Exercise 01

Saturday, October 14, 2023 1:57 PM

Exercise 1 (10 points)

(a) Show that the sets

$$B := \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2 \qquad B' := \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

are bases of $\mathbb{R}^2$, resp. $\mathbb{R}^3$.

[Hint: It is enough to show that the sets are linearly independent, because of Observation 1.1.13 in the manuscript – two linearly independent vectors in $\mathbb{R}^2$ automatically form a basis, same for three vectors in $\mathbb{R}^3$.]

For set B:

Assume that some linear combination of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is zero:

$$\lambda \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\lambda + \mu \\ 2\lambda + \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\lambda + \mu = 0 \\ 2\lambda + \mu = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ \mu = 0 \end{cases}$$

so vectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linear independent.

In $\mathbb{R}^2$ there cannot be more than 2 linear independent vectors, therefore $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ must be a basis of $\mathbb{R}^2$

For set B'

Assume that some linear combination of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ is zero:

$$a \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} a - b \\ a + c \\ b - 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} a = b \\ a = -c \\ b = 2c \end{cases} \Rightarrow a = b = c = 0$$

so vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ are linear independent.

so vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\-2\end{pmatrix}$ are linear independent.

In $\mathbb{R}^3$ there cannot be more than 3 linear independent vectors, therefore $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$ must be a basis of $\mathbb{R}^3$

(b) Consider the linear map

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad \begin{pmatrix}a\\b\end{pmatrix} \mapsto \begin{pmatrix}a-b\\b+2a\\3b+2a\end{pmatrix}.$$

Compute the matrices ${}_{S'}M(f)_S$, ${}_{S'}M(f)_B$ and ${}_{B'}M(f)_B$ where S, resp. S', denotes the standard basis of $\mathbb{R}^2$, resp. $\mathbb{R}^3$.

[Remark: If you get some slightly ugly fraction like $\frac{8}{3}$ as matrix entry, don't doubt yourself: That actually happens. Do not write floating point numbers – write fractions!]

$$S=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}, \quad S'=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$f\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}1\\2\\2\end{pmatrix}=1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$f\left(\begin{pmatrix}0\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\1\\3\end{pmatrix}=-1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+3\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\boxed{\text{Therefore} \quad {}_{S'}M(f)_S=\begin{pmatrix}1&-1\\2&1\\2&3\end{pmatrix}}$$

$$B=\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}, \quad S'=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$f\left(\begin{pmatrix}-1\\2\end{pmatrix}\right)=\begin{pmatrix}-1-2\\2+2\cdot(-1)\\3\cdot2+2\cdot(-1)\end{pmatrix}=\begin{pmatrix}-3\\0\\4\end{pmatrix}=-3\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+0\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+4\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$f\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)=\begin{pmatrix}1+1\\-1+2\\-3+2\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}=2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+(-1)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\boxed{\text{Therefore} \quad {}_{S'}M(f)_B=\begin{pmatrix}-3&2\\0&1\\4&-1\end{pmatrix}}$$

$$B=\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}, \quad B'=\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$$

Assume $a,b,c\in\mathbb{R}$

$$f\left(\begin{pmatrix}-1\\2\end{pmatrix}\right)=\begin{pmatrix}-3\\0\\4\end{pmatrix}=a\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}+b\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}+c\cdot\begin{pmatrix}0\\1\\-2\end{pmatrix}$$

$$f\left(\begin{pmatrix}-1\\2\end{pmatrix}\right) = \begin{pmatrix}-3\\0\\4\end{pmatrix} = a \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} + b \cdot \begin{pmatrix}-1\\0\\1\end{pmatrix} + c \cdot \begin{pmatrix}0\\1\\-2\end{pmatrix}$$

$$= \begin{pmatrix}a-b\\a+c\\b-2c\end{pmatrix} \qquad \begin{cases}a-b=-3\\a+c=0\\b-2c=4\end{cases} \qquad \begin{matrix}a=-c\\a-2c=1\end{matrix} \qquad \begin{matrix}-3c=1\\c=-\frac{1}{3}\end{matrix}$$

$$a=\frac{1}{3},\ b=a+3=\frac{8}{3}$$

so $f\left(\begin{pmatrix}-1\\2\end{pmatrix}\right) = \frac{1}{3} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{8}{3} \cdot \begin{pmatrix}-1\\0\\1\end{pmatrix} + \left(-\frac{1}{3}\right) \cdot \begin{pmatrix}0\\1\\-2\end{pmatrix}$

Assume $a', b', c' \in \mathbb{R}$

$$f\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}a'-b'\\a'+c'\\b'-2c'\end{pmatrix} \qquad \begin{cases}a'-b'=2\\a'+c'=1\\b'-2c'=-1\end{cases} \qquad \begin{cases}a'-2c'=1\\a'+c'=1\end{cases} \Rightarrow c'=0,\ a'=1.$$

$$b'=-1$$

so $f\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = 1 \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} + (-1) \cdot \begin{pmatrix}-1\\0\\1\end{pmatrix} + 0 \cdot \begin{pmatrix}0\\1\\2\end{pmatrix}$

$$\boxed{\text{Therefore } {}_{B'}M(f)_B = \begin{pmatrix}\frac{1}{3} & 1\\ \frac{8}{3} & -1\\ -\frac{1}{3} & 0\end{pmatrix}}$$

Exercise 2 (10 points)

Find a polynomial $p(x)$ of degree at most three that satisfies $p(0) = 1$, $p(1) = 1$, $p(2) = 0$, $p(-1) = 1$. Is there more than one such polynomial?

[Hint: A polynomial looks like $p(x) = ax^3 + bx^2 + cx + d$, and you are asked to find the coefficients a, b, c, d. The above values of the polynomial give you four linear equations with the variables a, b, c, d. Again, some slightly ugly fractions may occur.]

Assume $a, b, c, d \in \mathbb{R}$

$$\begin{cases}p(0) = d = 1\\ p(1) = a+b+c+d = 1\\ p(2) = 8a+4b+2c+d = 0\\ p(-1) = -a+b-c+d = 1\end{cases}$$

$$\Rightarrow \begin{cases}a+b+c=1 & ①\\ 8a+4b+2c=-1 & ②\\ -a+b-c=0 & ③\end{cases} \Rightarrow \begin{pmatrix}1 & 1 & 1\\ 8 & 4 & 2\\ -1 & 1 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

① + ③, $2b=1$, $b=\frac{1}{2}$.

$a+c=\frac{1}{2}$ $\qquad$ $a+c=\frac{1}{2}$ $\qquad$ $a=-\frac{2}{3}$

① + ③, $2b = 1$, $b = \frac{1}{2}$.

$$\begin{cases} a + c = \frac{1}{2} \\ 8a + 2c = -3 \end{cases} \Rightarrow \begin{cases} a + c = \frac{1}{2} \\ 4a + c = -\frac{3}{2} \end{cases} \Rightarrow \begin{cases} a = -\frac{2}{3} \\ c = \frac{7}{6} \end{cases}$$

$\Rightarrow a = -\frac{2}{3}$, $b = \frac{1}{2}$, $c = \frac{7}{6}$, $d = 1$

$p(x) = -\frac{2}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}x + 1$, all the coefficients are determined.

There is only one such polynominal.