Stability of LPV

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Consider LTI

$$\dot{x}(t) = Ax(t)$$
$$x(0) = x_0$$

The LTI is globally asymptotically stable if and if there is a $P > 0$ such that

$$-A^T P - PA > 0$$

Slight modification of the stability definitions for LPV

$$\dot{x}(t) = A(\rho(t))x(t)$$
$$x(0) = x_0$$

with $\rho \in \mathcal{P}$

- stable: $\forall \epsilon > 0$, there is $\delta = \delta(x) > 0$ such that $\forall \rho \in \mathcal{P}$

$$||x_0|| < \delta \implies ||x(x_0, \rho(\cdot), t)|| < \epsilon$$

- attractive:

$$||x_0|| < \delta \implies \lim_{t \to \infty} ||x(x_0, \rho(\cdot), t)|| = 0$$

- asymptotic stabile = stable + attractive
- exponential stabile: $\exists \delta, \alpha, \beta > 0$ such that $\forall \rho \in \mathcal{P}$

$$||x_0|| < \delta \implies ||x(x_0, \rho(\cdot), t)|| < \beta e^{-\alpha t} ||x_0||$$

Quadratic stability

Definition

LPV is **quadratically stable** if the quadratic form

$$V(x) = x^T Px, \quad P > 0$$

is a (common) Lyapunov function

Proposition

If LPV is quadratically stable then the spectrum of $A(\rho)$ is Hurwitz for $\rho \in \Delta_\rho$, i.e., it satisfies

$$\sigma(A(\rho)) \subset \{z \in \mathbb{C} | \operatorname{Re}(z) < 0\}$$

The converse does not hold

- Example

$$A(\rho) = \begin{bmatrix} 1 & \rho \\ -\frac{4}{\rho} & -3 \end{bmatrix}$$

for $\rho \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$

Robust Stability

Definition

We say that LPV is **robustly stable** if there exists a positive definite quadratic form

$$V(x, \rho) = x^T P(\rho)x, \quad P(\rho) > 0, \forall \rho \in \Delta_\rho$$

is a Lyapunov function. We shall call such a quadratic form V a parameter dependent Lyapunov function.

Proposition

Suppose that ρ is time invariant. Then the following statements are equivalent

1. *The system is robustly stable*
2. *$\sigma(A(\rho))$ is Hurwitz for all $\delta \in \Delta_\rho$.*

For time-varying ρ

Proposition

If LPV is robustly stable for all $\rho : \mathbb{R}_{\geq 0} \to \Delta_\rho$*, then for all* $\lambda \in \sigma(A(\rho))$, $\mathrm{Re}(\lambda) < 0$ *for all* $\rho \in \Delta_\rho$

Proposition

Let Δ_ρ *be compact. If for all* $\rho \in \Delta_\rho$*, the system* $A(\rho)$ *is Hurwitz, then the system is robustly stable provided that the rate of variation of* ρ *is sufficiently small.*

Consider LTI

$$\dot{x}(t) = A(\rho)x(t)$$
$$x(0) = x_0$$

The LTI is quadratically stable stable if and if there is a $P > 0$ such that **LMI**

$$-A(\rho)^T P - P(\rho)A > 0$$

holds for all $\rho \in \Delta_\rho$

Stability of Polyhedral LPV

Theorem

Let Δ_ρ be a (convex) polyhedron with vertices in V_ρ. Suppose $A(\rho)$ is affine in ρ. Then the following statements are equivalent

- $$\exists P > 0 \ s.t. \ -A(\rho)^T P - PA(\rho) > 0, \ \ \forall \rho \in \Delta_\rho$$
- $$\exists P > 0 \ s.t. \ -A(v)^T P - PA(v) > 0, \ \ \forall v \in V_\rho$$

Descriptor system

Theorem

The descriptor LPV

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = A(\rho(t)) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$x(0) = x_0$$

with $x(t) \in \mathbb{R}^n$ *and* $y(t) \in \mathbb{R}^m$,
is quadratically stable if and only if there exist $n \times n$ *matrix* $P_1 > 0$, $P_2 : \Delta_\rho \to \mathbb{R}^{m\times n}$ *and* $P_3 : \Delta_\rho \to \mathbb{R}^{m\times m}$ *such that for all* $\rho \in \Delta_\rho$, *it holds*

$$-A(\rho)^T P - PA(\rho) > 0 \text{ with } P(\rho) = \begin{bmatrix} P_1 & 0 \\ P_2(\rho) & P_3(\rho) \end{bmatrix}.$$

Consider again the LPV

$$\dot{x}(t) = \left(\frac{\rho(t)}{\rho(t)^2 + 1} - 3 \right) x(t) \text{ for } \rho \in [-1, 1]$$

with the following descriptor equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 & 1 \\ \rho & 1 & 0 & 0 \\ 0 & -\rho & 1 & \rho \\ 0 & 0 & -\rho & 1 \end{bmatrix} \begin{bmatrix} x \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Consider again the LPV

$$\dot{x}(t) = \left(\frac{\rho(t)}{\rho(t)^2+1} - 3\right) x(t) \text{ for } \rho \in [-1, 1]$$

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Verify that

$$-A(\rho)^T P - PA(\rho) > 0$$

with $P = \begin{bmatrix} 4 & 0 & 0 & 0 \\ -5 & -7 & -1 & 1 \\ -6 & -1 & -7 & -1 \\ -4 & 1 & -1 & -7 \end{bmatrix}$ holds on the vertices $\{-1\}$ and $\{1\}$.

Robust Stability

Theorem

Let $(\rho, \dot{\rho}) \in \Delta_\rho \times \mathrm{co}\{V_1, \ldots, V_M\}$. The LPV is robustly stable if there is a differentiable (matrix) map $P : \Delta_\rho \to S^n_{>0}$ (n by n positive matrices) such that

$$-A(\rho)^T P(\rho) - P(\rho)A(\rho) - \sum_i^N v_i \frac{\partial P}{\partial \rho_i}(\rho) > 0$$

for all $(\rho, v) \in \Delta_\rho \times \{V_1, \ldots, V_M\}$.

Theorem

Let $(\rho, \dot{\rho}) \in \Delta_\rho \times \mathrm{co}\{V_1, \ldots, V_M\}$. The descriptor LPV is robustly stable if there are $P_1 : \Delta_\rho \to S^n_{>0}$, $P_2 : \Delta_\rho \to \mathbb{R}^{m\times n}$, and $P_3 : \Delta_\rho \to \mathbb{R}^{m\times m}$ such that

$$-A(\rho)^T P(\rho) - P(\rho)A(\rho) - \sum_i^N \begin{bmatrix} v_i \frac{\partial P_1}{\partial \rho_i}(\rho) & 0 \\ 0 & 0 \end{bmatrix} > 0$$

with

$$P(\rho) = \begin{bmatrix} P_1(\rho) & 0 \\ P_2(\rho) & P_3(\rho) \end{bmatrix}$$

holds for all $(\rho, v) \in \Delta_\rho \times \{V_1, \ldots, V_M\}$.