

Advanced Gain-Scheduling Techniques for Uncertain Systems

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Abstract—This paper is concerned with the design of gain-scheduled controllers for uncertain linear parameter-varying systems. Two alternative design techniques for constructing such controllers are discussed. Both techniques are amenable to linear matrix inequality problems via a gridding of the parameter space and a selection of basis functions. These problems are then readily solvable using available tools in convex semidefinite programming. When used together, these techniques provide complementary advantages of reduced computational burden and ease of controller implementation. The problem of synthesis for robust performance is then addressed by a new scaling approach for gain-scheduled control. The validity of the theoretical results are demonstrated through a two-link flexible manipulator design example. This is a challenging problem that requires scheduling of the controller in the manipulator geometry and robustness in face of uncertainty in the high-frequency range.

Index Terms—Flexible manipulator, gain-scheduling control, linear matrix inequalities (LMI), linear parameter-varying (LPV) systems, nonlinear control, robust control, uncertain systems.

I. INTRODUCTION

THE gain-scheduling problem has been the subject of a great deal of research over recent years, both from theoretical and practical viewpoints. This renewed interest probably stems from the development of new techniques and software which allow for a more rigorous and systematic treatment of the gain-scheduling problem. The classical approach to this problem essentially consists of repeated design syntheses associated with some scheduling strategy connecting locally designed controllers. Such schemes, however, lack supporting theories that guarantee the behavior of the scheduled controller. A significant contribution toward the elimination of such weaknesses is the formulation of the gain-scheduling problem in the context of convex semidefinite programming [1], an elegant and solidly based branch of optimization theory [2]–[4]. Expressed in terms of linear matrix inequalities (LMI's), the gain-scheduling problem is readily and globally solved using currently available efficient optimization software [5]. LMI techniques now appear as very natural mechanisms for the formulation of gain-scheduling problems as well as for a vast array of other problems in the control field. Reference [6] gives an overview of the scope of application of such techniques.

As emphasized in H_∞ control theory, a key stage in the characterization of gain-scheduled controllers is the search

for adequate Lyapunov functions that establish stability and a performance bound for the closed-loop system. The LFT gain-scheduling techniques in [7]–[10] or the so-called quadratic gain-scheduled techniques in [11] and [12] make use of a fixed Lyapunov function, as opposed to one which depends on the scheduled variables, to characterize stability and performance. According to [13], such approaches are potentially very conservative because they allow for arbitrary rates of variation in the scheduled variables. More dramatically, it has been shown in [13] that some systems are not even quadratically stabilizable, that is, are not stabilizable on the basis of a single Lyapunov function. A significant improvement over such techniques can be obtained by exploiting the concept of parameter-dependent Lyapunov functions. This is discussed in the context of robustness analysis and synthesis in [14]–[16] and for the gain-scheduling problem in [13] and [16]. Parameter-dependent Lyapunov functions allow the incorporation of knowledge on the rate of variation in the analysis or synthesis technique, and therefore lead to much less conservative answers. The reader is referred to [13] and [17] for earlier work related to the approaches considered here. The discretization of continuous-time gain-scheduled controllers is considered in [18].

In this paper we investigate two different techniques: [19]–[21] and an extension of [22] and [23] to the gain-scheduling problem. These techniques impose no restriction on the plant and provide a simple and streamlined treatment of the gain-scheduling problem. Moreover, the technique in [19] and [20] allows the incorporation of multiple specifications into the design problem such as $H_2 - H_\infty$, pole clustering, or control effort constraints. The second technique is more restrictive but offers computational advantages. The focus of this work is on the computational effort for controller calculation and on the practical issues of controller implementation. A special emphasis is placed on the development of scaling techniques which take advantage of the problem's structural properties and thus reduce the conservatism of the gain-scheduling approach. It is further shown that combining the capabilities of both techniques provides a comprehensive and effective methodology, encompassing all components of the gain-scheduling task from theoretical constructions to real-time implementations.

This paper is structured as follows. Sections II to IV give a thorough discussion of gain-scheduling synthesis techniques together with some refinements and improvements in Section V. Finally, the validity and applicability of concepts and techniques are demonstrated for a two-link flexible manipulator application in Section VI.

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The notation used in the paper is fairly conventional. For real symmetric matrices M , $M > 0$ stands for “positive definite” and means that all the eigenvalues of M are positive. Similarly, $M < 0$ means “negative definite” (all the eigenvalues of M are negative) and $M \geq 0$ stands for “nonnegative definite” (the smallest eigenvalue of M is nonnegative). In large symmetric matrix expressions, terms denoted \star will be induced by symmetry. For instance, with S symmetric

$$\begin{bmatrix} S + M + N + (\star) & \star \\ Q & P \end{bmatrix} := \begin{bmatrix} S + M + M^T + N + N^T & Q^T \\ Q & P \end{bmatrix}.$$

We shall also use the matrix notations

$$\text{diag}_{i=1}^N X_i := \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_N \end{bmatrix}$$

$$[L_{jk}]_{j,k} := \begin{bmatrix} L_{11} & \cdots & L_{1m} \\ \vdots & \ddots & \vdots \\ L_{m1} & \cdots & L_{mm} \end{bmatrix}.$$

For appropriately dimensioned matrices K and $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and assuming the inverse exists, the upper LFT is defined as

$$F_u(M, K) = M_{22} + M_{21}K(I - M_{11}K)^{-1}M_{12}. \quad (1)$$

II. OUTPUT-FEEDBACK SYNTHESIS WITH GUARANTEED L_2 -GAIN PERFORMANCE

In this section we recap some known results on the gain-scheduling technique with bounded parameter variation rates and point out connections between different approaches. We first give a general characterization of gain-scheduled controllers, the solution to which involves both intermediate controller matrices and Lyapunov variables X and Y . This formulation will be referred as the *basic* characterization, emphasizing the fact that it can be easily extended to multiple objective problems, pole clustering problems, etc. [19], [20]. Next, a second formulation of gain-scheduled controllers is presented. It will be referred as the *projected* characterization, as the intermediate controller matrices have been eliminated through projections [22]. Reconstructing the controller state-space data from the projected conditions has been addressed in [22], [23] for the customary H_∞ control problem. The reconstruction procedure is again described here, in the case of the gain-scheduling problem, for completeness of the discussion. The reader is referred to [13], [17], and [24], for details, insights, and applications of analogous gain-scheduling techniques.

The problem addressed throughout the paper is the following. Suppose we are given a linear parameter-varying (LPV) plant $G(\theta)$ with state-space realization

$$\begin{aligned} \dot{x} &= A(\theta)x + B_1(\theta)w + B_2(\theta)u \\ z &= C_1(\theta)x + D_{11}(\theta)w + D_{12}(\theta)u \\ y &= C_2(\theta)x + D_{21}(\theta)w \end{aligned} \quad (2)$$

where

$$A \in \mathbb{R}^{n \times n}, \quad D_{12} \in \mathbb{R}^{p_1 \times m_2}, \quad \text{and} \quad D_{21} \in \mathbb{R}^{p_2 \times m_1}$$

define the problem dimension. The time-varying parameter $\theta := (\theta_1, \dots, \theta_L)^T$ as well as its rates of variation $\dot{\theta}$ are assumed bounded as follows.

- 1) Each parameter θ_i ranges between known extremal values $\underline{\theta}_i$ and $\bar{\theta}_i$

$$\theta_i(t) \in [\underline{\theta}_i, \bar{\theta}_i], \quad \forall t \geq 0. \quad (3)$$

- 2) The rate of variation $\dot{\theta}_i$ is assumed well-defined at all times and satisfies

$$\dot{\theta}_i(t) \in [\underline{\nu}_i, \bar{\nu}_i], \quad \forall t \geq 0 \quad (4)$$

where $\underline{\nu}_i \leq \bar{\nu}_i$ are known lower and upper bounds on $\dot{\theta}_i$.

