Kalman filter - for linear systems

CA9/CA3 course Nonlinear Control Systems

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Outline

State estimation part of the course

State estimation problem for linear systems

Estimators based on a probabilistic models

Kalman filter for the linear DD problem

Possibilities with the Kalman filter

Challenges using the Kalman filter

State estimation part of the course
Introduction

- *State estimation* is part of the CA9/CA3 course: Nonlinear Control Systems.
- State estimation is used for estimating something which is not directly measured, or which is measured with lots of noise, from something else which is measured.
- State estimates can be used for diagnostics, fault detection or control as e.g. state space control.
- State estimators can be based on *deterministic models* and are then called observers.
- Here the state estimators are based on *stochastic models*.

Literature and lectures

- Good literature for this course is hard to find.
- We will use the same literature as last year: Mohinder S. Grewal and Angus P. Andrews, Kalman Filtering, Wiley, 2008.
- The lectures will not follow the book closely but will try to give alternative expositions as well and sometimes use other notation.
- The focus is on discrete time models. Continuous time models will only be covered where appropriate.

Given a state space model and *input/output* measurement up to and including time t estimate the *state* at time s. The estimation are given specific names depending on the relation between t and s.

- $s < t$ State *smoothing*.
- $s = t$ State *filtering*.
- $s > t$ State *prediction*.

The discrete-discrete (DD) estimation problem

Assume the system:

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + w_k \ , \ w_k \in \mathrm{NID}(\underline{0}, Q_k) \ ,$$
$$y_k = H_k x_k + D_k u_k + v_k \ , \ v_k \in \mathrm{NID}(\underline{0}, R_k) \ ,$$
$$\mathrm{E}(w(k)v(l)^{\mathtt{T}}) = \underline{\underline{0}}$$

Given measurements and initial values:

$$\mathcal{Y}_k \triangleq y_0, y_1, y_2 \ldots y_k \ , \ u_0, u_1, u_2 \ldots u_k$$
$$x_0 \in \mathrm{N}(\hat{x}_0, P_0)$$

Find: $\hat{x}_k$ which minimized the mean square error

$$\mathrm{E}((x_k - \hat{x}_k)^{\mathtt{T}} M (x_k - \hat{x}_k)) \ , \ M > 0$$

The continuous-discrete (CD) estimation problem

The state model is in continuous time i.e. a stochastic differential equation (SDE):

$$
\begin{aligned}
dx(t) &= (F(t)x(t) + B(t)u(t))dt + d\omega(t) \ , \\
&\qquad \omega(t) \in \mathrm{W}(Q(t)) \ , \\
y(t_k) &= H_k x(t_k) + D_k u(t_k) + v_k \ , \ v_k \in \mathrm{NID}(\underline{0}, R_k)
\end{aligned}
$$

Given measurements and initial values:

$$
\begin{aligned}
&\mathcal{Y}_k \triangleq y(t_0), y(t_1), y(t_2) \ldots y(t_k) \ , \ u(s), t_0 \leq s \leq t_k \\
&x(t_0) \in \mathrm{N}(\hat{x}(t_0), P(t_0))
\end{aligned}
$$

Find: $\hat{x}_k$ which minimized the mean square error

$$
\mathrm{E}((x_k - \hat{x}_k)^{\mathtt{T}} M (x_k - \hat{x}_k)) \ , \ M > 0
$$

Using standard calculus for ODE the SDE can be rewritten as follows:

$$
\begin{aligned}
&dx(t) = (F(t)x(t) + B(t)u(t))dt + d\omega(t) \Rightarrow \\
&\dot{x}(t) = F(t)x(t) + B(t)u(t) + w(t)\ , \\
&w(t) = \frac{d\omega(t)}{dt}
\end{aligned}
$$

- This simplified formulation is wrong from a strict mathematical point of view.
- To make any sense $w(t)$ must be *continuous white noise* which is a unphysical signal with constant spectrum for all frequencies and consequently an infinite variance.

The continuous-continuous (CC) estimation problem

- The problem where continuous measurements is assumed is called the continuous-continuous estimation problem and the solution is called the Kalman-Busy filter.
- In practice the measurements are always in discrete time.
- Continuous measurements with continuous time white noise is also unphysical in some sense.

Notice: You do not need to worry about continuous time white noise and details on SDE's in this course.

