LPV systems. Descriptor form

Lecture 1

- From (1.20)

$$0 = A_{21}(\cdot)x + A_{22}(\cdot)y$$

Hence $y = -A_{22}^{-1}(\cdot)\,A_{21}(\cdot)x$

specifically

$$y(0) = -A_{22}^{-1}(\varrho(0))\,A_{21}(\varrho(0))\,x_0$$

and

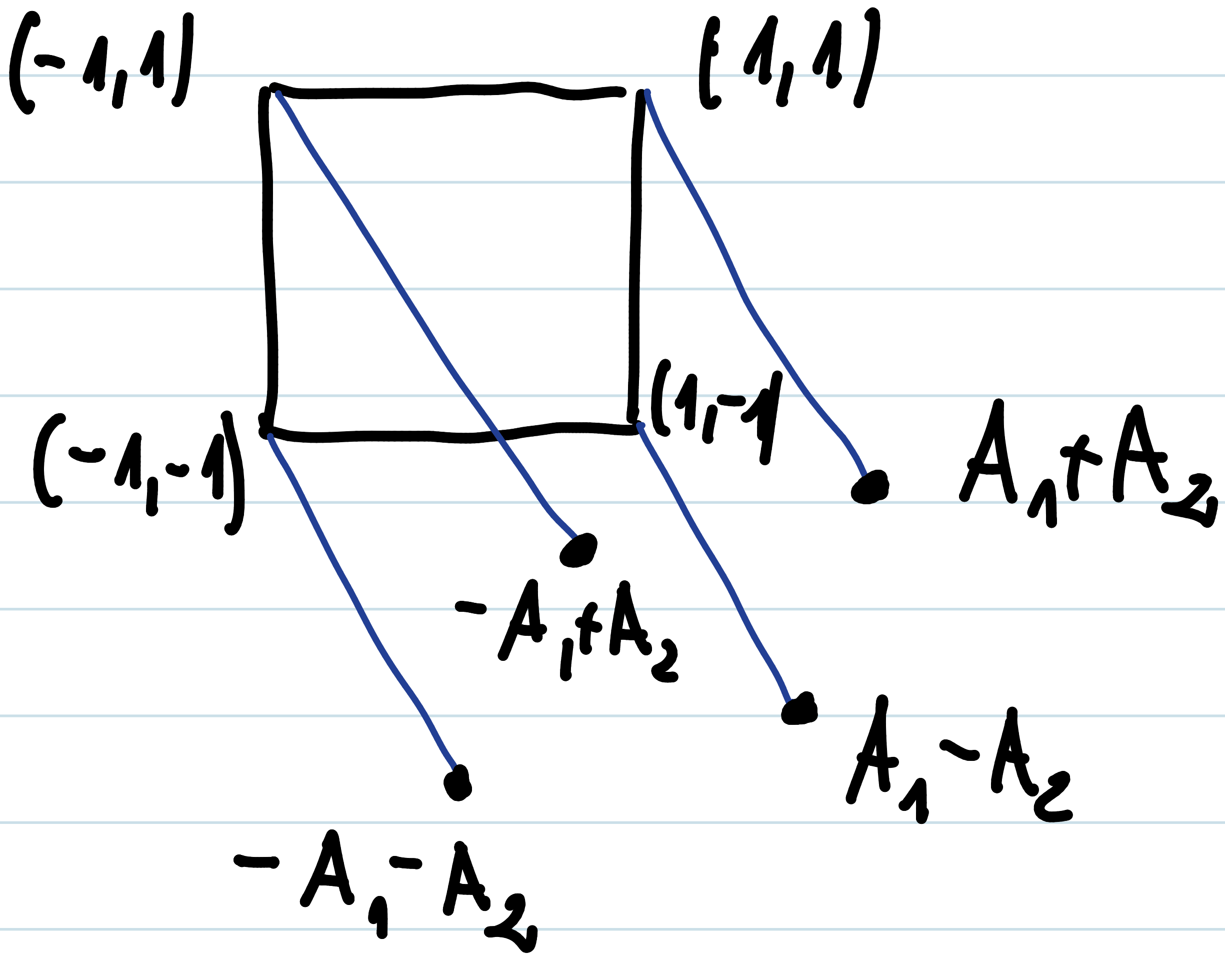
$$\begin{aligned}\dot{x} &= A_{11}(\cdot)x + A_{12}(\cdot)y\\ &= A_{11}(\cdot) - A_{12}(\cdot)A_{22}^{-1}(\cdot)A_{21}(\cdot)\end{aligned}$$

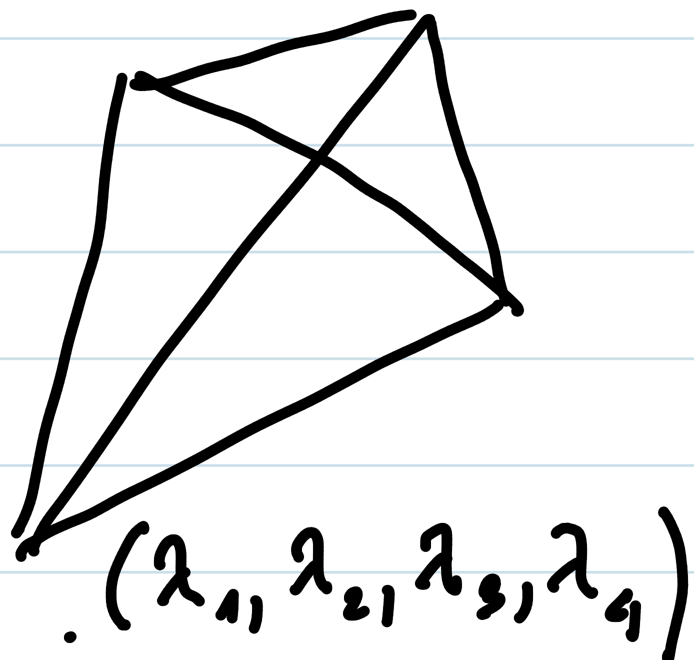
- $$-3 - \begin{bmatrix}-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -\varrho & 1 & \varrho\\ 0 & -\varrho & 1\end{bmatrix}^{-1}\begin{bmatrix}\varrho\\ 0\\ 0\end{bmatrix}$$

