

# Control-oriented modeling of thermodynamic systems

Tom S. Pedersen, Palle Andersen

December 13, 2016

Automation and Control, Department of Electronic Systems  
Aalborg University, Fredrik Bajers Vej 7, DK-9220 Aalborg Ø, Denmark

## Preamble

This note has been written for a course on thermodynamic systems for control engineers at the School Information and Communication Technology. In this form it has been used as a part of the course 'Modeling of Mechanical and Thermodynamic Systems' on 8th semester Control and Automation.

The purpose is to give a basic introduction to thermodynamics as it can be used for developing models for analysis, design and simulation of control systems. The notes take as a point of departure law of conservation of mass and the first law of thermodynamics. From these laws mass and energy balances in lumped and distributed form are developed. Heat transfer and two phase systems are introduced. For phase shift systems moving boundary models are shown. From second law of thermodynamics bounds on efficiency as i.e. shown from a Carnot cycle is formulated.

## Key Words

Modeling, differential equation, mass balance, energy balance, heat transfer, lumped parameters, distributed parameters, moving boundary.

## Affiliation of the Authors

The author are affiliated with Automation and Control, Department of Electronic Systems, Aalborg University, DK-9220 Aalborg Ø, Denmark.

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# Chapter 1

## First law of thermodynamics

In this chapter basic terms like energy, heat, work, power, heat capacity, specific heat, latent heat and enthalpy will be defined, explained and discussed. The first fundamental law of thermodynamics will be defined. Energy balances will be derived from the first law, and together with the mass balance these balances will be the backbone in dynamic modelling of thermodynamic systems. Some of the terms and laws are further explained in textbooks like [JS08].

### 1.1 First law, energy and mass balances

The first law of thermodynamics claims that for a system of particles

$$\Delta U_{int} = Q + W \quad (1.1)$$

where  $U_{int}$  is the internal energy of the system,  $Q$  is the heat supplied to the system and  $W$  is the work done on the system.  $Q$  and  $W$  come from other systems. It should be noticed that all terms may be positive or negative. The first law of thermodynamics is a law of conservation of energy meaning that the internal energy only may be changed by adding or subtracting heat or work from another system; energy can not emerge/disappear in a system.

**Internal energy** is all the energy of a system that is associated with its microscopic components (atoms and molecules) when viewed from a reference frame at rest with respect to the center of mass of the system [JS08].

**Heat** is defined as a process of transferring energy across the boundary of a system because of a temperature difference between the system and its surroundings. It is also the amount of energy  $Q$  transferred by this process [JS08].

**Work** will be defined later in the note.

It should be noticed that energy, heat and work all have the unit Joule. Heat is Joule transferred by a temperature difference, all other transfer of energy into the system e.g. electrical energy is covered by the term work.

When the internal energy of a system is changed, this can be observed as a change in the temperature of the system. An exception to this statement is the case in which a system undergoes a change of state (also called a phase transition), this will be discussed later on.

If the change in internal energy only causes a temperature change the heat capacity of the system is of great importance. The **heat capacity**  $C$  of a particular sample of particles is defined as the amount of energy needed to raise the temperature of that sample by  $1^\circ\text{C}$ . So if the energy  $\Delta U_{int}$  gives a raise in the temperature of  $\Delta T$ ,  $C$  is given as

$$C = \frac{\Delta U_{int}}{\Delta T} \quad (1.2)$$

For a given substance (copper, iron etc.) the specific heat capacity is given as

$$c = \frac{\Delta U_{int}}{M\Delta T} \quad (1.3)$$

where  $M$  is the mass. For specific heat capacity, often just named specific heat, values can be found using the internet.

For fluids like water the specific heat depends on the temperature and the pressure, and in a phase transition, for example when water goes from the fluid to the steam phase, the specific heat is not a usable term.

If we have a system where the only change in internal energy can be observed by a temperature change the first law gives

$$\Delta U_{int} = Mc\Delta T = Q + W \quad (1.4)$$

The heat  $Q$  in the first law, represents the energy transferred to the system by temperature differences; this can be described by thermal conduction, radiation or convection. It should be noticed that the system is a collection of particles. When the system in question is a volume in space for example a fixed volume in a tube instead of a collection of particles, particles goes in and out of this fixed volume space. This problem is handled later in this section.

**Thermal conduction** is a process where at an atomic scale exchange of kinetic energy between microscopic particles, molecules, atoms, and free electrons, in which less energetic particles gain energy in collisions with more energetic particles takes place. Conduction occurs only if there is a difference in temperature between two parts of the conducting medium. It is found that **power**  $\mathcal{P}$ , defined as

$$\mathcal{P}(t) = \frac{dQ(t)}{dt}, \quad (1.5)$$

is proportional to the cross-sectional area  $A$  and the temperature difference  $dT$  and inverse proportional to the thickness  $dx$

$$\mathcal{P}_{conduction} = kA \left| \frac{dT}{dx} \right| \quad (1.6)$$

where  $k$  is called thermal conductivity of the material and values may be found on the internet.

In many thermal systems conduction is not the dominating way of transferring energy into a system.

**Radiation** is energy transfer in the form of electromagnetic waves produced by thermal vibrations of the molecules. The power is transferred between two surfaces given by

$$\mathcal{P}_{radiation} = A\sigma(T^4 - T_o^4) \quad (1.7)$$

where  $A$  is the area,  $\sigma$  is a value that depends e.g. of the color of the surfaces and  $T$  respective  $T_o$  are the temperatures of the surfaces.

Radiation normally only plays an important role if the temperatures are high for example in a combustion chamber.

**Convection** is a mechanism where energy is transferred from a wall to a moving fluid/gas. It could be from a radiator to the air or from the water in the radiator to the radiator metal; the first is called natural convection the latter is called forced convection. The transfer is a combination of motion of particles and conduction. The power is given by

$$\mathcal{P}_{convection} = A\alpha(T - T_o) \quad (1.8)$$

where  $A$  is the area, the  $T$ 's are mentioned in radiation and  $\alpha$  is the heat transfer coefficient.

The heat transfer coefficients  $\alpha$  play an important role, but unfortunately, they can be problematic to find, because they depend on very many factors such as the form of the wall (flat, curved etc.), the state of the fluid (density, viscosity, specific heat, conductivity, temperature, pressure etc.), the velocity of the fluid (laminar, turbulent), phase shifts along the wall. To find the heat transfer coefficients consult textbooks like [Cha84]

Until now the terms energy and power have been used. The relation is

$$\mathcal{P}(t) = \frac{dQ(t)}{dt} \quad Q(t) = \int_0^t \mathcal{P}(\tau) d\tau + Q(t=0) \quad (1.9)$$

where the unit of energy  $Q$  is Joule and power  $\mathcal{P}$  is Joule/sec = Watt.

The first law is for a system of particles where the only change in internal energy can be observed by a temperature change

$$\Delta U_{int} = Mc\Delta T = Q + W \Rightarrow \quad (1.10)$$

$$\frac{dU_{int}}{dt} = \frac{d(Mc\Delta T)}{dt} = \frac{dQ}{dt} + \frac{dW}{dt} \quad (1.11)$$

Assuming constant mass and specific heat capacity we find

$$Mc \frac{d(\Delta T)}{dt} = \mathcal{P}_{convection} + \mathcal{P}_{radiation} + \mathcal{P}_{conduction} + \frac{dW}{dt} \quad (1.12)$$

### Example 1.1 (Heating of metal block)

*The particle system is a block of metal. This block is heated by an electrical resistor in thermal contact with the block. We will model the total system so that the input is the voltage  $U$  to the resistor, the output is the metal temperature  $T$  and the ambient temperature  $T_o$  is a disturbance. We assume that the metal block has the same spatial temperature, this is the same as assuming that the conductivity of the metal block is infinite. It is also assumed that all energy produced in the resistor is transferred directly to the block. We can now use the first law to find a model*

$$Mc \frac{dT}{dt} = \mathcal{P}_{convection} + \frac{dW}{dt} \quad (1.13)$$

where we assume no radiation from the block and no conduction. The term  $\frac{dW}{dt}$  with the unit Watt is the electrical power from the resistor and this is given as  $\mathcal{P}_{elec} = \frac{U^2}{R}$ , where  $R$  is the resistance. The term  $\mathcal{P}_{convection}$  is the heat loss from the block to the ambient.

The differential equation describing the system is

$$Mc \frac{dT(t)}{dt} = \alpha A(T(t) - T_o(t)) + \frac{U^2(t)}{R} \quad (1.14)$$

where it can be seen which terms are considered to vary in time and which are constants. The differential equation is non-linear and if the aim is to find a transfer function the equation must be linearised.  $\square$

Until now we have assumed that the internal energy of the system of particles can be observed by just measuring the temperature. But assume that the system consist of a mixture of water and steam at 100 °C, kept at a constant pressure of 1 atm. The particles can be in a cylinder with a weightless piston, then the pressure inside the cylinder will be the same as the ambient pressure. If we add more heat  $Q$  to the system the internal energy  $U_{int}$  will raise, but the temperature will stay at 100 °C, instead the internal energy raise by changing some water to steam. In this case the temperature can not be used to observe the energy.

**Latent heat** is a useful term to describe this phase change. The latent heat (literally, the 'hidden' heat) is the energy that is needed to phase shift 1 kg from one phase to another phase; it is assumed that the material is in a saturated state. Latent heat  $L$  is defined by

$$L = \frac{Q}{\Delta M} \quad (1.15)$$

where  $\Delta M$  is mass that changes phase.

### Example 1.2 (Latent heat)

In figure 1.1 1 gram of ice is added energy until the ice has been converted to 120 deg. steam; the pressure is 1 atm. In part A the ice temperature is raising due to the added energy, here there is no phase shift and the temperature will follow the equation  $T = \frac{Q}{Mc_{ice}}$ . The spec. heat capacity for ice is 2.090 J/(kg · °C); giving that 62.7 Joule must be added to reach 0 °C. In part B the ice is phase shifted to water. It is seen that the temperature stays at 0 °C; the internal energy is here given by the amount of water in the mixture.

The latent heat for ice is 3.33e5 J/kg and using this it is seen that 333 Joules are needed for melting all the ice. In part C the temperature of the water is raised by adding heat; the specific heat is 4190 J/(kg · °C). 419 Joule is needed to raise the temperature to 100 °C. In part D another phase shift takes place the mechanism is similar to part B; the latent heat here is 2.2565 J/kg. Finally in part D the steam temperature is raised; the spec. heat is 2010 J/(kg · °C).

The state where part C meet part D is called **saturation** state; the state where part D meet part E is also called a saturation state. For water the saturation state is a function of temperature and pressure.

The phase shift is normally modelled using a term called *enthalpy*, this will be defined later in the note.  $\square$

**Work** in thermodynamic processes is defined using the definition from mechanical systems where it is defined as force multiplied with displacement.

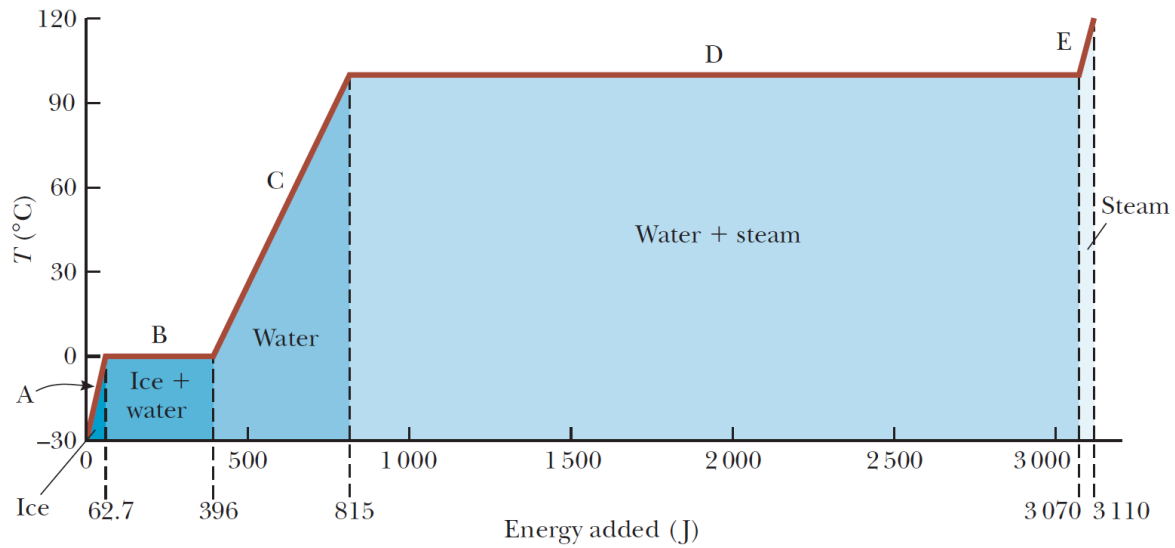


Figure 1.1: *Temperature versus energy added when the system initially consists of 1 gram ice at  $-30\text{ }^{\circ}\text{C}$ . Pressure is  $1\text{ atm}$ . [JS08]*

In figure 1.2 a force  $F$  gives rise to a displacement  $dy$  so that the work done on the cylinder system from a to b is  $dW = -Fdy$ . (The negative sign is because  $dy$  is negative). The force is given by the pressure  $P$  multiplied with the cross-sectional area  $A$  so  $dW = -Fdy = -PA dy$ . Because  $A dy$  is the change in volume of the gas  $dV$  the work on the gas is

$$dW = -PdV \quad (1.16)$$

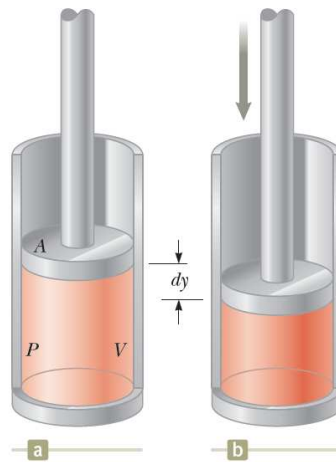


Figure 1.2: *Work is done on a gas contained in a cylinder, [JS08]*

The total work done on the gas as its volume changes from  $V_i$  to  $V_f$  is the given by the integral



$$W = - \int_{V_i}^{V_f} P dV \quad (1.17)$$

In a pressure – volume (PV) diagram as shown in figure 1.3 the work is the negative area under the curve.

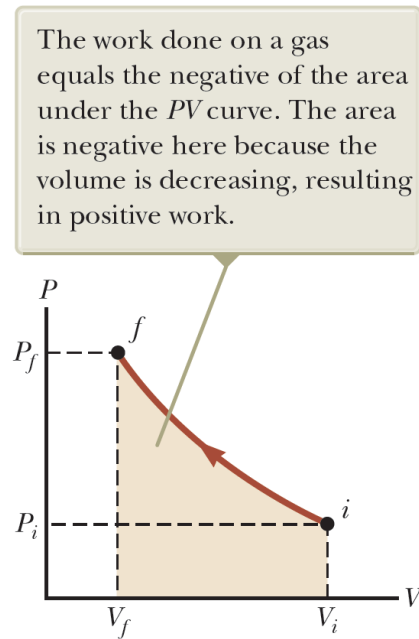


Figure 1.3: *Work in a PV diagram [JS08]*

It can also be seen that the work depends on how pressure changes on the path between  $V_i$  and  $V_f$ .

For an ideal gas with constant temperature we have that  $PV = nRT$  is constant; it can be proved [JS08] that the work associated with isothermal expansion is given by

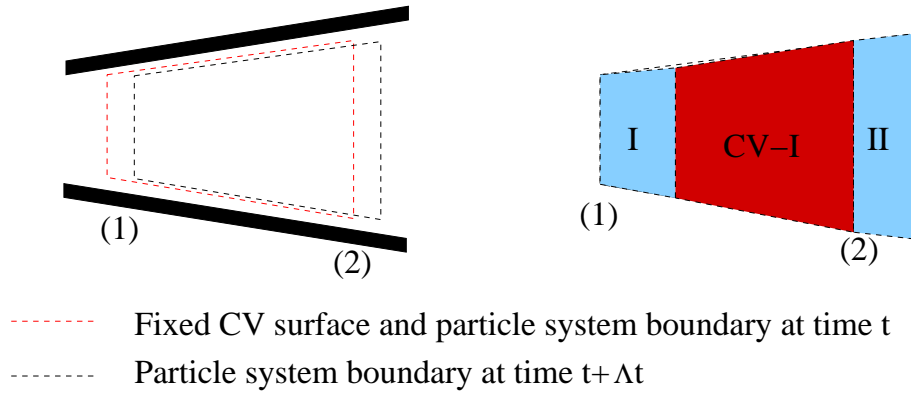
$$W = nRT \ln\left(\frac{V_i}{V_f}\right) \quad (1.18)$$

As a second property for an ideal gas under adiabatic (isolated) expansion, the curve in the PV diagram is given as  $PV^\gamma = \text{constant}$ , where  $\gamma$  depends on the ideal gas and  $\gamma > 1$ . These results will play a role in a proof later on in the note.