The first assumption means that the parameter vector θ is valued in a hypercube Θ . Similarly, (4) defines a hypercube Θ_d of \mathbb{R}^L with vertices in

$$\mathcal{T} := \{(\tau_1, \dots, \tau_L)^T : \tau_i \in \{\underline{\nu}_i, \bar{\nu}_i\}\}. \quad (5)$$

The gain-scheduled output-feedback control problem consists of finding a dynamic LPV controller, $K(\theta)$, with state-space equations

$$\begin{aligned} \dot{x}_K &= A_K(\theta, \dot{\theta})x_K + B_K(\theta, \dot{\theta})y \\ u &= C_K(\theta, \dot{\theta})x_K + D_K(\theta, \dot{\theta})y \end{aligned} \quad (6)$$

which ensures internal stability and a guaranteed L_2 -gain bound γ for the closed-loop operator (2)–(6) from the disturbance signal w to the error signal z , that is

$$\int_0^T z^T z d\tau \leq \gamma^2 \int_0^T w^T w d\tau, \quad \forall T \geq 0$$

and all admissible trajectories $(\theta, \dot{\theta})$ and zero-state initial conditions. Note that A and A_K have the same dimensions, since we restrict the discussion to the full-order case. The formulation of such controllers can be handled via an extension of the bounded real lemma with quadratic parameter-dependent Lyapunov functions $V(x_{cl}, \theta) = x_{cl}^T P(\theta) x_{cl}$ where x_{cl} stands for the state vector of the closed-loop system. See [13]–[15] and [19] for details. Note that the controller state-space matrices are allowed to depend explicitly on the derivative of the time-varying parameter θ . Different techniques to remove the dependence on $\dot{\theta}$ will be extensively discussed in Section III, see also [24].

Except the usual smoothness assumptions on the dependence on θ , the problem data and variables will be unrestricted in the subsequent derivations. The basic characterization of gain-scheduled controllers with guaranteed L_2 -gain performance is presented in the next theorem where the dependence of data and variables on θ and $\dot{\theta}$ has been dropped for simplicity.

Theorem 2.1—Basic Characterization: Consider the LPV plant governed by (2), with parameter trajectories constrained by (3), (4). There exists a gain-scheduled output-feedback controller (6) enforcing internal stability and a bound γ on the L_2 gain of the closed-loop system (2) and (6), whenever there exist parameter-dependent symmetric matrices Y and X and a parameter-dependent quadruple of state-space data $(\hat{A}_K, \hat{B}_K, \hat{C}_K, D_K)$ such that for all pairs $(\theta, \dot{\theta})$ in $\Theta \times$

Θ_d the following infinite-dimensional LMI problem holds as shown in (7) at the bottom of the page, together with

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \quad (8)$$

In such case, a gain-scheduled controller of the form (6) is readily obtained with the following two-step scheme.

- Solve for N, M , the factorization problem

$$I - XY = NM^T.$$

- Compute A_K, B_K, C_K with

$$A_K = N^{-1}(X\dot{Y} + N\dot{M}^T + \hat{A}_K - X(A - B_2D_KC_2)Y - \hat{B}_KC_2Y - XB_2\hat{C}_K)M^{-T} \quad (9)$$

$$B_K = N^{-1}(\hat{B}_K - XB_2D_K) \quad (10)$$

$$C_K = (\hat{C}_K - D_KC_2Y)M^{-T}. \quad (11)$$

Proof: See [19] and [20]. ■

Note that since all variables are involved linearly, the constraints (7) and (8) constitute an LMI system. This system is, however, infinite due to its dependence on $(\theta, \hat{\theta})$ ranging over $\Theta \times \Theta_d$. Using the projection lemma, detailed in [22], the controller variables can be eliminated, leading to a characterization involving the variables X and Y only. This is presented in the next theorem.

Theorem 2.2—Projected Solvability Conditions: Consider the LPV plant governed by (2), with parameter trajectories constrained by (3) and (4). There exists a gain-scheduled output-feedback controller (6) enforcing internal stability and a bound γ on the L_2 gain of the closed-loop system (2) and (6), whenever there exist parameter-dependent symmetric matrices $Y(\theta)$ and $X(\theta)$ such that for all pairs $(\theta, \hat{\theta})$ in $\Theta \times \Theta_d$ the following infinite-dimensional LMI problem holds:

$$\begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \dot{X} + XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \cdot \begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix} < 0 \quad (12)$$

$$\begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} -\dot{Y} + YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \cdot \begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix} < 0 \quad (13)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \quad (14)$$

where \mathcal{N}_X and \mathcal{N}_Y designate any bases of the null spaces of $[C_2 \ D_{21}]$ and $[B_2^T \ D_{12}^T]$, respectively.

Proof: This is a straightforward application of the projection lemma [22] to the LMI (7), with respect to the matrix variable

$$\begin{bmatrix} \hat{A}_K + (A + B_2D_KC_2)^T & \hat{B}_K \\ \hat{C}_K & D_K \end{bmatrix}.$$

Theorem 2.2 only provides existence conditions for controllers of the form (6). These conditions become necessary and sufficient if we confine the involved Lyapunov functions to the set of quadratic forms

$$V(x_{cl}, \theta) := x_{cl}^T P(\theta) x_{cl}, \quad \text{with } x_{cl} := \begin{bmatrix} x \\ x_K \end{bmatrix}.$$

As an immediate extension of the results in [23], the next theorem provides for controller construction. Once again, the dependence on θ has been dropped to facilitate manipulations. It is further assumed that

- **(H1)** D_{12} and D_{21} are full-column and full-row rank, respectively.

This assumption is without restriction and greatly simplifies the presentation. The construction is easily extended to the singular case along the lines of [23].

Theorem 2.3. Controller Construction from Projections: Assume the conditions of Theorem 2.2 hold for a pair (X, Y) and some performance level γ . Then a gain-scheduled controller can be constructed for any pair $(\theta, \hat{\theta})$ in $\Theta \times \Theta_d$ by the following sequential scheme:

- Compute D_K solution to

$$\sigma_{\max}(D_{11} + D_{12}D_KD_{21}) < \gamma \quad (15)$$

and set $D_{cl} := D_{11} + D_{12}D_KD_{21}$.

- Compute \hat{B}_K and \hat{C}_K solutions to the linear matrix equations

$$\begin{bmatrix} 0 & D_{21} & 0 \\ D_{21}^T & -\gamma I & D_{cl}^T \\ 0 & D_{cl} & -\gamma I \end{bmatrix} \begin{bmatrix} \hat{B}_K^T \\ \star \end{bmatrix} = - \begin{bmatrix} C_2 \\ B_1^T X \\ C_1 + D_{12}D_KC_2 \end{bmatrix} \quad (16)$$

$$\begin{bmatrix} 0 & D_{12}^T & 0 \\ D_{12} & -\gamma I & D_{cl} \\ 0 & D_{cl}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \hat{C}_K \\ \star \end{bmatrix} = - \begin{bmatrix} B_2^T \\ C_1 Y \\ (B_1 + B_2D_KD_{21})^T \end{bmatrix}. \quad (17)$$

- Compute

$$\begin{aligned} \hat{A}_K = & -(A + B_2D_KC_2)^T \\ & + [XB_1 + \hat{B}_KD_{21} \quad (C_1 + D_{12}D_KC_2)^T] \\ & \cdot \begin{bmatrix} -\gamma I & D_{cl}^T \\ D_{cl} & -\gamma I \end{bmatrix}^{-1} \begin{bmatrix} (B_1 + B_2D_KD_{21})^T \\ C_1 Y + D_{12}\hat{C}_K \end{bmatrix}. \end{aligned} \quad (18)$$

$$\begin{bmatrix} \dot{X} + XA + \hat{B}_KC_2 + (\star) & \star & \star & \star \\ \hat{A}_K^T + A + B_2D_KC_2 & -\dot{Y} + AY + B_2\hat{C}_K + (\star) & \star & \star \\ (XB_1 + \hat{B}_KD_{21})^T & (B_1 + B_2D_KD_{21})^T & -\gamma I & \star \\ C_1 + D_{12}D_KC_2 & C_1 Y + D_{12}\hat{C}_K & D_{11} + D_{12}D_KD_{21} & -\gamma I \end{bmatrix} < 0 \quad (7)$$

TABLE I
SELECTION OF VARIABLES IN THE GAIN-SCHEDULED CONTROL PROBLEM

	Variables X, Y	Variables N, M	Practical Validity
$\frac{d\theta}{dt} = 0$	$X := X(\theta), Y := Y(\theta)$	$NM^T = I - X(\theta)Y(\theta)$	Yes
$\frac{d\theta}{dt} \in \Theta_d$	$X := X(\theta), Y := Y(\theta)$	$NM^T = I - X(\theta)Y(\theta)$	No
$\frac{d\theta}{dt} \in \Theta_d$	$X := X(\theta), Y := Y_0$	$N := I - X(\theta)Y_0, M := I$	Yes
$\frac{d\theta}{dt} \in \Theta_d$	$X := X_0, Y := Y(\theta)$	$N := I, M := I - Y(\theta)X_0$	Yes
$\frac{d\theta}{dt}$ unbounded	$X := X_0, Y := Y_0$	$NM^T = I - X_0Y_0$	Yes

- Solve for N, M , the factorization problem

$$I - XY = NM^T.$$

- Finally, compute A_K, B_K , and C_K with the help of (9)–(11). ■

It should be noted that in spite of their different structures, the characterizations given in Theorems 2.1–2.3 are equivalent and can virtually be used interchangeably for controller synthesis. In contrast, when the focus is on computational complexity or practical implementation, these techniques exhibit significant differences. This is discussed in Section IV. Finally, the case where only some parameters θ_i are subject to constraints on their derivatives is easily handled by removing the unconstrained parameters from the matrix functions $X(\cdot)$ and $Y(\cdot)$.