- Polytopic example

$$\dot{x} = [A_1\varrho_1 + A_2\varrho_2]x$$

$$\varrho = (\varrho_1, \varrho_2) \in \square \subset \mathbb{R}^2$$





$$A(\lambda) = \lambda_1 \cdot (-A_1 - A_2) + \lambda_2 (A_1 - A_2) + \lambda_3 (A_1 + A_2) + \lambda_4 \cdot (-A_1 + A_2)$$

$$= A_1 (-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4) + A_2 (-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4)$$

$$= [(\lambda_2 + \lambda_3) - (\lambda_1 + \lambda_4)] A_1 + [(\lambda_3 + \lambda_4) - (\lambda_1 + \lambda_2)] A_2$$

Consider polynomial dependent LPV

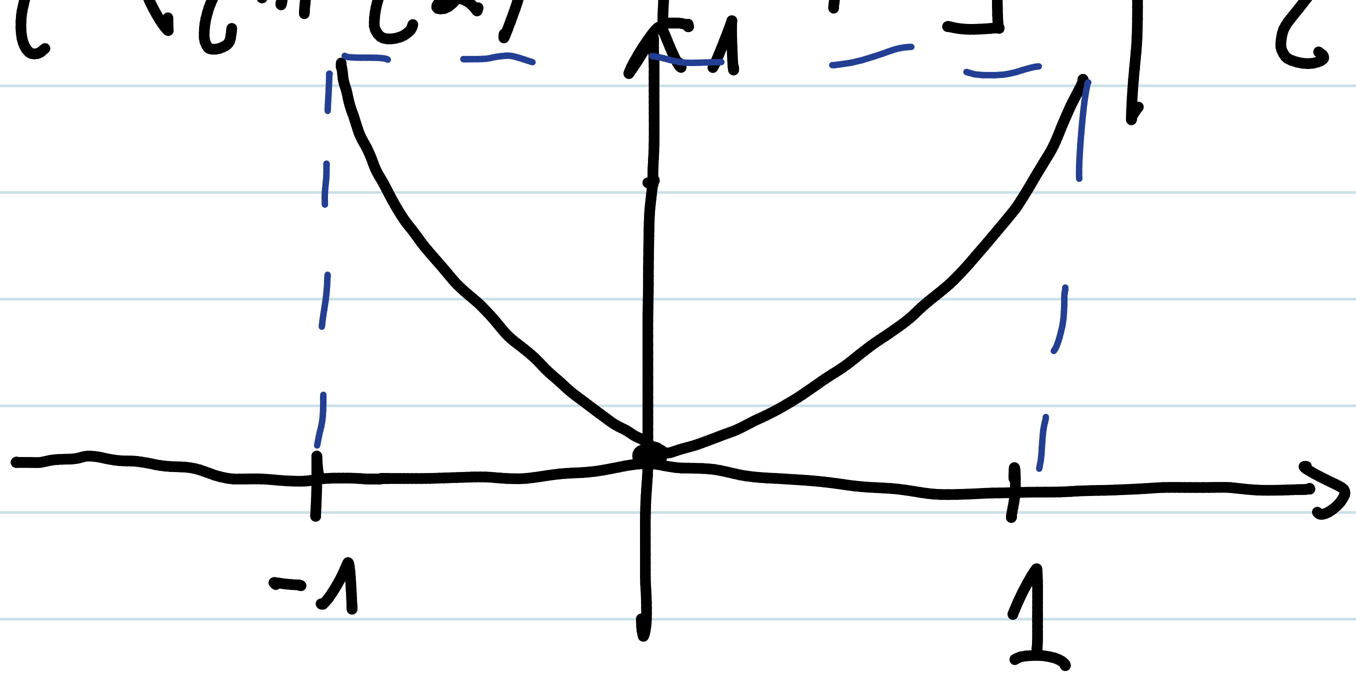
$$\dot{x} = (A_0 + A_1\varrho + A_2\varrho^2)x$$

