Ex 13 $\quad A = \begin{pmatrix} 2 & -2 & 1 \\ -4 & -8 & -8 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 2 & -4 \\ -2 & -8 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 \\ -4 & -8 & -8 \end{pmatrix} = \begin{pmatrix} 20 & 28 & 34 \\ 28 & 68 & 62 \\ 34 & 62 & 65 \end{pmatrix}$$

$$\det(A^T A - \lambda I) = \begin{vmatrix} 20-\lambda & 28 & 34 \\ 28 & 68-\lambda & 62 \\ 34 & 62 & 65-\lambda \end{vmatrix}$$

$$= (20-\lambda)(68-\lambda)(65-\lambda) + 28\times 62\times 34 + 28\times 62\times 34 - 34\times(68-\lambda)\times 34$$
$$- (20-\lambda)\times 62\times 62 - 28\times 28\times(65-\lambda)$$
$$= -\lambda^3 + 153\lambda^2 - 1296\lambda$$
$$= -\lambda(\lambda^2 - 153\lambda + 1296) = -\lambda(\lambda-9)(\lambda-144) = 0 \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 9 \\ \lambda_3 = 144 \end{cases}$$

① when $\lambda = 0$

the matrix: $\left(\begin{array}{ccc|c} 20 & 28 & 34 & 0 \\ 28 & 68 & 62 & 0 \\ 34 & 62 & 65 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{17}{10} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -\frac{1}{2}x_3 \end{cases}$

Let $x_3 = 1$, $V_1 = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

② when $\lambda = 9$

the matrix: $\left(\begin{array}{ccc|c} 11 & 28 & 34 & 0 \\ 28 & 59 & 62 & 0 \\ 34 & 62 & 56 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = -2x_3 \end{cases}$ Let $x_3 = 1$, $V_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Let $\lambda = 144$

the matrix: $\left(\begin{array}{ccc|c} -124 & 28 & 34 & 0 \\ 28 & -76 & 62 & 0 \\ 34 & 62 & -78 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}x_3 \\ x_2 = x_3 \end{cases}$

Let $x_3 = 1$, $V_3 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$

Find the square roots of the nonzero eigenvalues

$\Rightarrow \Sigma = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

And we can now find normalized vector:

$V = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

$u_1 = \frac{1}{12} \cdot \begin{bmatrix} 2 & -2 & 1 \\ -4 & -8 & -8 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$u_2 = \frac{1}{3} \cdot \begin{bmatrix} 2 & -2 & 1 \\ -4 & -8 & -8 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Therefor $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A = U \Sigma V^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Ex 14

Assume the polynoimal: $p(x) = ax^2 + bx + c$

Force the model to fit the data

$$\begin{cases} a - b + c = 0 \\ 4a + 2b + c = 1 \\ a + b + c = -1 \\ c = 1 \end{cases} \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

Compute A^TA : $\begin{pmatrix} 1 & 4 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{pmatrix}$

$$A^Tb = \begin{pmatrix} 1 & 4 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

The normal equations are: $A^TA\bar{x} = A^Tb$

$$\begin{pmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{11} & \frac{5}{22} \\ 0 & 1 & -\frac{3}{11} & -\frac{3}{22} \\ 0 & 0 & 1 & -\frac{1}{20} \end{array}\right) \longrightarrow \begin{cases} a = \frac{1}{4} \\ b = -\frac{3}{20} \\ c = -\frac{1}{20} \end{cases}$$

So, the polynomial $p(x) = \frac{1}{4}x^2 - \frac{3}{20}x - \frac{1}{20}$