Lindeberg-Feller CLT and Lyapunov's Condition

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Contents

1 Lindeberg-Feller Central Limit Theorem.

1.1 About the Theorem.

Sufficiency is proved by Lindeberg in 1922 and necessity by Feller in 1935. *Lindeberg-Feller CLT* is one of the most far-reaching results in probability theory. Nearly all generalizations of various types of central limit theorems spin from Lindeberg-Feller CLT.

The insights of the Lindeberg condition are that the “wild” values of the random variables, compared with s_n , the standard deviation of S_n as the normalizing constant, are insignificant and can be truncated off without affecting the general behavior of the partial sum S_n.

1.2 Lindeberg - Feller CLT.

Suppose $X_1, X_2, \ldots, X_n$ are independent r.v.s with mean 0 and variance σ_n^2. Let $s_n^2 = \sum_{i=1}^n \sigma_j^2$ denote the variance of partial sum $S_n = X_1 + X_2 + \ldots + X_n$. If, for every $\epsilon > 0$,

$$\frac{1}{s_n^2} \sum_{j=1}^{n} \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j| > \epsilon s_n\}}\right) \to 0,$$

then $\frac{S_n}{s_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.

Conversely, if $\max\limits_{1 \le j \le n} \frac{\sigma_j^2}{s_n^2} \to 0$ and $\frac{S_n}{s_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, then Lindeberg Condition holds.

1.3 Prerequisites

1. $$\forall x > 0, \left|e^{-x} - 1 + x\right| \le x^2/2$$

2. For complex z_j and w_j with $|z_j| \le 1$ and $|w_j| \le 1$, $\left|\prod_{j=1}^n z_j - \prod_{j=1}^n w_j\right| \le \sum_{j=1}^n |z_j - w_j|$

3. $$\forall x \in \mathbb{R} \ \cos x - 1 \ge -\frac{x^2}{2}$$

4. $$|\log(x+1) - x| \le x^2 \quad \text{for } x > 0$$

5. $\forall\, k \ge 1$

$$\left|\varphi_X(\lambda) - 1 - \sum_{j=1}^{k} \frac{(i\lambda)^j}{j!} \mathbb{E}\left(X^j\right)\right| \le \min\left\{\frac{2\,|\lambda|^k\, \mathbb{E}\,|X|^k}{k!}, \frac{|\lambda|^{k+1}\, \mathbb{E}\,|X|^{k+1}}{(k+1)!}\right\} \quad \forall \lambda \in \mathbb{R}$$

2 If part

2.1 Proof of Lindeberg-Feller CLT contd.

"$\Leftarrow$" The Lindeberg Condition implies,

$$\sigma_j^2 = \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right) + \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j|\le\epsilon s_n\}}\right)$$

$$\le \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right) + \epsilon^2 s_n^2 \;\; \forall j$$

$$\le \sum_{i=1}^{n} \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right) + \epsilon^2 s_n^2 \; \forall \; j$$

$$\implies \frac{\sigma_j^2}{s_n^2} \le \frac{1}{s_n^2}\sum_{i=1}^{n} \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right) + \epsilon^2 \;\; \forall j$$

$$\therefore \max_{1\le j\le n}\left(\frac{\sigma_j^2}{s_n^2}\right) \le \underbrace{\frac{1}{s_n^2}\sum_{i=1}^{n} \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right)}_{\longrightarrow 0} + \epsilon^2$$

by letting $n \to \infty$ and then $\epsilon \downarrow 0$,

$$\max_{1\le j\le n}\left(\frac{\sigma_j^2}{s_n^2}\right) \longrightarrow 0$$

For any $\epsilon > 0$, then $\forall\, j$

$$\left|\mathbb{E}\left(e^{itX_j/s_n}\right) - e^{-t^2\sigma_j^2/2s_n^2}\right|$$

$$= \left|\mathbb{E}\left(e^{itX_j/s_n}\right) - \mathbb{E}\left(1 + it\frac{X_j}{s_n} - \frac{t^2}{2}\frac{X_j^2}{s_n^2}\right) + 1 + it\mathbb{E}\left(\frac{X_j}{s_n}\right) - \frac{t^2}{2}\mathbb{E}\left(\frac{X_j^2}{s_n^2}\right) - e^{-t^2\sigma_j^2/2s_n^2} + 1 - \frac{t^2\sigma_j^2}{2s_n^2} - 1 + \frac{t^2\sigma_j^2}{2s_n^2}\right|$$

