

Discrete Mathematics Lecture Note

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1 Introduction

1.1 Mobius Inversion

Let D_n = all divisor set of positive integer n .

$a, b \in D_n, a \leq b \Leftrightarrow a|b$

Define $\mu : D_n \times D_n \rightarrow \mathbb{R}$

$$\mu(a, b) = \begin{cases} (-1)^r & \text{if } a|b \text{ and } \frac{b}{a} \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(b) = \mu(1, b) = \begin{cases} (-1)^r & \text{square free} \\ 0 & \text{not square free} \end{cases}$$

Lemma 1. For $m > 1$,

$$\sum_{d \in D_m} \mu(d) = \begin{cases} 1 & (m = 1) \\ 0 & (m \neq 1) \end{cases}$$

$$\sum_{a \leq c \leq b} \mu(a, c) = \begin{cases} 1 & (a = b) \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_d \mu(d) = \sum_{i=0}^r \binom{r}{i} (-1)^i = 0$$

Theorem 1. For function $f, g : D_m \rightarrow \mathbb{R}$,

$$g(n) = \sum_{d \in D_n} f(d) \Leftrightarrow f(n) = \sum_{d \in D_n} \mu(d, n) g(n) = \sum \mu(d) g\left(\frac{n}{d}\right)$$

Proof. (\Rightarrow)

$$\begin{aligned} & \sum_{d \in D_n} \mu(d, n) \sum_{e \in D_d} f(e) \\ &= \sum_{e \in D_n} \sum_{e \leq d \leq d_n} \mu(d, n) f(e) \\ &= \sum_{e \in D_n} f(e) \sum_{e \leq d \leq D_n} \mu(d, n) = f(n) \end{aligned}$$

□

Corollary 1.

$$\phi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) \Leftrightarrow n = \sum_{d \in D_n} \phi(d)$$

show that

$$\phi(n) = \sum_{d \in D_n} \mu(d) \frac{n}{d}$$

1.2 Generating Functions

Define power series ring $\mathbb{C}[[x]] = \left\{ \sum_{n \geq 0} a_n x^n \mid a_n \in \mathbb{C} \forall n \right\}$. ‘Formal’ means evaluation or radius of convergence is ignored. We call x^n is a ‘placeholder’ of a_n .
 a_n = coefficient of x^n

For a sequence $f : N_0 \rightarrow \mathbb{C}$ ($f \in \mathbb{C}[[x]]$)

The ordinary generating function is

$$\sum f(n) x^n$$

The exponential generating function is

$$\sum f(n) \frac{x^n}{n!}$$

Let $A(x), B(x)$ ordinary. Then $AB = C$ is ordinary, where $c_n = \sum_{i=0}^n a_i b_{n-i}$ convolution.

Notation. Define $e^x = \sum_{n \geq 0} \frac{x^n}{n!}$ e.g.f for $(1, 1, 1, \dots)$
 Define $e^{-x} = \sum_{n \geq 0} (-1)^n \frac{x^n}{n!}$
 Show that $e^x e^{-x} = \left(\sum_{n \geq 0} \frac{x^n}{n!} \right) \left(\sum_{n \geq 0} (-1)^n \frac{x^n}{n!} \right) = \sum_{n \geq 0} \left(\sum_{i=0}^n (-1)^i \binom{n}{i} \right) \frac{x^n}{n!}$
 $= 1$.
 Define $\log 1 + x = \sum_{n \geq 1} (-1)^{n-1} \frac{x^n}{n}$

Example 1. Generating function for $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ with $H_0 = 0$

$$\begin{aligned} \sum_{n \geq 0} H_n x^n &= \left(\sum_{n \geq 1} \frac{1}{n} x^n \right) \left(\sum_{n \geq 0} x^n \right) \\ &= -\log(1-x) \cdot (1-x)^{-1} \\ &= \frac{1}{1-x} \log \frac{1}{1-x} \end{aligned}$$

1.3 Infinite Sums and Products in $\mathbb{C}[[x]]$

Example 2. $p(n) = \#$ of partitions of n

$$\begin{aligned} \sum_{n \geq 0} p(n) x^n &= \prod_{i \geq 1} \frac{1}{1-x^i} \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdots \\ &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots) \cdots \end{aligned}$$

Note that the coefficient of x^n in 첫번째줄 requires finite number of factors.

$$\begin{aligned} & \text{coefficient of } x^n \\ &= \# \text{ of solutions } (a_1, a_2, \dots, a_n) \text{ to } n_1 + 2n_2 + \dots = n \\ &= p(n) \end{aligned}$$

Definition 1. Let $A_0, A_1, \dots \in \mathbb{C}[[x]]$

$A \in \mathbb{C}[[x]]$, $\deg(A)$ = first power with nonzero coefficient.

The sum $\sum_{i \geq 0} A_i$ exists iff $\deg A_i \rightarrow \infty$.

$$\begin{aligned} A_1 &= (a_{10}, a_{11}, a_{12}, a_{13}, \dots) \\ A_2 &= (0, a_{21}, a_{22}, a_{23}, \dots) \\ A_3 &= (9, 0, a_{32}, a_{33}, \dots) \end{aligned}$$

We can make each row sum is finite.

