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Discussion of Stability in a Class of Models on Recurrent Radial Basis Function Neural Networks *

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Abstract: Based on radial basis function neural networks(RBFNNs) and recurrent neural networks (RNNs), a class of new models on recurrent radial basis function neural networks (RRBFNNs) are proposed. These new networks possess the advantages of RBFNNs and RNNs. In this paper, asymptotic stability and learning algorithms of RRBFNNs are researched, and some theorems and formulae are given. Simulation results show that RRBFNNs possess great potentiality in controlling unstable nonlinear system(e. g. the cart and inverted pendulum system).

Key words: recurrent radial basis function neural network; asymptotic stability; nonlinear system; Lyapunov function

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一类递归 RBF 神经网络模型的稳定性讨论

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摘要：在径向基函数神经网络(RBFNN)和递归神经网络(RNN)的基础上，提出了一类新的递归径向基函数神经网络(RRBFNN)模型，它具有两种网络模型的优点．文中对它的渐近稳定性和学习算法进行了研究，并给出相关的定理和公式．仿真结果表明了该神经网络模型在控制不稳定非线性系统(如小车-倒摆系统)具有巨大潜力．

关键词：递归径向基函数神经网络；渐近稳定性；非线性系统；Lyapunov 函数

1 Introduction

Powell[1] proposed radial basis function (RBF) method for strict multivariable function interpolation in 1985. Broomhead and Lowe[2] firstly applied RBF with the neural network design to produce radial basis function neural network (RBFNN) in 1988. Unlike the backpropagation feedforward networks (BPFNs) whose parameters are acquired by nonlinear optimization methods or gradient descent algorithms, the RBFNNs provide some powerful techniques for nonlinear mapping by changing the nonlinear transfer functions of hidden-layer neurons, with the result that the connection weights from the hidden layers to the output layers are linear. Compared with BPFNs, the RBFNNs require less computation time for learning ($10^3 \sim 10^4$ times faster than the BPFNS) and also have more compact topologies. However, when applying RBFNNs to problems involving complicated nonlinear dynamical systems, the number of hidden-layer nodes will increase rapidly, resulting in the network complexity.

Recurrent neural networks (RNNs) with feedback can in some cases provide significant advantages over purely feedforward networks. RNNs can approximate arbitrarily well any dynamical systems[3]. In some cases a small system with feedback is equivalent to a much larger and possibly infinite feedforward system[4]. The feedback allows for recursive computation and the ability to repres-

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ent state information. However, the learning algorithms of RNNs are more complicated and their stability must be seriously considered. Therefore, RNNs has not been nearly as extensively used as feedforward networks.

In this paper, a new class of network models – recurrent radial basis function neural networks (RRBFNNs) that maintain the advantages of RBFNNs and RNNs and overcome their disadvantages discussed above are propo-sed. Some analytical methods and techniques which are simpler than those adopted in conventional RNNs are employed to investigate the sufficient conditions for stability of RRBFNNs. A learning algorithm for the RRBFNNs is deduced from the discussion of their stability.

2 Description of model on RRBFNN

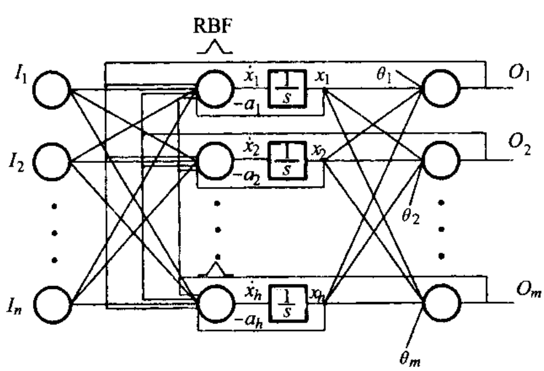

Fig. 1 Structure ofcontinuous time RRBFNN

A RRBFNN is depicted as in Fig. 1. The architecture consists of an input layer, a hidden layer and an output layer. The input layer and the output layer are similar to those of the basic RBFNN. The signals or samples enter the network through the input nodes. The network output is a linear combination of the output of the hidden nodes. The hidden layer which consists of a set of radial basis function neurons is in the closest analogy to that of the basic RNN, but they are not completely the same. The inputs of the hidden layer come from the input layer and the output layer.