$$\le \left|\mathbb{E}\left(e^{itX_j/s_n}\right) - \mathbb{E}\left(1 + it\frac{X_j}{s_n} - \frac{t^2}{2}\frac{X_j^2}{s_n^2}\right)\right| + \left|e^{-t^2\sigma_j^2/2s_n^2} - 1 + \frac{t^2\sigma_j^2}{2s_n^2}\right|$$

$$\le \mathbb{E}\left[\min\left(\frac{t^2X_j^2}{s_n^2}, \frac{|tX_j|^3}{6s_n^3}\right)\right] + \frac{t^4\sigma_j^4}{8s_n^4}$$

$$\le \mathbb{E}\left[\min\left(\frac{t^2X_j^2}{s_n^2}, \frac{|tX_j|^3}{6s_n^3}\right)\mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right] + \mathbb{E}\left[\min\left(\frac{t^2X_j^2}{s_n^2}, \frac{|tX_j|^3}{6s_n^3}\right)\mathbf{1}_{\{|X_j|\le\epsilon s_n\}}\right] + \frac{t^4\sigma_j^4}{8s_n^4}$$

$$\le \mathbb{E}\left[\frac{t^2X_j^2}{s_n^2}\mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right] + \mathbb{E}\left[\frac{|tX_j|^3}{6s_n^3}\mathbf{1}_{\{|X_j|\le\epsilon s_n\}}\right] + \frac{t^4\sigma_j^4}{8s_n^4}$$

$$\le \frac{t^2}{s_n^2}\mathbb{E}\left[X_j^2\mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right] + \frac{|t|^3\epsilon}{6s_n^2}\mathbb{E}\left[X_j^2\right] + \frac{t^4\sigma_j^2}{s_n^2}\max_{1\le k\le n}\left(\frac{\sigma_k^2}{s_n^2}\right)$$

2.2 Proof of Lindeberg-Feller CLT contd.

Then for any fixed t,

$$\left|\mathbb{E}\left(e^{itS_n/s_n}\right) - e^{-t^2/2}\right|$$

$$= \left|\prod_{j=1}^{n}\mathbb{E}\left(e^{itX_j/s_n}\right) - \prod_{j=1}^{n}\left(e^{-t^2\sigma_j^2/2s_n^2}\right)\right|$$

$$\leq \sum_{j=1}^{n}\left|\mathbb{E}\left(e^{itX_j/s_n}\right) - e^{-t^2\sigma_j^2/2s_n^2}\right|$$

$$\leq \sum_{j=1}^{n}\left(\frac{t^2}{s_n^2}\mathbb{E}\left[X_j^2\mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right] + \frac{|t|^3\epsilon}{6s_n^2}\mathbb{E}\left[X_j^2\right] + \frac{t^4\sigma_j^2}{s_n^2}\max_{1\leq k\leq n}\left(\frac{\sigma_k^2}{s_n^2}\right)\right)$$

$$\leq \left(\frac{t^2}{s_n^2}\sum_{j=1}^{n}\mathbb{E}\left[X_j^2\mathbf{1}_{\{|X_j|>\epsilon s_n\}}\right] + \frac{|t|^3\epsilon}{6} + t^4\max_{1\leq k\leq n}\left(\frac{\sigma_k^2}{s_n^2}\right)\right)$$

$$\to \frac{\epsilon\,|t|^3}{6}\ \text{ as } n\to\infty$$

Since, $\epsilon > 0$ is arbitrary, it follows that, $\mathbb{E}\left(e^{itS_n/s_n}\right) \to e^{-t^2/2}$ for all t . Levy's continuity theorem implies,

$$\frac{S_n}{s_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

3 Only If Part.

3.1 Proof of the only if part.

" $\implies$ " Let ψ_j be the Characteristic Function of X_j . The asymptotic normality is equivalent to,

$$\prod_{j=1}^{n}\psi_j\left(\frac{t}{s_n}\right) \to e^{-\frac{t^2}{2}}$$

Notice that,

$$\left|\psi_j\left(\frac{t}{s_n}\right) - 1\right| \leq \frac{t^2\sigma_j^2}{2s_n^2}$$