Estimation from a one dimensional distribution

Given the density function $f(x)$ for a scalar stochastic variable x find the estimate $\hat{x}$ that minimize the mean square error (MSE) $\mathrm{E}((x - x_e)^2)$ where x_e is any estimate.

$$
\begin{aligned}
L(x_e) &= \mathrm{E}\left((x - x_e)^2\right) \\
&= \mathrm{E}\left(((x - \mu) - (x_e - \mu))^2\right) \ , \ \mu \triangleq \mathrm{E}(x) \\
&= \mathrm{E}\left((x - \mu)^2\right) + \mathrm{E}\left((x_e - \mu)^2\right) \qquad \Rightarrow \\
&\quad - 2\,\mathrm{E}\left((x - \mu)(x_e - \mu)\right) \\
&= \mathrm{V}(x) + (x_e - \mu)^2
\end{aligned}
$$

$$
\hat{x} \triangleq \arg\min_{x_e} L(x_e) = \mu
$$

Notice: MSE is not the only possible criteria.

Estimation from a multi dimensional distribution

Given the density function $f(x)$ for a stochastic vector x find the estimate $\hat{x}$ that minimize the mean square error (MSE) $\mathrm{E}((x - x_e)^\mathsf{T} M (x - x_e))$ where x_e is any estimate and $M > 0$.

$$
\begin{aligned}
L(x_e) &= \mathrm{E}\left((x - x_e)^\mathsf{T} M (x - x_e)\right) \\
&= \mathrm{E}\left(((x - \mu) - (x_e - \mu))^\mathsf{T} M ((x - \mu) - (x_e - \mu))\right) \\
&= \mathrm{E}\left((x - \mu)^\mathsf{T} M (x - \mu)\right) + (x_e - \mu)^\mathsf{T} M (x_e - \mu) \Rightarrow
\end{aligned}
$$

$$
\hat{x} \triangleq \arg\min_{x_e} L(x_e) = \mu
$$

Estimation from a conditional distribution

Given the *conditional* density function $f(x|y)$ for a stochastic vector x given y find the estimate $\hat{x}$ that minimize the *conditional* mean square error (MSE) $\mathrm{E}((x-x_e)^\mathsf{T} M(x-x_e)|y)$ where x_e is any estimate and $M > 0$.

- The *conditional* distribution is still just a distribution.
- The solution is therefore the same as for other distribution i.e. the mean value which now is a conditional mean

$$L(x_e) = \mathrm{E}\left((x-x_e)^\mathsf{T} M(x-x_e)|y\right) \quad \Rightarrow$$
$$\hat{x} \triangleq \arg\min_{x_e} L(x_e) = \mathrm{E}(x|y)$$

Using unconditional loss

The conditional mean estimator will also minimize the *unconditional* loss

$$
\begin{aligned}
L_u(x_e) &\triangleq \mathrm{E}\left((x - x_e)^\mathsf{T} M (x - x_e)\right) \\
&= \mathrm{E}\left(\mathrm{E}\left((x - x_e)^\mathsf{T} M (x - x_e)|y\right)\right)
\end{aligned}
$$

The estimator $\hat{x} = \mathrm{E}(x|y)$ that minimizes the conditional MSE for any given y will also minimize the mean (with respect to y) MSE.
Notice: Above the general rule
$\mathrm{E}(g(x, y)) = \mathrm{E}(\mathrm{E}(g(x, y)|y))$ was used.

The error is unbiased

$$\begin{aligned}
\mathrm{E}\left((x-\hat{x})\right) &= \mathrm{E}\left(\mathrm{E}\left((x-\hat{x})|y\right)\right) \\
&= \mathrm{E}\left(\mathrm{E}\left((x-\mathrm{E}(x|y))|y\right)\right) \\
&= \mathrm{E}\left(\underline{0}\right) \\
&= \underline{0}
\end{aligned}$$

Notice:

- An estimator with bias would be a bad estimator.

