Discrete Mathematics Lecture Note

TaeYoung Rhee

October 2022

1 Introduction

1.1 Mobius Inversion

Let D_n = all divisor set of positive integer n.
$a, b \in D_n, a \leq b \Leftrightarrow a|b$
Define $\mu : D_n \times D_n \to \mathbb{R}$

$$\mu(a,b) = \begin{cases} (-1)^r & \text{if } a \mid b \text{ and } \frac{b}{a} \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(b) = \mu(1,b) = \begin{cases} (-1)^r & \text{square free} \\ 0 & \text{not square free} \end{cases}$$

Lemma 1. For $m > 1$,

$$\sum_{d \in D_m} \mu(d) = \begin{cases} 1 & (m = 1) \\ 0 & (m \neq 0) \end{cases}$$

$$\sum_{a \leq c \leq b} \mu(a,c) = \begin{cases} 1 & (a = b) \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{d} \mu(d) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i = 0$$

Theorem 1. For function $f, g : D_m \to \mathbb{R}$,

$$g(n) = \sum_{d \in D_n} f(d) \Leftrightarrow f(n) = \sum_{d \in D_n} \mu(d, n) g(n) = \sum \mu(d) g(\frac{n}{d})$$

Proof. ($\Rightarrow$)

$$\begin{aligned} &\sum_{d \in D_n} \mu(d, n) \sum_{e \in D_d} f(e) \\ &= \sum_{e \in D_n} \sum_{e \leq d \leq d_n} \mu(d, n) f(e) \\ &= \sum_{e \in D_n} f(e) \sum_{e \leq d \leq D_n} \mu(d, n) = f(n) \end{aligned}$$

$\square$

Corollary 1.

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \Leftrightarrow n = \sum_{d \in D_n} \phi(d)$$

show that

$$\phi(n) = \sum_{d \in D_n} \mu(d) \frac{n}{d}$$

1.2 Generating Functions

Define power series ring $\mathbb{C}[[x]] = \left\{ \sum_{n \geq 0} a_n x^n | a_n \in \mathbb{C} \ \forall n \right\}$. 'Formal' means evalutation or radius of convergence is ignored. We call x^n is a 'placeholder' of a_n.
$a_n =$ coefficient of x^n
For a sequence $f : N_0 \to \mathbb{C}$ ($f \in \mathbb{C}[[x]]$)
The ordinary generating function is

$$\sum f(n) x^n$$

The exponential generating function is

$$\sum f(n)\frac{x^n}{n!}$$

Let $A(x), B(x)$ ordinary. Then $AB = C$ is ordinary, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ convolution.

Notation. Define $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$ e.g.f for $(1,1,1,\cdots)$
Define $e^{-x} = \sum_{n\geq 0}(-1)^n \frac{x^n}{n!}$
Show that $e^x e^{-x} = \left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{n\geq 0}(-1)^n \frac{x^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n}(-1)^i \binom{n}{i}\right)\frac{x^n}{n!}$ $= 1$.
Define $\log 1 + x = \sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n}$

Example 1. Generating function for $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ with $H_0 = 0$

$$\begin{aligned}\sum_{n\geq 0} H_n x^n &= \left(\sum_{n\geq 1}\frac{1}{n}x^n\right)\left(\sum_{n\geq 0} x^n\right)\\ &= -\log(1-x)\cdot(1-x)^{-1}\\ &= \frac{1}{1-x}\log\frac{1}{1-x}\end{aligned}$$

1.3 Infinite Sums and Products in $\mathbb{C}[[x]]$

Example 2. $p(n) = \#$ of partitions of n

$$\begin{aligned}\sum_{n\geq 0} p(n)x^n &= \prod_{i\geq 1}\frac{1}{1-x^i}\\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdots\\ &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots\end{aligned}$$

Note that the coefficient of x^n in 첫번째줄 requires finite number of factors.

$$\begin{aligned}
&\text{coefficient of } x^n \\
&= \#\text{of solutions}(a_1, a_2, \cdots, a_n) \text{ to } n_1 + 2n_2 + \cdots + = n \\
&= p(n)
\end{aligned}$$

Definition 1. Let $A_0, A_1, \cdots \in \mathbb{C}[[x]]$
$A \in \mathbb{C}[[x]]$, $\deg(A) =$ first power with nonzero coefficient.

The sum $\sum_{i\geq 0} A_i$ exists iff $\deg A_i \to \infty$.

$$\begin{aligned}
A_1 &= (a_{10}, a_{11}, a_{12}, a_{13}, \cdots) \\
A_2 &= (0, a_{21}, a_{22}, a_{23}, \cdots) \\
A_3 &= (9, 0, a_{32}, a_{33}, \cdots)
\end{aligned}$$

We can make each row sum is finite.