Example 3. e^{1+x} is not well-defined.

$$e^{1+x} = 1 + (1+x) + \frac{(1+x)^2}{2!} + \frac{(1+x)^3}{3!} + \dots$$

$$\begin{aligned} e^{e^x - 1} &= \sum_{n \geq 0} B(n) \frac{x^n}{n!} \\ &= 1 + \left(\sum_{n \geq 1} \frac{x^n}{n!} \right) + \frac{\left(\sum_{n \geq 1} \frac{x^n}{n!} \right)^2}{2!} + \dots \end{aligned}$$

Assume the constant term of each $A_i = 1$. $\prod_{i \geq 1} A_i$ exists iff $\deg(A_i - 1) \rightarrow \infty$

Example 4. $(1+x)(1+x^2)(1+x^3) \dots$ is well defined. $= \sum_{n \geq 0} p_d(n)x^n$

Propositions

(1) $\prod_{i \geq 0} A_i$ and $\prod_{i \geq 0} B_i$ are well-defined $\Rightarrow \prod_{i \geq 0} A_i B_i = \left(\prod_{i \geq 0} A_i \right) \left(\prod_{i \geq 0} B_i \right)$

Proof. $\deg(AB - 1) \geq \min\{\deg(A - 1), \deg(B - 1)\}$ The factors that contribute to x^n are the same on both sides \square

$$(2) \left(\prod_{i \geq 0} A_i \right)^{-1} = \prod_{i \geq 0} A_i^{-1}$$

Proof. dhotldqkfs

$$\deg(A_i - 1) = \deg(A_i^{-1} - 1)$$

example

$$A = \prod_{i \geq 0} (i - x^i) \Rightarrow A^{-1} = \prod_{i \geq 0} \frac{1}{1 - x^i}$$

□

$$(3) \frac{\prod_{i \geq 0} A_i}{\prod_{i \geq 0} B_i} = \prod_{i \geq 0} \frac{A_i}{B_i}$$

Example 5. $\frac{\prod_{i \geq 1} (1 - x^{2^i})}{\prod_{i \geq 1} (1 - z^i)} = \prod_{i \geq 1} \frac{1 - x^{2^i}}{1 - z^i} = \prod_{i \geq 1} (1 + z^i) = \sum_{n \geq 0} p_d(n) x^n = \prod_{i \geq 1} \frac{1}{1 - z^{2^i - 1}} = \sum_{n \geq 0} p_0(n) x^n$

$p_0(n)$ is number of partitions of n where parts are odd.

1.4 Compositions in $\mathbb{C}[[x]]$

$$A(x), B(x) \in \mathbb{C}[[x]]$$

$A(B(x))$ is well-defined if either (1) $A(x)$ is a polynomial or (2) the constant term in $B(x) = 0$

Example 6.

$$A(x) = e^x$$

$$B(x) = e^x - 1$$

$$A(B(x)) = e^{e^x - 1}.$$

When $C(x) = x + 1$, $A(C(x)) = e^{x+1}$ is not well defined.

1.5 General Powers

Propositions. Given any $\lambda \in \mathbb{C}$, define $\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}$ with $\binom{\lambda}{0} = 1$

Define $(1+x)^\lambda = \sum_{n \geq 0} \binom{\lambda}{n} x^n \in \mathbb{C}[[x]]$

If λ is a positive integer, this is just the binomial theorem.

If $A(x) \in \mathbb{C}[[x]]$ with $A(0) = 0$, then $(1+A(x))^\lambda = \sum_{n \geq 0} \binom{\lambda}{n} A(x)^n$.

Example 7.

$$(1-x)^{-k} \stackrel{?}{=} \frac{1}{(1-x)^k}$$

Note that

$$\begin{aligned} \binom{-k}{n} &= \frac{-k(-k-1)(-k-2)\cdots(-k-n+1)}{n!} \\ &= \frac{(-1)^n(n+k-1)_n}{n!} \\ &= (-1)^n \binom{n+k-1}{n} = (-1)^n \binom{n+k-1}{k-1} \end{aligned}$$

$$\begin{aligned} (1-x)^{-k} &= \sum_{n \geq 0} \binom{-k}{n} (-1)^n x^n \\ &= \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k} \end{aligned}$$

Proposition. $(1+x)^\lambda(1+x)^\mu = (1+x)^{\lambda+\mu}$

Example 8.

$$(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = 1+x$$

Proof. Need to show

$$\sum_{i=0}^n \binom{\lambda}{i} \binom{\mu}{n-i} = \binom{\lambda+\mu}{n} \text{ for all } n \geq 0$$

We can prove it in algebra. Otherwise, we can show it by proving coefficient of x^n for LHS = coefficient of x^n for RHS.

Check that

$$\binom{x+y}{n} = \sum_{i=0}^n \binom{x}{i} \binom{y}{n-i} \text{ for all positive integers } x, y$$

Let LHS = $f(x, y)$ and RHS = $g(x, y)$.

$f(x, y) = g(x, y)$ for infinitely many x, y

$f = g$ as polynomials. □

1.6 Catalan Numbers

$$c_0 = 0, c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 5$$

$$c_5 = c_1 c_4 + c_2 c_3 + c_3 c_2 + c_4 c_1$$

$$c_n = \sum_{i=1}^{n-1} c_i c_{n-i}$$

Let $C(x) = \sum_{n \geq 0} c_n x^n$.

$$\begin{aligned} C(x) &= C(x)^2 + x \\ \Rightarrow C(x)^2 - C(x) &= -x \\ \Rightarrow C(x)^2 - C(x) + \frac{1}{4} &= \frac{1}{4} - x \\ \Rightarrow \left(C(x) - \frac{1}{2}\right)^2 &= \frac{1}{4} - x \\ \Rightarrow C(x) - \frac{1}{2} &= \pm \frac{1}{2}(1 - 4x)^{\frac{1}{2}} \end{aligned}$$

Since $C(x) = c_0 = 0$, we get

$$\begin{aligned} C(x) - \frac{1}{2} &= -\frac{1}{2}(1 - 4x)^{\frac{1}{2}} \\ \therefore C(x) &= \frac{1}{2} - \frac{1}{2}(1 - 4x)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} (1 - 4x)^{\frac{1}{2}} &= 1 + \sum_{n \geq 1} \binom{\frac{1}{2}}{n} (-1)^n 4^n x^n \\ &= 1 - 2 \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1} x^n \end{aligned}$$

Check that

$$\binom{1\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{2^{2n-1}} \frac{1}{n} \binom{2n-2}{n-1}$$

Thus

$$c_n = \frac{1}{n} \binom{2n-2}{n-1}$$

1.7 Other interpretations of Catalan Number

c_n = number of ways to parenthesize a product $x_1x_2\cdots x_n$

Example 9. For $x_1x_2x_3x_4$, there are 5 ways.