The error is uncorrelated with the known information

This is also called the orthogonality principle.

$$\begin{aligned} \mathrm{E}\left((x-\hat{x})y^{\mathsf{T}}\right) &= \mathrm{E}\left(\mathrm{E}\left((x-\hat{x})y^{\mathsf{T}}|y\right)\right) \\ &= \mathrm{E}\left(\mathrm{E}\left((x-\mathrm{E}(x|y))|y\right)y^{\mathsf{T}}\right) \\ &= \mathrm{E}\left(\underline{0}y^{\mathsf{T}}\right) \\ &= \underline{\underline{0}} \end{aligned}$$

Notice:

- correlation between $(x-\hat{x})$ and y would mean that there were more information to extract.
- correlation between $(x-\hat{x})$ and x is not zero.
- but correlation between $(x-\hat{x})$ and $\hat{x}$ is zero because $\hat{x}$ is a function of y.

Conditional Gaussian distributions

Assume x, y is jointly Gaussian then $x|y$ is Gaussian with mean and covariance

$$\mathrm{E}(x|y) = \mathrm{E}(x) + \mathrm{Cov}(x,y)\,\mathrm{Cov}(y)^{-1}(y - \mathrm{E}(y)) \ ,$$

$$\mathrm{Cov}(x|y) = \mathrm{Cov}(x) - \mathrm{Cov}(x,y)\,\mathrm{Cov}(y)^{-1}\,\mathrm{Cov}(y,x) \ ,$$

$$\mathrm{Cov}(x|y) \triangleq \mathrm{E}(\tilde{x}\tilde{x}^{\mathsf{T}}|y) = \mathrm{E}(\tilde{x}\tilde{x}^{\mathsf{T}}) \ , \ \tilde{x} \triangleq x - \mathrm{E}(x|y)$$

Independent measurements

Assume x, u, v is jointly Gaussian and that u, v is independent then

$$\mathrm{E}(x|u,v) = \mathrm{E}(x|u) + \mathrm{E}(x|v) - \mathrm{E}(x)$$

Notation

The notation used is not always the same as in Grewal.

$$
\begin{aligned}
\hat{x}_{k|k} &\triangleq \hat{x}_k^{(+)} \triangleq \mathrm{E}(x_k|\mathcal{Y}_k) \ , \\
\hat{x}_{k|k-1} &\triangleq \hat{x}_k^{(-)} \triangleq \mathrm{E}(x_k|\mathcal{Y}_{k-1}) \ , \\
\tilde{x}_{k|k} &\triangleq \tilde{x}_k^{(+)} \triangleq x_k - \hat{x}_{k|k} \ , \\
\tilde{x}_{k|k-1} &\triangleq \tilde{x}_k^{(-)} \triangleq x_k - \hat{x}_{k|k-1} \ , \\
P_{k|k} &\triangleq P_k^{(+)} \triangleq \mathrm{E}(\tilde{x}_{k|k}\tilde{x}_{k|k}^{\mathtt{T}}) \ , \\
P_{k|k-1} &\triangleq P_k^{(-)} \triangleq \mathrm{E}(\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^{\mathtt{T}})
\end{aligned}
$$

Main idea

The derivation builds on the following (which would be hard to figure out on your own):

- The LMS estimator is $\hat{x}_k = \mathrm{E}(x_k|\mathcal{Y}_k)$
- The state space model is recursive in time then the KF should also be recursive in time.
- Given the current estimate $\hat{x}_{k|k}$ it is easy to find predictions $\hat{x}_{k+j|k}$, $j \geq 1$ including the error covariance.
- The KF is split into a *measurement update* using y_k to update from $\hat{x}_{k|k-1}$ to $\hat{x}_{k|k}$ and a *time update* to update from $\hat{x}_{k|k}$ to $\hat{x}_{k+1|k}$ using no new measurements.

Time update

Assume $\hat{x}_{k|k}$ and $P_{k|k}$ known then

$$
\begin{aligned}
x_{k+1} &= \Phi_k x_k + \Gamma_k u_k + w_k \Rightarrow \\
\hat{x}_{k+1|k} &= \Phi_k \hat{x}_{k|k} + \Gamma_k u_k \Rightarrow \\
\tilde{x}_{k+1|k} &= \Phi_k \tilde{x}_{k|k} + w_k \Rightarrow \\
P_{k+1|k} &= \Phi_k P_{k|k} \Phi_k^{\mathsf{T}} + Q_k \ .
\end{aligned}
$$