Example 3. e^{1+x} is not well-defined.

$$e^{1+x} = 1 + (1+x) + \frac{(1+x)^2}{2!} + \frac{(1+x)^3}{3!} + \cdots$$

$$\begin{aligned}
e^{e^x - 1} &= \sum_{n\geq 0} B(n) \frac{x^n}{n!} \\
&= 1 + \left(\sum_{n\geq 1} \frac{x^n}{n!} \right) + \frac{\left(\sum_{n\geq 1} \frac{x^n}{n!} \right)^2}{2!} + \cdots
\end{aligned}$$

Assume the constant term of each $A_i = 1$. $\prod_{i\geq 1} A_i$ exists iff $\deg(A_i - 1) \to \infty$

Example 4. $(1+x)(1+x^2)(1+x^3)\cdots$ is well defined. $= \sum_{n\geq 0} p_d(n) x^n$

Propositions
(1) $\prod_{i\geq 0} A_i$ and $\prod_{i\geq 0} B_i$ are well-defined $\Rightarrow \prod_{i\geq 0} A_i B_i = \left(\prod_{i\geq 0} A_i\right)\left(\prod_{i\geq 0} B_i\right)$

Proof. $\deg(AB-1) \geq \min\{\deg(A-1), \deg(B-1)\}$ The factors that contribute to x^n are the same on both sides $\square$

(2) $\left(\prod_{i\geq 0} A_i\right)^{-1} = \prod_{i\geq 0} A_i^{-1}$

Proof. dhotldqkfs

$$\deg(A_i - 1) = \deg(A_i^{-1} - 1)$$

example

$$A = \prod_{i\geq 0}(i - x^i) \Rightarrow A^{-1} = \prod_{i\geq 0}\frac{1}{1-x^i}$$

$\square$

(3) $\frac{\prod_{i\geq 0} A_i}{\prod_{i\geq 0} B_i} = \prod_{i\geq 0}\frac{A_i}{B_i}$

Example 5. $\frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-z^i)} = \prod_{i\geq 1} = \prod_{i\geq 1}(1+z^i) = \sum_{n\geq 0} p_d(n)x^n = \prod_{i\geq 1}\frac{1}{1-z^{2i-1}} = \sum_{n\geq 0} p_0(n)x^n$

$p_0(n)$ is number of partitions of n where parts are odd.

1.4 Compositions in $\mathbb{C}[[x]]$

$A(x), B(x) \in \mathbb{C}[[x]]$

$A(B(x))$ is well-defined if either (1) $A(x)$ is a polynomial or (2) the constant term in $B(x) = 0$

Example 6.

$$A(x) = e^x$$
$$B(x) = e^x - 1$$

$A(B(x)) = e^{e^x-1}$.

When $C(x) = x+1$, $A(C(x)) = e^{x+1}$ is not well defined.

1.5 General Powers

Propositions. Given any $\lambda \in \mathbb{C}$, define $\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}$ with $\binom{\lambda}{0} = 1$

Define $(1+x)^\lambda = \sum_{n\geq 0}\binom{\lambda}{x^n} \in \mathbb{C}[[x]]$

If λ is a positive integer, this is just the binomial theorem.

If $A(x) \in \mathbb{C}[[x]]$ with $A(0) = 0$, then $(1 + A(x))^\lambda = \sum_{n\geq 0}\binom{\lambda}{n}A(x)$.

Example 7.

$$(1-x)^{-k} \stackrel{?}{=} \frac{1}{(1-x)^k}$$

Note that

$$\begin{aligned}\binom{-k}{n} &= \frac{-k(-k-1)(-k-2)\cdots(-k-n+1)}{n!} \\ &= \frac{(-1)^n(n+k-1)_n}{n!} \\ &= (-1)^n\binom{n+k-1}{n} = (-1)^n\binom{n+k-1}{k-1}\end{aligned}$$

$$\begin{aligned}(1-x)^{-k} &= \sum_{n\geq 0}\binom{-k}{n}(-1)^n x^n \\ &= \sum_{n\geq 0}\binom{n+k-1}{k-1}x^n = \frac{1}{(1-x)^k}\end{aligned}$$

Proposition. $(1+x)^{\lambda}(1+x)^{\mu} = (1+x)^{\lambda+\mu}$

Example 8.