Next we will go from considering a **system of particles** and instead focus on a **control volume** called CV [YMO97]. A simple version of relating the system concept to the control volume concept can be obtained from the one-dimensional flow through a fixed control volume as shown in figure 1.4.

The CV is fixed in the tube and it is marked by (1) and (2). The particle system is marked by the dashed line. We see that at time  $t$  the system and the CV coincide. At time  $t + \Delta t$  the system has moved and now occupies section (CV – I)+II (see figure 1.4 (b)).

We can now look at a certain property of the system described by the general parameter  $B$ , where  $B$  can be mass or energy. Then at time  $t$  we have  $B_{sys}(t) = B_{CV}(t)$

Figure 1.4: *System and Control Volume at two times*

At time  $t + \Delta t$  the system is

$$B_{sys}(t + \Delta t) = B_{CV}(t + \Delta t) - B_I(t + \Delta t) + B_{II}(t + \Delta t) \quad (1.19)$$

now if  $B_{sys}(t)$  is subtracted and we divide by  $\Delta t$

$$\frac{B_{sys}(t + \Delta t) - B_{sys}(t)}{\Delta t} = \frac{B_{CV}(t + \Delta t) - B_I(t + \Delta t) + B_{II}(t + \Delta t) - B_{sys}(t)}{\Delta t} \quad (1.20)$$

The two terms

$$\frac{B_I(t + \Delta t) - B_I(t)}{\Delta t} = \frac{B_I(t + \Delta t) - B_I(t)}{\Delta t} \quad \text{and} \quad \frac{B_{II}(t + \Delta t) - B_{II}(t)}{\Delta t} = \frac{\Delta B_{II}(t + \Delta t) - B_{II}(t)}{\Delta t} \quad (1.21)$$

Because  $B_I(t) = B_{II}(t) = 0$ , for  $\Delta t \rightarrow 0$  we find the inflow  $b_I = b_{in}$  and outflow  $b_{II} = b_{out}$  as

$$\lim_{\Delta t \rightarrow 0} \frac{B_I(t + \Delta t) - B_I(t)}{\Delta t} = b_{in}(t) \quad \text{and} \quad \lim_{\Delta t \rightarrow 0} \frac{B_{II}(t + \Delta t) - B_{II}(t)}{\Delta t} = b_{out}(t) \quad (1.22)$$

We further note that  $B_{sys}(t) = B_{CV}(t)$  and with  $\Delta t \rightarrow 0$  equation 1.20 is rewritten as

$$\frac{d(B_{sys}(t))}{dt} = \frac{d(B_{CV}(t))}{dt} - b_{in}(t) + b_{out}(t) \quad (1.23)$$

We can for instance consider the mass of system of particles  $M_{sys}$  vs the mass a control volume  $M_{CV}$ . Conservation of mass implies

$$\frac{d(M_{sys}(t))}{dt} = 0 = \frac{d(M_{CV}(t))}{dt} - m_{in}(t) + m_{out}(t) \quad (1.24)$$

$$\frac{dM_{CV}(t)}{dt} = m_{in}(t) - m_{out}(t) \quad (1.25)$$

Then the **mass balance** is

$$\boxed{\frac{dM_{CV}(t)}{dt} = m_{in}(t) - m_{out}(t)} \quad (1.26)$$

where  $M_{CV}$  is the mass in the CV,  $m_{in}$  is the mass flow going into the CV and  $m_{out}$  is the mass flow going out of the CV. The unit of mass flow is kg/sec.

If  $B$  in the general equation 1.23 is energy we can define energy flow associated with the streams in and out as  $\mathcal{P}_{in,flow}$  and  $\mathcal{P}_{out,flow}$

$$\frac{d(U_{sys}(t))}{dt} = \frac{d(U_{CV}(t))}{dt} - \mathcal{P}_{in,flow}(t) + \mathcal{P}_{out,flow}(t) \quad (1.27)$$

The left side of the equation is given by the first law 1.11

$$\frac{dQ}{dt} + \frac{dW}{dt} = \frac{d(U_{CV}(t))}{dt} - \mathcal{P}_{in,flow}(t) + \mathcal{P}_{out,flow}(t) \quad (1.28)$$

The first term is the power to the system from conduction, radiation and convection, we can denote this as  $P_{ext}$ .

$$\frac{d(U_{CV}(t))}{dt} = P_{ext}(t) + \mathcal{P}_{in,flow}(t) - \mathcal{P}_{out,flow}(t) + \frac{dW(t)}{dt} \quad (1.29)$$

The internal energy in the CV is given as the specific internal energy  $u$  multiplied with the mass ( $U_{CV} = M_{CV}u_{CV}$ ). The power in the flow are given as the flow multiplied by the internal energy ( $\mathcal{P}_{in,flow} = m_{in}u_{in}$ ,  $\mathcal{P}_{out,flow} = m_{out}u_{out}$  )

$$\frac{d(M_{CV}u_{CV}(t))}{dt} = P_{ext}(t) + m_{in}u_{in}(t) - m_{out}u_{out}(t) + \frac{dW(t)}{dt} \quad (1.30)$$

The work term  $dW$  is the nett work done on the system and associated to the inflow and outflow,  $dW = dW_{in} - dW_{out}$ . Remembering that the work is force multiplied by displacement

$$dW = dW_{in} - dW_{out} = F_{in}dy_{in} - F_{out}dy_{out} \quad (1.31)$$

The force equals pressure multiplied with cross-sectional area  $A$ . The displacement is velocity multiplied with time

$$dW = F_{in}dy_{in} - F_{out}dy_{out} = P_{in}A_{in}v_{in}dt - P_{out}A_{out}v_{out}dt \quad (1.32)$$

The velocity perpendicular to  $A$  is given by  $v = m \frac{1}{\rho} \frac{1}{A}$ , where  $\rho$  is density.

$$dW = P_{in}A_{in}v_{in}dt - P_{out}A_{out}v_{out}dt = P_{in}\frac{1}{\rho_{in}}m_{in}dt - P_{out}\frac{1}{\rho_{out}}m_{out}dt \quad (1.33)$$

giving

$$\frac{dW}{dt} = P_{in}\frac{1}{\rho_{in}}m_{in} - P_{out}\frac{1}{\rho_{out}}m_{out} \quad (1.34)$$

Putting this in equation 1.30 gives

$$\frac{d(M_{CV}(t)u_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + m_{in}(t)u_{in}(t) - m_{out}(t)u_{out}(t) + \frac{P_{in}(t)}{\rho_{in}(t)}m_{in}(t) - \frac{P_{out}(t)}{\rho_{out}(t)}m_{out}(t) \quad (1.35)$$

Defining **specific enthalpy**  $h$  as  $h = u + \frac{P}{\rho}$  gives

$$\frac{d(M_{CV}(t)u_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + m_{in}(t)h_{in}(t) - m_{out}(t)h_{out}(t) \quad (1.36)$$

In some textbooks this is called the energy balance.

If the term  $\frac{d(M_{CV}\frac{P}{\rho})}{dt} = \frac{d(PV_{CV})}{dt}$  is added ( $V_{CV}$  is volume)

$$\frac{d(M_{CV}(t)h_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + m_{in}(t)h_{in}(t) - m_{out}(t)h_{out}(t) + V_{CV}\frac{d(P(t))}{dt} \quad (1.37)$$

and the term  $V_{CV}\frac{d(P(t))}{dt}$  is negligible we find the **energy balance**

$$\boxed{\frac{d(M_{CV}(t)h_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + m_{in}(t)h_{in}(t) - m_{out}(t)h_{out}(t)} \quad (1.38)$$

The unit of specific enthalpy is Joule/kg.

Enthalpy is a function of temperature and pressure  $h = f(T, P)$ . In saturation state enthalpy is only a function of temperature or pressure. For many matters like for example water the enthalpy is measured and tables of enthalpy as function of pressure and temperature can be found. In figure 1.5 the relationship between enthalpy, pressure and temperature for water/steam is shown. Programme subroutines to calculate enthalpy can be can be downloaded (e.g. XSTEAM, search on Google for 'xsteam steam tables').

For further information look at the youtube video

<https://www.youtube.com/watch?v=QqEbN4dDp8M>

The specific heat for constant pressure is

$$c_p = \left. \frac{\partial h}{\partial T} \right|_{P=\text{constant}} \quad (1.39)$$

Because  $c_p$  in certain temperature intervals is nearly constant, (this is the case where only one phase exist) a good approximation for the enthalpy is

$$h \approx c \cdot T + \text{constant} \quad (1.40)$$

This mean that for a matter that do not change phase the **One phase energy balance** is

$$\boxed{\frac{d(c_{CV}M_{CV}(t)T_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + cm_{in}(t)T_{in}(t) - cm_{out}(t)T_{out}(t)} \quad (1.41)$$

Normally the specific heat  $c$  is assumed to be constant

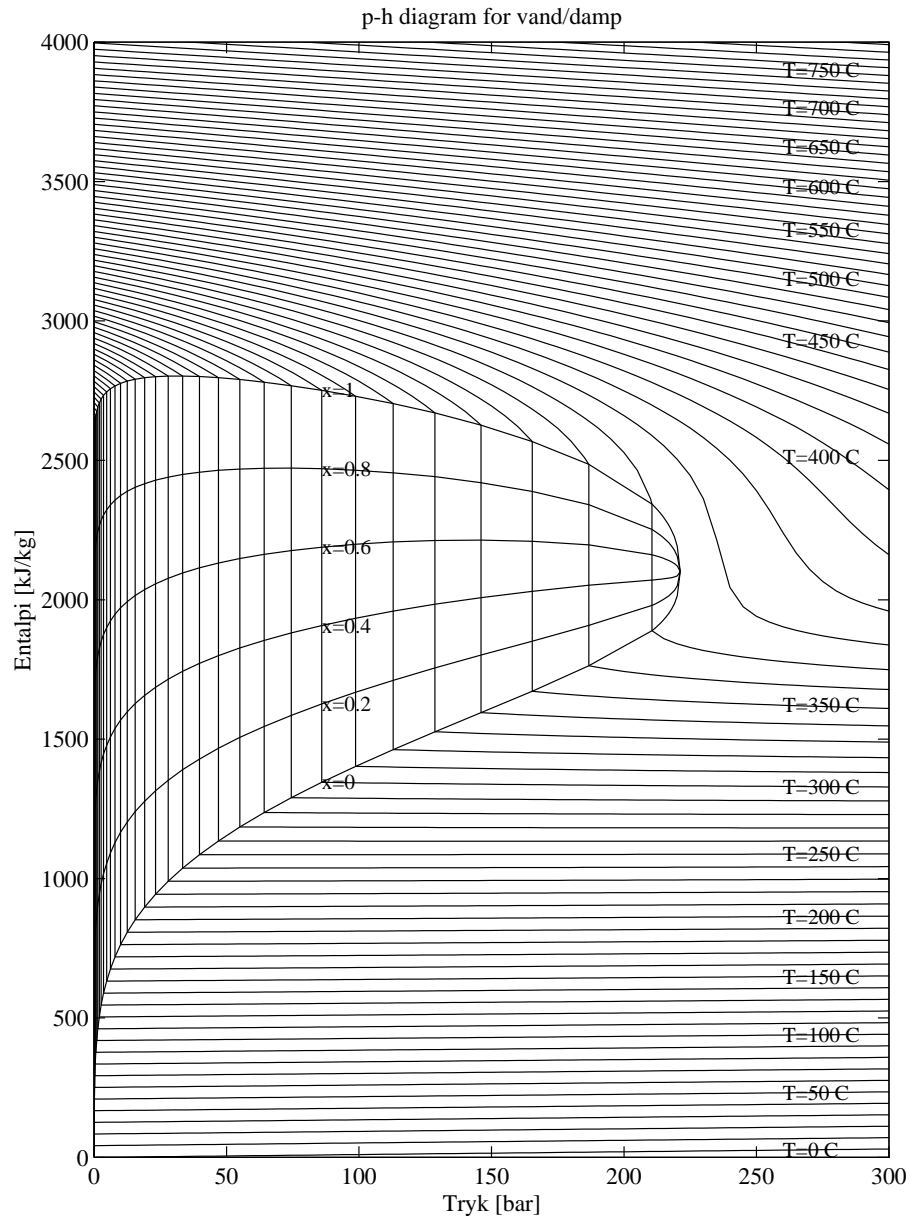


Figure 1.5: *Enthalpy pressure diagram for water/steam*

## 1.2 Examples using mass and energy balances

In this section there will be examples using the balance equations. There will also be more general considerations in the examples.

### Example using mass balances

In the example is considered a tank containing a mixture of the two fluids water and lye. Two flows are applied to the tank, one is a pure lye flow  $m_{in,l}$ , and the other is a flow  $m_{in,w+l}$  containing a mixture of water and lye, the lye concentration in this flow is  $K_{in,w+l}$ . The

concentration is defined as

$$K = \frac{M_{lye}}{M_{lye} + M_{water}}$$

The tank is assumed having a constant cross-sectional area  $A$ . The fluid in the tank is stirred so the lye concentration  $K_{tank}$  is the same all-over the tank. The fluid mass in the tank is called  $M$ . There is an output flow  $m_{out}$  where the concentration is the same as in the tank. It is assumed that there is no chemical reaction between water and lye.

The aim is to make a controller that can control the tank mass and the concentration in the tank. The manipulated variables of the controller are the flow of pure lye and the tank output flow. The input flow to the tank containing the mixture is a disturbance as well as the concentration in this flow.

In the modelling both the disturbances and the controller manipulated variables are model inputs.

When making a model of this system we can use three approaches I) Mass balances for the lye and water in the tank. II) Mass balance for the water and a total mass balance III) Mass balance for the lye and a total mass balance. All the three models can be reformulated to the same model. We will use III.

Total mass balance:

$$\frac{dM(t)}{dt} = m_{in,l}(t) + m_{in,w+l}(t) - m_{out}(t) \quad (1.42)$$

Lye mass balance:

$$\frac{d(M(t)K_{tank}(t))}{dt} = m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t) \quad (1.43)$$

Often the equations should be put in the **non-linear state space form**  $\dot{x} = f(x, u, t)$  to make them suitable for i.e. simulation programs. In this case we use **the chain rule** and find

$$\begin{aligned} \frac{d(M(t)K_{tank}(t))}{dt} &= M(t)\frac{dK_{tank}(t)}{dt} + K_{tank}(t)\frac{dM(t)}{dt} \\ &= m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t) \end{aligned} \quad (1.44)$$

Using equation 1.42

$$\begin{aligned} M(t)\frac{dK_{tank}(t)}{dt} + K_{tank}(t)[m_{in,l}(t) + m_{in,w+l}(t) - m_{out}(t)] \\ = m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t) \end{aligned} \quad (1.45)$$

Rearranging

$$\begin{aligned} \frac{dK_{tank}(t)}{dt} &= \frac{1}{M(t)} \left[ m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t) \right. \\ &\quad \left. - K_{tank}(t)[m_{in,l}(t) + m_{in,w+l}(t) - m_{out}(t)] \right] \end{aligned} \quad (1.46)$$

Equations 1.42 and 1.46 are the two model equations on the non-linear state space form.

### Example using mass balance and the one phase energy balance

In a tank hot and cold water is mixed. The inflow to the tank consist of hot water and cold water. Hot water has the temperature  $T_h$  and flow  $m_h$ ; cold water the temperature  $T_c$  and flow  $m_c$ . The tank is assumed having a cross-sectional area of  $A$ , the level is called  $L$ . The temperature in the tank is  $T_t$  the is an outflow of the tank, this is proportional with the square root of the level  $m_{out} = K_o\sqrt{L}$ . The density of water is  $\rho$  and the spec. heat  $c$ , both assumed constant.

The model has as output the level and the temperature in the tank. The input temperatures are assumed constant, the flows are time varying.