RRBFNN's differential equation is described as follows

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \phi(\|\Sigma(t) - C_i\|, \sigma_i) = q_i(x, t), \\ O_j(t) = \sum_{i=1}^{h} w_{ji} x_i(t) + \theta_j, \end{cases} \tag{1}$$

$$k = 1, \cdots, n;\ i = 1, \cdots, h;\ j = 1, \cdots, m,$$

$$\begin{cases} C_i = [C_{Ii} \quad C_{Oi}]^{\mathrm{T}} = \\ \qquad [C_{Ii1} \ \cdots \ C_{Iin} \quad C_{Oi1} \ \cdots \ C_{Oim}]^{\mathrm{T}}, \\ \|\Sigma(t) - C_i\| = \\ \sqrt{\sum_{k=1}^{n}(I_k(t) - C_{Iik})^2 + \sum_{j=1}^{m}(O_j(t) - C_{Oij})^2}, \end{cases} \tag{2}$$

where $x_i(t)$ is the inner state of RRBFNN, a_i is the positive time constant, $\phi(\cdot)$ is the radial basic function, $C_i \in \mathbb{R}^{n+m}$ is the center of the ith hidden node which consists of the input center C_I and the output feedback center C_O, σ_1 is the width, and θ is the threshold. $\|\cdot\|$ denotes the Euclidean norm. $\Sigma \in \mathbb{R}^{n+m}$ includes the network input $I(t)$ and the network output feedback $O(t)$. w_{ji} is the weight connecting the ith hidden neuron with the jth output neuron. For the convenience of writing, we will omit t in the following formulae.

Suppose that there exists one equilibrium point of RRBFNN $X^e = (x_1^e, x_2^e, \cdots, x_h^e)$. It is notable that in this paper we will not discuss the existence of the equilibrium points. Substituting X^e into Eq. (1), yields

$$\begin{cases} -a_i x_i^e + \phi(\|\Sigma^e - C_i\|, \sigma_i) = 0, \\ O_j^e = \sum_{i=1}^{h} w_{ji} x_i^e + \theta_j, \end{cases} \tag{3}$$

$$\|\Sigma^e - C_i\| = \sqrt{\sum_{k=1}^{n}(I_k - C_{Iik})^2 + \sum_{j=1}^{m}(O_j^e - C_{Oij})^2}.$$

By defining

$$z_i = x_i - x_i^e,$$

Eq. (1) is rewritten as follows

$$\begin{cases} \dot{z}_i = (x_i - x_i^e)' = \dot{x}_i = -a_i(z_i + x_i^e) + \phi(\|\Sigma - C_i\|, \sigma_i), \\ O_j = \sum_{i=1}^{h} w_{ji} x_i + \theta_j. \end{cases} \tag{4}$$

Defining

$$\varphi(z_i + x_i^e) = \phi(\|\Sigma - C_i\|, \sigma_i) - \phi(\|\Sigma^e - C_i\|, \sigma_i)$$

and using Eq. (1), Eq. (3) and Eq. (4), yields

$$\begin{cases} \dot{z}_i = -a_i z_i + \varphi(z_i + x_i^e), \\ O_j - O_j^e = \sum_{i=1}^{h} w_{ji} z_i. \end{cases}$$

We can write the above equation more compactly in the following form

$$\begin{cases} \dot{Z} = -TZ + \Psi, \\ O - O^e = WZ, \end{cases} \tag{5}$$

where

$$Z = [z_1 \quad z_2 \quad \cdots \quad z_h]^{\mathrm{T}}, T = \mathrm{diag}[a_1 \quad a_2 \quad \cdots \quad a_h],$$
$$O = [O_1 \quad O_2 \quad \cdots \quad O_m]^{\mathrm{T}}, O^e = [O_1^e \quad O_2^e \quad \cdots \quad O_m^e]^{\mathrm{T}},$$
$$\Psi = [\varphi(z_1 + x_1^e) \quad \varphi(z_2 + x_2^e) \quad \cdots \quad \varphi(z_h + x_h^e)]^{\mathrm{T}},$$
$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1h} \\ w_{21} & w_{22} & \cdots & w_{2h} \\ \square & \square & \square & \square \\ w_{m1} & w_{m2} & \cdots & w_{mh} \end{bmatrix} = [W_1 \quad W_2 \quad \cdots \quad W_h].$$

By the above transformation, the equilibrium point of Eq. (5) is $Z^e = 0$.