$$(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = 1+x$$

Proof. Need to show

$$\sum_{i=0}^{n}\binom{\lambda}{i}\binom{\mu}{n-i} = \binom{\lambda+\mu}{\mu} \text{ for all } n \geq 0$$

We can prove it in algebra. Otherwise, we can show it by proving coefficient of x^n for LHS = coefficient of x^n for RHS.
Check that

$$\binom{x+y}{n} = \sum_{i=0}^{n}\binom{x}{i}\binom{y}{n-i} \text{ for all positive integers } x, y$$

Let LHS = $f(x,y)$ and RHS = $g(x,y)$.
$f(x,y) = g(x,y)$ for infinitely many x, y
$f = g$ as polynomials. $\square$

1.6 Catalan Numbers

$$
\begin{aligned}
&c_0 = 0, c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 5 \\
&c_5 = c_1c_4 + c_2c_3 + c_3c_3 + c_4c_1
\end{aligned}
$$

$$c_n = \sum_{i=1}^{n-1} c_i c_{n-i}$$

Let $C(x) = \sum_{n\geq 0} c_n x^n$.

$$
\begin{aligned}
&C(x) = C(x)^2 + x \\
&\Rightarrow C(x)^2 - C(x) = -x \\
&\Rightarrow C(x)^2 - C(x) + \frac{1}{4} = \frac{1}{4} - x \\
&\Rightarrow \left(C(x) - \frac{1}{2}\right)^2 = \frac{1}{4} - x \\
&\Rightarrow C(x) - \frac{1}{2} = \pm\frac{1}{2}(1-4x)^{\frac{1}{2}}
\end{aligned}
$$

Since $C(x) = c_0 = 0$, we get

$$
\begin{aligned}
C(x) - \frac{1}{2} &= -\frac{1}{2}(1-4x)^{\frac{1}{2}} \\
\therefore C(x) &= \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
(1-4x)^{\frac{1}{2}} &= 1 + \sum_{n\geq 1} \binom{\frac{1}{2}}{n} (-1)^n 4^n x^n \\
&= 1 - 2\sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1} x^n
\end{aligned}
$$

Check that

$$\binom{1\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{2^{2n-1}}\frac{1}{n}\binom{2n-2}{n-1}$$

Thus

$$c_n = \frac{1}{n}\binom{2n-2}{n-1}$$

1.7 Other interpretations of Catalan Number

c_n = number of ways to parenthesize a product $x_1 x_2 \cdots x_n$

Example 9. For $x_1x_2x_3x_4$, there are 5 ways.
Key observation is the outermost parenthesis multiples two terms. The first is a product involving $x_1, x_2, \cdots, x_r \to a_r$ ways to do this. The second is a product involving $x_r + 1, x_r + 2, \cdots, x_n \to a_{n-r}$ ways.

number of binary trees with n leaves and 1 root. A tree is binary if every vertex has degree 1 or 3.
A bijections among these sets (triangulations of $n+1$-gon, parenthesized prod-

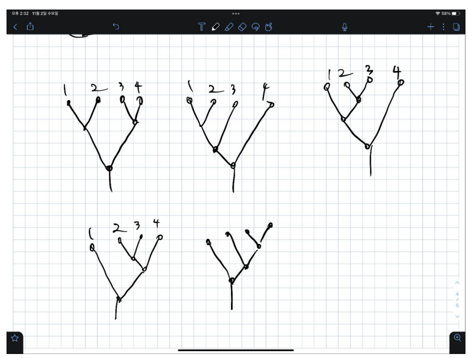

Figure 1: binary tree

ucts of n vatiables, binary trees with n leaves)

Example 10. Figure 2 is the bijection between the sets.

1.8 Product of exponential generating functions

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$
$$B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$$

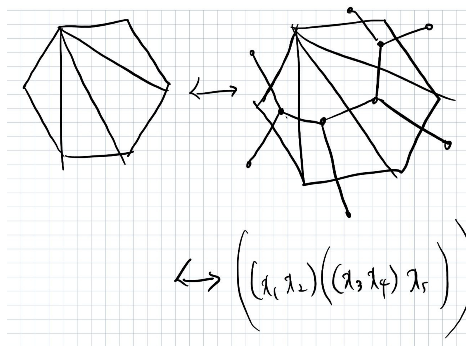

Figure 2: bijections

$$A(x)B(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!} \text{ where } c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$$

It means number of ways to partition on n-set into two ordered blocks and create structure "A" in the first block and structure "B" in the second block.
Remark If "A" = "B", then $\frac{A(x)^2}{2!}$ = e.g.f for the number of ways to create two unordered blocks in an n-set and create structure "A" in each block.
Remark If both blocks of an n-set has to be non-empty then let $a_0 = b_0 = 0$