Mass balance:

$$\frac{dM(t)}{dt} = A\rho\frac{dL(t)}{dt} = m_h(t) + m_c(t) - K_o\sqrt{L(t)} \quad (1.47)$$

One phase energy balance

$$c\frac{dM(t)T_t(t)}{dt} = cm_h(t)T_h(t) + cm_c(t)T_c(t) - cK_o\sqrt{L(t)}T_t(t) \quad (1.48)$$

To get the equations on non-linear state space form the chain rule can be used

$$\begin{aligned} \frac{dM(t)T_t(t)}{dt} &= M(t)\frac{dT_t(t)}{dt} + T_t(t)\frac{dM(t)}{dt} \\ &= M(t)\frac{dT_t(t)}{dt} + T_t(t)[m_h(t) + m_c(t) - K_o\sqrt{L(t)}] \\ &= m_h(t)T_h + m_c(t)T_c - K_o\sqrt{L(t)}T_t(t) \end{aligned} \quad (1.49)$$

giving

$$\frac{dT_t(t)}{dt} = \frac{1}{M(t)}[m_h(t)T_h + m_c(t)T_c - K_o\sqrt{L(t)}T_t(t) - T_t(t)[m_h(t) + m_c(t) - K_o\sqrt{L(t)}] \quad (1.50)$$

Inserting  $M = \rho AL$  we find the two equations describing the system

$$\begin{aligned} \frac{dT_t(t)}{dt} &= \frac{1}{\rho AL(t)}[m_h(t)T_h + m_c(t)T_c - T_t(t)[m_h(t) + m_c(t)]] \\ \frac{dL(t)}{dt} &= \frac{1}{A\rho}[m_h(t) + m_c(t) - K_o\sqrt{L(t)}] \end{aligned} \quad (1.51)$$

### Example: Different approximations for the CV temperature.

In the previous examples perfect mixing was assumed in the control volume CV. In other cases, for instance energy balance for fluid flowing through a tube mixing may not be a good assumption. We will consider two ways to approximate the total energy in a control volume with a one phase energy balance as an example.

$$\frac{d(c_{CV}M_{CV}(t)T_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + cm_{in}(t)T_{in}(t) - cm_{out}(t)T_{out}(t) \quad (1.52)$$

The mean temperature in the CV,  $T_{CV}$ , can be approximated in different ways, it could be approximated as the output temperature  $T_{CV} = T_{out}$ , this is correct if the control volume is perfectly stirred. It could also be the mean value between the input and output temperature  $T_{CV} = \frac{T_{in}+T_{out}}{2}$ . The first approximation is called **backward difference**, the second is called **central difference**. A simple example is a CV where the inflow is equal to the outflow and constant, there is no external power, but the input temperature  $T_{in}$  can change. This gives the balance equation

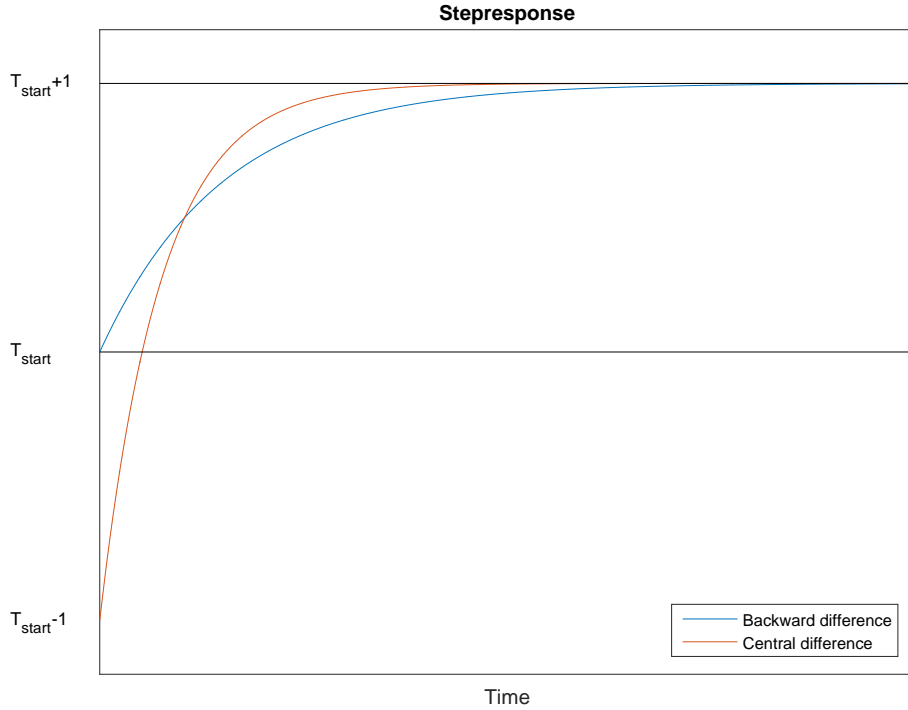


Figure 1.6: *Step response for backward respectively central difference.*

$$M \frac{dT_{CV}(t)}{dt} = mT_{in}(t) - mT_{out}(t) \quad (1.53)$$

If the backward difference approximation is used

$$M \frac{dT_{out}(t)}{dt} = mT_{in}(t) - mT_{out}(t) \quad (1.54)$$

If the central difference approximation is used

$$M \frac{d(\frac{T_{out}(t)+T_{in}(t)}{2})}{dt} = mT_{in}(t) - mT_{out}(t) \quad (1.55)$$

In this example both models are linear and it is possible to determine the transfer functions from the input  $T_{in}$  to the output  $T_{out}$



$$G_{backward} = \frac{T_{out}(s)}{T_{in}(s)} = \frac{1}{\frac{M}{m}s + 1} \quad (1.56)$$

$$G_{central} = \frac{T_{out}(s)}{T_{in}(s)} = \frac{1 - \frac{M}{2m}s}{\frac{M}{2m}s + 1} \quad (1.57)$$

The backward difference gives a first order system, the central difference gives a first order system with a right-half plane zero. The unit step responses of these systems can be seen in figure 1.6. Here it is seen, that a  $1^\circ\text{C}$  step will cause the output temperature to step down  $1^\circ\text{C}$  in the beginning of the step response, if the central approximation method is used. The two methods result in the same steady state in this example. Of course in this example the central difference is not suitable in the transient, this is often the case when central difference approach is used. In other examples in specific a counter flow heat exchanger the central difference approximation has the advantage to give much more precise steady state values and can sometimes be preferred.

Rarely a third approach called **forward difference**, where  $T_{CV} = T_{in}$ , is used.

### Example: Energy balance with convective heat transfer

In this example we have water flowing through a metal tube. It is assumed, that in the tube perimeter, is placed an electrical heat element, and all power  $\mathcal{P}_{elec}$  from this goes without any loss into tube, and the power is equally distributed along the tube. The input water temperature is  $T_i$ . The input flow as well as the output flow is  $m$ . It is assumed that the system can be described by two CV's one for the metal part where the temperature is  $T_m$  and one from the water parts where the temperature is  $T_w$ . The transfer of heat from metal to water can be characterized as convective.

The model is given by the two energy balances

$$c_m M_m \frac{dT_m(t)}{dt} = \mathcal{P}_{elec}(t) - \alpha A (T_m(t) - T_w(t)) \quad (1.58)$$

$$c_w M_w \frac{dT_w(t)}{dt} = c_w m(t) (T_{in}(t) - T_w(t)) + \alpha A (T_m(t) - T_w(t)) \quad (1.59)$$

In the following we will discuss considerations made by engineers for finding a value of the heat transfer coefficient  $\alpha$ .

By experiments it has been shown that all the parameter mentioned in Table 1.1 have influence on  $\alpha$ . Unfortunately it has not been possible to find physical laws which can relate these quantities, instead empirical relation have been found.

$$\alpha = f(v, D, \lambda, \rho, c, \mu) \quad (1.60)$$

It has been found that product functions of the form

$$\alpha = K v^{a1} D^{a2} \rho^{a3} c^{a4} \mu^{a5} \lambda^{a6} \quad (1.61)$$

can give satisfactory descriptions. To achieve an expression in fewer dimensionless variables the usual way is to seek an expression, where the fundamental dimensions fit

$$\text{Dim}(\alpha) = \text{Dim}(v)^{a1} \text{Dim}(D)^{a2} \dots \text{Dim}(\lambda)^{a6} = [\text{kg s}^{-3} \text{ C}^{-1}] \quad (1.62)$$

variable	notation	unit
Water velocity	$v$	m $s^{-1}$
Tube diameter	$D$	m
Water density	$\rho$	kg $m^{-3}$
Water heat	$c$	$m^2 s^{-2} {}^{\circ}C^{-1}$
Water viscosity	$\mu$	kg $m^{-1} s^{-1}$
Water conductivity	$\lambda$	kg m $s^{-3} {}^{\circ}C^{-1}$
Heat transfer	$\alpha$	kg $s^{-3} {}^{\circ}C^{-1}$

Table 1.1: *Variable influencing  $\alpha$ , determined by experiments.*

Inserting the fundamental units gives

$$(\text{kg s}^{-3} {}^{\circ}\text{C}^{-1}) = (\text{m s}^{-1})^{a1} (\text{m})^{a2} (\text{kg m}^{-3})^{a3} (\text{m}^2 \text{s}^{-2} {}^{\circ}\text{C}^{-1})^{a4} (\text{kg m}^{-1} \text{s}^{-1})^{a5} (\text{kg m s}^{-3} {}^{\circ}\text{C}^{-1})^{a6} \quad (1.63)$$

Equation 1.63 is solved with respect to  $a_1 \cdots a_6$ , giving

$$\text{kg : } 1 = a3 + a5 + a6 \quad (1.64)$$

$$\text{m : } 0 = a1 + a2 - 3a3 + 2a4 - a5 + a6 \quad (1.65)$$

$$\text{s : } -3 = -a1 - 2a4 - a5 - 3a6 \quad (1.66)$$

$$\text{C : } -1 = -a4 - a6 \quad (1.67)$$

Choosing  $a1 = a$  and  $a4 = b$  gives  $a2 = a - 1$ ,  $a3 = a$ ,  $a5 = b - a$  and  $a6 = 1 - b$ . Inserting in equation 1.61 gives

$$\alpha = K v^a D^{a-1} \rho^a c^b \mu^{b-a} \lambda^{1-b} = K v^a D^a D^{-1} \rho^a c^b \mu^b \mu^{-a} \lambda \lambda^{-b} \Rightarrow \quad (1.68)$$

$$\left( \frac{\alpha D}{\lambda} \right) = K \left( \frac{D \rho v}{\mu} \right)^a \left( \frac{\mu c}{\lambda} \right)^b \quad (1.69)$$

where the three terms in parenthesis are dimensionless called Nusselts, Reynolds and Prandtl numbers.

$$Nu = \frac{\alpha D}{\lambda} \quad (1.70)$$

$$Re = \frac{D \rho v}{\mu} \quad (1.71)$$

$$Pr = \frac{\mu c}{\lambda} \quad (1.72)$$

ending in

$$Nu = K Re^a Pr^b \quad (1.73)$$

Results for heat transfer is often presented in a form where the mentioned numbers appear. The method where the relevant parameters are put in a product form and the exponents are fitted using unit considerations is called **unit analysis**; more about this method may be found in e.g. [JW76].

Now returning to the example, following relation for forced convection for fluids is given in [Cha84]

$$Nu = 0.023 Re^{0.8} Pr^n \quad (1.74)$$

where

$$n = 0.4 \quad \text{at heating} \quad (1.75)$$

$$n = 0.3 \quad \text{at cooling} \quad (1.76)$$

$$0.7 < Pr < 160 \quad (1.77)$$

$$10^4 < Re < 10^6 \quad (1.78)$$

Now the definition of Nusselts, Reynolds and Prandtls numbers may be inserted in the equation and  $\alpha$  can be determined.

### Example: Phase shift energy balance

In this example we have a boiler containing a mixture of water and water-steam, both considered in saturation state. There is a water flow  $m_{in}$  with a temperature  $T_{in}$  entering the boiler. The boiler is supplied with electrical power  $\mathcal{P}_{elec}$ . The volume of water in the boiler is called  $V_w$ , the total volume of the boiler is  $V_{tot}$  and is constant this gives that the steam volume in the boiler is  $V_{tot} - V_w$ .

The saturation temperature in the boiler is  $T_{sat}$ . Out of the boiler flows steam  $m_{out}$  with saturation temperature. The boiler is loaded with a tube system described by the equation

$$P - P_d = R m_{out}^2 \quad (1.79)$$

a standard equation for a tube system with turbulent flow, flow resistance  $R$  and pressure drop  $P - P_d$ . It is assumed that  $R$  and  $P_d$  are known.

The inputs to the model is  $m_{in}$ ,  $T_{in}$  and  $\mathcal{P}_{elec}$ . All physical dimensions of the boiler is known. The desired outputs of the model is the water volume  $V_w$  and the pressure  $P$  in the boiler.

In this example properties of steam/water are important. The main properties of water/steam are pressure  $P$ , density  $\rho$  and temperature  $T$ . Knowing two of the properties the third can be found in a steam table, see [Sch82].

$$P = f_p(T, \rho); \quad \rho = f_\rho(P, T); \quad T = f_T(P, \rho) \quad (1.80)$$

If we assume that the thermodynamic state is saturation, the temperature is a function of pressure only

$$T = f_{sat}(P); \quad P = f_{sat}^{-1}(T) \quad (1.81)$$

There are two densities one for saturated water and one for saturated steam

$$\rho_w = f_{\rho,w}(P); \quad \rho_{st} = f_{\rho,st}(P) \quad (1.82)$$

The specific enthalpy  $h$  is in general a function of pressure and temperature

$$h = u + \frac{P}{\rho} = f_{ent}(T, P) \quad (1.83)$$

If two of the three thermodynamic states,  $h, P$  and  $T$ , are known, the third can always be found in a steam table.

In saturation the spec. enthalpy is only a function of pressure

$$h_w = h' = f_{ent,sat,w}(P); \quad h_{st} = h'' = f_{ent,sat,st}(P) \quad (1.84)$$

The notation "water"="steam" is often used in steam tables.

All the functions  $f$  mentioned above can be found in the steam table.

We now return to the modelling.

The energy balance for the water volume plus the steam volume of the boiler can be developed

$$\frac{d(M_{CV}(t)h_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + m_{in}(t)h_{in}(t) - m_{out}(t)h_{out}(t) \quad (1.85)$$

$$\frac{d(\rho_w(P(t))V_w(t)h_w(P(t)) + \rho_{st}(P(t))(V_{tot} - V_w(t))h_{st}(P(t)))}{dt} = \mathcal{P}_{elec}(t) + m_{in}(t)cT_{in}(t) - m_{out}(t)h_{st}(P(t)) \quad (1.86)$$

Here it is utilized that the enthalpy in the control volume  $M_{CV}h_{CV}$ , is the sum of steam and water enthalpies in the CV.

The mass balance is given by

$$\frac{d(\rho_w(P(t))V_w(t) + \rho_{st}(V_{tot} - V_w(t)))}{dt} = m_{in}(t) - m_{out}(t) \quad (1.87)$$

From equation 1.79 it is seen that  $m_{out}$  only depends on pressure  $P$ , the other terms in equation 1.79 are constants and known.

$$\frac{d(\rho_w(P(t))V_w(t)h_w(P(t)) + \rho_{st}(P(t))(V_{tot} - V_w(t))h_{st}(P(t)))}{dt} = \mathcal{P}_{elec}(t) + m_{in}(t)cT_{in}(t) - m_{out}(t)h_{st}(P(t)) \quad (1.88)$$

$$\frac{d(\rho_w(P(t))V_w(t) + \rho_{st}(P(t))(V_{tot} - V_w(t)))}{dt} = m_{in}(t) - m_{out}(t) \quad (1.89)$$

There are two states in the equation namely  $P(t)$  and  $V_w(t)$ . All terms that solely depend on the  $P$  state are red, and those which depend on  $V_w$  state are green. Black terms are known inputs or constants.

To bring the equations on a standard form, intensive use of the two rules for derivatives are necessary

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{for example} \quad (1.90)$$

$$\frac{d}{dt}(\rho(P(t))V(t)) = \rho(P(t))\frac{dV(t)}{dt} + V(t)\frac{d\rho(P(t))}{dt} \quad (1.91)$$

$$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx} \quad \text{for example} \quad (1.92)$$

$$\frac{d\rho(P(t))}{dt} = \frac{d\rho(P(t))}{dP(t)} \cdot \frac{dP(t)}{dt} = \left. \frac{d\rho(P)}{dP} \right|_{P=P(t)} \cdot \frac{dP(t)}{dt} \quad (1.93)$$

Note that  $\frac{d\rho(P)}{dP}$  is a number that can be found in the water/steam table.

The equations 1.87 and 1.89 can, after using the rules for derivatives, be written on the **descriptor form**

$$\mathbf{M}(\mathbf{x}, \mathbf{z})\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \quad (1.94)$$

where state vector is  $\mathbf{x}(t) = [P(t) \ V_w(t)]^T$ .  $\mathbf{M}$  is a 2x2 matrix where the elements are states or  $\mathbf{z}$  elements.  $\mathbf{z}$  contains constants and terms that can be found in the steam table for example  $h_{st}(P(t))$ . Finally  $\mathbf{u}(t) = [\mathcal{P}_{elec}(t) \ m_{in}(t) \ T_{in}(t)]^T$  is input.

If  $\mathbf{M}$  is invertible the system can be written on the standard non-linear state space form

$$\dot{\mathbf{x}} = \mathbf{M}^{-1}(\mathbf{x}, \mathbf{z})\mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \quad (1.95)$$

## Chapter 2

# Second law of thermodynamics

In this chapter we will focus on efficiency of thermodynamic engines. The idea is how to take heat energy from a heat reservoir and convert this energy to mechanical energy in an efficient way; this is a problem in for example a power plant. It will turn out that the theory is also applicable to engines where energy is taken from a cold reservoir and transferred to a hot reservoir; this is the case in e.g. a refrigerator or a heat pump. This chapter is mainly build on [Fey66] and [JS08].