A. Extensions to Multiobjective Problems

A useful practical advantage of the basic technique is that it easily extends to multiobjective problems. Various channels of the closed-loop system can be specified independently with a rich list of specifications. See [21] and [25] for a thorough discussion. As an example, it is possible to specify an L_2 gain bound with regional pole constraints on the closed-loop dynamics of the underlying LTI systems (θ frozen). Such constraints consist of vertical and horizontal strips, disks, conic sectors, parabolas, ..., or intersections of such regions. The LMI's (7) and (8) must then be complemented with

$$\begin{aligned} & \left[\lambda_{jk} \begin{bmatrix} Y & I \\ I & X \end{bmatrix} + \mu_{jk} \begin{bmatrix} AY + B_2\hat{C}_K & A + B_2D_KC_2 \\ \hat{A}_K & XA + \hat{B}_KC_2 \end{bmatrix} \right. \\ & \left. + \mu_{kj} \begin{bmatrix} AY + B_2\hat{C}_K & A + B_2D_KC_2 \\ \hat{A}_K & XA + \hat{B}_KC_2 \end{bmatrix}^T \right]_{j,k} < 0. \quad (19) \end{aligned}$$

where the data λ_{jk} and μ_{kj} defines the geometry of the region.

III. PRACTICAL VALIDITY OF GAIN-SCHEDULED CONTROLLERS

It must be stressed out that an LPV controller derived from Theorems 2.1–2.3 is not gain-scheduled in the usual sense of the term. Its implementation requires not only the real-time measurement of the parameter θ , but also of its time-derivative $\dot{\theta}$. This is generally prohibitive, since parameter derivatives either are not available or are difficult to estimate during system operation. Gain-scheduled controllers that do not require a measurement of $\dot{\theta}$ will be called *practically valid* hereafter. As discussed in [17], there is no systematic and

tractable approach for removing the dependence on $\dot{\theta}$ while maintaining the generality of Theorems 2.1–2.3. As suggested by the controller formula (9), a simple but conservative approach has been proposed in [24]. It consists of restricting the variable $Y(\theta)$ to $\dot{Y} = 0$, that is, Y not depending on θ . This operation amounts to using a fixed Lyapunov function for the parameter-dependent control problem described in (13). It thereby sacrifices some performance, resulting in a higher γ .

Keeping in mind that the dependence of the controller data on $\dot{\theta}$ stems from the term $X\dot{Y} + N\dot{M}^T$ in (9), the general characterization of Theorem 2.3 offers additional freedom that is worth pointing out. The discussion is summarized in the Table I.

Row 1 of the table simply says that if the scheduled variable is assumed constant in time, a practically valid gain-scheduled controller can theoretically be constructed using Theorem 2.1 or alternatively Theorems 2.2 and 2.3, for any matrix functions $X(\cdot)$ and $Y(\cdot)$ of θ . Such an approach ignores possible time variations of θ and provides neither performance nor stability guarantees for the closed-loop system in the face of time-variations. With the same choice of matrix functions $X(\cdot)$ and $Y(\cdot)$, but the rate of variations of θ being confined to a compact Θ_d , row 2 says that there is no known techniques to compute a practically valid gain-scheduled controller. In rows 3 and 4, we have assumed the conservative choices that X or Y are constant matrix variables. In both cases, the gain-scheduling problem with bounded rate of variations admits practically valid controller solutions, provided the variables N and M are adequately selected in Theorems 2.1 and 2.3. With further conservatism, that is, $\dot{\theta}$ is unbounded, row 5 says that the problem is again tractable and solvable using the same techniques. The case of time-varying parameters with bounds on the rate of variation can be constructively handled by the choices of rows 3 and 4. However, due to the loss of duality in the variables X and Y , such choices are not equivalent. As a consequence, there are some problems for which it is better to take a parameter-dependent X and a constant Y while others will require the converse. Hence, both alternatives must be tried to get a less conservative design. In the controller construction scheme, the variables N and M are subject to the algebraic constraint $I - XY = NM^T$ from which one easily infers the identity

$$\dot{X}Y + \dot{N}M^T = -(X\dot{Y} + N\dot{M}^T).$$

In light of this identity, a practically valid gain-scheduled controller in the cases of rows 3 and 4 can be derived using the same formulas (10) and (11), but with A_K suitably updated to

$$\begin{aligned} A_K = & N^{-1}[\hat{A}_K - X(A - B_2D_KC_2)Y \\ & - \hat{B}_KC_2Y - XB_2\hat{C}_K]M^{-T}. \quad (20) \end{aligned}$$

The same formulas are still valid for the case of frozen-in-time parameters, row 1, and for arbitrarily varying parameters, row 5, the variables X and Y being replaced by their constant values X_0 and Y_0 , in the latter case. Summing up, Table I displays all options to handle any situations from the frozen-in-time parameters to arbitrarily time-varying parameters. However, the case in which both X and Y depend on θ with a bounded $\dot{\theta}$ still resists a convex formulation for a practically valid controller.

IV. REDUCTION TO FINITE-DIMENSIONAL PROBLEMS

Even with the simplifications of Table I in place, the characterizations of Theorems 2.1–2.3 involve the solution of a convex but infinite-dimensional and infinitely constrained problem. This is the price to pay for allowing a general parameter dependence in the plant (2). Generally speaking, there is no systematic rule for selecting the functional dependence of the matrix functions X and Y on θ . We are therefore led to some simple heuristics in order to simplify the computation of solutions to the LMI problems (7), (8), or (12)–(14). A simple but practical technique has been proposed in [13]. The key idea is to “mimic” the parameter dependence of the plant in the Lyapunov function variables X and Y . Interestingly, the same idea can be used in the more general context of the basic characterization of Theorem 2.1. In return, this offers new potential approaches for the synthesis of gain-scheduled controllers with multiple objective constraints (mixed $H_2 - H_\infty$, pole clustering, and others still to find). To be more specific, consider the class of plants (2) having an LFT dependence on nonlinear functions of the scheduled variable, that is, whose state-space data further satisfy

$$G(\theta) := F_u \left\{ \begin{bmatrix} A_\rho & B_\rho & B_1 & B_2 \\ C_\rho & D_{\rho\rho} & D_{\rho 1} & D_{\rho 2} \\ C_1 & D_{1\rho} & D_{11} & D_{12} \\ C_2 & D_{2\rho} & D_{21} & 0 \end{bmatrix}, \text{diag}_{i=1}^N [\rho_i(\theta) I_{r_i}] \right\} \quad (21)$$

where $\rho_i(\cdot)$, $i = 1, \dots, N$ are differentiable functions of θ . Note that such a description encompasses many practical situations, since most systems in aeronautics and robotics can be represented as an LFT in nonlinear functions of the time-varying parameters. Copies of the plant’s nonlinear functions, $\rho_i(\cdot)$, can be introduced into the quadruple $[\hat{A}_K(\cdot), \hat{B}_K(\cdot), \hat{C}_K(\cdot), D_K(\cdot)]$ and the pair $[X(\cdot), Y(\cdot)]$ in an affine fashion

$$\begin{aligned} \hat{A}_K(\theta) &:= \hat{A}_{K,0} + \sum_{i=1}^N \rho_i(\theta) \hat{A}_{K,i} \\ \hat{B}_K(\theta) &:= \hat{B}_{K,0} + \sum_{i=1}^N \rho_i(\theta) \hat{B}_{K,i} \\ \hat{C}_K(\theta) &:= \hat{C}_{K,0} + \sum_{i=1}^N \rho_i(\theta) \hat{C}_{K,i} \\ D_K(\theta) &:= D_{K,0} + \sum_{i=1}^N \rho_i(\theta) D_{K,i} \end{aligned} \quad (22)$$

and

$$\begin{aligned} X(\theta) &:= X_0 + \sum_{i=1}^N \rho_i(\theta) X_i \\ Y(\theta) &:= Y_0 + \sum_{i=1}^N \rho_i(\theta) Y_i. \end{aligned} \quad (23)$$

The functional dependence of X and Y being fixed, the matrices $\hat{A}_{K,0}, \hat{A}_{K,i}, \dots$, play the role of decision variables in the infinitely constrained LMI problems (7), (8), or (12)–(14). A simple remedy for turning such problems into a finite set of LMI’s is to grid the value set of θ [13]. Since the derivative $\dot{\theta}$ appears linearly in the LMI’s (7), (12), and (13), there is only need to check the extreme points of the set Θ_d , denoted \mathcal{T} , for all admissible values of θ . The overall procedure can be described as follows.

