Discrete Mathematics Lecture Note

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1 Introduction

Notation. Define $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$ e.g.f for $(1, 1, 1, \cdots)$
Define $e^{-x} = \sum_{n\geq 0}(-1)^n \frac{x^n}{n!}$
Show that $e^x e^{-x} = \left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{n\geq 0}(-1)^n \frac{x^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n}(-1)^i \binom{n}{i}\right) \frac{x^n}{n!}$
$= 1.$
Define $\log 1 + x = \sum_{n\geq 1}(-1)^{n-1} \frac{x^n}{n}$

Example 1. Generating function for $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ with $H_0 = 0$

$$\begin{aligned}\sum_{n\geq 0} H_n x^n &= \left(\sum_{n\geq 1} \frac{1}{n} x^n\right)\left(\sum_{n\geq 0} x^n\right)\\ &= -\log(1-x)\cdot(1-x)^{-1}\\ &= \frac{1}{1-x}\log\frac{1}{1-x}\end{aligned}$$

1.1 Infinite Sums and Products in $\mathbb{C}[[x]]$

Example 2. $p(n) = \#$ of partitions of n

$$\begin{aligned}\sum_{n\geq 0} p(n) x^n &= \prod_{i\geq 1} \frac{1}{1-x^i}\\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdots\\ &= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)\cdots\end{aligned}$$

Note that the coefficient of x^n in 첫번째줄 requires finite number of factors.

$$
\begin{aligned}
&\text{coefficient of } x^n \\
&= \#\text{of solutions}(a_1, a_2, \cdots, a_n) \text{ to } n_1 + 2n_2 + \cdots + = n \\
&= p(n)
\end{aligned}
$$

Definition 1. Let $A_0, A_1, \cdots \in \mathbb{C}[[x]]$
$A \in \mathbb{C}[[x]]$, $\deg(A) =$ first power with nonzero coefficient.

The sum $\sum_{i\geq 0} A_i$ exists iff $\deg A_i \to \infty$.

$$
\begin{aligned}
A_1 &= (a_{10}, a_{11}, a_{12}, a_{13}, \cdots) \\
A_2 &= (0, a_{21}, a_{22}, a_{23}, \cdots) \\
A_3 &= (9, 0, a_{32}, a_{33}, \cdots)
\end{aligned}
$$

We can make each row sum is finite.

Example 3. e^{1+x} is not well-defined.

$$
e^{1+x} = 1 + (1+x) + \frac{(1+x)^2}{2!} + \frac{(1+x)^3}{3!} + \cdots
$$

$$
\begin{aligned}
e^{e^x - 1} &= \sum_{n\geq 0} B(n) \frac{x^n}{n!} \\
&= 1 + \left(\sum_{n\geq 1} \frac{x^n}{n!} \right) + \frac{\left(\sum_{n\geq 1} \frac{x^n}{n!} \right)^2}{2!} + \cdots
\end{aligned}
$$

Assume the constant term of each $A_i = 1$. $\prod_{i\geq 1} A_i$ exists iff $\deg(A_i - 1) \to \infty$

Example 4. $(1+x)(1+x^2)(1+x^3)\cdots$ is well defined. $= \sum_{n\geq 0} p_d(n) x^n$

Propositions
(1) $\prod_{i\geq 0} A_i$ and $\prod_{i\geq 0} B_i$ are well-defined $\Rightarrow \prod_{i\geq 0} A_i B_i = \left(\prod_{i\geq 0} A_i\right)\left(\prod_{i\geq 0} B_i\right)$

Proof. $\deg(AB-1) \geq \min\{\deg(A-1), \deg(B-1)\}$ The factors that contribute to x^n are the same on both sides □

(2) $\left(\prod_{i\geq 0} A_i\right)^{-1} = \prod_{i\geq 0} A_i^{-1}$

Proof. dhotldqkfs

$$\deg(A_i - 1) = \deg(A_i^{-1} - 1)$$

example

$$A = \prod_{i\geq 0}(i - x^i) \Rightarrow A^{-1} = \prod_{i\geq 0} \frac{1}{1 - x^i}$$

□

(3) $\frac{\prod_{i\geq 0} A_i}{\prod_{i\geq 0} B_i} = \prod_{i\geq 0} \frac{A_i}{B_i}$

Example 5. $\frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-z^i)} = \prod_{i\geq 1} = \prod_{i\geq 1}(1+z^i) = \sum_{n\geq 0} p_d(n)x^n = \prod_{i\geq 1} \frac{1}{1-z^{2i-1}} = \sum_{n\geq 0} p_0(n)x^n$

$p_0(n)$ is number of partitions of n where parts are odd.