A **heat reservoir** is large pool of matter with a constant temperature; the temperature is not changed if heat is removed or added to the reservoir.

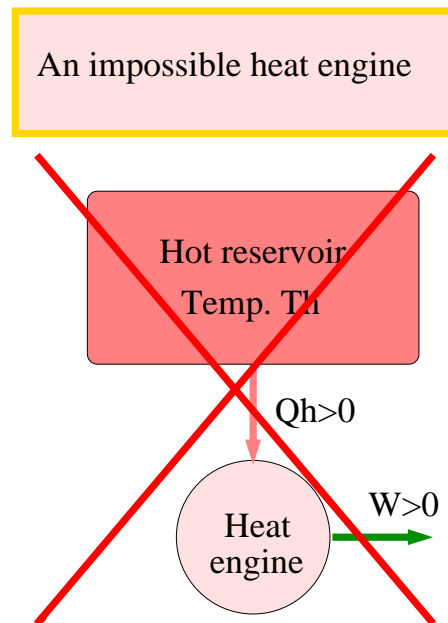


Figure 2.1: *Illustration of the second law. An impossible engine.*

A **heat engine** or **engine** is a device that takes in energy by heat and, operating in a cyclic process, expels a fraction of that energy by means of work, [JS08].

The **second law of thermodynamics**: It is impossible to construct a heat engine, that,

with no other change in the universe, takes heat energy from a reservoir and convert this energy to an equal amount of work.

The phrase 'with no other change in the universe' is important. If we have compressed air in a cylinder at a certain temperature; the cylinder is placed in a water tank with the same temperature. If we let the air do work by expansion, it will take in heat from the water and do work; then we take heat from a reservoir and transform it to work, but, we have changed something in the universe, namely the pressure in the cylinder.

On the other hand it is possible to take energy from a heat reservoir, transform some of this energy to work and the rest of the energy is going to a colder reservoir; this do not contradict the second law, because we change the energy amount in the cold reservoir. The interesting is now what is the optimal heat engine?; is there something in nature that stops an optimization so it is not possible to let  $\frac{W}{Q_h} \rightarrow 1$  ? Notice that the second law claims that  $\frac{W}{Q_h} < 1$ ; how much is it smaller than 1?

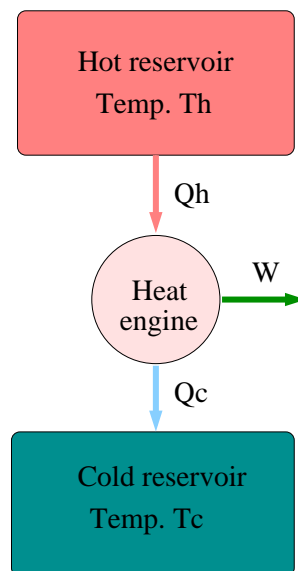


Figure 2.2: *Illustration of a possible heat engine*

We will define the **optimal heat engine** as an engine that optimize  $\frac{W}{Q_h}$ . For more than 200 years ago an upper limit for a heat engine was found. This was claimed to be the reversible engine. A **reversible engine** is an engine where the direction of  $Q_h, Q_c, W$  can be changed; all are either as given in figure 2.2 or they have all the opposite direction. It is assumed that the arrows in the figures give the direction of energy flows; the values on the arrows are always positive.

To prove that the **reversible engine is an optimal engine** we look at figure 2.3. Here two engines are operating in parallel; a reversible and a non-reversible engine. First we adjust (or scale) the two engines so they take in equal amount of heat,  $Q_{h,rev} = Q_{h,non} = Q_h$ . We now claim that  $W_{non} > W_{rev}$  meaning that the non-reversible engine is the most efficient. Next we reverse the reversible heat engine as shown in figure 2.4. Because we claim that  $W_{non} > W_{rev}$ , we can take a part of this work to drive the reversible engine; still there will be excess work  $W_{non} - W_{rev} > 0$ , that can drive another device. There is no change in the hot reservoir because it receives/releases the same amount of energy. What we now have is an

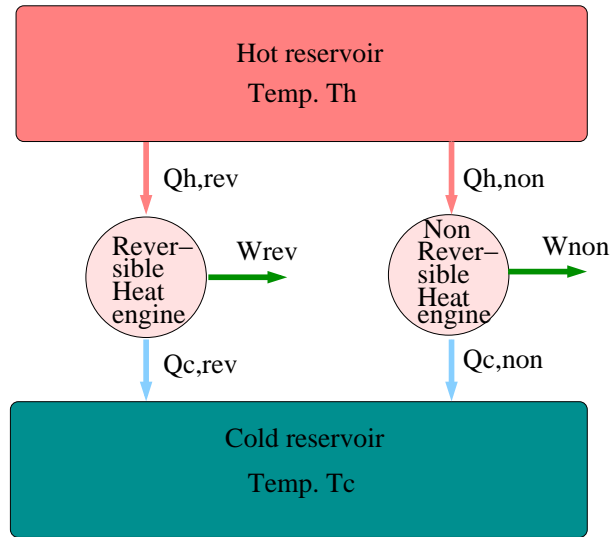


Figure 2.3: *A reversible engine A and a not-reversible engine B*

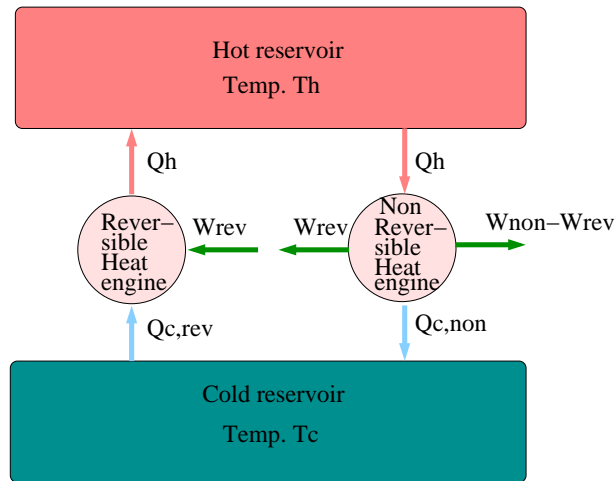


Figure 2.4: *A reversible engine A and a no-reversible engine B. The reversible engine has been reversed.*

engine (the combination of the two engines), that takes heat from a reservoir and convert this to work, BUT this is impossible due to the second law. So the reversible engine is an optimal engine.  $\square$

Later on in the note there will be an example of a reversible engine, although this is not implementable, it will show characteristics of an optimal heat engine. An important characteristic is (proved in the example)

$$\frac{Q_h}{T_h} = \frac{Q_c}{T_c} \quad (2.1)$$

If the heat engine is a **steam engine** the best performance is



$$\eta_{max} = \max \frac{W}{Q_h} \quad (2.2)$$

Using eq. 2.1 and the first law we find

$$\begin{aligned} \eta_{max} &= \max \frac{W}{Q_h} = \max \frac{Q_h - Q_c}{Q_h} = \frac{Q_h - \frac{Q_h}{T_h} T_c}{Q_h} \\ &= 1 - \frac{T_c}{T_h} = \frac{T_h - T_c}{T_h} \end{aligned} \quad (2.3)$$

So the max efficiency for a steam engine is

$$\boxed{\eta_{max, steam-engine} = \max \frac{W}{Q_h} = 1 - \frac{T_c}{T_h} = \frac{T_h - T_c}{T_h}} \quad (2.4)$$

Note that the temperatures are in Kelvin; also that the nature gives a limit to efficiency only defined by temperatures. A power plant working between 650 °C and 0 °C has a theoretical max. efficiency of 70 %. Note that the second law claimed that  $\frac{W}{Q_h} < 1$  and Carnot found, by using the second law, that the max is  $\frac{W}{Q_h} = 1 - \frac{T_c}{T_h}$

For a **heat pump** a coefficient of performance (COP) is defined as

$$COP = \frac{Q_h}{W} \quad (2.5)$$

The max COP is

$$COP_{max} = \max \frac{Q_h}{W} \quad (2.6)$$

Using the first law and eq. 2.1 it is found

$$\boxed{COP_{max, heat-pump} = \frac{T_h}{T_h - T_c}} \quad (2.7)$$

Note that this is the inverse of the steam-engine. As an example an air to air heat pump operating at an ambient temperature of 2 °C and outlet temperature of 40 °C has a max COP of 8.

Finally for a **refrigerator** the COP is

$$COP = \frac{Q_c}{W} \quad (2.8)$$

and the best COP is

$$\boxed{COP_{max, refrigerator} = \frac{T_c}{T_h - T_c}} \quad (2.9)$$

An refrigerator operating between 4 °C and 50 °C has a max efficiency of 6.

Going back to one of the characteristic of the optimal engine

$$\frac{Q_h}{T_h} = \frac{Q_c}{T_c} \quad (2.10)$$

we see that the fraction heat per temperature is that same at the heat input and output of the optimal engine. This fraction is called **entropy**.

If we go from one equilibrium to another in a reversible system with a very small amount of heat transferred  $dQ_{rev}$  then the change in entropy is

$$dS = \frac{dQ_{rev}}{T} \quad (2.11)$$

To calculate the change in entropy we must integrate the entropy from the initial state to the final state along a reversible path.

$$\Delta S = \int_a^b \frac{dQ_{rev}}{T} \quad (2.12)$$

If we integrate the entropy all along a reversible cycle the change in entropy is zero.

An interesting thing is that curves with same entropy (isentropics) can be mapped on a state diagram for a given fluid. This is often the case in a P - h diagram where isentropics also can be seen. This is useful, because in a cyclic process, this gives information on, how far away we are from the theoretical best cycle. On figure 2.5 isentropics for water/and steam are shown.

At the end of the day all work will turn into heat and go into a heat reservoir; this increases the entropy. The entropy is defined so it follows the heat flow, if the heat flows into a reservoir the entropy will increase, and if it flows out of a reservoir the entropy will decrease. When two reservoirs are in thermal contact there will flow heat from the hot to the cold, the entropy out of the hot is  $\frac{Q}{T_h}$ , the entropy into the cold is  $\frac{Q}{T_c}$  ( $Q$  is the same), because  $\frac{Q}{T_c} > \frac{Q}{T_h}$  the total entropy of the two reservoirs will increase. The facts that work will increase entropy and heat transfer will increase entropy, means that the second law also can be formulated as: The entropy in the universe is always increasing.

Locally the entropy can decrease, but if this is the case, there will always be another location, where the entropy increases more. The only engine that do not increase the entropy is the optimal engine; unfortunately this can not be implemented in practice.

## 2.1 Examples: Second law of thermodynamic

In this section there is first an example of a reversible engine based on an ideal gas; the engine is not implementable, but it gives upper limits of efficiency. Then there are examples of implementable engines, namely a cycle in a power plant (Rankine cycle), and a refrigeration cycle.

### Example of a reversible engine (an optimal engine)

In this example there will be a suggestion for a reversible engine working optimal between two heat reservoirs. The engine uses an ideal gas. In the literature this is called a Carnot engine.

First the engine is assumed frictionless. Then we assume: when a hot or a cold reservoir is in contact with the engine, the heat flow between the engine and the reservoir can be reversed by a small change in the work. This is only possible if the temperature difference between a reservoir and the engine is very small. So any heat transport must take place between two very close temperatures to ensure reversibility.

The engine works in a cycle between four thermodynamic states called a,b,c and d. The engine consist of a cylinder with a piston containing an ideal gas. The cylinder is in some parts of

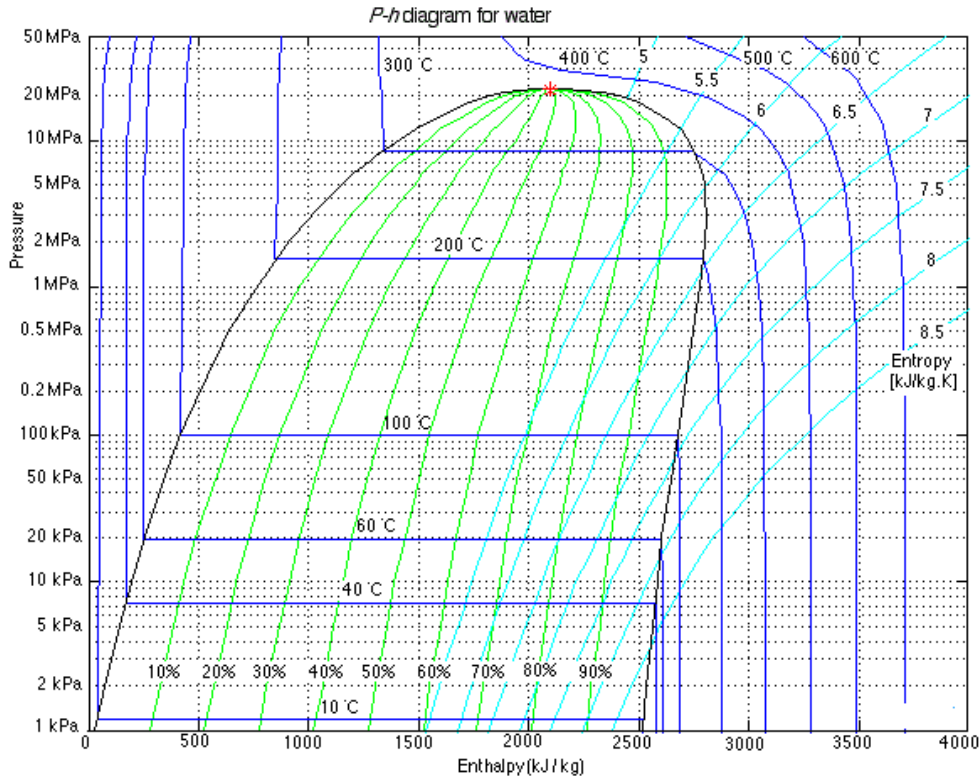


Figure 2.5:  $p - h$  diagram for water/steam, including isentropics.

the cycle totally isolated so no heat is transferred, in other parts of the cycle the cylinder is in contact with a reservoir so heat can be transferred to or from the ideal gas. In figure 2.6 the engine is illustrated in four situations. In  $a \rightarrow b$  the cylinder is in contact with the hot reservoir, we start at a high pressure and let the gas do work, this means that the pressure drops and the volume increase. If we do this very slowly, the temperature in the gas will be equal to the reservoir temperature, and the process is described by

$$PV = nRT_h = K_{ab} \Rightarrow P = nRT_h \frac{1}{V} \quad (2.13)$$

This curve is seen in the P-V diagram figure 2.7.

At some point (b) we make a total thermal insulation and let the piston do work; this means that the pressure will drop further and the volume will increase. The gas will now be described by the equation for the adiabatic condition

$$PV^\gamma = K_{bc} \Rightarrow P = K_{bc} \frac{1}{V^\gamma} \quad (2.14)$$

where  $\gamma$  is a constant larger than 1, so the adiabatic curve from b to c has a more negative slope than the isothermal curve from a to b; this is seen at figure 2.7. Under adiabatic expansion the temperature in the gas decreases, and the point c is where the gas temperature, is the same, as the temperature in the cold reservoir.

