Discrete Mathematics Lecture Note

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1 Introduction

Notation. Define $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$ e.g.f for $(1, 1, 1, \cdots)$
Define $e^{-x} = \sum_{n\geq 0}(-1)^n \frac{x^n}{n!}$
Show that $e^x e^{-x} = \left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{n\geq 0}(-1)^n \frac{x^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n}(-1)^i \binom{n}{i}\right)\frac{x^n}{n!}$
$= 1$.
Define $\log 1 + x = \sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n}$

Example 1. Generating function for $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ with $H_0 = 0$

$$\begin{aligned}\sum_{n\geq 0} H_n x^n &= \left(\sum_{n\geq 1}\frac{1}{n}x^n\right)\left(\sum_{n\geq 0} x^n\right)\\ &= -\log(1-x)\cdot(1-x)^{-1}\\ &= \frac{1}{1-x}\log\frac{1}{1-x}\end{aligned}$$

1.1 Infinite Sums and Products in $\mathbb{C}[[x]]$

Example 2. $p(n) = \#$ of partitions of n

$$\begin{aligned}\sum_{n\geq 0} p(n)x^n &= \prod_{i\geq 1}\frac{1}{1-x^i}\\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdots\\ &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots\end{aligned}$$

Note that the coefficient of x^n in 첫번째줄 requires finite number of factors.

$$\begin{aligned}&\text{coefficient of } x^n\\&= \#\text{of solutions}(a_1, a_2, \cdots, a_n) \text{ to } n_1 + 2n_2 + \cdots + = n\\&= p(n)\end{aligned}$$

Definition 1. Let $A_0, A_1, \cdots \in \mathbb{C}[[x]]$
$A \in \mathbb{C}[[x]]$, $\deg(A) =$ first power with nonzero coefficient.

The sum $\sum_{i \geq 0} A_i$ exists iff $\deg A_i \to \infty$.

$$\begin{aligned}A_1 &= (a_{10}, a_{11}, a_{12}, a_{13}, \cdots)\\A_2 &= (0, a_{21}, a_{22}, a_{23}, \cdots)\\A_3 &= (9, 0, a_{32}, a_{33}, \cdots)\end{aligned}$$

We can make each row sum is finite.

Example 3. e^{1+x} is not well-defined.

$$e^{1+x} = 1 + (1+x) + \frac{(1+x)^2}{2!} + \frac{(1+x)^3}{3!} + \cdots$$

$$\begin{aligned}e^{e^x - 1} &= \sum_{n \geq 0} B(n) \frac{x^n}{n!}\\&= 1 + \left(\sum_{n \geq 1} \frac{x^n}{n!} \right) + \frac{\left(\sum_{n \geq 1} \frac{x^n}{n!} \right)^2}{2!} + \cdots\end{aligned}$$

Assume the constant term of each $A_i = 1$. $\prod_{i \geq 1} A_i$ exists iff $\deg(A_i - 1) \to \infty$

Example 4. $(1+x)(1+x^2)(1+x^3)\cdots$ is well defined. $= \sum_{n \geq 0} p_d(n) x^n$

Propositions
(1) $\prod_{i \geq 0} A_i$ and $\prod_{i \geq 0} B_i$ are well-defined $\Rightarrow \prod_{i \geq 0} A_i B_i = \left(\prod_{i \geq 0} A_i \right) \left(\prod_{i \geq 0} B_i \right)$

Proof. $\deg(AB - 1) \geq \min\{\deg(A-1), \deg(B-1)\}$ The factors that contribute to x^n are the same on both sides $\square$

(2) $\left(\prod_{i\geq 0} A_i\right)^{-1} = \prod_{i\geq 0} A_i^{-1}$

Proof. dhotldqkfs

$$\deg(A_i - 1) = \deg(A_i^{-1} - 1)$$

example

$$A = \prod_{i\geq 0}(i - x^i) \Rightarrow A^{-1} = \prod_{i\geq 0} \frac{1}{1-x^i}$$

□

(3) $\frac{\prod_{i\geq 0} A_i}{\prod_{i\geq 0} B_i} = \prod_{i\geq 0} \frac{A_i}{B_i}$

Example 5. $\frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-z^i)} = \prod_{i\geq 1} = \prod_{i\geq 1}(1+z^i) = \sum_{n\geq 0} p_d(n)x^n = \prod_{i\geq 1} \frac{1}{1-z^{2i-1}} = \sum_{n\geq 0} p_0(n)x^n$

$p_0(n)$ is number of partitions of n where parts are odd.

