Stability of LPV

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Consider LPV in LFT form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \\ w(t) &= \theta(\rho(t))z(t). \end{aligned}$$

LPV in LFT

Suppose that $\hat{G}(s) = C(sI - A)^{-1}B + D$.

Proposition

The following statements are equivalent:

1. *The LTI system G is asymptotically stable.*
2. *The system G has finite L_2-norm.*
3. *The system $\hat{G}$ has finite H_∞-norm.*

Proposition (Bounded-Real Lemma)

$||\hat{G}||_{H_\infty} < \gamma$ if and only if

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ * & -\gamma I & D^T \\ * & * & -\gamma I \end{bmatrix} < 0$$

Furthermore,the above LMI is equivalent to

$$\dot{V}(x(t)) - \gamma u(t)^T u(t) + \gamma^{-1} y(t)^T y(t) < 0$$

for $V(x) = x^TPx$

Small Gain Theorem

Theorem

Suppose that $||M||_{H_\infty} < 1$ *and* $||\Delta||_{H_\infty} \leq 1$, *then the interconnection on the blackboard is asymptotically stable.*

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Proposition

The following statements are equivalent:

1. *The interconnection on the blackboard is stable for all* $\Delta(s)$ *with* $||\Delta||_{H_\infty} \leq 1$.
2. *The transfet function* $M(s)$ *is such that* $||M||_{H_\infty} < 1$

For LPV, we will work with the induced L_2 norm of linear operators (systems).

Suppose that $\theta(\rho)$ is defined for $\rho \in \Delta_\rho$, and let

$w(t) = \theta_\rho(z)(t) = \theta(\rho(t))z(t)$

Let $\mathcal{P} \equiv \{\rho : \mathbb{R}_{\geq 0} \to \Delta_\rho\}$. The L_2-norm of the operator θ_ρ is

$$\sup_{\rho \in \mathcal{P}} ||\theta_\rho||_{L_2 - L_2} = \max_{\xi \in \Delta_\rho} ||\theta(\xi)||_2.$$

Proposition

Suppose that $||\theta_\rho||_2 \leq 1$ for all $\rho \in \mathcal{P}$, then the LPV system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \\ w(t) &= \theta(\rho(t))z(t) \end{aligned}$$

is asymptotically stable if

$$||\hat{G}||_{H_\infty} < 1.$$

- **Generalised form of LPV** for performance specification (from the input signal w to the output signal z):

$$
\begin{aligned}
\dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) + E(\rho(t))w(t) \\
z(t) &= C(\rho(t))x(t) + D(\rho(t))u(t) + F(\rho(t))w(t) \\
y(t) &= \bar{C}(\rho(t))x(t) + \bar{F}(\rho(t))w(t) \\
x(0) &= x_0
\end{aligned}
$$

- Control types:
  1. Unconstrained velocities of parameters

$$\mathcal{P}^{\infty} = \{\rho : \mathbb{R}_{\geq 0} \to \Delta_\rho\}$$

  2. Constrained velocity $\dot{\rho}$

$$\mathcal{P}^{v} = \{\rho : \mathbb{R}_{\geq 0} \to \Delta_\rho | \ \dot{\rho}(t) \in \Delta_v, \ t \geq 0\}$$

- Gain-scheduled state feedback:

$$u(t) = K(\rho(t))x(t).$$

State-feedback with L_2-gain performance constraint

- Recall that that quadratic stability (with Lyapunov function $V(x) = x^T Px$) provides stability for all $u \in \mathcal{P}^\infty$

Proposition (Quadratic Stabilisation by State Feedback)

The LPV in the generalized form is **quadratically** *stabilizable by the state feedback* $u(t) = K(\rho(t))x(t)$. *If and only if there is an n by n matrix* $X > 0$ *and a matrix valued function* $Y : \Delta_\rho \to \mathbb{R}^{m\times n}$ *such that the following LMI holds for all* $\rho \in \Delta_\rho$

$$\begin{bmatrix} \mathrm{He}[A(\rho)X + B(\rho)Y(\rho)] & E(\rho) & [C(\rho)X + D(\rho)Y(\rho)]^T \\ * & -\gamma I & F(\rho)^T \\ * & * & -\gamma I \end{bmatrix} < 0$$

Moreover, the state-feedback control given by

$$u(x) = Y(\rho)X^{-1}x$$

ensures that $||z||_{L_2} \leq \gamma||w||_{L_2} + (\gamma x_0^T X^{-1} x_0)^{1/2}$ *for all* $w \in L_2$ *and all* $\rho \in \mathcal{P}^\infty$.

