

Examples: mass and energy balances

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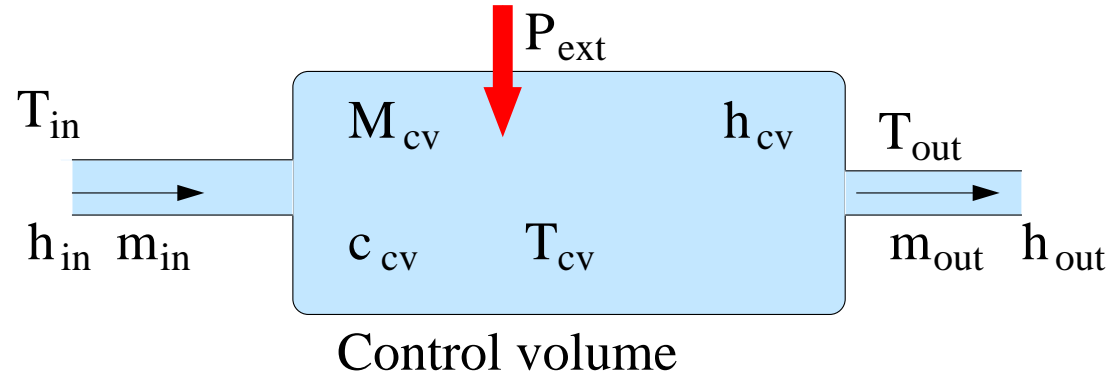
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Mass and energy balances

The **mass balance**

$$\frac{dM_{CV}(t)}{dt} = m_{in}(t) - m_{out}(t)$$



The general **energy balance**

$$\frac{d(M_{CV}(t)h_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + m_{in}(t)h_{in}(t) - m_{out}(t)h_{out}(t)$$

The **one phase energy balance**

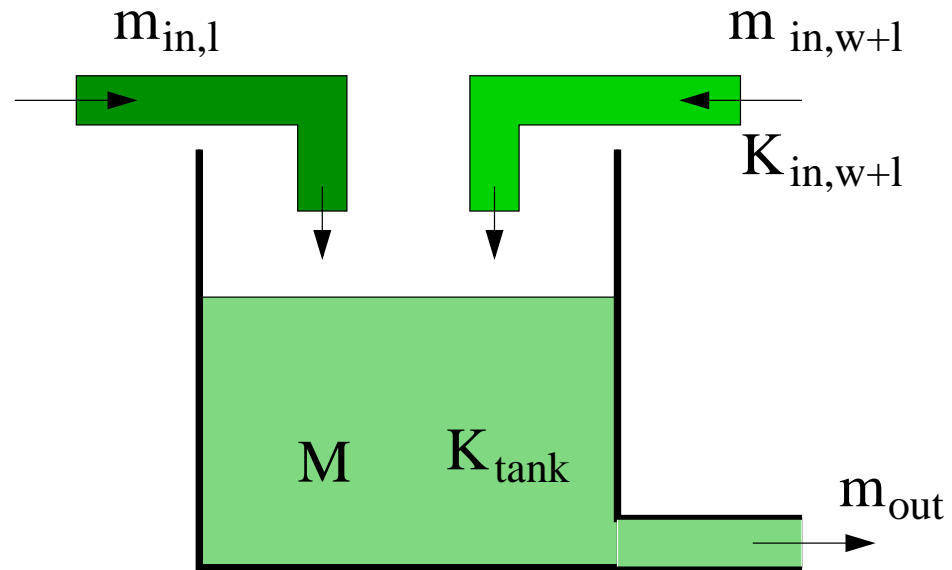
$$\frac{d(c_{CV}M_{CV}(t)T_{CV}(t))}{dt} = \mathcal{P}_{ext}(t) + cm_{in}(t)T_{in}(t) - cm_{out}(t)T_{out}(t)$$

Example: Mass balances (1)

Concentration lye

$$K = \frac{M_{lye}}{M_{lye} + M_{water}}$$

Massbalance for total, water or lye ?