Write as $n\to\infty$,

$$\sum_{j=1}^{n}\left(\psi_j\left(\frac{t}{s_n}\right) - 1\right) + \frac{t^2}{2}$$

$$= \sum_{j=1}^{n} \left(\psi_j \left(\frac{t}{s_n} \right) - 1 - \log \psi_j \left(\frac{t}{s_n} \right) \right) + \underbrace{\sum_{i=1}^{n} \log \psi_j \left(\frac{t}{s_n} \right) + \frac{t^2}{2}}_{=o(1)}$$

$$\leq \sum_{j=1}^{n} \left| \psi_j \left(\frac{t}{s_n} \right) - 1 \right|^2 + o\,(1)$$

$$\leq \max_{1 \leq j \leq n} \left| \psi_j \left(\frac{t}{s_n} \right) - 1 \right| \times \sum_{j=1}^{n} \left| \psi_j \left(\frac{t}{s_n} \right) - 1 \right| + o\,(1)$$

$$\leq \max_{1 \leq j \leq n} \frac{t^2 \sigma_j^2}{2 s_n^2} \times \sum_{j=1}^{n} \frac{t^2 \sigma_j^2}{2 s_n^2} + o\,(1) = o\,(1)$$

by the assumption, $\max_{1 \leq k \leq n} \left(\frac{\sigma_k^2}{s_n^2} \right) \to 0$

3.2 Proof of the only if part contd.

On the other hand, by definition of characteristic function, the above expression is, as $n \to \infty$,

$$o(1) = \sum_{j=1}^{n} \left(\psi_j \left(\frac{t}{s_n} \right) - 1 \right) + \frac{t^2}{2}$$

$$= \sum_{j=1}^{n} \mathbb{E} \left(e^{itX_j/s_n} - 1 \right) + \frac{t^2}{2}$$

$$= \sum_{j=1}^{n} \mathbb{E} \left(\cos \left(\frac{tX_j}{s_n} \right) - 1 \right) + \frac{t^2}{2} + i \sum_{j=1}^{n} \mathbb{E} \left(\sin \left(\frac{tX_j}{s_n} \right) \right)$$

$$= \sum_{j=1}^{n} \mathbb{E} \left\{ \left(\cos \left(\frac{tX_j}{s_n} \right) - 1 \right) \mathbf{1}_{\{|X_j| > \epsilon s_n\}} \right\} + \sum_{j=1}^{n} \mathbb{E} \left\{ \left(\cos \left(\frac{tX_j}{s_n} \right) - 1 \right) \mathbf{1}_{\{|X_j| \leq \epsilon s_n\}} \right\} + \frac{t^2}{2} + \text{imaginary part (immaterial)}$$

Since, $\cos x - 1 \geq -\frac{x^2}{2}$ for all real x,

$$\frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E} \left(X_j^2 \mathbf{1}_{\{|X_j| > \epsilon s_n\}} \right) = 1 - \frac{2}{t^2} \sum_{j=1}^{n} \mathbb{E} \left(\frac{t^2 X_j^2}{2 s_n^2} \mathbf{1}_{\{|X_j| \leq \epsilon s_n\}} \right)$$

$$\leq \frac{2}{t^2} \left(\frac{t^2}{2} + \sum_{j=1}^{n} \mathbb{E} \left\{ \left(\cos \left(\frac{tX_j}{s_n} \right) - 1 \right) \mathbf{1}_{\{|X_j| \leq \epsilon s_n\}} \right\} \right)$$

$$\leq \frac{2}{t^2} \left(\left| \sum_{j=1}^{n} \mathbb{E} \left\{ \left(\cos \left(\frac{tX_j}{s_n} \right) - 1 \right) \mathbf{1}_{\{|X_j| > \epsilon s_n\}} \right\} \right| + o(1) \right)$$

$$\leq \frac{2}{t^2} \sum_{j=1}^{n} 2\mathbb{P}\left(|X_j| > \epsilon s_n\right) + o(1)$$

$$\leq \frac{4}{t^2} \sum_{j=1}^{n} \left(\frac{\sigma_j^2}{\epsilon^2 s_n^2}\right) + o(1) \quad \text{by Chebyshev's Inequality}$$

$$\leq \frac{4}{t^2 \epsilon^2} + o(1).$$

Since t can be chosen arbitrarily large, Lindeberg condition holds.

