Context-Free Languages & Grammars (CFLs & CFGs)

Reading: Chapter 5

Not all languages are regular

- So what happens to the languages which are not regular?

- Can we still come up with a language recognizer?
  - i.e., something that will accept (or reject) strings that belong (or do not belong) to the language?

Context-Free Languages

- A language class larger than the class of regular languages

- Supports natural, recursive notation called "context-free grammar"

- Applications:
  - Parse trees, compilers
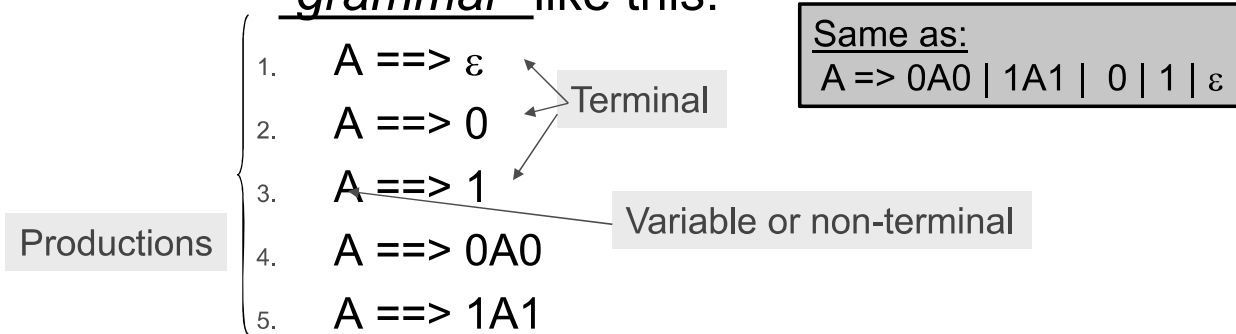 - XML
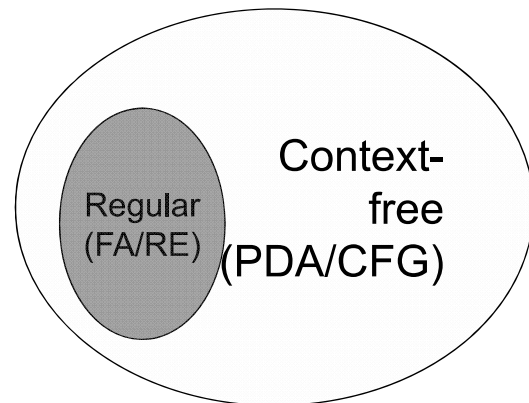
An Example

- A palindrome is a word that reads identical from both ends
  - E.g., $\overrightarrow{ma}\overleftarrow{dam}$, $\overrightarrow{redi}\overleftarrow{vider}$, $\overrightarrow{mala}\overleftarrow{yalam}$, $\overrightarrow{01001}\overleftarrow{0010}$
- Let L = { w | w is a binary palindrome}
- Is L regular?
  - No.
  - <u>Proof:</u>
    - Let $w=0^N 1 0^N$ (assuming N to be the p/l constant)
    - By Pumping lemma, w can be rewritten as xyz, such that $xy^k z$ is also L (for any k≥0)
    - But |xy|≤N and y≠ε
    - ==> $y=0^+$
    - ==> $xy^k z$ *will NOT* be in L for k=0
    - ==> Contradiction

But the language of palindromes…

is a CFL, because it supports recursive substitution (in the form of a CFG)

- This is because we can construct a "*grammar*" like this:

Same as:
A => 0A0 | 1A1 | 0 | 1 | ε

Productions
1. A ==> ε
2. A ==> 0   → Terminal
3. A ==> 1
4. A ==> 0A0   → Variable or non-terminal
5. A ==> 1A1

How does this grammar work?

How does the CFG for palindromes work?

An input string belongs to the language (i.e., accepted) iff it can be generated by the CFG

G:
A => 0A0 | 1A1 | 0 | 1 | ε

- Example: w=01110
- G can generate w as follows:

1. A   => 0A0
2.   => 01A10
3.   => 01110

Generating a string from a grammar:
1. Pick and choose a sequence of productions that would allow us to generate the string.
2. At every step, substitute one variable with one of its productions.