1.2 Compositions in $\mathbb{C}[[x]]$

$A(x), B(x) \in \mathbb{C}[[x]]$

$A(B(x))$ is well-defined if either (1) $A(x)$ is a polynomial or (2) the constant term in $B(x) = 0$

Example 6.

$$A(x) = e^x$$
$$B(x) = e^x - 1$$

$A(B(x)) = e^{e^x - 1}$.

When $C(x) = x + 1$, $A(C(x)) = e^{x+1}$ is not well defined.

1.3 General Powers

Propositions. Given any $\lambda \in \mathbb{C}$, define $\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}$ with $\binom{\lambda}{0} = 1$

Define $(1 + x)^\lambda = \sum_{n\geq 0} \binom{\lambda}{x^n} \in \mathbb{C}[[x]]$

If λ is a positive integer, this is just the binomial theorem.

If $A(x) \in \mathbb{C}[[x]]$ with $A(0) = 0$, then $(1 + A(x))^\lambda = \sum_{n\geq 0} \binom{\lambda}{n} A(x)$.

Example 7.

$$(1-x)^{-k} \stackrel{?}{=} \frac{1}{(1-x)^k}$$

Note that

$$\begin{aligned}\binom{-k}{n} &= \frac{-k(-k-1)(-k-2)\cdots(-k-n+1)}{n!}\\ &= \frac{(-1)^n(n+k-1)_n}{n!}\\ &= (-1)^n\binom{n+k-1}{n} = (-1)^n\binom{n+k-1}{k-1}\end{aligned}$$

$$\begin{aligned}(1-x)^{-k} &= \sum_{n\geq 0}\binom{-k}{n}(-1)^n x^n\\ &= \sum_{n\geq 0}\binom{n+k-1}{k-1}x^n = \frac{1}{(1-x)^k}\end{aligned}$$

Proposition. $(1+x)^{\lambda}(1+x)^{\mu} = (1+x)^{\lambda+\mu}$

Example 8.

$$(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} = 1+x$$

Proof. Need to show

$$\sum_{i=0}^{n}\binom{\lambda}{i}\binom{\mu}{n-i} = \binom{\lambda+\mu}{\mu} \text{ for all } n \geq 0$$

We can prove it in algebra. Otherwise, we can show it by proving coefficient of x^n for LHS = coefficient of x^n for RHS.
Check that

$$\binom{x+y}{n} = \sum_{i=0}^{n}\binom{x}{i}\binom{y}{n-i} \text{ for all positive integers } x, y$$

Let LHS $= f(x,y)$ and RHS $= g(x,y)$.
$f(x,y) = g(x,y)$ for infinitely many x, y
$f = g$ as polynomials. □

1.4 Catalan Numbers

$$
\begin{aligned}
&c_0 = 0, c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 5 \\
&c_5 = c_1c_4 + c_2c_3 + c_3c_3 + c_4c_1
\end{aligned}
$$

$$
c_n = \sum_{i=1}^{n-1} c_i c_{n-i}
$$

Let $C(x) = \sum_{n\geq 0} c_n x^n$.

$$
\begin{aligned}
&C(x) = C(x)^2 + x \\
&\Rightarrow C(x)^2 - C(x) = -x \\
&\Rightarrow C(x)^2 - C(x) + \frac{1}{4} = \frac{1}{4} - x \\
&\Rightarrow \left(C(x) - \frac{1}{2}\right)^2 = \frac{1}{4} - x \\
&\Rightarrow C(x) - \frac{1}{2} = \pm\frac{1}{2}(1-4x)^{\frac{1}{2}}
\end{aligned}
$$

Since $C(x) = c_0 = 0$, we get

$$
\begin{aligned}
C(x) - \frac{1}{2} &= -\frac{1}{2}(1-4x)^{\frac{1}{2}} \\
\therefore C(x) &= \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
(1-4x)^{\frac{1}{2}} &= 1 + \sum_{n\geq 1} \binom{\frac{1}{2}}{n} (-1)^n 4^n x^n \\
&= 1 - 2\sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1} x^n
\end{aligned}
$$

Check that

$$
\binom{1\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{2^{2n-1}} \frac{1}{n} \binom{2n-2}{n-1}
$$