Total mass balance:

$$\frac{d(M(t))}{dt} = m_{in,l}(t) + m_{in,w+l}(t) - m_{out}(t)$$

Lye mass balance:

$$\frac{d(M(t)K_{tank}(t))}{dt} = m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t)$$

Often the equations should be put in the **non-linear state space form** $\dot{x} = f(x, u, t)$ to make them suitable for i.e. simulation programs.

Example: Mass balances (2)

In this case we use **the chain rule** and find

$$\begin{aligned}\frac{d(M(t)K_{tank}(t))}{dt} &= M(t)\frac{dK_{tank}(t)}{dt} + K_{tank}(t)\frac{dM(t)}{dt} \\ &= m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t)\end{aligned}$$

Using the total balance

$$\begin{aligned}M(t)\frac{dK_{tank}(t)}{dt} + K_{tank}(t)[m_{in,l}(t) + m_{in,w+l}(t) - m_{out}(t)] \\ = m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t)\end{aligned}$$

Rearranging

$$\begin{aligned}\frac{dK_{tank}(t)}{dt} &= \frac{1}{M(t)} \left[m_{in,l}(t) + m_{in,w+l}(t)K_{in,w+l}(t) - m_{out}(t)K_{tank}(t) \right. \\ &\quad \left. - K_{tank}(t)[m_{in,l}(t) + m_{in,w+l}(t) - m_{out}(t)] \right]\end{aligned}$$

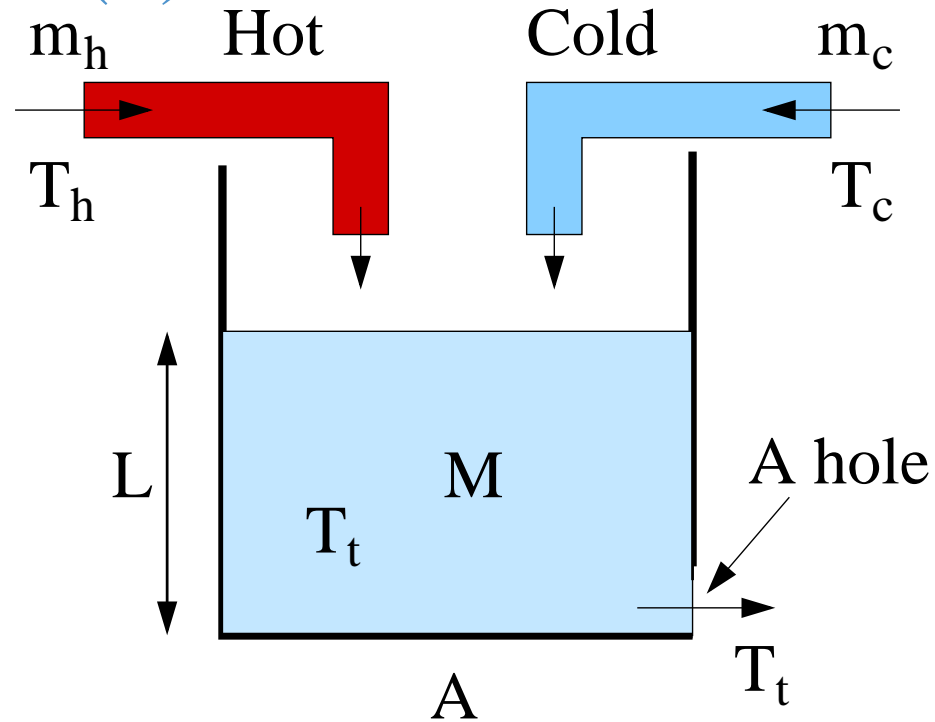
The total balance and the last equation are the two model equations on the non-linear state space form. This is very suitable for simulating using ODE-functions in Matlab.