- Step 1) Define a grid \mathcal{G} for the value set of θ .
- Step 2) Minimize γ subject to the LMI constraints associated with $\mathcal{G} \times \mathcal{T}$.
- Step 3) Check the constraints with a denser grid.
- Step 4) If Step 3) fails, increase the grid density and return to Step 2).

Computing solutions (22) and (23) to the LMI system associated with $\mathcal{G} \times \mathcal{T}$ is a convex optimization that can be solved by polynomial-time algorithms [2], [26], and the software [5]. Such problems generally require a large number of variables and constraints that today limit the scope of application of such techniques. With this considered, available solvers are still efficient for two or three scheduled variables. LMI-based gain-scheduling techniques have proven very powerful in a number of delicate applications [12], [13], [17], [24].

When restricted to the parameterization (22) and (23), the basic and projected characterizations are no longer equivalent. In the first one, we have further restrictions on the structure of the quadruple $[\hat{A}_K(\cdot), \hat{B}_K(\cdot), \hat{C}_K(\cdot), D_K(\cdot)]$. As a result, the first approach is generally more conservative, although we have observed very little difference in practice. See the application Section VI for comparisons. From a complexity viewpoint, the first technique requires a larger number of scalar variables to be optimized; the number of additional variables being approximately $n(n+m_2+p_2)L$, where L is the number of blocks in the LFT (21). Its scope of application is therefore more restricted. In contrast, the controller equations resulting from the basic characterization are significantly less complex than those resulting from the projected characterization. Note that such controller constructions are essentially dominated by matrix inversions and QR decompositions in (9)–(11) and (16)–(18). At each sampling time, the basic characterization essentially requires

- one matrix inversion;

whereas the projected characterization will require

- two QR decompositions and three matrix inversions for problem (15);
- two matrix inversions for the computation of A_K, B_K, C_K by exploiting partitioning.

Thus, in the light of these comments and because the expressions (22) essentially reduce to scalar-by-matrix multi-

plications, controllers resulting from the first technique are more easily implemented for rapidly varying LPV systems. In addition, these controllers have an LFT representation in terms of the nonlinear functions, $\rho_i(\cdot)$, and hence are computationally comparable to those of the LFT gain-scheduling approaches in [7], [8]. Note also that for both techniques, the most computationally demanding step comes from the inversion of the term $I - X(\theta)Y(\theta)$, typically a large matrix. It is sometimes possible to exploit rank deficiencies in the X_i 's and the Y_i 's to further reduce computational efforts, for instance, by using inversion with rank correction formulas. A simple case that does not require the inversion of $I - X(\theta)Y(\theta)$ is when the LFT system (21) depends on a single nonlinear function $\rho_1(\theta)$. Noting that

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix}^{-1} = \begin{bmatrix} (X - Y^{-1})^{-1} & (I - YX)^{-1} \\ (I - XY)^{-1} & (Y - X^{-1})^{-1} \end{bmatrix}$$

the inverse of $I - X(\theta)Y(\theta)$ can be computed from the lower-left block in the above expression. Moreover, if we assume without restriction that zero is in the image set of $\rho_1(\cdot)$, then

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} = \begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix} + \rho_1 \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix} > 0, \quad (24)$$

The positive condition in (24) ensures that the matrices

$$\begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix}; \quad \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}$$

are simultaneously diagonalizable. Hence there exists a congruence transformation T and a diagonal matrix Λ_1 , computed off-line, such that for any value of the map $\rho_1(\cdot)$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix}^{-1} = T(I + \rho_1 \Lambda_1)^{-1} T^T. \quad (25)$$

Therefore, a cheap way of computing $(I - XY)^{-1}$ at each sample of time is simply to invert the diagonal matrix in (25) and perform multiplications of corresponding blocks.

Since they offer complementary advantages, the techniques described above can be used together to yield a more effective methodology. Confirmed by practical experience, the following rules have proven useful.

- 1) All necessary tunings, requiring repeated computations should be based on the less costly projected technique.
- 2) The procedure is completed by running the basic technique, for controller implementation purposes.

Though the last phase may be very slow, it is run only once in the whole design process.

A. Bypassing the Gridding Phase

As discussed earlier, there is no direct technique to overpass the gridding phase, hence making the design more direct. Under special circumstances, LFT, affine, or polynomial parameter dependence of data and variables or polytopic approximation of the original plant, techniques such as the \mathcal{S} -procedure or multiconvexity concepts can be used repeatedly

to get a *finite* number of (sufficient) LMI conditions. See the conference version of the present paper [27] and also [14], [15], and references therein.

V. REDUCING CONSERVATISM BY SCALING

As is common in robust control theory, it is possible to further enhance the design procedure by exploiting structural information on the operator relating the signals w and z . Yet, the conditions of Theorems 2.1 or 2.2 also provides robust stability conditions in face of

- multiplicative memoryless time-varying uncertainties

$$w(t) = \Delta(t)z(t), \quad \sigma_{\max}[\Delta(t)] \leq \frac{1}{\gamma}, \quad t \geq 0;$$

- nonexpansive dynamic uncertainty (for instance LTI) [6]

$$w = \Delta(z), \quad \gamma \int_0^T w^T w d\tau \leq \frac{1}{\gamma} \int_0^T z^T z d\tau, \quad T \geq 0.$$

This description, however, ignores potential structures of the operator Δ . We therefore assume, hereafter, that the plant is governed by (2) with w and z subject to

$$[w_1(t)^T, \dots, w_N(t)^T]^T = \Delta(t) [z_1(t)^T, \dots, z_N(t)^T]^T \quad (26)$$

where Δ is a multiplicative memoryless time-varying operator with structure

$$\Delta(t) := \text{diag}[\Delta_1(t), \dots, \Delta_N(t)] \quad (27)$$

and confined to the compact set

$$\sigma_{\max}[\Delta(t)] \leq \frac{1}{\gamma}, \quad \forall t \geq 0. \quad (28)$$

The set of scalings associated with the structure (27) is defined as

$$\mathcal{S}_\Delta := \{S : S > 0, S\Delta(t) = \Delta(t)S, \forall t \geq 0\}. \quad (29)$$

We have implicitly assumed, without restriction, that the problem has been squared so that $m_1 = p_1$ in the subsequent derivations. With these notations in mind, a scaled version of the bounded real lemma can be established.

Lemma 5.1. Structured Robust Stability: The LPV system governed by

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)w \\ z &= C(\theta)x + D(\theta)w \end{aligned} \quad (30)$$

and (26)–(28), with parameter trajectories $\theta(t)$ constrained with (3) and (4), is internally stable whenever there exists a parameter-dependent symmetric matrix $P(\theta)$ and a parameter-dependent scaling $S(\theta)$ in \mathcal{S}_Δ such that $P(\theta) > 0$ and

$$\begin{bmatrix} \dot{P}(\theta) + A(\theta)^T P(\theta) + P(\theta)A(\theta) & P(\theta)B(\theta) & C(\theta)^T \\ B(\theta)^T P(\theta) & -\gamma S(\theta) & D(\theta)^T \\ C(\theta) & D(\theta) & -\gamma S(\theta)^{-1} \end{bmatrix} < 0 \quad (31)$$

holds for all admissible values of the parameter vector θ and of its time derivative $\dot{\theta}$.