Example 11. c_n = number of ways to color n labeled balls with red and blue so that an even number of balls are colored red and an odd number of balls are colored blue.
$c_n = 0$ if n is even. $c_n = 2^{n-1}$ if n is odd.
$R(x)$ = e.g.f for $(1, 0, 1, 0, \cdots) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} = \frac{e^x + e^{-x}}{2}$
$B(x)$ = e.g.f for $(0, 1, 0, 1, \cdots) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2}$

$$\begin{aligned}
\sum_{n\geq 0} c_n \frac{x^n}{n!} &= C(x) = R(x) \cdot B(x) \\
&= \frac{e^{2x} - e^{-2x}}{4} \\
&= \frac{1}{4}\left[\left(1 + 2x + \frac{(2x)^2}{2!} + \cdots\right) - \left(1 - 2x + \frac{(2x)^2}{2!} - \cdots\right)\right] \\
&= \sum_{n\geq 0 \ n \text{ odd}} 2^{n-1} \frac{x^n}{n!}
\end{aligned}$$

Example 12. Dearangement (revisited)

$$P(x) = \sum_{n\geq 0} n!\frac{x^n}{n!} = \sum_{n\geq 0} x^n = \frac{1}{1-x} = \text{ e.g.f for } \{|S_n| = n!\}$$

$$\begin{aligned} I(X) &= \sum_{n\geq 0} 1\frac{x^n}{n!} = e^x \\ &= \text{ e.g.f for the number of identity permutations on an } n\text{-set} \\ &= \text{ e.g.f for the sequence } (1, 1, 1, \cdots) \end{aligned}$$

$$D(x) = \sum_{n\geq 0} d_n\frac{x^n}{n!} \text{ where } d_n = \text{ number of dearangements on an } n\text{-set}$$

$P(x) = I(x)D(x)$ because a permutations on an n-set is obtained by

1. partitioning the n-set into two ordered blocks.
2. create the identity permutations on the first block and create a dearangement on the second block.

$$\therefore D(x) = P(x)I(x)^{-1} = \frac{e^{-x}}{1-x}$$

PS3 #7

number of involutions in S_n = number of partial matchings with n verticies = a_n
partial matching = degree 1 graphs = collection of disjoint edges and vertices.
$\sigma \in S_n$ is an involution if $\sigma = id$, i.e., σ is a product of disjoint transpositions cycles of length 2.
partial matchings is

1. Split $[n]$ into two blocks.
2. Create a perfect matching in the first block and leave the second block untouched.

For PS3 #7 (a), check that $a_n = a_{n-1} + (n-1)a_{n-2}$ with $a_0 = a_1 = 1$.
(b)

$$f(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!} = e^{x+\frac{x^2}{2}}$$

$$= 1 + (x + \frac{x^2}{2}) + \frac{\left(x+\frac{x^2}{2}\right)}{2!} + \cdots$$

$$= 1 + x + \sum_{n\geq 2}(a_{n-1} + (n-1)a_{n-2})\frac{x^n}{n!}$$

$$f'(x) = 1 + \sum_{n\geq 2} a_{n-1}\frac{x^{n-1}}{(n-1)!} + \sum_{n\geq 2} a_{n-2}\frac{x^{n-1}}{(n-2)!}$$

$$= 1 + (f(x) - 1) + xf(x)$$

$$\Rightarrow f'(x) = (1+x)f(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = 1 + x$$

$$\Rightarrow f(x) = e^{x+\frac{x^2}{2}}$$

Let $g(x) = e^{x+\frac{x^2}{2}} = \sum_{n\geq 0} b_n \frac{x^n}{n!}$.
Since $g'(x) = (1+x)g(x)$.
We can check that $b_n = b_{n-1} + (n-1)b_{n-2}$ with $b_0 = b_1 = 1$.
Then $\{a_n\}$ and $\{b_n\}$ satisfies the same recurrance relations

$$\therefore \forall n \;\; a_n = b_n$$

Review: $S(n,k)$ = Stirling numbers of the second kind.
$k!S(n,k)$ = number of all surjective functions $f : [n] \to k$ (k fixed)
$= \sum_{(n_1,n_2,\cdots,n_k)} \binom{n}{n_1,n_2,\cdots,n_k}$ where the sum is over all compositions $(n_1, \cdots, n_k)$

of n.