Example: mass balance and one phase energy balance (1)

A tank where hot and cold water is mixed. There is a 'hole' in the tank, the outflow is given as

$$m_{out} = K_o \sqrt{L}$$

where K_o is a constant. (This equation is a special case of the bernoulli equation)



Mass balance:

$$\frac{dM(t)}{dt} = A\rho \frac{dL(t)}{dt} = m_h(t) + m_c(t) - K_o \sqrt{L(t)}$$

One phase energy balance

$$c \frac{dM(t)T_t(t)}{dt} = cm_h(t)T_h(t) + cm_c(t)T_c(t) - cK_o \sqrt{L(t)}T_t(t)$$

Again this set of equations can be brought on the non-linear state space form by using the chain rule.

Example: mass balance and one phase energy balance (2)

Mass balance:

$$\frac{dM(t)}{dt} = A\rho \frac{dL(t)}{dt} = m_h(t) + m_c(t) - K_o \sqrt{L(t)}$$

One phase energy balance

$$c \frac{dM(t)T_t(t)}{dt} = cm_h(t)T_h(t) + cm_c(t)T_c(t) - cK_o \sqrt{L(t)}T_t(t)$$

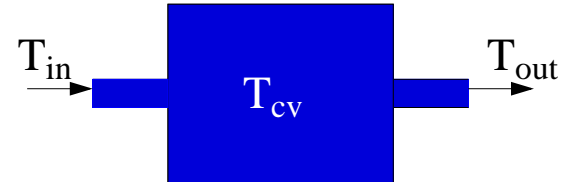
Using the chain rule

$$\begin{aligned} \frac{dT_t(t)}{dt} &= \frac{1}{\rho AL(t)} \left[m_h(t)T_h + m_c(t)T_c - T_t(t)[m_h(t) + m_c(t)] \right] \\ \frac{dL(t)}{dt} &= \frac{1}{A\rho} [m_h(t) + m_c(t) - K_o \sqrt{L(t)}] \end{aligned}$$

This is very suitable for simulating using ODE-functions in Matlab.

Different approximations for the CV temperature (1)

The mean temperature in the CV, T_{CV} , can be approximated in different ways, it could be approximated as the output temperature $T_{CV} = T_{out}$, this is correct if the control volume is perfectly stirred. It could also be the mean value between the input and output temperature $T_{CV} = \frac{T_{in} + T_{out}}{2}$. The first approximation is called **backward difference**, the second is called **central difference**.



$$M \frac{dT_{CV}(t)}{dt} = mT_{in}(t) - mT_{out}(t)$$

If the backward difference approximation is used

$$M \frac{dT_{out}(t)}{dt} = mT_{in}(t) - mT_{out}(t)$$

If the central difference approximation is used

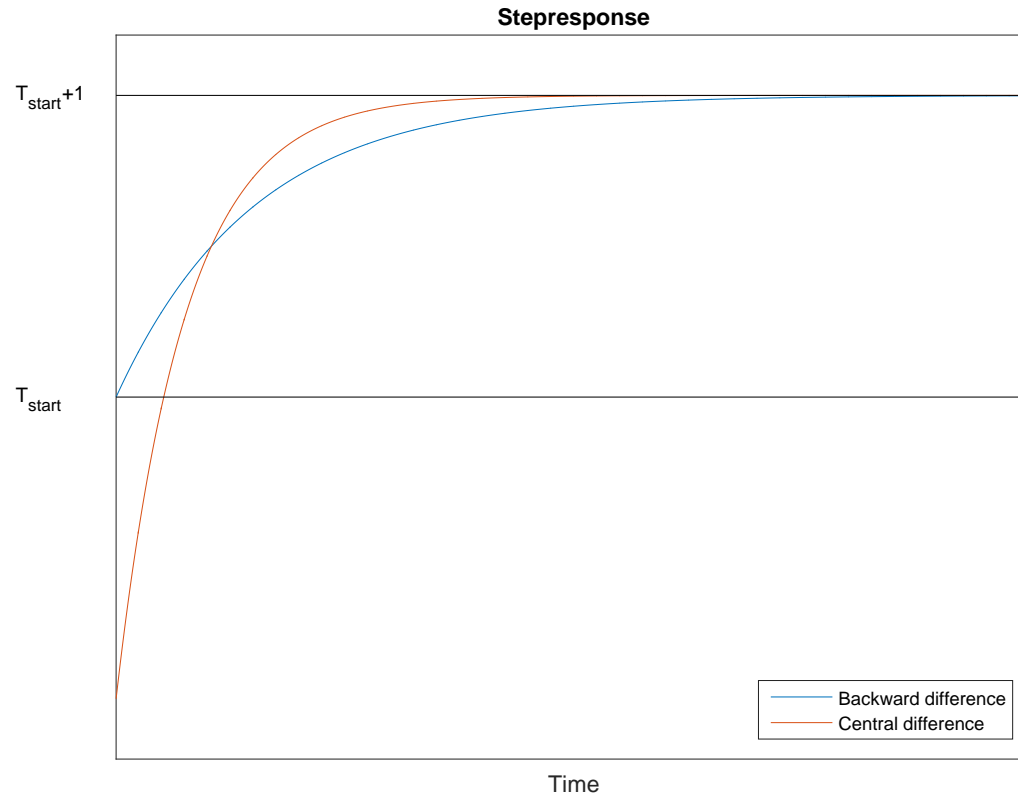
$$M \frac{d\left(\frac{T_{out}(t) + T_{in}(t)}{2}\right)}{dt} = mT_{in}(t) - mT_{out}(t)$$