$\varrho \in [-1,1]$

view ϱ and ϱ^2 as independent

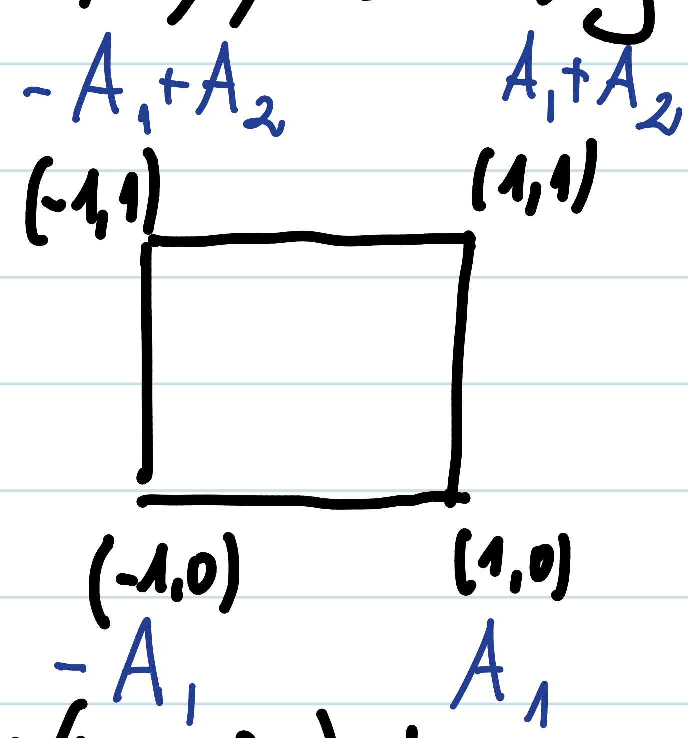
$$\{(\varrho,\varrho^2)\in\mathbb{R}^2 \mid \varrho\in[-1,1]\}$$

$$= \{(\varrho_1,\varrho_2)\in[-1,1]^2 \mid \varrho_1^2 = \varrho_2\}$$



$$\subset \mathrm{co}\,\{(-1,0),(1,0),(-1,1),(1,1)\}$$

$$= [-1,1]\times[0,1]$$



$$A(\lambda) = -\lambda_1 A_1 + \lambda_2 A_1 + \lambda_3(A_1+A_2) + \lambda_4(-A_1+A_2) = \sum_{i=1}^{4}\lambda_i A_1(-1)^i + (\lambda_3+\lambda_4)A_2$$

$$\begin{cases} w = (I - \Theta(\rho)D)^{-1}\,\Theta(\rho)Cx \\ \dot{x} = \left(A + (I - \Theta(\rho)D)^{-1}\,\Theta(\rho)C\right)x \end{cases}$$

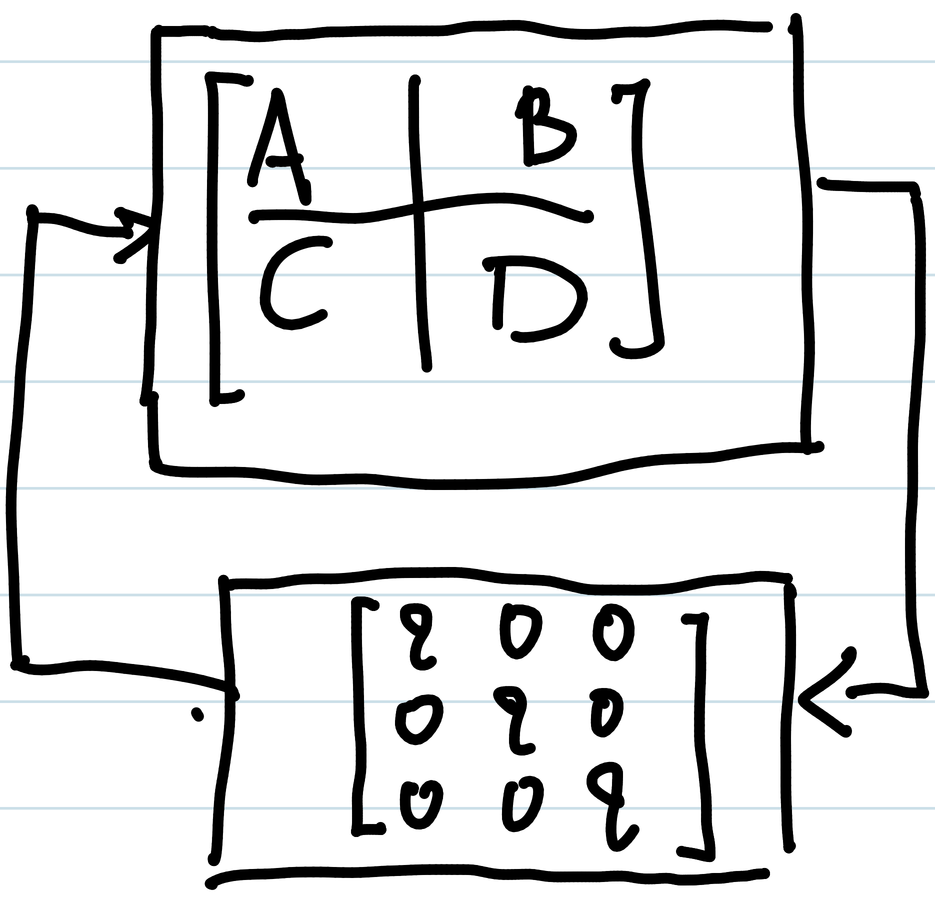
$$\dot{x} = Ax + Bw \quad (1)$$
$$z = Cx + Dw \quad (2)$$
$$w = \Theta(\rho)z \quad (3)$$

from (2), $\Theta(\rho)z = \Theta(\rho)Cx + \Theta(\rho)Dw$

from (3) $w = \Theta(\rho)Cx + \Theta(\rho)Dw$

$$0 = \Theta(\rho)Cx + (\Theta(\rho)D - I)w$$

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ \Theta(\rho)C & (\Theta(\rho)D - I) \end{bmatrix}}_{(*)} \begin{bmatrix} x \\ w \end{bmatrix}$$



with

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|ccc} -3 & -1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