3 Asymptotic stability conditions for RRBFNN

Theorem 1 (Global asymptotic stability) The equilibrium point $X = X^e$ of RRBFNN (1) is globally asymptotically stable if the jth element of the ith output feedback center $C_{\mathrm{O}ij}$ satisfies

$$C_{\mathrm{O}ij} = \frac{1}{2}(O_j + O_j^e), \quad (i = 1, \cdots, h; j = 1, \cdots, m)$$

i.e. it equals the mean value of the nonce output and stable output of RRBFNN (1).

Proof For the RRBFNN (5), we choose the positive Lyapunov function $V(Z) = Z^{\mathrm{T}} \cdot Z/2$. The derivative of $V(Z)$ along the solution of Eq. (5) is given by

$$\dot{V}(Z) = Z^{\mathrm{T}}\dot{Z} = Z^{\mathrm{T}}(-TZ + \Psi) = -Z^{\mathrm{T}}TZ + Z^{\mathrm{T}}\Psi,$$

the last term of the right side of the above equation is written as

$$Z^{\mathrm{T}}\Psi = \sum_{i=1}^{h} Z_i [\phi(\| \Sigma - C_i \|, \sigma_i) - \phi(\| \Sigma^e - C_i \|, \sigma_i)] =$$
$$\sum_{i=1}^{h} Z_i [\phi(\| \sqrt{\sum_{k=1}^{n}(I_k - C_{\mathrm{I}ik})^2 + \sum_{j=1}^{m}(O_j - C_{\mathrm{O}ij})^2} \|, \sigma_i) -$$
$$\phi(\| \sqrt{\sum_{k=1}^{n}(I_k - C_{\mathrm{I}ik})^2 + \sum_{j=1}^{m}(O_j^e - C_{\mathrm{O}ij})^2} \|, \sigma_i).$$

Using the condition $C_{\mathrm{O}ij} = (O_j + O_j^e)/2$, i.e. $(O_j - C_{\mathrm{O}ij})^2 = (O_j^e - C_{\mathrm{O}ij})^2 (i = 1, \cdots, h; j = 1, \cdots, m)$, yields $Z^{\mathrm{T}}\Psi = 0$ and $\dot{V}(Z) = -Z^{\mathrm{T}}TZ \leqslant 0$ (only while $Z = 0$, $\dot{V}(Z) = 0$). By the Lyapunov asymptotic stability theorem[5], RRBFNN (1) whose initial state $X^0 \in \Omega = \mathbb{R}^h$ is globally asymptotically stable at $X = X^e$, and thus we complete the proof. Q.E.D.

Remark 1 Physically, when the states of the network are away from the equilibrium point, the systemic energy is larger. By the condition in Theorem 1 and Eq. (2), we can see that the output feedback is strong, thus the states of the network will transfer to the equilibrium point gradually. When the states are very close to the equilibrium point, the output feedback is almost near to zero.

Remark 2 Theorem 1 provides a partly online learning algorithm for RRBFNN. The algorithm possesses great potentiallty in shortening the learning process and strengthening the robustness of the network, which is greatly beneficial to real-time control over fast system. We describe the learning algorithm in two steps: on-line and off-line.

Step 1 Off-line: After we disconnect the output feedback, the input centers C_{I} can be determined by the conventional algorithms for RBFNNs (e. g. Kmeans cluster etc.) Once C_{I} is known, we connect the feedback, thus RRBFNN (1) equals to the basic RNN, so the residual parameters can be trained by the learning algorithms for RNNs (e. g. backpropagation through time and recursive backpropagation etc).

Step 2 On-line: During the real-time control process, the output feedback centers C_{O} are tuned online by the condition of Theorem 1, and the weights w_{ji} are adjusted by recursive LS.