Different approximations for the CV temperature (2)

In this example both models are linear and it is possible to determine the transfer functions from the input T_{in} to the output T_{out}

$$G_{backward} = \frac{1}{\frac{M}{m}s + 1}$$

$$G_{central} = \frac{1 - \frac{M}{2m}s}{\frac{M}{2m}s + 1}$$

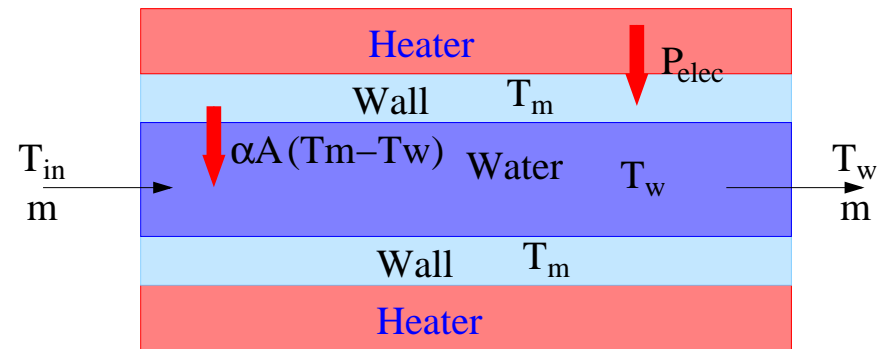


The backward difference gives a first order system, the central difference gives a first order system with a right-half plane zero. The unit step responses of these systems can be seen in the figure. Here it is seen, that a $1^{\circ}C$ step will cause the output temperature to step down $1^{\circ}C$ in the beginning of the step response, if the central approximation method is used. The two methods result in the same steady state in this example. Of course in this example the central difference is not suitable in the transient, this is often the case when central difference approach is used. In other examples in specific a counter flow heat exchanger the central difference approximation has the advantage to give much more precise steady state values and can sometimes be preferred.

Rarely a third approach called **forward difference**, where $T_{CV} = T_{in}$, is used.

Example: Energy balance with convective heat transfer (1)

In this example we have water flowing through a metal tube. It is assumed, that in the tube perimeter, is placed an electrical heat element, and all power \mathcal{P}_{elec} from this goes without any loss into tube, and the power is equally distributed along the tube. The input water temperature is T_i . The input flow as well as the output flow is m . It is assumed that the system can be described by two CV's one for the metal part where the temperature is T_m and one from the water parts where the temperature is T_w . The transfer of heat from metal to water can be characterized as convective.