LFT

LPV in Descriptor form

$$(*) = \begin{bmatrix} -3 & -1 \quad 0 \quad 1 \\ \rho\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ \rho & -1 & -\rho \\ 0 & \rho & -1 \end{pmatrix} \end{bmatrix}$$

Spectrum of :

$$A(q) = \begin{bmatrix} 1 & q \\ -\frac{4}{q} & -3 \end{bmatrix}$$

$$\det(Is - A(q)) = \det\begin{pmatrix} s-1 & -q \\ \frac{4}{q} & s+3 \end{pmatrix} = s^2 + 2s - 3 + 4$$

$$= s^2 + 2s + 1 = (s+1)^2 \Rightarrow \sigma(A(q)) = \{-1\}$$

But Let $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix} > 0$

$$M(q) = A(q)^T P + P A(q)$$

$$\begin{bmatrix} 1 & -\frac{4}{q} \\ q & -3 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix}\begin{bmatrix} 1 & q \\ -\frac{4}{q} & -3 \end{bmatrix}$$

$$= \begin{bmatrix} P_1 - \frac{4P_2}{q} & P_2 - \frac{4P_4}{q} \\ qP_1 - 3P_2 & qP_2 - 3P_4 \end{bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix}^T$$

$$= \begin{bmatrix} 2P_1 + \frac{8P_2}{q} & qP_1 - 2P_2 - \frac{4P_4}{q} \\ * & 2qP_2 - 6P_4 \end{bmatrix} < 0$$

Specifically for $q \in [\frac{1}{2}, 1]$ $P(q) < 0$ and $P(-q) < 0$

Hence $P(q) + P(-q) < 0$

$$P(\varrho)+P(-\varrho)=\begin{bmatrix} 4p_1 & -4p_2 \\ * & -12p_3 \end{bmatrix}$$

but $p_1>0$ hence $P(\varrho)+P(-\varrho)$ cannot be negative definite.

$\square$

$$x^T\left(A(\varrho)^T P(\varrho)+P(\varrho)A(\varrho)\right)x<0$$

specifically let $e_i(\varrho)$ be an eigenvector of $P(\varrho)$

$$e_i(\varrho)^* \quad -\shortparallel- \quad)e_i(\varrho)<0$$

$$\underbrace{(\lambda_i+\lambda_i^*)}_{=2\mathrm{Re}(\lambda_i)}\left(e_i^T(\varrho)Pe_i(\varrho)\right)<0$$

$$\Downarrow \quad \lambda_i<0$$

$\square$

$$\underbrace{A(\varrho)^T P(\varrho)+P(\varrho)A(\varrho)}_{*}+\underbrace{\sum_{k=1}^{N}\frac{\partial P(\varrho)}{\partial \varrho_k}\dot{\varrho}_k}_{\text{if sufficently small}}$$

by compactness

$\exists\ Q>0$ s.t $*<-Q$

$\Rightarrow$ robustly stable

Proof Thm 2.4.2

$1 \Rightarrow 2$ if $-A(\varrho)^T P - P A(\varrho) > 0 \; (*)$ holds for any $\varrho \in \Delta_\varrho \Rightarrow$ specifically $(*)$ holds for $\varrho \in V_\varrho$

$2 \Rightarrow 1$ $\quad \Delta_\varrho$ convex, hence

Let $\varrho \in \Delta_\varrho \Rightarrow \exists\, \lambda_1, \dots, \lambda_N \in \Lambda_N$

$$\varrho = \lambda_1 V_1 + \dots + \lambda_N V_N$$

A affine, i.e.,

$$A(\varrho) = \lambda_1 A(V_1) + \dots + \lambda_N A(V_N)$$

Hence

$$(*) = \lambda_1\left(-A(V_1)^T P - P A(V_1)\right) + \dots + \lambda_N\left(-A(V_N)^T P - P A(V_N)\right) > 0$$

$\square$

Consider

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \dot{x} \\ \hline \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{array}\right] = \left[\begin{array}{c|ccc} -3 & -1 & 0 & 1 \\ \hline \delta & 1 & 0 & 0 \\ 0 & -\delta & 1 & \delta \\ 0 & 0 & -\delta & 1 \end{array}\right] \left[\begin{array}{c} x \\ \hline y_1 \\ y_2 \\ y_3 \end{array}\right]$$

with $\delta \in [-1, 1]$

Check that $P = \left[\begin{array}{c|ccc} 4 & 0 & 0 & 0 \\ \hline -5 & -7 & -1 & 1 \\ -6 & -1 & -7 & -1 \\ -4 & 1 & -1 & -7 \end{array}\right]$

on the vertices $\{-1\}$ and $\{1\}$

$$-A(-1)^T P - P A(-1) > 0$$
$$-A(1) P - P A(1) > 0$$

verify that

$$-\begin{bmatrix} -3 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ -5 & -7 & -1 & 1 \\ -6 & -1 & -7 & -1 \\ -4 & 1 & -1 & -7 \end{bmatrix} - \dots > 0$$

$$-\begin{bmatrix} -3 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad —||— \quad - \quad \dots > 0$$

Proof 2.4.4.