Key observation is the outermost parenthesis multiples two terms. The first is a product involving $x_1, x_2, \dots, x_r \rightarrow a_r$ ways to do this. The second is a product involving $x_{r+1}, x_{r+2}, \dots, x_n \rightarrow a_{n-r}$ ways.

number of binary trees with n leaves and 1 root. A tree is binary if every vertex has degree 1 or 3.

A bijections among these sets (triangulations of $n+1$ -gon, parenthesized prod-

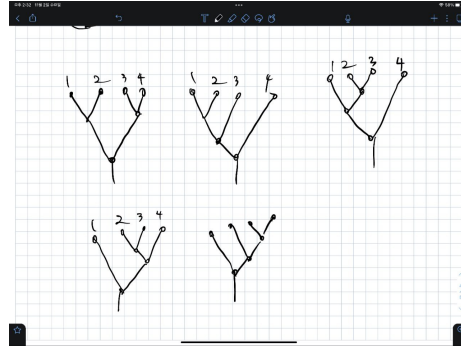


Figure 1: binary tree

ucts of n variables, binary trees with n leaves)

Example 10. Figure 2 is the bijection between the sets.

1.8 Product of exponential generating functions

$$A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$

$$B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$$

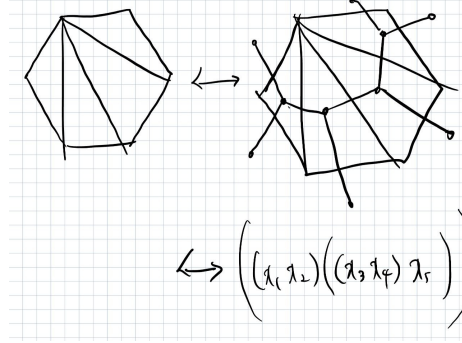


Figure 2: bijections

$$A(x)B(x) = \sum_{n \geq 0} c_n \frac{x^n}{n!} \text{ where } c_n = \sum_{i=0}^n \binom{n}{i} a_i b_{n-i}$$

It means number of ways to partition on n -set into two ordered blocks and create structure “A” in the first block and structure “B” in the second block.

Remark If “A” = “B”, then $\frac{A(x)^2}{2!}$ = e.g.f for the number of ways to create two unordered blocks in an n -set and create structure “A” in each block.

Remark If both blocks of an n -set has to be non-empty then let $a_0 = b_0 = 0$

Example 11. c_n = number of ways to color n labeled balls with red and blue so that an even number of balls are colored red and an odd number of balls are colored blue.

$c_n = 0$ if n is even. $c_n = 2^{n-1}$ if n is odd.

$$R(x) = \text{e.g.f for } (1, 0, 1, 0, \dots) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} = \frac{e^x + e^{-x}}{2}$$

$$B(x) = \text{e.g.f for } (0, 1, 0, 1, \dots) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \frac{e^x - e^{-x}}{2}$$

$$\begin{aligned} \sum_{n \geq 0} c_n \frac{x^n}{n!} &= C(x) = R(x) \cdot B(x) \\ &= \frac{e^{2x} - e^{-2x}}{4} \\ &= \frac{1}{4} \left[\left(1 + 2x + \frac{(2x)^2}{2!} + \dots \right) - \left(1 - 2x + \frac{(2x)^2}{2!} - \dots \right) \right] \\ &= \sum_{n \geq 0, n \text{ odd}} 2^{n-1} \frac{x^n}{n!} \end{aligned}$$

Example 12. Dearangement (revisited)

$$P(x) = \sum_{n \geq 0} n! \frac{x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x} = \text{e.g.f for } \{|S_n| = n!\}$$

$$I(X) = \sum_{n \geq 0} 1 \frac{x^n}{n!} = e^x$$

= e.g.f for the number of identity permutations on an n -set

= e.g.f for the sequence $(1, 1, 1, \dots)$

$$D(x) = \sum_{n \geq 0} d_n \frac{x^n}{n!} \text{ where } d_n = \text{number of dearangements on an } n\text{-set}$$

$P(x) = I(x)D(x)$ because a permutations on an n -set is obtained by

1. partitioning the n -set into two ordered blocks.
2. create the identity permutations on the first block and create a dearangement on the second block.

$$\therefore D(x) = P(x)I(x)^{-1} = \frac{e^{-x}}{1-x}$$

PS3 #7

number of involutions in S_n = number of partial matchings with n verticies = a_n

partial matching = degree 1 graphs = collection of disjoint edges and vertices.

$\sigma \in S_n$ is an involution if $\sigma = id$, i.e., σ is a product of disjoint transpositions cycles of length 2.

partial matchings is

1. Split $[n]$ into two blocks.
2. Create a perfect matching in the first block and leave the second block untouched.

For PS3 #7 (a), check that $a_n = a_{n-1} + (n-1)a_{n-2}$ with $a_0 = a_1 = 1$.
(b)

$$\begin{aligned}
 f(x) &= \sum_{n \geq 0} a_n \frac{x^n}{n!} = e^{x + \frac{x^2}{2}} \\
 &= 1 + \left(x + \frac{x^2}{2}\right) + \frac{\left(x + \frac{x^2}{2}\right)^2}{2!} + \dots \\
 &= 1 + x + \sum_{n \geq 2} (a_{n-1} + (n-1)a_{n-2}) \frac{x^n}{n!}
 \end{aligned}$$

$$\begin{aligned}
 f'(x) &= 1 + \sum_{n \geq 2} a_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n \geq 2} a_{n-2} \frac{x^{n-1}}{(n-2)!} \\
 &= 1 + (f(x) - 1) + xf(x) \\
 &\Rightarrow f'(x) = (1+x)f(x) \\
 &\Rightarrow \frac{f'(x)}{f(x)} = 1+x \\
 &\Rightarrow f(x) = e^{x + \frac{x^2}{2}}
 \end{aligned}$$

Let $g(x) = e^{x + \frac{x^2}{2}} = \sum_{n \geq 0} b_n \frac{x^n}{n!}$.

