Compositional Stability Analysis

Systems of Systems

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Stability Analysis

We consider two types of autonomous systems

1. Nonlinear system

$$\dot{x} = f(x)$$

   where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$.

2. Linear system

$$\dot{x} = Ax$$

   where $A \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix.

Stability Analysis
Definition: Stability

The equilibrium $x = 0$ of a dynamical system is

- *stable* if, for each $\epsilon > 0$, there is δ such that

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$$||x(0)|| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

- *exponentially stable* if it is stable and there exist constants $c, k, \lambda > 0$ such that

$$||x(t)|| \leq k||x(0)||e^{-\lambda t} \qquad \forall\, ||x(0)|| < c.$$

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let $V : \mathbb{R}^n \to \mathbb{R}$ be a differentiable. The equilibrium point $x = 0$ is

- *stable* if V satisfies

$$V(0) = 0 \tag{1a}$$

$$V(x) > 0 \quad \forall x \in \mathbb{R}^n \backslash \{0\} \tag{1b}$$

$$\frac{\partial V}{\partial x}(x) f(x) \leq 0 \quad \forall x \in \mathbb{R}^n. \tag{1c}$$

Stability Analysis
Lyapunov Stability

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let $V : \mathbb{R}^n \to \mathbb{R}$ be a differentiable. The equilibrium point $x = 0$ is

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- *asymptotically stable* if V satisfies (1) and

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$$\frac{\partial V}{\partial x}(x) f(x) < 0 \qquad \forall x \in \mathbb{R}^n \backslash \{0\}$$

- *exponentially stable* if V satisfies

$$k_1 ||x||^a \leq V(x) \leq k_2 ||x||^a \qquad \forall x \in \mathbb{R}^n$$
$$\frac{\partial V}{\partial x}(x) f(x) \leq -k_3 ||x||^a \qquad \forall x \in \mathbb{R}^n$$

The system

$$\dot{x} = Ax$$

has an asymptotically stable equilibrium point $x = 0$ ***if and only if*** for any positive definite Q there exists a positive definite P such that

$$A^T P + PA = -Q.$$

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The solution to the Lyapunov equation is

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

The following code requires the package yalmip.

```
% System matrix
A = [-1 0;2 -2];
% Lyapunov matrix
P = sdpvar(2,2,'symmetric');
Objective = 1;
Constraint1 = A'*P+P*A<=0;
Constraint2 = P>=0;
optimize([Constraint1,Constraint2],Objective)
```

Stability Analysis
Reflection

Why bother about Lyapunov stability for linear systems, when we can check if A is stable via eigenvalue analysis?

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Reflection

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The optimization-based approach allows composition of requirements - an example is $\mathcal{D}$-stability.

$\mathcal{D}$-Stability
Definition

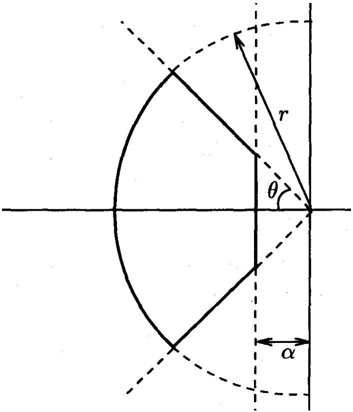
A linear system

$$\dot{x} = Ax$$

is said to be $\mathcal{D}$-stable if all eigenvalues of A lies in $\mathcal{D}$, and $\mathcal{D}$ is a subset of the complex left-half plane.

$\mathcal{D}$-Stability

$S(\alpha, r, \theta)$



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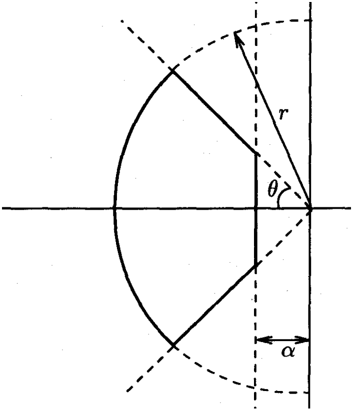

$$x < \alpha < 0, |x + jy| < r, \tan(\theta) < -|y|.$$

A linear system

$$\dot{x} = Ax$$

is $\mathcal{D}$-stable with LMI region $S(\alpha, r, \theta)$ if there exists symmetric matrix X such that

$$\begin{aligned} AX + XA^T + 2\alpha X \prec 0 \\ \begin{bmatrix} -rX & AX \\ XA^T & -rX \end{bmatrix} \prec 0 \\ \begin{bmatrix} \sin\theta(AX + XA^T) & \sin\theta(AX - XA^T) \\ \cos\theta(XA^T - AX) & \sin\theta(AX + XA^T) \end{bmatrix} \prec 0. \end{aligned}$$

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Note that we can impose constraints on the *closed-loop poles*.