Theorem 2 (Local asymptotic stability) The equilibrium point $X = X^e$ of RRBFNN (1) whose initial state $X^0 \in S(X^e, \varepsilon) \in \mathbb{R}^h$ is locally asymptotically stable if the following inequalities are simultaneously satisfied

1) $$\left| \sum_{j=1}^{m} (O_j^e - C_{\mathrm{O}kj}) w_{jk} \right| \geqslant \left| \sum_{\substack{i=1 \\ i \neq k}}^{h} \sum_{j=1}^{m} (O_j^e - C_{\mathrm{O}ij}) w_{ji} \right|;$$

2) $$\frac{\phi'(\| \Sigma^e - C_k \|, \sigma_k)}{\| \Sigma^e - C_k \|} \sum_{i=1}^{m} (O_k^e - C_{\mathrm{O}ki}) w_{ik} \leqslant 0,$$
$k = 1, \cdots, h$.

Proof $q(x, t)$ in Eq. (1) is continuous and differentiable and there exists arbitrary order derivative in the neighborhood $S(X^e, \varepsilon) \in \mathbb{R}^h$ of the equilibrium point. Then any element $x_i (i = 1, \cdots, h)$ of X satisfies

$$\dot{x}_i = q_i(X^e) + \sum_{j=1}^{h} \frac{\partial q_i}{\partial x_j} |_{x_j = x_j^e} (x_j - x_j^e) + Q(n).$$

It is obvious that high-order differential terms will tend to zero while the states approximate the equilibrium point. Thus the above equation can be written as

$$q_i(X^e) = 0, \quad \dot{x}_i = \sum_{j=1}^{h} \frac{\partial q_i}{\partial x_j} |_{x_j = x_j^e} (x_j - x_j^e),$$

where $\sum_{j=1}^{h}\frac{\partial q_i}{\partial x_j}\mid_{x_j=x_j^e}$ is an element of Jacobian matrix $J(X)$. The nonlinear Eq. (1) is transformed into a linear equation.

$$J(X^e)=\begin{bmatrix}-a_1+b_1^eD_1^eW_1 & b_1^eD_1^eW_2 & \cdots & b_1^eD_1^eW_h\\ b_2^eD_2^eW_1 & -a_2+b_2^eD_2^eW_2 & \cdots & b_2^eD_2^eW_h\\ \square & \square & \square & \square\\ b_h^eD_h^eW_1 & b_h^eD_h^eW_2 & \cdots & -a_h+b_h^eD_h^eW_h\end{bmatrix}$$

where

$$b_i^e=\frac{\phi'(\|\Sigma^e-C_i\|,\sigma_i)}{\|\Sigma^e-C_i\|},$$

$$D_i^e=[O_1^e-C_{Oi1}\quad O_2^e-C_{Oi2}\quad\cdots\quad O_m^e-C_{Oim}],\quad i=1,\cdots,h.$$

Applying Condition (1), yields

$$a_k+\left|-b_k^e\sum_{j=1}^{m}(O_j^e-C_{Okj})w_{jk}\right|>\left|-b_k^e\sum_{\substack{i=1\\i\neq k}}^{h}\sum_{j=1}^{m}(O_j^e-C_{Oij})w_{ji}\right|,\quad k=1,\cdots,h.$$

Applying Condition (2), yields

$$\left|a_k-b_k^e\sum_{j=1}^{m}(O_j^e-C_{Okj})w_{jk}\right|>\left|-b_k^e\sum_{\substack{i=1\\i\neq k}}^{h}\sum_{j=1}^{m}(O_j^e-C_{Oij})w_{ji}\right|,$$

$$a_k>b_k^e\sum_{j=1}^{m}(O_j^e-C_{Okj})w_{jk},\quad k=1,\cdots,h.$$

$-J(X^e)$ is a strictly diagonally dominant matrix, and all its diagonal elements are positive. The real parts of all the eigenvalues of $-J(X^e)$ are positive, thus those of $J(X^e)$ are negative. So RRBFNN (1) is asymptotically stable in the neighborhood $S(X^e,\varepsilon)\in\mathbb{R}^h$. Q.E.D.

Remark 3 The Conditions (1) and (2) can be easy to satisfy. When we design the control system, we wish the attractive region which could be estimated by $|Q(n)|<\delta$ (δ is required precision) would be as large as possible.