Thus

$$c_n = \frac{1}{n}\binom{2n-2}{n-1}$$

1.5 Other interpretations of Catalan Number

c_n = number of ways to parenthesize a product $x_1 x_2 \cdots x_n$

Example 9. For $x_1x_2x_3x_4$, there are 5 ways.
Key observation is the outermost parenthesis multiples two terms. The first is a product involving $x_1, x_2, \cdots, x_r \to a_r$ ways to do this. The second is a product involving $x_r + 1, x_r + 2, \cdots, x_n \to a_{n-r}$ ways.

number of binary trees with n leaves and 1 root. A tree is binary if every vertex has degree 1 or 3.
A bijections among these sets (triangulations of $n+1$-gon, parenthesized products of n vatiables, binary trees with n leaves)

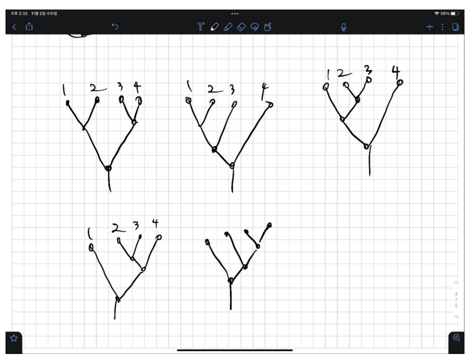

Figure 1: binary tree

Example 10. Figure 2 is the bijection between the sets.

1.6 Product of exponential generating functions

$$A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$

$$B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$$

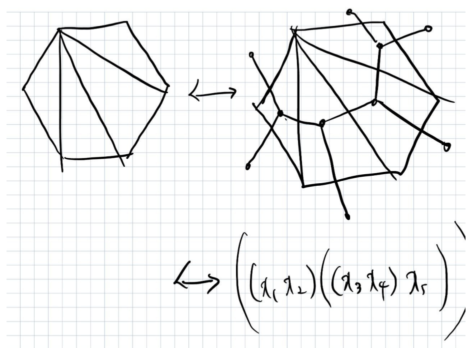

Figure 2: bijections

$$A(x)B(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!} \text{ where } c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$$

It means number of ways to partition on n-set into two ordered blocks and create structure "A" in the first block and structure "B" in the second block.
Remark If "A" = "B", then $\frac{A(x)^2}{2!}$ = e.g.f for the number of ways to create two unordered blocks in an n-set and create structure "A" in each block.
Remark If both blocks of an n-set has to be non-empty then let $a_0 = b_0 = 0$

Example 11. c_n = number of ways to color n labeled balls with red and blue so that an even number of balls are colored red and an odd number of balls are colored blue.
$c_n = 0$ if n is even. $c_n = 2^{n-1}$ if n is odd.
$R(x) =$ e.g.f for $(1, 0, 1, 0, \cdots) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} = \frac{e^x + e^{-x}}{2}$
$B(x) =$ e.g.f for $(0, 1, 0, 1, \cdots) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2}$

$$\begin{aligned}
\sum_{n\geq 0} c_n \frac{x^n}{n!} &= C(x) = R(x) \cdot B(x) \\
&= \frac{e^{2x} - e^{-2x}}{4} \\
&= \frac{1}{4}\left[\left(1 + 2x + \frac{(2x)^2}{2!} + \cdots\right) - \left(1 - 2x + \frac{(2x)^2}{2!} - \cdots\right)\right] \\
&= \sum_{n\geq 0 \; n \text{ odd}} 2^{n-1} \frac{x^n}{n!}
\end{aligned}$$

Example 12. Dearangement (revisited)

$$P(x) = \sum_{n \geq 0} n! \frac{x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x} = \text{ e.g.f for } \{|S_n| = n!\}$$

$$\begin{aligned} I(X) &= \sum_{n \geq 0} 1 \frac{x^n}{n!} = e^x \\ &= \text{ e.g.f for the number of identity permutations on an } n\text{-set} \\ &= \text{ e.g.f for the sequence } (1, 1, 1, \cdots) \end{aligned}$$

$$D(x) = \sum_{n \geq 0} d_n \frac{x^n}{n!} \text{ where } d_n = \text{ number of dearangements on an } n\text{-set}$$

$P(x) = I(x)D(x)$ because a permutations on an n-set is obtained by

1. partitioning the n-set into two ordered blocks.
2. create the identity permutations on the first block and create a dearangement on the second block.

$$\therefore D(x) = P(x)I(x)^{-1} = \frac{e^{-x}}{1-x}$$