Context-Free Grammar: Definition

- A context-free grammar G=(V,T,P,S), where:
    - V: set of variables or non-terminals
    - T: set of terminals (= alphabet U {ε})
    - P: set of *productions,* each of which is of the form
        V ==> α_1 | α_2 | …
        - Where each α_i is an arbitrary string of variables and terminals
    - S ==> start variable

CFG for the language of binary palindromes:
  G=({A},{0,1},P,A)
  P: A ==> 0 A 0 | 1 A 1 | 0 | 1 | ε

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More examples

- Parenthesis matching in code
- Syntax checking
- In scenarios where there is a general need for:
    - Matching a symbol with another symbol, or
    - Matching a count of one symbol with that of another symbol, or
    - Recursively substituting one symbol with a string of other symbols

Example #2

- **Language of balanced paranthesis**
  e.g., ()((((()))))((())).…
- **CFG?**

G:
S => (S) | SS | ε

How would you "interpret" the string "((((()))()())" using this grammar?

Example #3

- **A grammar for $L = \{0^m 1^n \mid m \geq n\}$**

- **CFG?**

G:
S => 0S1 | A
A => 0A | ε

How would you interpret the string "00000111"
using this grammar?

Example #4

A program containing **if-then(-else)** statements

      if *Condition* **then** *Statement* **else** *Statement*

      (Or)

      if *Condition* **then** *Statement*

CFG?

More examples

- $L_1 = \{0^n \mid n \geq 0\}$
- $L_2 = \{0^n \mid n \geq 1\}$
- $L_3 = \{0^i 1^j 2^k \mid i=j \text{ or } j=k, \text{ where } i,j,k \geq 0\}$
- $L_4 = \{0^i 1^j 2^k \mid i=j \text{ or } i=k, \text{ where } i,j,k \geq 1\}$

Applications of CFLs & CFGs

- Compilers use parsers for syntactic checking
- Parsers can be expressed as CFGs

1. Balancing paranthesis:
   - B ==> BB | (B) | *Statement*
   - *Statement* ==> …
2. If-then-else:
   - S ==> SS | *if Condition then Statement else Statement* | *if Condition then Statement* | *Statement*
   - *Condition* ==> …
   - *Statement* ==> …
3. C paranthesis matching { … }
4. Pascal *begin-end* matching
5. YACC (Yet Another Compiler-Compiler)

More applications

- **Markup languages**
  - **Nested Tag Matching**
    - **HTML**
      - <html> …<p> … … </p> … </html>
    - **XML**
      - <PC> … <MODEL> … </MODEL> .. <RAM> … </RAM> … </PC>

Tag-Markup Languages

Roll ==> <ROLL> Class Students </ROLL>

Class ==> <CLASS> Text </CLASS>

Text ==> Char Text | Char

Char ==> a | b | … | z | A | B | .. | Z

Students ==> Student Students | ε

Student ==> <STUD> Text </STUD>

Here, the left hand side of each production denotes one non-terminals
(e.g., "Roll", "Class", etc.)
Those symbols on the right hand side for which no productions (i.e.,
substitutions) are defined are terminals (e.g., 'a', 'b', '|', '<', '>', "ROLL",
etc.)

Structure of a production

head   derivation   body

A   =======> $\alpha_1 \mid \alpha_2 \mid \ldots \mid \alpha_k$
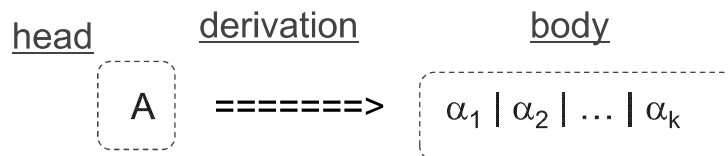
The above is same as:

1. A ==> α_1
2. A ==> α_2
3. A ==> α_3
…
K. A ==> α_k

CFG conventions

- Terminal symbols <== a, b, c…

- Non-terminal symbols <== A,B,C, …

- Terminal <u>or</u> non-terminal symbols <== X,Y,Z

- Terminal strings <== w, x, y, z

- Arbitrary strings of terminals and non-terminals <== $\alpha, \beta, \gamma, ..$

Syntactic Expressions in Programming Languages

result = a*b + score + 10 * distance + c

terminals   variables   Operators are also terminals

Regular languages have only terminals
- Reg expression = [a-z][a-z0-1]*
- If we allow only letters a & b, and 0 & 1 for constants (for simplification)
  - Regular expression = (a+b)(a+b+0+1)*

String membership

How to say if a string belong to the language defined by a CFG?

1. ## Derivation
    - Head to body
2. ## Recursive inference
    - Body to head

Both are equivalent forms

Example:
- w = 01110
- Is w a palindrome?

G:
A => 0A0 | 1A1 | 0 | 1 | ε

A => 0A0
  => 01A10
  => 01110

Simple Expressions…

- We can write a CFG for accepting simple expressions
- G = (V,T,P,S)
    - V = {E,F}
    - T = {0,1,a,b,+,*,(,)}
    - S = {E}
    - P:
        - E ==> E+E | E*E | (E) | F
        - F ==> aF | bF | 0F | 1F | a | b | 0 | 1

Generalization of derivation

- Derivation is *head ==> body*

- A==>X   (A derives X in a single step)
- A ==>$^*_G$ X   (A derives X in a multiple steps)

- <u>Transitivity:</u>
  IFA ==>$^*_G$B, and B ==>$^*_G$C, THEN A ==>$^*_G$ C

Context-Free Language

- The language of a CFG, G=(V,T,P,S), denoted by L(G), is the set of terminal strings that have a derivation from the start variable S.
  - L(G) = { w in T* | S ==>$^*_G$ w }

□

Left-most & Right-most Derivation Styles

G:
E => E+E | E*E | (E) | F
F => aF | bF | 0F | 1F | ε

Derive the string a*(ab+10) from G: $E =\!\!\overset{*}{=}\!\!>_G a*(ab+10)$

Left-most derivation:

Always substitute leftmost variable

```
■E
■==> E * E
■==> F * E
■==> a * E
■==> a * (E)
■==> a * (E + E)
■==> a * (F + E)
■==> a * (aF + E)
■==> a * (abF + E)
■==> a * (ab + E)
■==> a * (ab + F)
■==> a * (ab + 1F)
■==> a * (ab + 10F)
■==> a * (ab + 10)
```

Right-most derivation:

Always substitute rightmost variable

```
■E
■==> E * E
■==> E * (E)
■==> E * (E + E)
■==> E * (E + F)
■==> E * (E + 1F)
■==> E * (E + 10F)
■==> E * (E + 10)
■==> E * (F + 10)
■==> E * (aF + 10)
■==> E * (abF + 0)
■==> E * (ab + 10)
■==> F * (ab + 10)
■==> aF * (ab + 10)
■==> a * (ab + 10)
```

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Leftmost vs. Rightmost derivations

Q1) For every leftmost derivation, there is a rightmost derivation, and vice versa. True or False?

True - will use parse trees to prove this

Q2) Does every word generated by a CFG have a leftmost and a rightmost derivation?

Yes – easy to prove (reverse direction)

Q3) Could there be words which have more than one leftmost (or rightmost) derivation?

Yes – depending on the grammar

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How to prove that your CFGs are correct?

(using induction)

CFG & CFL

G_{pal}:
$A \Rightarrow 0A0 \mid 1A1 \mid 0 \mid 1 \mid \varepsilon$

- **Theorem:** A string w in (0+1)* is in $L(G_{pal})$, if and only if, w is a palindrome.