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Note that we can impose constraints on the *closed-loop poles*.

This can be combined with, e.g., the minimization of the quadratic cost known from linear quadratic regulation.

Sum of Squares Polynomials

Sum of Squares Polynomials
Question

Consider the polynomial

$$12x^2 - 6.3x^4 + x^6 + 3xy - 12y^2 + 12y^4$$

How can optimization be used to find the minimum value of the polynomial?

Theorem

Fix $p \in \mathcal{R}_{n,2d}$. $p \in \Sigma_{n,2d}$ *if and only if there exists a* $Q \geq 0$ *such that*

$$p(x) = z_{n,d}(x)^T Q z_{n,d}(x).$$

Sum of Squares Polynomials
Example: Decomposition

We consider the polynomial

$$p(x) = x_1^4 + x_1^2 x_2^2 + x_2^4$$

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Is it a sum of squares?

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We consider the polynomial

$$p(x) = x_1^4 + x_1^2 x_2^2 + x_2^4$$

Is it a sum of squares?

Yes, $p(x) = (x_1^2)^2 + (x_1 x_2)^2 + (x_2^2)^2$.

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We consider the polynomial

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To generate a SOS decomposition, we need the following monomials

$$z_{2,2} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

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We must write $p(x)$ on the following form

$$p(x) = z_{n,d}^T Q z_{n,d}$$

with $Q \succeq 0$.

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What is going on?

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Since the SOS decomposition is not unique, we must parameterize the set of SOS decompositions.

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and let $\{Q_i\}_{i=1}^m$ be the set of matrices such that

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Combined we get the set

$$Q_p = \{Q \mid z_{n,d}^T(x) Q z_{n,d}(x) = p(x)\} = \{Q_0 + \sum_{i=1}^m \lambda_i Q_i \mid \lambda_i \in \mathbb{R},\ i = 1, \ldots, m\}$$

Sum of Squares Polynomials
Example: Decomposition

For

$$p(x) = x_1^4 + x_1^2 x_2^2 + x_2^4$$

we have

$$\begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}^T \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ q_2 & q_7 & q_8 & q_9 & q_{10} & q_{11} \\ q_3 & q_8 & q_{12} & q_{13} & q_{14} & q_{15} \\ q_4 & q_9 & q_{13} & q_{16} & q_{17} & q_{18} \\ q_5 & q_{10} & q_{14} & q_{17} & q_{19} & q_{20} \\ q_6 & q_{11} & q_{15} & q_{18} & q_{20} & q_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

Sum of Squares Polynomials
Example: Decomposition

Monomial	Coefficient
1	q_1
x_1	$2q_2$
x_2	$2q_3$
x_1^2	$q_7 + 2q_4$
x_1x_2	$2q_8 + 2q_5$
x_2^2	$q_{12} + 2q_6$
x_1^3	$2q_9$
$x_1^2x_2$	$2q_{10} + 2q_{13}$
$x_1x_2^2$	$2q_{11} + 2q_{14}$
x_2^3	$2q_{15}$
x_1^4	q_{16}
$x_1^3x_2$	$2q_{17}$
$x_1^2x_2^2$	$q_{19} + 2q_{18}$
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x_2^4	q_{21}

Note that the constraints are linear.

Sum of Squares Polynomials

Example: Decomposition

Given polynomial p of degree $2d$, find the affine subspace

$$Q_p = \{Q_0 + \sum_{i=1} \lambda_i Q_i |\ \lambda_i \in \mathbb{R}\}.$$

There exists a SOS decomposition of p if and only if the following LMI is feasible

$$\exists \lambda_i$$

$$\text{s.t. } Q_0 + \sum_{i=1} \lambda_i Q_i \succeq 0.$$

Sum of Squares Polynomials

Finding a Decomposition

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To find the decomposition

- Factorize $Q = L^T L$. Then

$$f(x) = z(x)^T L^T L z(x) = \sum_i (Lz(x))_i^2.$$

- The terms in the SOS decomposition are given by $f_i(x) = (Lz(x))_i$.
- The number of squares is equal to the rank of Q.

