Model uncertainty and validation

CA9/CA3 course Nonlinear Control Systems

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Outline

Model uncertainty

Model uncertainty will always be a problem in practice, at least to some extend

Many approaches exist:

- Model validation
- Multi model estimation
- Estimating uncertain parameters
- Alternatives to KF technique e.g. H-infinity based filter design (which will not be further discussed here)

Model validation

There are two very different questions regarding model validation:

- Is the model sufficient?
- Can the model be improved?

The means to answer the questions are also very different:

- To judge if the model is sufficient it must be used in the application it was intended for e.g. model based prediction, filtering or control design. The final result e.g. controller performance must then be assessed to give an answer.
- To see if the model can be improved or not statistical test must be used.

Can the model be improved?

Basic principle: If the model structure is correct and observable then the output prediction error is white noise.

Test for white noise:

- Plot the *residuals* (output prediction errors) $\tilde{y}_{k|k-1}$ by time to inspect for whiteness, instationarity and outliers.
- Test for white noise using a run test based on:

$$
\begin{aligned}
\text{Number of sign changes} &\in B\left(N-1, \frac{1}{2}\right) \\
&\in_{\text{approx.}} \mathrm{N}\left(\frac{N-1}{2}, \frac{N-1}{4}\right)
\end{aligned}
$$

- Plot the residuals in a normal plot (normplot in matlab) to judge the departure from normality and also to see outliers. (This is not about whiteness.)
- Plot the estimated auto correlations function $\hat{\rho}_{\tilde{y}}(k)$ with confidence limits. For approximate test of single (pre-determined) lags use:

$$\hat{\rho}_{\tilde{y}}(k) \in_{\text{approx.}} \mathrm{N}\left(0, \frac{1}{N}\right)$$

- For a test of whiteness use the *Portmanteau* test based on;

$$N \sum_{i=1}^{m} \hat{\rho}_{\tilde{y}}(i)^2 \in_{\text{approx.}} \chi^2(m)$$

Normally m can be chosen between 15 and 25 but less than $N/10$.

Multiple potential models can exist due to:

- A mechanical devise can use one unknown brand of e.g. damper out of a small number of brands.
- A number of potential values of a unknown parameter.
- Models of different complexity normally with different number of states.
- Changing dynamics e.g. in hybrid systems.

Basic assumptions:

- The basic assumptions for KF applies i.e. linearity and Gaussianity.
- Exactly one model out of J is know to be correct.
- An initial a priory probability $p_0(m^j)$ for model j being correct is assumed known.

The probability of correct models after having received measurement $\mathcal{Y}_k$ can be defined by

$$p(m^j|\mathcal{Y}_k) \triangleq \mathrm{P}(\text{Model } j \text{ correct}|\mathcal{Y}_k)$$

Recursive time update of $p(m^j|\mathcal{Y}_k)$

$$
\begin{aligned}
p(m^j|\mathcal{Y}_k) &= \frac{p(m^j,\mathcal{Y}_k)}{p(\mathcal{Y}_k)} = \frac{p(m^j,\mathcal{Y}_{k-1},y_k)}{p(\mathcal{Y}_k)} \\
&= \frac{p(m^j,\mathcal{Y}_{k-1},\tilde{y}_k)}{p(\mathcal{Y}_k)} \\
&= \frac{p(\tilde{y}_k|m^j,\mathcal{Y}_{k-1})p(m^j,\mathcal{Y}_{k-1})}{p(\mathcal{Y}_k)} \\
&= \frac{p(\tilde{y}_k|m^j)p(m^j|\mathcal{Y}_{k-1})p(\mathcal{Y}_{k-1})}{p(\mathcal{Y}_k)} \\
&= \frac{p(\tilde{y}_k|m^j)p(m^j|\mathcal{Y}_{k-1})}{p(\tilde{y}_k)} \\
&= \frac{p(\tilde{y}_k|m^j)}{\sum_{j=1}^J p(m^j|\mathcal{Y}_{k-1})p(\tilde{y}_k|m^j)} p(m^j|\mathcal{Y}_{k-1})
\end{aligned}
$$

Interpretation of time update for $p(m^j|\mathcal{Y}_k)$

$$p(m^j|\mathcal{Y}_k) = \frac{p(\tilde{y}_k|m^j)}{\sum_{j=1}^{J} p(m^j|\mathcal{Y}_{k-1})p(\tilde{y}_k|m^j)} p(m^j|\mathcal{Y}_{k-1})$$

- The update factor is a fraction between the output prediction error probability (density) for model j in question and a weighted average of all output prediction probabilities.
- This factor will be > 1 for the superior models and < 1 for the inferior models.
- Eventually one single model will have $p(m^j|\mathcal{Y}_k) \to 1$ while the rest will have $p(m^j|\mathcal{Y}_k) \to 0$ as $k \to \infty$.