The model is given by the two energy balances

$$c_m M_m \frac{dT_m(t)}{dt} = \mathcal{P}_{elec}(t) - \alpha A (T_m(t) - T_w(t))$$

$$c_w M_w \frac{dT_w(t)}{dt} = c_w m(t) (T_{in}(t) - T_w(t)) + \alpha A (T_m(t) - T_w(t))$$

How do we find a value of the heat transfer coefficient α ?

Example: Energy balance with convective heat transfer (2)

By **experiments** it has been shown that all the parameter mentioned in the table have influence on α . Unfortunately it has not been possible to find physical laws which can relate these quantities, instead empirical relation have been found.

variable	notation	unit			
Water velocity	v	m	s^{-1}		
Tube diameter	D	m			
Water density	ρ	kg	m^{-3}		
Water heat	c	m^2	s^{-2}	$^{\circ}C^{-1}$	
Water viscosity	μ	kg	m^{-1}	s^{-1}	
Water conductivity	λ	kg	m	s^{-3}	$^{\circ}C^{-1}$
Heat transfer	α	kg	s^{-3}	$^{\circ}C^{-1}$	

$$\alpha = f(v, D, \lambda, \rho, c, \mu)$$

It has been found that **product functions** of the form

$$\alpha = K v^{a1} D^{a2} \rho^{a3} c^{a4} \mu^{a5} \lambda^{a6}$$

can give satisfactory descriptions. To achieve an expression in fewer dimensionless variables the usual way is to seek an expression, where the fundamental dimensions fit

$$Dim(\alpha) = Dim(v)^{a1} Dim(D)^{a2} \dots Dim(\lambda)^{a6} = [kg \, s^{-3} \, C^{-1}]$$

Example: Energy balance with convective heat transfer (3)

Inserting the fundamental units gives

$$(kg s^{-3} C^{-1}) = (m s^{-1})^{a_1} (m)^{a_2} (kg m^{-3})^{a_3} (m^2 s^{-2} C^{-1})^{a_4} (kg m^{-1} s^{-1})^{a_5} (kg m s^{-3} C^{-1})^{a_6}$$

Equation is solved with respect to $a_1 \cdots a_6$, giving

$$\text{kg : } 1 = a_3 + a_5 + a_6$$

$$\text{m : } 0 = a_1 + a_2 - 3a_3 + 2a_4 - a_5 + a_6$$

$$\text{s : } -3 = -a_1 - 2a_4 - a_5 - 3a_6$$

$$\text{C : } -1 = -a_4 - a_6$$

Choosing $a_1 = a$ and $a_4 = b$ gives $a_2 = a - 1$, $a_3 = a$, $a_5 = b - a$ and $a_6 = 1 - b$. Inserting gives

$$\alpha = K v^a D^{a-1} \rho^a c^b \mu^{b-a} \lambda^{1-b} = K v^a D^a D^{-1} \rho^a c^b \mu^b \mu^{-a} \lambda \lambda^{-b} \quad \Rightarrow$$

$$\frac{\alpha D}{\lambda} = K \left(\frac{D \rho v}{\mu} \right)^a \left(\frac{\mu c}{\lambda} \right)^b$$

Example: Energy balance with convective heat transfer (4)

$$\frac{\alpha D}{\lambda} = K \left(\frac{D \rho v}{\mu} \right)^a \left(\frac{\mu C}{\lambda} \right)^b$$

where the three terms in parenthesis are dimensionless called Nusselts, Reynolds and Prandtls numbers.

$$Nu = \frac{\alpha D}{\lambda}$$

$$Re = \frac{D \rho v}{\mu}$$

$$Pr = \frac{\mu C}{\lambda}$$

ending in

$$Nu = K Re^a Pr^b$$

Results for heat transfer is often presented in a form where the mentioned numbers appear. The method where the relevant parameters are put in a product form and the exponents are fitted using unit considerations is called **unit analysis**; more about this method may be found in e.g. (Welty). *Measurements are often described using dimensionless numbers like Nusselts, Reynolds and Prandtls numbers.*

Example: Energy balance with convective heat transfer (5)

Now returning to the example, following relation for forced convection for fluids is given in Chapman.