1.2 Compositions in $\mathbb{C}[[x]]$

$A(x), B(x) \in \mathbb{C}[[x]]$

$A(B(x))$ is well-defined if either (1) $A(x)$ is a polynomial or (2) the constant term in $B(x) = 0$

Example 6.

$$A(x) = e^x$$
$$B(x) = e^x - 1$$

$A(B(x)) = e^{e^x - 1}$.

When $C(x) = x + 1$, $A(C(x)) = e^{x+1}$ is not well defined.

1.3 General Powers

Propositions. Given any $\lambda \in \mathbb{C}$, define $\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}$ with $\binom{\lambda}{0} = 1$

Define $(1+x)^\lambda = \sum_{n\geq 0} \binom{\lambda}{x^n} \in \mathbb{C}[[x]]$

If λ is a positive integer, this is just the binomial theorem.

If $A(x) \in \mathbb{C}[[x]]$ with $A(0) = 0$, then $(1 + A(x))^\lambda = \sum_{n\geq 0} \binom{\lambda}{n} A(x)$.

Example 7.

$$(1-x)^{-k} \stackrel{?}{=} \frac{1}{(1-x)^k}$$

Note that

$$\begin{aligned}\binom{-k}{n} &= \frac{-k(-k-1)(-k-2)\cdots(-k-n+1)}{n!} \\ &= \frac{(-1)^n(n+k-1)_n}{n!} \\ &= (-1)^n\binom{n+k-1}{n} = (-1)^n\binom{n+k-1}{k-1}\end{aligned}$$

$$\begin{aligned}(1-x)^{-k} &= \sum_{n\geq 0}\binom{-k}{n}(-1)^n x^n \\ &= \sum_{n\geq 0}\binom{n+k-1}{k-1}x^n = \frac{1}{(1-x)^k}\end{aligned}$$

Proposition. $(1+x)^\lambda(1+x)^\mu = (1+x)^{\lambda+\mu}$

Example 8.

$$(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = 1+x$$

Proof. Need to show

$$\sum_{i=0}^{n}\binom{\lambda}{i}\binom{\mu}{n-i} = \binom{\lambda+\mu}{\mu} \text{ for all } n \geq 0$$

We can prove it in algebra. Otherwise, we can show it by proving coefficient of x^n for LHS = coefficient of x^n for RHS.
Check that

$$\binom{x+y}{n} = \sum_{i=0}^{n}\binom{x}{i}\binom{y}{n-i} \text{ for all positive integers } x, y$$

Let LHS = $f(x,y)$ and RHS = $g(x,y)$.
$f(x,y) = g(x,y)$ for infinitely many x, y
$f = g$ as polynomials. $\square$

1.4 Catalan Numbers

$$\begin{aligned}
&c_0 = 0, c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 5 \\
&c_5 = c_1c_4 + c_2c_3 + c_3c_3 + c_4c_1
\end{aligned}$$

$$c_n = \sum_{i=1}^{n-1} c_i c_{n-i}$$

Let $C(x) = \sum_{n\geq 0} c_n x^n$.

$$\begin{aligned}
&C(x) = C(x)^2 + x \\
&\Rightarrow C(x)^2 - C(x) = -x \\
&\Rightarrow C(x)^2 - C(x) + \frac{1}{4} = \frac{1}{4} - x \\
&\Rightarrow \left(C(x) - \frac{1}{2}\right)^2 = \frac{1}{4} - x \\
&\Rightarrow C(x) - \frac{1}{2} = \pm\frac{1}{2}(1-4x)^{\frac{1}{2}}
\end{aligned}$$

Since $C(x) = c_0 = 0$, we get

$$\begin{aligned}
C(x) - \frac{1}{2} &= -\frac{1}{2}(1-4x)^{\frac{1}{2}} \\
\therefore C(x) &= \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
(1-4x)^{\frac{1}{2}} &= 1 + \sum_{n\geq 1} \binom{\frac{1}{2}}{n} (-1)^n 4^n x^n \\
&= 1 - 2\sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1} x^n
\end{aligned}$$

Check that

$$\binom{1\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{2^{2n-1}} \frac{1}{n} \binom{2n-2}{n-1}$$

Thus

$$c_n = \frac{1}{n}\binom{2n-2}{n-1}$$

1.5 Other interpretations of Catalan Number

$c_n =$ number of ways to parenthesize a product $x_1 x_2 \cdots x_n$

Example 9. For $x_1 x_2 x_3 x_4$, there are 5 ways.
Key observation is the outermost parenthesis multiples two terms. The first is a product involving $x_1, x_2, \cdots, x_r \to a_r$ ways to do this. The second is a product involving $x_r + 1, x_r + 2, \cdots, x_n \to a_{n-r}$ ways.