Exercise sheet 2

Exercise 3

$$A = \begin{pmatrix} 2 & 12 & 17 \\ 0 & 0 & 3 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\det\left(\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & 12 & 17 \\ 0 & 0 & 3 \\ 0 & 2 & -1 \end{pmatrix}\right) = \det\begin{pmatrix} \lambda-2 & -12 & -17 \\ 0 & \lambda & -3 \\ 0 & -2 & \lambda+1 \end{pmatrix}$$

$$= (\lambda-2)\det\begin{pmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{pmatrix} + 0\det\begin{pmatrix} -12 & -17 \\ -2 & \lambda+1 \end{pmatrix} - 0\begin{pmatrix} -12 & -17 \\ \lambda & -3 \end{pmatrix}$$

$$= (\lambda-2)(\lambda(\lambda+1)-6) = 0$$

$\Rightarrow \lambda = 2$ or $\lambda^2+\lambda-6=0$

$\Rightarrow (\lambda-2)(\lambda+3)=0$

$\lambda = -3$ or 2

So, the eigenvalues of A are 2 and -3.

when $\lambda=2$

$$\left(\begin{pmatrix} 2 & 12 & 17 \\ 0 & 0 & 3 \\ 0 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right) = \begin{pmatrix} 0 & 12 & 17 \\ 0 & -2 & 3 \\ 0 & 2 & -3 \end{pmatrix} X = 0$$

I.e. solve. $\begin{cases} 12x_2 + 17x_3 = 0 \\ -2x_2 + 3x_3 = 0 \\ 2x_2 - 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = x_3 = 0 \\ x_1 = x_1 \end{cases}$

general solution : $\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$ and the solution set : $\{x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$

Let $x_1 = 1$, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

when $\lambda = -3$

$$\left(\begin{pmatrix} 2 & 12 & 17 \\ 0 & 0 & 3 \\ 0 & 2 & -1 \end{pmatrix} - \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}\right) = \begin{pmatrix} 5 & 12 & 17 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \end{pmatrix} Y = 0$$

I.e. solve $\begin{cases} 5y_1 + 12y_2 + 17y_3 = 0 \\ 3y_2 + 3y_3 = 0 \\ 2y_2 + 2y_3 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = -y_3 \\ y_2 = -y_3 \\ y_3 = y_3 \end{cases}$

general solution: $\begin{pmatrix} -y_3 \\ -y_3 \\ y_3 \end{pmatrix}$ and the solution set : $\{ y_3 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \}$

Let $y_3 = 1$, $V_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

So, the two eigenvectors of A are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Exercise 4

(a) Assume $\exists y$, s.t. $A^T x = A^T y \Rightarrow A^T x - A^T y = 0$

$\Rightarrow A^T (x - y) = 0$

since A is an $n \times m$ - matrix of rank $n \Rightarrow A^T \neq 0$

$\Rightarrow x = y$

So, $x \mapsto A^T x$ is an injective map.

(b) A is an $n \times m$ - matrix $\Rightarrow A^T$ is an $m \times n$ - matrix

$\Rightarrow A \cdot A^T$ is an $n \times n$ matrix

To prove $A \cdot A^T$ is invertible, we try to show if the null space of $A \cdot A^T$ is trivial:

Assume $\exists V$, s.t. $(A \cdot A^T) V = 0$

$\Rightarrow V^T (A \cdot A^T) V = 0 \Rightarrow (A^T V)^T \cdot (A^T V) = 0$

$\Rightarrow A^T V = 0 \Rightarrow V = 0$

Hence $A \cdot A^T$ is invertible.

(c) B is an $n \times m$ matrix of rank m, B^T is $m \times n$ matrix

Assume $\exists x$ s.t. $(B^T B) x = 0 \Rightarrow x^T (B^T B) x = 0$

$\Rightarrow (Bx)^T \cdot (Bx) = 0 \Rightarrow Bx = 0 \Rightarrow x = 0$

Therefore $B^T B$ is an invertible $m \times m$ - matrix.

Exercise 5.

(i) $A(A^TA)^{-1}A^TA$

$= (a_{ij})((a_{ji})(a_{ij}))^{-1}(a_{ji})\, a_{ij}$

$= (a_{ij})(a_{jj})^{-1}(a_{ji})(a_{ij})$

$= (a_{ij})(a_{jj})^{-1}(a_{jj})$

$= (a_{ij}) = A$

(ii) $(A^TA)^{-1}A^T\, A\, (A^TA)^{-1}A^T$

$= ((a_{ji})(a_{ij}))^{-1}(a_{ij})((a_{ji})(a_{ij}))^{-1}$

$= (a_{jj})^{-1}(a_{ji})(a_{ij})((a_{ji})(a_{ij}))^{-1}$

$= (a_{jj})^{-1}(a_{jj})((a_{ji})(a_{ij}))^{-1}$

$= ((a_{ji})(a_{ij}))^{-1} = (A^TA)^{-1}A^T$

(iii) $(A(A^TA)^{-1}A^T)^T$

$= [(A^TA)^{-1}A^T]^T A^T$

$= [(a_{jj})^{-1}(a_{ji})]^T(a_{ji})$

$= a_{ij}\,(a_{jj})^{-1}(a_{ji})$

$= A(A^TA)^{-1}A^T$

? (iv) $((A^TA)^{-1}A^TA)^T$

$= ((a_{jj})^{-1}(a_{ji})(a_{ij}))^T$

$= ((a_{jj})^{-1}(a_{jj}))^T$

$= I^T$

$= I$

$= (a_{jj})^{-1}(a_{jj})$

$= (a_{jj})^{-1}(a_{ji})(a_{ij})$

$= (A^TA)^{-1}A^TA$

As B is invertible $\Rightarrow BB^{-1} = I$

Taking transpose of both sides $(BB^{-1})^T = I^T$

Since $I^T = I$,

we get $(B^T)(B^{-1})^T = I$

$(B^T)^{-1}(B^T)(B^{-1})^T = (B^T)^{-1} I$

Since B^T is invertible $\Rightarrow (B^T)^{-1}(B^T) = I$

So $(B^{-1})^T = (B^T)^{-1}$