$$Nu = 0.023Re^{0.8}Pr^n$$

where

$$n = 0.4 \quad \text{at heating}$$

$$n = 0.3 \quad \text{at cooling}$$

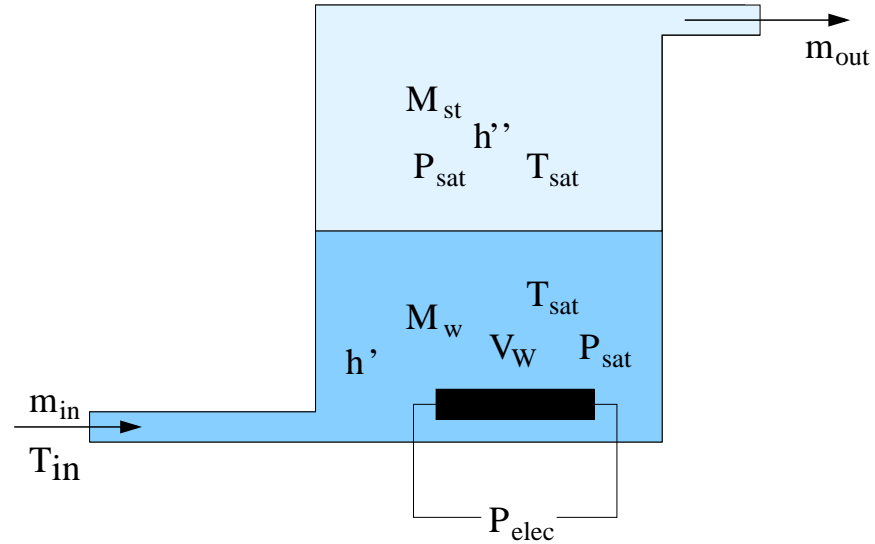
$$0.7 < Pr < 160$$

$$10^4 < Re < 10^6$$

Now the definition of Nusselts, Reynolds and Prandtls numbers may be inserted in the equation and α can be determined.

Phase shift energy balance (1)

In this example we have a boiler containing a mixture of water and water-steam, both considered in saturation state. There is a water flow m_{in} with a temperature T_{in} entering the boiler. The boiler is supplied with electrical power \mathcal{P}_{elec} . The volume of water in the boiler is called V_w , the total volume of the boiler is V_{tot} and is constant this gives that the steam volume in the boiler is $V_{tot} - V_w$. The saturation temperature in the boiler is T_{sat} . Out of the boiler flows steam m_{out} with saturation temperature.



The boiler is loaded with a tube system described by the equation

$$P - P_d = Rm_{out}^2$$

a standard equation for a tube system with turbulent flow, flow resistance R and pressure drop $P - P_d$. It is assumed that R and P_d are known.

The inputs to the model is m_{in} , T_{in} and \mathcal{P}_{elec} . All physical dimensions of the boiler is known. The desired outputs of the model is the water volume V_w and the pressure P in the boiler.

Phase shift energy balance (2)

In this example properties of steam/water are important. The main properties of water/steam are pressure P , density ρ and temperature T . Knowing two of the properties the third can be found in a steam table.

$$P = f_p(T, \rho); \quad \rho = f_\rho(P, T); \quad T = f_T(P, \rho)$$

If we assume that the thermodynamic state is saturation, the temperature is a function of pressure only

$$T = f_{sat}(P); \quad P = f_{sat}^{-1}(T)$$

There are two densities one for saturated water and one for saturated steam

$$\rho_w = f_{\rho,w}(P); \quad \rho_{st} = f_{\rho,st}(P)$$

The specific enthalpy h is in general a function of pressure and temperature

$$h = u + \frac{P}{\rho} = f_{ent}(T, P)$$

If two of the three thermodynamic states, h, P and T , are known, the third can always be found in a steam table.

Phase shift energy balance (3)

In saturation the spec. enthalpy is only a function of pressure

$$h_w = h' = f_{ent,sat,w}(P); \quad h_{st} = h'' = f_{ent,sat,st}(P)$$

The notation '=water '=steam is often used in steam tables.