Take $V(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T P(q)^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

and observe

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} P_1 & P_2^T(q) \\ 0 & P_3^T(q) \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{= E} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} P_1 x \\ 0 \end{bmatrix} = x P_1 x \qquad \text{i.e. } \{(x,y) \in \mathbb{R}^{n \times m} \mid V(x,y) = 0\}$$

$$= \{(0,y) \in \mathbb{R}^{n \times m}\}$$

$$\dot{V}(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}^T P(q)^T E \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}^T P^T(q) E \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}^T E^T P \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x^T \\ y^T \end{bmatrix} P^T(q) A \begin{bmatrix} x \\ y \end{bmatrix}$$

we have used $\quad P^T E = E^T P = EP$

$$\begin{bmatrix} P_1 & P_2^T \\ 0 & P_3^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \square$$

Proof 2.4.6.

$$A(q)^T P(q) + P(q) A(q) + \sum_{i=1}^{N} \underset{\substack{\uparrow \\ \text{affine of } \dot{q}_i}}{\dot{q}_i} \frac{\partial P(q)}{\partial q_i} < 0$$

therefore enough to check on the corners of $\Delta_v = co\{v_1, \dots, v_M\}$ $\square$

Proof 2.4.7

$$A(q)^T \bar{P}(q) + \bar{P}(q) A(q) + \sum_{i=1}^{N} \dot{q}_i \frac{\partial \bar{P}(q)}{\partial q_i} \quad (*)$$

$$v(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \underbrace{\begin{bmatrix} P_1(q) & 0 \\ P_2(q) & P_3(q) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{\bar{P}} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\bar{P}(q) = \begin{bmatrix} P_1(q) & 0 \\ 0 & 0 \end{bmatrix} \text{ hence } \frac{\partial \bar{P}}{\partial q} = \begin{bmatrix} \frac{\partial P_1(q)}{\partial q_i} & 0 \\ 0 & 0 \end{bmatrix}$$

and since $\dot{q}_i \in co\{v_1, \dots, v_M\}$ it is enough to evaluate $(*)$ on the vertices v_i.

$\square$

MIL 3&4

Lecture 2

Robust stability refers to $\dot{q}$. Suppose that $\dot{q} \in \Delta_v = co\{d_1, \dots, d_N\}$

$$q(t) = \sum_{i=1}^{N} \lambda_i(t) v_i$$ position polytope

$$\dot{q}(t) = \sum_{i=1}^{N} \dot{\lambda}_i(t) v_i$$

at the same time $\dot{q}(t) = \sum_{i=1}^{N} \xi_i(t) d_i$ velocity polytop

Let $\mathcal{V} = [v_1, \dots, v_N]$ and $\mathcal{D} = [d_1, \dots, d_N]$

$$\mathcal{V} \dot{\lambda}(t) = \mathcal{D} \xi$$

and $\sum_{i=1}^{N} \dot{\lambda}_i = 0$ $\sum_{i=1}^{N} \xi_i = 1$

$$\begin{bmatrix} V \\ \mathbb{1}_N^T \\ 0 \end{bmatrix} \dot{\lambda}(t) = \begin{bmatrix} \mathcal{D} \\ 0 \\ 1_N^T \end{bmatrix} \xi(t) - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$Fx = e$, a solution exists if $e \in \operatorname{im} F \Leftrightarrow \langle e, v \rangle = 0$ $\exists v \neq 0$
$F^T v = 0$

Notice that $\begin{bmatrix} V \\ \mathbb{1}_N^T \end{bmatrix}$ full row rank ; hence,

$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has the property $F^T = [V^T \ \ 1_N \ \ 0] v = 0$

and $\langle v, e \rangle = \langle \begin{bmatrix} \mathcal{D} \\ 0 \\ \mathbb{1}^T \end{bmatrix} \xi(t) - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle = 0$

$[\xi_0]$

Therefore $\dot{\lambda}(t) = \begin{bmatrix} V \\ \mathbb{1}_N^T \\ 0 \end{bmatrix}^{+} \left(\begin{pmatrix} D \\ 0 \\ \mathbb{1}_N^T \end{pmatrix} \xi(t) - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \ker \begin{bmatrix} V \\ \mathbb{1}_N^T \\ 0 \end{bmatrix}$

$$\dot{\lambda}(t) = \begin{bmatrix} V \\ \mathbb{1}_N^T \\ 0 \end{bmatrix}^{+} \left(\begin{bmatrix} D \\ 0 \\ \mathbb{1}_N^T \end{bmatrix} \xi(t) - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

but $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \ker \begin{bmatrix} V \\ \mathbb{1}_N^T \\ 0 \end{bmatrix}$, $\begin{bmatrix} V \\ \mathbb{1}_N^T \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