From $\hat{x}_{k|k-1}$ to $\hat{y}_{k|k-1}$

$$
\begin{aligned}
y_k &= H_k x_k + D_k u_k + v_k \Rightarrow \\
\hat{y}_{k|k-1} &= H_k \hat{x}_{k|k-1} + D_k u_k \Rightarrow \\
\tilde{y}_{k|k-1} &= H_k \tilde{x}_{k|k-1} + v_k \Rightarrow \\
\mathrm{P}^y_{k|k-1} &= H_k P_{k|k-1} H_k^{\mathsf{T}} + R_k \ .
\end{aligned}
$$

Measurements update - estimate

Assume $\hat{x}_{k|k-1}$ and $P_{k|k-1}$ and $y(k)$ known then

$$\begin{aligned}\hat{x}_{k|k} &= \mathrm{E}(x_k|\mathcal{Y}_k)\\ &= \mathrm{E}(x_k|y_k,\mathcal{Y}_{k-1})\\ &= \mathrm{E}(x_k|\tilde{y}_{k|k-1},\mathcal{Y}_{k-1})\\ &= \hat{x}_{k|k-1} + \mathrm{E}(x_k|\tilde{y}_{k|k-1}) - \mathrm{E}(x_k)\ .\end{aligned}$$

$$\mathrm{E}(x_k|\tilde{y}_{k|k-1}) = \mathrm{E}(x_k) + \\ \mathrm{Cov}(x_k,\tilde{y}_{k|k-1})\,\mathrm{Cov}(\tilde{y}_{k|k-1})^{-1}\tilde{y}_{k|k-1}$$

$$
\begin{aligned}
&\mathrm{Cov}(x_k, \tilde{y}_{k|k-1}) \\
&\quad = \mathrm{Cov}(\hat{x}_{k|k-1} + \tilde{x}_{k|k-1}, \tilde{y}_{k|k-1}) \\
&\quad = \mathrm{Cov}(\hat{x}_{k|k-1}, \tilde{y}_{k|k-1}) + \mathrm{Cov}(\tilde{x}_{k|k-1}, \tilde{y}_{k|k-1}) \\
&\quad = \mathrm{Cov}(\tilde{x}_{k|k-1}, \tilde{y}_{k|k-1}) \\
&\quad = \mathrm{E}(\tilde{x}_{k|k-1}\tilde{y}_{k|k-1}) \\
&\quad = \mathrm{E}(\tilde{x}_{k|k-1}(H_k\tilde{x}_{k|k-1} + v_k)^{\mathtt{T}}) \\
&\quad = \mathrm{E}(\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^{\mathtt{T}}H_k^{\mathtt{T}}) \\
&\quad = P_{k|k-1}H_k^{\mathtt{T}}
\end{aligned}
$$

Collecting the above

$$
\hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}H_k^{\mathtt{T}}(H_kP_{k|k-1}H_k^{\mathtt{T}} + R_k)^{-1}\tilde{y}_{k|k-1}
$$

Measurements update - error variance

$$
\begin{aligned}
K_k &\triangleq P_{k|k-1}H_k^\mathsf{T}(H_kP_{k|k-1}H_k^\mathsf{T}+R_k)^{-1} \Rightarrow \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1}+K_k\tilde{y}_{k|k-1} \Rightarrow \\
\tilde{x}_{k|k} &= \tilde{x}_{k|k-1}-K_k\tilde{y}_{k|k-1} \\
&= \tilde{x}_{k|k-1}-K_k(H_k\tilde{x}_{k|k-1}+v_k) \\
&= (I-K_kH_k)\tilde{x}_{k|k-1}-K_kv_k \quad \Rightarrow \\
P_{k|k} &= (I-K_kH_k)P_{k|k-1}(I-K_kH_k)^\mathsf{T}+K_kR_kK_k^\mathsf{T} \\
&= (I-K_kH_k)P_{k|k-1}
\end{aligned}
$$

Kalman filter algorithm

Initial conditions:

$$x_0 \in \mathrm{N}(\hat{x}_{0|-1}, P_{0|-1})$$

Measurements update after receiving y_k and u_k:

$$\hat{y}_{k|k-1} = H_k \hat{x}_{k|k-1} + D_k u_k \ ,$$

$$\tilde{y}_{k|k-1} = y_k - \hat{y}_{k|k-1} \ ,$$

$$K_k \triangleq P_{k|k-1} H_k^{\mathsf{T}} (H_k P_{k|k-1} H_k^{\mathsf{T}} + R_k)^{-1} \ ,$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_{k|k-1} \ ,$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^{\mathsf{T}} + K_k R_k K_k^{\mathsf{T}} \ .$$