Between the points c and d work is done on the gas. The isothermal process is described by

$$PV = nRT_c = K_{cd} \Rightarrow P = nRT_c \frac{1}{V} \quad (2.15)$$

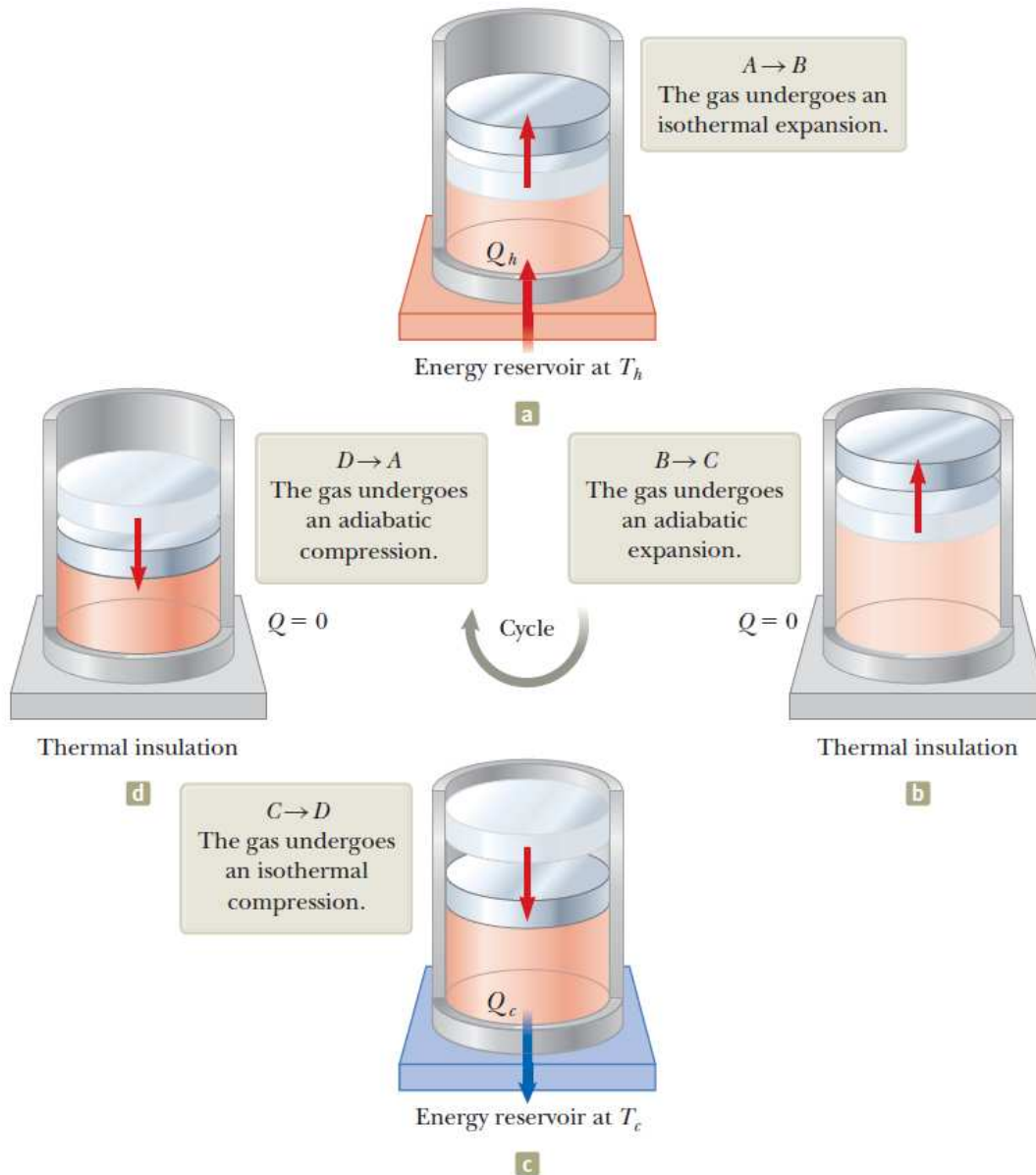


Figure 2.6: A reversible (Carnot) engine [JS08].

Finally when we come to the point d the cylinder is isolated and under adiabatic conditions we press the gas until the point a is reached; this is described by

$$PV^\gamma = K_{ad} \Rightarrow P = K_{ad} \frac{1}{V^\gamma} \quad (2.16)$$

We observe that in the two paths  $a \rightarrow b$  and  $b \rightarrow c$  the gas is doing work on the surrounding (given by the area below the curves) and in the paths  $c \rightarrow d$  and  $d \rightarrow a$  the surrounding is doing work on the gas (given as the area below the curves). When subtracting the two areas we find the area inside the graph, this is the total work done by the gas.

Another property of an ideal gas is that the internal energy of the gas only depends on the

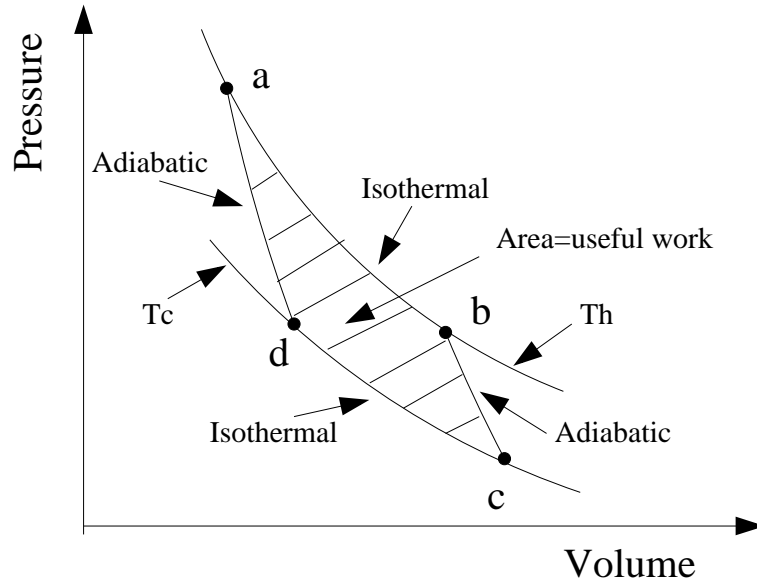


Figure 2.7: *The Carnot cycle in a P-V diagram.*

temperature; then using first law the work from  $a \rightarrow b$  is

$$W_{a \rightarrow b} = Q_h = \int_a^b P dV = \int_a^b nRT_h \frac{1}{V} dV = nRT_h \ln\left(\frac{V_b}{V_a}\right) \quad (2.17)$$

Similar from  $c \rightarrow d$

$$W_{c \rightarrow d} = Q_c = nRT_c \ln\left(\frac{V_c}{V_d}\right) \quad (2.18)$$

Note in the ln-function that this is negative.

To finish our analyses we need only to find a relation between  $V_c/V_d$  and  $V_b/V_a$ . This is done by noting that under adiabatic expansion we have that  $PV^\gamma$  is a constant. Since  $PV = nRT$ , we can write this as  $(PV)V^{\gamma-1} = \text{constant}$  or in terms of  $T$  and  $V$ , as  $TV^{\gamma-1} = \text{constant}$ , or

$$T_h V_b^{\gamma-1} = T_c V_c^{\gamma-1} \quad (2.19)$$

Likewise we also find

$$T_h V_a^{\gamma-1} = T_c V_d^{\gamma-1} \quad (2.20)$$

If we divide this equation with the previous one, we find that  $V_b/V_a$  must equal  $V_c/V_d$ , so the ln's in equation 2.17 and 2.18 are equal, and that

$$\boxed{\frac{Q_c}{T_c} = \frac{Q_h}{T_h}} \quad (2.21)$$

This relation is true for all reversible engines and then also for all optimal heat engines. This also shows that an optimal engine must have the same entropy input and output, so it is isentropic. It is also useful to draw the Carnot engine in a  $P - h$  diagram instead of a  $P - V$  diagram.

This Carnot engine is impossible to construct because it must be friction less, and because the temperature difference between the reservoirs and the ideal gas must go to zero implying that the time it takes to move Joules goes to infinity; indeed a long time.

**A Carnot cycle in a P-h diagram.** This cycle is optimal for both power generation and for heat pumps/refrigeration system, because the engine is reversible. In these cases we do not use an ideal gas, but use a fluid that can make a phase shift; water/steam in a power plant and a special fluid like R134a in a refrigerator. The reason for using phase shift is that a large amount of heat can be absorbed or released without changing the temperature and pressure; this is useful in the isothermal paths. The path b  $\rightarrow$  c is implemented as a turbine in the power plant, and c  $\rightarrow$  b is a pump (compressor) in the refrigeration. The path d  $\rightarrow$  a is a feedwater pump in the power plant and a reduction valve in the fridge. The Carnot cycle is shown in the left graph figure 2.8.

### Example: Refrigeration system

A refrigeration system is illustrated in the P-h diagram figure 2.8 right graph. The fridge consists of four elements. An evaporator placed inside the fridge, where heat is taken from the food stuff and transferred into the evaporator. In the evaporator we have a refrigerant (e.g. R134a) that is kept on a low pressure, for example 3 bar where the boiling point is 0 °C. If the temperature inside is a bit higher than 0 °C heat will be transferred into the evaporator by convection; in steady state the temperature will be 0 °C. The compressor is controlled so the evaporator pressure is 3 bar. In the compressor the pressure is raised, for example to 7 bar. Unfortunately compressors are not isentropic so the efficiency is lower than in a Carnot cycle; this is seen in figure 2.8 where the enthalpy added from the compressor is larger than the Carnot value. In the point b the vapour/gas is led to the condenser. The condenser is placed outside the fridge. In the condenser the refrigerant will transfer heat to the ambient; if the pressure in the condenser is 7 bar the condensing point is 25 °C. The final component in the fridge is the valve where the pressure is reduced. This operation is nearly isentropic so there is no loss in efficiency here.

The efficiency for the refrigerator is

$$COP = \frac{Q_{evaporation}}{W_{compressor}} = \frac{m(h_2 - h_1)}{m(h_4 - h_2)} = \frac{h_2 - h_1}{h_4 - h_2} \quad (2.22)$$

where  $m$  is the mass flow, the enthalpies can be seen on the figure.

A booklet with a down to earth presentation of refrigeration systems is found in [Dan78].

### Example: Calculation of entropy for an ideal gas

For an ideal gas given by

$$PV = nRT \quad (2.23)$$

the entropy change is given by equation (2.12)

$$\Delta S = \int_a^b \frac{dQ_{rev}}{T} \quad (2.24)$$

from state a to state b.

Using the first law

$$dU = dQ + dW \Rightarrow dU = dQ - PdV \Rightarrow dQ = dU + PdV \quad (2.25)$$

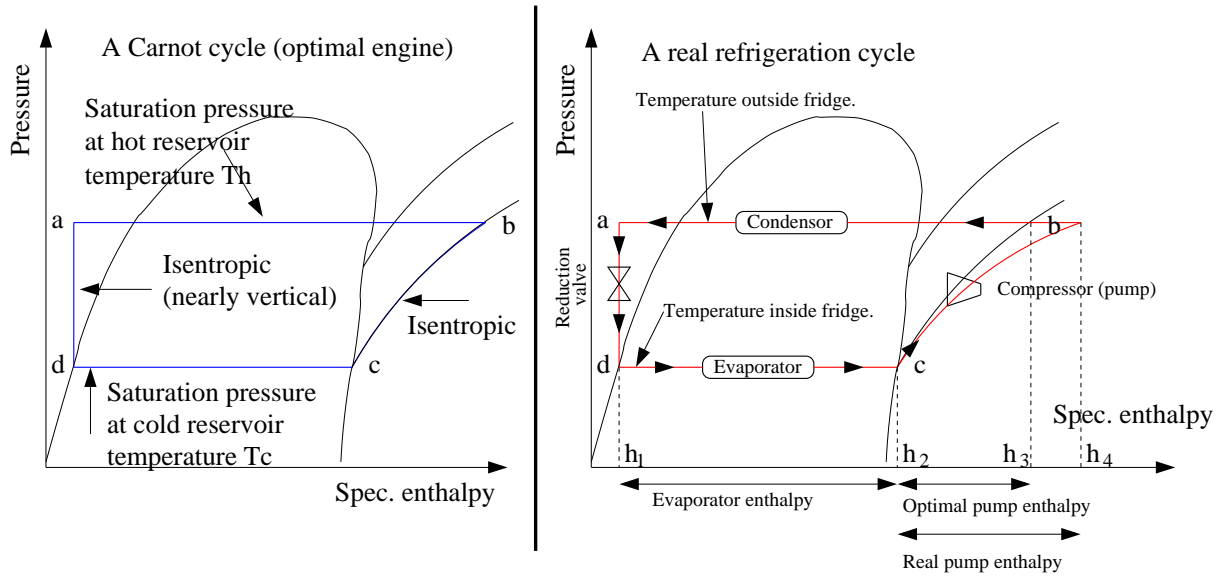


Figure 2.8: A Carnot cycle and a real refrigeration cycle.

The expression  $dW = -PdV$  is found in the section about work.

Now inserting gives

$$\Delta S = \int_a^b \frac{dQ_{rev}}{T} = \int_a^b \frac{dU}{T} + \int_a^b \frac{PdV}{T} \quad (2.26)$$

defining

$$c_v = \frac{dU}{dT} \quad \text{and using} \quad P = \frac{nRT}{V} \quad (2.27)$$

gives

$$\Delta S = \int_a^b \frac{c_v}{T} dT + \int_a^b \frac{nR}{V} dV \quad (2.28)$$

Assuming constant  $c_v$  gives

$$\Delta S = c_v \ln\left(\frac{T_b}{T_a}\right) + nR \ln\left(\frac{V_b}{V_a}\right) \quad (2.29)$$

It is also possible to include pressure in a  $\Delta S$  expression by first using the definition of enthalpy

$$dH = dU + d(PV) = dU + PdV + VdP = dQ - PdV + PdV + VdP \Rightarrow \quad (2.30)$$

$$dQ = dH - VdP = c_p dT - VdP \Rightarrow dS = \frac{dQ}{T} = c_p \frac{dT}{T} - \frac{nR}{P} dP \quad (2.31)$$

Integrating

$$\Delta S = \int_a^b c_p \frac{dT}{T} - \int_a^b \frac{nR}{P} dP \quad (2.32)$$

giving for constant  $c_p$

$$\Delta S = c_p \ln\left(\frac{T_b}{T_a}\right) - nR \ln\left(\frac{P_b}{P_a}\right) \quad (2.33)$$

(it can be proven for an ideal gas that  $c_p = c_v + R$ ).

Using the equations (2.29) and (2.33) and one known value of entropy, all values of entropy in

a (P,V), (P,T) or (T,V) diagram can be calculated. Furthermore curves where the entropy is constant (the Carnot engine) may be calculated. Sometimes the expressions are also used in the steam area for water/steam; more correct measured values can be found in a steam table.



## Chapter 3

# Distributed parameter models

In modelling a system, one often distinguishes between the two model types

- Models using lumped parameters
- Models using distributed parameters

Models using lumped parameters are characterized by the fact that the variables only depend on time, whereas in models using distributed parameters, the variables will depend on both time and position.

Models using lumped parameters are described by ordinary differential equations. Models using distributed parameters are described by partial differential equations.

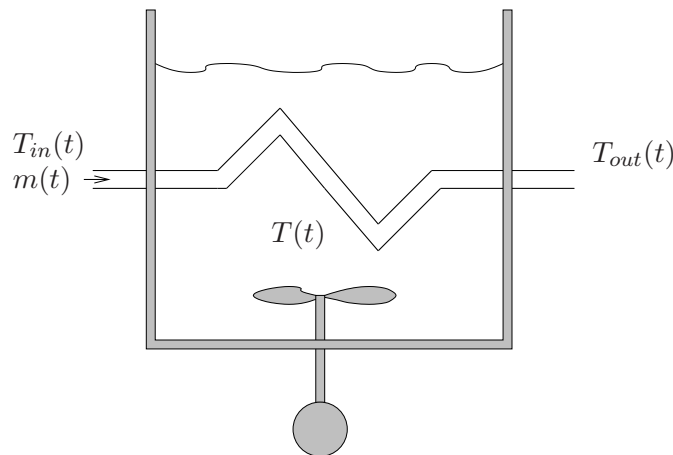


Figure 3.1: *Simple heat exchanger*

Figure 3.1 shows a heat exchanger where a fluid in the pipe is heating the fluid in the tank. The tank fluid is stirred so the temperature all over is the same. The flow is  $m(t)$  and the input temperature is  $T_{in}(t)$ ; the output temperature is  $T_{out}(t)$  and the temperature of the fluid in the tank is  $T(t)$ .

In a lumped parameter description with one control volume in the tube, the fluid in the tube has the same temperature all over, giving a model of the form:

$$f\left(\frac{dT_{out}(t)}{dt}, T_{in}(t), T(t), T_{out}(t), m(t)\right) = 0$$

In a distributed parameter description, the temperature in the tube will depend on the position, giving a model of the form

$$f\left(\frac{\partial T(x, t)}{\partial t}, \frac{\partial T(x, t)}{\partial x}, T(t), m(t)\right) = 0$$

$$T(0, t) = T_{in}(t) \quad \text{with} \quad T(L, t) = T_{out}(t)$$

where  $L$  is the length of the tube.

A model using distributed parameters gives a more detailed description than a model using lumped parameters; but a distributed parameter model is not easy to handle, for instance no control design method has been developed for distributed parameter systems.

The model type must be chosen with respect to

- field of application (controller design, simulation etc.)
- steady state demands
- transient demands (bandwidth)

Most of the description will be based on energy balances, but for completeness we start with a mass balance.

