of X_n is uniform on $\mathcal{S}$ for all $n \geq 0$.

If the initial dist. π_0 is uniform over S,

then $\pi_0 = (\frac{1}{k}, \frac{1}{k}, \cdots \frac{1}{k})$.

when $n = 0$,

by definition, the initial dist. π_0 is uniform.

Assume $\pi_n = (\frac{1}{k}, \frac{1}{k}, \cdots \frac{1}{k})$ is uniform for some $n \geq 0$.

we need to show that π_{n+1} is also uniform.

$$\pi_{n+1} = \pi_n \cdot P$$

$$= (\frac{1}{k}, \frac{1}{k}, \cdots, \frac{1}{k}) P$$

The j-th element of π_{n+1} can be expressed as

$$(\pi_{n+1})_j = \sum_{i=1}^{k} \frac{1}{k} P_{ij}$$

Using the property of doubly stochastic matrices

(column sums to 1):

$$(\pi_{n+1})_j = \frac{1}{k} \sum_{i=1}^{k} P_{ij} = \frac{1}{k} \times 1 = \frac{1}{k}$$

This shows that each element of π_{n+1} is $\frac{1}{k}$,

conforming that π_{n+1} remains uniform.