• **14.5** Let $x_1 = \theta$, $x_2 = \dot{\theta}$, $u = T$, $a = g/\ell$, $b = k/m$, $c = 1/m\ell^2$, and $\zeta(t) = h(t)/\ell$, to obtain

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a\sin x_1 - bx_2 + cu + \zeta(t)\cos x_1$$

Take $s = x_1 + x_2$. Then,

$$\dot{s} = x_2 - a\sin x_1 - bx_2 + cu + \zeta(t)\cos x_1 = c[u + \delta]$$

where

$$\delta = \frac{1}{c}\left[x_2 - a\sin x_1 - bx_2 + \zeta(t)\cos x_1\right]$$

$$|\delta| \le \left|\frac{a}{c}\right| |x_1| + \left|\frac{1-b}{c}\right| |x_2| + \left|\frac{\zeta(t)}{c}\right| \le 16.1865|x_1| + 1.815|x_2| + 1.1111$$

Take

$$u = -[16.1865|x_1| + 1.815|x_2| + 2]\ \mathrm{sat}\left(\frac{s}{\varepsilon}\right)$$

The trajectory reaches the boundary layer $\{|s| \le \varepsilon\}$ in finite time. Inside the boundary layer, we have $\dot{x}_1 = -x_1 + s$. Taking $V_1 = x_1^2/2$, we obtain

$$\dot{V}_1 = -x_1^2 + x_1 z \le -x_1^2 + |x_1|\varepsilon \le -(1-\theta)x_1^2, \quad \forall\ |x_1| \ge \frac{\varepsilon}{\theta}$$

where $0 < \theta < 1$. Thus, the trajectory reaches the set $\Omega_\varepsilon = \{|x_1| \le \varepsilon/\theta,\ |x_1 + x_2| \le \varepsilon\}$ in finite time. Inside this set,

$$|x_2| = |x_1 + x_2 - x_1| \le |x_1 + x_2| + |x_1| \le (1 + 1/\theta)\varepsilon$$

For $\theta = 0.9$, we have $|x_2| \le 2.11\varepsilon$. Choose ε small enough that $2.11\varepsilon \le 0.01$. In particular, take $\varepsilon = 0.004$.