### Mass balance for distributed systems

The mass balance for a slice from  $x$  to  $x + \delta x$  as shown in figure 3.2 can be written using

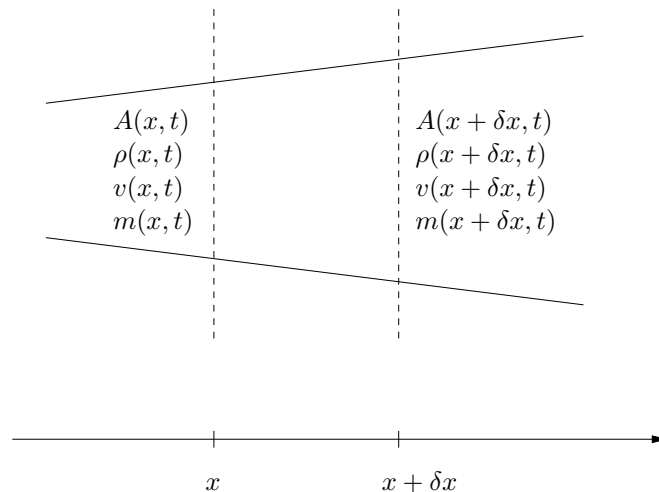


Figure 3.2: *Flow in a tube with distributed parameters*

equation 1.26, where we express the mass held in the control volume of finite size as an integral

$$\frac{d}{dt}(M(t)) = \frac{d}{dt}\left(\int_x^{x+\delta x} A(\xi, t)\rho(\xi, t)d\xi\right) = m(x, t) - m(x + \delta x, t) \quad (3.1)$$

Usually we have to use for instance backward approximation for the mass. If we look at an infinitesimal 'slice' (length  $dx$ ) the integral becomes  $A(x, t)\rho(x, t)dx$  and the balance will be.

$$\frac{\partial[A(x, t)\rho(x, t)]}{\partial t}dx = m(x, t) - m(x + dx, t) \quad (3.2)$$

If we divide equation 3.1 by  $\delta x$  and let  $\delta x \rightarrow 0$  we get the partial differential equation of the **one dimensional mass balance**

$$\boxed{\frac{\partial[A(x, t)\rho(x, t)]}{\partial t} + \frac{\partial m(x, t)}{\partial x} = 0} \quad (3.3)$$

We can express mass flow like

$$m(x, t) = A(x, t)v(x, t)\rho(x, t) \quad (3.4)$$

and get the **one dimensional mass balance** in terms of  $A$ ,  $\rho$  and  $v$

$$\boxed{\frac{\partial[A(x, t)\rho(x, t)]}{\partial t} + \frac{\partial[A(x, t)v(x, t)\rho(x, t)]}{\partial x} = 0} \quad (3.5)$$

### Energy balance for a counter current heat exchanger.

In figure 3.3 is a sketch of a counter current heat exchanger.

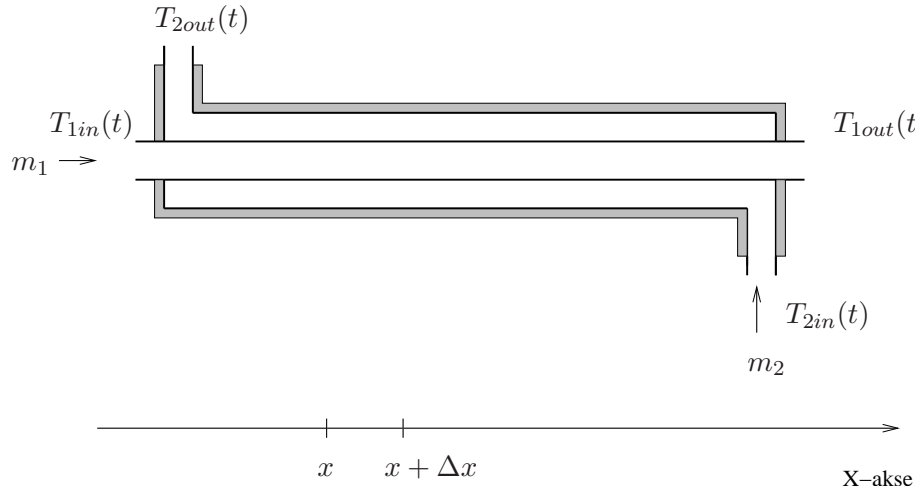


Figure 3.3: *Counter current heat exchanger*

We want a model with  $T_{1in}$  and  $T_{2in}$  as input and  $T_{2out}$  and  $T_{1out}$  as output. The assumptions are:

- no energy accumulation in the tube walls.
- the temperature is the same in a cross-sectional area.
- the velocity is the same in a cross-sectional area.
- heat capacities and densities are constant.
- the mass flows  $m_1$  and  $m_2$  are constant.
- the power between the two fluids is proportional to the temperature differences

For fluid 1 the energy balance for the small CV, shown in figure 3.3, is

$$c_1 A_1 \rho_1 \Delta x \frac{\partial T_1(x, t)}{\partial t} = m_1 c_1 (T_1(x, t) - T_1(x + \Delta x, t)) + \alpha U \Delta x (T_2(x, t) - T_1(x, t))$$

where

- $c_1$  is the spec. heat capacity of fluid 1 [ $J/(Kkg)$ ]
- $\alpha$  is heat transfer coef. between fluid 1 and fluid 2 [ $J/(sKm^2)$ ]
- $U$  is circumference of the inner tube [ $m$ ]
- $A_1$  is cross-sectional area of inner tube [ $m^2$ ].
- $\rho_1$  is density of fluid 1 [ $kg/m^3$ ]

The equation is divided by  $\Delta x$ , and for  $\Delta x \rightarrow 0$  we find the partial differential equation

$$-m_1 c_1 \frac{\partial T_1(x, t)}{\partial x} - c_1 A_1 \rho_1 \frac{\partial T_1(x, t)}{\partial t} + \alpha U (T_2(x, t) - T_1(x, t)) = 0 \quad (3.6)$$

Similar the energy balance for fluid 2 gives

$$m_2 c_2 \frac{\partial T_2(x, t)}{\partial x} - c_2 A_2 \rho_2 \frac{\partial T_2(x, t)}{\partial t} - \alpha U (T_2(x, t) - T_1(x, t)) = 0 \quad (3.7)$$

### General one dimensional energy balance.

Very similar to the previous example, the energy balance for a small CV, is given by

$$\Delta x \frac{\partial A \rho(x, t) (h(x, t) - \frac{P(x, t)}{\rho(x, t)})}{\partial t} = m(x, t) h(x, t) - m(x + \Delta x, t) h(x + \Delta x, t) + \alpha U \Delta x (T_2(x, t) - T_1(x, t))$$

The equation is divided by  $\Delta x$ , and for  $\Delta x \rightarrow 0$  we find the partial differential equation (**one dimensional energy balance**)

$$\boxed{\frac{\partial [A \rho(x, t) (h(x, t) - \frac{P(x, t)}{\rho(x, t)})]}{\partial t} + \frac{\partial [m(x, t) h(x, t)]}{\partial x} + \alpha U (T_1(x, t) - T_2(x, t)) = 0}$$

## Flow in an open channel

On figure 3.4 is shown a fluid streaming in an open channel. We want a model with mass

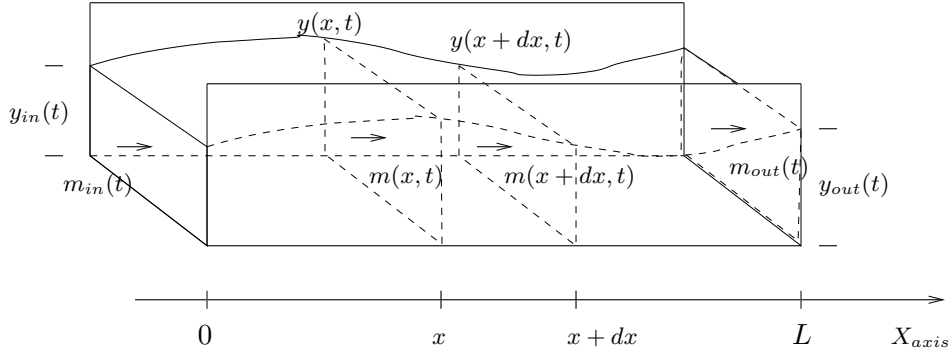


Figure 3.4: *Flow in an open channel*

flow,  $m_{in}(t)$ , and fluid level,  $y_{in}(t)$ , as inputs, and as outputs,  $m_{out}(t)$ , and fluid level,  $y_{out}(t)$ . Assumptions:

- the channel has a rectangular cross-sectional area.
- the channel is in level.
- the velocity of the fluid is uniform in a cross-sectional area.
- the fluid density is constant.
- the fluid is friction-less.

For a thin slice with length  $\Delta x$  the mass balance gives

$$\frac{\partial(B\rho y(x, t)\Delta x)}{\partial t} = m(x, t) - m(x + \Delta x, t)$$

where

- $m(x, t)$  is mass flow [ $kg/s$ ]
- $y(x, t)$  is level [ $m$ ]
- $B$  is channel width [ $m$ ]
- $\rho$  is density of fluid [ $kg/m^3$ ]

Division by  $\Delta x$  and  $\Delta x \rightarrow 0$  gives

$$\frac{\partial m(x, t)}{\partial x} + B\rho \frac{\partial y(x, t)}{\partial t} = 0 \quad (3.8)$$

The momentum balance for the slice gives

$$\frac{d(m(x,t)\Delta x)}{dt} = \frac{m(x,t)^2}{\rho B y(x,t)} - \frac{m(x+\Delta x,t)^2}{\rho B y(x+\Delta x,t)} + \frac{1}{2}B\rho g y(x,t)^2 - \frac{1}{2}B\rho g y(x+\Delta x,t)^2$$

Re-organizing and letting  $\Delta x \rightarrow 0$  gives

$$\frac{2m(x,t)}{\rho B y(x,t)} \frac{\partial m(x,t)}{\partial x} + \left[ \frac{-m(x,t)^2}{\rho B y(x,t)^2} + B\rho y(x,t) \right] \frac{\partial y(x,t)}{\partial x} + \frac{\partial m(x,t)}{\partial t} = 0 \quad (3.9)$$

The two partial differential equations, are together with the initial conditions

$$m_{ind}(t) = m(0,t) \quad y_{ind}(t) = y(0,t) \quad m_{ud}(t) = m(L,t) \quad y_{ud}(t) = y(L,t)$$

the model ( $L$  is the channel length).

### Transmission lines

A number of problems can be solved using the theory for transmission lines, a topic that has been profound analysed in the field of telecommunication.

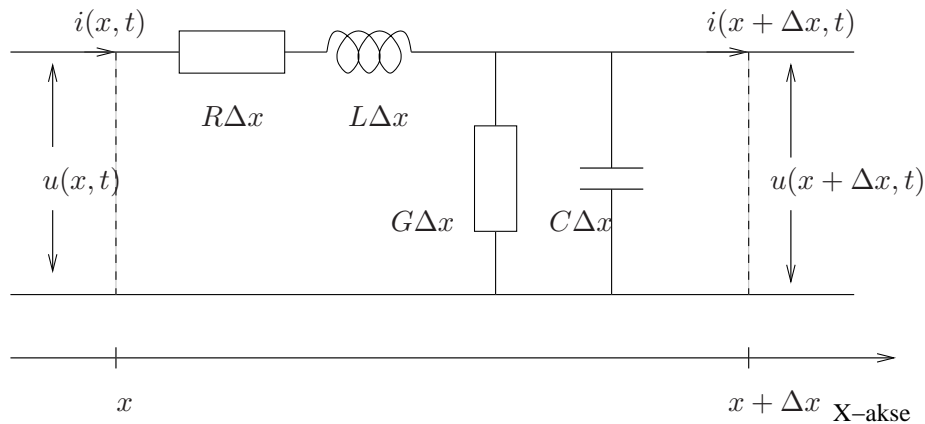


Figure 3.5: *An electric transmission line*

A section of the line is shown in figure 3.5, where

- $R$  resistance per length [ $Ohm/m$ ]
- $L$  induction per length [ $H/m$ ]
- $G$  conductance per length [ $1/(Ohmm)$ ]
- $C$  capacity per length [ $F/m$ ]

. Kirchhoff's laws give

$$u(x+\Delta x,t) - u(x,t) = -R\Delta x i(x,t) - L\Delta x \frac{\partial i(x,t)}{\partial t}$$

$$i(x + \Delta x, t) - i(x, t) = -G\Delta x u(x, t) - C\Delta x \frac{\partial u(x, t)}{\partial t}$$

By dividing  $\Delta x$  and  $\Delta x \rightarrow 0$

$$\frac{\partial u(x, t)}{\partial x} + Ri(x, t) + L \frac{\partial i(x, t)}{\partial t} = 0 \quad (3.10)$$

$$\frac{\partial i(x, t)}{\partial x} + Gu(x, t) + C \frac{\partial u(x, t)}{\partial t} = 0 \quad (3.11)$$

The two partial differential equations 3.10 og 3.11 are the model of the transmission line.  $\square$

Above we have shown examples of models with distributed parameter described by partial differential equations. Solving partial differential equations is more difficult than solving ordinary differential equations. In the next chapter we will shown how the counter current heat exchanger partial differential equations can be solved.

### 3.1 Solving partial differential equations

Solving partial differential equations is a comprehensive mathematical topic and in this note we will only look at solutions using a simple example, namely a counter current heat exchanger.

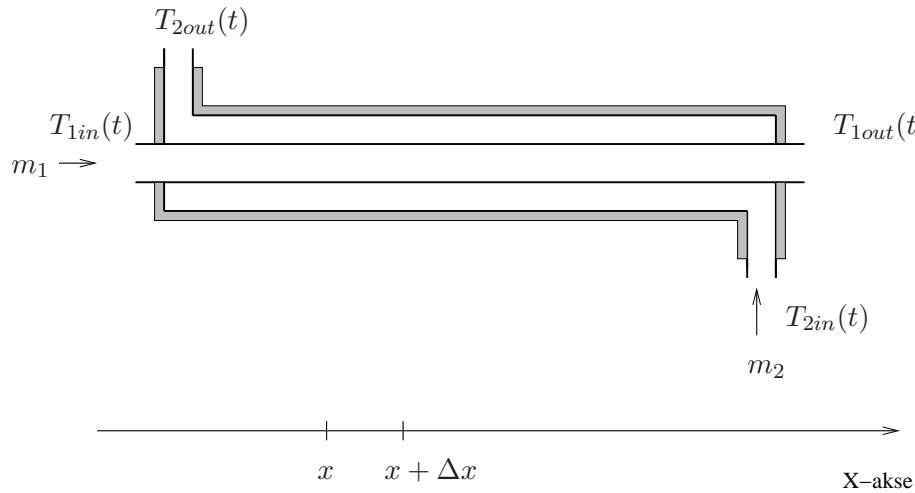


Figure 3.6: *Counter current heat exchanger*

The equations were

$$-m_1 c_1 \frac{\partial T_1(x, t)}{\partial x} - c_1 A_1 \rho_1 \frac{\partial T_1(x, t)}{\partial t} + \alpha U (T_2(x, t) - T_1(x, t)) = 0 \quad (3.12)$$

$$m_2 c_2 \frac{\partial T_2(x, t)}{\partial x} - c_2 A_2 \rho_2 \frac{\partial T_2(x, t)}{\partial t} - \alpha U (T_2(x, t) - T_1(x, t)) = 0 \quad (3.13)$$

We will only study the symmetric case where

$$\frac{m_1}{A_1 \rho_1} = \frac{m_2}{A_2 \rho_2} = v \quad \text{and} \quad \frac{\alpha U}{c_1 A_1 \rho_1} = \frac{\alpha U}{c_2 A_2 \rho_2} = \frac{1}{\tau}$$

giving the model equations

$$\frac{\partial T_1(x, t)}{\partial t} + v \frac{\partial T_1(x, t)}{\partial x} = \frac{1}{\tau} (T_2(x, t) - T_1(x, t)) \quad (3.14)$$

$$\frac{\partial T_2(x, t)}{\partial t} - v \frac{\partial T_2(x, t)}{\partial x} = \frac{1}{\tau} (T_1(x, t) - T_2(x, t)) \quad (3.15)$$

The two equations are linear and it is possible to use **Laplace transform** and find an analytic solution. However the Laplace transform is complex to derive and the result difficult to use for control design. We will only show an example of such transfer functions to give an impression of the nature the transfer function for the symmetric heat exchanger.

As an example the transfer function from  $T_{1in}$  to  $T_{2out}$  is found to

$$T_{2out}(s) = \frac{1}{(s^2\tau^2 + 2s\tau)^{1/2} \coth(v^{-1}\tau^{-1}L(s^2\tau^2 + 2s\tau)^{1/2}) + s\tau + 1} T_{1in}(s) \quad (3.16)$$

where the hyperbolic cotangent  $\coth$  is given as

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad (3.17)$$

As seen the transfer function is not as usual given by polynomial fraction, because of terms as  $\sqrt{s}$  and  $\exp(\sqrt{s})$ . This makes it difficult to calculate poles; this is also known from delay systems which can also be described using partial differential equations.