Since $g'(x) = (1+x)g(x)$.

We can check that $b_n = b_{n-1} + (n-1)b_{n-2}$ with $b_0 = b_1 = 1$.

Then $\{a_n\}$ and $\{b_n\}$ satisfies the same recurrence relations

$$\therefore \forall n \quad a_n = b_n$$

Review: $S(n, k)$ = Stirling numbers of the second kind.

$k!S(n, k)$ = number of all surjective functions $f : [n] \rightarrow k$ (k fixed)

$= \sum_{(n_1, n_2, \dots, n_k)} \binom{n}{n_1, n_2, \dots, n_k}$ where the sum is over all compositions (n_1, \dots, n_k)

of n .

$$\begin{aligned}\Rightarrow S(n, k) &= \frac{1}{k!} \sum_{(n_1, \dots, n_k)} \binom{n}{n_1, n_2, \dots, n_k} \\ &= \frac{1}{k!} \times \left(\text{coefficient of } \frac{x^n}{n!} \text{ in } (e^x - 1)^k \right) \\ &= \frac{1}{k!} \sum_{(n_1, \dots, n_k)} \binom{n}{n_1, n_2, \dots, n_k} \overbrace{1 \cdot 1 \cdots 1}^{k \text{ times}}\end{aligned}$$

$$\sum_{n \geq 0} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$$

Corollary 2.

$f_n =$ number of surjections $[n] \rightarrow [k]$

$$\sum_{n \geq 0} f_n \frac{x^n}{n!} = (e^x - 1)^k$$

Exercise. Let $g_n =$ number of surjections with $|f^{-1}(i)| \geq 3 \ \forall i \in [k]$. Find $\sum_{n \geq 0} g_n \frac{x^n}{n!}$.

Proposition 1. Let $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$ where $a_n =$ number of ways to create structure “ A ” on an n -set.

Then

$$\frac{A(x)^k}{k!} = \sum_{n \geq 0} b_n \frac{x^n}{n!}$$

where $b_n =$ number of ways to

1. partition an n -set into unordered k blocks
2. create structure “ A ” in each block.

Proof. clear. □

Example 13. $c(n, k) =$ number of ways to create k cycles on $[n]$.

Let $A(x) = \sum_{n \geq 1} (n-1)! \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n} = -\log(1-x) = \log\left(\frac{1}{1-x}\right)$.

$(n-1)!$ is the number of ways to create one cycle of length n

$$\therefore \sum_{n \geq 0} c(n, k) \frac{x^n}{n!} = \frac{\log \left(\frac{1}{1-x} \right)^k}{k!}$$

Theorem 2. Let $A(x) = \sum_{n \geq 1} a_n \frac{x^n}{n!}$ ($n \geq 1$)

$$e^{A(x)} = \exp \left(\sum_{n \geq 1} a_n \frac{x^n}{n!} \right) = \sum_{n \geq 0} b_n \frac{x^n}{n!} \quad (b_0 = 1)$$

where b_n = number of ways to partition an n -set into unordered non-empty blocks and create structure “ A ” in each block.

Proof.

$$e^{A(x)} = 1 + A(x) + \frac{A(x)^2}{2!} + \cdots + \frac{A(x)^k}{k!} + \cdots$$

coefficient of $\frac{x^n}{n!}$ in $e^{A(x)} = \sum_{k \geq 1} \text{coefficient of } \frac{x^n}{n!} \text{ in } \frac{A(x)^k}{k!}$

which means the number of ways to partition an n -set into unordered non-empty blocks and create structure “ A ” in each block. \square

Example 14.

$$\begin{aligned} \sum_{n \geq 0} B(n) \frac{x^n}{n!} &= \sum_{n \geq 0} \left(\sum_{k \geq 0} S(n, k) \right) \frac{x^n}{n!} \\ &= \sum_{k \geq 0} \left(\sum_{n \geq 0} S(n, k) \frac{x^n}{n!} \right) \\ &= \sum_{k \geq 0} \frac{(e^x - 1)^k}{k!} = e^{e^x - 1} \end{aligned}$$

where $B(n)$ is Bell number.

Example 15.

$$A(x) = \sum_{n \geq 1} \frac{x^n}{n!} = \log \left(\frac{1}{1-x} \right)$$

$$\begin{aligned}
\frac{1}{1-x} &= \sum n! \frac{x^n}{n!} \\
&= \sum_{n \geq 0} |S_n| \frac{x^n}{n!} \\
&= e^{A(x)} = e^{\log(\frac{1}{1-x})} = \frac{1}{1-x}
\end{aligned}$$

Example 16. number of simple graphs on the vertex set $[n] = 2^{\binom{n}{2}}$. Simple graph is a compound structure given by connected graphs.

graph figure

Let $f(x) = \sum_{n \geq 1} d_n \frac{x^n}{n!}$ where d_n = number of connected graphs on an n -vertex set.