$$\begin{aligned}\Rightarrow S(n,k) &= \frac{1}{k!}\sum_{(n_1,\cdots,n_k)}\binom{n}{n_1,n_2,\cdots,n_k}\\ &= \frac{1}{k!}\times\left(\text{coefficient of } \frac{x^n}{n!} \text{ in } (e^x-1)^k\right)\\ &= \frac{1}{k!}\sum_{(n_1,\cdots,n_k)}\binom{n}{n_1,n_2,\cdots,n_k}\overbrace{1\cdot 1\cdots 1}^{k \text{ times}}\end{aligned}$$

$$\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$$

Corollary 2.

$$\begin{aligned}f_n &= \text{ number of surjections } [n]\to[k]\\ \sum_{n\geq 0} f_n\frac{x^n}{n!} &= (e^x-1)^k\end{aligned}$$

Exercise. Let g_n = number of surjections with $|f^{-1}(i)| \geq 3\ \forall i \in [k]$. Find $sum_{n\geq 0} g_n \frac{x^n}{n!}$.

Proposition 1. Let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ where a_n = number of ways to create structure "A" on an n-set.
Then

$$\frac{A(x)^k}{k!} = \sum_{n\geq 0} b_n \frac{x^n}{n!}$$

where b_n = number of ways to

1. partition an n-set into unordered k blocks
2. create structure "A" in each block.

Proof. clear. ☐

Example 13. $c(n,k)$ = number of ways to create k cycles on $[n]$.
Let $A(x) = \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} = \sum_{n\geq 1}\frac{x^n}{n} = -\log(1-x) = \log(\frac{1}{1-x})$.

$(n-1)!$ is the number of ways to create one cycle of length n

$$\therefore \sum_{n\geq 0} c(n,k)\frac{x^n}{n!} = \frac{\log\left(\frac{1}{1-x}\right)^k}{k!}$$

Theorem 2. Let $A(x) = \sum_{n\geq 1} a_n \frac{x^n}{n!}$ $(n \geq 1)$

$$e^{A(x)} = \exp\left(\sum_{n\geq 1} a_n \frac{x^n}{n!}\right) = \sum_{n\geq 0} b_n \frac{x^n}{n!}\ (b_0 = 1)$$

where b_n = number of ways to partition an n-set into unordered non-empty blocks and create structure "A" in each block.

Proof.

$$e^{A(x)} = 1 + A(x) + \frac{A(x)^2}{2!} + \cdots + \frac{A(x)^k}{k!} + \cdots$$

coefficient of $\frac{x^n}{n!}$ in $e^{A(x)} = \sum_{k\geq 1}$ coefficient of $\frac{x^n}{n!}$ in $\frac{A(x)^k}{k!}$
which means the number of ways to partition an n-set into unordered non-empty blocks and create structure "A" in each block. $\square$

Example 14.

$$\begin{aligned}\sum_{n\geq 0} B(n)\frac{x^n}{n!} &= \sum_{n\geq 0}\left(\sum_{k\geq 0} S(n,k)\right)\frac{x^n}{n!} \\ &= \sum_{k\geq 0}\left(\sum_{n\geq 0} S(n,k)\frac{x^n}{n!}\right) \\ &= \sum_{k\geq 0}\frac{(e^x-1)^k}{k!} = e^{e^x-1}\end{aligned}$$

where $B(n)$ is Bell number.

Example 15.

$$A(x) = \sum_{n\geq 1}\frac{x^n}{n!} = \log\left(\frac{1}{1-x}\right)$$

$$
\begin{aligned}
\frac{1}{1-x} &= \sum n! \frac{x^n}{n!} \\
&= \sum_{n\geq 0} |S_n| \frac{x^n}{n!} \\
&= e^{A(x)} = e^{\log\left(\frac{1}{1-x}\right)} = \frac{1}{1-x}
\end{aligned}
$$

Example 16. number of simple graphs on the vertext set $[n] = 2^{\binom{n}{2}}$. Simple graph is a compound structure given by connected graphs.
graph figre
Let $f(x) = \sum_{n\geq 1} d_n \frac{x^n}{n!}$ where d_n = number of connected graphs on an n-vertext set.
We know that $2^{\binom{n}{2}}$ = number of simple graphs on n-vertices.

$$
\begin{aligned}
e^{f(x)} &= \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!} \quad \left(\binom{0}{2} = \binom{1}{2} = 0\right) \\
f(x) &= \log\left(\sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}\right)
\end{aligned}
$$

$$
\overset{diff}{\Rightarrow} f'(x) = \frac{\mathrm{aaa}'}{\mathrm{aaa}}
$$

$$
\left(\sum_{n\geq 1} n d_n \frac{x^{n-1}}{n!}\right)\left(\sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^{n-1}}{n!}\right) = \sum_{n\geq 0} n 2^{\binom{n}{2}} \frac{x^n}{n!}
$$