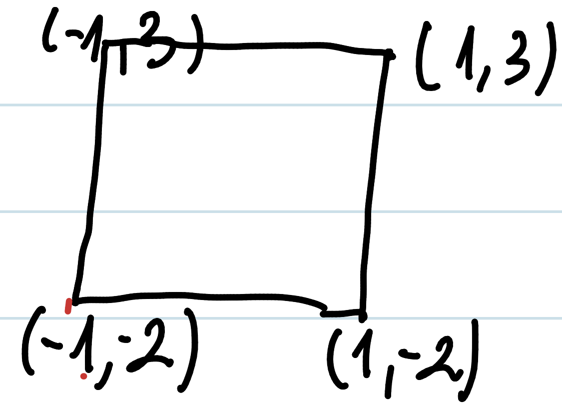
hence a particular solution

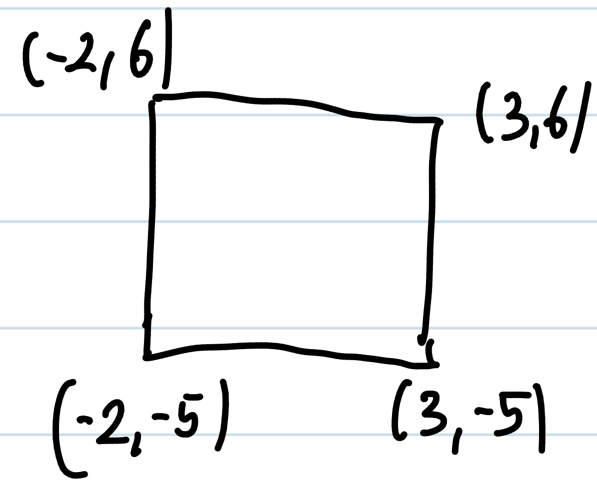
$$\dot{\lambda}(t) = \begin{bmatrix} V \\ \mathbb{1}_N^T \\ 0 \end{bmatrix}^{T} \begin{bmatrix} D \\ 0 \\ \mathbb{1}_N^T \end{bmatrix} \xi(t)$$

Consider

$$(q_1, q_2) = [-1, 1] \times [-2, 3]$$

$$(\dot{q}_1, \dot{q}_2) = [-2, 3] \times [-5, 6]$$





$$V = \begin{bmatrix} -1 & 1 & 1 & -1 \\ -2 & -2 & 3 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 3 & 3 & -2 \\ -5 & -5 & 6 & 6 \end{bmatrix}$$

$$\dot{\lambda}(t) = \begin{bmatrix} -1 & 1 & 1 & -1 \\ -2 & -2 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^+ \begin{bmatrix} -2 & 3 & 3 & -2 \\ -5 & -5 & 6 & 6 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xi(t)$$

$$A^T P + PA < 0$$

$$L = \begin{bmatrix} A^T P + PA & 0 \\ 0 & -A \end{bmatrix} < 0$$

For $\{P_i\}$ basis of symmetric matrices of dim. n

$$P = \sum_{i=1}^{\frac{(n+1)n}{2}} x_i P_i$$

Thus find $X = (x_1, \dots, x_{\frac{(n+1)n}{2}})$ such that

$$A = \sum x_i A_i > 0 \quad \text{for } A_i = \begin{bmatrix} A^T P_i + P_i A & 0 \\ 0 & -P_i \end{bmatrix}$$

Solving infinite dimensional problem

$$\dot{x} = A(\varrho)$$

Suppose $P(\varrho) = \sum P_i \varrho^i$ $\Big/ \text{or} = \sum_\alpha P_\alpha \varrho^\alpha$

$$\text{He}\left(\sum P_i \varrho^i A(\varrho)\right) + \sum_{i=1}^{N} v_i \left(\sum P_i \, i \, \varrho^{i-1}\right) < 0$$

holds for all $(\varrho, v) \in \Delta_p \times V_v$
vertices

3 methods:
- affine
- gridding (will not talk about)
- Sum of squares

Affine. Start with $M(x,\delta) = M_0(x) + \sum_{i=1}^{N} \delta_i M_i(x)$

$\delta_i \in [\delta_i^-, \delta_i^+]$ can be changed to

$$\delta_i = \frac{\delta_i^+ - \delta_i^-}{2} \tilde{\delta}_i + \frac{\delta_i^+ + \delta_i^-}{2}, \quad \tilde{\delta}_i \in [-1, 1]$$

Proof B.3.1

$1 \Rightarrow 2$ "trivial"

$2 \Rightarrow 1$

Notice first that any $\delta \in [-1, 1]^N$ can be written

$$\delta = \sum_{i=1}^{2^N} \lambda_i v_i \quad , v_i \in \{-1, 1\}^N$$

and $\lambda \in \Lambda_N$ specifically $\delta_j = \sum_{i=1}^{2^N} \lambda_i v_{ij}$

$v \in \{-1,1\}^N$

$$M_0 + \sum_{i=1}^{N} \delta_i M_i = M_0 + \sum_{i=1}^{N} \sum_{j=1}^{2^N} \lambda_j v_{ij} M_i = \sum_{j=1}^{2^N} \lambda_j \left(M_0 + \sum_{i=1}^{N} v_{ij} M_i\right) *$$

but $M_0 + \sum_{i=1}^{N} v_{ij} M_i \succ 0 \quad \forall j$

Hence (*) positive definite.

Proof B.3.2 (Matrix Cube Theorem)

- $-X_i \pm M_i(x) \preceq 0 \qquad i=1,\dots,N$

$$\begin{aligned} X_i - M_i &\succeq 0 \\ X_i + M_i &\succeq 0 \end{aligned} \Rightarrow X_i \succeq 0$$

- $-X_i \pm M_i(x) \preceq 0 \Rightarrow \sum_{i=1}^{N} |\delta_i| \left[-X_i \pm M_i(x)\right] \preceq 0$

$$\sum_{i=1}^{N} X_i \mp |\delta_i| M_i(x) = *$$

$$\sum_{i=1}^{N} |\delta_i| X_i \mp |\delta_i| M_i(x) = **$$

$$\sum_{i=1}^{N} (1-|\delta_i|) X_i \succeq 0 \Rightarrow * \succeq **$$

$$-* \preceq -**$$

$$\sum_{i=1}^{N} -X_i \pm |\delta_i| M_i(x) \preceq \sum |\delta_i| (-X_i \pm M_i(x)) \preceq 0$$