It is possible to draw bode plot for the transfer function, just by replacing  $s = j\omega$ . In the below plots the transfer function is sketched as functions of  $\Omega = \omega\tau$ , because this gives  $L/(v\tau)$  as the only variable in the transfer function.  $\tau$  is the thermal time constant of the heat exchanger and  $t_d = L/v$  is the transport delay through the exchanger. The ratio  $t_d/\tau$  is important when analyzing the dynamic properties of the exchanger as seen in figure 3.7. As seen in figure 3.7 the plots have an unusual shape for high frequencies when the ratio  $t_d/\tau$  is small. It should be noticed that bode plot could be used directly for controller design using gain and phase margin criteria. As mentioned in the beginning Laplace transform of partial differential equations is however rarely used maybe because they are difficult to derive.  $\square$

**Sectioning** of the partial differential equations is an often used way to analyze a model and is also often used in controller design. Here the exchanger is divided in a final number of sections. The equations are

$$\frac{\partial T_1(x, t)}{\partial t} + v \frac{\partial T_1(x, t)}{\partial x} = \frac{1}{\tau} (T_2(x, t) - T_1(x, t)) \quad (3.18)$$

$$\frac{\partial T_2(x, t)}{\partial t} - v \frac{\partial T_2(x, t)}{\partial x} = \frac{1}{\tau} (T_1(x, t) - T_2(x, t)) \quad (3.19)$$

Using the Backward Euler transformation for the x-component, following the direction of the flow gives

$$sT_1(x, s) + v \frac{T_1(x, s) - T_1(x - \Delta x, s)}{\Delta x} = \frac{1}{\tau} (T_2(x - \Delta x, s) - T_1(x, s)) \quad (3.20)$$

$$sT_2(x - \Delta x, s) - v \frac{T_2(x, s) - T_2(x - \Delta x, s)}{\Delta x} = \frac{1}{\tau} (T_1(x, s) - T_2(x - \Delta x, s)) \quad (3.21)$$



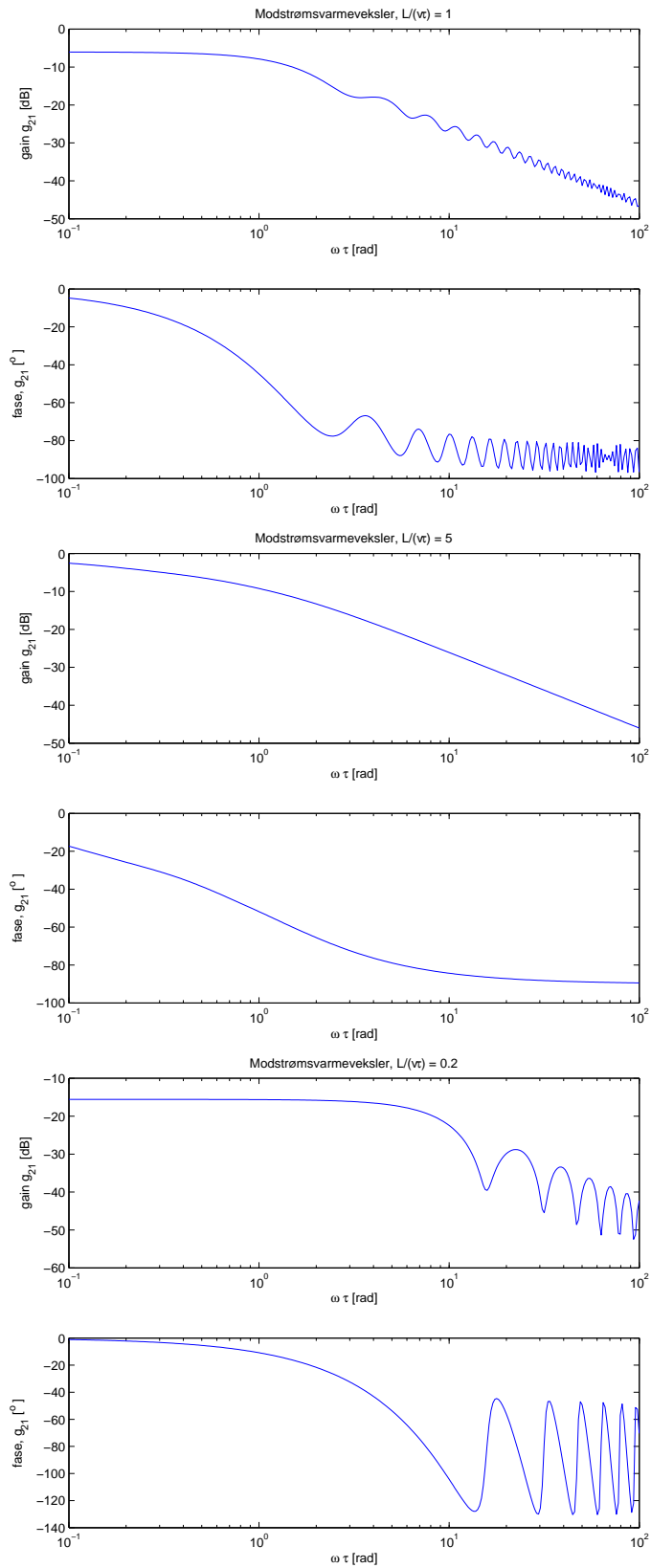


Figure 3.7: Bode plot for a counter flow heat exchanger for three values of  $L/(v\tau)$ .

From these equations the transfer function for one section is

$$\begin{bmatrix} T_1(x, s) \\ T_2(x - \Delta x, s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} T_1(x - \Delta x, s) \\ T_2(x, s) \end{bmatrix} \quad (3.22)$$

where

$$G_{11}(s) = G_{22}(s) = \frac{\tau v}{\Delta x} \frac{s\tau + v\tau/\Delta x + 1}{(s\tau + v\tau/\Delta x + 1)^2 - 1} \quad (3.23)$$

$$G_{12}(s) = G_{21}(s) = \frac{\tau v}{\Delta x} \frac{1}{(s\tau + v\tau/\Delta x + 1)^2 - 1} \quad (3.24)$$

If the exchanger is described by one section,  $\Delta x = L$  must be put in the equations 3.23 and 3.24

$$T_1(L, s) = T_{1out}(s) \quad T_1(0, s) = T_{1in}(s) \quad T_2(L, s) = T_{2in}(s) \quad T_2(0, s) = T_{2out}(s)$$

If we want an analytical model using  $n$  sections the equations 3.20 and 3.21 must be solved with respect to  $T_1(x, s)$  og  $T_2(x, s)$  giving

$$\begin{bmatrix} T_1(x, s) \\ T_2(x, s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{bmatrix} T_1(x - \Delta x, s) \\ T_2(x - \Delta x, s) \end{bmatrix} \quad (3.25)$$

where

$$H_{11}(s) = G_{11}(s) - G_{12}(s)G_{21}(s)/G_{22}(s) \quad (3.26)$$

$$H_{12}(s) = G_{12}(s)/G_{22}(s) \quad (3.27)$$

$$H_{21}(s) = -G_{21}(s)/G_{22}(s) \quad (3.28)$$

$$H_{22}(s) = 1/G_{22}(s) \quad (3.29)$$

giving the transfer matrix for  $n$  sections as

$$\begin{bmatrix} T_{1out}(s) \\ T_{2in}(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}^n \begin{bmatrix} T_{1in}(s) \\ T_{2out}(s) \end{bmatrix} \quad (3.30)$$

The transfer functions should have  $T_{1out}(s)$  og  $T_{2out}(s)$  as output og the final transfer matrix is found by solving 3.30 with respect to the two outputs.

As seen it is possible to find an approximate usual transfer function described as polynomial fractions in  $s$  using sectioning. The advantage is that the usual control design methods are applicable. Furthermore the method gives analytical transfer functions without and the influence of parameters can be studied. Each section can be investigated for possible simplifications. A disadvantage is the complex calculation work included, if a large number of sections should be included.

Note that the sectioning method can also be used to derive a state space formulation of the model.  $\square$

## Computer simulation of distributed systems

In the previous an analytical solution of systems of partial differential equations was investigated for the symmetric counter flow heat exchanger. Numeric solution of partial differential equations is, like the case with the analytical solutions, a comprehensive mathematical topic; this note can not cover this plethora of methods. Instead we will continue with the counter flow heat exchange example given by

$$\frac{\partial T_1(x, t)}{\partial t} + v \frac{\partial T_1(x, t)}{\partial x} = \frac{1}{\tau} (T_2(x, t) - T_1(x, t)) \quad (3.31)$$

$$\frac{\partial T_2(x, t)}{\partial t} - v \frac{\partial T_2(x, t)}{\partial x} = \frac{1}{\tau} (T_1(x, t) - T_2(x, t)) \quad (3.32)$$

The simulation will be implemented using a division in sections, where the exchanger will be split into a number of sections. For each section a physical meaningful model will be used.

We will use the sectioning shown in the previous where a Backward Euler approximation of equations 3.31 and 3.32 in position giving the two ordinary differential equations

$$\frac{dT_1(x, t)}{dt} + v \frac{T_1(x, t) - T_1(x - \Delta x, t)}{\Delta x} = \frac{1}{\tau} (T_2(x - \Delta x, t) - T_1(x, t)) \quad (3.33)$$

$$\frac{dT_2(x - \Delta x, t)}{dt} - v \frac{T_2(x, t) - T_2(x - \Delta x, t)}{\Delta x} = \frac{1}{\tau} (T_1(x, t) - T_2(x - \Delta x, t)) \quad (3.34)$$

The equations are normalized using the dimensionless quantity  $t \rightarrow t/\tau$

$$\frac{dT_1(x, t)}{dt} + v\tau \frac{T_1(x, t) - T_1(x - \Delta x, t)}{\Delta x} = T_2(x - \Delta x, t) - T_1(x, t) \quad (3.35)$$

$$\frac{dT_2(x - \Delta x, t)}{dt} - v\tau \frac{T_2(x, t) - T_2(x - \Delta x, t)}{\Delta x} = T_1(x, t) - T_2(x - \Delta x, t) \quad (3.36)$$

Now using Backward Euler approximation of the time-derivatives gives

$$\frac{T_1(x, t) - T_1(x, t - \Delta t)}{\Delta t} + v\tau \frac{T_1(x, t) - T_1(x - \Delta x, t)}{\Delta x} = T_2(x - \Delta x, t) - T_1(x, t) \quad (3.37)$$

$$\frac{T_2(x - \Delta x, t) - T_2(x - \Delta x, t - \Delta t)}{\Delta t} - v\tau \frac{T_2(x, t) - T_2(x - \Delta x, t)}{\Delta x} = T_1(x, t) - T_2(x - \Delta x, t) \quad (3.38)$$

It is seen in the equations 3.37 and 3.38, that for each section  $T_1(x - \Delta x, t)$  and  $T_2(x, t)$  are inputs.  $T_1(x, t)$  and  $T_2(x - \Delta x, t)$  are outputs (unknown). If we use  $n$  sections the unknown (state variable) will be

$$x(t) = \begin{bmatrix} T_1(L/n, t) \\ T_2(0, t) \\ T_1(2L/n, t) \\ T_2(L/n, t) \\ \vdots \\ T_1(L, t) \\ T_2((n-1)L/n, t) \end{bmatrix} \quad (3.39)$$

The system of equations for  $n$  sections is then

$$Ax(t) = \frac{1}{\Delta t} x(t - \Delta t) + \frac{v\tau n}{L} u(t) \quad (3.40)$$

where  $u(t)$  is the input vector given by

$$u(t) = \begin{bmatrix} T_1(0,t) \\ 0 \\ \cdot \\ \cdot \\ 0 \\ T_2(L,t) \end{bmatrix} \quad (3.41)$$

and  $A$  is a  $2n \times 2n$  matrix given by

$$A = \begin{bmatrix} a & -1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & a & 0 & b & 0 & \cdot & \cdot & \cdot & \cdot \\ b & 0 & a & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & a & 0 & b & 0 & \cdot & \cdot \\ 0 & 0 & b & 0 & a & -1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & 0 & a & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & -1 & a \end{bmatrix} \quad (3.42)$$

where

$$a = \frac{1}{\Delta t} + \frac{v\tau n}{L} + 1 \quad (3.43)$$

$$b = -\frac{v\tau n}{L} \quad (3.44)$$

It is seen that  $A$  is constructed so uneven rows contain the sequence  $b \ 0 \ a \ -1$  and even row numbers the sequence  $-1 \ a \ 0 \ b$ ;  $a$  is the term in the diagonal. The simulation is now implemented by at time  $t$  solve the equation system 3.40 with respect to  $x(t)$ . In the program, a Matlab procedure able to solve the equation system, is shown.

`% Simulation of a symmetric counter flow heat exchanger by sectioning`

```
r      =      input('L/v * \tau = r                               =');
dt     =      input('time step      dt                               =');
n      =      input('number of sections n                           =');
tmax   =      input('number of time steps tmax                       =');
```

`% Calculation of A,C and D matrices`

```
a = 1/dt + n/r + 1;
b = -n/r;
A = zeros(2*n,2*n);
C = zeros(2*n,1);
C(1,1) = n/r;
D = zeros(2*n,1);
D(2*n,1) = n/r;
for i = 1:n
    A(2*i-1,2*i-1) = a;
```

```

    if i>1 A(2*i-1,2*i-3)= b; end;
    A(2*i-1,2*i) = -1;
    A(2*i,2*i) = a;
    A(2*i,2*i-1) = -1;
    if i<n A(2*i,2*i+2) = b; end;
    end

% Initialising the state vector

x = zeros(2*n,1);

% Chosing the input sequence

T1ind = ones(tmax+1,1);
T2ind = zeros(tmax+1,1);

% Simulation of the sectioned equation system

for t = 1:tmax+1
    tid(1,t) = t*dt-dt;
    B = x/dt + T1ind(t,1)*C + T2ind(t,1)*D;
    x = A\B;
    T1ud(t) = x(2*n-1,1);
    T2ud(t) = x(2,1);
end

% Plot

subplot(2,1,1)
plot(tid,T2ud)
title(['Counter flow heat exchanger, L/(v \tau) = ', num2str(r)])
xlabel('time/\tau')
ylabel('T_{2ud} [^\circ C]')
subplot(2,1,2)
plot(tid,T1ud)
title(['Counter flow heat exchanger, L/(v \tau) = ', num2str(r)])
xlabel('time/\tau')
ylabel('T_{1ud} [^\circ C]')

```

There are plots for  $L/(v\tau)$  on 1, 5 and 0.2 similar to the amplitude and phase plots in the previous. As time step  $\Delta t/\tau = 0.01, 0.05$  and  $0.002$ ; the number of sections,  $n$  is 16.

On figure 3.8 is  $T_{2ud}$  the step response corresponding to the transfer function  $G_{21}(s)$ , whose amplitude and phase plots are shown in figure 3.7. Furthermore the step response for  $T_{1ud}$  corresponding to the transfer function  $G_{11}(s)$  is shown. It is seen that for  $T_{2ud}$  the step response is very similar to a first order response; this was also expected from the bode plots. For  $T_{1ud}$  is seen when  $L/v\tau$  is large a distinct time delay.

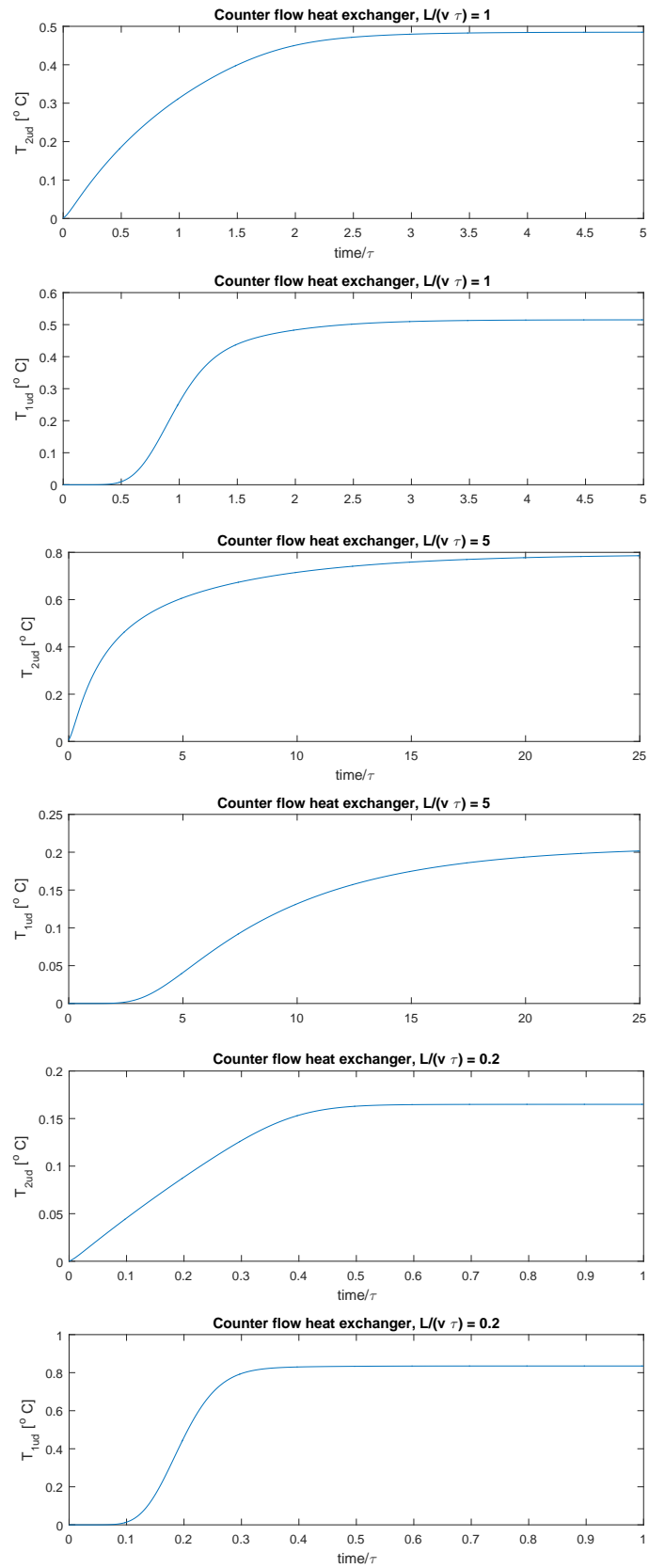


Figure 3.8: *Stepresponses for symmetric counter flow heat exchanger with three values of  $L/(v\tau)$ .*

# Chapter 4

## Moving boundary models

In this chapter we describe a modelling method that can be used in systems where there is a phase shift. As an example we have an evaporator in a refrigeration system; this can be operated such that pure liquid is entering and pure gas (vapor) is leaving. This gives three zones in the evaporator: in the first the temperature is increasing until the boiling point is reached, in the second the evaporation takes place until all liquid is evaporated and finally the third where the temperature of the gas is increased. The chapter is mainly based on [Jen03] and [JT02]; other references are [RA06], [RA04] and [WPPS98].