Proof: See Appendix \blacksquare

Note that Lemma 5.1 provides robust stability conditions for nonexpansive uncertain operators Δ_i . These conditions also guarantee a robust L_2 -gain performance bound γ with respect to any input–output channel (w_i, z_i) , with the remaining channels corresponding to uncertainties of the form described earlier. See, for instance, [6] for details. Other Extensions to repeated-scalar uncertainties $\Delta_i = \delta_i I_{s_i}$ are straightforward. Also important is the fact that the scaling matrix $S(\theta)$ is allowed to vary with θ . Hence, the conservatism of the gain-scheduling technique can potentially be reduced for all values of θ independently. As an immediate consequence of the above Lemma, the gain-scheduling techniques in Theorems 2.1–2.3 are readily extended to handle the structural constraint (26) and (27). Such extensions simply follow from the following substitutions:

$$\begin{bmatrix} -\gamma I & \star \\ D_{11} + D_{12}D_KD_{21} & -\gamma I \end{bmatrix} \rightarrow \begin{bmatrix} -\gamma S & \star \\ D_{11} + D_{12}D_KD_{21} & -\gamma S^{-1} \end{bmatrix}$$

in Theorems 2.1 and 2.3 and

$$\begin{bmatrix} -\gamma I & D_{11}^T \\ D_{11} & -\gamma I \end{bmatrix} \rightarrow \begin{bmatrix} -\gamma S & D_{11}^T \\ D_{11} & -\gamma S^{-1} \end{bmatrix}$$

$$\begin{bmatrix} -\gamma I & D_{11} \\ D_{11}^T & -\gamma I \end{bmatrix} \rightarrow \begin{bmatrix} -\gamma S^{-1} & D_{11} \\ D_{11}^T & -\gamma S \end{bmatrix}$$

in Theorem 2.2.

Owing to the dependence of the modified characterizations on both S and S^{-1} , they are no longer standard LMI problems. Such problems are in the class of LMI problems with rank reduction constraints hence difficult to solve. Here, we adopted a simple computational scheme in the spirit of μ synthesis techniques.

Recall that the LMI (7), with the scaling $S(\theta)$ in place, can take the form given in (32). Note that this is an LMI in the variables S^{-1} , Y , \hat{A}_K , and \hat{C}_K provided that the variables X , \hat{B}_K , and D_K are maintained fixed. Therefore, coupled with Theorem 2.1, this result suggests a first scheme to compute a best possible S^{-1} . Now, our goal is to obtain the counterpart of the projected characterization Theorem 2.2 for computing S^{-1} . In view of the discussion of Section IV, this will be at a reduced computational cost. Interestingly, such a scheme cannot be directly derived from the LMI's (12) and (13). We shall make use of a partially projected characterization, as shown in (32) at the bottom of the page.

Applying the Projection Lemma [22], with respect to the (2, 1) free term in (32), equivalent LMI conditions are shown in (33) and (34). Assuming now that (33) holds and projecting again with respect to the controller variable \hat{C}_K in (34), an equivalent LMI condition is obtained in (35). Gathered together with (14), the matrix inequalities (33) and (35) form an LMI system in the variable Y and S^{-1} with fixed X , \hat{B}_K , and D_K . It turns out that (8) and (32) on one side or the conditions (14), (33), and (35) on the other side form the basis of two possible schemes for iteratively reducing the conservatism of the gain scheduling techniques of Section II. Such schemes proceed as follows.

- Step 0) Setting $S(\theta) = I$, minimize γ with the basic or the projected technique.
- Step 1) Compute a scaling $S(\theta)^{-1}$ minimizing γ with the help of (8) and (32) or alternatively (14), (33), and (35).
- Step 2) With $S(\theta)$ being fixed, perform a gain-scheduling synthesis with the basic or projected technique.
- Step 3) Iterate over Steps 1) to 3) until convergence.

Then we have (33)–(35), shown at the bottom of the page.

As before, we are using a grid of the set Θ to perform the optimizations and a user-defined functional dependence of

$$\begin{bmatrix} \dot{X} + XA + \hat{B}_K C_2 + (\star) & \star & \star & \star \\ \hat{A}_K^T + A + B_2 D_K C_2 & -\dot{Y} + AY + B_2 \hat{C}_K + (\star) & \star & \star \\ S^{-1}(XB_1 + \hat{B}_K D_{21})^T & S^{-1}(B_1 + B_2 D_K D_{21})^T & -\gamma S^{-1} & \star \\ C_1 + D_{12} D_K C_2 & C_1 Y + D_{12} \hat{C}_K & (D_{11} + D_{12} D_K D_{21}) S^{-1} & -\gamma S^{-1} \end{bmatrix} < 0. \quad (32)$$

$$\begin{bmatrix} \dot{X} + XA + \hat{B}_K C_2 + (\star) & \star & \star \\ S^{-1}(XB_1 + \hat{B}_K D_{21})^T & -\gamma S^{-1} & \star \\ C_1 + D_{12} D_K C_2 & (D_{11} + D_{12} D_K D_{21}) S^{-1} & -\gamma S^{-1} \end{bmatrix} < 0 \quad (33)$$

$$\begin{bmatrix} -\dot{Y} + AY + B_2 \hat{C}_K + (\star) & \star & \star \\ S^{-1}(B_1 + B_2 D_K D_{21})^T & -\gamma S^{-1} & \star \\ C_1 Y + D_{12} \hat{C}_K & (D_{11} + D_{12} D_K D_{21}) S^{-1} & -\gamma S^{-1} \end{bmatrix} < 0 \quad (34)$$

$$\left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array} \right]^T \left[\begin{array}{cc|c} -\dot{Y} + YA^T + AY & Y C_1^T & \star \\ C_1 Y & -\gamma S^{-1} & \star \\ \hline S^{-1}(B_1 + B_2 D_K D_{21})^T & S^{-1}(D_{11} + D_{12} D_K D_{21})^T & -\gamma S^{-1} \end{array} \right] \left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array} \right] < 0 \quad (35)$$

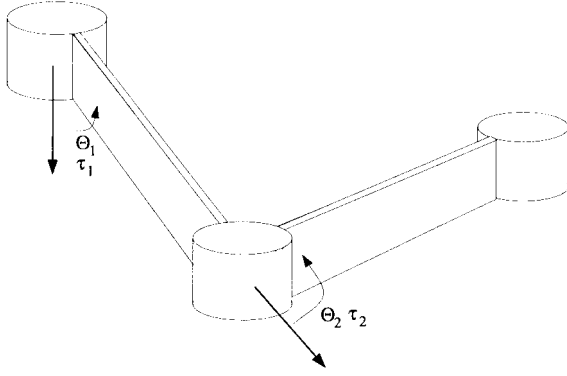


Fig. 1. Two-link flexible manipulator.

S^{-1} on θ . For LFT plants of the form (21), a practical choice is the affine expansion

$$S^{-1}(\theta) := \Sigma_0 + \sum_{i=1}^L \rho_i(\theta) \Sigma_i.$$

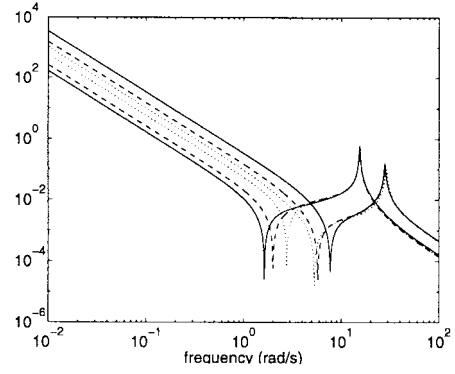
It is important to mention that the previously described iterative procedure, though not giving a global solution to the problem, has proven very efficient in practice. A demonstration is given in the following section. Unlike the standard $D-K$ iteration procedure, it involves not only scalings and Lyapunov variables, but also some controller variables in the same optimization step. In our opinion, this is a central factor favoring convergence, both in speed and accuracy.

VI. CONTROL OF A TWO-LINK FLEXIBLE MANIPULATOR

A. Problem Description

The gain-scheduled control of a two-link flexible manipulator is a nontrivial problem. The dynamics of such a system include both rigid body and lightly damped structural modes. The problem is complicated by uncertainty in the high-frequency dynamics of the system and by the variation of dynamics with manipulator geometry. The first of these complications drives the requirement for closed-loop robust stability while the second drives the requirement for gain-scheduling. In addition, a rapid closed-loop response to position commands is desired. The ability of a control synthesis approach to handle the tradeoffs between robustness, performance, and gain scheduling with the least possible conservatism is thus critical for such a system. For example, if the gain-scheduling parameters are allowed to vary infinitely quickly, closed-loop performance and robustness will suffer. If the uncertainty structure of the design model is not considered, it will be impossible to find a controller which meets design objectives. These tradeoffs are studied and addressed in this section.