By comparing the coefficient of $\frac{x^n}{n!}$ on both sides.

$$
\sum_{i=1}^{n} \binom{n}{i} i \cdot d_i 2^{\binom{n-i}{2}} = n 2^{\binom{n}{2}}
$$

$$
\begin{aligned}
& n = 1 : d_1 = 1 \\
& n = 2 : \binom{2}{1} d_1 + \binom{2}{2} d_2 = 2 \cdot 2 \\
& \Rightarrow d_2 = 1, d_3 = 4
\end{aligned}
$$

Definition 2. Tree = a connected acyclic group

Cayley: number of spanning trees in $K_n = n^{n-2}$

Example 17. $n = 3 : 3^1 = 3$
3 spanning tree grim
$n = 4 : 4^2 = 16$
4 spanning tree grim

Definition 3. Forest = a disjoint union of trees

Example 18. For $n = 3$,
forest grim

Define tree generating function.

$$T(x) = \sum_{n\geq 1} n^{n-2} \frac{x^n}{n!}$$

$$e^{T(x)} = \sum_{n\geq 0} r_n \frac{x^n}{n!}$$

where r_n is the number of spanning forests in K_n

$$e^{T(x)} = 1 + T(x) + \frac{T(x)^2}{2!} + \cdots$$

We can think coefficient of $\frac{x^n}{n!}$ in $\frac{T(x)^k}{k!}$ $(k \geq 0)$
Define f-vector of K_n

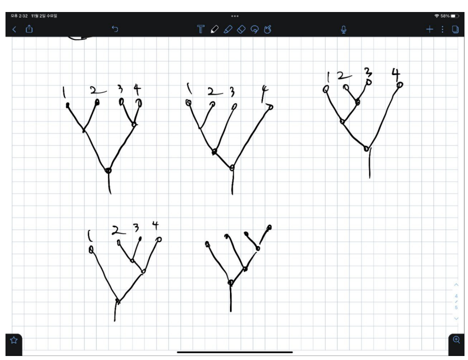

Figure 3: binary tree

$$f_{K_n} = (f_0, f_1, \cdots, f_{n-1})$$
$$f_{K_4} = (1, 6, 15, 16) \ \ (15^2 \geq 6 \cdot 16)$$
$$f_{K_5} = (1, 10, 45, 110, 123) \ \ (45^2 \geq 10 \cdot 110)$$
$$f_{K_6} = (1, 15, 105, 435, 1080, 1296)$$

Log-cocavity of f_{K_n} (proved by Heo)
Define an alternating sum

$n \backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	3	3	1			
4	16	15	6	1		
5	125	110	45	10	1	
6	1296	1080	435	105	15	1

Figure 4

$$\alpha(K_n) = f_{n-1} - f_{n-2} + f_{n-3} - \cdots \pm f_0$$

n	2	3	4	5	6
$\alpha(K_n)$	0	1	6	61	560

$$-e^{-T(x)} = 1 - T(x) + \frac{T(x)^2}{2!} - \cdots = \sum_{n \geq 0} \alpha(K_n) \frac{x^n}{n!}$$

1.9 Generating Functions for Number Partitons

We know $p(n)$ = number of partitons $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$ of n = number of weak solutions to $a_1 + 2a_2 + 3a_3 + \cdots + na_n = n$

Theorem 3.

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1} \frac{1}{1-x^i}$$

Proof.

$$\begin{aligned}\prod_{i\geq 1}\frac{1}{1-x^i} &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdots \\ &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots\end{aligned}$$

coefficient of x^n = number of sequence $(a_1, a_2, \cdots, a_n)$ satisfying $a_1+2a_2+\cdots+na_n = n$ is $p(n)$ □

$p_d(n)$ = number of partitions of n with distinct part.

$$\sum_{n\geq 0} p_d(n)x^n = \prod_{i\geq 1}(1+x^i)$$

$p_o(n)$ = number of partitions of n with parts that are odd only.

$$\sum_{n\geq 0} p_o(n)x^n = \prod i \geq 1 \frac{1}{1-x^{2i-1}}$$

Euler. $p_d(n) = p_o(n)$

Proof.

$$\begin{aligned}\prod_{i\geq 1}(1+x^i) &= (1+x)(1+x^2)(1+x^3)\cdots \\ &= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots \\ &= \prod_{i\geq 1}\frac{1}{1-x^{2i-1}}\end{aligned}$$

□

Combinatirocial proof of $p_d(n) = p_o(n)$

Lemma 2. t_n = number of partitions of n into distinct powers of $2 = 1$ $\forall n$

Proof.