All the functions f mentioned above can be found in the steam table.

We now return to the modelling.

The energy balance for the water volume plus the steam volume of the boiler can be developed

$$\begin{aligned} \frac{d(M_{CV}(t)h_{CV}(t))}{dt} &= \mathcal{P}_{ext}(t) + m_{in}(t)h_{in}(t) - m_{out}(t)h_{out}(t) \\ \frac{d(\rho_w(P(t))V_w(t)h_w(P(t)) + \rho_{st}(P(t))(V_{tot} - V_w(t))h_{st}(P(t))}{dt} &= \\ \mathcal{P}_{elec}(t) + m_{in}(t)cT_{in}(t) - m_{out}(t)h_{st}(P(t)) \end{aligned}$$

Here is it utilized that the enthalpy in the control volume $M_{CV}h_{CV}$, is the sum of steam and water enthalpies in the CV.

The mass balance is given by

$$\frac{d(\rho_w(P(t))V_w(t) + \rho_{st}(V_{tot} - V_w(t))}{dt} = m_{in}(t) - m_{out}(t)$$

From the output equation it is seen that m_{out} only depends on pressure P , the other terms are constants and known.

Phase shift energy balance (4)

$$\frac{d(\rho_w(P(t))V_w(t)h_w(P(t)) + \rho_{st}(P(t))(V_{tot} - V_w(t))h_{st}(P(t)))}{dt} = \mathcal{P}_{elec}(t) + m_{in}(t)cT_{in}(t) - m_{out}(t)h_{st}(P(t))$$

$$\frac{d(\rho_w(P(t))V_w(t) + \rho_{st}(P(t))(V_{tot} - V_w(t)))}{dt} = m_{in}(t) - m_{out}(t)$$

There are two states in the equation namely $P(t)$ and $V_w(t)$. All terms that solely depend on the P state are red, and those which depend on V_w state are green. Black terms are known inputs or constants.

To bring the equations on a standard form, intensively use of the two rules for derivatives are necessary

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \quad \text{for example}$$

$$\frac{d}{dt}(\rho(P(t))V(t)) = \rho(P(t)) \frac{dV(t)}{dt} + V(t) \frac{d\rho(P(t))}{dt}$$

$$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx} \quad \text{for example}$$

$$\frac{d\rho(P(t))}{dt} = \frac{d\rho(P(t))}{dP(t)} \cdot \frac{dP(t)}{dt} = \left. \frac{d\rho(P)}{dP} \right|_{P=P(t)} \cdot \frac{dP(t)}{dt}$$

Phase shift energy balance (4)

$$\frac{d(\rho_w(P(t))V_w(t)h_w(P(t)) + \rho_{st}(P(t))(V_{tot} - V_w(t))h_{st}(P(t)))}{dt} = \mathcal{P}_{elec}(t) + m_{in}(t)cT_{in}(t) - m_{out}(t)h_{st}(P(t))$$

$$\frac{d(\rho_w(P(t))V_w(t) + \rho_{st}(P(t))(V_{tot} - V_w(t)))}{dt} = m_{in}(t) - m_{out}(t)$$

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Phase shift energy balance (5)

The equations can, after using the rules for derivatives, be written on the **descriptor form**

$$\mathbf{M}(\mathbf{x}, \mathbf{z})\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

where state vector is $\mathbf{x}(t) = [P(t) \ V_w(t)]^T$. \mathbf{M} is a 2x2 matrix where the elements are states or \mathbf{z} elements. \mathbf{z} contains constants and terms that can be found in the steam table for example $h_{st}(P(t))$.

Finally $\mathbf{u}(t) = [\mathcal{P}_{elec}(t) \ m_{in}(t) \ T_{in}(t)]^T$ is input.

If \mathbf{M} is invertible the system can be written on the standard non-linear state space form

$$\dot{\mathbf{x}} = \mathbf{M}^{-1}(\mathbf{x}, \mathbf{z})\mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$