We know that $2^{\binom{n}{2}}$ = number of simple graphs on n -vertices.

$$\begin{aligned}
e^{f(x)} &= \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!} \quad \left(\binom{0}{2} = \binom{1}{2} = 0 \right) \\
f(x) &= \log \left(\sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!} \right) \\
\stackrel{diff}{\Rightarrow} f'(x) &= \frac{aaa'}{aaa}
\end{aligned}$$

$$\left(\sum_{n \geq 1} n d_n \frac{x^{n-1}}{n!} \right) \left(\sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^{n-1}}{n!} \right) = \sum_{n \geq 0} n 2^{\binom{n}{2}} \frac{x^n}{n!}$$

By comparing the coefficient of $\frac{x^n}{n!}$ on both sides.

$$\sum_{i=1}^n \binom{n}{i} i \cdot d_i 2^{\binom{n-i}{2}} = n 2^{\binom{n}{2}}$$

$$n = 1 : d_1 = 1$$

$$n = 2 : \binom{2}{1} d_1 + \binom{2}{2} d_2 = 2 \cdot 2$$

$$\Rightarrow d_2 = 1, d_3 = 4$$

Definition 2. Tree = a connected acyclic group

Cayley: number of spanning trees in $K_n = n^{n-2}$

Example 17. $n = 3 : 3^1 = 3$

3 spanning tree grim

$n = 4 : 4^2 = 16$

4 spanning tree grim

Definition 3. Forest = a disjoint union of trees

Example 18. For $n = 3$,

forest grim

Define tree generating function.

$$T(x) = \sum_{n \geq 1} n^{n-2} \frac{x^n}{n!}$$

$$e^{T(x)} = \sum_{n \geq 0} r_n \frac{x^n}{n!}$$

where r_n is the number of spanning forests in K_n

$$e^{T(x)} = 1 + T(x) + \frac{T(x)^2}{2!} + \dots$$

We can think coefficient of $\frac{x^n}{n!}$ in $\frac{T(x)^k}{k!}$ ($k \geq 0$)

Define f -vector of K_n

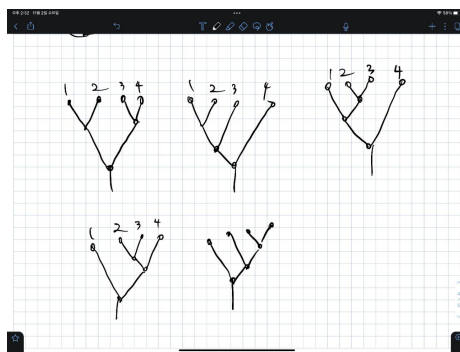


Figure 3: binary tree

$$\begin{aligned}
f_{K_n} &= (f_0, f_1, \dots, f_{n-1}) \\
f_{K_4} &= (1, 6, 15, 16) \quad (15^2 \geq 6 \cdot 16) \\
f_{K_5} &= (1, 10, 45, 110, 123) \quad (45^2 \geq 10 \cdot 110) \\
f_{K_6} &= (1, 15, 105, 435, 1080, 1296)
\end{aligned}$$

Log-cocavity of f_{K_n} (proved by Heo)

Define an alternating sum

$n \backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	3	3	1			
4	16	15	6	1		
5	125	110	45	10	1	
6	1296	1080	435	105	15	1

Figure 4

$$\alpha(K_n) = f_{n-1} - f_{n-2} + f_{n-3} - \dots \pm f_0$$

n	2	3	4	5	6
$\alpha(K_n)$	0	1	6	61	560

$$-e^{-T(x)} = 1 - T(x) + \frac{T(x)^2}{2!} - \dots = \sum_{n \geq 0} \alpha(K_n) \frac{x^n}{n!}$$

1.9 Generating Functions for Number Partitons

We know $p(n)$ = number of partitons $\lambda = \lambda_1 \geq \lambda_2 \geq \dots$ of n = number of weak solutions to $a_1 + 2a_2 + 3a_3 + \dots + na_n = n$

Theorem 3.

$$\sum_{n \geq 0} p(n)x^n = \prod_{i \geq 1} \frac{1}{1-x^i}$$

Proof.

$$\begin{aligned} \prod_{i \geq 1} \frac{1}{1-x^i} &= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdots \\ &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots) \cdots \end{aligned}$$

coefficient of x^n = number of sequence (a_1, a_2, \dots, a_n) satisfying $a_1 + 2a_2 + \cdots + na_n = n$ is $p(n)$ \square

$p_d(n)$ = number of partitions of n with distinct part.

$$\sum_{n \geq 0} p_d(n)x^n = \prod_{i \geq 1} (1+x^i)$$

$p_o(n)$ = number of partitions of n with parts that are odd only.

$$\sum_{n \geq 0} p_o(n)x^n = \prod_{i \geq 1} \frac{1}{1-x^{2i-1}}$$

Euler. $p_d(n) = p_o(n)$

Proof.

$$\begin{aligned} \prod_{i \geq 1} (1+x^i) &= (1+x)(1+x^2)(1+x^3) \cdots \\ &= \frac{1-x^2}{1-x} \frac{1-x^4}{1-x^2} \frac{1-x^6}{1-x^3} \cdots \\ &= \prod_{i \geq 1} \frac{1}{1-x^{2i-1}} \end{aligned}$$

\square

Combinatorial proof of $p_d(n) = p_o(n)$

Lemma 2. t_n = number of partitions of n into distinct powers of 2 = 1 $\forall n$

Proof.

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = 1+x+x^2+\cdots$$

□

Example 19.

$$\lambda = (12, 9, 7, 6, 3, 2) \in Par_d(39)$$

$$12 = 2^2 \cdot 3$$

$$9 = 2^0 \cdot 9$$

$$7 = 2^0 \cdot 7$$

$$6 = 2^1 \cdot 3$$

$$3 = 2^0 \cdot 3$$

$$2 = 2^1 \cdot 1$$

We can get $\tilde{\lambda} = (9, 7, 3, 3, 3, 3, 3, 3, 1, 1)$. We know how many 3 used and decompose $7 = 2^2 + 2^1 + 2^0$.

Example 20. number of $\lambda \vdash n$ s.t. only odd parts may be repeated = number of $\lambda \vdash n$ s.t. no part appears more than 3 times.