4 Lyapunov's Condition.

4.1 Lyapunov's Condition.

Suppose $X_1, X_2, \ldots, X_n$ are independent r.v.s with mean 0 and variance σ_n^2. Let $s_n^2 = \sum_{i=1}^{n} \sigma_j^2$ denote the variance of partial sum $S_n = X_1 + X_2 + \ldots + X_n$ and $\mathbb{E}\left(|X_j|^{2+\delta}\right) < \infty\ \forall j$. Then,

$$\frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \mathbb{E}\left(|X_j|^{2+\delta}\right) \longrightarrow 0 \ \text{ as}\, n \to \infty$$

for some $\delta > 0$.

4.2 Lyapunov's Condition $\implies$ Lindeberg's Condition.

Note that,

$$\frac{1}{s_n^2} \sum_{j=1}^{n} \mathbb{E}\left(X_j^2 \mathbf{1}_{\{|X_j| > \epsilon s_n\}}\right) = \frac{1}{s_n^2} \sum_{j=1}^{n} \mathbb{E}\left(X_j^2 \mathbf{1}_{\left\{\left|\frac{X_j}{\epsilon s_n}\right| > 1\right\}}\right)$$

$$= \frac{1}{s_n^2} \sum_{j=1}^{n} \mathbb{E}\left(X_j^2 \mathbf{1}_{\left\{\frac{|X_j|^\delta}{\epsilon^\delta s_n^\delta} > 1\right\}}\right)$$

$$\leq \frac{1}{s_n^2} \sum_{j=1}^{n} \mathbb{E}\left(X_j^2 \frac{|X_j|^\delta}{\epsilon^\delta s_n^\delta}\right) \leq \frac{1}{\epsilon^\delta s_n^\delta} \sum_{j=1}^{n} \mathbb{E}\left(|X_j|^{2+\delta}\right) \longrightarrow 0$$

if Lyapunov's Condition holds.

5 Multivariate Version.

5.1 Multivariate Lindeberg-Feller CLT.

Let $\boldsymbol{X_1}, \boldsymbol{X_2}, \ldots, \boldsymbol{X_n}$ be independent random vectors with $\mathbb{E}\left(\boldsymbol{X_j}\right) = \mathbf{0}$ and $\operatorname{Var}\left(\boldsymbol{X_j}\right) = \boldsymbol{\Sigma_j}$. Suppose that, $\frac{1}{n}\left(\boldsymbol{\Sigma_1} + \boldsymbol{\Sigma_2} + \ldots + \boldsymbol{\Sigma_n}\right) \to \boldsymbol{\Sigma}$ as $n \to \infty$ and,

$$\forall \epsilon > 0 \ \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left(\left\|\boldsymbol{X_j}\right\|^2 \mathbf{1}_{\left\{\left\|\boldsymbol{X_j}\right\| > \epsilon\sqrt{n}\right\}}\right) \longrightarrow 0 \ \text{ as } n \to \infty$$

Then,

$$\frac{\boldsymbol{S_n}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}_d\left(\mathbf{0}, \boldsymbol{\Sigma}\right) \ \text{ as } n \to \infty$$

where $\boldsymbol{S_n} = \boldsymbol{X_1} + \boldsymbol{X_2} + \ldots + \boldsymbol{X_n}$.

