Problem 1 *(6 points)*

Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $S = \{1,2,3\}$ and transition probability matrix

$$P = \begin{pmatrix} 0.4 & 0 & 0.6 \\ 0.3 & 0.1 & 0.6 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Compute the stationary distribution π.

$$\begin{cases} 0.4\pi_1 + 0.3\pi_2 + 0.2\pi_3 = \pi_1 \\ 0\pi_1 + 0.1\pi_2 + 0.2\pi_3 = \pi_2 \\ 0.6\pi_1 + 0.6\pi_2 + 0.6\pi_3 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{4}{15} \\ \pi_2 = \frac{2}{15} \\ \pi_3 = \frac{2}{3} \end{cases}$$

$$\pi = \left(\frac{4}{15}, \frac{2}{15}, \frac{2}{3}\right)$$

Problem 2 *(8 points)*

Let $(X_n)_{n\geq 0}$ denote a Markov chain with state space $\mathcal{S} = \{1,2,3,4\}$ and transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & p & 1-p & 0 \end{pmatrix}$$

where $p \in [0,1]$ is some arbitrary probability.

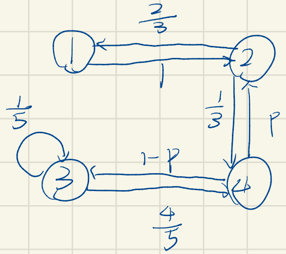
(a) Provide the transition graph.

(b) Determine all communication classes. Which communication classes are closed?

Hint: You need to do a case-by-case analysis depending on the value of p.

(c) Fix $p = 0$. The initial distribution is given by $\alpha^T = (0, 0, \frac{5}{9}, \frac{4}{9})$. Compute the distribution of X_1. What is the distribution of X_{103}?

(a)



(b) case 1: $p = 0$

state 3 and state 4 are communication and closed

state 1 and state 2 are communication.

case 2: $0 < p < 1$

These 4 states are communication and closed.

case 3: $p = 1$

state 1, 2 and 4 are communication

state 3 is recurrent.

(c) if $p=0$, then $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 \end{pmatrix}$

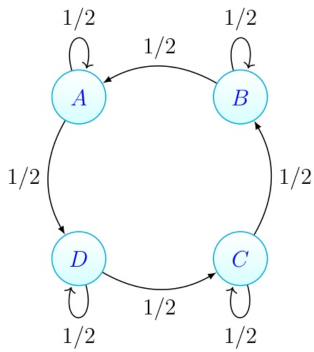
$$X_1 = \alpha^T P = (0 \quad 0 \quad \tfrac{5}{9} \quad \tfrac{4}{9}) \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= (0 \quad 0 \quad \tfrac{5}{9} \quad \tfrac{4}{9}) = \alpha^T$$

α^T is stationary distribution, so $X_{103} = (0 \quad 0 \quad \frac{5}{9} \quad \frac{4}{9})$

Problem 3 *(8 points)*

A Markov chain $X_0, X_1, X_2, \ldots$ has the following transition graph:



(a) Provide the transition matrix for the Markov chain.

(b) Classify all states (recurrent/transient).

(c) Find the communication classes. Is the chain irreducible?

(d) Find the stationary distribution.

(e) What do you know about the limiting distribution?

(a)

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b) Each state in this Markov Chain can be reached from any other state. So, all states are recurrent.

(c) Since every state can reach every other state, all states belong to a single communication class. So it is irreducible.

(d)
$$\begin{cases} \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + 0\pi_3 + 0\pi_4 = \pi_1 \\ 0\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 + 0\pi_4 = \pi_2 \\ 0\pi_1 + 0\pi_2 + \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 = \pi_3 \\ \frac{1}{2}\pi_1 + 0\pi_2 + 0\pi_3 + \frac{1}{2}\pi_4 = \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{1}{4} \\ \pi_2 = \frac{1}{4} \\ \pi_3 = \frac{1}{4} \\ \pi_4 = \frac{1}{4} \end{cases}$$

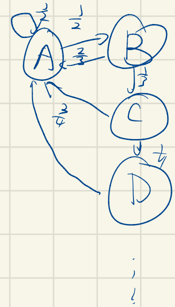
So $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

(e) Since the Markov Chain is irreducible and every state is recurrent, the limiting distribution also exists and is equal to the stationary distribution, which means the probability of being in any of the states A, B, C or D converges to $\frac{1}{4}$.

Problem 4 *(8 points)*
Consider a Markov chain on the state space $\mathcal{S} = \{1, 2, 3, 4, \ldots\}$ with the following transition matrix:

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & & \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 2/3 & 0 & 1/3 & 0 & 0 & \cdots \\ 3/4 & 0 & 0 & 1/4 & 0 & \cdots \\ 4/5 & 0 & 0 & 0 & 1/5 & \cdots \\ 5/6 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

That is, $P_{ij} = i/(i+1)$ if $j = 1$, $P_{ij} = 1/(i+1)$ if $j = i+1$, and $P_{ij} = 0$ otherwise.



(a) Classify all states of the Markov chain (transient, recurrent)

(b) Determine if the Markov chain is irreducible.

(a) For this Markov Chain, consider the transition probabilities:

$P_{ij} = \frac{i}{i+1}$ if $j=1$, when i increase, P_{ij} increase

$P_{ij} = \frac{1}{i+1}$ if $j=i+1$, when i increase, P_{ij} decrease.

However, each state has a non-zero probability of returning to state 1, indicating that the chain can revisit any state after a series of transitions. So this chain is recurrent.

(b) From any state $i > 1$, there are non-zero probabilities of transitioning back to 1 and to j $(i+1)$.

For state 1, there's a non-zero probability of transitioning to any other state j (via state i).

Hence, the Markov Chain is irreducible.