The generating function of first part is

$$\begin{aligned} & \frac{1}{1-x}(1+x^2)\frac{1}{1-x^3}(1+x^4)\cdots \\ &= \frac{1}{1-x}\frac{(1+x^2)(1-x^2)}{1-x^2}\frac{1}{1-x^2}\frac{(1+x^4)(1-x^4)}{(1-x^4)}\cdots \\ &= \frac{1}{1-x}\frac{(1-x^4)}{1-x^2}\frac{1}{1-x^2}\frac{(1-x^8)}{(1-x^4)}\cdots \\ &= (1+x+x^2+x^3)(1+x^2+x^4+x^6)\cdots \end{aligned}$$

1.10 Durfee Square

$p_{sc}(n)$ = number of self-conjugate partitions of n

= number of n with all parts odd and distinct

$p_{sc}(n : d)$ = number of self-conjugate partitions of n with $d \times d$ Durfee square

= number of partitions of $n - d^2$ with the largest part $\geq d$

and each part appearing even number of times

$n - d^2$ is the coefficient of x^n in $\frac{x^{d^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2d})}$

$$\therefore p_{sc}(n) = \sum_{d \geq 0} p_{sc}(n : d)$$

$$\sum_{n \geq 0} p_{sc}(n) x^n = \prod_{i \geq 1} (1 + x^{2i-1}) = (1+x)(1+x^3)(1+x^5) \cdots$$

$$= \sum_{d \geq 0} \frac{x^{d^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2d})}$$

Suppose $\lambda \vdash n$ is self-conjugate, i.e., $\lambda = \lambda^*$. The Durfee square of λ = the largest square of the form $d \times d$ in the upper left corner of λ

Example 21. $\lambda = 7, 5, 3, 2, 2, 1, 1 \vdash 21$

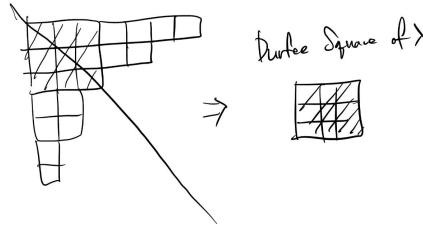


Figure 5: durfee square

1.11 Two Variable Generating Function

Theorem 4.

$$\sum_{n,k \geq 0} p(n,k) x^n t^k = \prod_{i \geq 1} \frac{1}{(1 - x^i t)} \\ = (1 + xt + x^2 t^2 + \dots)(1 + x^2 t + x^4 t^2 + \dots)(1 + x^3 t + x^6 t^2 + \dots) \dots$$

$p(n,k)$ means the number of partitions of n with k parts.

Proof. The coefficient of $x^n t^k$ in the theorem is the number of solutions to $e_1 + 2e_2 + \dots + ne_n = n$ which is $p(n,k)$ with $e_1 + e_2 + \dots + e_n = k$ \square

Corollary 3.

$$\prod_{i \geq 1} \frac{1}{1 + x^i t} = \sum_{n,k \geq 0} p(n,k) x^n (-t)^k \\ = \sum_{n \geq 0} (p_e(n) - p_o(n)) x^n \text{ with } t = 1$$

Euler's pentagonal number theorem

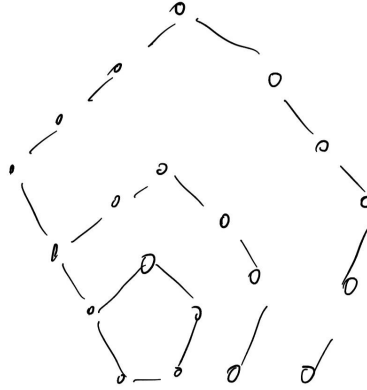


Figure 6: pentagonal number

$$\frac{1}{2}m(3m-1)$$

m	$\frac{1}{2}m(3m-1)$	$\frac{1}{2}m(3m+1)$
1	1	2
2	5	7
3	12	15

$$D(x, t) = \prod_{i \geq 1} (1 + x^i t) = \sum_{n \geq 0} p_d(n, k) x^n t^k$$

where $p_d(n, k)$ is the number of $\lambda \vdash n$ with k parts and all parts are distinct.

$$\begin{aligned} D(x, -1) &= \prod_{i \geq 1} (1 - x^i) = \sum_{n \geq 0} p_d(n, k) x^n (-1)^k \\ &= \sum_{n \geq 0} (e_n - o_n) x^n \end{aligned}$$

where e_n is $p_d(n, k)$ when k = even, vice versa.

$$\begin{aligned} P(x) &= \prod_{i \geq 1} \frac{1}{1 - x^i} \\ &= D(x, -1)^{-1} \end{aligned}$$

Theorem 5.

$$e_n - o_n = \begin{cases} (-1)^m & \text{if } n = \frac{1}{2}m(3m \pm 1) \ (m \geq 1) \\ 0 & \text{otherwise} \end{cases}$$

i.e.

$$\begin{aligned} P(x)^{-1} &= D(x, -1) = Q(x) \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \dots \end{aligned}$$

$$\therefore P(x) \cdot Q(x) = 1$$

$$\Rightarrow p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \dots = 0 \quad (n \geq 1)$$

Proof. Let E_n = the set of all $\lambda \vdash n$ with even number of parts and all parts are distinct.

Idea. Define $f : E_n \rightarrow O_n$ which is a bijection if $n \neq \frac{1}{2}m(3m \pm 1)$ and “almost” a bijection if $n = \frac{1}{2}(3m \pm 1)$

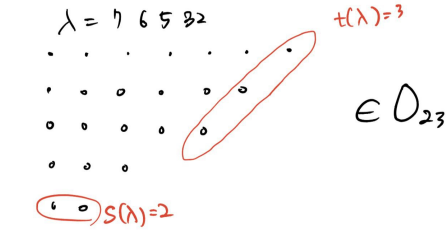


Figure 7: diag

Define $S(\lambda)$ = the smallest part in λ , $t(\lambda)$ = length of the longest NE \rightarrow SW diagonal starting at the tip of the largest part.