Sum of Squares Polynomials
Example: Finding a Decomposition

Given polynomial

$$p(x, y) = 2x^4 + 5y^4 - x^2y^2 + 2x^3y$$

Sum of Squares Polynomials
Example: Finding a Decomposition

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$$
\begin{aligned}
p(x,y) &= 2x^4 + 5y^4 - x^2y^2 + 2x^3y \\
&= \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}
\end{aligned}
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&= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3
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\end{aligned}
$$

The primal SDP is $\exists Q$ s.t.

$$
\begin{aligned}
Q &\succeq 0 \\
q_{11} &= 2 \\
q_{22} &= 5 \\
2q_{23} &= 0 \\
2q_{13} &= 2 \\
q_{33} + 2q_{12} &= -1.
\end{aligned}
$$

We obtain

$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = L^T L$$

Sum of Squares Polynomials
Example: Finding a Decomposition

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Thus

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

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Finally

$$p(x, y) = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2$$

Sum of Squares Polynomials
Example: Finding a Decomposition

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Finally

$$p(x, y) = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2$$

This is a certificate of positivity.

Sum of Squares Polynomials

Relation between SOS and positiveness

Let $P_{n,d}$ be the set of positive polynomials in n variables of degree at most d, and let $\Sigma_{n,d}$ be the set of SOS polynomials in n variables of degree at most d. Then $\Sigma_{n,d} \subseteq P_{n,d}$, with equality holding only in the following cases:

- Bivariate forms: $n = 2$.
- Quadratic forms: $d = 2$.
- Ternary quartics: $n = 3$, $d = 4$.

Algorithmic Stability Analysis

Consider a dynamical system

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^n$ and f_i is polynomial for $i = 1, \ldots, n$.

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where $x \in \mathbb{R}^n$ and f_i is polynomial for $i = 1, \dots, n$.Then the system is stable if

$$\begin{aligned} V(x) &\in \Sigma_{n,d} \\ -\frac{\partial V}{\partial x}(x) f(x) &\in \Sigma_{n,d} \end{aligned}$$

```
% Define state variable
x = sdpvar(1,1);
% Specify maximal degree of Lyapunov function
dV = 2;
% Define the unknown Lyapunov function
[V, cV] = polynomial(x,dV);
% Define the vector field
f = -x^3;
% Specify Lyapunov conditions
con = [ sos(V) ; sos( -jacobian(V,x)*f) ];
options = sdpsettings('solver','mosek');
optimize(con,1,options,[cV]);
% Get coefficients of the Lyapunov function
cV = double(cV);
X = sdpvar(1,1);
vv = monolist(X,dV);
% Compute the symbolic Lyapunov function
Vpoly = vectorize(sdisplay(cV'*vv));
```

Let

$$\mathbb{K} = \{x \in \mathbb{R}^n |\, h_1(x) \geq 0, \dots, h_s(x) \geq 0\}$$

Based on Putinar's Positivestellensatz, we know that the polynomial f is positive on $\mathbb{K}$ if

$$f - \sum_{i=1}^{s} h_i f_i \in \Sigma_n$$

where $f_0, \dots, f_s \in \Sigma_n$.

Compositional Stability Analysis

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Problem Formulation

It may be difficult to find a Lyapunov function of a system with many states. However, a compositional approach to the stability analysis may ease the computations.

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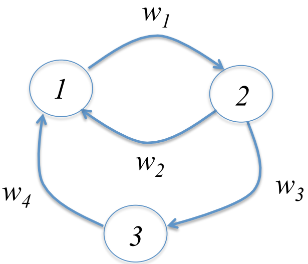
Problem

Given an interconnection of subsystems, find a Lyapunov function that certifies the stability of the interconnected system by the use of distributed optimization.

Compositional Stability Analysis
Example System

To ease the presentation, we consider the following interconnected system throughout the lecture.



The interconnected dynamical system is defined by

$$\begin{cases} \dot{x}_1 = f_1(x_1, w_2, w_4) \\ w_1 = h_1(x_1) \end{cases}, \quad \begin{cases} \dot{x}_2 = f_2(x_2, w_1) \\ (w_2, w_3) = h_2(x_2) \end{cases}, \quad \begin{cases} \dot{x}_3 = f_3(x_3, w_3) \\ w_4 = h_3(x_3) \end{cases}$$

Compositional Stability Analysis

Lyapunov Stability Condition

If there exists a positive definite continuous differentiable function $V : \mathbb{R}^{n_1+n_2+n_3} \to \mathbb{R}$ such that

$$V(0) = 0$$
$$\frac{\partial V}{\partial x}(x) f(x) < 0 \quad \forall x \in \mathbb{R}^{n_1+n_2+n_3} \setminus \{0\}$$

where $x = (x_1, x_2, x_3)$ then the *Example System* is internally stable.