4 Simulation experiments

For several decades, the cart and inverted pendulum system has served as an excellent test bed for control theory, because it exhibits fast, multivariable, nonlinear, unstable and non-minimum phase dynamics. The system is shown in Fig. 2. According to dynamic theory and ignoring the friction of cart on table and the friction of pole on cart, we can use the following single input state-space representation to describe the system.

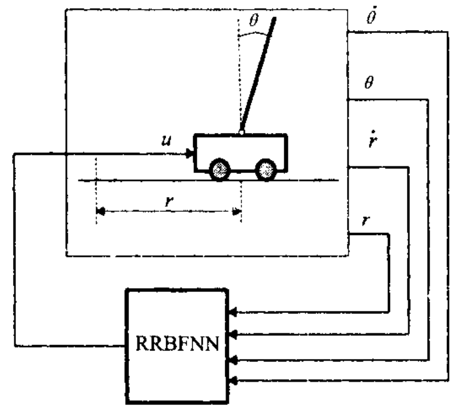

Fig. 2 The cart and inverted pendulum system with RRBFNN controller

$$\begin{cases}\dot X=f(X)+g(X)u,\\ Y=h(X),\end{cases}\tag{6}$$

where

$$X=[x_1\quad x_2\quad x_3\quad x_4]=[r\quad \dot r\quad \theta\quad \dot\theta],$$

$r,\dot r,\theta,\dot\theta$ are respectively the position and velocity of the cart, angle of the pole with the vertical and rate of change of the angle. $f(X)$, $g(X)$ and $h(X)$ are expressed as follows

$$f(X)=\begin{bmatrix}x_2\\ \dfrac{\frac{4}{3}m_plx_4^2\sin x_3-\frac{m_pg}{2}\sin(2x_3)}{\frac{4}{3}(m_c+m_p)-m_p\cos^2x_3}\\ x_4\\ \dfrac{(m_c+m_p)g\sin x_3-\frac{m_pl}{2}x_4^2\sin(2x_3)}{l\left[\frac{4}{3}(m_c+m_p)-m_p\cos^2x_3\right]}\end{bmatrix},$$

$$g(X)=\begin{bmatrix}0\\ \dfrac{4}{3}\dfrac{1}{\frac{4}{3}(m_c+m_p)-m_p\cos^2x_3}\\ 0\\ -\dfrac{\cos x_3}{l\left[\frac{4}{3}(m_c+m_p)-m_p\cos^2x_3\right]}\end{bmatrix},$$

$$h(X)=X.$$

The values used for the constants are $g=9.8\mathrm{m/s^2}$, acceleration due to gravity; $m_c=1.0\mathrm{kg}$, mass of cart; $m_p=0.1\mathrm{kg}$, mass of pole; $l=0.5\mathrm{m}$, distance between the centroid and axis of pole. The force applied to the cart is limited in the range of [−10, 10] Newtons. The RRBFNN controller with 4 input nodes, 4 hidden nodes and 1 output node is shown in Fig. 1. Control objectives are that the pole can vertically stand on the cart and the cart can approach zero position.

By Linearizing Eq. (6) around $X = 0$, linearized system is

$$\begin{cases} \dot{X} = f_0 X + g(0) u, \\ Y = h_0 X. \end{cases}$$

The matrices f_0 and h_0 represent the matrices $f_X = \frac{\partial f_i}{\partial x_j}(i, j = 1, \cdots, n)$ and $h_X = \frac{\partial h_i}{\partial x_j}(i = 1, \cdots, m; j = 1, \cdots, n)$ evaluated at $X = X^e = 0$. Linear dynamic output feedback

$$\begin{cases} \dot{Z} = EZ + FY, \\ u = GZ + HY, \end{cases} \quad (7)$$

applied on the linearized system leads to the closed loop system

$$\begin{cases} \dot{Z} = EZ + Fh_0 X, \\ \dot{X} = g(0) GZ + (f_0 + g(0) Hh_0) X, \end{cases}$$

which is stable if

$$\mathrm{Re}\left\{ \lambda_i \left(\begin{bmatrix} E & Fh_0 \\ g(0) G & f_0 + g(0) Hh_0 \end{bmatrix} \right) \right\} < 0, \quad (8)$$