Case 1. $s(\lambda) \leq t(\lambda)$

Remove $s(\lambda)$ and attach it to the right of $t(\lambda)$

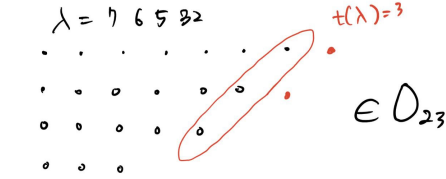


Figure 8: case 1

Case 2. $s(\lambda) \geq t(\lambda)$

Remove $t(\lambda)$ and attach it below $s(\lambda)$

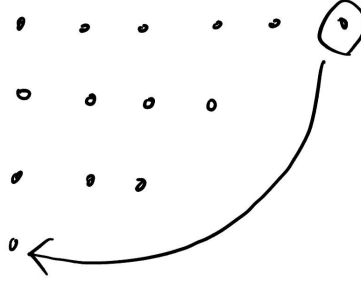


Figure 9: case 2

However, we cannot apply case 1 when $S(\lambda) \cap T(\lambda) \neq \emptyset$ and $s(\lambda) = t(\lambda)$

$$\begin{aligned} \Rightarrow n &= m + (m + 1) + \cdots + (2m - 1) \\ &= \frac{1}{2}m(3m - 1) \end{aligned}$$

Also, we cannot apply case 2 when $S(\lambda) = m + 1$ and $t(\lambda) = m$ and $S(\lambda) \cap T(\lambda) \neq \emptyset$

$$\begin{aligned} \Rightarrow n &= (m + 1) + (m + 2) + \cdots + 2m \\ &= \frac{1}{2}m(3m + 1) \end{aligned}$$

□

2 Graphs

2.1 Königberg's Bridges

Reduced Euler characteristic

$$\tilde{\chi} = 1 - \text{number of vertices} + \text{number of edges}$$

Definition 4. A finite graph $G = (V, E)$ consists of

- A finite vertex set $V = V(G)$
- A finite edge (multi)set $E = E(G)$

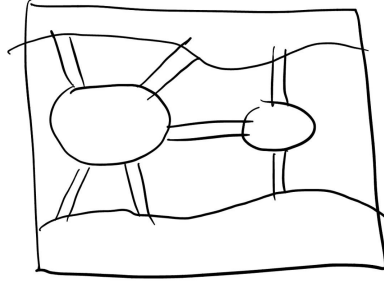


Figure 10: bridges

An edge $e \in E$ is an unordered pair of vertices. We write $e = \{v, w\} = vw$ where $v, w \in V$

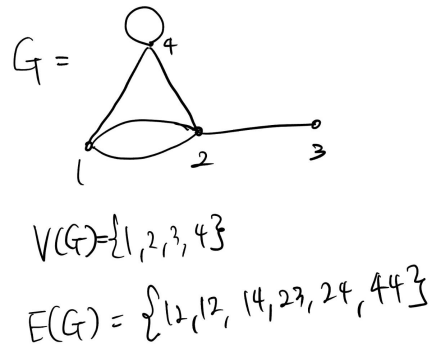


Figure 11: graph example

Example 22. We call $\{12, 12\}$ parallel edges, $\{44\}$ loop, and $\{14, 23, 34\}$ simple edge.

$$e = vw \in E \Leftrightarrow \begin{cases} v \text{ and } w \text{ are adjacent} \\ e \text{ is incident with } v \text{ and } w \\ v \text{ and } w \text{ are endpoints of } e \\ e \text{ joins } v \text{ and } w \end{cases}$$

Definition 5. Adjacency matrix $A(G) = (a_{ij})$ is $n \times n$ where $n = |V(G)|$

$a_{ij} = a_{ji}$ = number of edges joining vertices i and j ($i \neq j$)

$a_{ii} = 0$ (if G is loopless) ($a_{ii} = 2 \times$ number of loops at i)

Definition 6. Degree matrix $D(G) = (d_{ij})$. d_{ii} = degree of vertex i . $d_{ij} = 0$

Example 23. For finite graph G , Adjacency matrix is

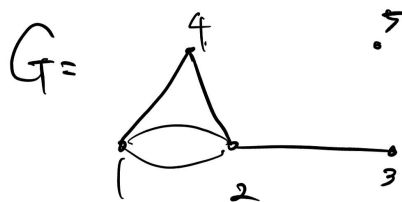


Figure 12: graph example 2

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

v	1	2	3	4	5
$\deg v$	3	4	1	4	0

Theorem 6. A finite graph G has an even number of odd degree vertices.

Proof. Suffices to show

$$\sum_{v \in V(G)} \deg v = \text{even integer}$$

Indeed

$$\sum_{v \in G} \deg v = 2|E|$$

because every edge $e \in E$ contributes 2. □

Definition 7. A finite graph G is simple if G has no loops and no multiple edges.

Example 24. The complete graph K_n

$$V(K_n) = [n]$$

$$E(K_n) = \{ij \mid 1 \leq i < j \leq n\} = \binom{[n]}{2}$$

$$|E(K_n)| = \binom{n}{2}$$

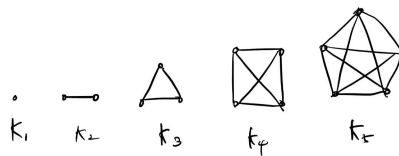


Figure 13: complete graph

We can order edges lexicographically.

$$E(K_5) = 12, 13, 14, 15, 23, 24, 25, 34, 35, 45$$

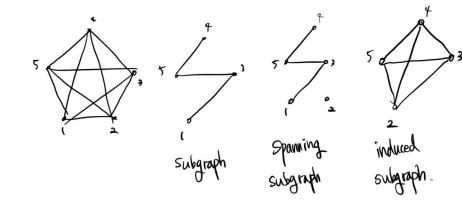


Figure 14: subgraphs

Definition 8. • H is a subgraph of G , $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