Calculation of $\hat{x}_{k|k}$

Using

$$\mathrm{E}(x) = \mathrm{E}\left(\mathrm{E}(x|y)\right) \Rightarrow \mathrm{E}(x|y) = \mathrm{E}\left(\mathrm{E}(x|y,z)|y\right)$$

we obtain

$$\begin{aligned}\hat{x}_{k|k} = \mathrm{E}(x_k|\mathcal{Y}_k) &= \sum_{j=1}^{J} p(m^j|\mathcal{Y}_k)\,\mathrm{E}(x_k|\mathcal{Y}_k, m^j) \\ &= \sum_{j=1}^{J} p(m^j|\mathcal{Y}_k)\hat{x}^j_{k|k}\end{aligned}$$

Calculation of $P_{k|k}$

Using

$$
\begin{aligned}
\mathrm{V}(x) &= \mathrm{E}(\mathrm{V}(x|y)) + \mathrm{V}(\mathrm{E}(x|y)) \Rightarrow \\
\mathrm{V}(x|y) &= \mathrm{E}(\mathrm{V}(x|y,z)|y) + \mathrm{V}(\mathrm{E}(x|y,z)|y)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
P_{k|k} &= \mathrm{V}(x_k|\mathcal{Y}_k) \\
&= \sum_{j=1}^{J} p(m^j|\mathcal{Y}_k) \left(P^j_{k|k} + (\hat{x}^j_{k|k} - \hat{x}_{k|k})(\hat{x}^j_{k|k} - \hat{x}_{k|k})^{\mathtt{T}} \right)
\end{aligned}
$$

Calculation of $p(\tilde{y}_k|m^j)$

$\tilde{y}_k|m^j \triangleq y_k - \mathrm{E}(y_k|\mathcal{Y}, m^j)$ is simply the output prediction error from the KF based on model j then with P_k^{yj} being the associated covariance we obtain:

$$p(\tilde{y}_k|m^j) = \frac{1}{(2\pi)^{n/2}|P_k^{yj}|^{\frac{1}{2}}} e^{-\frac{1}{2}\tilde{y}_k \left(P_k^{yj}\right)^{-1} \tilde{y}_k^{\mathsf{T}}}\ , \ n = \dim(y)\ ,$$

$$P_k^{yj} = H P_{k|k-1}^j H^{\mathsf{T}} + R_k$$

Model changes with time - adaptive solution

Instead of one model being correct all the time the correct one might switch between the models from time to time.

Approaches to adaptive model selection:

- Use the static method but limit $p(m^j|\mathcal{Y}_k)$ to some suitable small value significantly less than $^1/_J$. This will keep all filters "alive" when a new model becomes superior.
- Introduce models for the model switching e.g. Markov chains and use this in the estimation method. This is quite complicated and approximate methods are normally used.

Parameter estimation

If some parameters are unknown or changes slowly with time they can be estimated.

Main methods:

- Include unknown parameters as states and use EKF/UKF.
- Use system identification (SI) methods preferable maximum likelihood (ML).
  If θ is the parameter vector to be estimated the ML methods is:

  $$\hat{\theta}_{ML} = \arg\max_{\theta} L(\theta) \ , \ L(\theta) = p(\mathcal{Y}_k, \theta) \ ,$$

  where $p(\mathcal{Y}_k, \theta)$ is the probability density function for the output using θ for the unknown parameters.

Include unknown parameters as states and use EKF/UKF

A parameter (vector) θ to be estimated can be given the below state model which is simply added to the original state space model.

$$\theta_{k+1} = A\theta_k + \nu_k \ , \ \nu_k \in \text{NID}(\underline{0}, \Sigma) \ , \ \text{Cov}(\theta_0) = \Pi$$

A, Σ, Π must be chosen by the user to model the behavior of the parameter variation.

- A, Σ, Π can be tailored to any behavior e.g. dependence between parameters.
- A, Σ, Π diagonal simplifies the design.
- $A = I, \Sigma = \underline{\underline{0}}, \Pi > \underline{\underline{0}}$ means parameters are unknown but constant.
- $A = I, \Sigma > \underline{\underline{0}}, \Pi > \underline{\underline{0}}$ means parameters are varying and can end up anywhere.
- $A < I, \Sigma > \underline{\underline{0}}, \Pi > \underline{\underline{0}}$ means parameters are varying but with limiting range.

System identification methods

There exists many methods for different model structures:

Discrete time:

- Linear
  - Black box - estimating all parameters with no particular state space
    - Prediction error methods
    - Sub space methods
  - Grey box - estimating only unknown parameters in a specific state model.
    - Prediction error methods
    - Maximum likelihood methods
- Nonlinear - mostly gray box models

Continuous time:

Mostly gray box models for linear and mildly non linear systems using ML methods or approximations to ML. The likelihood function can be calculated for given parameters using the KF as below.

$$p(\mathcal{Y}_N|x_0) = \prod_{k=1}^{N} p(\tilde{y}_{k|k-1}) \ ,$$

$$p(\tilde{y}_k) = \frac{1}{(2\pi)^{n/2}|P_k^y|^{\frac{1}{2}}} e^{-\frac{1}{2}\tilde{y}_k \left(P_k^y\right)^{-1} \tilde{y}_k^{\mathsf{T}}} \ ,$$

$$P_k^y = H P_{k|k-1} H^{\mathsf{T}} + R_k$$

Start by estimating as few parameters as possible.