Time update from k to $k+1$:

$$\hat{x}_{k+1|k} = \Phi_k \hat{x}_{k|k} + \Gamma_k u_k \ ,$$

$$P_{k+1|k} = \Phi_k P_{k|k} \Phi_k^{\mathsf{T}} + Q_k \ .$$

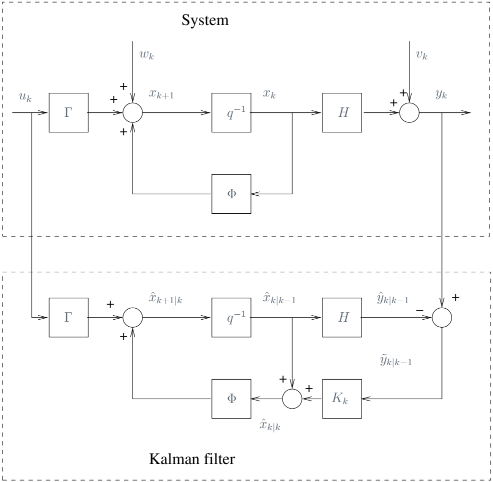
Initial conditions:
Assume $\hat{x}_{k|k}, P_{k|k}$ know,

Multi step prediction is then obtained for $j \geq 0$ by:

$$\hat{x}_{k+j+1|k} = \Phi_{k+j}\hat{x}_{k+j|k} + \Gamma_{k+j}u_{k+j} \ ,$$
$$P_{k+j+1|k} = \Phi_{k+j}P_{k+j|k}\Phi_{k+j}^{\mathsf{T}} + Q_{k+j} \ ,$$
$$\hat{y}_{k+j+1|k} = H_{k+j+1}\hat{x}_{k+j+1|k} + D_{k+j+1}u_{k+j+1} \ ,$$
$$P_{k+j+1|k}^{y} = H_{k+j+1}P_{k+j+1|k}H_{k+j+1}^{\mathsf{T}} + R_{k+j+1} \ .$$

Multi step prediction can be interpreted as: simulation starting with initial state $\hat{x}_{k|k}$

Block diagram interpretation



Other important features of Kalman filter

- The filter gain K_k and error covariances $P_{k|k}, P_{k|k-1}, P^y_{k|k-1}$ does not depend on measurements.
- If the model is time invariant the Kalman filter also converges to a time invariant filter given by the solution to the *discrete time Riccati equation*

$$P \triangleq \lim_{k\to\infty} P_{k|k-1} \ ,$$
$$P = \Phi P \Phi^{\mathsf{T}} - \Phi P H^{\mathsf{T}}(HPH^{\mathsf{T}} + R)^{-1} HP\Phi^{\mathsf{T}} + Q$$

- The measurement errors $\tilde{y}_{k|k-1}, \tilde{y}_{j|j-1}, k \neq j$ are independent and uncorrelated because for e.g. $j < k$, $\tilde{y}_{k|k-1}$ is independent of $\mathcal{Y}_{k-1}$ and $\tilde{y}_{j|j-1}$ is a linear function of $\mathcal{Y}_j \subseteq \mathcal{Y}_{k-1}$ for $j < k$.

The Kalman filter is a fantastic flexible tool

- The kalman filter can serve many purposes.
- Creative use of the parameters makes it possible to tailor the Kalman filter to the problem.
- State estimation, smoothing and prediction is the basic application.
- Parameters estimation for ARX models is a less standard problem that can be solved.

Challenges using the Kalman filter

- If the process and measurement noise is not white the filter is not optimal.
  - Colored noise is included by augmenting the original state space model.
  - Correlation between process and measurement noise can also be included in the Kalman filter.
- All parameters must be known.
- Especially noise covariances are difficult to get.
  - Some times they can be found from *first principles*.
  - If input, output and states can be measured they can be estimated.
  - If only input and output, but not states, can be measured only a *innovation* model can be estimated.
  - Noise covariances can also be interpreted as tuning parameters.
- The system must of course be *observable*.