Systems of systems
The Hamilton-Jacobi-Bellman Equation
and
Pontryagin's Maximum Principle

Lecturer:

- John Leth

In this lecture we will study two fundamental concept from optimal control theory:

- The Hamilton-Jacobi-Bellman (HJB) equation (dynamic programming)
- Pontryagin's Maximum Principle

The HJB equation, a nonlinear first-order partial differential equation, is a sufficient (and necessary) conditions for optimality, while Pontryagin's Maximum Principle in general only give necessary conditions. We will see that under certain regularity conditions these conditions will also be sufficient.

Headlines:

- The Hamilton-Jacobi-Bellman Equation
- Pontryagin's Maximum Principle

Literature:

- Literature: [ST00] Ch. 2 page 23-38

Exercises (prioritize 1, 4, 6 and 7):

1. Discuss the relation between min, max, argmin and argmax.
2. Complete the argumentation that the Bolza, Lagrange and Mayer form are all equivalent. That is, prove that $(L) \subset (M)$ and that $(B), (M) \subset (L)$.
3. Write the Mayer problem

$$\max_u z(T) + x(T)'Gx(T)$$

subject to

$$\begin{aligned}\dot{z}(t) &= x(t)'Qx(t) + u(t)'Ru(t),\ z(0) = 0\\ \dot{x}(t) &= Ax(t) + Bu(t),\ x(0) = x_0,\end{aligned}$$

in Bolza form.

4. Control of Chemical Reaction with Nonlinear Cost: Consider a chemical mixture A which is added to a tank at a constant rate for a fixed positive time $t \in [0, T]$. Assume that the pH value $x > 0$ at which the reaction occurs determines the quality of the final product and that this pH value can be controlled by the strength u of some component of A.

   Suppose that the reaction take place so that the rate of change of pH value is (positive) proportional to the sum of the current pH value and the strength of the controlling ingredient. Furthermore, suppose that the cost is

   $$J_T(u) = \int_0^T (ax^2 + u^2)dt, \qquad a > 0.$$

   With a specified initial pH value, $x(0)$, find a controller $\bar{u}$ on $[0, T]$ which minimize the cost by using the HJB equation and then the maximum principle.

   HINT: Use $V(t, x) = c(t)h(x)$ and try to guess the expression for h. You do not need to determine c explicit, it is enough if you at some point can argue that you CAN determine c explicit.

5. Redo the above exercises with the use of the minimum version of the HJB equation

   $$0 = \min_u \{H(x, u, W_x(x, t), t)\} + W_t(x, t),$$

   where now $W(x, t) = \min_u J_t(u)$. Compare the result and discuss.

6. Use the maximum principle to solve the optimal control problem

   $$\max_u \int_0^2 -x(t)dt,$$

   subject to

   $$\begin{aligned} &\dot{x}(t) = u(t),\ x(0) = 1 \\ &-1 \leq u(t) \leq 1 \end{aligned}$$

7. Find the value function corresponding to problem in exercise 6.

Solutions:

ad.1:

$$\min_x\{f(x)\} = -\max_x\{-f(x)\}$$
$$\text{argmin}_x\{f(x)\} = \text{argmax}_x\{-f(x)\}$$

ad.2: $(L) \subset (M)$ is as $(B) \subset (M)$ from the slides. For $(B),(M) \subset (L)$ we argue as follows: Introduce a new variable $z \in \mathbb{R}$ by

$$\dot{z} = 0,\ z(0) = z_0 = S/T, \quad S = S(x(T),T).$$

Then (for the case $(B) \subset (L)$)

$$J = \int_0^T F\ dt + S = \int_0^T F\ dt + Tz_0 = \int_0^T (F + z_0)\ dt = \int_0^T (F + z)\ dt,$$

and (for the case $(M) \subset (L)$)

$$J = S = Tz_0 = \int_0^T z_0\ dt = \int_0^T z\ dt,.$$

Both subject to

$$\dot{y} = \bar{F}(y,u,t),\ y(0) = (z_0, x_0),$$

with $y = (z,x)$ and $\bar{F} = (0,f)$.