$\Sigma -X_i \pm |\delta_i| M_i(x) \preceq 0$

$$\Rightarrow \sum_{i=1}^{N} X_i \succeq \sum_{i=1}^{N} \delta_i M_i(x)$$

$$M_0(x) + \sum_{i=1}^{N} \delta_i M_i(x) \preceq M_0(x) + \sum_{i=1}^{N} X_i \prec 0$$

from the hypothesis

$\Rightarrow$ equivalent to feasibility of LMI

$\blacksquare$

More standard version of relaxation lemma is
The following are equivalent

- LMI
$$M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0$$
holds
- $\exists N$ s.t.
$$M_{11} + N^T M_{12}^T + M_{12} N + N^T M_{22} N < 0$$
holds

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other formulation of Finsler's lemma

- $x^T M x < 0$ on $\mathcal{X} = \{x \mid Bx = 0,\ x \neq 0\}$
- $\exists \tau \in \mathbb{R}$
$$M - \tau B^T B < 0$$
- There is $X = X^T$ s.t.
$$M - B^T X B < 0$$
- There is a matrix
$$M + N^T B + B^T N < 0$$
- $B_T^T M B_T < 0, \quad B_T = \ker B$

directly

Proof of theorem 2.5.6

We will show that

$$* = \begin{bmatrix} -(X+X)^T & P(\lambda)+X^T A(\lambda) & X^T \\ * & -\xi P(\lambda)+P(\dot\lambda) & 0 \\ * & * & -P(\lambda)/\xi \end{bmatrix} < 0$$

implies $A(\lambda)^T P(\lambda) + P(\lambda)A(\lambda) + P(\theta) < 0$
$(\lambda,\theta) \in \Lambda_N \times \text{vert}(\dot\Lambda_N)$

$$* = \underbrace{\begin{bmatrix} 0 & P(\lambda) & 0 \\ * & -\xi P(\lambda)+P(\dot\lambda) & 0 \\ * & * & -P(\lambda)/\xi \end{bmatrix}}_{\Psi} + He\left(\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X^T \begin{bmatrix} -I & A(\lambda) & I \end{bmatrix}\right)$$

$P^T \Omega^T Q$

$$\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} -X^T & X^T A(\lambda) & X^T \end{bmatrix} = \begin{bmatrix} -X^T & H^T A(\lambda) & X^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

by projection lemma

$$P = \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} -I & A(\lambda) & I \end{bmatrix}$$

$$P_\perp = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \quad \text{for } \underbrace{P P_\perp}_{\text{base of the null-space of } P} = 0$$

$$Q_\perp = \begin{bmatrix} A(\lambda) & I \\ I & 0 \\ 0 & I \end{bmatrix} \quad \text{for } Q Q_\perp = \begin{bmatrix} -A(\lambda)+A(\lambda); & -I+I \end{bmatrix} = 0$$

Hence $* < 0$ equivalent to

$$\underbrace{P_\perp^T \Psi P_\perp}_{\Delta_1} < 0 \quad \text{and} \quad \underbrace{Q_\perp^T \Psi Q_\perp}_{\Delta_2} < 0$$

$$\Delta_1 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & P(\lambda) & 0 \\ * & -\xi P(\lambda) + P(\dot\lambda) & 0 \\ * & * & -\frac{P(\lambda)}{\xi} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P(\lambda) & 0 \\ -\xi P(\lambda) + P(\dot\lambda) & 0 \\ 0 & -\frac{P(\lambda)}{\xi} \end{bmatrix} = \begin{bmatrix} -\xi P(\lambda) + P(\dot\lambda) & 0 \\ 0 & -\frac{P(\lambda)}{\xi} \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} A(\lambda)^T & I & 0 \\ I & 0 & I \end{bmatrix} \begin{bmatrix} 0 & P(\lambda) & 0 \\ * & -\xi P(\lambda) + P(\dot\lambda) & 0 \\ * & * & -\frac{P(\lambda)}{\xi} \end{bmatrix} \begin{bmatrix} A(\lambda) & I \\ I & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A(\lambda)^T & I & 0 \\ I & 0 & I \end{bmatrix} \begin{bmatrix} P(\lambda) & 0 \\ P(\lambda)A(\lambda) - \xi P(\lambda) + P(\dot\lambda) & P(\lambda) \\ 0 & -\frac{P(\lambda)}{\xi} \end{bmatrix}$$

$$= \begin{bmatrix} A(\lambda)^T P(\lambda) + P(\lambda) A(\lambda) + \xi P(\lambda) + P(\dot\lambda) & P(\lambda) \\ * & -\frac{P(\lambda)}{\xi} \end{bmatrix}$$

Schur complement on $\Delta_2 < 0$ $\quad M<0$

one of the conditions $\underbrace{M_{11}<0}_{*}$ and $M_{22} - M_{12}^T M_{11}^{-1} M_{12}$

$$A(\lambda)^T P(\lambda) + P(\lambda) A(\lambda) + P(\dot\lambda) < -\frac{1}{\xi} P(\lambda) < 0$$

affine in $\dot\lambda$ $\quad (\lambda, \dot\lambda) \in \Lambda_N \times \dot\Lambda_N$

hence equivalent to

$$A(\lambda)^T P(\lambda) + P(\lambda) A(\lambda) + \sum P_i \dot\lambda_i < 0 \quad \forall \lambda_i \in \dot\Lambda_N$$

$$\Updownarrow$$

$$A(\lambda)^T P(\lambda) + P(\lambda) A(\lambda) + \sum v_i P_i < 0 \quad \forall v_i \in \mathrm{ver}(\dot\Lambda_N)$$