- **Proof:**
  - Use induction
    - on string length for the IF part
   - On length of derivation for the ONLY IF part
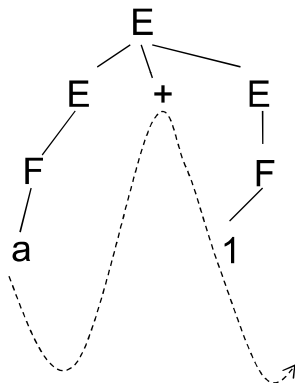
Parse trees

Parse Trees

- Each CFG can be represented using a *parse tree:*
    - Each <u>internal node</u> is labeled by a variable in V
    - Each <u>leaf</u> is terminal symbol
    - For a production, A==>$X_1 X_2 \ldots X_k$, then any internal node labeled A has k children which are labeled from $X_1, X_2, \ldots X_k$ from left to right

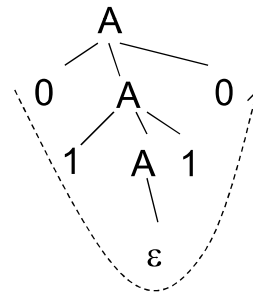<u>Parse tree for production and all other subsequent productions:</u>

$A ==> X_1 .. X_i .. X_k$

Examples

E
├─ E
│ └─ F
│   └─ a
├─ +
└─ E
   └─ F
      └─ 1
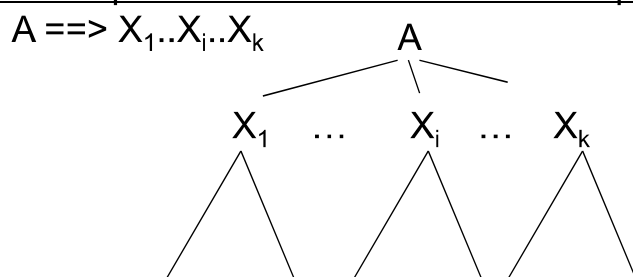
Parse tree for *a + 1*

G:
 E => E+E | E*E | (E) | F
 F => aF | bF | 0F | 1F | 0 | 1 | a | b

Recursive inference

A
├─ 0
├─ A
│ ├─ 1
│ ├─ A
│ │ └─ ε
│ └─ 1
└─ 0

Parse tree for *0110*

Derivation

G:
 A => 0A0 | 1A1 | 0 | 1 | ε

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Parse Trees, Derivations, and Recursive Inferences

Production:
$$A ==> X_1..X_i..X_k$$

Recursive inference

A
├─ X_1 … X_i … X_k

Derivation

Left-most derivation ← Parse tree

Derivation ← Right-most derivation → Recursive inference
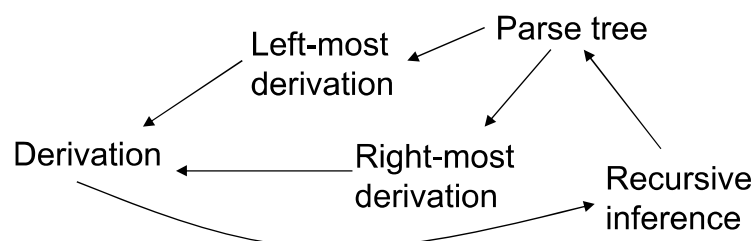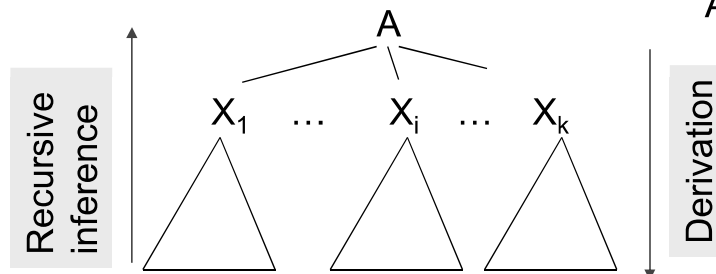
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Interchangeability of different CFG representations