SECAFLEX is a two-link flexible planar manipulator driven by geared dc motors, used as a laboratory platform for control-structure interaction experiments at CERT-ONERA in Toulouse, France. The two flexible members are homogeneous beams. There is a concentrated mass at the elbow due to the dc motor and a concentrated mass at the tip of the second

Fig. 2. $\sigma_i[G(j\omega, \theta_2)]$ for different manipulator geometries $\theta_2 = 0$: solid; $\theta_2 = \frac{\pi}{2}$: dashed; $\theta_2 = \pi$: dotted.

beam which is the payload. The modeling of the manipulator has been studied extensively [28]. A simple drawing of the two-link manipulator is shown in Fig. 1.

θ_1 and θ_2 are the shoulder and elbow joint angles, respectively. τ_1 and τ_2 are the corresponding control torques. The second-order form of the manipulator equations of motion are

$$M(\theta_2)\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Fu(t) \quad (36)$$

where M is the inertia matrix, D is the damping matrix, and K is the stiffness matrix. $u(t)$ is the input vector, $u = (\tau_1 \ \tau_2)^T$. Due to the variable geometry of the system, the inertia matrix is a function of the second joint angle, θ_2 . This dependence causes significant changes in the response of the system to input torques over the range of possible configurations, $\theta_2 \in [0, \pi]$ (rad). If we consider an output vector, $y = (\theta_1 \ \theta_2)^T$, then we can define the transfer function from u to y as $G(s, \theta_2)$. With the parameter θ_2 frozen in time, Fig. 2 illustrates the variation of the manipulator dynamics with geometry by showing the singular values of $G(s, \theta_2)$ at three different values of θ_2 .

The numerical values which define this system are as follows. The dependence of the inertia matrix on θ_2 can be expressed as

$$M(\theta_2) = M(\pi/2) + \cos(\theta_2)[M(\pi/2) - M(\pi)] \quad (37)$$

where

$$M(\pi/2) = \begin{bmatrix} 34.7077 & 9.7246 & 23.6398 & 5.9114 \\ 9.7246 & 9.8783 & 9.7246 & 5.9114 \\ 23.6398 & 9.7246 & 17.5711 & 5.9114 \\ 5.9114 & 5.9114 & 5.9114 & 3.7233 \end{bmatrix},$$

$$M(\pi) = \begin{bmatrix} 17.0296 & 0.8856 & 9.7776 & 0.8430 \\ 0.8856 & 9.8783 & 4.7016 & 5.9114 \\ 9.7776 & 4.7016 & 7.5249 & 3.0311 \\ 0.8430 & 5.9114 & 3.0311 & 3.7233 \end{bmatrix} \quad (38)$$

and the damping, stiffness, and control effectiveness matrices are, respectively

$$D = \text{diag}\{0, 0, 0.09, 0.05\},$$

$$K = \text{diag}\{0, 0, 89.1473, 45.6434\}, \quad F = \begin{bmatrix} I_{2 \times 2} \\ 0_{2 \times 2} \end{bmatrix}. \quad (39)$$

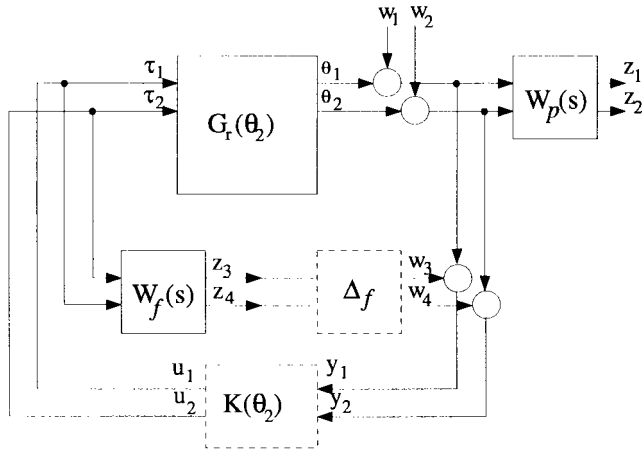


Fig. 3. Synthesis model.

The manipulator equations of motion can be rewritten in first-order LFT form in θ_2 [29]

$$\dot{x}(t) = \begin{bmatrix} 0 & I \\ -M(\theta_2)^{-1}K & -M(\theta_2)^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M(\theta_2)^{-1}F \end{bmatrix} u(t). \quad (40)$$

To provide closed-loop command tracking, a simple weighted minimization of the sensitivity function is used. A frequency-dependent weight W_p penalizes the error e between angular position commands w_1, w_2 , and the system response y . By forcing this weighted sensitivity function to be less than unity, the complementary sensitivity function approaches identity at low frequencies, thus providing good command tracking.

In order to account for uncertainties in high-frequency dynamics, an additive uncertainty model is incorporated into the synthesis model. The additive uncertainty weight is formulated by considering the difference between the full-order geometry dependent model $G(s, \theta_2)$ and some reduced order design model $G_r(s, \theta_2)$, of lower order but still dependent on manipulator geometry. Consider W_f , the additive uncertainty weighting function and Δ_f , a complex uncertainty block, scaled such that $\|\Delta_f\|_\infty < 1$. We can define the error between the full-order and reduced-order models as $E(s, \theta_2)$

$$E(s, \theta_2) = G(s, \theta_2) - G_r(s, \theta_2). \quad (41)$$

The additive uncertainty weight must then provide a frequency domain bound on this error, that is

$$|W_f(j\omega)| \geq \max_{\theta_2} \sigma_{\max}[E(j\omega, \theta_2)], \quad \forall \omega \in [0, \infty]. \quad (42)$$

The LPV design model has the form (2) and is built by combining the above performance and robustness formulations into a single multiobjective synthesis model, Fig. 3.

The design weights used for the following examples are:

$$W_f = \frac{4(s + 0.1)^2}{(s + 100)^2} I_{2 \times 2} \\ W_p = \frac{0.1075(s + 1.066)}{(s + 0.03)} I_{2 \times 2}. \quad (43)$$

TABLE II PERFORMANCE COMPARISONS (γ) WITH IGNORED STRUCTURE	
Basic Technique with $X(\theta_2)$ and $Y(\theta_2)$	3.82
Projected Technique with $X(\theta_2)$ and $Y(\theta_2)$	3.82
LFT Technique in [8]	3.82

It should be noted here that, in the manipulator example, $\theta = \theta_2$ is not an independently evolving external parameter; it is a state of the system. By treating θ_2 as an external parameter, we are actually immersing the “quasilinear” dynamics (40) into the larger class of LPV dynamics (30). Therefore, in guaranteeing stability and performance for a class of parameter trajectories (through bounds on θ_2 and $\dot{\theta}_2$), we ignore the fact that the trajectories themselves are defined by the plant dynamics. The result is thus a degree of conservatism in the design. The resulting controller is in the class of nonlinear controllers since $\theta := \theta_2$ is here a subvector of the measurement $y := (\theta_1, \theta_2)^T$. It is usually referred to as a *scheduled on the plant output* controller [30]. It captures the nonlinearity of the plant (40) in θ_2 . The controller matrices (A_K, B_K, C_K, D_K) are instantaneously updated with respect to the plant’s state θ_2 , as described in Section II.

B. Numerical Examples

Different cases have been investigated which highlight the merits and some of the concerns in using the gain-scheduling methodologies detailed in Sections II–V. Comparisons with existing gain-scheduling techniques are also given.

Case 1: In this first case, the two-block structure associated with robustness and performance is ignored. The basic and projected techniques are applied with both X and Y depending on θ_2

$$X(\theta_2) := X_0 + \cos(\theta_2)X_1 \\ Y(\theta_2) := Y_0 + \cos(\theta_2)Y_1.$$

As discussed earlier, this amounts to assuming that the scheduled variable θ_2 is frozen in time. The corresponding γ levels are compared with the LFT gain-scheduling technique in [8], a technique that puts no bound on the parameter variation rates. The results are presented in Table II. Surprisingly, all techniques give the same result. The achieved value of γ is actually a lower bound that can be checked by performing an H_∞ synthesis on a nominal model ($\theta_2 = \pi/2$). Therefore, if the structure of the problem is ignored, the techniques in this paper may offer no advantages over existing LFT gain-scheduling techniques [7], [8].