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = 1+x+x^2+\cdots$$

$\square$

Example 19.

$$\begin{aligned}
\lambda &= (12,9,7,6,3,2) \in Par_d(39)\\
12 &= 2^2\cdot 3\\
9 &= 2^0\cdot 9\\
7 &= 2^0\cdot 7\\
6 &= 2^1\cdot 3\\
3 &= 2^0\cdot 3\\
2 &= 2^1\cdot 1
\end{aligned}$$

We can get $\widetilde{\lambda} = (9,7,3,3,3,3,3,3,3,1,1)$. We know how many 3 used and decompose $7 = 2^2+2^1+2^0$.

Example 20. number of $\lambda \vdash n$ s.t. only odd parts may be repeated = number of $\lambda \vdash n$ s.t. no part appears more than 3 times.

The generating function of first part is

$$\begin{aligned}
&\frac{1}{1-x}(1+x^2)\frac{1}{1-x^3}(1+x^4)\cdots\\
&= \frac{1}{1-x}\frac{(1+x^2)(1-x^2)}{1-x^2}\frac{1}{1-x^2}\frac{(1+x^4)(1-x^4)}{(1-x^4)}\cdots\\
&= \frac{1}{1-x}\frac{(1-x^4)}{1-x^2}\frac{1}{1-x^2}\frac{(1-x^8)}{(1-x^4)}\cdots\\
&= (1+x+x^2+x^3)(1+x^2+x^4+x^6)\cdots
\end{aligned}$$

1.10 Durfee Square

$$
\begin{aligned}
p_{sc}(n) &= \text{ number of self-conjugate partitions of } n \\
&= \text{ number of } n \text{ with all parts odd and distinct} \\
p_{sc}(n:d) &= \text{ number of self-conjugate partitions of } n \text{ with } d \times d \text{ Durfee square} \\
&= \text{ number of partitions of } n-d^2 \text{ with the largest part } \geq d \\
&\quad \text{and each part appearing even number of times}
\end{aligned}
$$

$n - d^2$ is the coefficient of x^n in $\frac{x^{d^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2d})}$

$$\therefore p_{sc}(n) = \sum_{d \geq 0} p_{sc}(n:d)$$

$$
\begin{aligned}
\sum_{n \geq 0} p_{sc}(n)x^n &= \prod_{i \geq 1}\left(1 + x^{2i-1}\right) = (1+x)(1+x^3)(1+x^5)\cdots \\
&= \sum_{d \geq 0} \frac{x^{d^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2d})}
\end{aligned}
$$

Suppose $\lambda \vdash n$ is self-conjugate, i.e., $\lambda = \lambda^*$. The Durfee square of λ = the largest square of the form $d \times d$ in the upper left corner of λ

Example 21. $\lambda = 7, 5, 3, 2, 2, 1, 1 \vdash 21$

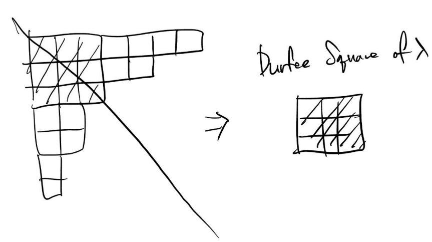

Figure 5: durfee square

1.11 Two Variable Generating Function

Theorem 4.

$$\sum_{n,k\geq 0} p(n,k)x^n t^k = \prod_{i\geq 1} \frac{1}{(1-x^i t)}$$
$$= (1+xt+x^2t^2+\cdots)(1+x^2t+x^4t^2+\cdots)(1+x^3t+x^6t^2+\cdots)\cdots$$

$p(n,k)$ means the number of partitions of n with k parts.