If there exist positive continuous differentiable functions $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, $V_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, $V_3 : \mathbb{R}^{n_3} \to \mathbb{R}$ such that $V_1(0) = 0$, $V_2(0) = 0$, $V_3(0) = 0$ and positive real numbers $\gamma_{11}, \gamma_{12}, \gamma_{14}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{33}, \gamma_{34}$ such that

$$\begin{aligned}
\frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, w_2, w_4) &< -\gamma_{11} w_1^T w_1 + \gamma_{12} w_2^T w_2 + \gamma_{14} w_4^T w_4 \\
\frac{\partial V_2}{\partial x_2}(x_2) f_2(x_2, w_1) &< -\gamma_{22} w_2^T w_2 - \gamma_{23} w_3^T w_3 + \gamma_{21} w_1^T w_1 \\
\frac{\partial V_3}{\partial x_3}(x_3) f_3(x_3, w_3) &< -\gamma_{34} w_4^T w_4 + \gamma_{33} w_3^T w_3
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{21} - \gamma_{11} \le 0, &\quad \gamma_{12} - \gamma_{22} \le 0 \\
\gamma_{33} - \gamma_{23} \le 0, &\quad \gamma_{14} - \gamma_{34} \le 0
\end{aligned}$$

then $(x_1, x_2, x_3) = (0, 0, 0)$ is an asymptotically stable equilibrium point.

Reformulate the problem as

$$\max_{\lambda_1,\ldots,\lambda_4 \geq 0} \min_{\gamma_{ij}, V_i} \lambda_1(\gamma_{21} - \gamma_{11}) + \lambda_2(\gamma_{12} - \gamma_{22}) + \lambda_3(\gamma_{33} - \gamma_{23}) + \lambda_4(\gamma_{14} - \gamma_{34})$$

subject to

$$\frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, w_2, w_4) < -\gamma_{11} h_1(x_1)^T h_1(x_1) + \gamma_{12} w_2^T w_2 + \gamma_{14} w_4^T w_4$$

$$\frac{\partial V_2}{\partial x_2}(x_2) f_2(x_2, w_1) < -h_2(x_2)^T \begin{bmatrix} \gamma_{22} & 0 \\ 0 & \gamma_{23} \end{bmatrix} h_2(x_2) + \gamma_{21} w_1^T w_1$$

$$\frac{\partial V_3}{\partial x_3}(x_3) f_3(x_3, w_3) < -\gamma_{34} h_3(x_3)^T h_3(x_3) + \gamma_{33} w_3^T w_3$$

Procedure:

0. Initialization: Let $k := 0$ and $\lambda^k = \lambda_{\text{init}}$.
1. Solve the subproblems (example given for Subsystem 1)

$$
\begin{aligned}
\text{minimize} \;\; & -\lambda_1\gamma_{11} + \lambda_2\gamma_{12} + \lambda_4\gamma_{14} \\
\text{subject to} \;\; & V_1 > 0 \\
& \frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, w_2, w_4) < -\gamma_{11} h_1(x_1)^T h_1(x_1) + \gamma_{12} w_2^T w_2 + \gamma_{14} w_4^T w_4 \\
& \gamma_{11} > 0,\ \gamma_{12} > 0,\ \gamma_{14} > 0
\end{aligned}
$$

2. Update dual variables $\lambda_i^{k+1} := \lambda_i^k + \alpha_k \delta_i(\lambda^k)$, where α_k is the step size and

$$
\begin{aligned}
\delta_1(\lambda^k) &= \gamma_{11}^*(\lambda^k) - \gamma_{21}^*(\lambda^k), \quad & \delta_2(\lambda^k) &= \gamma_{22}^*(\lambda^k) - \gamma_{12}^*(\lambda^k) \\
\delta_3(\lambda^k) &= \gamma_{23}^*(\lambda^k) - \gamma_{33}^*(\lambda^k), \quad & \delta_4(\lambda^k) &= \gamma_{34}^*(\lambda^k) - \gamma_{14}^*(\lambda^k)
\end{aligned}
$$