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Markov Chains

Problem sheet 1

Review concepts probability, conditional expectation, random walk, markov chains

Problems to be discussed (in parts) during the exercise sections:

Problem 1

We chose a number from the set $\{1, 2, 3, \ldots, 100\}$ uniformly at random and denote this number by X. For each of the following choices decide whether the two events in question are independent or not.

(a) $A = \{X \text{ is even}\}, B = \{X \text{ is divisible by } 5\}$

(b) $C = \{X \text{ has two digits}\}, D = \{X \text{ is divisible by } 3\}$

(c) $E = \{X \text{ is prime}\}, F = \{X \text{ has a digit } 5\}$. Note that 1 is not considered a prime number.

Problem 2

Suppose there are two secretaries working as typists in a law firm. The number of typos per page made by secretary A is a Poisson random variable with parameter $\lambda_A = 3$. The number of typos per page made by secretary B is also a Poisson random variable with an average of 7 typos per page.

Assume a letter is typed up. From experience, this work will be done with $1/3$ probability by secretary A and with $2/3$ probability by secretary B.

(a) What is the probability that the typewritten letter will contain **exactly one typo**?

(b) It turns out that the typewritten letter does **not** contain **any** typos. Given this information, what is the probability that secretary B typewrote this letter?

Problem 3

Assume the number B of blossoms on an apple tree is Poisson distributed with parameter λ and suppose that, independently from all the others, each blossom will yield a fruit with probability $p \in [0, 1]$.

(a) Compute the distribution of the number of apples A on the tree.

(b) Given the number of apples A, how is the original number of blossoms B distributed?

Hint: Both answers can be phrased in terms of Poisson distributions. You also need to use the fact that

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} \qquad (\lambda \in \mathbb{R}).$$

Problem 4

The so-called *Wald's Identity* states the following: Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with finite mean. Let further N be a nonnegative integer-valued random variable independent of the $X_1, X_2, \dots$ and with finite mean. We define the random sum $S_N := X_1 + \dots + X_N$. Then

$$\mathbb{E}[S_N] = \mathbb{E}[N] \cdot \mathbb{E}[X_1].$$

Prove Wald's identity by conditioning on $\{N = n\}$ and by using the law of total expectation.

Problem 5

Let X_1, X_2 be independent geometric random variables with the same success probability $p \in (0,1)$, i.e. $\mathbb{P}(X_i = k) = (1-p)^{k-1}p$ for $i \in \{1,2\}$. Calculate $\mathbb{E}[X_1^2 | X_1 + X_2]$.

Hint: You will need the formula

$$1^2 + 2^2 + \dots n^2 = \frac{n(n+1)2n+1}{6}.$$

Problems to be handed in by:

Thursday, April 25th, 11:59 p.m., online via Ilias.

Problem 1 *(10 points)*

Let $(X_n)_{n\geq 0}$ be a simple random walk starting in $X_0 = 0$ with probability $p \in (0,1)$ to move up. Determine the following probabilities:

$\mathbb{P}(X_6 = 4 | X_5 = 3, X_4 = 2)$,
$\mathbb{P}(X_6 = 4 | X_5 = 3)$,
$\mathbb{P}(X_6 = 4 | X_5 = 3, X_4 \text{ is even})$,
$\mathbb{P}(X_6 = 4 | X_5 \text{ is odd}, X_4 = 2)$,
$\mathbb{P}(X_6 = 4 | X_5 \text{ is odd})$.

Problem 2 *(10 points)*

Let $(X_n)_{n\in\mathbb{N}_0}$ be a simple random walk with probability $p \in (0,1)$ to move up. Determine

(a) $\mathbb{P}(X_n - X_0 = k)$ for $k \in \mathbb{Z}$,

(b) $\mathbb{E}[X_n|X_{n-1}]$,

(c) $\mathbb{E}[|X_n||X_{n-1}]$.

Problem 3 *(10 points)*

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. $B(1,p)$-distributed random variables. Let

$$Y_i := 1_{\{X_i=1\}} + 1_{\{X_i=X_{i-1}=X_{i-2}=1\}},\ i \geq 3$$

$$Z_i := 1_{\{X_i=1\}} + 1_{\{X_i=X_{i-1}=1\}} + 1_{\{X_i=X_{i-1}=X_{i-2}=1\}},\ i \geq 3$$

(a) Determine $\mathbb{P}(Z_{n+1} = j_1|Z_n = i_1)$ and $\mathbb{P}(Y_{n+1} = j_2|Y_n = i_2)$ for $i_1, j_1 \in \{0,1,2,3\}$ and for $i_2, j_2 \in \{0,1,2\}$ and $n \geq 3$.

(b) Is $(Y_n)_{n\geq 3}$ respectively $(Z_n)_{n\geq 3}$ a Markov Chain? Justify your answer.