Proof. The coefficient of $x^n t^k$ in the thoerem is the number of solutions to $e_1 + 2e_2 + \cdots + ne_n = n$ which is $p(n,k)$ with $e_1 + e_2 + \cdots + e_n = k$ $\square$

Corollary 3.

$$\prod_{i\geq 1} \frac{1}{1+x^i t} = \sum_{n,k\geq 0} p(n,k)x^n(-t)^k$$
$$= \sum_{n\geq 0} (p_e(n) - p_o(n))\, x^n \text{ with } t = 1$$

Euler's pentagonal number theorem

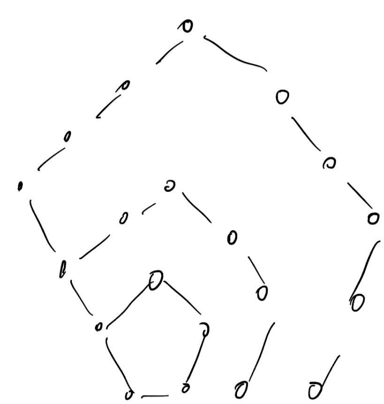

Figure 6: pentagonal number

$$\frac{1}{2}m(3m-1)$$

m	$\frac{1}{2}m(3m-1)$	$\frac{1}{2}m(3m+1)$
1	1	2
2	5	7
3	12	15

$$D(x,t) = \prod_{i\geq 1}(1+x^i t) = \sum_{n\geq 0} p_d(n,k)x^n t^k$$

where $p_d(n,k)$ is the number of $\lambda \vdash n$ with k parts and all parts are distinct.

$$\begin{aligned} D(x,-1) = \prod_{i\geq 1}(1-x^i) &= \sum_{n\geq 0} p_d(n,k)x^n(-1)^k \\ &= \sum_{n\geq 0}(e_n - o_n)x^n \end{aligned}$$

where e_n is $p_d(n,k)$ when $k =$ even, vice versa.

$$\begin{aligned} P(x) &= \prod_{i\geq 1}\frac{1}{1-x^i} \\ &= D(x,-1)^{-1} \end{aligned}$$

Theorem 5.

$$e_n - o_n = \begin{cases} (-1)^m & \text{if } n = \frac{1}{2}m(3m\pm 1)\ (m\geq 1) \\ 0 & \text{otherwise} \end{cases}$$

i.e.

$$\begin{aligned} P(x)^{-1} &= D(x,-1) = Q(x) \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots \end{aligned}$$

$$\therefore P(x)\cdot Q(x) = 1$$

$$\Rightarrow p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots = 0 \ (n \geq 1)$$

Proof. Let E_n = the set of all $\lambda \vdash n$ with even number of parts and all parts are distinct.
Idea. Define $f : E_n \to O_n$ which is a bijection if $n \neq \frac{1}{2}m(3m \pm 1)$ and "almost" a bijection if $n = \frac{1}{2}(3m \pm 1)$

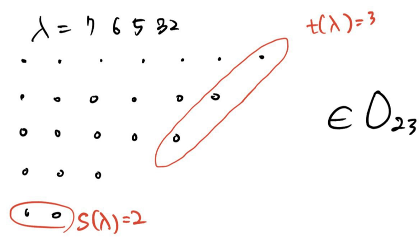

Figure 7: diag

Define $S(\lambda)$ = the smallest part in λ, $t(\lambda)$ = length of the longest NE $\to$ SW diagonal starting at the tip of the largest part.
Case 1. $s(\lambda) \leq t(\lambda)$
Remove $s(\lambda)$ and attach it to the right of $t(\lambda)$

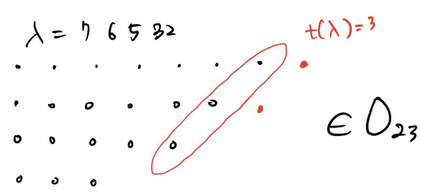

Figure 8: case 1

Case 2. $s(\lambda) \geq t(\lambda)$
Remove $t(\lambda)$ and attach it below $s(\lambda)$

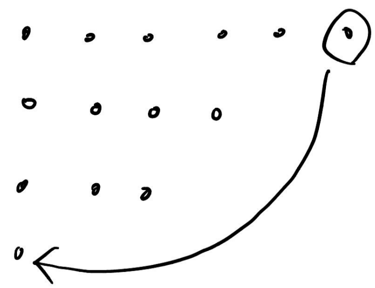

Figure 9: case 2

However, we cannot apply case 1 when $S(\lambda) \cap T(\lambda) \neq \emptyset$ and $s(\lambda) = t(\lambda)$

$$\begin{aligned}\Rightarrow n &= m + (m+1) + \cdots + (2m-1) \\ &= \frac{1}{2}m(3m-1)\end{aligned}$$

Also, we cannot apply case 2 when $S(\lambda) = m+1$ and $t(\lambda) = m$ and $S(\lambda) \cap T(\lambda) \neq \emptyset$

$$\begin{aligned}\Rightarrow n &= (m+1) + (m+2) + \cdots + 2m \\ &= \frac{1}{2}m(3m+1)\end{aligned}$$

$\square$