There are two ways to model this evaporator. One method use a fixed **sectioning** of the evaporator and use mass and the enthalpy in each section as variables. Using this method one of the sections will contain the point, where the liquid starts boiling, and another section will contain the point, where the transition between the mixed phase and the pure gas phase takes place. These two positions can over time move from section to section and the section model will change character depending on this. If we are interested in an accurate position of the two mentioned points a rather large number of sections could be required. If we are interested in the two mentioned positions a **moving boundary** model could be the solution. In this solution the control volume is not fixed to the position of the tube, but to the three mentioned zones. This gives three sections, one for the pure liquid, one for the mixed phase and one for the pure gas phase; these sections have moving boundaries.

The idea is now to write mass and energy balances for each subsection and as result have a model where the positions of the zones are part of state vector in the control model (of course there are other variables in the state vector).

One example of a moving boundary model of an evaporator takes its point of origin in figure 4.1. Here the lengths of the zones are  $L_1, L_2$  and  $L_3$  and it is assumed that  $L_1 + L_2 + L_3 = L$ . In the paper [JT02] and the Ph.D. thesis [Jen03] the modelling of the total evaporator is shown; in this note we will not give details of all zones. To give an impression of the model, the model if the mixed zone will be derived later, here the mass balance ends like

$$\mathcal{A} \frac{dL_1(t)}{dt} + \mathcal{B} \frac{dL_2(t)}{dt} + \mathcal{C} L_2(t) \frac{dP(t)}{dt} = m_{12} - m_{23} \quad (4.1)$$

The energy balance for the mixed zone gives

$$\mathcal{D} \frac{dL_1(t)}{dt} + \mathcal{F} \frac{dL_2(t)}{dt} + \mathcal{G} L_2(t) \frac{dP(t)}{dt} = m_{12} h_l - m_{23} h_g - \alpha_{mixed} L_2(t) (T_{w2} - T_{r2}) \quad (4.2)$$

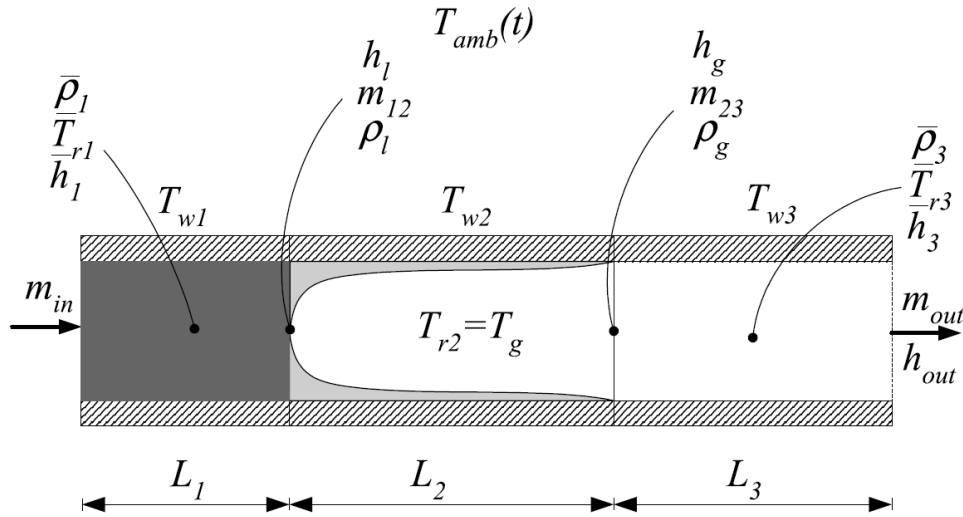


Figure 4.1: An evaporator shown with moving boundaries [JT02]

where  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$  depends on geometry, densities found in the steam table and a term called the average void fraction;  $\mathcal{D}$ ,  $\mathcal{F}$  and  $\mathcal{G}$  additionally include enthalpies found in the steam table. A detailed derivation follows later.

Notice that both equations have the three quantities  $L_1$ ,  $L_2$  and  $P$  as states. It should be noticed that the first terms depend on the length  $L_1$ , is a state variable of the previous zone; if the fluid enters the evaporator at the boiling point this term will be zero. Otherwise if all zones are relevant the equations must be solved together using the state variables for all zones.

With this type of model we can obtain a model which in most cases good approximation with low order, where models using sectioning of the distributed parameter system give high order and difficulties with handling the sections with the shift of zones.

The factors  $\mathcal{A}, \mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$ ,  $\mathcal{F}$  and  $\mathcal{G}$  may contain unpleasant expressions, but the equation system is of second order per zone are relatively easy to use in a control design context or for simulation.

### Derivation of mass balance for the mixed zone

For the derivation we take as the starting point the one dimensional mass balance in the form

$$\frac{\partial[A(x,t)\rho(x,t)]}{\partial t} = -\frac{\partial m(x,t)}{\partial x} \quad (4.3)$$

When we use this equation for a section with varying boundaries we integrate over the zone from  $L_1$  to  $L_1 + L_2$ , see figure 4.1:

$$\int_{L_1}^{L_1+L_2} \left( \frac{\partial[A(x,t)\rho(x,t)]}{\partial t} \right) dx = - \int_{L_1}^{L_1+L_2} \frac{\partial m(x,t)}{\partial x} dx \quad (4.4)$$

or with constant cross sectional area

$$A \int_{L_1}^{L_1+L_2} \left( \frac{\partial[\rho(x,t)]}{\partial t} \right) dx = -m(L_1 + L_2, t) + m(L_1, t) = -m_{23} + m_{12} \quad (4.5)$$



When we evaluate left side of this integral we must use Leibniz's rule for differentiation of integrals with varying limits:

$$\int_{z_1}^{z_2} \frac{\partial f(z, t)}{\partial t} dz = \frac{d}{dt} \int_{z_1}^{z_2} f(z, t) dz + f(z_1, t) \frac{dz_1}{dt} - f(z_2, t) \frac{dz_2}{dt} \quad (4.6)$$

Using this gives the equation

$$A \frac{d}{dt} \int_{L_1}^{L_1+L_2} \rho(x, t) dx + A\rho(L_1, t) \frac{dL_1}{dt} - A\rho(L_1 + L_2, t) \frac{d(L_1 + L_2)}{dt} = -m_{23} + m_{12} \quad (4.7)$$

A mean density  $\rho = \bar{\gamma}\rho_g + (1 - \bar{\gamma})\rho_l$  is assumed. The void fraction,  $\gamma : 0 < \gamma < 1$  describes the mean density. This quantity can be estimated using considerations about the distribution of evaporation, gas volume liquid volume and gas/liquid velocities along the zone. A thorough investigation of this can be found in [JT02]. The mass balance for the two-phase zone becomes

$$A \frac{d}{dt} (\rho L_2) + A(\rho_l - \rho_g) \frac{dL_1}{dt} - A\rho_g \frac{dL_2}{dt} = m_{12} - m_{23} \quad (4.8)$$

The time derivative of  $\rho$  is

$$\frac{d\rho}{dt} = (\bar{\gamma} \frac{d\rho_g}{dP} + (1 - \bar{\gamma}) \frac{d\rho_l}{dP}) \frac{dP}{dt} \quad (4.9)$$

which inserted in the mass balance gives the final mass balance for the two phase zone

$$A(\rho_l - \rho_g) \frac{dL_1(t)}{dt} + A(1 - \bar{\gamma})(\rho_l - \rho_g) \frac{dL_2(t)}{dt} + A(\bar{\gamma} \frac{d\rho_g}{dP} + (1 - \bar{\gamma}) \frac{d\rho_l}{dP}) L_2(t) \frac{dP(t)}{dt} = m_{12} - m_{23} \quad (4.10)$$

**Example** We will try to give a slightly more intuitive way to obtain equation 4.7 with reference to figure 4.2

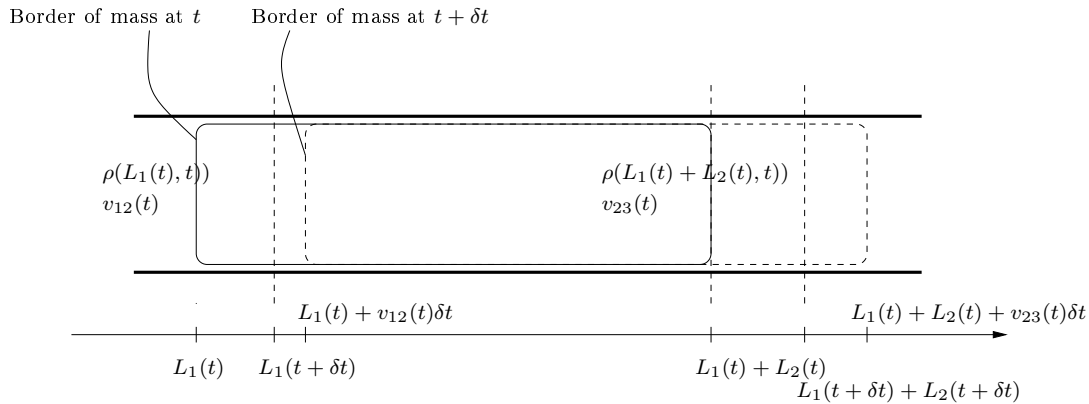


Figure 4.2: Sketch for explaining mass balance of moving boundary

We will set up a mass equation for a mass which we can denote  $M_{sys}$  which at time  $t$  is between positions  $L_1(t)$  and  $L_1(t) + L_2(t)$

$$M_{sys}(t) = \int_{L_1(t)}^{L_1(t)+L_2(t)} \rho(x, t) A dx \quad (4.11)$$

The same mass will at time  $t + \delta t$  be between positions  $L_1(t) + v_{12}(t)\delta t$  and  $L_1(t) + L_2(t) + v_{23}(t)\delta t$

$$M_{sys}(t + \delta t) = \int_{L_1(t) + v_{12}(t)\delta t}^{L_1(t) + L_2(t) + v_{23}(t)\delta t} \rho(x, t + \delta t) A dx \quad (4.12)$$

By letting the two right sides equal ( $M_{sys}(t) = M_{sys}(t + \delta t)$ ) we get

$$\int_{L_1(t)}^{L_1(t) + L_2(t)} \rho(x, t) A dx = \int_{L_1(t) + v_{12}(t)\delta t}^{L_1(t) + L_2(t) + v_{23}(t)\delta t} \rho(x, t + \delta t) A dx \quad (4.13)$$

We develop the right side such that the integral is taken from  $L_1(t + \delta t)$  to  $L_1(t + \delta t) + L_2(t + \delta t)$  and additional terms represent corrections because of the change in boundaries of the integral

$$\begin{aligned} \int_{L_1(t)}^{L_1(t) + L_2(t)} \rho(x, t) A dx &= \int_{L_1(t + \delta t)}^{L_1(t + \delta t) + L_2(t + \delta t)} \rho(x, t + \delta t) A dx \\ &+ \rho(L_1(t + \delta t), t + \delta t) A (L_1(t + \delta t) - L_1(t)) \\ &- \rho(L_1(t + \delta t) + L_2(t + \delta t), t + \delta t) A (L_1(t + \delta t) + L_2(t + \delta t) - L_1(t) - L_2(t)) \\ &- \rho(L_1(t + \delta t), t + \delta t) A v_{12}(t) \delta t \\ &+ \rho(L_1(t + \delta t) + L_2(t + \delta t), t + \delta t) A v_{23}(t) \delta t \end{aligned} \quad (4.14)$$

Now by dividing by  $\delta t$  and letting the  $\delta t \rightarrow 0$  and rearrange we obtain

$$\begin{aligned} \frac{d}{dt} \int_{L_1(t)}^{L_1(t) + L_2(t)} \rho(x, t) A dx + \rho(L_1(t), t) A \frac{dL_1(t)}{dt} \\ - \rho(L_1(t) + L_2(t), t) A \left( \frac{dL_1(t)}{dt} + \frac{dL_2(t)}{dt} \right) \\ - \rho(L_1(t), t) A v_{12}(t) \\ + \rho(L_1(t) + L_2(t), t) A v_{23}(t) = 0 \end{aligned} \quad (4.15)$$

inserting  $\rho A v = m$  we simply find equation 4.7.

□

### Derivation of energy balance for the mixed zone

We use the energy balance on the from equation 3

$$\frac{\partial [A\rho(x, t)h(x, t) - AP(t)]}{\partial t} + \frac{\partial [m(x, t)h(x, t)]}{\partial x} + \alpha U(T_1(x, t) - T_2(x, t)) = 0 \quad (4.16)$$

Using the notation on figure 4.1 we find for the mixed zone

$$\frac{\partial [A\rho(x, t)h(x, t) - AP(t)]}{\partial t} + \frac{\partial [m(x, t)h(x, t)]}{\partial x} = \pi D \alpha L_2 (T_w - T_r) \quad (4.17)$$

The energy balance is integrated over the two-phase zone from  $L_1$  to  $L_1 + L_2$ . Applying Leibniz's rule gives for a constant cross-sectional area

$$\begin{aligned} A \frac{d}{dt} \int_{L_1}^{L_1 + L_2} \rho h dx + A \rho(L_1) h(L_1) \frac{dL_1}{dt} - A L_1 \frac{dP}{dt} \\ - A \rho(L_1 + L_2) h(L_1 + L_2) \frac{d(L_1 + L_2)}{dt} - A L_2 \frac{dP}{dt} + m_{23} h_g - m_{12} h_l \\ = \pi D \alpha L_2 (T_w - T_r) \end{aligned} \quad (4.18)$$

The first term is evaluated as

$$\begin{aligned}
\frac{d}{dt} \int_{L_1}^{L_1+L_2} \rho h dx &= \frac{d}{dt} \int_{L_1}^{L_1+L_2} (\gamma \rho_g h_g + (1 - \gamma) \rho_l h_l) dx \\
&= \frac{d}{dt} \left( (\bar{\gamma} \rho_g h_g + (1 - \bar{\gamma}) \rho_l h_l) L_2 \right) \\
&= L_2 \left( \bar{\gamma} \frac{d(\rho_g h_g)}{dP} + (1 - \bar{\gamma}) \frac{d(\rho_l h_l)}{dP} \right) \frac{dP}{dt} \\
&\quad + \left( \bar{\gamma} \rho_g h_g + (1 - \bar{\gamma}) \rho_l h_l \right) \frac{dL_2}{dt}
\end{aligned} \tag{4.19}$$

Inserting equation 4.20 in equation 4.19 gives the final energy balance for the mixed zone

$$\begin{aligned}
&A(\bar{\gamma} \rho_g h_g + (1 - \bar{\gamma}) \rho_l h_l) \frac{dL_1(t)}{dt} + A(1 - \bar{\gamma}) (\rho_l h_l - \rho_g h_g) \frac{dL_2(t)}{dt} \\
&+ AL_2 \left( \bar{\gamma} \frac{\rho_g h_g}{dP} + (1 - \bar{\gamma}) \frac{\rho_l h_l}{dP} - 1 \right) \frac{dP(t)}{dt} \\
&= m_{12} h_l - m_{23} h_g - \pi D \alpha L_2 (T_{w2} - T_{r2})
\end{aligned} \tag{4.20}$$

□

We have now derived mass and energy balances and comparison with equations 4.1 and 4.2 show the same form and the possibility to find expressions for the factors  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$ ,  $\mathcal{F}$  and  $\mathcal{G}$ . For simulation of the equations the factors can be calculated in each simulation step from tube geometry and properties of the fluid. The only quantity missing is  $\bar{\gamma}$ ; we refer to [Jen03] and [JT02] for a calculation of this  $\bar{\gamma}$ .

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