$$i = 1, \cdots, n + h.$$

Let us assume such a linear stabilizing controller (Eq. (7)) is available, obtained by means of classical, modern or modern robust control techniques, including e. g. PID, LQG, H_2, H_∞ and μ controllers. By virtue of Theorem 2 and Eq. (8), parameters of RRBFNN are constrained as follows:

$$G = [w_1 \quad w_2 \quad \cdots \quad w_h],$$

$$E = -T + \mathrm{diag}[b_1^0 \quad b_2^0 \quad \cdots \quad b_h^0] \cdot \begin{bmatrix} d_1^0 \\ d_2^0 \\ \Box \\ d_h^0 \end{bmatrix} \cdot G,$$

$$F = -\mathrm{diag}[b_1^0 \quad b_2^0 \quad \cdots \quad b_h^0] \cdot [C_{I1} \quad C_{I2} \quad \cdots \quad C_{Ih}]^{\mathrm{T}},$$

$$\theta g_0 = g(0) \cdot H \cdot h_0.$$

Observe that the parameters of RRBFNN are not completely determined by the above constraints. A certain degree of freedom is left, which means that additional requirements could be achieved by the RRBFNN controller, such as enlarging the attractive region and improving the control precision. We will use genetic algorithm to search for the parameters of RRBFNN which are constrained by the above equation. From the same initial state $X^0 = [0, 0, \pi/6, 0]$, Fig. 3 shows the courses of the position r (dotted lines) and the horizontal position of the tip of the pole $X_\theta = r + 2l\sin\theta$ (solid lines) with time under the state feedback controller, RBFNN controller and RRBFNN controller respectively. We see that under RRBFNN controller the transient process of the system is the shortest and stabilized region is the widest.

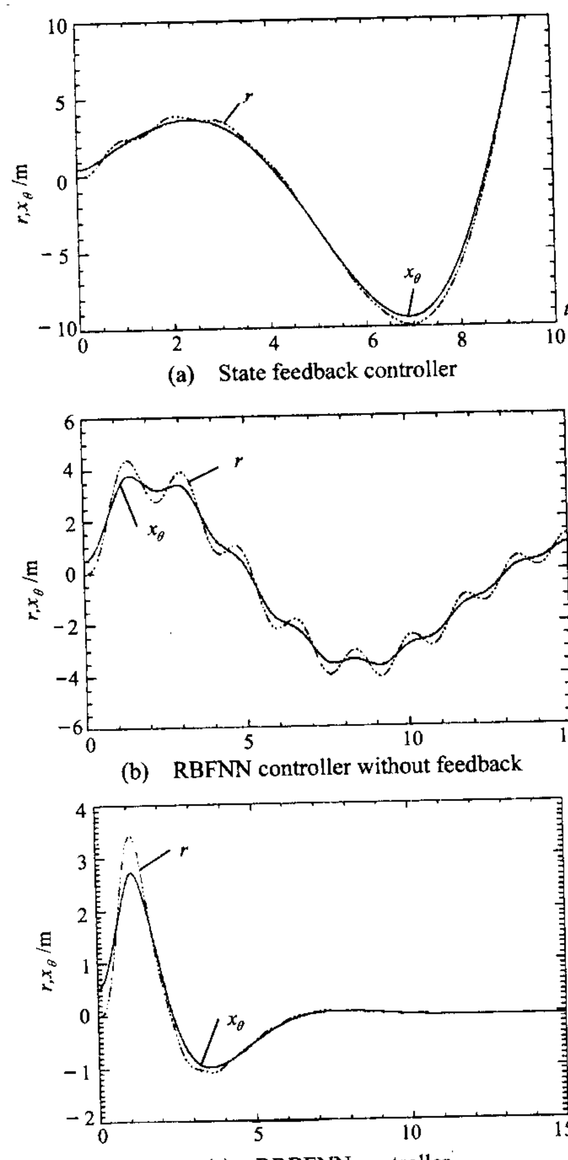

Fig. 3 Response of the cart and inverted pendulum system under different controollers

5 Conclusion

A new class of modes on RRBFNNs are advanced and their asymptotic stability is discussed in this paper. In the inherently unstable nonlinear system, RRBFNN controller

(Continued on page 928)