$\square$

- L_2 space

$$\|w\|_2 = \|w\|_{L_2} = \sqrt{\int_0^\infty w(s)^T w(s)\, ds}$$

$$G : L_2 \to L_2$$

$$\|G\|_2 = \sup_{\|w\|_2 = 1} \|Gw\|_2$$

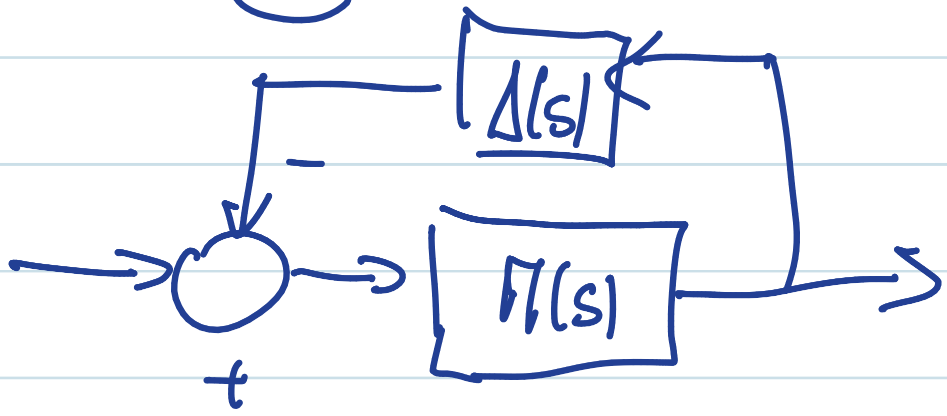
- H_∞ space

Let $G : \mathbb{C} \to \mathbb{C}^{m \times p}$ analytic in closed right-half plane

$$\|\hat G\|_\infty = \|\hat G\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} \hat G(j\omega)$$

- $\|G\|_2 = \|\hat{G}\|_{H_\infty}$

o Interconnection



$$M_\Delta = \frac{M(s)}{1+M(s)\Delta(s)}$$

o Proof of Thm 7.6.7 (Small gain theorem)

$$\|M\Delta\|_\infty \leq \|M\|_\infty \|\Delta\|_\infty < 1$$

$$\|M_\Delta\|_\infty = \left\|\frac{M}{1+M\Delta}\right\| \leq \frac{\|M\|}{1-\|M\Delta\|} < \infty$$

bounded thus M_Δ assymptotically stable

Closed system is o by

$$\dot{x} = [A(q) + B(q)K(q)]x + E(q)w$$
$$z = C(q) + D(q)K(q)]x + F(q)w$$

By bounded - real lemma

$$\begin{bmatrix} He[PA_q + PB_qK_q] + \sum v_i \frac{\partial P}{\partial q_i} & PE_q & [C_q + D_qK_q]^T \\ * & -\gamma I & F_q \\ * & * & -\gamma I \end{bmatrix} < 0$$

$$\forall q \in \Delta_2$$

Congruence transformation

Multiply by $\mathrm{diag}(X \; I \; I)$ on the left with $X = P^{-1}$

$\mathrm{diag}(X^T \; I \; I)$ on the right $\Leftrightarrow$ X is symmetric

$(X \; I \; I)$

$$\begin{bmatrix} R^T & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix} \begin{bmatrix} R & & \\ & I & \\ & & I \end{bmatrix} =$$

$$\begin{bmatrix} R^T & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} A_{11}R & A_{12} & A_{13} \\ A_{12}^TR & A_{22} & A_{23} \\ A_{13}^TR & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} R^TA_{11}R & R^TA_{12} & R^TA_{13} \\ A_{12}^TR & A_{22} & A_{23} \\ A_{13}^TR & A_{23} & A_{33} \end{bmatrix}$$

$$
\begin{bmatrix}
\mathrm{He}\left[A_2 X + B_2 \overbrace{K_2 X}^{Y_2}\right] + X \underbrace{\sum v_i \frac{\partial P}{\partial q_i}}_{Q} X & E_2 & X\left[C_2 + D K_2\right]^T \\
* & -\gamma I & F_2^T \\
* & * & -\gamma I
\end{bmatrix}
$$

$$\left[C_2 X + D_2 \overbrace{K_2 X}^{Y_2}\right]^T$$

To prove the bounds

Let $V(x) = x^T P x$

$$P^{-1} \frac{\partial P}{\partial t} P^{-1} = -\frac{d}{dt}\left(P^{-1}\right)$$

$$P P^{-1} = I \qquad \frac{\partial P}{\partial t} P^{-1} + P \frac{\partial P^{-1}}{\partial t} = 0$$

$$\dot V(x(t)) - \gamma w(t)^T w(t) + \gamma^{-1} z(t)^T z(t) < 0$$

by integrating

$$\underset{\downarrow\, 0}{V(x(t))} - V(x_0) \leq \gamma \|w\|_{L_2} - \gamma^{-1} \|z\|_{L_2}$$

$$\|z\|_{L_2}^2 \leq \gamma^2 \|w\|_{L_2}^2 + \gamma \underbrace{V(x_0)}_{x_0^T P x_0,\ P = X^{-1}}$$

$\square$

$$a + b = \sqrt{a}^2 + \sqrt{b}^2 \leq \left(\sqrt{a} + \sqrt{b}\right)^2$$