ad.3: Either take $F = 0$ or

$$\max_u \int (x'Qx + u'Ru)\ dt + x(T)'Gx(T)$$
subject to
$$\dot{x} = Ax + Bu,\ x(0) = x_0,$$

ad.4: From the text we deduce the following optimal control problem:

$$\min_u J_T(u)$$
subject to
$$\dot{x} = \alpha x + \beta u,\ x(0) = x_0,$$

with $\alpha, \beta > 0$. Since $\text{argmin}_u\{J_T(u)\} = \text{argmax}_u\{-J_T(u)\}$ we proceed as follows:

For the HJB-equation we have

$$\begin{aligned}
-V_t &= \max_u\{-ax^2 - u^2 + V_x(\alpha x + \beta u)\} \\
&= -ax^2 + V_x\alpha x + \max_u\{-u^2 + V_x\beta u\} \\
&= -ax^2 + V_x\alpha x - (V_x\beta/2)^2 + (V_x\beta)^2/2) \\
&= -ax^2 + V_x\alpha x + (V_x\beta)^2/4),
\end{aligned}$$

where $\bar{u} = V_x\beta/2$ was used in the third equation. Using the hint $V = ch$ we obtain (with c', h' denoting the derivative)

$$-c'h = -ax^2 + \alpha xch' + (\beta^2/4)(ch')^2,$$

which leads to the guess $h(x) = x^2$ yielding

$$-c'h = (-a + 2\alpha xc + \beta^2 c^2)h.$$

Hence

$$\begin{aligned}
&\bar{u} = \beta xc(t) \\
&c' = a - 2\alpha c - \beta^2 c^2, \quad c(T) = V(x,T)/h(x) = 0/h(x) = 0.
\end{aligned}$$

Alternatively, since this is a LQ problem you could use the general procedure presented during the lecture.
For the maxumum principle we have

$$H = -ax^2 - u^2 + \lambda(\alpha x + \beta u) \tag{2}$$

leading to $\bar{u} = \frac{1}{2}\beta\lambda$ with $\lambda = 2xc(t)$.
Alternatively, since this is a LQ problem you could use the general procedure presented during the lecture.

ad.5: Similar to ad.4 we have

$$\begin{aligned}
-W_t &= \min_u\{ax^2 + u^2 + W_x(\alpha x + \beta u)\} \\
&= ax^2 + W_x\alpha x + \min_u\{u^2 + V_x\beta u\} \\
&= ax^2 + W_x\alpha x + (W_x\beta/2)^2 - (V_x\beta)^2/2) \\
&= ax^2 + W_x\alpha x - (W_x\beta)^2/4),
\end{aligned}$$

where $\bar{u} = -W_x\beta/2$ was used in the third equation. Hence with $W(x,t) = -c(t)h(x)$ we obtain

$$\begin{aligned}
&\bar{u} = \beta xc(t) \\
&c' = a - 2\alpha c - \beta^2 c^2, \quad c(T) = V(x,T)/h(x) = 0/h(x) = 0,
\end{aligned}$$

which is the same as in ad.4 (which of course it should be). The reason for this is as follows

$$\begin{aligned} V_t + \max\{-F + V_x f\} &= V_t - \min\{F - V_x f\} \\ &= -W_t - \min\{F + W_x f\} \\ &= W_t + \min\{F + W_x f\}, \end{aligned}$$

where we have used

$$\begin{aligned} V &= \max\{-J\} = -\min\{J\} = -W \\ 0 &= W_t + \min\{F + W_x f\}. \end{aligned}$$

ad.6: See example 2.2 in the literature.

ad.7: Using the optimal input $u^*(t) = -1$ from exercise 6 we obtain, with initial condition $x(s) = z$, the optimal state trajectory $x^*(t) = -t + s + z$. Hence the value function $V = V(z, s)$ is

$$V(z, s) = -\int_s^2 -t + s + z \; dt$$