Polytopic LPV Systems

$$
\begin{aligned}
\dot{x}(t) &= A(\lambda(t))x(t) + Bu(t) + E(\lambda(t))w(t) \\
z(t) &= C(\lambda(t))x(t) + Du(t) + F(\lambda(t))w(t) \\
x(0) &= x_0
\end{aligned}
$$

$$
A(\lambda) = \sum_{i=1}^{N} \lambda_i A_i, \quad E(\lambda) = \sum_{i=1}^{N} \lambda_i E_i,
$$

$$
C(\lambda) = \sum_{i=1}^{N} \lambda_i C_i, \quad F(\lambda) = \sum_{i=1}^{N} \lambda_i F_i
$$

Proposition

The polytopic LPV system is **quadratically** *stabile using a gain-scheduled state-feedback if there are a matrix* $X \in \mathbf{S}^N_{>0}$*, amtrices* $Y_i \in \mathbb{R}^{m \times n}$*,* $i = 1, \dots, N$ *and a scalar* $\gamma > 0$ *such that the LMIs*

$$\begin{bmatrix} \mathrm{He}[A_i X + BY_i] & E_i & [C_i X + DY_i]^T \\ * & -\gamma I & F_i^T \\ * & * & -\gamma I \end{bmatrix} < 0$$

hold for all $i = 1, \dots, N$*. Furthermore, the gain given by*

$$K_i = Y_i X^{-1}$$

ensures that the L_2*-gain of the transfer function* $w \mapsto z$ *is smaller than* γ *for all* $\lambda : \mathbb{R} : > 0 \to \Lambda_N$*.*

Proposition (Robust Stabilisation by State Feedback)

The LPV in the generalized form is **robustly** *stabilizable by the state feedback* $u(t) = K(\rho(t))x(t)$*. If and only if there is a matrix function* $\Delta_\rho \to \mathbf{S}^n_{>0}$ *and a matrix function* $Y : \Delta_\rho \to \mathbb{R}^{m\times n}$ *such that the following LMI holds for all* $(\rho, \nu) \in \Delta_\rho \times \mathrm{vert}(\Delta_v)$

$$\begin{bmatrix} \Sigma(\rho,\nu) & E(\rho) & [C(\rho)X(\rho) + D(\rho)Y(\rho)]^T \\ * & -\gamma I & F(\rho)^T \\ * & * & -\gamma I \end{bmatrix} < 0$$

with

$$\Sigma(\rho,\nu) \equiv \mathrm{He}[A(\rho)X(\rho) + B(\rho)Y(\rho)] + \sum_{i=1}^{N} \nu_i \frac{\partial X(\rho)}{\partial \rho_i}.$$

Moreover, the state-feedback control given by

$$u(x) = Y(\rho)X(\rho)^{-1}x$$

Proposition (Polytopic LPV)

The LPV system is **robustly** *stabilizable using state-feedback if there exist matrices* $Q_i \in \mathbf{S}^N_{>0}$, $i = 1, \ldots, N$, *a matrix* $W \in \mathbb{R}^{n \times n}$ *and a sufficiently large* $\xi > 0$ *such that LMIs*

$$\begin{bmatrix} -\mathrm{He}[W] & Q_i + A_i W + BY_i & W & E_i & (C_i W + DY_i)^T \\ * & -\xi Q_i + \sum_{j=1}^N Q_j \theta_j & 0 & 0 & 0 \\ * & * & -Q_i/\xi & 0 & 0 \\ * & * & * & -\gamma I & F_i^T \\ * & * & * & * & -\gamma I \end{bmatrix} < 0$$

hold for all $i = 1, \ldots, N$, *and all* $\theta \in \mathrm{vert}(\dot{\Lambda}_N)$. *Furthermore the gain-scheduled controller*

$$K_i = Y_i W^{-1}$$

ensures that the L_2*-gain of the transfer function* $w \mapsto z$ *is smaller than* γ *for all* $\lambda \in \Lambda_N$ *and* $\dot{\lambda} \in \dot{\Lambda}_N$.

Dynamic output feedback control

There exists a gain-scheduled dynamic output feedback control

$$\begin{aligned} \dot{x}_c(t) &= A_c(\rho(t))x_c(t) + B_c(\rho(t))y(t) \\ u(t) &= C_c(\rho(t))x_c(t) + D_c(\rho(t))y(t) \end{aligned}$$

of order n that **quadratically** stabilizes the generalized LPV and ensures the L_2 gain of the transfer function $w \mapsto z$ is less than $\gamma > 0$ if and only if there are matrices $X_1 > 0$, $Y_1 > 0$ such that

$$N_Y(\rho)^T \begin{bmatrix} A(\rho)Y_1 + Y_1 A(\rho(t))^T & Y_1 C(\rho)^T & E(\rho) \\ * & -\gamma I & F(\rho) \\ * & * & -\gamma I \end{bmatrix} N_Y(\rho) < 0,$$

$$N_X(\rho)^T \begin{bmatrix} X_1 A(\rho) + A(\rho(t))^T X_1 & X_1 E(\rho)^T & C(\rho) \\ * & -\gamma I & F(\rho)^T \\ * & * & -\gamma I \end{bmatrix} N_X(\rho) < 0,$$

and ...

...

$$\begin{bmatrix} X_1 & I \\ * & Y_1 \end{bmatrix} > 0$$

hold for all $\rho \in \Delta_\rho$ and for full-rank matrices $N_X(\rho), N_Y(\rho)$ defined as

$$\begin{bmatrix} \bar{C}(\rho) & \bar{F}(\rho) & 0 \end{bmatrix} N_X(\rho) = 0 \quad \text{and} \quad N_Y(\rho)^T \begin{bmatrix} B(\rho) \\ D(\rho) \\ 0 \end{bmatrix} = 0.$$