5.2 Proof of Multivariate Lindeberg-Feller CLT.

We have $\boldsymbol{S_n} = \sum_{j=1}^{n} \boldsymbol{X_j}$, where $\boldsymbol{X_j} \sim \left(\mathbf{0}, \boldsymbol{\Sigma_j}\right)$ independently. We shall use *Cramer-Wold Device* to establish the theorem. We have to show, $\forall \boldsymbol{a} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $(0, \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a})$.

$$\frac{\boldsymbol{a}'\boldsymbol{S_n}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}\right)$$

Define, $Y_j = \boldsymbol{a}'\boldsymbol{X_j} \ \forall j = 1(1)n$. Then, $Y_j \sim \left(0, \sigma_j^2\right)$ where $\sigma_j^2 = \boldsymbol{a}'\boldsymbol{\Sigma_j}\boldsymbol{a}, \ \forall j = 1(1)n$. Say,

$$\tilde{S}_n = \sum_{j=1}^{n} Y_j, \quad s_n^2 = \sum_{j=1}^{n} \sigma_j^2 = \boldsymbol{a}'\left(\sum_{j=1}^{n} \boldsymbol{\Sigma_j}\right)\boldsymbol{a}$$

For any $\epsilon > 0$

$$\frac{1}{s_n^2}\sum_{j=1}^{n} \mathbb{E}\left(Y_j^2 \mathbf{1}_{\{|Y_j| > \epsilon s_n\}}\right)$$

$$\frac{1}{s_n^2}\sum_{j=1}^{n} \mathbb{E}\left(\left(\boldsymbol{a}'\boldsymbol{X_j}\boldsymbol{X_j}'\boldsymbol{a}\right)\mathbf{1}_{\{(\boldsymbol{a}'\boldsymbol{X_j}\boldsymbol{X_j}'\boldsymbol{a}) > \epsilon^2 s_n^2\}}\right)$$

$$\leq \frac{\|\boldsymbol{a}\|^2}{s_n^2}\sum_{j=1}^{n} \mathbb{E}\left(\left\|\boldsymbol{X_j}\right\|^2 \mathbf{1}_{\left\{\left\|\boldsymbol{X_j}\right\|^2 \geq \epsilon'^2 s_n^2\right\}}\right) \quad \text{where } \epsilon'^2 = \frac{\epsilon^2}{\|\boldsymbol{a}\|^2}$$

$$= \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left(\left\|\boldsymbol{X_j}\right\|^2 \mathbf{1}_{\left\{\left\|\boldsymbol{X_j}\right\|^2 \geq \epsilon'^2 s_n^2\right\}}\right)\frac{\|\boldsymbol{a}\|^2}{s_n^2/n}$$

Now,

$$\frac{s_n}{\sqrt{n}} \longrightarrow \sqrt{\boldsymbol{a}'\Sigma\boldsymbol{a}} = c \text{ (say)}$$

Then, $\exists K \in \mathbb{N} \ni \forall n \geq K,\ \frac{s_n}{\sqrt{n}} > \frac{c}{2}$

$$\implies \quad \forall n \geq K, \quad \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left(\|\boldsymbol{X_j}\|^2 \, \mathbf{1}_{\left\{\|\boldsymbol{X_j}\|^2 \geq \epsilon'^2 s_n^2\right\}}\right) \leq \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left(\|\boldsymbol{X_j}\|^2 \, \mathbf{1}_{\left\{\|\boldsymbol{X_j}\|^2 \geq \epsilon''^2 n\right\}}\right) \text{ where } \epsilon'' = \epsilon' c/2$$

Taking $\limsup$ as $n \to \infty$ on both side,we get,

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left(\|\boldsymbol{X_j}\|^2 \, \mathbf{1}_{\left\{\|\boldsymbol{X_j}\|^2 \geq \epsilon'^2 s_n^2\right\}}\right) \longrightarrow 0 \ \text{ as } n \to \infty$$

from the given conditions.

$$\therefore \sum_{j=1}^{n} \frac{Y_j}{s_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

also,

$$\frac{s_n}{\sqrt{n}} \longrightarrow \sqrt{\boldsymbol{a}'\Sigma\boldsymbol{a}}$$

Combining, by *Slutsky's Theorem*, we get,

$$\sum_{j=1}^{n} \frac{Y_j}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \boldsymbol{a}'\Sigma\boldsymbol{a})$$

i.e.

$$\frac{\boldsymbol{a}'\boldsymbol{S_n}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a})$$