Case 2: In this second case, the problem’s uncertainty structure is explicitly taken into account. The performance and robustness objectives are relaxed by introducing a fixed scaling $S := \text{diag}(0.5e-3 I_{2 \times 2}, I_{2 \times 2})$, found by performing a standard μ synthesis with constant scaling on the nominal plant ($\theta_2 = \pi/2$). The application of gain-scheduling techniques to this scaled problem leads to the bound, $\gamma = 0.58$. Results are presented in Table III for different selections of the Lyapunov variables X and Y and assuming θ_2 fixed in time ($\dot{\theta}_2 = 0$).

Both the basic and projected techniques guarantee the same performance bound for all admissible values of θ_2 , provided

TABLE III
PERFORMANCE COMPARISONS OF BASIC AND PROJECTED TECHNIQUES ($\dot{\theta} = 0$)

γ	$X(\theta_2), Y(\theta_2)$	$X(\theta_2), Y_0$	$X_0, Y(\theta_2)$	X_0, Y_0
Basic Technique	0.58	0.95	25.51	380.60
Projected Technique	0.58	0.95	25.35	380.60

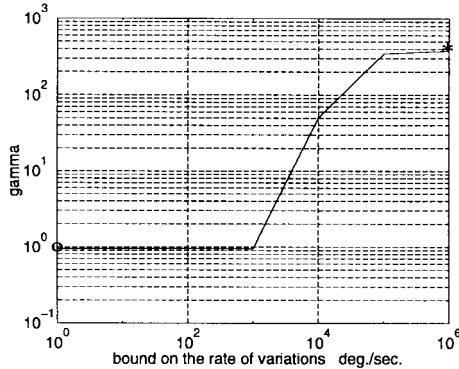


Fig. 4. Performance level γ versus bound on the rate of variations *: value of γ for unbounded derivatives, o: value of γ for frozen parameter.

that both X and Y depend on θ_2 . A slight degradation occurs when the dependence on θ_2 is restricted to X , but the design is still acceptable. Conversely, when Y only depends on θ_2 , both techniques appear very conservative. As predicted in Section III, the selections of columns 2 and 3 are not equivalent and both must be tested to reduce conservatism. The designs in column 4 use fixed X and Y matrices and thus yield much more conservative answers. The corresponding value, $\gamma = 380.60$, was also obtained using the LFT technique in [8]. The consequences of this discussion are threefold.

- Ignoring the uncertainty structure introduces undesirable limitations on the power of advanced gain-scheduling techniques. One must compromise between robustness/performance requirements and gain-scheduling objectives in order to fully benefit from such techniques.
- Tradeoffs can be systematically handled through the use of scalings. Good initial guesses for such scalings can often be found on the basis of a nominal system issued from the LPV plant.
- The basic and the projected techniques give about the same results, the first one being preferable in a real-time implementation perspective.

Case 3: We investigate here the effects of a bound on the rate of variation of θ_2 on the achievable robustness/performance level with the projected synthesis technique. The scaling S is as defined previously, and we assume symmetric bounds

$$|\dot{\theta}_2| < \alpha.$$

It is intuitively clear that increasing the bound on the variation of θ_2 degrades both robustness and performance. For our manipulator system a realistic bound is 100°/s. As can be observed in Fig. 4 which describes γ as a function of α , the proposed techniques perform well for variations of up to an order of magnitude greater than those expected.

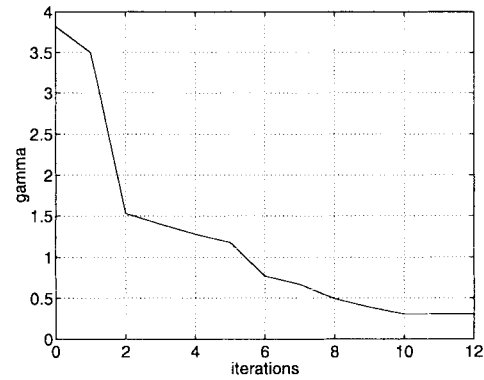


Fig. 5. Evolution of γ versus alternate iterations.

Case 4: In this last case, the iterative schemes proposed in Section V are used to further improve robustness and performance. We exploit the scheme based on the matrix inequalities (14), (33), and (35). Recall that such schemes take advantage of parameter-dependent scalings. According to the definition of S_Δ in (29), the scaling assumes here the special form

$$S^{-1}(\theta_2) := \Sigma_0 + \cos(\theta_2)\Sigma_1.$$

The Lyapunov variables have been selected in the form $X = X_0 + \cos(\theta_2)X_1$ and $Y = Y_0$. Practical validity of the controller is thus ensured from the analysis in Section III. The evolution of γ during the alternate iterations is depicted in Fig. 5. Convergence required 12 elementary steps as described in Section V and led to a best γ value of 0.30. This is obviously the best design among those attempted. The result in Fig. 5 was achieved with the projected technique and the characterizations given in (33) and (35) for the scaling computation. The resulting scaling matrix is

$$S^{-1}(\theta_2) := \begin{bmatrix} 3.363I_2 & 0 \\ 0 & 0.020I_2 \end{bmatrix} + \cos(\theta_2) \begin{bmatrix} 0.074I_2 & 0 \\ 0 & 2e-4I_2 \end{bmatrix}.$$

Finally, using the above scaling, we recomputed the gain-scheduled controller with the basic technique. The same value, $\gamma = 0.30$, was obtained. Since it is of a simpler form and provides satisfactory performance, this last gain-scheduled controller is used in the subsequent analysis and simulations.

C. Frequency and Time-Domain Validations

In this section, the best- γ controller from Case 4 is analyzed. In Fig. 2 we saw how the manipulator dynamics changed with θ_2 . It is now interesting to examine how the resulting gain-scheduled controller varies with manipulator geometry. Fig. 6 shows the singular values of the underlying LTI controllers $K(s, \theta_2)$ at three different values of θ_2 .

We see that the gain-scheduled controller evolves as physically expected, applying higher gains when the manipulator inertias are greater ($\theta_2 = 0$) and reduced gains when the inertias are smaller ($\theta_2 = \pi$). Notice that at frequencies above 10 rad/s, both the manipulator system and the controller are relatively independent of the parameter.

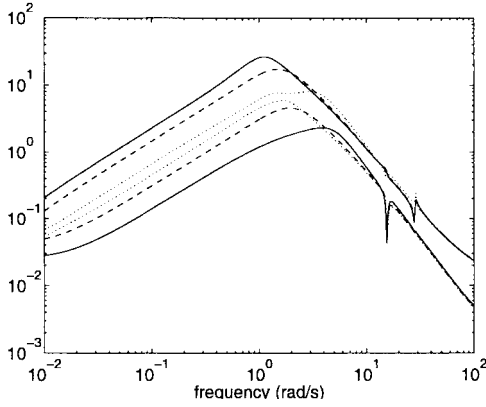


Fig. 6. $\sigma_i[K(j\omega, \theta_2)]$ for different manipulator geometries $\theta_2 = 0$: solid; $\theta_2 = \pi/2$: dashed; $\theta_2 = \pi$: dotted.

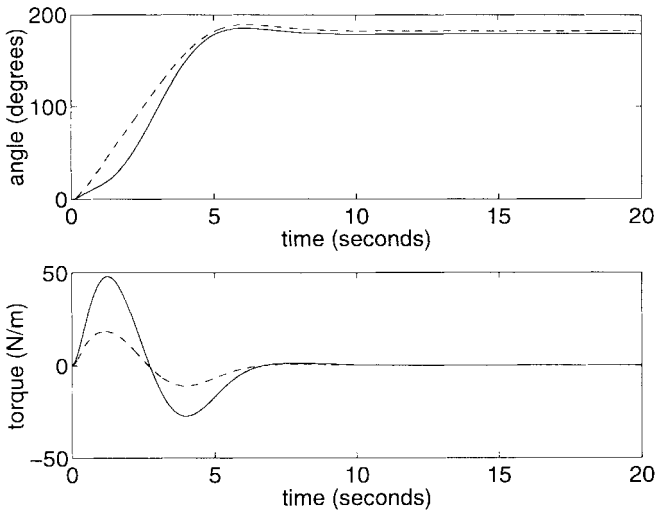


Fig. 7. Nonlinear simulation results solid: θ_1, τ_1 ; dashed: θ_2, τ_2 .