- H is a spanning subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$
- H is an induced subgraph of G if $V(H) \subseteq V(G)$ and $ab \in E(H) \Leftrightarrow ab \in E(G)$

A simple graph has two vertices of the same degree (PHP)

Definition 9. A path in a graph G from a to b ($a, b \in V(G)$) is a subgraph consisting of

1. distinct vertices $a = v_0, v_1, \dots, v_k = b$
2. distinct edges $v_0v_1, v_1v_2, \dots, v_{k-1}v_k$

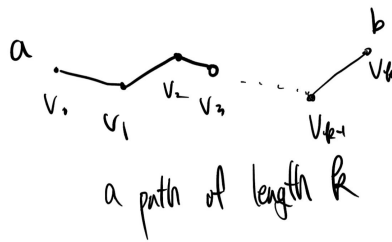


Figure 15: path

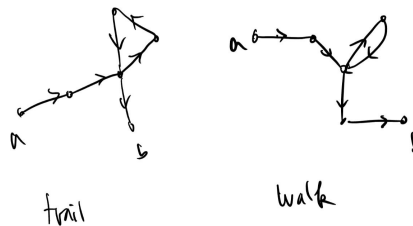


Figure 16: trails and walk

Definition 10. G is connected if a path exists between any pair of vertices in G .

Definition 11. A cycle is a “closed” path, i.e., $a = b$.

Definition 12. A component of G is a connected induced subgraph that is maximal under inclusion.

component grim

Definition 13. Eulerian circuit in G is a circuit that contains all of the edges in G (exactly once).

Definition 14. Eulerian graph is a graph with an Eulerian circuit.

Definition 15. Hamiltonian cycle in G is a cycle that contains all of the vertices in G .

Example 25. K_4 has 3 Hamiltonian cycle.

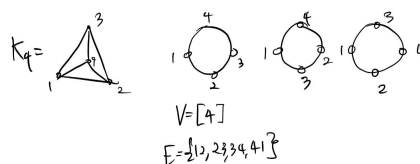


Figure 17: Hamiltonian cycle

Theorem 7. A finite graph G with no isolated vertex is Eulerian if and only If G is connected and every vertex of G has an even degree.

Proof. (\Rightarrow) Let G Eulerian.

There exists a circuit that uses every edge in G . Every vertex of G has an even degree.

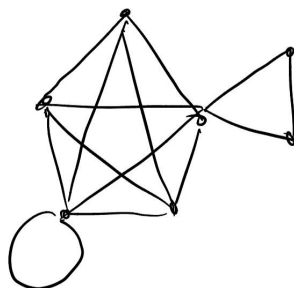


Figure 18

(\Leftarrow) Pick a vertex v . Construct a circuit starting and ending at v using all edges incident to v . Call it C .

Then remove all the edges in C from G . Let $G \setminus C = G_1 \sqcup G_2 \sqcup \dots \sqcup G_m$ when C_i 's are components in $G \setminus C$. Assume G_1, G_2, \dots, G_m appear in this order along the circuit C . Note that in each G_i , every vertex has an even degree. By induction on the number of edges in G , each G_i contains an Eulerian circuit

call it C_i .

Then $C \cup C_1 \cup C_2 \cup \dots \cup C_m$ is an Eulerian circuit in G . □

Definition 16. A graph is acyclic if it contains no cycle.

Example 26. Acyclic graph

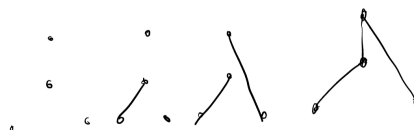


Figure 19: acyclic graph

A connected acyclic graph is a tree. Hence an acyclic graph is a disjoint union of trees. An acyclic graph is also called a forest.

Remark. In an acyclic graph, there exists a unique path between two distinct vertices.

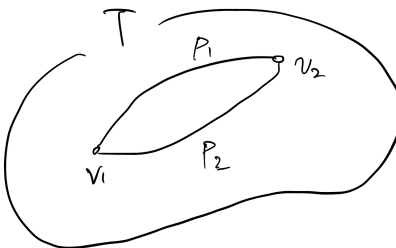


Figure 20: unique path

A tree with at least one edge has a leaf (vertex of degree 1).

G is acyclic iff every $e \in E(G)$ is a isthmus(=bridge). An edge is a bridge if it connects two components.

Definition 17. For any finite graph H , $c(H)$ = number of components in H and $e(H)$ = number of edges in H .

Theorem 8. Let G be a finite graph with n vertices and H a spanning subgraph of G , i.e., $V(H) = V(G)$.

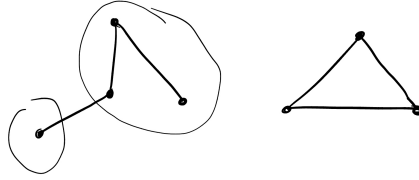


Figure 21: isthmus

1. $c(H) + e(H) \geq n$
2. $c(H) + e(H) = n$ iff H is acyclic.

Example 27. Let $G = K_4$.

H						
(c, e)	$(4, 0)$	$(3, 1)$	$(2, 2)$	$(1, 3)$	$(2, 3)$	$(1, 4)$
	$c+e=4$				$c+e > 4$	

Figure 22: $c+e$

Proof. Suppose $H = H' \cup e$. $c(H) = c(H')$ iff e is not an isthmus. $c(H) = c(H') - 1$ iff e is an isthmus. Therefore $c(H) \geq c(H') - 1$. \square

Definition 18. Let G be a finite graph with n vertices. A spanning tree of G is a spanning subgraph T of G (i.e., $V(T) = V(G)$) satisfying any two of the following conditions.

1. T is acyclic. : $c(T) + e(T) = n$
2. T is connected. : $c(T) = 1$
3. T has $n - 1$ edges. $e(T) = n - 1$

Hence, every spanning tree of G with $|V(G)| = n$ has $n - 1$ edges. Every connected graph has a spanning tree.