- Parse tree ==> left-most derivation
    - DFS left to right
- Parse tree ==> right-most derivation
    - DFS right to left
- ==> left-most derivation == right-most derivation
- Derivation ==> Recursive inference
    - Reverse the order of productions
- Recursive inference ==> Parse trees
    - bottom-up traversal of parse tree
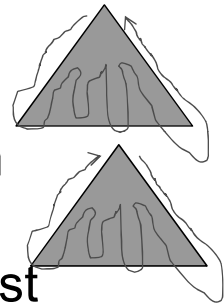
Connection between CFLs and RLs

CFLs & Regular Languages

- A CFG is said to be *right-linear* if all the productions are one of the following two forms: *A ==> wB (or) A ==> w*
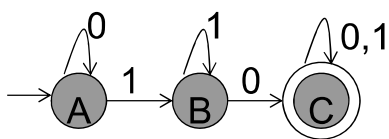
  Where:
  - A & B are variables,
  - w is a string of terminals
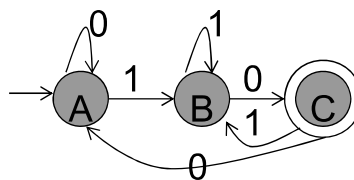
- <u>Theorem 1:</u> Every right-linear CFG generates a regular language
- <u>Theorem 2:</u> Every regular language has a right-linear grammar
- <u>Theorem 3:</u> Left-linear CFGs also represent RLs

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Some Examples

Right linear CFG?

Right linear CFG?

> A => 01B | C
> B => 11B | 0C | 1A
> C => 1A | 0 | 1

Finite Automaton?

Ambiguity in CFGs and CFLs

Ambiguity in CFGs

- A CFG is said to be *ambiguous* if there exists a string which has more than one left-most derivation

Example:
    S ==> AS | ε
    A ==> A1 | 0A1 | 01

LM derivation #1:
    S => AS
       => 0A1S
       =>0A11S
       => 00111S
       => 00111

LM derivation #2:
    S => AS
       => A1S
       => 0A11S
       => 00111S
       => 00111

Input string: 00111
    Can be derived in two ways

Why does ambiguity matter?

E ==> E + E | E * E | (E) | a | b | c | 0 | 1

Values are different !!!

*string = a * b + c*

• LM derivation #1:
  • E => E + E => E * E + E
    ==>* a * b + c
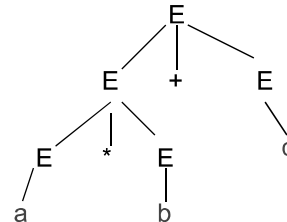
(a*b)+c

• LM derivation #2
  • E => E * E => a * E =>
    a * E + E ==>* a * b + c
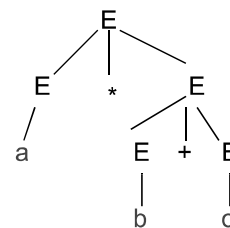
a*(b+c)

The calculated value depends on which of the two parse trees is actually used.

Removing Ambiguity in Expression Evaluations

- **It MAY be possible to remove ambiguity for some CFLs**
    - E.g.,, in a CFG for expression evaluation by imposing rules & restrictions such as precedence
    - This would imply rewrite of the grammar

- Precedence: (), * , +

Modified unambiguous version:

E => E + T | T
T => T * F | F
F => I | (E)
I => a | b | c | 0 | 1

Ambiguous version:

E ==> E + E | E * E | (E) | a | b | c | 0 | 1

How will this avoid ambiguity?

Inherently Ambiguous CFLs

- However, for some languages, it may not be possible to remove ambiguity

- A CFL is said to be *inherently ambiguous* if every CFG that describes it is ambiguous

<u>Example:</u>

- $L = \{ a^n b^n c^m d^m \mid n,m \geq 1\} \cup \{a^n b^m c^m d^n \mid n,m \geq 1\}$
- L is inherently ambiguous
- Why?

  Input string: $a^n b^n c^n d^n$

Summary

- Context-free grammars
- Context-free languages
- Productions, derivations, recursive inference, parse trees
- Left-most & right-most derivations
- Ambiguous grammars
- Removing ambiguity
- CFL/CFG applications
  - parsers, markup languages