Extensive nonlinear simulations of the response of the closed-loop system to various commands have been performed. For the sake of brevity, we will only present one of these. The high-frequency flexible modes that were removed from the synthesis model are reintroduced in the simulation model. With initial conditions of $\theta_1(0) = 0$ and $\theta_2(0) = 0$, a step command of 180° in both angles is given. This maneuver was chosen to take the manipulator through the entire range of possible dynamics as quickly as possible. The angular responses and corresponding control inputs are depicted in Fig. 7. The rise time (3.5 s), settling time (5 s), and overshoot ($<5\%$) are markedly superior to those observed using either robust LTI controllers or heuristically motivated gain-scheduling approaches [29]. The maneuver is completed without violating the limits on control authority, $|\tau_1| \leq 100$ N/m and $|\tau_2| \leq 20$ N/m.

VII. CONCLUSIONS

Advanced gain-scheduling design approaches for LPV systems have been presented with emphasis on the practical goals of reduced computational burden and ease of implementation. Two complementary techniques for the calculation of such

controllers have been investigated which, when used together, achieve these two objectives. The methodology is completed with a new scaling technique that takes into account the uncertainty structure of multiobjective synthesis problems. The challenging problem of the control of a two-link flexible manipulator is introduced in this context and used to demonstrate the validity of the theoretical solutions.

APPENDIX

Proof of Lemma 5.1: The system governed by (26)–(28) and (30) is stable whenever there exists a quadratic Lyapunov function $V(x, \theta) = x^T P(\theta)x$ such that $P(\theta) > 0$ and

$$\frac{d}{dt} V(x, \theta) < 0 \quad (44)$$

for all admissible trajectories $x(t)$, $w(t)$, $z(t)$, $\theta(t)$, and $\Delta(t)$.

The signals w and z related by (26)–(28) satisfy the quadratic constraints

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} \gamma^{-1} S(\theta) & 0 \\ 0 & -\gamma S(\theta) \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \geq 0, \quad \forall t \geq 0 \quad (45)$$

where S is in \mathcal{S}_Δ and may depend on θ . Therefore, stability is ensured whenever (44) holds for all trajectories $x(t)$, $\theta(t)$, and any pair $[w(t), z(t)]$ characterized by (45). By a \mathcal{S} -procedure argument [31], [32], this is guaranteed whenever

$$\frac{d}{dt} V(x, \theta) - \gamma w^T S w + \gamma^{-1} z^T S z < 0.$$

This expression can be rewritten in the form

$$2(Ax + Bw)^T P x + x^T \dot{P} x + \gamma^{-1}(Cx + Dw)^T \cdot S(Cx + Dw) - \gamma w^T S w < 0$$

which is nothing else than the quadratic form condition associated with the matrix inequality in (31). This completes the proof of the lemma. ■

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REFERENCES

- [1] A. Packard, K. Zhou, P. Pandey, and G. Becker, "A collection of robust control problems leading to LMI's," in *Proc. IEEE Conf. Decision Contr.*, Brighton, U.K., 1991, pp. 1245–1250.
- [2] Y. E. Nesterov and A. S. Nemirovski, *Interior Point Polynomial Methods in Convex Programming: Theory and Applications*, vol. 13 of SIAM Studies in Applied Mathematics. Philadelphia, PA: SIAM, 1994.
- [3] A. Nemirovski and P. Gahinet, "The projective method for solving linear matrix inequalities," in *Proc. Amer. Contr. Conf.*, 1994, pp. 840–844.
- [4] L. Vanderberghe and S. Boyd, "Primal-dual potential reduction method for problems involving matrix inequalities," *Math. Programming Series B*, pp. 205–236, 1995.
- [5] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*. The MathWorks Inc., 1994.
- [6] S. Boyd, L. ElGhaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, vol. 15 of SIAM Studies in Applied Mathematics. Philadelphia: SIAM, 1994.
- [7] A. Packard, "Gain scheduling via linear fractional transformations," *Syst. Contr. Lett.*, vol. 22, pp. 79–92, Oct. 1994.
- [8] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled H_∞ controllers," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 853–864 and p. 1681, May 1995.

- [9] W. M. Lu and J. C. Doyle, "H_∞ control of LFT systems: An LMI approach," in *Proc. IEEE Conf. Decision Contr.*, Tucson, AZ, 1992, pp. 1997–2001.
- [10] ———, "H_∞ control of nonlinear systems: A convex characterization," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1668–1674, Sept. 1995.
- [11] G. Becker, A. Packard, D. Philbrick, and G. Balas, "Control of parametrically-dependent linear systems: A single quadratic Lyapunov approach," in *Proc. Amer. Contr. Conf.*, San Francisco, CA, 1993, pp. 2795–2799.
- [12] P. Apkarian, P. Gahinet, and G. Becker, "Self-scheduled H_∞ control of linear parameter-varying systems: A design example," *Automatica*, vol. 31, pp. 1251–1261, Sept. 1995.
- [13] F. Wu, X. Yang, A. Packard, and G. Becker, "Induced L₂-norm control for LPV system with bounded parameter variations rates," in *Proc. Amer. Contr. Conf.*, Seattle, WA, 1995, pp. 2379–2383.
- [14] P. Gahinet, P. Apkarian, and M. Chilali, "Parameter-dependent Lyapunov functions for real parametric uncertainty," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 436–442, 1996.
- [15] E. Feron, P. Apkarian, and P. Gahinet, "Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1041–1046, July 1996.
- [16] A. Helmersson, "Methods for robust gain-scheduling," Ph.D. dissertation, Linköping University, Sweden, 1995.
- [17] G. Becker, "Parameter-dependent control of an under-actuated mechanical system," in *Proc. IEEE Conf. Decision Contr.*, 1995.
- [18] P. Apkarian, "On the discretization of LMI-synthesized linear parameter-varying controllers," *Automatica*, vol. 33, pp. 655–661, 1997.
- [19] C. Scherer, "Mixed H₂/H_∞ control," *Trends in Control: A European Perspective*, Special Contribution to the ECC'95 ed.
- [20] M. Chilali and P. Gahinet, "H_∞ design with pole placement constraints: An LMI approach," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 358–367, Mar. 1995.
- [21] I. Masubuchi, A. Ohara, and N. Suda, "Robust multiobjective controller design via convex optimization," in *Proc. IEEE Conf. Decision Contr.*, Dec. 1996, pp. 263–264.
- [22] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *Int. J. Robust and Nonlinear Contr.*, vol. 4, pp. 421–448, 1994.
- [23] P. Gahinet, "Explicit controller formulas for LMI-based H_∞ synthesis," in *Proc. Amer. Contr. Conf.*, 1994, pp. 2396–2400.
- [24] G. Becker, "Additional results on parameter-dependent controllers for LPV systems," in *IFAC World Congr.*, San Francisco, CA, 1996.
- [25] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Automat. Contr.*, to be published.
- [26] S. Boyd and L. El Ghaoui, "Method of centers for minimizing generalized eigenvalues," *Linear Algebra Appl.*, vol. 188, pp. 63–111, Apr. 1992.
- [27] P. Apkarian and R. Adams, "Advanced gain-scheduling techniques for uncertain systems," in *Proc. Amer. Contr. Conf.*, June 1997, pp. 3331–3335.
- [28] D. Alazard and J. P. Chretien, "The impact of local masses and inertias on the dynamic modeling of flexible manipulators," in *Int. Symp. Artificial Intell., Robot., Automat. in Space*, Toulouse, France, 1992.
- [29] R. J. Adams, P. Apkarian, and J. P. Chretien, "Robust control approaches for a two-link flexible manipulator," in *3rd Int. Conf. Dynamics Contr. Structures in Space*, Cranfield, U.K., May 1996, pp. 101–116.
- [30] J. F. Shamma and M. Athans, "Analysis of gain scheduled control for nonlinear plants," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 898–907, Aug. 1990.
- [31] V. A. Yakubovich, "The {S}-procedure in nonlinear control theory," *Vestnik Leningrad Univ. Math.*, vol. 4, pp. 73–93, 1977 (in Russian), 1971.
- [32] ———, "Nonconvex optimization problem: The infinite-horizon linear-quadratic control problem with quadratic constraints," *Syst. Contr. Lett.*, vol. 19, pp. 13–22, 1992.



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